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Existence of positive solutions of some semilinear elliptic equations with singular coefficients

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In this paper we consider the semilinear elliptic problem in a bounded domain $\Omega \subseteq \mathbb{R}^n$,

$$\begin{aligned} -\Delta u &= \frac{\mu}{|x|^\alpha} u^{2^*_\alpha - 1} + f(x)g(u) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where $\mu \geqslant 0$, $0 \leqslant \alpha \leqslant 2$, $2^*_\alpha := 2(n-\alpha)/(n-2)$, $f : \Omega \to \mathbb{R}^+$ is measurable, $f > 0$ a.e, having a lower-order singularity than $|x|^{-2}$ at the origin, and $g : \mathbb{R} \to \mathbb{R}$ is either linear or superlinear. For $1 < p < n$, we characterize a class of singular functions $\Im_p$ for which the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$ is compact. When $p = 2$, $\alpha = 2$, $f \in \Im_2$ and $0 \leqslant \mu < (\frac{1}{2}(n-2))^2$, we prove that the linear problem has H_0^1-discrete spectrum. By improving the Hardy inequality we show that for f belonging to a certain subclass of $\Im_2$, the first eigenvalue goes to a positive number as μ approaches $(\frac{1}{2}(n-2))^2$. Furthermore, when g is superlinear, we show that for the same subclass of $\Im_2$, the functional corresponding to the differential equation satisfies the Palais–Smale condition if $\alpha = 2$ and a Brezis–Nirenberg type of phenomenon occurs for the case $0 \leqslant \alpha < 2$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 3$, with $0 \in \Omega$. We are concerned with the existence of weak solutions of the following semilinear elliptic problem,

$$\left.\begin{aligned} -\Delta u &= \frac{\mu}{|x|^\alpha} u^{2^*_\alpha - 1} + f(x)g(u) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\mu \geqslant 0$, $0 \leqslant \alpha \leqslant 2$, $2^*_\alpha = 2(n-\alpha)/(n-2)$, $f : \Omega \to \mathbb{R}^+$ is measurable, $f > 0$ a.e. and $g : \mathbb{R} \to \mathbb{R}$ is either linear or superlinear. For $0 \leqslant \alpha \leqslant 2$, 2^*_α is the limiting exponent of the Hardy–Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*_\alpha}(\Omega, |x|^{-\alpha})$. After the pioneering work of Brezis and Nirenberg on the critical exponent problem [4] (the case $\alpha = 0$), it is now well understood that certain lower-order terms can reverse the non-existence and cause positive solutions to exist. Our aim here is to understand how certain singular coefficients $f(x)$ of the lower-order terms can cause the existence of positive solutions.

When $\alpha = 2$, it is well known that for any star-shaped domain Ω and any $\mu \geqslant 0$, the problem

$$\begin{aligned} -\Delta u &= \frac{\mu}{|x|^2}u && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

has no weak solution in $H_0^1(\Omega)$ (see theorem 3.3 in [8] for bounded domains and § 3 in [11] for $\mathbb{R}^n$). The operator $L_\mu := -(\Delta + \mu/|x|^2)$ on $H_0^1(\Omega)$ has discrete spectrum if and only if $\mu < \beta_{n,2} := (\frac{1}{2}(n-2))^2$. This can be seen by using the Hardy–Sobolev inequality

$$\int_\Omega \frac{|u|^2}{|x|^2}\,\mathrm{d}x \leqslant \left(\frac{2}{n-2}\right)^2 \int_\Omega |\nabla u|^2\,\mathrm{d}x \quad \forall u \in H_0^1(\Omega). \tag{1.2}$$

For such L_μ, we study the eigenvalue problem with a singular weight $f(x)$ and the behaviour of the first eigenvalue $\lambda_\mu^1(f)$ as $\mu \to \beta_{n,2}$. Notice that the coercivity of L_μ is lost at $\beta_{n,2}$. Since (1.2) is still valid even after adding the integral $\int_\Omega |u|^2/|x|^\beta\,\mathrm{d}x$ to the left-hand side (see § 3, the lemma on improved Hardy inequality), we deduce that when $f(x)$ is like $|x|^{-\beta}$ near 0 for some $0 \leqslant \beta < 2$, the first eigenvalue $\lambda_\mu^1(f)$ goes to a positive number as $\mu \to \beta_{n,2}$. For the case $\beta = 0$, this inequality has been proved by Brezis and Vazquez in [5]. To establish the non-existence of a positive eigenfunction in $H_0^1(\Omega)$, we show more generally the non-existence of any positive $H_0^1(\Omega)$ supersolution of $-\Delta u \geqslant (\mu/|x|^2)u$ for $\mu > \beta_{n,2}$.

We also study a semilinear problem for L_μ, with singular coefficient. The coefficient $f(x)$ should have lower-order singularity than $|x|^{-2}$ at $x = 0$. If $f(x) = |x|^{-2}$, then the problem (1.1) for $\alpha = 2$ does not admit any weak solution, at least for continuous functions $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$2G(u) \leqslant g(u)u \quad \forall u \geqslant 0$$

in any bounded star-shaped domain (see theorem 3.3 in [8]), where

$$G(u) = \int_0^u g(s)\,\mathrm{d}s.$$

For the case $0 < \alpha < 2$, positive solutions in $H_0^1(\Omega)$ exist for the problem

$$-\Delta u = \frac{u}{|x|^\alpha}|u|^{2_\alpha^* - 2} \quad \text{in } \Omega$$

only when Ω is invariant under scalings centred at $x = 0$, for example, $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$ (see [11, theorem 2]). Using this, an existence result for the Brezis–Nirenberg-type problem

$$\begin{aligned} -\Delta u &= \frac{u^{2^*_\alpha - 1}}{|x|^\alpha} + \lambda u && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

for any bounded domain Ω is also proved in [11]. We study a similar problem with a superlinear lower-order term $g(u)$ and a singular coefficient, where an interesting balance between the singularity and nonlinearity is essential for the existence of a solution.

Our main results are the following.

THEOREM 1.1. *Let $0 \leqslant \mu < \beta_{n,2}$, $\lambda \in \mathbb{R}^+$, $f \in \Im_2$, where*

$$\Im_2 = \Big\{ f : \Omega \to \mathbb{R}^+ \mid \lim_{|x|\to 0} |x|^2 f(x) = 0 \text{ with } f \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\}) \Big\}.$$

The eigenvalue problem

$$\left.\begin{aligned} -\Delta u &= \frac{\mu}{|x|^2} u + \lambda u f(x) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \tag{1.3}$$

admits non-trivial weak solutions in $H^1_0(\Omega)$, corresponding to $\lambda \in (\lambda^k_\mu(f))^\infty_{k=1}$, where

$$0 < \lambda^1_\mu(f) < \lambda^2_\mu(f) \leqslant \lambda^3_\mu(f) \leqslant \cdots \to +\infty.$$

If Ω is $C^{1,1}$, then any weak solution of (1.3) is in $H^1_0(\Omega) \cap W^{2,r}(\Omega)$ for all r, $1 < r < 2n/(n+2)$.

Furthermore, if $f \in \Re_{2,\beta} := \{ f \in \Im_2 \mid 0 < \lim_{|x|\to 0} |x|^\beta f(x) < \infty \}$ for some $0 \leqslant \beta < 2$, then $\lambda^1_\mu(f) \to \lambda(f) > 0$ as $\mu \to \beta_{n,2}$.

REMARK 1.2. In a recent paper [1], it is shown that $\lambda^1_\mu(f) \to \lambda(f) > 0$ for a larger class of weight functions than $\Re_{2,\beta}$. Furthermore, the borderline behaviour of f, for which $\lambda(f)$ is positive or zero, is determined.

THEOREM 1.3. *For $\mu > \beta_{n,2}$, there exists no $u \in H^1_0(\Omega)$ with $u \geqslant 0$, $u \not\equiv 0$, such that*

$$-\Delta u \geqslant \frac{\mu}{|x|^2} u \quad \text{in } \Omega,$$

i.e. there does not exist any non-negative $u \in H^1_0(\Omega)$ with $u \not\equiv 0$ such that

$$\int_\Omega \nabla u \cdot \nabla \phi \, \mathrm{d}x \geqslant \mu \int_\Omega \frac{u}{|x|^2} \phi \, \mathrm{d}x$$

for all $\phi \in H^1_0(\Omega)$, $\phi \geqslant 0$.

