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HIGHER-RANK GRAPHS AND THEIR C^*-ALGEBRAS

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Abstract We consider the higher-rank graphs introduced by Kumjian and Pask as models for higher-rank Cuntz–Krieger algebras. We describe a variant of the Cuntz–Krieger relations which applies to graphs with sources, and describe a local convexity condition which characterizes the higher-rank graphs that admit a non-trivial Cuntz–Krieger family. We then prove versions of the uniqueness theorems and classifications of ideals for the C^*-algebras generated by Cuntz–Krieger families.

Keywords: graph algebra; Cuntz–Krieger algebra; uniqueness

AMS 2000 *Mathematics subject classification:* Primary 46L05

1. Introduction

The C^*-algebras of higher-rank graphs of Kumjian and Pask [**7**] generalize the higher-rank Cuntz–Krieger algebras of Robertson and Steger [**10–12**] in the same way that the C^*-algebras of infinite graphs generalize the original Cuntz–Krieger algebras [**4**,**5**]. In [**7**], Kumjian and Pask analysed higher-rank graph algebras using a groupoid model like that used in [**8**] and [**9**] to analyse graph algebras. The results in [**8**] and [**9**] were sharpened in [**2**] using a direct analysis based on the original arguments used by Cuntz and Krieger in [**4**] and [**5**]; the analysis of [**2**] applies to the algebras of quite general row-finite graphs, and in particular the graphs can have sinks or sources.

Here we carry out a direct analysis of the C^*-algebras of row-finite higher-rank graphs. One interesting new feature is the difficulty in extending results to higher-rank graphs with sources: the paths λ in higher-rank graphs have degrees $d(\lambda)$ in $\mathbb{N}^k$ rather than lengths in $\mathbb{N}$, and vertices may receive edges of some degrees and not of others. To overcome this difficulty we modify the Cuntz–Krieger relation to ensure that Cuntz–Krieger algebras have a spanning family of the usual sort, and identify a local convexity condition under which a higher-rank graph admits a non-trivial Cuntz–Krieger family. We then prove versions of the gauge-invariant uniqueness theorem and the Cuntz–Krieger uniqueness theorem for locally convex higher-rank graphs, extending the results of [**7**], and use them to investigate the ideal structure.

The Cuntz–Krieger relation of [**7**] involves the spaces Λ^p of paths of degree $p \in \mathbb{N}^k$. Our key technical innovation is the introduction of path spaces $\Lambda^{\leqslant p}$ consisting of the paths λ with $d(\lambda) \leqslant p$ which cannot be extended to paths $\lambda\mu$ with $d(\lambda\mu) \leqslant p$; the key Lemmas 3.6 and 3.7 show that the spaces $\Lambda^{\leqslant p}$ have combinatorial properties like those of the spaces Λ^p, and ensure that the C^*-algebras behave like Cuntz–Krieger algebras (see Proposition 3.5). These new path spaces would also have simplified the analysis of the core in the C^*-algebras of graphs with sinks in [**2**, § 2]. Indeed, the $\Lambda^{\leqslant q}$ notation works so smoothly that arguments sometimes appear deceptively easy.

2. Higher-rank graphs

Definitions 2.1. Given $k \in \mathbb{N}$, a *graph of rank* k (or k-graph) (Λ, d) consists of a countable category $\Lambda = (\mathrm{Obj}(\Lambda), \mathrm{Hom}(\Lambda), r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, which satisfies the *factorization property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$. Elements $\lambda \in \Lambda$ are called *paths*. For $m \in \mathbb{N}^k$ and $v \in \mathrm{Obj}(\Lambda)$, we define $\Lambda^m := \{\lambda \in \Lambda : d(\lambda) = m\}$ and $\Lambda^m(v) := \{\lambda \in \Lambda^m : r(\lambda) = v\}$. A morphism between two k-graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \to \Lambda_2$ which respects the degree maps. (Λ, d) is *row finite* if for each $v \in \mathrm{Obj}(\Lambda)$ and $m \in \mathbb{N}^k$, the set $\Lambda^m(v)$ is finite; (Λ, d) has *no sources* if $\Lambda^m(v) \neq \emptyset$ for all $v \in \mathrm{Obj}(\Lambda)$ and $m \in \mathbb{N}^k$.

The factorization property says that there is a unique path of degree 0 at each vertex, and hence allows us to identify $\mathrm{Obj}(\Lambda)$ with Λ^0.

Examples 2.2.

(i) Let E be a directed graph, and let $l : E^* \to \mathbb{N}$ be the length function on the path space. Then (E^*, l) is a 1-graph.

(ii) Let $k \in \mathbb{N}$, let $m \in (\mathbb{N} \cup \{\infty\})^k$, and define a partial ordering on $\mathbb{N}^k$ by $m \leqslant n \iff m_i \leqslant n_i$ for all i. $(\Omega_{k,m}, d)$ is the k-graph with category $\Omega_{k,m}$ defined by

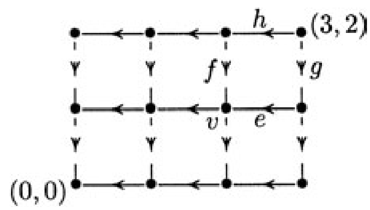
$$\begin{aligned}\mathrm{Obj}(\Omega_{k,m}) &:= \{p \in \mathbb{N}^k : p \leqslant m\},\\ \mathrm{Hom}(\Omega_{k,m}) &:= \{(p,q) \in \mathrm{Obj}(\Omega_{k,m}) \times \mathrm{Obj}(\Omega_{k,m}) : p \leqslant q\},\end{aligned}$$

$r(p,q) := p$, $s(p,q) := q$, and degree map $d(p,q) := q - p$.

When each $m_i = \infty$, the resulting k-graph is the main example Ω_k used in [**7**].

When $m \in \mathbb{N}^k$, the resulting k-graphs are important because every path λ of degree m in a k-graph Λ determines a graph morphism $x_\lambda : \Omega_{k,m} \to \Lambda$: set $x_\lambda(p,q) := \lambda''$, where $\lambda = \lambda'\lambda''\lambda'''$ and $d(\lambda') = p$, $d(\lambda''') = m - q$. Indeed, this sets up a bijection between Λ^m and the graph morphisms $x : \Omega_{k,m} \to \Lambda$.

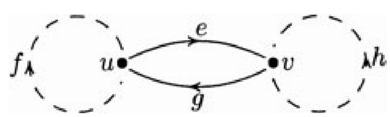
To visualize a k-graph, we draw its 1-*skeleton*, which is the graph with vertex set Λ^0, edge set $\bigcup_{i=1}^k \Lambda^{e_i}$, range and source maps inherited from Λ, and with the edges of different degrees distinguished using k different colours. (In the pictures here, we imagine that solid lines are green, dashed lines are red and dotted lines are blue.) For example, the 1-skeleton of $\Omega_{2,(3,2)}$ is



(2.1)

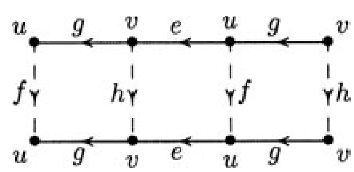
Because the edges represent morphisms in a category, we write eg for the path in the 1-skeleton which consists of g followed by e.

The 1-skeleton of a k-graph does not always suffice to determine the k-graph: we have to say how the edges in Λ^{e_i} fit together to give elements of Λ^n. We interpret elements of Λ^n as commuting diagrams of shape n in which the morphisms correspond to edges in the given 1-skeleton. Thus, for example, in a 2-graph with 1-skeleton



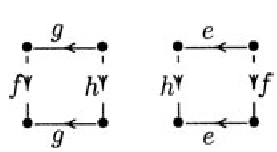
(2.2)

where the dashed lines have degree $(0, 1)$, the unique example of a $(3, 1)$ path λ with $r(\lambda) = u$ and $s(\lambda) = v$ is

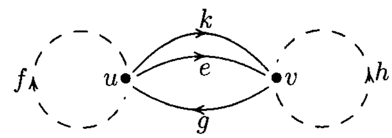


From such a picture we can read off the factorizations of λ: $\lambda = gegh = gefg = gheg = fgeg$.

