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Nirmalendu Chaudhuri

University of Wollongong, chaudhur@uow.edu.au

Kunnath Sandeep

Universite Pierre et Marie Curie (Paris VI), sandeep@math.tifrbng.res.in

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On a heat problem involving the perturbed Hardy–Sobolev operator

Nirmalendu Chaudhuri

Centre for Mathematics and its Applications,
Australian National University, Canberra, ACT 0200, Australia
(chaudhur@maths.anu.edu.au)

Kunnath Sandeep

Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie,
Boite courrier 187, 75252 Paris Cedex 05, France
(kunnath.sandeep@math.u-cergy.fr)

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Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 3$ and $0 \in \Omega$. It is known that the heat problem $\partial u/\partial t + L_{\lambda^*} u = 0$ in $\Omega \times (0, \infty)$, $u(x, 0) = u_0 \geq 0$, $u_0 \not\equiv 0$, where $L_{\lambda^*} := -\Delta - \lambda^*/|x|^2$, $\lambda^* := \frac{1}{4}(n-2)^2$, does not admit any H_0^1 solutions for any $t > 0$. In this paper we consider the perturbation operator $L_{\lambda^* q} := -\Delta - \lambda^* q(x)/|x|^2$ for some suitable bounded positive weight function q and determine the border line between the existence and non-existence of positive H_0^1 solutions for the above heat problem with the operator $L_{\lambda^* q}$. In dimension $n = 2$, we have similar phenomena for the critical Hardy–Sobolev operator $L^* := -\Delta - (1/4|x|^2)(\log R/|x|)^{-2}$ for sufficiently large R .

1. Introduction

Let Ω be a bounded open connected subset in \mathbb{R}^n , $n \geq 3$ and $0 \in \Omega$. Baras and Goldstein, in their classical paper [3], considered the following heat problem,

$$\left. \begin{aligned} u_t - \Delta u &= V(x)u && \text{in } \Omega \times (0, \infty), \\ u(x, 0) &= u_0(x) && \text{in } \Omega, \\ u(x, t) &= 0 && \text{on } \partial\Omega \times (0, \infty), \end{aligned} \right\} \quad (1.1)$$

with $u_0 \geq 0$ and $u_0 \not\equiv 0$ (i.e. u_0 not equal to zero a.e.). For the critical potential $V(x) = \mu/|x|^2$, they proved that the Cauchy–Dirichlet problem (1.1) has a global solution (in the sense of distributions) for $0 < \mu \leq \lambda^* := \frac{1}{4}(n-2)^2$, provided $\int_{\Omega} |x|^{-\alpha_1} u_0(x) dx < \infty$, where α_1 is the smallest root of $(n-2-\alpha)\alpha = \mu$ and does not admit any solution for $t > 0$, whenever $\mu > \lambda^*$. It is well understood that the existence of solutions for problem (1.1) is deeply connected to the Hardy–Sobolev inequality

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4}(n-2)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx \quad \text{for all } u \in H_0^1(\Omega) \quad (1.2)$$

and the associated spectral problem $-\Delta u - (\mu/|x|^2)u = \lambda u$. By inequality (1.2), it is easy to show that problem (1.1) admits a solution in the function space $C([0, \infty); L^2(\Omega)) \cap L^2(0, \infty; H_0^1(\Omega))$ if $\mu < \lambda^*$. But a strong change of behaviour in the space variable takes place at the transition value $\mu = \lambda^*$ (see [10]). In this case, the behaviour of the solution is intimately related to the improved form of the classical Hardy inequality (1.2). Adimurthi *et al.* [2] showed that one can add as many lower-order terms on the right-hand side of (1.2) with each term being optimal. Here we will only use the special case, which states that (see [2, corollary 1.1]), for $n \geq 3$ and for any $R \geq e \sup_{\Omega} |x|$, there exist positive constants C_1 and C_2 such that

$$\int_{\Omega} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx + C_1 \int_{\Omega} \frac{|u|^2}{|x|^2} \left(\log \frac{R}{|x|}\right)^{-2} dx + C_2 \int_{\Omega} |u|^2 dx \tag{1.3}$$

holds for all $u \in H_0^1(\Omega)$.