REMARK 1.4. Theorem 1.3 is still valid even at the critical level $\mu = \beta_{n,2}$, and a proof can be found in [7].

THEOREM 1.5. *The problem*

$$\left.\begin{aligned} -\Delta u &= \frac{\mu}{|x|^2}u + u^{q-1}f(x) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.4}$$

where $0 \leqslant \mu < \beta_{n,2}$, $f \in \Re_{2,\beta}$ *for some* $0 \leqslant \beta < 2$ *and* $2 < q < 2^*_\beta$, *admits a weak solution in* $H^1_0(\Omega)$.

REMARK 1.6. If Ω is a bounded star-shaped domain with respect to 0 and $q \geqslant 2^*_\beta$, $f(x) = |x|^{-\beta}$ in Ω, $0 \leqslant \beta < 2$, then, by using a Pohozaev-type identity, one can easily show that the problem (1.4) does not admit any solution. Thus 2^*_β is critical for this problem.

REMARK 1.7. It is therefore natural to ask that if one relaxes the condition on the local behaviour of $f(x)$, can there exist a solution to the problem (1.4)? The counterexample below shows that the relaxation of the condition may not be possible. Consider the domain Ω to be the ball $B(0, r)$, $0 < r \ll 1$, $q > 2$, and let $f(x) = |x|^{-2}/|\log|x||$. Then, by using a Pohozaev-type identity, one can easily show that the problem (1.4) does not admit any solution.

THEOREM 1.8. *The problem*

$$\left.\begin{aligned} -\Delta u &= \frac{u^{2^*_\alpha - 1}}{|x|^\alpha} + \lambda u^{q-1}f(x) && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.5}$$

where $0 \leqslant \alpha < 2$, $f \in \Re_{2,\beta}$, *for some* $0 \leqslant \beta < 2$, $2 < q < 2^*_\beta$ *and* $\lambda \in \mathbb{R}^+$, *admits a weak solution*

(a) *for all* $n \geqslant 4$ *and* $\lambda > 0$,

(b) *when* $n = 3$, *in the following different cases:*

  (i) *for all* $1 \leqslant \beta < 2$ *and* $\lambda > 0$;
  (ii) *for all* $0 \leqslant \beta < 1$ *with* $2(2 - \beta) < q < 2^*_\beta$ *and for all* $\lambda > 0$;
  (iii) *for all* $0 \leqslant \beta < 1$ *with* $2 < q \leqslant 2(2 - \beta)$ *and for sufficiently large* $\lambda > 0$.

REMARK 1.9. The main difference between the cases $\alpha = 2$ and $\alpha < 2$ for the problem (1.1) is that the functional corresponding to the partial differential equation satisfies the Palis–Smale (PS) condition at any energy level when $\alpha = 2$, but for the case $\alpha < 2$ it satisfies the PS condition only below a certain energy level.

2. Preliminary results

Let us first recall the well-known Hardy–Sobolev inequality. For $1 < p < n$, $D^{1,p}(\mathbb{R}^n)$ is embedded continuously in $L^p(\mathbb{R}^n, |x|^{-p})$ and

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p}\,\mathrm{d}x \leqslant \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |\nabla u|^p\,\mathrm{d}x \quad \forall u \in D^{1,p}(\mathbb{R}^n), \tag{2.1}$$

where $D^{1,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n;\mathbb{R})$ in the norm $\|u\|_{D^{1,p}} = \|\nabla u\|_{L^p}$ (see [9], for example). If $\Omega = B_R$, a ball, $R > 0$, then also (2.1) is true for any $u \in W_0^{1,p}(B_R)$. If Ω is any general bounded domain, using Schwartz's symmetrization, one can verify the validity of (2.1) for $W_0^{1,p}(\Omega)$. The constant $\beta_{n,p} := ((n-p)/p)^p$ is the best for the Hardy–Sobolev embedding for any Ω.

To prove our main results we will make use of the following two propositions, which are immediate consequences of Hardy and Sobolev embeddings. Although we use only the case $p = 2$, we prove here for p, $1 < p < n$, just to indicate the possible generalization of our results to the p-Laplacian case.

Proposition 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $1 < p < n$. Let $f \in \Im_p$, where*

$$\Im_p := \bigl\{ f : \Omega \to \mathbb{R}^+ \mid \lim_{|x|\to 0} |x|^p f(x) = 0 \textit{ with } f \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\}) \bigr\}.$$

Then the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f\,dx)$ is compact.

Proof. Since $\lim_{|x|\to 0} |x|^p f(x) = 0$, for any $\epsilon > 0$, $\exists \delta > 0$ such that

$$\sup_{B_\delta \subseteq \Omega} |x|^p f(x) \leqslant \epsilon \quad \text{and} \quad f|_{(\Omega \setminus B_\delta)} \quad \text{is bounded,}$$

where $B_\delta = B(0,\delta)$. Now the continuity of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$ will follow from the Hardy, Sobolev and Poincaré inequalities. To show the compactness of the embedding, let $(g_m) \subseteq W_0^{1,p}(\Omega)$ be a bounded sequence. By reflexivity of the space and the Sobolev embedding, we have, for a subsequence (g_{m_k}) of (g_m) as $k \to \infty$,

$$\left.\begin{aligned} g_{m_k} &\rightharpoonup g \quad \text{weakly in } W_0^{1,p}(\Omega), \\ g_{m_k} &\to g \quad \text{strongly in } L^p(\Omega). \end{aligned}\right\} \tag{2.2}$$

Let $C_\delta = \|f\|_{L^\infty(\Omega \setminus B_\delta)}$, so we have

$$\begin{aligned} \int_\Omega |g_{m_k} - g|^p f \, \mathrm{d}x &\leqslant \int_{B_\delta} |g_{m_k} - g|^p f(x) \, \mathrm{d}x + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)} \\ &\leqslant \epsilon \int_{B_\delta} |g_{m_k} - g|^p |x|^{-p} \, \mathrm{d}x + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)}. \end{aligned}$$

Hence, by the Hardy inequality, we have

$$\int_\Omega |g_{m_k} - g|^p f \, \mathrm{d}x \leqslant \epsilon \left(\frac{p}{n-p} \right)^p \|\nabla (g_{m_k} - g)\|^p_{L^p(\Omega)} + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)}.$$

Since $(g_{m_k}) \subseteq W_0^{1,p}(\Omega)$ is bounded,

$$\int_\Omega |g_{m_k} - g|^p f(x) \, \mathrm{d}x \leqslant \epsilon M + C_\delta \|g_{m_k} - g\|^p_{L^p(\Omega)},$$

where $M > 0$ is a constant depending on n and p. By (2.2), we have

$$\lim_{k\to\infty} \int_\Omega |g_{m_k} - g|^p f \, \mathrm{d}x \leqslant \epsilon M.$$

As $\epsilon > 0$ is arbitrary,

$$\lim_{k\to\infty} \int_\Omega |g_{m_k} - g|^p f \,\mathrm{d}x = 0.$$

Hence $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$ is compact. □

REMARK 2.2. The class of functions $f \in \Im_p$ has lower-order singularity than $|x|^{-p}$ at the origin. Here we give some examples of such functions.

(a) Any bounded function.

(b) In a small neighbourhood of 0, f is $|x|^{-\beta}$, $0 < \beta < p$.

(c) $f(x) = |x|^{-p}/|\log|x||$ in a small neighbourhood of 0.

REMARK 2.3. If either $\lim_{|x|\to 0} |x|^p f(x)$ does not exist or it is different from zero, then the embedding need not be compact. For example, take $f(x)$ to be either $c|x|^{-p}|\sin(1/|x|)|$ or $c|x|^{-p}/(1+|x|^p)$, where $c > 0$, then the required limit does not exist in the first case and is $c > 0$ in the second case. Since $|\sin(1/|x|)|$ and $1/(1+|x|^p) \in L^\infty(\Omega)$ and $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, |x|^{-p})$ is not compact, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, f)$, is also non-compact.

PROPOSITION 2.4. *Let Ω be a bounded domain and $1 < p < n$. Let $f \in \Re_{p,\beta}$, where*

$$\Re_{p,\beta} := \Big\{ f \in \Im_p \mid 0 < \lim_{|x|\to 0} |x|^\beta f(x) < \infty \Big\}$$

for some $0 \leqslant \beta < p$. Let $p_\beta^* := p(n-\beta)/(n-p)$. Then the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega, f)$ is

(i) *continuous* $\forall p \leqslant q \leqslant p_\beta^*$,

(ii) *compact* $\forall p \leqslant q < p_\beta^*$.