When $k = 2$, it suffices to specify the factorizations of paths ef of length 2 in the 1-skeleton for which e and f have different colours. Any collection S of squares which contains each such bi-coloured path exactly once determines a unique 2-graph Λ with the given 1-skeleton and $\Lambda^{(1,1)} = S$ (see [**7**, § 6]); there may be no such collection, or there may be many. For the 1-skeleton in (2.2), the factorization property implies that $\Lambda^{(1,1)}$ consists of the two squares



and hence there is exactly one 2-graph with this 1-skeleton. However, if we add one extra edge to the 1-skeleton in (2.2), we have to make a choice. For example, in the 1-skeleton

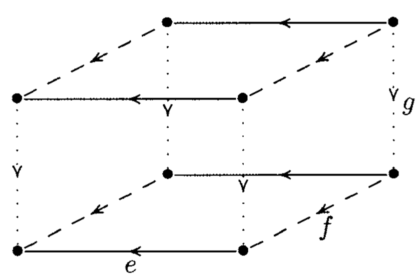


there are four possible bi-coloured paths from u to v, and we have to decide how to pair these off into paths of degree $(1,1)$: either

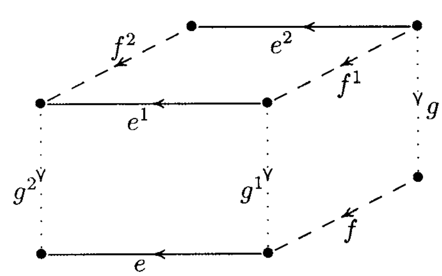
$$\begin{tikzcd} \bullet & \bullet \arrow[l, "e"'] \arrow[d, dashed, "f"] \\ \bullet & \bullet \arrow[l, "e"] \arrow[from=1-1, to=2-1, dashed, "h"'] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet & \bullet \arrow[l, "e"'] \arrow[d, dashed, "f"] \\ \bullet & \bullet \arrow[l, "k"] \arrow[from=1-1, to=2-1, dashed, "h"'] \end{tikzcd}$$

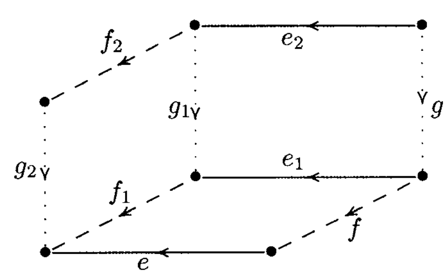
is a path of degree $(1,1)$, and once we have decided which, the other pairing is determined by the factorization property.

For $k > 2$, a collection S of squares may not be the set of paths of degree $(1,1)$ for any k-graph with the given 1-skeleton. However, [**6**, Theorem 2.1] tells us that it suffices to know that for every tri-coloured path efg in the 1-skeleton, the six squares on the sides of the cube



give a well-defined path of degree $(1,1,1)$. More precisely, we need to know that the path $g^2f^2e^2$ with reverse colouring obtained by successively filling in the three visible squares agrees with the path $g_2f_2e_2$ obtained by filling in the three invisible squares:





(In the left-hand diagram, we first use the right-hand face to determine g^1f^1, then use the front to find g^2e^1, and then the top to find f^2e^2; in the right-hand diagram, we use the bottom first.) A family of squares which contains each bi-coloured path exactly once and which satisfies this condition on cubes determines a unique k-graph (see Theorem 2.1 and Remark 2.3 in [**6**]).

3. Higher-rank graphs and their C^*-algebras

For a row-finite k-graph Λ with no sources, the authors of [**7**] define a Cuntz–Krieger Λ-family to be a family of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ such that $\{s_v : v \in \Lambda^0\}$ are mutually orthogonal projections, $s_\lambda s_\mu = s_{\lambda\mu}$ for all $\lambda, \mu \in \Lambda$ with $r(\mu) = s(\lambda)$, $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$, and

$$s_v = \sum_{\lambda \in \Lambda^m(v)} s_\lambda s_\lambda^* \quad \text{for all } v \in \Lambda^0 \text{ and } m \in \mathbb{N}^k. \tag{3.1}$$

When Λ has sources, there is a problem with relation (3.1) in that $\Lambda^m(v)$ may be non-empty for some values of m and empty for others. If a vertex $v \in \Lambda^0$ receives no edges, we could just impose no relation for that vertex, as is done for directed graphs in [**9**]; if v receives edges of some degrees but not others, then we have to change (3.1) in more subtle ways. The obvious strategy is to observe that when $\Lambda^{e_i}(v) \neq \emptyset$ for all $i \in \{1, \dots, k\}$ and $v \in \Lambda^0$, (3.1) is equivalent to (3.2), and then to replace (3.1) with

$$s_v = \sum_{\lambda \in \Lambda^{e_i}(v)} s_\lambda s_\lambda^* \quad \text{for } v \in \Lambda^0 \text{ and } 1 \leqslant i \leqslant k \text{ with } \Lambda^{e_i}(v) \neq \emptyset. \tag{3.2}$$

While this works for a large class of k-graphs (see Proposition 3.11), in general there are problems. We consider the following key example:

$$\begin{tikzcd} z \arrow[d, dashed, "f"'] & \\ v & w \arrow[l, "e"] \end{tikzcd} \tag{3.3}$$

In the 2-graph of (3.3), relation (3.2) would say $s_e s_e^* = s_v = s_f s_f^*$. But then $s_e^* s_f$ would be a partial isometry with source projection s_w and range projection s_z, and consequently would not be expressible as a sum of partial isometries of the form $s_\lambda s_\mu^*$; thus the C^*-algebra generated by $\{s_f, s_e, s_v\}$ would not look like a Cuntz–Krieger algebra. This problem does not arise in the 2-graph given by (2.1): the compositions eg and fh define the same path in $\Lambda^{(1,1)}$, so $s_e s_g = s_f s_h$ and $s_e^* s_f = s_g s_h^*$.

Our adaptation of (3.1) imposes a relation for each $m \in \mathbb{N}^k$, but involves sums over paths which extend as far as possible in the direction m. Formally, this relation introduces the following definition.

Definition 3.1. Let (Λ, d) be a k-graph. For $q \in \mathbb{N}^k$ and $v \in \Lambda^0$ we define

$$\Lambda^{\leqslant q} := \{\lambda \in \Lambda : d(\lambda) \leqslant q \text{ and } \Lambda^{e_i}(s(\lambda)) = \emptyset \text{ when } d(\lambda) + e_i \leqslant q\},$$

and

$$\Lambda^{\leqslant q}(v) := \{\lambda \in \Lambda^{\leqslant q} : r(\lambda) = v\}.$$

Remarks 3.2. Notice that $\Lambda^{\leqslant q}(v)$ is never empty: if there are no non-trivial paths of degree less than or equal to q, then $\Lambda^{\leqslant q}(v) = \{v\}$; in particular, if $r^{-1}(v) = \emptyset$, then $\Lambda^{\leqslant q}(v) = \{v\}$ for all $q \in \mathbb{N}^k$. The sets $\Lambda^{\leqslant q}$ and $\Lambda^{\leqslant q}(v)$ are used in arguments where the Λ^q and $\Lambda^q(v)$ may have been used in [**7**]; when Λ has no sources, $\Lambda^{\leqslant q} = \Lambda^q$.

