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Abstract
Mackey's imprimitivity theorem characterizes the unitary representations of a locally compact group $G$ which have been induced from representations of a closed subgroup $K$; Rieffel's influential reformulation says that the group $C^*$-algebra $C^*(K)$ is Morita equivalent to the crossed product $C_0(G/K) \times G$ [14]. There have since been many important generalizations of this theorem, especially by Rieffel [15, 16] and by Green [3, 4]. These are all special cases of the symmetric imprimitivity theorem of [11], which gives a Morita equivalence between two crossed products of induced $C^*$-algebras.

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Regularity of induced representations and a theorem of Quigg and Spielberg

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Mackey’s imprimitivity theorem characterizes the unitary representations of a locally compact group $G$ which have been induced from representations of a closed subgroup $K$; Rieffel’s influential reformulation says that the group $C^*$-algebra $C^*(K)$ is Morita equivalent to the crossed product $C_0(G/K) \times G$ [14]. There have since been many important generalizations of this theorem, especially by Rieffel [15, 16] and by Green [3, 4]. These are all special cases of the symmetric imprimitivity theorem of [11], which gives a Morita equivalence between two crossed products of induced $C^*$-algebras.

Quigg and Spielberg proved in [10], by ingenious but indirect methods, that the symmetric imprimitivity theorem, and hence all the other generalizations of Rieffel’s imprimitivity theorem, have analogues for reduced crossed products. A different Morita equivalence between the same reduced crossed products was obtained by Kasparov [7, theorem 3.15].

Here we identify the representations which induce to regular representations under the Morita equivalence of the symmetric imprimitivity theorem (see Theorem 1 and Corollary 6), and thus obtain a direct proof of the theorem of Quigg and Spielberg (see Corollary 3). We discovered Theorem 1 while trying to understand why Rieffel’s theory of proper actions in [17] gives an equivalence involving reduced crossed products rather than full ones.

Theorem 1 has several other interesting applications. We can use it to see, albeit somewhat indirectly, that regular representations themselves nearly always induce to regular representations (see Corollary 7), and it gives a new proof of the main theorem of [5, section 4] which avoids a complicated argument involving a composition of crossed-product Morita equivalences (see Corollary 11). It also sheds light on constructions in [2] and [8], which, in various special cases of the symmetric imprimitivity theorem, yield pairs of regular representations which induce to each other (see Remarks 10).

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Notation. We shall use left Haar measure, because we always do, and the right-
regular representation $\rho$, because this is what seems to come naturally out of our con-
structions; this means that modular functions intrude, so that $\rho = \rho^G : G \to U(L^2(G))$
is defined by $(\rho \xi)(t) = \xi(st)\Delta_G(t)^{1/2}$. We also write $\rho$ for the representation on
$L^2(G, \mathcal{M})$ defined by the same formula. With our conventions, the regular repre-
sentation of $A \times \alpha G$ induced from a representation $\pi : A \to B(\mathcal{M})$ is the integrated
form $\pi \times \rho$ of the covariant representation $(\pi, \rho)$ of $(A, G, \alpha)$ on $L^2(G, \mathcal{M})$ in which
$(\pi(a)\xi)(s) = \pi(\alpha_s(a))(\xi)(s))$. Given a non-degenerate representation $\mu$ of $A$ we denote
by $\mu$ its extension to the multiplier algebra $M(A)$ of $A$. We shall denote by $\text{lt}$ and $\text{rt}$
any actions of groups by, respectively, left and right translation; thus if $H$ acts on
the left of a locally compact space $P$ and $K$ acts on the right, we have $\text{lt}_s(f)(p) = f(s^{-1}p)$
and $\text{rt}_t(f)(p) = f(pt)$ for $f \in C_0(P)$.