Proof. (i) The cases $\beta = 0$ and $q = p$ follow from the Sobolev and Hardy's inequalities, respectively. Let $0 < \beta < p$ and $p < q \leqslant p_\beta^*$. Since $f \in \Re_{p,\beta}$, there exists $\delta(f) > 0$ and $C_\delta > 0$ such that $f(x) \leqslant C_\delta/|x|^\beta$ on $B(0,\delta)$. Then for $u \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}\int_\Omega |u|^q f \,\mathrm{d}x &\leqslant C_\delta \int_{B_\delta} \frac{|u|^q}{|x|^\beta} \,\mathrm{d}x + C(\delta) \int_{\Omega\setminus B_\delta} |u|^q \,\mathrm{d}x \\ &\leqslant C_\delta \int_{B_\delta} \left(\frac{|u|^\beta}{|x|^\beta}\right) |u|^{q-\beta} \,\mathrm{d}x + C(\delta) \int_\Omega |u|^q \,\mathrm{d}x.\end{aligned}$$

By the Hölder, Hardy and Poincaré inequalities, we have

$$\begin{aligned}&\int_\Omega |u|^q f \,\mathrm{d}x \\ &\quad\leqslant C_\delta \bigg(\int_{B_\delta} \frac{|u|^p}{|x|^p} \,\mathrm{d}x\bigg)^{\beta/p} \bigg(\int_{B_\delta} |u|^{p(q-\beta)/(p-\beta)} \,\mathrm{d}x\bigg)^{(p-\beta)/p} + C_{n,p,\delta} \|\nabla u\|_{L^p(\Omega)}^q \\ &\quad\leqslant C_{n,p,q,\delta} \|\nabla u\|_{L^p(\Omega)}^\beta \|\nabla u\|_{L^p(\Omega)}^{q-\beta} + C_{n,p,\delta} \|\nabla u\|_{L^p(\Omega)}^q.\end{aligned}$$

Here we have used the fact that

$$\frac{p(q-\beta)}{p-\beta} \leqslant p^* = \frac{np}{n-p}$$

if and only if $q \leqslant p^*_\beta$. Hence

$$\biggl(\int_\Omega |u|^q f \,\mathrm{d}x\biggr)^{1/q} \leqslant C\|\nabla u\|_p,$$

where $C > 0$ depends on n, p, q, δ and Ω. Hence $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega, f)$ is continuous.

(ii) For $q = p$, the embedding is compact by proposition 2.1. So assume $0 < \beta < p$ and $p < q < p^*_\beta$. Let $(g_m) \subseteq W_0^{1,p}(\Omega)$ be bounded. Then there exists a subsequence (g_{m_k}) of (g_m) such that, as $k \to \infty$,

$$\left.\begin{aligned} g_{m_k} &\rightharpoonup g \quad \text{weakly in } W_0^{1,p}(\Omega), \\ g_{m_k} &\to g \quad \text{strongly in } L^r(\Omega) \quad \forall p \leqslant r < p^*. \end{aligned}\right\} \tag{2.3}$$

We have

$$\int_\Omega |g_{m_k} - g|^q f(x)\,\mathrm{d}x \leqslant C_\delta \int_{B_\delta} \frac{|g_{m_k} - g|^q}{|x|^\beta} + C_\delta \|g_{m_k} - g\|^q_{L^p(\Omega)}. \tag{2.4}$$

Choose $r > 0$ such that $q < r < p^*$ and let $p_1 = r/q$. Then $p_1' = r/(r-q)$. By Sobolev embedding, $(g_{m_k} - g) \in L^{p_1}(\Omega)$. If $|x|^{-\beta} \in L^{p_1'}(\Omega)$, then we can use Hölder's inequality in (2.4). Now $|x|^{-\beta} \in L^{p_1'}(\Omega)$ if and only if $\beta p_1' < n$, i.e. if and only if $\beta r/(r-q) < n$, i.e. if and only if $r > nq/(n-\beta)$. Since $q < p(n-\beta)/(n-p)$, we have $nq/(n-\beta) < np/(n-p) = p^*$. Choose $r > q$ such that $nq/(n-\beta) < r < p^*$.

Hence, from (2.3) and (2.4), we have

$$\begin{aligned} &\int_\Omega |g_{m_k} - g|^q f(x)\,\mathrm{d}x \\ &\quad \leqslant C_\delta \biggl(\int_{B_\delta} |g_{m_k} - g|^r\biggr)^{q/r} \biggl(\int_{B_\delta} \frac{1}{|x|^{\beta r/(r-q)}}\biggr)^{(r-q)/r} + C_\delta \|g_{m_k} - g\|^q_{L^q(\Omega)} \\ &\quad \leqslant C\|g_{m_k} - g\|^q_{L^r(\Omega)} \\ &\quad \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Hence the proof follows. □

REMARK 2.5. If $\Omega = \mathbb{R}^n$ and $f(x) = |x|^{-\alpha}$, for $0 \leqslant \alpha < p$, then the continuity of the embedding

$$D^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha}) \tag{2.5}$$

follows exactly as part (a) of proposition 2.4.

REMARK 2.6. In the above proposition, p^*_β is critical in the sense that if $q > p^*_\beta$, then we will not have any such embedding and for $q = p^*_\beta$ we can show easily the non-compactness of this embedding in the same way as in the case of $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

3. The case $\alpha = 2$

Here we prove theorems 1.1 and 1.3.

Proof of theorem 1.1.

Existence. By Hardy's inequality, it follows that the operator $L_\mu = -(\Delta + \mu/|x|^2)$ on $H_0^1(\Omega)$ is positive definite and self-adjoint for all $0 \leqslant \mu < \beta_{n,2}$. Hence, by the Lax–Milgram lemma, for any $g \in H^{-1}(\Omega)$,

$$\begin{aligned} L_\mu u &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

has a unique solution, and $L_\mu^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ exists and is continuous. By proposition 2.1, it follows that

$$\begin{aligned} M_f : H_0^1(\Omega) &\to H^{-1}(\Omega), \\ u &\mapsto uf \end{aligned}$$

is a compact operator. Since L_μ^{-1} is positive definite and self-adjoint,

$$S_\mu = L_\mu^{-1} \circ M_f : H_0^1(\Omega) \to H_0^1(\Omega)$$

is a compact self-adjoint positive-definite operator. Hence the problem can be written as

$$\begin{aligned} S_\mu u &= \lambda^{-1} u \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

and has non-trivial solutions corresponding to $\lambda \in (\lambda_\mu^k(f))$, where

$$0 < \lambda_\mu^1(f) \leqslant \lambda_\mu^2(f) \leqslant \lambda_\mu^3(f) \leqslant \cdots \to +\infty,$$

and they are characterized by the usual Rayleigh quotients.

Let $u \in H_0^1(\Omega)$ be a first eigenfunction, i.e.

$$L_\mu u = \lambda_\mu^1 uf.$$

Since $u^+, u^- \in H_0^1(\Omega)$, we have

$$\begin{aligned} -\Delta u^+ &\geqslant 0 \quad \text{in } \Omega, \\ -\Delta u^- &\geqslant 0 \quad \text{on } \Omega. \end{aligned}$$

Here, f need not be smooth enough. So we use the strong maximum principle for weak solutions (theorem 8.19 in [10]) to conclude that either $u^+ \equiv 0$ a.e. on Ω or $u^+ > 0$ a.e. on Ω. Thus u cannot change sign.

If possible, let $u, v \in H_0^1(\Omega)$ be two orthogonal eigenfunctions corresponding to $\lambda = \lambda_\mu^1$. Using the equations for u and v, we have

$$\int_\Omega uv\left(\frac{\mu}{|x|^2} + \lambda_\mu^1 f\right) \mathrm{d}x = 0,$$

which is a contradiction, because u, v do not change sign in Ω. Hence λ_μ^1 is simple.