But for $n = 2$ and $R \geq e^{e^e} \sup_{\Omega} |x|$, there exists a constant $C > 0$ such that the following two-dimensional Hardy–Sobolev inequality holds for all $u \in H_0^1(\Omega)$:

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{4} \int_{\Omega} \frac{|u|^2}{|x|^2} \left(\log \frac{R}{|x|}\right)^{-2} \left[1 + \left(\log \log \frac{R}{|x|}\right)^{-2}\right] dx + C \int_{\Omega} |u|^2 dx. \tag{1.4}$$

For the case $n \geq 3$ and $\mu = \lambda^*$, Vazquez and Zuazua [10] proved that the solution $u(t)$ of problem (1.1) with $u_0 \in L^2(\Omega)$ is in the Hilbert space $H(\Omega)$ for all time $t > 0$, but not in $H_0^1(\Omega)$ for $t > 0$, where $H(\Omega)$ is the completion of $H_0^1(\Omega)$ with respect to the *energy norm*

$$\|u\|_H := \left(\int_{\Omega} \left\{|\nabla u|^2 - \lambda^* \frac{u^2}{|x|^2}\right\} dx\right)^{1/2}.$$

It is interesting to note that

$$H_0^1(\Omega) \subset H(\Omega) \subset \bigcap_{q < 2} W_0^{1,q}(\Omega)$$

and the inclusions are proper. So a natural question arises: is it possible to perturb the *Hardy–Sobolev operator* $L_{\lambda^*} := -\Delta - \lambda^*/|x|^2$ in such a way that the heat equation admits a non-negative solution $u(t) \in H_0^1(\Omega)$ for all $t > 0$ with $u_0 \geq 0$? Or can we characterize perturbations of the stationary operator L_{λ^*} determining the border line between existence and non-existence of H_0^1 solutions? In this paper we answer these questions. For the stationary operator L_{λ^*} , such questions were answered by Adimurthi and Sandeep in [1] and we state their results in lemmas 3.1 and 4.1. For more on the stationary problem involving the Hardy–Sobolev operator, see [4–6, 8].

Throughout the paper, we assume that $0 \leq q \leq 1$ and $\eta \geq 0$ are measurable functions on Ω such that $\eta \in L_{\text{loc}}^{\infty}(\Omega \setminus \{0\})$, and it satisfies

$$\left. \begin{aligned} \lim_{x \rightarrow 0} |x|^2 \left(\log \frac{1}{|x|}\right)^2 \eta(x) &= 0, & n \geq 3, \\ \lim_{x \rightarrow 0} |x|^2 \left(\log \frac{1}{|x|}\right)^2 \left(\log \log \frac{1}{|x|}\right)^2 \eta(x) &= 0, & n = 2. \end{aligned} \right\} \tag{1.5}$$

Now, let us define the following constant:

$$C(q, \eta) := \begin{cases} \inf_{H_0^1(\Omega)} \left\{ \int_{\Omega} \left[|\nabla u|^2 - \lambda^* \frac{|u|^2 q}{|x|^2} \right] \mid \int_{\Omega} |u|^2 \eta = 1 \right\} & \text{for } n \geq 3, \\ \inf_{H_0^1(\Omega)} \left\{ \int_{\Omega} \left[|\nabla u|^2 - \frac{1}{4} \frac{|u|^2 q(x)}{(|x| \log R/|x|)^2} \right] \mid \int_{\Omega} |u|^2 \eta = 1 \right\} & \text{for } n = 2. \end{cases}$$

From inequalities (1.3) and (1.4), it follows that $C(q, \eta) > 0$.

THEOREM 1.1. *Let $n \geq 3$, $u_0 \in L^\infty(\Omega)$ non-negative and $u_0 \not\equiv 0$. Then the problem*

$$\left. \begin{aligned} u_t - \Delta u - \lambda^* \frac{q(x)}{|x|^2} u &= \lambda \eta u && \text{in } \Omega \times (0, \infty), \\ u(x, 0) &= u_0(x) && \text{in } \Omega, \\ u(x, t) &= 0 && \text{on } \partial\Omega \times (0, \infty) \end{aligned} \right\} \quad (1.6)$$

admits a unique global non-negative solution $u \in C([0, \infty); L^2(\Omega)) \cap L^2(0, \infty; H_0^1(\Omega))$ with $u' \in L^2(0, \infty; H^{-1}(\Omega))$, $u \not\equiv 0$ for all λ , $0 \leq \lambda \leq C(q, \eta)$, provided q satisfies

$$\liminf_{x \rightarrow 0} \left(\log \frac{1}{|x|} \right)^2 (1 - q(x)) > \frac{3}{(n-2)^2}. \quad (1.7)$$

But, for any $u_0 \in L^2(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$ and for any $\lambda \geq 0$ with q satisfying

$$\sup_{|x| < R} \left(\log \frac{R}{|x|} \right)^2 (1 - q(x)) \leq \frac{3}{(n-2)^2}, \quad R > 0, \quad (1.8)$$

problem (1.6) does not admit any non-negative H^1 solution, even locally in time t (i.e. for any $0 < t < T \leq \infty$).