1. The main theorem

We start with commuting free and proper actions of two locally compact groups
$H$ and $K$ on the left and right of a locally compact space $P$, and commuting actions
$\tau : H \to \text{Aut} C$ and $\sigma : K \to \text{Aut} C$ on a $C^*$-algebra $C$. The induced $C^*$-algebra
$\text{Ind}(C, \sigma)$ is the $C^*$-subalgebra of $C_0(P, C)$ consisting of the functions $f$ such that
$f(pt) = \sigma_t^{-1}(f(p))$ for all $t \in K$ and $p \in P$, and such that the function $pK \mapsto \|f(p)\|
vanishes at infinity on $P/K$. The diagonal action $\text{lt} \otimes \tau$ on $C_0(P, C) \subset M(C_0(P, C))$
restricts to a well-defined strongly continuous action of $H$ on $\text{Ind}(C, \sigma)$, which is
classified by $(\text{lt} \otimes \tau)_s(f)(p) = \tau_s(f(s^{-1}p))$. (The continuity of this action was
established in [5, lemma 5.1].) Likewise, $\text{Ind}(C, \tau)$ consists of the bounded continuous
functions $f : P \to C$ such that $f(sp) = \tau_s(f(p))$ for $s \in H$ and $Hp \mapsto \|f(p)\|
vanishes at infinity on $H \setminus P$, and we have a natural action $\text{rt} \otimes \sigma$ of $K$ on $\text{Ind}(C, \tau)$
given by $(\text{rt} \otimes \sigma)_t(f)(p) = \sigma_t(f(pt))$.

The symmetric imprimitivity theorem of [11, theorem 1.1] shows how to make
$X_0 := C_c(P, C)$ into a pre-imprimitivity bimodule whose completion $X$ implements a
Morita equivalence between $\text{Ind}(C, \sigma) \times_{\text{lt} \otimes \tau} H$ and $\text{Ind}(C, \tau) \times_{\text{rt} \otimes \sigma} K$. Since a shortage of Greek letters and left-right ambivalence have previously led to conflicts of
notation, it is worthwhile to record the formulas we use:

$$
\begin{align*}
a \cdot x(p) &= \int_H a(s, p)\tau_s(x(s^{-1}p))\Delta_H(s)^{1/2} \, ds \\
x \cdot d(p) &= \int_K \sigma_t(x(pt)d(t^{-1}, pt))\Delta_K(t)^{-1/2} \, dt \\
\langle x, y \rangle_{\text{Ind}(C, \sigma) \times H} &= \int_K \sigma_t(x(pt)\tau_s(y(s^{-1}pt)^*) \, dt \Delta_H(s)^{-1/2} \\
\langle x, y \rangle_{\text{Ind}(C, \tau) \times K} &= \int_H \tau_s(x(s^{-1}p)^* \sigma_t(y(s^{-1}pt))) \, ds \Delta_K(t)^{-1/2},
\end{align*}
$$

where $a \in C_c(H, \text{Ind}(C, \sigma)) \subset \text{Ind}(C, \sigma) \times H$, $d \in C_c(K, \text{Ind}(C, \tau)) \subset \text{Ind}(C, \tau) \times K$, and
$x, y$ belong to $X_0 = C_c(P, C)$.

We shall also need the one-sided version of this bimodule which is based on the
same space $Z_0 := C_c(P, C)$ but omits all mention of the group $H$; this bimodule is the
dual of the bimodule first considered in [12]. Thus we denote by $Z$ the completion
of $Z_0 = C_c(P, C)$ as an $\text{Ind}(C, \sigma)^{\ast} \otimes (C_0(P, C) \times \text{rt} \otimes \sigma) K$ imprimitivity bimodule. We use
that which intertwines the given representations.

We shall prove that as desired.

\[ \langle x, y \rangle_{C_0(P, C) \times K(t, p)} = x(p)^* \sigma_x(y(p)t) \Delta_K(t)^{-1/2}. \]

**Theorem 1.** Suppose \((\mu, U)\) is a covariant representation of \((C_0(P, C), K, \mu \otimes \sigma)\) on \(\mathcal{H}\). Then the representation

\[ X \cdot \text{Ind}_{C_0(P, C) \times K} \left( \mu \times \text{Ind}_{C_1} \right) \times U): \text{Ind}(C, \sigma) \times H \rightarrow B(X \otimes \text{Ind}_{C_1} \times K \mathcal{H}) \]

is unitarily equivalent to the right-regular representation

\[ (Z \cdot \text{Ind}_{C_0(P, C) \times K} \left( \mu \times U \right))^{-1} \times \rho: \text{Ind}(C, \sigma) \times H \rightarrow B(L^2(H, Z \otimes C_0(P, C) \times K \mathcal{H})). \]

**Proof.** We shall write \(D := \text{Ind}(C, \sigma) \times H \) and \(B := C_0(P, C) \times H \); we are going to view \(X_0 \equiv C_c(K, \text{Ind}(C, \sigma))\) as the pre-Hilbert \(C_c(K, \text{Ind}(C, \sigma))\)-module described above, and also as a dense subspace of the Hilbert \(B\)-module \(Z\).