Regularity. Without loss of generality, let $v \in H_0^1(\Omega)$ be a solution of (1.3) corresponding to $\lambda = \lambda_\mu^1$. Let

$$h(x) = \frac{\mu}{|x|^2} v(x) + \lambda_\mu^1 v(x) f(x) \quad \text{on } \Omega.$$

Since $|x|^{-2} \in L^r \ \forall r < \frac{1}{2}n$ and $v(x) \in L^{2^*}$, we have

$$v(x)|x|^{-2} \in L^r \quad \forall 1 < r < 2n/(n+2).$$

There exists a constant $c > 0$ such that $|f(x)| < c|x|^{-2}$. Therefore, it follows that $h(x) \in L^r$ for all $r \in (1, 2n/(n+2))$. By theorem 9.15 in [10], it follows that $v \in W^{2,r}(\Omega)$ for all r $1 < r < 2n/(n+2)$. □

REMARK 3.1. This H_0^1 eigenfunction need not be in L^∞. In fact, it need not be even in L^p for $p > 2^*$. The eigenfunctions in a ball, calculated in [6] for the case $f \equiv 1$ and μ in $(0, ((n-2)/2)^2)$, behave like $r^{1-(n/2)+\nu}$ near 0, with $\nu = \sqrt{(((n-2)/2)^2 - \mu)}$. Such eigenfunctions lie in L^p for $p < 2n/(n-2-2\nu)$ only. However, if $f \in C^\infty(\Omega \setminus \{0\})$, then by the standard elliptic regularity theory (see corollary 8.11 in [10]) any eigenfunction will be in $C^\infty(\Omega \setminus \{0\})$.

Limit of λ_μ^1 as $\mu \to \beta_{n,2}$

For $f \in \Im_2$, we have, for all $0 \leqslant \mu < \beta_{n,2}$,

$$\lambda_\mu^1(f) = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega (|\nabla u|^2 - \mu(|u|^2/|x|^2))\,\mathrm{d}x}{\int_\Omega |u|^2 f\,\mathrm{d}x}. \tag{3.1}$$

Let μ_k be increasing to $\beta_{n,2}$ as $k \to \infty$. Then $(\lambda_{\mu_k}^1(f))$ is a decreasing sequence bounded above by λ_0^1, the (first eigenvalue of $-\Delta$ with zero Dirichlet boundary value) and below by 0. Let

$$\lambda_{\mu_k}^1(f) \to \lambda(f) \quad \text{as } \mu_k \to \beta_{n,2}. \tag{3.2}$$

CLAIM 3.2. $\lambda(f) > 0$.

We prove this claim through the following improved Hardy inequality.

LEMMA 3.3 (improved Hardy inequality). *If $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 3$, is a bounded domain, then*

$$\beta_{n,2} \int_\Omega \frac{|u|^2}{|x|^2}\,\mathrm{d}x + C_{p,\beta,\Omega}\left(\int_\Omega \frac{|u|^p}{|x|^\beta}\,\mathrm{d}x\right)^{2/p} \leqslant \int_\Omega |\nabla u|^2\,\mathrm{d}x \tag{3.3}$$

for all $u \in H_0^1(\Omega)$ and for all $0 \leqslant \beta < 2$, $1 < p < 2(n-\beta)/(n-2) = 2_\beta^*$ with $C_{p,\beta,\Omega} > 0$.

Proof. We replace Ω by a ball B_R with $R = (n|\Omega|/\omega_n)^{1/n}$, where $\omega_n = |S^{n-1}|$ and the function u by its symmetric rearrangement. It is well known that the rearrangement of u does not change the L^2-norm, decreases the H_0^1-norm and increases the integral $\int_\Omega u|^2/|x|^\beta$ for all $0 < \beta \leqslant 2$ (see [3]). Hence it is enough to prove (3.3) only for radial functions. And, moreover, by rescaling, we can also take $R = 1$. For $u \in C_0^1(\Omega)$ and u radial, we define

$$v(r) = u(r) r^{(n-2)/2}, \qquad r = |x|$$

to reduce the dimension from n to 2. Now we observe that

$$\int_B |\nabla u|^2 \, \mathrm{d}x - \beta_{n,2} \int_B \frac{u^2}{|x|^2} \, \mathrm{d}x = \omega_n \left[\int_0^1 (v')^2 r \, \mathrm{d}r - (n-2) \int_0^1 v(r) v'(r) \, \mathrm{d}r \right]. \quad (3.4)$$

Since $v \in C_0^1(0,1)$, the last integral in (3.4) is zero and hence we have

$$\int_B |\nabla u|^2 \, \mathrm{d}x - \beta_{n,2} \int_B u^2 |x|^{-2} \, \mathrm{d}x = \omega_n \int_0^1 (v')^2 r \, \mathrm{d}r. \quad (3.5)$$

Now, for $0 \leqslant \beta < 2$, we have

$$\int_B \frac{|u|^p}{|x|^\beta} \, \mathrm{d}x = \omega_n \int_0^1 |v(r)|^p r^{n-\beta-1-p(n-2)/2} \, \mathrm{d}r. \quad (3.6)$$

Notice that for the C^1-radial function $v(r)$,

$$\begin{aligned} |v(r)| &= \left| \int_r^1 v'(t) \, \mathrm{d}t \right| \\ &= \left| \int_r^1 (v'(t) t^{1/2}) t^{-1/2} \, \mathrm{d}t \right| \\ &\leqslant \left(\int_r^1 |v'(t)|^2 t \, \mathrm{d}t \right)^{1/2} \left(\log \frac{1}{r} \right)^{1/2}, \end{aligned}$$

and hence

$$\begin{aligned} &\int_0^1 |v|^p r^{n-\beta-1-p(n-2)/2} \, \mathrm{d}r \\ &\qquad \leqslant \left(\int_0^1 |v'(r)|^2 r \, \mathrm{d}r \right)^{p/2} \int_0^1 r^{n-\beta-1-p(n-2)/2} \left(\log \frac{1}{r} \right)^{p/2} \mathrm{d}r \\ &\qquad \leqslant \left(\int_0^1 |v'(r)|^2 r \, \mathrm{d}r \right)^{p/2} \int_0^\infty \mathrm{e}^{-\{n-\beta-p(n-2)/2\} r} r^{p/2} \, \mathrm{d}r. \end{aligned} \quad (3.7)$$

The second integral on the right-hand side is convergent if and only if

$$p < \frac{2(n-\beta)}{n-2}.$$

Call

$$C_{p,\beta} = (\omega_n)^{1-2/p} \left(\int_0^\infty \mathrm{e}^{-\{n-\beta-p(n-2)/2\} r} r^{p/2} \, \mathrm{d}r \right)^{-2/p}.$$

Then, from (3.6) and (3.7), we get

$$\omega_n \int_0^1 |v'(r)|^2 r \, \mathrm{d}r \geqslant C_{p,\beta} \left(\int_B \frac{|u(x)|^p}{|x|^\beta} \, \mathrm{d}x \right)^{2/p}.$$

Combining this with (3.5), the inequality (3.3) holds for all $u \in C_0^1(\Omega)$. Hence, by the density argument, the inequality (3.3) is true for all $u \in H_0^1(\Omega)$. □

Proof of the claim. Since $2_\beta^* > 2$, as $\beta < 2$, we can take $p = 2$ in (3.3), which reduces to

$$\beta_{n,2} \int_\Omega \frac{|u|^2}{|x|^2} \,\mathrm{d}x + C_{2,\beta} \int_\Omega \frac{|u|^2}{|x|^\beta} \,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

If $f \in \Re_{2,\beta}$, then $0 < l := \lim_{|x|\to 0} |x|^\beta f(x) < \infty$. As $f \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\})$, there exists $\alpha(f) > 0$ such that

$$\alpha(f) \int_\Omega u^2 f(x) \,\mathrm{d}x \leqslant \int_\Omega \frac{u^2}{|x|^\beta} \,\mathrm{d}x.$$

Thus

$$\beta_{n,2} \int_\Omega \frac{|u|^2}{|x|^2} \,\mathrm{d}x + \alpha(f) C_{2,\beta} \int_\Omega |u|^2 f(x) \,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

Hence

$$C(f) = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega (|\nabla u|^2 - \beta_{n,2}(|u|^2/|x|^2)) \,\mathrm{d}x}{\int_\Omega |u|^2 f(x) \,\mathrm{d}x} \geqslant \alpha(f) C_{2,\beta} > 0. \tag{3.8}$$