Definition 3.3. Let Λ be a row-finite k-graph. A *Cuntz–Krieger Λ-family* in a C^*-algebra B consists of a family of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ satisfying the *Cuntz–Krieger relations*:

(1) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;

(2) $s_{\lambda\mu} = s_\lambda s_\mu$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$;

(3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$; and

(4) $s_v = \sum_{\lambda \in \Lambda^{\leqslant m}(v)} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

Examples 3.4. For the 2-graph given by (2.1), the relations at v say $s_v = s_e s_e^*$ for $m = (1,0)$, $s_v = s_f s_f^*$ for $m = (0,1)$, and $s_v = s_{eg} s_{eg}^* = s_{fh} s_{fh}^*$ for $m = (1,1)$. For any $m \in \mathbb{N}^2$ with $m \geqslant (1,1)$, the relation at v for m reduces to that for $(1,1)$.

In the 2-graph given by (3.3), the relation $s_v = s_e s_e^* + s_f s_f^*$ for $m = (1,1)$ combines with $s_e s_e^* = s_v = s_f s_f^*$ to force everything to be zero.

The following proposition shows that our Cuntz–Krieger relations yield the usual type of spanning family (see Remark 3.8 (1)).

Proposition 3.5. *Let (Λ, d) be a row-finite k-graph and let $\{s_\lambda : \lambda \in \Lambda\}$ be a Cuntz–Krieger Λ-family. Then for $\lambda, \mu \in \Lambda$ and $q \in \mathbb{N}^k$ with $d(\lambda), d(\mu) \leqslant q$ we have*

$$s_\lambda^* s_\mu = \sum_{\substack{\lambda\alpha = \mu\beta \\ \lambda\alpha \in \Lambda^{\leqslant q}}} s_\alpha s_\beta^*. \tag{3.4}$$

To prove this, we need some properties of $\Lambda^{\leqslant q}$.

Lemma 3.6. *Let (Λ, d) be a k-graph, $\lambda \in \Lambda^{\leqslant m}$, and $\alpha \in \Lambda^{\leqslant n}(s(\lambda))$. Then $\lambda\alpha \in \Lambda^{\leqslant m+n}$.*

Proof. We know $d(\lambda\alpha) \leqslant m+n$. Suppose there exists i such that $d(\lambda\alpha)+e_i \leqslant m+n$. If $d(\alpha)+e_i \leqslant n$, then $\Lambda^{e_i}(s(\lambda\alpha)) = \Lambda^{e_i}(s(\alpha)) = \emptyset$, so suppose $\langle d(\alpha), e_i\rangle = \langle n, e_i\rangle$. Then $d(\lambda\alpha)+e_i \leqslant m+n$ implies that $d(\lambda)+e_i \leqslant m$. But $\lambda \in \Lambda^{\leqslant m}$, so $\Lambda^{e_i}(s(\lambda)) = \emptyset$. Now suppose that there exists $\beta \in \Lambda^{e_i}(s(\lambda\alpha)) = \Lambda^{e_i}(s(\alpha))$. Then by the factorization property there exist $\mu_1, \mu_2 \in \Lambda$ such that $\mu_1\mu_2 = \alpha\beta$ and $d(\mu_1) = e_i$. But then $\mu_1 \in \Lambda^{e_i}(s(\lambda))$, a contradiction. Therefore, $\Lambda^{e_i}(s(\lambda\alpha)) = \emptyset$, and $\lambda\alpha \in \Lambda^{\leqslant m+n}$. □

Lemma 3.7. *Let (Λ, d) be a row-finite k-graph and let $\{s_\lambda : \lambda \in \Lambda\}$ be a Cuntz–Krieger Λ-family. Then for $q \in \mathbb{N}^k$ and $\lambda, \mu \in \Lambda^{\leqslant q}$, $s_\lambda^* s_\mu = \delta_{\lambda,\mu} s_{s(\lambda)}$.*

Proof. The Cuntz–Krieger relations (1) and (4) tell us that the projections $\{s_\alpha s_\alpha^* : \alpha \in \Lambda^{\leqslant q}\}$ are mutually orthogonal. Cuntz–Krieger relations (2) and (3) then give

$$s_\lambda^* s_\mu = (s_\lambda^* s_\lambda) s_\lambda^* s_\mu (s_\mu^* s_\mu) = s_\lambda^* (s_\lambda s_\lambda^*)(s_\mu s_\mu^*) s_\mu = \delta_{\lambda,\mu} s_{s(\lambda)}.$$

□

Proof of Proposition 3.5.

$$\begin{aligned}
s_\lambda^* s_\mu &= s_{s(\lambda)} s_\lambda^* s_\mu s_{s(\mu)} \quad \text{using Definition 3.3 (2)} \\
&= \Big(\sum_{\alpha \in \Lambda^{\leqslant q-d(\lambda)}(s(\lambda))} s_\alpha s_\alpha^* \Big) s_\lambda^* s_\mu \Big(\sum_{\beta \in \Lambda^{\leqslant q-d(\mu)}(s(\mu))} s_\beta s_\beta^* \Big) \\
&\qquad\qquad\qquad\qquad \text{using Definition 3.3 (4)} \\
&= \Big(\sum_{\alpha \in \Lambda^{\leqslant q-d(\lambda)}(s(\lambda))} s_\alpha s_{\lambda\alpha}^* \Big) \Big(\sum_{\beta \in \Lambda^{\leqslant q-d(\mu)}(s(\mu))} s_{\mu\beta} s_\beta^* \Big) \\
&\qquad\qquad\qquad\qquad \text{using Definition 3.3 (2)} \\
&= \sum_{\alpha \in \Lambda^{\leqslant q-d(\lambda)}(s(\lambda))} \sum_{\beta \in \Lambda^{\leqslant q-d(\mu)}(s(\mu))} s_\alpha s_{\lambda\alpha}^* s_{\mu\beta} s_\beta^* \\
&= \sum_{\substack{\lambda\alpha = \mu\beta \\ \lambda\alpha \in \Lambda^{\leqslant q}(r(\lambda))}} s_\alpha s_{s(\alpha)} s_\beta^* \quad \text{by Lemma 3.6 and Lemma 3.7} \\
&= \sum_{\substack{\lambda\alpha = \mu\beta \\ \lambda\alpha \in \Lambda^{\leqslant q}(r(\lambda))}} s_\alpha s_\beta^* \quad \text{using Definition 3.3 (2).}
\end{aligned}$$

□

Remarks 3.8.

(1) Proposition 3.5 implies that

$$C^*(\{s_\lambda : \lambda \in \Lambda\}) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : s(\beta) = s(\alpha)\}.$$

(2) When $q = d(\lambda) \vee d(\mu)$, if $\lambda\alpha = \mu\beta$ and $\lambda\alpha \in \Lambda^{\leqslant q}(r(\lambda))$, then $d(\lambda\alpha) = d(\mu\beta) = q$. Notice that if we take $\lambda = \mu$ in Proposition 3.5, then the lemma reduces to relation (4) of Definition 3.3 at the vertex $s(\lambda)$. Hence (4) could be replaced with relation (3.4) from Proposition 3.5—this relation would be harder to verify in examples, but might provide more insight.

Given a row-finite k-graph (Λ, d), there is a C^*-algebra $C^*(\Lambda)$ generated by a universal Cuntz–Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ (see [**3**, § 1]); in other words, for each Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$, there is a homomorphism $\pi : C^*(\Lambda) \to C^*(\{t_\lambda\})$ such that $\pi(s_\lambda) = t_\lambda$ for every $\lambda \in \Lambda$. Contrary to our experience with directed graphs and their C^*-algebras, it is not straightforward to construct Cuntz–Krieger Λ-families $\{t_\lambda\}$ in which all the partial isometries t_λ are non-zero; in fact, as we saw in Examples 3.4, the 2-graph of (3.3) admits no such families. We will show that the existence of non-trivial Cuntz–Krieger families is characterized by a local convexity condition on the k-graph.