For \(x \otimes_d h \in X_0 \otimes \mathcal{H}\), we define \(W(x \otimes_d h): H \rightarrow Z \otimes_B \mathcal{H}\) by

\[ W(x \otimes_d h)(s) := (\text{lt} \otimes \tau)_s(x) \otimes_B h. \]

We shall prove that \(W\) extends to a unitary operator of \(X \otimes_d \mathcal{H}\) onto \(L^2(H, Z \otimes_B \mathcal{H})\) which intertwines the given representations.

We first prove that \(W\) is well-defined and isometric: for both, it suffices to show that

\[ (W(x \otimes_d h) | W(y \otimes_d \tilde{\mu}(f)k)) = (x \otimes_d h | y \otimes_d \tilde{\mu}(f)k) \]

for \(x, y \in X_0, h, k \in \mathcal{H}\) and \(f \in C_c(P) \subset M(C_0(P, C))\). (Inserting the function \(f\) allows us to deduce from the properness of the actions that the integrands in the following calculations have compact support.) To prove (1), we note that

\[ (x \otimes_B h | y \otimes_B \tilde{\mu}(f)k) = (\mu \times U(\langle y, x \rangle_B)h | \tilde{\mu}(f)k) = \int_K (\mu(\langle y, x \rangle_B(t))u_t h | \tilde{\mu}(f)k) \, dt \]

\[ = \int_K (\mu(p \mapsto \overline{\langle x, y \rangle D}(t))u_t h | \tilde{\mu}(f)k) \, dt. \]

From this and an application of Fubini’s Theorem, we deduce that

\[ (W(x \otimes_d h) | W(y \otimes_d \tilde{\mu}(f)k)) = \int_{H} \left( (\text{lt} \otimes \tau)_s(x) \otimes_B h | (\text{lt} \otimes \tau)_s(y) \otimes_B \tilde{\mu}(f)k \right) \, ds \]

\[ = \int_{H} \int_{K} (\mu(p \mapsto \overline{\langle x, y \rangle D}(t))u_t h | \tilde{\mu}(f)k) \, dt \, ds \]

\[ = \int_{K} (\tilde{\mu}(\overline{\langle x, y \rangle D}(t))u_t h | \tilde{\mu}(f)k) \, dt \]

\[ = (x \otimes_d h | y \otimes_d \tilde{\mu}(f)k), \]

as desired.
We next prove that $W$ is surjective. We begin by observing that, with the pointwise action $w \cdot b(s) := w(s) \cdot b$ and the inner product

$$
\langle v, w \rangle_B := \int_H \langle v(s), w(s) \rangle_B \, ds,
$$

$C_c(H, Z_0)$ is a pre-Hilbert $B$-module; we denote its completion by $L^2(H, Z)$. (Although we shall not need this, it might help to observe that $L^2(H, Z)$ is naturally isomorphic to the external Hilbert-module tensor product $L^2(H)_C \otimes Z_B$ of, for example, \cite[3.4]{13}.) Writing out the formulas shows that the map $w \otimes_B h \mapsto (s \mapsto w(s) \otimes_B h)$ is inner-product preserving from $L^2(H, Z) \otimes_B \mathcal{H}$ to $L^2(H, Z \otimes_B \mathcal{H})$, and we can see by considering elementary tensors $w$ and $h$ that $L^2(H, Z \otimes_B \mathcal{H})$ is dense in $L^2(H, Z)$. Thus the map $\Phi: C_c(P \times P, C, \sigma) \to C_c(H \times P, C, \sigma)$ defined by $\Phi(w)(s, p) = \tau_s(w(s^{-1}p))$ is linear isometry which preserves the kind of approximation we require. For $v \in C_c(H \times P, C)$, we choose an extension $w$ of $\Phi^{-1}(v)$ to a function of compact support on $P \times P$, and now standard arguments show that we can approximate $w$ in $C_c(P \times P, C)$ by functions $\sum x_i \otimes f_i$ in $C_c(P, C) \otimes C_c(P)$; then $\Phi(\sum x_i \otimes f_i)$ is the required approximation to $v$. Thus $W$ is surjective.