Since, for all $u \in H_0^1(\Omega)$ and for every $0 \leqslant \mu < \beta_{n,2}$,

$$\lambda_\mu^1(f) \leqslant \frac{\int_\Omega (|\nabla u|^2 - \mu(|u|^2/|x|^2)) \,\mathrm{d}x}{\int_\Omega |u|^2 f(x) \,\mathrm{d}x}, \tag{3.9}$$

we have, for every $u \in H_0^1(\Omega)$,

$$0 \leqslant \lambda(f) \leqslant \frac{\int_\Omega (|\nabla u|^2 - \beta_{n,2}(|u|^2/|x|^2)) \,\mathrm{d}x}{\int_\Omega |u|^2 f(x) \,\mathrm{d}x}, \tag{3.10}$$

and hence

$$\lambda(f) \leqslant C(f). \tag{3.11}$$

To prove the other way, observe that for all $u \in H_0^1(\Omega)$, the right-hand side of (3.9) is greater than the right-hand side of (3.10), and therefore, for each $\mu < \beta_{n,2}$,

$$\lambda_\mu^1(f) \geqslant C(f).$$

Hence

$$\lambda(f) = C(f) > 0. \tag{3.12}$$

□

REMARK 3.4. From the above proof we have seen that in the improved Hardy inequality

$$\beta_{n,2} \int_\Omega \frac{|u|^2}{|x|^2} \,\mathrm{d}x + C_{2,\beta} \int_\Omega \frac{|u|^2}{|x|^\beta} \,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2 \,\mathrm{d}x \tag{3.13}$$

the constant $C_{2,\beta}$ is the best and is the limit of the first eigenvalues $\lambda_\mu^1(|x|^{-\beta})$ as $\mu \to \beta_{n,2}$. For the case $f = 1$ on Ω and for $\mu < \beta_{n,2}$, it is proved in [6] that the first eigenvalue of L_μ is the square of the first zero of the Bessel function J_ν, where $\nu = (\beta_{n,2} - \mu)^{1/2}$ and the limiting eigenvalue goes to the square of the first zero of the Bessel function J_0, which is nothing but $C(1)$ here.

REMARK 3.5. Recently in [1], a further improvement of (3.13), not only for $p = 2$ but for all $1 < p \leqslant n$, has been obtained. The precise statement is as follows.

THEOREM 3.6 (cf. [1]). *Let $R \geqslant \sup_{\Omega}(|x|\mathrm{e}^{2/p})$ and $1 < p \leqslant n$, then there exists $C > 0$, depending on n, p and R, such that*

$$\left(\frac{n-p}{p}\right)^p \int_{\Omega} \frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x + C\int_{\Omega} \frac{|u(x)|^p}{|x|^p}\left(\log\frac{R}{|x|}\right)^{-\gamma}\mathrm{d}x \leqslant \int_{\Omega} |\nabla u|^p\,\mathrm{d}x$$

for any $u \in W_0^{1,p}(\Omega)$ if and only if

(i) *$\gamma \geqslant 2$ when $1 < p < n$,*

(ii) *$\gamma \geqslant n$ when $p = n$.*

Proof of theorem 1.3. Let $\mu > \beta_{n,2}$ and, if possible, let there exist non-negative $u \in H_0^1(\Omega)$, $u \not\equiv 0$, such that, for all $\phi \in H_0^1(\Omega)$, $\phi \geqslant 0$,

$$\int_{\Omega} \nabla u \cdot \nabla\phi\,\mathrm{d}x \geqslant \mu\int_{\Omega} \frac{u}{|x|^2}\phi\,\mathrm{d}x. \tag{3.14}$$

Then, by the strong maximum principle, $u > 0$ a.e. in Ω. Since $0 \in \Omega$, there exists $0 < r < 1$ such that $B_r = B(0,r) \subseteq \Omega$. From theorem 1.1, for any $0 \leqslant \beta < 2$, there exists $\phi_\beta > 0$ in $H_0^1(B_r)$ and $\lambda(\beta) > 0$ such that

$$-\Delta\phi_\beta = \frac{\lambda(\beta)}{|x|^\beta}\phi_\beta \quad \text{in } B_r. \tag{3.15}$$

It is easy to check that $\lambda(\beta)$ is a decreasing function on $[0,2)$. By Hardy's inequality and (3.15), we have $\lambda(\beta) \geqslant \beta_{n,2}$ for any $\beta < 2$. Hence

$$\lim_{\beta\to 2} \lambda(\beta) \geqslant \beta_{n,2}. \tag{3.16}$$

Since (3.14) is true for any $\phi \in H_0^1(B_r)$, $\phi \geqslant 0$, we get

$$-\int_{B_r} u\Delta\phi_\beta\,\mathrm{d}x + \int_{\partial B(0,r)} \frac{\partial\phi_\beta}{\partial\nu}\,\mathrm{d}\sigma \geqslant \mu\int_{B_r} \frac{u}{|x|^2}\phi_\beta\,\mathrm{d}x.$$

From the above inequality and (3.15),

$$\lambda(\beta)\int_{B_r} \frac{u}{|x|^\beta}\phi_\beta\,\mathrm{d}x + \int_{\partial B(0,r)} \frac{\partial\phi_\beta}{\partial\nu}\,\mathrm{d}\sigma \geqslant \mu\int_{B_r} \frac{u}{|x|^2}\phi_\beta\,\mathrm{d}x.$$

By Hopf's lemma, $\partial\phi_\beta/\partial\nu < 0$ on $\partial B(0,r)$, and hence we have

$$\mu < \lambda(\beta) \quad \text{for all } \beta \in [0,2). \tag{3.17}$$

Since, for any $\beta < 2$,

$$\lambda(\beta) \leqslant \frac{\int_{B_r} |\nabla v|^2\,\mathrm{d}x}{\int_{B_r} v^2/|x|^\beta\,\mathrm{d}x}$$

for all $v \in H_0^1(B_r)$, we conclude that

$$\lim_{\beta\to 2} \lambda(\beta) \leqslant \beta_{n,2}. \tag{3.18}$$

Hence, from (3.16) and (3.18), we obtain

$$\lim_{\beta\to 2}\lambda(\beta)=\beta_{n,2}. \tag{3.19}$$

Since $\mu \geqslant \beta_{n,2}$, equations (3.17) and (3.19) lead to a contradiction. Hence the theorem. $\square$

Proof of theorem 1.5. For $0 \leqslant \mu < \beta_{n,2}$, we define $J_\mu : H_0^1(\Omega) \to \mathbb{R}$ by

$$J_\mu(u)=\frac{1}{2}\int_\Omega\left(|\nabla u|^2-\mu\frac{|u|^2}{|x|^2}\right)\mathrm{d}x-\frac{1}{q}\int_\Omega |u|^q f(x)\,\mathrm{d}x \quad \forall u\in H_0^1(\Omega).$$

By proposition 2.4 and Hardy's inequality, it follows that $J_\mu \in C^1(H_0^1(\Omega),\mathbb{R})$ and, for any $u, v \in H_0^1(\Omega)$, we have

$$\langle DJ_\mu(u),v\rangle=\int_\Omega\left[\nabla u\cdot\nabla v-\mu\frac{u}{|x|^2}v-|u|^{q-2}uvf(x)\right]\mathrm{d}x.$$

To get a non-trivial solution of (1.4), we look for critical points of J_μ. We first show that J_μ satisfies the geometric conditions of the mountain-pass lemma and the PS condition. We will take $\|\nabla u\|_{L^2(\Omega)}$ as the norm in $H_0^1(\Omega)$.

(i) Clearly, $J_\mu(0) = 0$.

(ii) To prove that there exists $\delta > 0$ and $\rho > 0$ such that $J_\mu(u) > \rho$ wherever $\|u\|_{H_0^1(\Omega)} = \delta$, observe that

$$J_\mu(u)\geqslant \tfrac{1}{2}(1-\beta_{n,2}^{-1}\mu)\|u\|^2_{H_0^1(\Omega)}-\frac{1}{q}\int_\Omega |u|^q f(x)\,\mathrm{d}x. \tag{3.20}$$

By proposition 2.4 and (3.20), we have, for all $u \in H_0^1(\Omega)$,

$$J_\mu(u)\geqslant C_1\|u\|^2_{H_0^1(\Omega)}-C_2\|u\|^q_{H_0^1(\Omega)}, \tag{3.21}$$

where $C_1 = \frac{1}{2}(1-\mu\beta_{n,2}^{-1}) > 0$ and $C_2 > 0$ are constants. So, for sufficiently small $\delta > 0$, there exists $\rho > 0$ such that $J_\mu(u) > \rho$ whenever $\|u\|_{H_0^1(\Omega)} = \delta$.