Definition 3.9. A k-graph (Λ, d) is *locally convex* if, for all $v \in \Lambda^0$, $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $\lambda \in \Lambda^{e_i}(v)$ and $\mu \in \Lambda^{e_j}(v)$, $\Lambda^{e_j}(s(\lambda))$ and $\Lambda^{e_i}(s(\mu))$ are non-empty.

Remark 3.10. The 2-graph of (3.3) is not locally convex since $r^{-1}(u) = r^{-1}(w) = \emptyset$. Every 1-graph is locally convex, as is every higher-rank graph without sources.

Proposition 3.11. *Let (Λ, d) be a locally convex row-finite k-graph. Then the Cuntz–Krieger relation (4) of Definition 3.3 is equivalent to (3.2).*

The crucial idea in the proof of Proposition 3.11 is that, when the k-graph is locally convex, the factorization property of paths extends to elements of $\Lambda^{\leqslant m}(v)$. This is not the case for the 2-graph of (3.3): the path f is in $\Lambda^{\leqslant(1,1)}(v)$, but does not factor as $\lambda'\lambda''$ with $\lambda' \in \Lambda^{\leqslant(1,0)}(v)$.

Lemma 3.12. *Let (Λ, d) be a k-graph, and suppose that (Λ, d) is locally convex. Then for all $v \in \Lambda^0$, $m \in \mathbb{N}^k \setminus \{0\}$, and $j \in \{1, \ldots, k\}$ with $\langle m, e_j\rangle \geqslant 1$, we have*

$$\Lambda^{\leqslant m}(v) = \{\lambda'\lambda'' \in \Lambda : \lambda' \in \Lambda^{\leqslant m-e_j}(v) \text{ and } \lambda'' \in \Lambda^{\leqslant e_j}(s(\lambda'))\}.$$

Proof. Fix $j \in \{1, \ldots, k\}$ with $\langle m, e_j\rangle \geqslant 1$; there is at least one such j since $m \neq 0$. If $\lambda'\lambda'' \in \Lambda$ satisfies $\lambda' \in \Lambda^{\leqslant m-e_j}(v)$ and $\lambda'' \in \Lambda^{\leqslant e_j}(s(\lambda'))$, then by Lemma 3.6 $\lambda'\lambda'' \in \Lambda^{\leqslant m}(v)$, so we have the containment

$$\Lambda^{\leqslant m}(v) \supseteq \{\lambda'\lambda'' \in \Lambda : \lambda' \in \Lambda^{\leqslant m-e_j}(v) \text{ and } \lambda'' \in \Lambda^{\leqslant e_j}(s(\lambda'))\}.$$

Now suppose $\lambda \in \Lambda^{\leqslant m}(v)$. Since $d(\lambda) \leqslant m$, we must have

(1) $\langle d(\lambda), e_j\rangle < \langle m, e_j\rangle$, or

(2) $\langle d(\lambda), e_j\rangle = \langle m, e_j\rangle$.

First suppose (1) holds. Then $d(\lambda) \leqslant m - e_j$, and hence $\lambda \in \Lambda^{\leqslant m-e_j}$. Also $\Lambda^{e_j}(s(\lambda)) = \emptyset$ (since $\lambda \in \Lambda^{\leqslant m}(v)$, and hence $\Lambda^{\leqslant e_j}(s(\lambda)) = \{s(\lambda)\}$). Taking $\lambda' = \lambda$ and $\lambda'' = s(\lambda)$ gives $\lambda = \lambda'\lambda''$ with $\lambda' \in \Lambda^{\leqslant m-e_j}(v)$ and $\lambda'' \in \Lambda^{\leqslant e_j}(s(\lambda'))$. Now suppose (2) holds. Then we can factorize $\lambda = \lambda'\lambda''$ with $d(\lambda'') = e_j$, so $\lambda'' \in \Lambda^{\leqslant e_j}(s(\lambda'))$. We claim that $\lambda' \in \Lambda^{\leqslant m-e_j}(v)$. To see this, suppose that $\lambda' \notin \Lambda^{\leqslant m-e_j}(v)$. Then there exists i such that $d(\lambda') + e_i \leqslant m - e_j$ and $\Lambda^{e_i}(s(\lambda')) \neq \emptyset$, say $\alpha \in \Lambda^{e_i}(s(\lambda'))$. By (2) we know that $\langle d(\lambda'), e_j\rangle = \langle m - e_j, e_j\rangle$, so $i \neq j$. Since Λ is locally convex there is a $\beta \in \Lambda^{e_i}(s(\lambda''))$, but this implies $d(\lambda\beta) \leqslant m$, a contradiction since $\lambda \in \Lambda^{\leqslant m}(v)$. Hence $\lambda = \lambda'\lambda''$ with $\lambda' \in \Lambda^{\leqslant m-e_j}(v)$ and $\lambda'' \in \Lambda^{\leqslant e_j}(s(\lambda'))$. □

Remark 3.13. The k-graphs (Λ, d) studied in [**7**] have no sources; that is, $\lambda^m(v) \neq \emptyset$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$. Then $\Lambda^{\leqslant m}(v) = \Lambda^m(v)$, and Lemma 3.12 just says that a path $\lambda \in \Lambda^m(v)$ can be factorized into $\lambda = \lambda'\lambda''$ with $d(\lambda') = m - e_j$ and $d(\lambda'') = e_j$. Thus local convexity ensures that $\Lambda^{\leqslant m}(v)$ has factorization properties like those of $\Lambda^m(v)$.

Proof of Proposition 3.11. Property (3.2) merely consists of specific cases of Cuntz–Krieger relation (4), so suppose (3.2) holds. Define a map $l : \mathbb{N}^k \to \mathbb{N}$ by $l(m) = \sum_{i=1}^k m_i$. We prove (4) by induction on $l(n)$.

The $n = 0$ case is trivial since it amounts to the tautology $s_v = s_v$, and the $n = 1$ case follows directly from (3.2) since $l^{-1}(1) = \{e_1, \dots, e_k\}$. Suppose (4) holds for all m such that $l(m) \leqslant n$. Let $m \in \mathbb{N}^k$ satisfy $l(m) = n + 1$ and choose j such that $m_j \geqslant 1$. Using Lemma 3.12 we have

$$\begin{aligned}
\sum_{\lambda \in \Lambda^{\leqslant m}(v)} s_\lambda s_\lambda^* &= \sum_{\lambda' \in \Lambda^{\leqslant m - e_j}(v)} \sum_{\lambda'' \in \Lambda^{\leqslant e_j}(s(\lambda'))} s_{\lambda'\lambda''} s_{\lambda'\lambda''}^* \\
&= \sum_{\lambda' \in \Lambda^{\leqslant m - e_j}(v)} s_{\lambda'} \Bigg(\sum_{\lambda'' \in \Lambda^{\leqslant e_j}(s(\lambda'))} s_{\lambda''} s_{\lambda''}^* \Bigg) s_{\lambda'}^* \\
&= \sum_{\lambda' \in \Lambda^{\leqslant m - e_j}(v)} s_{\lambda'} s_{s(\lambda')} s_{\lambda'}^* \quad \text{by (3.2)} \\
&= \sum_{\lambda' \in \Lambda^{\leqslant m - e_j}(v)} s_{\lambda'} s_{\lambda'}^* \\
&= s_v \quad \text{by inductive hypothesis.}
\end{aligned}$$

Hence (4) holds whenever $l(m) = n + 1$. □

Kumjian and Pask use the infinite path space Λ^∞ of a k-graph Λ with no sources to produce a Cuntz–Krieger Λ-family of non-zero partial isometries (see [**7**, Proposition 2.11]). In a k-graph which admits sources, however, not every finite path is contained in an infinite path of the form defined in [**7**], and hence the proof of [**7**, Proposition 2.11] does not carry over. To allow for sources, we replace Λ^∞ with a boundary-path space $\Lambda^{\leqslant\infty}$; for locally convex k-graphs, we can achieve this construction using the k-graphs $\Omega_{k,m}$ of Example 2.2 (ii).