It remains to check that $W$ intertwines the given representations as claimed. Let $a \in C_c(H, \text{Ind}(C, \sigma))$, and for this calculation denote the action $\text{Ind}(\mu \times U)(a) : H \to H$ on $\text{Ind}(C, \sigma)$ by $\alpha$. Then for $x \otimes_B h \in X_0 \otimes \mathcal{H}$, we have

$$
W \left( \text{Ind}(\mu \times U)(a)(x \otimes_B h) \right)(s) = W((a \cdot x) \otimes_B h)(s) = \alpha_s(a \cdot x) \otimes_B h. \tag{2}
$$
On the other hand,

\[
(Z-\text{Ind}(\mu \times U)^- \times \rho(a))(W(x \otimes_D h))(s)
= \int_H Z-\text{Ind}(\mu \times U)(\alpha_s(a(r)))\left(\rho_r(W(x \otimes_D h))(s)\right) \, dr
= \int_H Z-\text{Ind}(\mu \times U)(\alpha_s(a(r)))\left(W(x \otimes_D h)(sr)\Delta_H(r)^{1/2}\right) \, dr
= \int_H (\langle\alpha_s(a(r)) \cdot \alpha_{sr}(x)\rangle \otimes_B h)\Delta_H(r)^{1/2} \, dr.
\]

(3)

Since we have

\[
\alpha_s(a \cdot x)(p) = \tau_s(a \cdot x(s^{-1}p))
= \tau_s\left(\int_H a(r, s^{-1}p)\tau_r(x(r^{-1}s^{-1}p))\Delta_H(r)^{1/2} \, dr\right)
= \int_H (\alpha_s(a(r)) \cdot \alpha_{sr}(x))(p)\Delta_H(r)^{1/2} \, dr,
\]

the only difference between (2) and (3) is the location of the integral with respect to \(\otimes_B h\). But the integrands in both formulas are continuous and compactly supported, so there is no difficulty verifying that they have the same inner product with every vector of the form \(y \otimes_B k \in Z_0 \otimes \mathcal{H}\), and hence must be equal. We deduce that

\[
W(X-\text{Ind}(\bar{\mu} \times U)(a)) = (Z-\text{Ind}(\mu \times U)^- \times \rho)(a)W,
\]

and we have proved the theorem.

Remark 2. The referee pointed out that Theorem 1 suggests that there is, and would follow from, an isomorphism

\[
X \otimes_{\text{Ind}(C, \tau) \times K} \left(C_0(P, C) \times K\right) \cong Y \otimes_{\text{Ind}(C, \sigma)} Z
\]

(4)
of right-Hilbert \(\text{Ind}(C, \sigma) \times H\)-\(\left(C_0(P, C) \times K\right)\) bimodules, where \(Y\) is the Hilbert bimodule of Green which induces representations of \(\text{Ind}(C, \sigma)\) to regular representations of \(\text{Ind}(C, \sigma) \times H\). Such isomorphisms have proved to be a powerful tool for studying the duality between induction and restriction of representations [6]. If applications arise which require functorial properties of the equivalence in Theorem 1, then establishing such an isomorphism might be an efficient way to proceed; for our present applications, Theorem 1 suffices.

2. Applications

2.1. The theorem of Quigg and Spielberg

We retain the notation of the preceding section. We shall say that a dynamical system \((A, G, \alpha)\) is amenable if the regular representation induced from a faithful representation of \(A\) is faithful on \(A \times G\).