(iii) Since $q > 2$, from (3.21), it follows that $\exists v \in H_0^1(\Omega)$ with $\|v\|_{H_0^1(\Omega)} > \delta$ such that $J_\mu(v) < 0$. Hence J_μ satisfies the geometric conditions of the mountain-pass theorem.

To prove J_μ satisfies the PS condition, let $(u_m) \subseteq H_0^1(\Omega)$ be a PS sequence at some level $C \in \mathbb{R}^+$, i.e. as $m \to \infty$,

$$J_\mu(u_m)\to C,\quad DJ_\mu(u_m)\to 0\quad \text{strongly in } H^{-1}(\Omega). \tag{3.22}$$

Observe that

$$J_\mu(v)-\tfrac{1}{2}\langle DJ_\mu(v),v\rangle=\left(\frac{1}{2}-\frac{1}{q}\right)\int_\Omega |v|^q f\,\mathrm{d}x\quad \forall v\in H_0^1(\Omega). \tag{3.23}$$

This, together with (3.22), implies that (u_m) is bounded in $H_0^1(\Omega)$. By proposition 2.4 and reflexivity, there exists a subsequence (u_{m_k}) of (u_m) such that, as $k \to \infty$,

$$\left.\begin{aligned} u_{m_k} &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ u_{m_k} &\rightharpoonup u \quad \text{weakly in } L^2(\Omega, |x|^{-2}), \\ u_{m_k} &\to u \quad \text{strongly in } L^q(\Omega, f). \end{aligned}\right\} \tag{3.24}$$

Since $J_\mu(u_{m_k}) \to C$ as $k \to \infty$, we have

$$\begin{aligned} C &= J_\mu(u_{m_k}) + o(1), \quad \text{where } o(1) \to 0 \quad \text{as } k \to \infty \\ &= \tfrac{1}{2}(\|\nabla u_{m_k}\|_2^2 - \mu \|u_{m_k}\|^2_{L^2(|x|^{-2})}) - \frac{1}{q}\|u_{m_k}\|^q_{L^q(f)} + o(1) \\ &= \tfrac{1}{2}(\|\nabla u\|_2^2 - \mu \|u\|^2_{L^2(|x|^{-2})}) - \frac{1}{q}\|u\|^q_{L^q(f)} \\ &\qquad + \tfrac{1}{2}(\|\nabla (u_{m_k} - u)\|_2^2 - \mu \|u_{m_k} - u\|^2_{L^2(|x|^{-2})}) + o(1) \\ &\geqslant J_\mu(u) + \tfrac{1}{2}(1 - \mu\beta_{n,2}^{-1})\|\nabla(u_{m_k} - u)\|_2^2 + o(1). \end{aligned} \tag{3.25}$$

We will now prove that $J_\mu(u) = C$, so that from (3.25) we have

$$\|\nabla(u_{m_k} - u)\|^2_{L^2(\Omega)} \to 0 \quad \text{as } k \to \infty,$$

and hence J_μ satisfies the PS condition. From (3.22) and (3.24), we have

$$\langle DJ_\mu(u), v\rangle = 0 \quad \forall v \in H_0^1(\Omega). \tag{3.26}$$

Therefore,

$$\begin{aligned} C &= \lim_{k\to\infty} J_\mu(u_{m_k}) \\ &= \lim_{k\to\infty} [J_\mu(u_{m_k}) - \tfrac{1}{2}\langle DJ_\mu(u_{m_k}), u_{m_k}\rangle] && \text{by (3.22)} \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega |u|^q f \, dx && \text{by (3.23) and (3.24)} \\ &= J_\mu(u) - \tfrac{1}{2}\langle DJ_\mu(u), u\rangle && \text{by (3.23)} \\ &= J_\mu(u) && \text{by (3.26).} \end{aligned}$$

Now define

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = v\}.$$

Then $d = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} J_\mu(\gamma(t)) \geqslant \rho > 0$. By the mountain-pass theorem of Ambrosetti and Rabinowitz [2], d is a critical value, i.e. there exists $u \in H_0^1(\Omega)$ such that $J'_\mu(u) = 0$ and $J_\mu(u) = d > 0$. Since $J_\mu(u) = d = J_\mu(|u|)$, the problem (1.4) admits a solution. □

4. Proof of theorem 1.8 (the case $0 \leqslant \alpha < 2$)

For $\lambda \geqslant 0$, define $J_\lambda : H_0^1(\Omega) \to \mathbb{R}$ by

$$J_\lambda(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{2^*_\alpha}\int_\Omega |u|^{2^*_\alpha}|x|^{-\alpha} \, dx - \frac{\lambda}{q}\int_\Omega |u|^q f \, dx \quad \forall u \in H_0^1(\Omega).$$

By proposition 2.4, $J_\lambda \in C^1(H_0^1, \mathbb{R})$, and for any $u, v \in H_0^1(\Omega)$ we have

$$\langle DJ_\lambda(u), v\rangle = \int_\Omega (\nabla u \cdot \nabla v - |u|^{2^*_\alpha - 2} uv|x|^{-\alpha} - \lambda |u|^{q-2} uvf)\,\mathrm{d}x.$$

Any non-trivial critical point of J_λ will give us a solution to (1.5). For $\alpha = 0$ and $\beta = 0$, the existence of a solution to (1.5) was shown by Brezis and Nirenberg [4]. They showed that J_λ satisfies the PS condition below the energy level $(1/n)S^{n/2}$, where S is the best Sobolev constant. In our case, we will get a certain energy level and conclude that J_λ satisfies the PS condition below that level.

For $0 \leqslant \alpha < 2$, the embedding $D^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})$ is continuous (see remark 2.5). Define

$$S_\alpha := \inf_{u \in H^1(\mathbb{R}^n)} \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^n)}}{\|u\|^2_{L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})}}. \tag{4.1}$$

Since $f \in \Re_{2,\beta}$, there exist $r > 0$ and $K_1(f)$, $K_2(f) > 0$ such that

$$K_1|x|^{-\beta} \leqslant f(x) \leqslant K_2|x|^{-\beta} \quad \text{on } B(0, r).$$

For any $\epsilon > 0$, the family u_ϵ on $\mathbb{R}^n$, defined by

$$u_\epsilon(x) := \frac{C_\epsilon}{(\epsilon^{2-\alpha} + |x|^{2-\alpha})^{(n-2)/(2-\alpha)}}, \qquad C_\epsilon = [\epsilon^{2-\alpha}(n-2)(n-\alpha)]^{(n-2)/2(2-\alpha)}, \tag{4.2}$$

satisfies

$$-\Delta u_\epsilon = u_\epsilon^{2^*_\alpha - 1}|x|^{-\alpha} \quad \text{in } \mathbb{R}^n$$

and is a minimizer for (4.1). Hence we have

$$\|\nabla u_\epsilon\|^2_{L^2(\mathbb{R}^n)} = \|u_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\mathbb{R}^n, |x|^{-\alpha})} = S_\alpha^{(n-\alpha)/(2-\alpha)}. \tag{4.3}$$

Choose $\phi \in C_0^\infty(\Omega)$ such that $\phi \equiv 1$ on B_r. For $\epsilon > 0$, define the family $\tilde{u}_\epsilon$ on Ω by

$$\tilde{u}_\epsilon(x) := \phi(x) u_\epsilon(x).$$

Then, by (4.2) and (4.3),

$$\|\nabla \tilde{u}_\epsilon\|^2_{L^2(\Omega)} = S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-2}), \tag{4.4}$$

$$\|\tilde{u}_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega, |x|^{-\alpha})} = S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-\alpha}). \tag{4.5}$$

Define $v_\epsilon : \Omega \to \mathbb{R}$ by

$$v_\epsilon(x) = \frac{\tilde{u}_\epsilon(x)}{\|\tilde{u}_\epsilon\|_{L^{2^*_\alpha}(\Omega, |x|^{-\alpha})}}, \tag{4.6}$$

so from (4.4) and (4.5) we have

$$\|v_\epsilon\|_{L^{2^*_\alpha}(\Omega, |x|^{-\alpha})} = 1 \tag{4.7}$$

and

$$\|\nabla v_\epsilon\|^2_{L^2(\Omega)} = S_\alpha + O(\epsilon^{n-2}). \tag{4.8}$$

We can show that for $\lambda \geqslant 0$, J_λ satisfies the geometric conditions of the mountain-pass lemma, as in the proof of theorem 1.3. We will use the theorem 2.2 of Brezis and Nirenberg [4] to get a non-trivial critical point for J_λ. We need the following two lemmas, which are similar to theorem 2.1 and lemma 2.1 in [4]. We indicate the proof here, as the calculations are quite different and are important to understand the new phenomenon here, namely the balance between the singularity and the nonlinearity.