Definition 3.14. Let Λ be a locally convex k-graph. A *boundary path* in Λ is a graph morphism $x : \Omega_{k,m} \to \Lambda$ for some $m \in (\mathbb{N} \cup \{\infty\})^k$ such that, whenever $v \in \mathrm{Obj}(\Omega_{k,m})$ satisfies $(\Omega_{k,m})^{\leqslant e_i}(v) = \{v\}$, we also have $\Lambda^{\leqslant e_i}(x(v)) = \{x(v)\}$. We denote the collection of all boundary paths in Λ by $\Lambda^{\leqslant\infty}$. The range map of Λ extends naturally to $\Lambda^{\leqslant\infty}$ via $r(x) := x(0)$. For $v \in \Lambda^0$, we write $\Lambda^{\leqslant\infty}(v)$ for $\{x \in \Lambda^{\leqslant\infty} : r(x) = v\}$. We define a degree map $d_\infty : \Lambda^{\leqslant\infty} \to (\mathbb{N} \cup \{\infty\})^k$ by setting $d_\infty(x) := m$ when $x : \Omega_{k,m} \to \Lambda$.

As with the infinite paths of [**7**], a boundary path x is completely determined by the set of paths $\{x(0, p) : p \leqslant d_\infty(x)\}$. In fact, when the k-graph has no sources, $\Lambda^{\leqslant\infty}$ is exactly the infinite path space from [**7**]. If a k-graph Λ is locally convex, then for any vertex v

of Λ, the set $\Lambda^{\leqslant\infty}(v)$ is non-empty: even if v emits no paths of non-zero degree, we have $\Lambda^{\leqslant\infty}(v) = \{v\} \neq \emptyset$.

Theorem 3.15. *Let (Λ, d) be a row-finite k-graph. Then there is a Cuntz–Krieger Λ-family $\{S_\lambda : \lambda \in \Lambda\}$ with each S_λ non-zero if and only if Λ is locally convex.*

Proof. First suppose that Λ is not locally convex. Then there exists a vertex $v \in \Lambda^0$ and $\mu \in \Lambda^{e_i}(v)$ for some $i \in \{1, \dots, k\}$ such that $\Lambda^{e_j}(v) \neq \emptyset$ and $\Lambda^{e_j}(s(\mu)) = \emptyset$ for some $j \neq i$. Considering the partial isometry $s_\mu \in C^*(\Lambda)$, we have

$$s_\mu = s_v s_\mu = \sum_{\nu \in \Lambda^{e_j}(v)} s_\nu s_\nu^* s_\mu = \sum_{\nu \in \Lambda^{e_j}(v)} s_\nu \sum_{\substack{\nu\alpha = \mu\beta \\ d(\mu\beta) = e_i + e_j}} s_\alpha s_\beta^*,$$

but since $\Lambda^{e_j}(s(\mu)) = \emptyset$, no such β exists. Thus $s_\mu = 0$, and so by the universal property of $C^*(\Lambda)$ any Cuntz–Krieger Λ-family $\{S_\lambda : \lambda \in \Lambda\}$ must have $S_\mu = 0$.

Now suppose that Λ is locally convex. Let $\mathcal{H} := \ell^2(\Lambda^{\leqslant\infty})$, and for each $\lambda \in \Lambda$ define $S_\lambda \in B(\mathcal{H})$ by

$$S_\lambda u_x := \begin{cases} u_{\lambda x} & \text{if } s(\lambda) = r(x), \\ 0 & \text{otherwise,} \end{cases} \tag{3.5}$$

where $\{u_x : x \in \Lambda^{\leqslant\infty}\}$ is the usual basis for $\mathcal{H}$. Each $S_\lambda \neq 0$ because $\Lambda^{\leqslant\infty}(s(\lambda)) \neq \emptyset$. Cuntz–Krieger relations (1)–(3) follow directly from the definition of the operators S_λ; it remains to show that relation (4) is fulfilled. Since Λ is locally convex, by Proposition 3.11 it suffices to show that for each $v \in \Lambda^0$ and $i \in \{1, \dots, k\}$, $S_v = \sum_{\lambda \in \Lambda^{\leqslant e_i}(v)} S_\lambda S_\lambda^*$. If $\Lambda^{\leqslant e_i}(v) = \{v\}$, then the relation is trivially true, so suppose $\Lambda^{\leqslant e_i}(v) \neq \{v\}$. For $x \in \Lambda^{\leqslant\infty}$ we have

$$\begin{aligned}
\sum_{\lambda \in \Lambda^{\leqslant e_i}(v)} S_\lambda S_\lambda^* u_x &= \sum_{\lambda \in \Lambda^{\leqslant e_i}(v)} S_\lambda(\delta_{\lambda, x(0,e_i)} u_{x(e_i,\infty)}) \\
&= \sum_{\lambda \in \Lambda^{\leqslant e_i}(v)} \delta_{\lambda, x(0,e_i)} u_x \\
&= \begin{cases} u_x & \text{if there exists } \lambda \in \Lambda^{e_i}(v) \text{ such that } \lambda = x(0, e_i), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Taking $\lambda = v$ in (3.5), we can see that it suffices to show that $r(x) = v$ if and only if there exists $\lambda \in \Lambda^{e_i}(v)$ such that $\lambda = x(0, e_i)$. If $\lambda = x(0, e_i)$ for some $\lambda \in \Lambda^{e_i}(v)$, then $r(x) = r(\lambda) = v$. If $r(x) = v$, then $x(0, e_i) \in \Lambda^{e_i}(v)$ because $\Lambda^{\leqslant e_i}(v) \neq \{v\}$ and x is a boundary path. $\square$

4. The uniqueness theorems

4.1. The gauge-invariant uniqueness theorem

Our gauge-invariant uniqueness theorem extends [**7**, Theorem 3.4] to row-finite k-graphs with sources.

Let (Λ, d) be a row-finite k-graph. For $z \in \mathbb{T}^k$ and $n \in \mathbb{Z}^k$, let $z^n := z_1^{n_1} \cdots z_k^{n_k}$. Then $\{z^{d(\lambda)} s_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family which generates $C^*(\Lambda)$, and the universal property of $C^*(\Lambda)$ gives a homomorphism $\gamma_z : C^*(\Lambda) \to C^*(\Lambda)$ such that $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ for $\lambda \in \Lambda$; $\gamma_{\bar{z}}$ is an inverse for γ_z, so γ_z is an automorphism. An $\epsilon/3$-argument shows that γ is a strongly continuous action of $\mathbb{T}^k$ on $C^*(\Lambda)$, which is called the *gauge action*.

Theorem 4.1 (the gauge-invariant uniqueness theorem). *Let (Λ, d) be a locally convex row-finite k-graph, let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz–Krieger Λ-family, and let π be the representation of $C^*(\Lambda)$ such that $\pi(s_\lambda) = t_\lambda$. If each t_v is non-zero and there is a strongly continuous action $\beta : \mathbb{T}^k \to \operatorname{Aut}(C^*(\{t_\lambda : \lambda \in \Lambda\}))$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for $z \in \mathbb{T}^k$, then π is faithful.*

Remark 4.2. Strictly speaking, it is not necessary to assume that Λ is locally convex; if there is a Cuntz–Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ with each t_v non-zero, then Theorem 3.15 implies that Λ is locally convex.