THEOREM 1.2. *For $n = 2$, $u_0 \in L^\infty(\Omega)$ non-negative and $u_0 \not\equiv 0$, the problem*

$$\left. \begin{aligned} u_t - \Delta u - \frac{1}{4} \frac{q(x)}{(|x| \log R/|x|)^2} u &= \lambda \eta u && \text{in } \Omega \times (0, \infty), \\ u(x, 0) &= u_0(x) && \text{in } \Omega, \\ u(x, t) &= 0 && \text{on } \partial\Omega \times (0, \infty) \end{aligned} \right\} \quad (1.9)$$

admits a unique global non-negative solution $u \in C([0, \infty); L^2(\Omega)) \cap L^2(0, \infty; H_0^1(\Omega))$ with $u' \in L^2(0, \infty; H^{-1}(\Omega))$, $u \not\equiv 0$ for all λ , $0 \leq \lambda \leq C(q, \eta)$, provided q satisfies

$$\liminf_{x \rightarrow 0} \left(\log \log \frac{1}{|x|} \right)^2 (1 - q(x)) > 3. \quad (1.10)$$

But, for any $u_0 \in L^2(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$ and q satisfying

$$\sup_{|x| < R} \left(\log \log \frac{R}{|x|} \right)^2 (1 - q(x)) \leq 3, \quad R > 0, \quad (1.11)$$

problem (1.9) does not admit any non-negative H^1 solution, even locally in time t , for any $\lambda \geq 0$.

REMARK 1.3. For $n \geq 3$, by taking $q = 1 - C/(\log R/|x|)^\alpha$ and $\eta \equiv 1$, we can see that problem (1.6) has a unique global solution in $H_0^1(\Omega)$ if and only if $0 < \alpha < 2$, or $\alpha = 2$ and $C > 3/(n - 2)^2$.

We denote by $W = W(0, \infty; H_0^1(\Omega), H^{-1}(\Omega))$ the Hilbert space

$$W(0, \infty; H_0^1(\Omega), H^{-1}(\Omega)) := \{u \mid u \in L^2(0, \infty; H_0^1(\Omega)), u' \in L^2(0, \infty; H^{-1}(\Omega))\}$$

equipped with the norm

$$\|u\|_W = \left(\int_0^\infty (\|\nabla u(t)\|^2 + \|u'(t)\|_{H^{-1}}^2) dt \right)^{1/2}.$$

DEFINITION 1.4. A function $u \in W(0, \infty; H_0^1(\Omega), H^{-1}(\Omega))$ is said to be a solution to problem (1.6) (similar for (1.9)) if, for all $v \in H_0^1(\Omega)$,

$$\frac{d}{dt} \int_\Omega u(t)v \, dx + \langle L_{\lambda^*q} u(t), v \rangle_{H^1} = 0 \tag{1.12}$$

in the sense of $D'(0, \infty)$ and $u(0) = u_0$. Here,

$$L_{\lambda^*q} := -\Delta - \lambda^* \frac{q(x)}{|x|^2} - \lambda\eta,$$

and $\langle L_{\lambda^*q} \cdot, \cdot \rangle_{H^1} : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ is the bilinear functional defined by

$$\langle L_{\lambda^*q} u, v \rangle_{H^1} := \int_\Omega \langle \nabla u, \nabla v \rangle \, dx - \lambda^* \int_\Omega \frac{uq}{|x|^2} v \, dx - \lambda \int_\Omega uv\eta \, dx \quad \forall u, v \in H_0^1(\Omega).$$

2. Preliminaries

To prove the non-existence of local solutions, we need a theorem of Baras and Goldstein, theorem 2.2 and remark 7.1 of [3] and the two lemmas below.

THEOREM 2.1 (Baras and Goldstein). *Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, $0 \in \Omega$ and $\phi > 0$ such that $V(x) := -\Delta\phi/\phi \in L^1(\Omega)$ and $\phi\Delta\phi \in L^1(\Omega)$. Assume that u is a solution (in the sense of distributions) to the heat problem (1.1) with $0 \leq u_0 \in L^2(\Omega)$ and $u_0 \not\equiv 0$. Moreover, if ϕ is such that $\int_\Omega u_0\phi \, dx < \infty$ and, for $B_r := B(0, r)$, $r > 0$ such that $B_r \subset \Omega$, the following weighted Sobolev inequality holds for all radial functions $v \in H^1(B_r)$ for some $p > 2$:*

$$\left(\int_{B_r} |v|^p |\phi|^2 \, dx \right)^{2/p} \leq C(p, r) \int_{B_r} (|\nabla v|^2 + |v|^2) |\phi|^2 \, dx. \tag{2.1}$$

Then, for any $\Omega' \subset\subset \Omega$ and $0 < t_1 < t_2 < \infty$, there exists $C(t_1, t_2, \Omega') > 0$ such that

$$u(x, t) \geq C(t_1, t_2, \Omega') \phi(x) \quad \text{in } \Omega' \times [t_1, t_2]. \tag{2.2}$$