LEMMA 4.1. *Let $(u_m) \subseteq H_0^1(\Omega)$ be a sequence such that*

$$J_\lambda(u_m) \to \beta_\lambda < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)} \quad \text{and} \quad DJ_\lambda(u_m) \to 0 \quad \text{strongly in } H^{-1}. \tag{4.9}$$

Then (u_m) is relatively compact in $H_0^1(\Omega)$, i.e. J_λ satisfies the Palis–Smale condition.

Proof. Let us first prove that (u_m) is bounded. From (4.9), we have, for all $m \geqslant k$,

$$\left.\begin{aligned} J_\lambda(u_m) &= \beta_\lambda + o(1), \\ \langle DJ_\lambda(u_m), u_m\rangle &\leqslant o(1)\|u_m\|_{H_0^1(\Omega)}. \end{aligned}\right\} \tag{4.10}$$

Now we have

$$\begin{aligned} &2J_\lambda(u_m) - \langle DJ_\lambda(u_m), u_m\rangle \\ &\qquad = \left(1 - \frac{2}{2_\alpha^*}\right)\int_\Omega |u|^{2_\alpha^*}|x|^{-\alpha}\,\mathrm{d}x + \lambda\left(1 - \frac{2}{q}\right)\int_\Omega |u_m|^q f\,\mathrm{d}x. \end{aligned} \tag{4.11}$$

Since $\lambda \geqslant 0$, $2_\alpha^* > 2$, $q > 2$, by standard arguments (see, for example, lemma 2.3 in [12]), using (4.10) and (4.11), we conclude that (u_m) is bounded in $H_0^1(\Omega)$. By propositions 2.1 and 2.4 and the reflexivity of the spaces, there exists a subsequence (u_{m_k}) of (u_m) such that, as $k \to \infty$,

$$\left.\begin{aligned} u_{m_k} &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ u_{m_k} &\to u \quad \text{strongly in } L^p(\Omega, |x|^{-\alpha}) \quad \forall 2 \leqslant p < 2_\alpha^*, \\ u_{m_k} &\rightharpoonup u \quad \text{weakly in } L^{2_\alpha^*}(\Omega, |x|^{-\alpha}), \\ u_{m_k} &\to u \quad \text{strongly in } L^q(\Omega, f) \quad \forall 2 < q < 2_\beta^*, \\ u_{m_k} &\to u \quad \text{a.e. } [|x|^{-\alpha}\,\mathrm{d}x] \text{ in } \Omega. \end{aligned}\right\} \tag{4.12}$$

We write $\mathrm{d}\mu(x) = \mathrm{d}x/|x|^\alpha$. Since $DJ_\lambda(u_m) \to 0$ strongly in H^{-1}, by (4.12), we have

$$\langle DJ_\lambda(u), v\rangle = 0 \quad \forall v \in H_0^1(\Omega).$$

In particular, $\langle DJ_\lambda(u), u\rangle = 0$, and hence we have from (4.11)

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{2_\alpha^*}\right)\int_\Omega |u|^{2_\alpha^*}\,\mathrm{d}\mu(x) + \lambda\left(\frac{1}{2} - \frac{1}{q}\right)\int_\Omega |u|^q f\,\mathrm{d}x \geqslant 0. \tag{4.13}$$

Again by (4.12), we have

$$\int_\Omega |\nabla u_{m_k}|^2 \,\mathrm{d}x = \int_\Omega |\nabla(u_{m_k} - u)|^2 \,\mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x + o(1), \tag{4.14}$$

$$\int_\Omega |u_{m_k}|^{2^*_\alpha} \,\mathrm{d}\mu(x) = \int_\Omega |u_{m_k} - u|^{2^*_\alpha} \,\mathrm{d}\mu(x) + \int_\Omega |u|^{2^*_\alpha} \,\mathrm{d}\mu(x) + o(1), \tag{4.15}$$

and

$$\int_\Omega (u_{m_k}|u_{m_k}|^{2^*_\alpha-2} - u|u|^{2^*_\alpha-2})(u_{m_k} - u) \,\mathrm{d}\mu(x) = \int_\Omega |u_{m_k} - u|^{2^*_\alpha} \,\mathrm{d}\mu(x) + o(1). \tag{4.16}$$

By (4.14), (4.15), we have

$$J_\lambda(u_{m_k}) = J_\lambda(u) + J_0(u_{m_k} - u) + o(1), \tag{4.17}$$

and by (4.16),

$$\begin{aligned} o(1) &= \langle u_{m_k} - u, DJ_\lambda(u_{m_k})\rangle \\ &= \langle u_{m_k} - u, DJ_\lambda(u_{m_k}) - DJ_\lambda(u)\rangle \\ &= \int_\Omega |\nabla(u_{m_k} - u)|^2 \,\mathrm{d}x - \int_\Omega |u_{m_k} - u|^{2^*_\alpha} \,\mathrm{d}\mu(x) + o(1). \end{aligned} \tag{4.18}$$

Hence it follows from (4.18) that

$$J_0(u_{m_k} - u) = \frac{(2-\alpha)}{2(n-\alpha)} \int_\Omega |\nabla(u_{m_k} - u)|^2 \,\mathrm{d}x + o(1), \tag{4.19}$$

and thus from (4.9), (4.13) and (4.17),

$$J_0(u_{m_k} - u) \leqslant J_\lambda(u_{m_k}) + o(1) \leqslant \beta_\lambda < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha 0} \quad \forall k \geqslant k_0.$$

Combining (4.19) with this, we have $\forall k \geqslant k_0$,

$$\|\nabla(u_{m_k} - u)\|_2^2 < S_\alpha^{(n-\alpha)/(2-\alpha)}, \tag{4.20}$$

and hence

$$(1 - S_\alpha^{-2^*_\alpha/2} \|u_{m_k} - u\|_{H_0^1}^{2^*_\alpha-2}) > 0. \tag{4.21}$$

Now (4.1) implies

$$\|w\|^2_{L^{2^*_\alpha}(\Omega,|x|^{-\alpha})} \leqslant S_\alpha^{-1} \|\nabla w\|_2^2 \quad \forall w \in H_0^1(\Omega).$$

Thus

$$\begin{aligned} &\|u_{m_k} - u\|^2_{H_0^1}(1 - S_\alpha^{-2^*_\alpha/2}\|u_{m_k} - u\|_{H_0^1}^{2^*_\alpha-2}) \\ &\qquad \leqslant \int_\Omega |\nabla(u_{m_k} - u)|^2 \,\mathrm{d}x - \int_\Omega |u_{m_k} - u|^{2^*_\alpha} \,\mathrm{d}\mu(x) \\ &\qquad = o(1). \end{aligned} \tag{4.22}$$

From (4.21) and (4.22), it follows that $u_{m_k} \to u$ strongly in $H_0^1(\Omega)$. □

LEMMA 4.2. *There exists some $v \in H_0^1(\Omega)$, $v \neq 0$, such that*

$$\sup_{t \geqslant 0} J_\lambda(tv) < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)}$$

holds

(a) *for all $n \geqslant 4$ and $\lambda > 0$,*

(b) *when $n = 3$, in the following three cases:*

(i) *for all $1 \leqslant \beta < 2$, $2 < q < 2_\beta^*$ and $\lambda > 0$;*
(ii) *for all $0 \leqslant \beta < 1$ with $2(2-\beta) < q < 2_\beta^*$ and for all $\lambda > 0$;*
(iii) *for all $0 \leqslant \beta < 1$ with $2 < q \leqslant 2(2-\beta)$ and for sufficiently large $\lambda > 0$.*

Proof. Take $\{v_\epsilon\}$ for $\epsilon > 0$ as defined in (4.6). Call $X_\epsilon = \|\nabla v_\epsilon\|_{L^2(\Omega)}^2$,

$$J_\lambda(tv_\epsilon) = \frac{t^2}{2} X_\epsilon - \frac{t^{2_\alpha^*}}{2_\alpha^*} - \frac{\lambda t^q}{q} \int_\Omega |v_\epsilon|^q f \, \mathrm{d}x. \tag{4.23}$$