The first part of the proof, the analysis of the core $C^*(\Lambda)^\gamma$, is the same for both uniqueness theorems. Using $\Lambda^{\leqslant p}$ instead of Λ^p, and Lemmas 3.6 and 3.7, we can follow the argument of [**7**, § 3]. We consider the map $\Phi : C^*(\Lambda) \to C^*(\Lambda)$ defined by

$$\Phi(a) := \int_{\mathbb{T}^k} \gamma_z(a) \, \mathrm{d}z,$$

which is faithful on positive elements and whose range is the fixed-point algebra $C^*(\Lambda)^\gamma$. To identify the structure of $C^*(\Lambda)^\gamma$, we let $v \in \Lambda^0$, $q \in \mathbb{N}^k$, and define

$$\mathcal{F}_q(v) := \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda^{\leqslant q},\ d(\lambda) = d(\mu),\ s(\lambda) = s(\mu) = v\}.$$

It follows from Lemma 3.7 that $\mathcal{F}_q(v)$ is the direct sum of the subalgebras

$$\mathcal{F}_{q,p}(v) := \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda^{\leqslant q},\ d(\lambda) = d(\mu) = p,\ s(\lambda) = s(\mu) = v\},$$

that

$$\mathcal{F}_{q,p}(v) = \mathcal{K}(\ell^2(\{\lambda \in \Lambda^{\leqslant q} : d(\lambda) = p \text{ and } s(\lambda) = v\})), \tag{4.1}$$

and that for fixed q, the $\mathcal{F}_q(v)$ are mutually orthogonal. Since the elements $s_\lambda s_\mu^*$ span $C^*(\Lambda)$, and since $\gamma_z(s_\lambda s_\mu^*) = z^{d(\lambda)-d(\mu)} s_\lambda s_\mu^*$, the algebras $\mathcal{F}_q := \oplus_{v \in \Lambda^0} \mathcal{F}_q(v)$ span $C^*(\Lambda)^\gamma$. When $p \leqslant q$, we have $\mathcal{F}_p \subset \mathcal{F}_q$ by Lemma 3.6, so $C^*(\Lambda)^\gamma$ is the direct limit $\overline{\cup_{q \in \mathbb{N}^k} \mathcal{F}_q}$ of the algebras $\mathcal{F}_q$. In particular, $C^*(\Lambda)^\gamma$ is AF.

Proof of Theorem 4.1. Because each t_v is non-zero, and t_λ has initial projection $t_{s(\lambda)}$, each t_λ is non-zero, and hence the representation π is non-zero on each $\mathcal{F}_{q,p}(v)$. It follows from (4.1) that π is faithful on $\mathcal{F}_{q,p}(v)$, hence on $\mathcal{F}_q(v)$ and on $\mathcal{F}_q$. Since $C^*(\Lambda)^\gamma = \varinjlim \mathcal{F}_q$, it follows that π is faithful on $C^*(\Lambda)^\gamma$ (see [**1**, Lemma 1.3], for example). We can now use the argument of [**7**, p. 11]. □

4.2. The Cuntz–Krieger uniqueness theorem

Our Cuntz–Krieger uniqueness theorem extends [**7**, Theorem 4.6] to row-finite k-graphs with sources (see Remark 4.4 below).

Theorem 4.3. *Let (Λ, d) be a locally convex row-finite k-graph, and suppose that*

$$\text{for each } v \in \Lambda^0, \text{ there exists } x \in \Lambda^{\leqslant\infty}(v) \text{ such that } \alpha \neq \beta \text{ implies } \alpha x \neq \beta x. \tag{B}$$

Let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz–Krieger Λ-family, and let π be the representation of $C^*(\Lambda)$ such that $\pi(s_\lambda) = t_\lambda$. If each t_v is non-zero, then π is faithful.

Proof. For $\lambda \in \Lambda \cup \Lambda^{\leqslant\infty}$ and $p \in \mathbb{N}^k$, we write $\lambda(0,p)$ for the unique path such that $\lambda = \lambda(0,p)\lambda'$ with $d(\lambda(0,p)) = p \wedge d(\lambda)$.

We know from the analysis in § 4.1 that π is faithful on the fixed-point algebra $C^*(\Lambda)^\gamma$, and hence the standard argument will work once we know that

$$\|\pi(\Phi(a))\| \leqslant \|\pi(a)\| \quad \text{for } a \in C^*(\Lambda). \tag{4.2}$$

Recalling that $\operatorname{span}\{s_\lambda s_\mu^* : s(\lambda) = s(\mu)\}$ is dense in $C^*(\Lambda)$, we consider arbitrary $a = \sum_{(\lambda,\mu)\in F} \zeta_{(\lambda,\mu)} s_\lambda s_\mu^*$, where F is finite and $\zeta_{(\lambda,\mu)} \in \mathbb{C}$. Let l be the least upper bound of $\{d(\lambda) \vee d(\mu) : (\lambda,\mu) \in F\}$. Then

$$\Phi(a) = \sum_{\{(\lambda,\mu)\in F : d(\lambda)=d(\mu)\}} \zeta_{(\lambda,\mu)} s_\lambda s_\mu^* \in \mathcal{F}_l.$$

For $(\lambda,\mu) \in F$, we define

$$F_{(\lambda,\mu)} = \{(\lambda\nu, \mu\nu) : \nu \in \Lambda^{\leqslant l-d(\lambda)}(s(\lambda))\},$$

and $E = \cup_{(\lambda,\mu)\in F} F_{(\lambda,\mu)}$. For $\nu \in \Lambda^{\leqslant l-d(\lambda)}(s(\lambda))$, we define

$$\xi_{(\lambda\nu,\mu\nu)} = \sum_{\{(\lambda',\mu')\in F : (\lambda\nu,\mu\nu)=(\lambda'\nu',\mu'\nu') \text{ for some } \nu'\in\Lambda\}} \zeta_{(\lambda',\mu')},$$

and using Cuntz–Krieger relation (4) we then have

$$a = \sum_{(\alpha,\beta)\in E} \xi_{(\alpha,\beta)} s_\alpha s_\beta^*;$$

the point is that now $\alpha \in \Lambda^{\leqslant l}$ for all $(\alpha,\beta) \in E$.

Since $\mathcal{F}_l$ decomposes as a direct sum $\oplus_{v\in\Lambda^0} \mathcal{F}_l(v)$, so does its image under π, and there is a vertex $v \in \Lambda^0$ such that

$$\|\pi(\Phi(a))\| = \Bigg\| \sum_{\{(\alpha,\beta)\in E : d(\alpha)=d(\beta) \text{ and } s(\alpha)=v\}} \xi_{(\alpha,\beta)} t_\alpha t_\beta^* \Bigg\|. \tag{4.3}$$

Choose a boundary path $x \in \Lambda^{\leqslant\infty}(v)$ such that $\alpha x \neq \beta x$ for all $\alpha \neq \beta \in \Lambda$; then for each $\alpha \neq \beta \in \Lambda$, there exists $M_{\alpha,\beta} \geqslant d(\alpha) \vee d(\beta)$ such that $(\alpha x)(0,m) \neq (\beta x)(0,m)$ whenever $m \geqslant M_{\alpha,\beta}$. Let

$$T = \{\tau \in \Lambda^{\leqslant l} : \tau = \alpha \text{ or } \tau = \beta \text{ for some } (\alpha,\beta) \in E,\ s(\tau) = v\}.$$

Let M be the least upper bound of $\{M_{\tau,\beta} : \tau \in T,\ (\alpha,\beta) \in E \text{ for some } \alpha\}$. In particular,

$$(\beta x)(0,M) \neq (\tau x)(0,M) \tag{4.4}$$

when β is the second coordinate of an element of E, $\tau \in T$, and $\beta \neq \tau$. Write x_M for $x(0,M)$.