LEMMA 2.2. *Let $B_r \subset \mathbb{R}^n$, $n \geq 3$, be the ball of radius $r > 0$. Let $R > 2r$ and $p > 2$. Then there exists constant $C(p, r, R) > 0$ such that the following weighted*

Sobolev inequality holds for all radial functions $u \in H^1(B_r)$:

$$\left(\int_{B_r} |u|^p \frac{(\log R/|x|)^{-1}}{|x|^{n-2}} dx \right)^{2/p} \leq C(p, r, R) \int_{B_r} (|\nabla u|^2 + |u|^2) \frac{(\log R/|x|)^{-1}}{|x|^{n-2}} dx. \tag{2.3}$$

Proof. We first prove inequality (2.3) for any $u \in C^1[0, r]$. Let $v \in C^1[0, 2r]$ be such that $v(2r) = 0$, $R > 2r$ and let ϕ be the function defined by $\phi(\tau) := \tau^{-(n-2)/2}(\log R/\tau)^{-1/2}$ for $0 < \tau < R$. Now observe that

$$\begin{aligned} |v(s)| &= \left| - \int_s^{2r} v'(\tau) d\tau \right| \\ &\leq \int_s^{2r} |v'(\tau)| \phi \tau^{(n-1)/2} (\phi^{-1} \tau^{-(n-1)/2}) d\tau \\ &\leq \left(\int_s^{2r} |v'(\tau)|^2 \phi^2 \tau^{n-1} d\tau \right)^{1/2} \left(\int_s^{2r} \left(\log \frac{R}{\tau} \right) \frac{d\tau}{\tau} \right)^{1/2} \\ &= \frac{1}{2} \left(\int_s^{2r} |v'(\tau)|^2 \phi^2 \tau^{n-1} d\tau \right)^{1/2} \left[\left(\log \frac{R}{s} \right)^2 - \left(\log \frac{R}{2r} \right)^2 \right]^{1/2} \\ &\leq \frac{1}{2} \left(\int_0^{2r} |v'(\tau)|^2 \phi^2 \tau^{n-1} d\tau \right)^{1/2} \left(\log \frac{R}{s} \right). \end{aligned}$$

Therefore, for any $p > 2$, we have

$$\begin{aligned} \int_0^{2r} |v(\tau)|^p \phi^2 \tau^{n-1} d\tau &\leq \left(\frac{1}{2} \right)^p \left(\int_0^{2r} |v'(\tau)|^2 \phi^2 \tau^{n-1} d\tau \right)^{p/2} \int_0^{2r} \left(\log \frac{R}{\tau} \right)^p \phi^2 \tau^{n-1} d\tau \\ &= \left(\frac{1}{2} \right)^p \left(\int_0^{2r} |v'(\tau)|^2 \phi^2 \tau^{n-1} d\tau \right)^{p/2} \int_0^{2r} \left(\log \frac{R}{\tau} \right)^{p-1} \tau d\tau \\ &\leq R^2 \left(\frac{1}{2} \right)^p \left(\int_0^{2r} |v'(\tau)|^2 \phi^2 \tau^{n-1} d\tau \right)^{p/2} \int_0^\infty \tau^{p-1} e^{-2\tau} d\tau \\ &= \frac{\Gamma(p) R^2}{2^{2p}} \left(\int_0^{2r} |v'(\tau)|^2 \phi^2 \tau^{n-1} d\tau \right)^{p/2}. \end{aligned} \tag{2.4}$$

Now let $\xi \in C^1[r, 2r]$ be such that $0 \leq \xi \leq 1$, $\xi = 1$ on $[r, \frac{5}{4}r]$ and $\xi = 0$ on $[\frac{3}{2}r, 2r]$. For $u \in C^1[0, r]$, define the extension \tilde{u} on $[0, 2r]$ by

$$\tilde{u}(s) := \begin{cases} u(s) & \text{for } 0 \leq s \leq r, \\ u(2r - s)\xi(s) & \text{for } r \leq s \leq 2r. \end{cases}$$

Therefore, by inequality (2.4), we obtain

$$\begin{aligned} \left(\int_0^r |u(\tau)|^p \phi^2 \tau^{n-1} d\tau \right)^{2/p} &\leq \left(\int_0^{2r} |\tilde{u}(\tau)|^p \phi^2 \tau^{n-1} d\tau \right)^{2/p} \\ &\leq \frac{1}{16} (R^2 \Gamma(p))^{2/p} \int_0^{2r} |\tilde{u}'(\tau)|^2 \phi^2 \tau^{n-1} d\tau \end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^r |u'(\tau)|^2 \phi^2 \tau^{n-1} d\tau \\
&\quad + 2C \int_r^{2r} |u'(2r-\tau)|^2 \xi^2 \phi^2 \tau^{n-1} d\tau \\
&\quad + 2C \int_r^{2r} |u(2r-\tau)|^2 |\xi'|^2 \phi^2 \tau^{n-1} d\tau. \quad (2.5)
\end{aligned}$$