Corollary 3 ([10]). Denote by \(I\) and \(J\) the kernels of the quotient maps of \(\text{Ind}(C, \tau) \times_{\tau \otimes \sigma} K\) and \(\text{Ind}(C, \sigma) \times_{\tau \otimes \tau} H\) onto the reduced crossed products. Then \(X-\text{Ind} I = J\). In particular, this implies that \(\text{Ind}(C, \tau) \times_{\tau \otimes \tau} K\) is Morita equivalent to \(\text{Ind}(C, \sigma) \times_{\tau \otimes \tau} H\), and that the system \((\text{Ind}(C, \sigma), H, \text{rt} \otimes \tau)\) is amenable if and only if \((\text{Ind}(C, \tau), K, \text{rt} \otimes \sigma)\) is amenable.
Every regular representation \( \tilde{\pi} \times \rho \) induced from a faithful representation \( \pi \) of \( \text{Ind}(C, \tau) \) has the same kernel \( I \), and \( X\text{-Ind}(\tilde{\pi} \times \rho) \) has kernel \( X\text{-Ind}I \). We choose \( \pi \) to be the restriction \( \varepsilon|_{\text{Ind}(C, \tau)} \) of a faithful nondegenerate representation of \( C_0(P, C) \). Then \((\tilde{\pi}, \rho)\) is the restriction of the regular representation \((\tilde{\nu}, \rho)\) of \( (C_0(P, C), K, \text{rt} \otimes \sigma) \), and we can apply Theorem 1 with \((\mu, U) = (\tilde{\nu}, \rho)\). We deduce that, for this \( \pi \), \( X\text{-Ind}(\tilde{\pi} \times \rho) \) is equivalent to the right-regular representation of \( \text{Ind}(C, \sigma) \times_{K \otimes \tau} H \) induced from the representation \( Z\text{-Ind}(\tilde{\nu} \times \rho) \) of \( \text{Ind}(C, \sigma) \). Thus

\[
X\text{-Ind}I = \ker(X\text{-Ind}(\tilde{\pi} \times \rho)) = \ker(Z\text{-Ind}(\tilde{\nu} \times \rho) \rtimes \rho) \subset J. \tag{5}
\]

(If we knew that \( Z\text{-Ind}(\tilde{\nu} \times \rho) \) is faithful then we would have equality in 5 above; instead we show that equality holds and deduce that \( Z\text{-Ind}(\tilde{\nu} \times \rho) \) is faithful.) By symmetry \( X\text{-Ind} J \subset I \), and now applying \( X\text{-Ind} I \) we see that \( J \subset X\text{-Ind} I \).

It follows from standard properties of the Rieffel correspondence [13, proposition 3.25] that the reduced crossed products are Morita equivalent. Finally, if \((\text{Ind}(C, \tau), K, \text{rt} \otimes \sigma)\) is amenable, then \( I = \{0\} \), so \( J = \{0\} \) and the system \((\text{Ind}(C, \sigma), H, \text{lt} \otimes \tau)\) is amenable; the last part follows by symmetry.

One special case is worth mentioning because the possibility of such a result was specifically mooted in [17, page 171], and because the proof we have given is more direct than others.

**Corollary 4.** Suppose \( H \) acts freely and properly on \( P \) and \( \tau \) is an action of \( H \) on a \( C^* \)-algebra \( C \). Then

\[
C_0(P, C) \times_{K \otimes \tau} H = C_0(P, C) \times_{K \otimes \tau} \tau H.
\]

**Proof.** This is the special case of Corollary 3 in which \( K = \{e\} \); the dynamical system \((\text{Ind}(C, \tau), K, \text{rt} \otimes \sigma)\) is then trivially amenable.

2.2. Inducing regular representations

Because the symmetric imprimitivity theorem passes to reduced crossed products, it is natural to ask whether the symmetric imprimitivity theorem matches up the regular representations themselves. More precisely, if \( \tilde{\pi} \times \rho \) is a regular representation of \( \text{Ind}(C, \tau) \times_{K \otimes \sigma} K \), is the induced representation \( X\text{-Ind}(\tilde{\pi} \times \rho) \) of \( \text{Ind}(C, \sigma) \times_{K \otimes \tau} H \) regular? We can settle this question, though in a rather roundabout fashion (see Corollary 7 below).