For each $n \leqslant l$, we define

$$Q_n := \sum_{\{\tau \in T : d(\tau) = n\}} t_{\tau x_M} t^*_{\tau x_M}.$$

Now we define $Q : C^*(\{t_\lambda : \lambda \in \Lambda\}) \to C^*(\{t_\lambda : \lambda \in \Lambda\})$ by

$$Q(b) := \sum_{n \leqslant l} Q_n b Q_n.$$

Since the Q_n are mutually orthogonal projections, we have

$$\|Q(b)\| = \Big\| \sum_{n \leqslant l} Q_n b Q_n \Big\| \leqslant \|b\| \quad \text{for } b \in C^*(\{t_\lambda\}).$$

We aim to show that $\|Q(\pi(\Phi(a)))\| = \|\pi(\Phi(a))\|$, and that $Q(\pi(a)) = Q(\pi(\Phi(a)))$; this will give us

$$\|\pi(\Phi(a))\| = \|Q(\pi(\Phi(a)))\| = \|Q(\pi(a))\| \leqslant \|\pi(a)\|, \tag{4.5}$$

and the proof will be complete.

Write M_T for the matrix algebra spanned by $\{s_\lambda s^*_\mu : \lambda, \mu \in T,\ d(\lambda) = d(\mu)\}$. Notice that $M_T \subset \mathcal{F}_l(v)$. For $s_\lambda s^*_\mu \in M_T$ we have $\lambda, \mu \in \Lambda^{\leqslant l}$, so for $\tau \in T$, $t^*_\tau t_\lambda = 0$ unless $\tau = \lambda$, and $t^*_\mu t_\tau = 0$ unless $\tau = \mu$, and hence

$$\begin{aligned} Q(\pi(s_\lambda s^*_\mu)) &= \sum_{n \leqslant l} \Big(\sum_{\{\tau \in T : d(\tau) = n\}} t_{\tau x_M} t^*_{\tau x_M} \Big) t_\lambda t^*_\mu \Big(\sum_{\{\tau' \in T : d(\tau') = n\}} t_{\tau' x_M} t^*_{\tau' x_M} \Big) \\ &= t_{\lambda x_M} t^*_{x_M} t^*_\lambda t_\lambda t^*_\mu t_\mu t_{x_M} t^*_{\mu x_M} \\ &= t_{\lambda x_M} t^*_{\mu x_M} \\ &\neq 0. \end{aligned}$$

Using Lemma 3.7, it follows that $\{Q(\pi(s_\lambda s^*_\mu)) : s_\lambda s^*_\mu \in M_T\}$ is a family of non-zero matrix units, and from this we deduce that the map $b \mapsto Q(\pi(b))$ is a faithful representation of M_T. Since both π and $Q \circ \pi$ are faithful on M_T and since

$$\sum_{\{(\alpha,\beta) \in E : d(\alpha) = d(\beta) \text{ and } s(\alpha) = v\}} \xi_{(\alpha,\beta)} t_\alpha t^*_\beta \in M_T,$$

it follows from (4.3) that $\|\pi(\Phi(a))\| = \|Q(\pi(\Phi(a)))\|$.

To establish that $Q(\pi(a)) = Q(\pi(\Phi(a)))$, we show that $Q(t_\alpha t_\beta^*) = 0$ whenever $(\alpha, \beta) \in E$ and $d(\alpha) \neq d(\beta)$; this shows that Q kills those terms of $\pi(a)$ which are the images under π of terms of a killed by Φ. Notice that if $(\alpha, \beta) \in E$, then $\alpha \in \Lambda^{\leqslant l}$, so for $\tau \in T$, $t_\tau^* t_\alpha = 0$ unless $\tau = \alpha$. Hence, for $(\alpha, \beta) \in E$ with $d(\alpha) \neq d(\beta)$, we have

$$\begin{aligned} Q(t_\alpha t_\beta^*) &= \sum_{n \leqslant l} \Bigg(\sum_{\{\tau \in T : d(\tau) = n\}} t_{\tau x_M} t_{\tau x_M}^* \Bigg) t_\alpha t_\beta^* \Bigg(\sum_{\{\tau' \in T : d(\tau') = n\}} t_{\tau' x_M} t_{\tau' x_M}^* \Bigg) \\ &= \sum_{\{\tau' \in T : d(\tau') = d(\alpha)\}} t_{\alpha x_M} t_{\beta x_M}^* t_{\tau' x_M} t_{\tau' x_M}^* \\ &= \sum_{\{\tau' \in T : d(\tau') = d(\alpha)\}} t_{\alpha x_M} \Bigg(\sum_{\substack{\beta x_M \eta = \tau' x_M \zeta \\ d(\beta x_M \eta) = d(\beta x_M) \vee d(\tau' x_M)}} t_\eta t_\zeta^* \Bigg) t_{\tau' x_M}^*, \end{aligned}$$

which is non-zero if and only if there exist $\eta, \zeta \in \Lambda$ such that

$$(\beta x_M \eta)(0, M) = (\tau' x_M \zeta)(0, M). \tag{4.6}$$

But $(\beta x_M \eta)(0, M) = (\beta x_M)(0, M)$: if not, then there exists an i such that $d(\beta x_M)_i < M_i$ and $d(\eta)_i > 0$. But

$$d(\beta x_M)_i < M_i \implies d(x_M)_i < M_i \implies \Lambda^{e_i}(s(x_M)) = \emptyset,$$

since x is a boundary path. Likewise, $(\tau' x_M \zeta)(0, M) = (\tau' x_M)(0, M)$, and so (4.6) is equivalent to $(\beta x_M)(0, M) = (\tau' x_M)(0, M)$, which is impossible by (4.4). This proves (4.5), and the result follows. $\square$

Remark 4.4. Condition (B) in Theorem 4.3 is automatic if Λ has no sources and satisfies the aperiodicity condition (A) of [**7**, Definition 4.3]. To see this, let σ be the shift map on Λ^∞ defined as in [**7**] by $\sigma^p(x) = x(p, \infty)$. Suppose that Λ has no sources and that (B) does not hold. Then there is a vertex $v \in \Lambda^0$ such that for each $x \in \Lambda^\infty(v)$, there exist $\alpha_x \neq \beta_x$ such that $\alpha_x x = \beta_x x$. Then for $x \in \Lambda^\infty(v)$,

$$\begin{aligned} \sigma^{d(\alpha_x) \vee d(\beta_x) - d(\alpha_x)}(x) &= \sigma^{d(\alpha_x) \vee d(\beta_x)}(\alpha_x x) \\ &= \sigma^{d(\alpha_x) \vee d(\beta_x)}(\beta_x x) = \sigma^{d(\alpha_x) \vee d(\beta_x) - d(\beta_x)}(x). \end{aligned}$$

Hence condition (A) of [**7**, Definition 4.3] does not hold at v.

Thus Theorem 4.3 is formally stronger than [**7**, Theorem 4.6] even when Λ has no sources. We have been unable to decide whether it is equivalent: we do not know whether in graphs without sources, (B) implies (A). For 1-graphs with no sources, we can prove that (B) implies (A) because it is easy to construct aperiodic paths (see the proof of [**9**, Lemma 3.4]). In higher-rank graphs, it is hard to produce aperiodic paths, and we suspect that in practice (B) might be easier to check than (A).

5. The ideal structure

Let (Λ, d) be a locally convex row-finite k-graph. Define a relation on Λ^0 by setting $v \geqslant w$ if there is a path $\lambda \in \Lambda$ with $r(\lambda) = v$ and $s(\lambda) = w$. A subset H of Λ^0 is *hereditary* if $v \geqslant w$ and $v \in H$ imply $w \in H$; H is *saturated* if, for $v \in \Lambda^0$,

$$\{s(\lambda) : \lambda \in \Lambda^{\leqslant e_i}(v)\} \subset H \text{ for some } i \in \{1, \dots, k\} \implies v \in H.$$

The *saturation* of a set H is the smallest saturated subset $\bar{H}$ of Λ^0 containing H.