We notice that

$$\begin{aligned}
\int_r^{2r} |u'(2r-\tau)|^2 \xi^2 \phi^2 \tau^{n-1} d\tau &= \int_r^{2r} |u'(2r-\tau)|^2 \xi^2 \left(\log \frac{R}{\tau}\right)^{-1} \tau d\tau \\
&\leq \int_r^{3r/2} |u'(2r-\tau)|^2 \left(\log \frac{R}{\tau}\right)^{-1} \tau d\tau \\
&= \int_{r/2}^r |u'(\tau)|^2 \frac{2r-\tau}{\log[R/(2r-\tau)]} d\tau \\
&= \int_{r/2}^r |u'(\tau)|^2 \frac{1}{(\log R/\tau)} \frac{(2r-\tau) \log R/\tau}{\log[R/(2r-\tau)]} d\tau \\
&\leq 3 \left(\frac{\log 2R/r}{\log 2R/3r}\right) \int_0^r |u'(\tau)|^2 \frac{\tau}{\log R/\tau} d\tau \\
&= 3 \left(\frac{\log 2R/r}{\log 2R/3r}\right) \int_0^r |u'(\tau)|^2 \phi^2 \tau^{n-1} d\tau, \quad (2.6)
\end{aligned}$$

and since $|\xi'| \leq 4/r$, we have

$$\begin{aligned}
\int_r^{2r} |u(2r-\tau)|^2 |\xi'|^2 \phi^2 \tau^{n-1} d\tau &\leq \frac{16}{r^2} \int_r^{2r} |u(2r-\tau)|^2 \phi^2 \tau^{n-1} d\tau \\
&\leq \frac{48}{r^2} \left(\frac{\log 2R/r}{\log 2R/3r}\right) \int_0^r |u(\tau)|^2 \phi^2 \tau^{n-1} d\tau. \quad (2.7)
\end{aligned}$$

Now inequality (2.3) follows by substituting (2.6) and (2.7) in (2.5). \square

LEMMA 2.3. Let B_r , be a ball in \mathbb{R}^2 . Assume that $R \geq 2er$ and let us define

$$\phi(x) := \left(\log \frac{R}{|x|}\right)^{1/2} \left(\log \log \frac{R}{|x|}\right)^{-1/2}.$$

Then there exists constant $C(p, r, R) > 0$ such that the following weighted Sobolev inequality holds for any radial functions $u \in H^1(B_r)$ and for every $p > 2$:

$$\left(\int_{B_r} |u|^p |\phi|^2 dx\right)^{2/p} \leq C(p, r, R) \int_{B_r} (|\nabla u|^2 + |u|^2) |\phi|^2 dx. \quad (2.8)$$

Proof. This follows from the proof of lemma 2.2. \square

3. Proof of theorem 1.1

Existence part

For $0 < s \leq 1$, let us consider the operator $L_{\lambda^* q s} : H_0^1(\Omega) \rightarrow L^2(\Omega)$ defined by

$$L_{\lambda^* q s} := -\Delta - \lambda^* s \frac{q(x)}{|x|^2} - \lambda s \eta.$$

We observe that $L_{\lambda^* q s}^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$, for any $0 < s < 1$, is a compact self-adjoint positive-definite operator. By the standard semigroup theory [9], there exists a unique function $u^s \in W(0, \infty; H_0^1(\Omega), H^{-1}(\Omega))$, $u^s \geq 0$, $u^s \not\equiv 0$, satisfying the following heat equation

$$\frac{\partial u^s}{\partial t} + L_{\lambda^* q s} u^s = 0 \quad \text{in } \Omega \times (0, \infty), \tag{3.1}$$

with the initial data

$$u^s(x, 0) = u_0(x) \quad \text{in } \Omega.$$

Now, by taking $v = u^s(t)$ in (1.12), we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u^s(t)|^2 dx = - \int_{\Omega} |\nabla u^s(t)|^2 dx + \lambda^* s \int_{\Omega} \frac{q|u^s(t)|^2}{|x|^2} dx + \lambda s \int_{\Omega} \eta |u^s(t)|^2 dx. \tag{3.2}$$

Our aim is to show that the sequence $(u^s)_{0 < s < 1}$ is bounded in the function space $W(0, \infty; H_0^1(\Omega), H^{-1}(\Omega))$. But this does not follow immediately from (3.2). So we plan to find a suitable test function ϕ such that u^s is pointwise bounded above by the function ϕ . To obtain such a ϕ , we need the following lemma, which is essentially contained in the recent work of Adimurthi and Sandeep [1]. A proof can be found in [7].