Since $q, 2_\alpha^* > 2$, from (4.23) it follows that $J_\lambda(tv_\epsilon) \to -\infty$ on $t \to +\infty$. Hence $\sup_{t \geqslant 0} J_\lambda(tv_\epsilon)$ is achieved at some $t_\epsilon > 0$ (if $t_\epsilon = 0$, then we are done). Therefore,

$$X_\epsilon - t_\epsilon^{2_\alpha^*-2} - \lambda t_\epsilon^{q-2} \int_\Omega |v_\epsilon|^q f(x) \, \mathrm{d}x = 0. \tag{4.24}$$

Let $Y_\epsilon = \sup_{t \geqslant 0} J_\lambda(tv_\epsilon) = J_\lambda(t_\epsilon v_\epsilon)$. Therefore, from (4.24), we have

$$t_\epsilon \leqslant X_\epsilon^{1/(2_\alpha^*-2)}$$

and

$$Y_\epsilon \leqslant \frac{(2-\alpha)}{2(n-\alpha)} X_\epsilon^{(n-\alpha)/(2-\alpha)} - \frac{\lambda t_\epsilon^q}{q} \int_\Omega |v_\epsilon|^q f \, \mathrm{d}x.$$

Hence, by (4.8), we have

$$Y_\epsilon \leqslant \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)} + O(\epsilon^{n-2}) - \frac{\lambda t_\epsilon^q}{q} \int_\Omega |v_\epsilon|^q f \, \mathrm{d}x. \tag{4.25}$$

The aim is to show that the third term in (4.25) is small and dominates the second term. We will first show that

$$t_\epsilon \to S_\alpha^{1/(2_\alpha^*-2)} = S_\alpha^{(n-2)/2(2-\alpha)} \quad \text{as } \epsilon \to 0. \tag{4.26}$$

Since $X_\epsilon \to S_\alpha$ as $\epsilon \to 0$, it is enough to prove that

$$\int_\Omega |v_\epsilon|^q f(x) \, \mathrm{d}x \to 0 \quad \text{as } \epsilon \to 0. \tag{4.27}$$

From (4.2), (4.6) and (4.7), we have

$$\int_\Omega |v_\epsilon|^q f(x) \, \mathrm{d}x \leqslant C\epsilon^{q(n-2)/2} \int_0^r \frac{s^{n-1-\beta}}{[\epsilon^{2-\alpha} + s^{2-\alpha}]^{q(n-2)/(2-\alpha)}} \, \mathrm{d}s + O(\epsilon^{q(n-2)/2}), \tag{4.28}$$

where $C > 0$ is a constant independent of ϵ. Now, for $\epsilon < r$,

$$\begin{aligned}\int_0^r & \frac{s^{n-1-\beta}}{[\epsilon^{2-\alpha}+s^{2-\alpha}]^{q(n-2)/(2-\alpha)}}\,\mathrm{d}s \\ &= \int_0^\epsilon \frac{s^{n-1-\beta}}{[\epsilon^{2-\alpha}+s^{2-\alpha}]^{q(n-2)/(2-\alpha)}}\,\mathrm{d}s + \int_\epsilon^r \frac{s^{n-1-\beta}}{[\epsilon^{2-\alpha}+s^{2-\alpha}]^{q(n-2)/(2-\alpha)}}\,\mathrm{d}s \\ &\leqslant \epsilon^{n-1-\beta}\int_0^\epsilon \frac{\mathrm{d}s}{\epsilon^{q(n-2)}} + \int_\epsilon^r s^{n-1-\beta-q(n-2)}\,\mathrm{d}s \\ &= \epsilon^{n-\beta-q(n-2)} + I_{\epsilon,n}, \end{aligned} \tag{4.29}$$

where

$$I_{\epsilon,n} = \int_\epsilon^r s^{n-\beta-1-q(n-2)}\,\mathrm{d}s.$$

For all $n \geqslant 4$, we have $n-\beta-q(n-2) < 0$ and for $n = 3$, $n-\beta-q(n-2) = 3-\beta-q$. Therefore,

$$I_{\epsilon,n} = \begin{cases} |\log\epsilon| - |\log r| & \text{for } n = 3,\ 0 \leqslant \beta < 1 \text{ with } q = 3-\beta, \\ C_1(\epsilon^{3-\beta-q} - C_1') & \text{for } n = 3,\ q \neq 3-\beta, \\ C_2(\epsilon^{n-\beta-q(n-2)} - C_2') & \text{for all } n \geqslant 4, \end{cases} \tag{4.30}$$

where $C_1', C_2, C_2' > 0$ and $C_1 \geqslant 0$ are constants. Since $q < 2^*_\beta$, i.e. $\frac{1}{2}q(n-2) < (n-\beta)$, from (4.28)–(4.30), we have

$$\int_\Omega |v_\epsilon| f(x)\,\mathrm{d}x \to 0 \quad \text{as } \epsilon \to 0.$$

Now, to prove

$$Y_\epsilon < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)} \tag{4.31}$$

for sufficiently small $\epsilon > 0$, it is enough to prove that $\lambda\int_\Omega |v_\epsilon|^q f\,\mathrm{d}x$ goes to zero slower than ϵ^{n-2} as $\epsilon \to 0$ or, equivalently,

$$\lambda \lim_{\epsilon\to 0} \epsilon^{2-n} \int_\Omega |v_\epsilon|^q f\,\mathrm{d}x = +\infty. \tag{4.32}$$

From (4.2), (4.6), we have

$$\begin{aligned}\int_\Omega |v_\epsilon|^q f(x) &\geqslant \int_{B_\epsilon} |v_\epsilon|^q f(x)\,\mathrm{d}x \\ &= \int_{B_\epsilon} |v_\epsilon|^q |x|^{-\beta}\,\mathrm{d}x \\ &\geqslant C\epsilon^{q(n-2)/2}\epsilon^{n-\beta-q(n-2)} \\ &= C\epsilon^{n-\beta-q(n-2)/2}. \end{aligned}$$

Therefore,

$$\epsilon^{2-n}\int_\Omega |v_\epsilon|^q f(x)\,\mathrm{d}x \geqslant C\epsilon^{2-\beta-q(n-2)/2}, \tag{4.33}$$

where $C > 0$ is a constant. From (4.33), we have the following two cases.

Case a. When $n \geqslant 4$, we have $(2-\beta) - \frac{1}{2}q(n-2) < 0$, because $q > 2$ and hence (4.32) holds for any $\lambda > 0$.

Case b. When $n = 3$, $2 - \beta - \frac{1}{2}q(n-2) = 2 - \beta - \frac{1}{2}q$.

(i) Since for $1 \leqslant \beta < 2$ we have $2 - \beta - \frac{1}{2}q < 0$, equation (4.32) holds for all $\lambda > 0$.

(ii) For $0 \leqslant \beta < 1$ with $2(2-\beta) < q < 2^*_\beta$, we have $2 - \beta - \frac{1}{2}q < 0$, so (4.32) holds for all $\lambda > 0$.

(iii) For $0 \leqslant \beta < 1$ with $2 < q \leqslant 2(2-\beta)$, we have $2 - \beta - \frac{1}{2}q \geqslant 0$. Hence we can conclude the validity of (4.32) only for large $\lambda > 0$.

Hence the lemma follows. □

Proof of theorem 1.8. Choose t_0 sufficiently large such that

$$J_\lambda(t_0 v_\epsilon) < \rho = \inf_{\|u\|=\delta} J_\lambda(u).$$

Call $v_0 = t_0 v_\epsilon$ and define

$$\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = v_0\}$$

and

$$\beta_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in I} J_\lambda(\gamma(t)) \geqslant \rho > 0.$$

Therefore, we have

$$\beta_\lambda \leqslant \max_{t\geqslant 0} J_\lambda(t v_\epsilon) < \frac{(2-\alpha)}{2(n-\alpha)} S_\alpha^{(n-\alpha)/(2-\alpha)}. \tag{4.34}$$

By theorem 2.2 of [4], there exists a sequence $(u_m) \subseteq H_0^1(\Omega)$ such that, as $m \to \infty$,

$$J_\lambda(u_m) \to \beta_\lambda \quad \text{and} \quad DJ_\lambda(u_m) \to 0 \quad \text{strongly in } H^{-1}(\Omega).$$

Since β_λ satisfies (4.34), using lemmas 4.1 and 4.2, we complete the proof. □

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