Lemma 5.1. *Suppose Λ is a locally convex row-finite k-graph, and H is a hereditary subset of Λ^0. Then the saturation $\bar{H}$ is hereditary.*

Proof. We use an inductive construction of $\bar{H}$ like that used by Szymański for 1-graphs in [**13**]. For $F \subset \Lambda^0$, we define

$$\Sigma(F) := \bigcup_{i=1}^{k} \{v \in \Lambda^0 : s(\lambda) \in F \text{ for all } \lambda \in \Lambda^{\leqslant e_i}(v)\},$$

and write $\Sigma^n(F)$ for the set obtained by repeating the process n times. Notice that if F is hereditary, then $F \subset \Sigma(F)$. We will show that $\bigcup_{n=1}^{\infty} \Sigma^n(H)$ is hereditary and equal to $\bar{H}$.

We begin by showing that if F is hereditary, then $\Sigma(F)$ is hereditary. To see this, suppose that $v \in \Sigma(F)$ and that $v \geqslant w$. Then there exists $\lambda \in \Lambda$ such that $r(\lambda) = v$ and $s(\lambda) = w$. If $d(\lambda) = 0$, then $w = v \in F$, so suppose $d(\lambda)_j > 0$, and factor $\lambda = \lambda'\lambda''$, where $d(\lambda') = e_j$. We claim that $s(\lambda') \in \Sigma(F)$. To see this, choose i such that $\{s(\mu) : \mu \in \Lambda^{\leqslant e_i}(v)\} \subset F$. If $\Lambda^{\leqslant e_i}(v) = \{v\}$ or if $i = j$, then $s(\lambda') \in F \subset \Sigma(F)$. So suppose that $\Lambda^{\leqslant e_i}(v) \neq \{v\}$ and $i \neq j$. Since Λ is locally convex, $\Lambda^{e_i}(s(\lambda')) \neq \emptyset$, so it suffices to show that $\nu \in \Lambda^{e_i}(s(\lambda'))$ implies $s(\nu) \in F$. Let $\nu \in \Lambda^{e_i}(s(\lambda'))$. Then $\lambda'\nu = \mu\nu'$ for some $\mu \in \Lambda^{\leqslant e_i}(v)$. But now $s(\mu) \in F$ and hence $s(\nu') = s(\nu) \in F$ because F is hereditary. Thus $s(\lambda') \in \Sigma(F)$ as claimed. By induction on the length of λ, it follows that $w \in \Sigma(F)$, and hence $\Sigma(F)$ is hereditary.

We now know that $\Sigma^n(H) \subset \Sigma^{n+1}(H)$ for all n, and that $\Sigma^n(H)$ is hereditary for all n; thus $\bigcup_{n=1}^{\infty} \Sigma^n(H)$ is also hereditary. It remains to show that $\bar{H} = \bigcup_{n=1}^{\infty} \Sigma^n(H)$. Because applying Σ can never take us outside of a saturated set, we have $\bigcup \Sigma^n(H) \subset \bar{H}$, so it is enough to show that $\bigcup_{n=1}^{\infty} \Sigma^n(H)$ is saturated. To see this, suppose that $v \in \Lambda^0$ and $\{s(\lambda) : \lambda \in \Lambda^{\leqslant e_i}(v)\} \subset \bigcup_{n=1}^{\infty} \Sigma^n(H)$. Then, since Λ is row finite, we have $\{s(\lambda) : \lambda \in \Lambda^{\leqslant e_i}(v)\} \subset \Sigma^N(H)$ for some N and it follows that $v \in \Sigma^{N+1}(H)$. Thus $\bigcup_{n=1}^{\infty} \Sigma^n(H)$ is saturated. $\square$

Theorem 5.2. *Let (Λ, d) be a locally convex row-finite k-graph. For each subset H of Λ^0, let I_H be the closed ideal in $C^*(\Lambda)$ generated by $\{s_v : v \in H\}$.*

(a) *The map $H \mapsto I_H$ is an isomorphism of the lattice of saturated hereditary subsets of Λ^0 onto the lattice of closed gauge-invariant ideals of $C^*(\Lambda)$.*

(b) *Suppose H is saturated and hereditary. Then*

$$\Gamma(\Lambda \setminus H) := (\Lambda^0 \setminus H, \{\lambda \in \Lambda : s(\lambda) \notin H\}, r, s)$$

is a locally convex row-finite k-graph, and $C^*(\Lambda)/I_H$ is canonically isomorphic to $C^*(\Gamma(\Lambda \setminus H))$.

(c) *If H is any hereditary subset of Λ^0, then*

$$\Lambda(H) := (H, \{\lambda \in \Lambda : r(\lambda) \in H\}, r, s)$$

is a locally convex row-finite k-graph, $C^*(\Lambda(H))$ is canonically isomorphic to the subalgebra $C^*(\{s_\lambda : r(\lambda) \in H\})$ of $C^*(\Lambda)$, and this subalgebra is a full corner in I_H.

Proof. The proof of Theorem 5.2 is the same as the proof of [**2**, Theorem 4.1] once we establish that $\Gamma(\Lambda \setminus H)$ and $\Lambda(H)$ from parts (b) and (c) are locally convex row-finite k-graphs. This is easy to check for $\Lambda(H)$, and the row finiteness of $\Gamma(\Lambda \setminus H)$ follows from that of Λ. We need to check that $\Gamma(\Lambda \setminus H)$ is a k-graph and is locally convex. For convenience, write $\Gamma = \Gamma(\Lambda \setminus H)$.

To show that the factorization property holds for (Γ, d), take $\lambda \in \Gamma$ and suppose $d(\lambda) = p + q$. We know there exist unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = p$ and $d(\nu) = q$. Certainly $s(\nu) = s(\lambda) \notin H$, and if $s(\mu) \in H$, then $s(\nu) \in H$, a contradiction. Hence $\mu, \nu \in \Gamma$, and Γ is a sub-k-graph.

Now we show that Γ is locally convex. Consider an arbitrary vertex $v \in \Gamma^0$ which has $\lambda \in \Gamma^{e_i}(v)$ and $\mu \in \Gamma^{e_j}(v)$ for some $i \neq j$. We know that $s(\lambda), s(\mu) \notin H$, and since Λ is locally convex, we also know there exist $\alpha \in \Lambda^{e_i}(s(\mu))$ and $\beta \in \Lambda^{e_j}(s(\lambda))$. If $\Gamma^{e_i}(s(\mu)) = \emptyset$, then $\{s(\alpha) : \alpha \in \Lambda^{\leqslant e_i}(s(\mu))\} \subset H$, and similarly, if $\Gamma^{e_j}(s(\mu)) = \emptyset$, then $\{s(\beta) : \beta \in \Lambda^{\leqslant e_j}(s(\lambda))\} \subset H$; in either case, $s(\mu) \in H$ or $s(\lambda) \in H$ because H is saturated, and we have a contradiction. Hence $\Gamma^{e_i}(s(\mu)) \neq \emptyset$ and $\Gamma^{e_j}(s(\lambda)) \neq \emptyset$, so Γ is locally convex. □

The proof of the next theorem is the same as the first two paragraphs of the proof of [**2**, Theorem 4.1] except that in the first paragraph, we apply the Cuntz–Krieger uniqueness theorem rather than the gauge-invariant uniqueness theorem to show that $H \mapsto I_H$ is surjective.

Theorem 5.3. *Let (Λ, d) be a locally convex row-finite k-graph such that for every saturated hereditary subset H of Λ^0, $\Gamma(\Lambda \setminus H)$ satisfies condition (B) of Theorem 4.3. Then $H \mapsto I_H$ is an isomorphism of the lattice of saturated hereditary subsets of Λ^0 onto the lattice of closed ideals of $C^*(\Lambda)$.*

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