LEMMA 3.1. *Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 3$ with $0 \in \Omega$, and take $q, \eta, C(q, \eta)$ as defined in § 1, with the extra assumption (1.7) on q . Then, for any $0 \leq \lambda \leq C(q, \eta)$, there exists $u \in H_0^1(\Omega)$, $u > 0$, satisfying the following equation:*

$$-\Delta u - \lambda^* \frac{q(x)}{|x|^2} u - \lambda \eta u = \lambda^1(\Omega, q, \eta) u \quad \text{in } \Omega \tag{3.3}$$

for

$$\lambda^1(\Omega, q, \eta) = \inf_{H_0^1(\Omega)} \left\{ \int_{\Omega} |\nabla u|^2 dx - \lambda^* \int_{\Omega} \frac{|u|^2 q}{|x|^2} dx - \lambda \int_{\Omega} |u|^2 \eta dx \mid \int_{\Omega} |u|^2 dx = 1 \right\}. \tag{3.4}$$

Let Ω_1 be a bounded domain in \mathbb{R}^n such that $\bar{\Omega} \subset \Omega_1$. Take $v \in H_0^1(\Omega_1)$, $v > 0$ satisfying equation (3.3) in the domain Ω_1 , where $\lambda^1(\Omega_1, q, \eta)$ as in (3.4) and we denote $\lambda^1(\Omega_1, q, \eta)$ by $\lambda^1(\Omega_1)$. Define the function ϕ on $\Omega_1 \times (0, \infty)$ by

$$\phi(x, t) := e^{-\lambda^1(\Omega_1)t} v(x).$$

Then $\phi \in W(0, \infty; H_0^1(\Omega_1), H^{-1}(\Omega_1))$, $\phi > 0$ in $\Omega \times (0, \infty)$ and

$$\phi_t + L_{\lambda^* q} \phi = e^{-\lambda^1(\Omega_1)t} (-\lambda^1(\Omega_1)v + L_{\lambda^* q} v) = 0 \quad \text{in } \Omega_1 \times (0, \infty).$$

Hence ϕ is a solution to the problem

$$\left. \begin{aligned} \phi_t + L_{\lambda^* q} \phi &= 0 && \text{in } \Omega_1 \times (0, \infty), \\ \phi(x, 0) &= v(x) && \text{in } \Omega_1, \\ \phi(x, t) &= 0 && \text{on } \partial\Omega_1 \times (0, \infty). \end{aligned} \right\} \quad (3.5)$$

Now we define

$$w^s(x, t) := M\phi(x, t) - u^s(x, t) \quad \text{on } \Omega \times (0, \infty).$$

Since $u^s(x, 0) = u_0(x) \in L^\infty(\Omega)$, we can choose some $M > 0$ such that

$$w^s(x, 0) := w^s(0) = Mv(x) - u_0(x) \geq 0 \quad \text{in } \Omega$$

(we have taken $\Omega \subset\subset \Omega_1$ and $v \in H_0^1(\Omega_1)$, just to ensure that $v > 0$ on $\partial\Omega$, so that $w^s(0) \geq 0$ in Ω). From (3.1) and (3.5), we have $\partial w^s / \partial t + L_{\lambda^* q} w^s \geq 0$ in $\Omega \times (0, \infty)$, and hence $w^s \geq 0$ in $\Omega \times (0, \infty)$. In other words, we have

$$u^s \leq M e^{-\lambda^1(\Omega_1)t} v(x) \quad \text{in } \Omega \times (0, \infty). \quad (3.6)$$

To show that u^s is bounded in $L^2(0, \infty; H_0^1(\Omega))$, we observe that, by using (3.6) in (3.2) and integrating from 0 to t ,

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |u^s(t)|^2 dx + \int_0^t \int_{\Omega} |\nabla u^s(t)|^2 dx dt \\ &= \lambda^* s \int_0^t \int_{\Omega} \frac{q|u^s(t)|^2}{|x|^2} dx dt + \lambda s \int_0^t \int_{\Omega} \eta |u^s(t)|^2 dx dt + \frac{1}{2} \int_{\Omega} |u_0(x)|^2 dx \\ &\leq \frac{M^2}{\lambda^1(\Omega_1)} \left(\lambda^* \int_{\Omega} \frac{|v|^2}{|x|^2} dx + \lambda \int_{\Omega} \eta(x) |v|^2 dx \right) + \frac{1}{2} \int_{\Omega} |u_0(x)|^2 dx \\ &\leq \frac{M^2}{\lambda^1(\Omega_1)} \int_{\Omega} |\nabla v(t)|^2 dx + \frac{1}{2} \int_{\Omega} |u_0(x)|^2 dx, \end{aligned}$$

and hence u^s is bounded in $L^2(0, \infty; H_0^1(\Omega))$. Now, from (3.1), for all $w \in H_0^1(\Omega)$ with $\|\nabla w\|_2 \leq 1$, by the Hölder and Hardy inequality, we have

$$\begin{aligned} \left| \left\langle \frac{d}{dt} u^s(t), w \right\rangle_{H^1} \right| &\leq \left(\int_{\Omega} |\nabla u^s(t)|^2 dx \right)^{1/2} \\ &\quad + \lambda^* \left(\int_{\Omega} \frac{|u^s(t)|^2}{|x|^2} dx \right)^{1/2} \left(\int_{\Omega} \frac{|w|^2}{|x|^2} dx \right)^{1/2} \\ &\quad + \lambda \left(\int_{\Omega} \eta |u^s(t)|^2 dx \right)^{1/2} \left(\int_{\Omega} \eta |w|^2 dx \right)^{1/2} \\ &\leq C \left(\int_{\Omega} |\nabla u^s(t)|^2 dx \right)^{1/2} \end{aligned}$$

for some constant $C > 0$ and hence $(d/dt)u^s$ is bounded in $L^2(0, \infty; H^{-1}(\Omega))$.

Therefore, in the limit $s \rightarrow 1$ (passing through a subsequence if necessary), we have

$$\left. \begin{aligned} u^s &\rightharpoonup u && \text{weakly in } L^2(0, \infty; H_0^1(\Omega)), \\ \frac{d}{dt}u^s &\rightharpoonup \frac{d}{dt}u && \text{weakly in } L^2(0, \infty; H^{-1}(\Omega)). \end{aligned} \right\} \quad (3.7)$$

As $u \in W(0, \infty; H_0^1(\Omega), H^{-1}(\Omega))$, then $u(x, 0) = u(0) \in L^2(\Omega)$. Since u^s satisfies equation (3.1), we have, for all $w \in H_0^1(\Omega)$ and for all $\zeta \in C_c^\infty(0, \infty)$,

$$-\int_0^\infty \int_\Omega u^s(t)w(x)\zeta'(t) \, dxdt + \int_0^\infty \langle L_{\lambda^*q_s}u^s(t), w \rangle_{H^1} \zeta(t) \, dt = 0, \quad (3.8)$$

and hence, by (3.7), we have, as $s \rightarrow 1$,

$$-\int_0^\infty \int_\Omega u(t)w(x)\zeta'(t) \, dxdt + \int_0^\infty \langle L_{\lambda^*q}u, w \zeta(t) \rangle_{H^1} \, dt = 0 \quad (3.9)$$

for all $w \in H_0^1(\Omega)$ and for all $\zeta \in C_c^\infty(0, \infty)$. Therefore, u satisfies the equation

$$u_t + L_{\lambda^*q}u = 0 \quad \text{in } \Omega \times (0, \infty),$$

with the initial data $u(x, 0) = u(0)$ in Ω , and it is easy to show that $u(0) = u_0$. Since $u_0 \geq 0$, it follows that $u \geq 0$. Now we observe that, for any $t > 0$, the solution u satisfies

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |u(t)|^2 \, dx = - \int_\Omega |\nabla u(t)|^2 \, dx + \lambda^* \int_\Omega \frac{q|u(t)|^2}{|x|^2} \, dx + \lambda \int_\Omega \eta|u(t)|^2 \, dx.$$

By the definition of $\lambda^1(\Omega, q, \eta)$ in (3.4) and invoking Gronwall's lemma, we have

$$\int_\Omega |u(t)|^2 \, dx \leq e^{-2\lambda^1(\Omega, q, \eta)t} \int_\Omega |u_0(x)|^2 \, dx. \quad (3.10)$$

Hence the uniqueness follows from (3.10).

Non-existence part

Let us assume that q satisfies condition (1.8) and u is a solution of (1.6) in the time interval $(0, T)$ for some $T \leq \infty$. From condition (1.8), there exists $R_0 > 0$ such that $B_{R_0} := B(0, R_0) \subset \Omega$ and

$$q(x) \geq V_0(x) := \left(1 - \frac{3}{(n-2)^2} \left(\log \frac{R}{|x|} \right)^{-2} \right) \quad \text{in } B_{R_0}.$$

Let us assume that v satisfies

$$\left. \begin{aligned} v_t - \Delta v &= \lambda^* \frac{V_0(x)}{|x|^2} v && \text{in } B_{R_0} \times (0, T), \\ u(x, 0) &= u_0(x) && \text{in } B_{R_0}, \\ u(x, t) &= 0 && \text{on } \partial B_{R_0} \times (0, T). \end{aligned} \right\} \quad (3.11)$$

By using inequality (1.3), it is easy to show that $u \geq v$ in $B_{R_0} \times (0, T)$. Hence, without loss of generality, we can assume that $q(x) = V_0(x)$ in Ω and $\eta = 0$. Our

aim is prove that, for any $0 < t_1 < t_2 < T$ and $\Omega' \subset\subset \Omega$, there exists a constant $C(t_1, t_2, \Omega') > 0$ such that

$$u(x, t) \geq C(t_1, t_2, \Omega') |x|^{-(n-2)/2} \left(\log \frac{R}{|x|} \right)^{-1/2} \quad \text{in } \Omega' \times [t_1, t_2]. \tag{3.12}$$

To prove estimate (3.12), we follow the idea of Baras and Goldstein. In order to use theorem 2.1, we need to choose the test function ϕ such that $\phi \Delta \phi \in L^1(\Omega)$ and the weighted Sobolev inequality (2.1) holds with the weight ϕ . By lemma 2.2, we note that the function $\phi(x) := |x|^{-(n-2)/2} (\log R/|x|)^{-1/2}$ satisfies (2.1), but that $\phi \Delta \phi$ does not belong to $L^1(\Omega)$. So, we need to regularize the function ϕ such that all the conditions in theorem 2.1 hold. We notice that, for any $k > \frac{1}{2}$, the function $\phi_k(x) := |x|^{-(n-2)/2} (\log R/|x|)^{-k}$ satisfies all the conditions of theorem 2.1. Moreover, for any $k > \frac{1}{2}$, it satisfies the following equation:

$$\phi_k \Delta \phi_k = \frac{\lambda^* \phi_k^2}{|x|^2} \left(1 - \frac{4k(k+1)}{(n-2)^2} \left(\log \frac{R}{|x|} \right)^{-2} \right) \quad \text{in } \Omega. \tag{3.13}$$

Let

$$q_k := \left(1 - \frac{4k(k+1)}{(n-2)^2} \left(\log \frac{R}{|x|} \right)^{-2} \right) \quad \text{in } \Omega$$

and u_k be a solution to the problem

$$\begin{aligned} \frac{\partial u_k}{\partial t} - \Delta u_k &= \lambda^* \frac{q_k(x)}{|x|^2} u_k \quad \text{in } \Omega \times (0, \infty), \\ u_k(x, 0) &= u_0(x) \quad \text{in } \Omega, \\ u_k(x, t) &= 0 \quad \text{on } \partial\Omega \times (0, \infty). \end{aligned}$$

Therefore, by theorem 2.1, there exists a constant $C(t_1, t_2, \Omega') > 0$, independent of k , such that

$$u_k(x, t) \geq C(t_1, t_2, \Omega') |x|^{-(n-2)/2} \left(\log \frac{R}{|x|} \right)^{-k} \quad \text{in } \Omega' \times [t_1, t_2]. \tag{3.14}$$

Since $V_0(x) \geq q_k(x)$, we obtain $u(x, t) \geq u_k(x, t)$ in $\Omega \times (0, T)$ for all $k > \frac{1}{2}$. By considering the limit $k \rightarrow \frac{1}{2}$, from (3.14), we obtain the estimate (3.12) and it contradicts the fact that $u(t) \in H_0^1(\Omega)$, because $\int_{\Omega} |\phi|^2 / |x|^2 dx = \infty$.

4. Proof of theorem 1.2

Existence

In the proof of theorem 1.1, we chose the test function ϕ with the use of lemma 3.1. Here we also have a similar lemma, which will enable us to carry out the proof, and this lemma is again essentially contained in [1].

LEMMA 4.1. *Let Ω be a bounded domain in \mathbb{R}^2 , with $0 \in \Omega$ and $R \geq e^{e^e} \sup_{\Omega} |x|$. Take $q, \eta, C(q, \eta)$ as defined in the introduction, with the extra assumption (1.10)*

on q . Then, for any $0 \leq \lambda \leq C(q, \eta)$, there exists $u \in H_0^1(\Omega)$, $u > 0$, satisfying the following equation,

$$-\Delta u - \frac{1}{4} \frac{q(x)}{|x|^2 (\log R/|x|)^2} u - \lambda \eta u = \mu^1(\Omega, q, \eta) u \quad \text{in } \Omega, \quad (4.1)$$

where

$$\mu^1(\Omega, q, \eta) := \inf_{H_0^1(\Omega)} \left\{ \int_{\Omega} |\nabla u|^2 - \frac{1}{4} \int_{\Omega} \frac{q|u|^2}{|x|^2 (\log R/|x|)^2} - \lambda \int_{\Omega} |u|^2 \eta \mid \int_{\Omega} |u|^2 = 1 \right\}. \quad (4.2)$$

Now it is easy to complete the proof by following proof of theorem 1.1. Here, the only differences are that we have to use lemma 4.1 instead of lemma 3.1 and inequality (1.4) instead of (1.3) in the respective places.

Non-existence

By taking $\phi_k := (\log R/|x|)^{1/2} (\log \log R/|x|)^{-k}$, $k > \frac{1}{2}$, and using lemma 2.3 and theorem 2.1, the non-existence follows.

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