We begin our analysis of this question by observing that Theorem 1 can be used to characterize the representations of \( \text{Ind}(C, \tau) \times_{K \otimes \sigma} K \) which induce to regular representations of \( \text{Ind}(C, \sigma) \times_{K \otimes \tau} H \).

**Corollary 5.** Let \((\nu, V)\) be a covariant representation of \((\text{Ind}(C, \tau), K, \text{rt} \otimes \sigma)\) on \( \mathcal{H} \). Then the representation \( X\text{-Ind}(\nu \times V) \) is regular if and only if there is a covariant representation \((\mu, U)\) of \((C_0(P, C), K, \text{rt} \otimes \sigma)\) on \( \mathcal{H} \) such that \((\nu, V)\) is equivalent to \((\tilde{\mu}|_{\text{Ind}(C, \tau)}, U)\).

**Proof.** Theorem 1 immediately gives the ‘if’ direction. So suppose \( X\text{-Ind}(\nu \times V) \) is equivalent to the regular representation \( \tilde{\pi} \times \rho \) for some representation \( \pi \) of \( \text{Ind}(C, \sigma) \). Let \( \mu \times U := \tilde{Z}\text{-Ind}(\tilde{\pi}|_{\text{Ind}(C, \tau)} \times U) \), and note that \( Z\text{-Ind}(\mu \times U) \) is equivalent to \( \pi \). Theorem 1 implies that \( X\text{-Ind}(\tilde{\mu}|_{\text{Ind}(C, \tau)} \times U) \) is equivalent to \( \tilde{\pi} \times \rho \), and applying \( \tilde{X}\text{-Ind} \) shows that \((\nu, V)\) is equivalent to \((\tilde{\mu}|_{\text{Ind}(C, \tau)}, U)\).
Corollary 6. Let $\pi$ be a non-degenerate representation of $\text{Ind}(C, \tau)$. Then there is a covariant representation $(\mu, U)$ of $(C_0(P, C), H, \text{lt} \otimes \tau)$ such that

$$X \text{-Ind}_{\text{Ind}(C, \tau)}^{\text{Ind}(C, \sigma) \times K}(\tilde{\pi} \times \rho) = \tilde{\mu}|_{\text{Ind}(C, \sigma)} \times U.$$ 

Proof. This follows from Corollary 5 by taking advantage of the symmetry of the situation. Suppose that $\pi$ is a representation of $\text{Ind}(C, \sigma)$ instead. Let $(\nu, V) = X \text{-Ind}(\tilde{\pi} \times \rho)$ and note that $X \text{-Ind}(\nu \times V)$ is regular because it is equivalent to $\tilde{\pi} \times \rho$. By Corollary 5 $(\nu, V)$ is the restriction of a covariant representation $(\mu, U)$ of $(C_0(P, C), K, \text{rt} \otimes \sigma)$.

Corollary 7. Suppose, in addition to our standard assumptions on $H E_1$, that $P$ is a locally trivial principal $H$-bundle. Then for every non-degenerate representation $\pi$ of $\text{Ind}(C, \tau)$, the induced representation $X \text{-Ind}(\tilde{\pi} \times \rho)$ of $\text{Ind}(C, \sigma) \times_{R \otimes \tau} H$ is regular.

By Green’s imprimitivity theorem [4, theorem 6], it suffices to construct a non-degenerate representation $\nu$ of $C_0(H)$ on the Hilbert space $X \otimes D \mathcal{H}_\pi$ of $X \text{-Ind}(\tilde{\pi} \times \rho)$ which commutes with the action of $\text{Ind}(C, \sigma)$ and, together with the unitary part of $X \text{-Ind}(\tilde{\pi} \times \rho)$, is covariant for the action $\text{lt}: H \to \text{Aut}(C_0(H))$: the imprimitivity theorem then implies that $X \text{-Ind}(\tilde{\pi} \times \rho)$ is induced from the subgroup $\{e\}$ of $H$, and hence is regular. Let $(\mu, U)$ be the covariant representation of $(C_0(P, C), H, \text{lt} \otimes \tau)$ from Corollary 6. We shall use $\mu$ and the copies of $H$ inside the principal bundle $P$ to construct the required representation $\nu$ of $C_0(H)$. We make this precise in the following lemma:

Lemma 8. Suppose $P$ is a locally trivial principal $H$-bundle and $(\mu, U)$ is a covariant representation of $(C_0(P, C), H, \text{lt} \otimes \tau)$ on $\mathcal{H}$. Then there is a $(\mu, U)$-invariant subspace $\mathcal{H}_1$ of $\mathcal{H}$ and a representation $\nu_1: C_0(H) \to B(\mathcal{H}_1)$ such that $(\nu_1, U|_{\mathcal{H}_1})$ is a covariant representation of $(C_0(H), H, \text{lt})$ and each $\nu_1(f)$ commutes with each $\mu(g)|_{\mathcal{H}_1}$.

Proof. We can use a partition of unity on $H \setminus P$ to write every function in the dense subalgebra $C_0(P, C)$ as a sum of functions supported on $H$-saturated open subsets of $P$ which are trivial as $H$-bundles. Since $\mu$ is non-degenerate and in particular non-zero, $\mu$ must be non-zero on one of these sets. More formally, there is an $H$-saturated open set $N$ such that there is a bundle isomorphism $\phi: N \to (H \setminus N) \times H$, and such that $\mu$ is not identically zero on $I_N := \{g \in C_0(P, C): g(p) = 0 \text{ for } p \notin N\}$. Because $N$ is $H$-saturated, $I_N$ is invariant, and thus $\mathcal{H}_1 := \overline{\text{span}\{\mu(g)h: g \in I_N, h \in \mathcal{H}\}}$ is a non-zero $(\mu, U)$-invariant subspace of $\mathcal{H}$.

We now let $\phi_2: N \to (H \setminus N) \times H \to H$ denote the composition of $\phi$ with the projection onto the second factor $H$, and define $\iota: C_0(H) \to Z M(I_N)$ by

$$\iota(f)(g)(p) = \left\{ \begin{array}{ll} f(\phi_2(p))g(p) & \text{for } p \in N \\ 0 & \text{for } p \notin N, \end{array} \right.$$ 

where $g \in I_N$. Because $\mu|_{\mathcal{H}_1}$ is non-degenerate on $I_N$ it extends to a representation $\tilde{\mu}_1$ of $M(I_N)$. Since $\iota$ is non-degenerate as a homomorphism into $M(I_N)$ it follows that $\nu_1 := \tilde{\mu}_1 \circ \iota$ is a non-degenerate representation of $C_0(H)$ on $\mathcal{H}_1$ whose range commutes with every $\mu(g)|_{\mathcal{H}_1}$. Since $\phi$ is $H$-equivariant, $\iota$ is equivariant for the action $\text{lt}$ of $H$ on $C_0(H)$ and the action $\text{lt} \otimes \tau$ of $H$ on $I_N \subset C_0(P, C)$; thus the covariance of $(\mu, U)$ implies that $(\nu_1, U|_{\mathcal{H}_1})$ is a covariant representation of $(C_0(H), H, \text{lt})$. 

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Proof of Corollary 7. From Lemma 8 and a Zorn’s Lemma argument, we obtain a decomposition $\mathcal{H} = \bigoplus \mathcal{H}_i$ into $(\mu, U)$-invariant subspaces, each of which admits a suitable non-degenerate representation $\nu_i: C_0(H) \to B(\mathcal{H}_i)$. Now we just take $\nu := \bigoplus \nu_i$, and apply Green’s imprimitivity theorem as described above.

Remark 9. The local triviality hypothesis in Corollary 7 is a minor one, and is automatic if $H$ is a Lie group, for example. Indeed, because the action of $H$ is free and proper, $P$ is locally trivial if and only if the orbit map $q: P \to H \backslash P$ admits local continuous cross-sections [13, proposition 4.65], and a theorem of Palais says that $q$ always admits such sections when $H$ is a Lie group [9, section 4.1].

Remarks 10. One situation in which the induced representation is naturally regular is that considered by Kirchberg and Wassermann in [8]. Suppose $K$ is a closed subgroup of a locally compact group $G$, $P = G$, $H = G$ and $\tau_s = \text{id}$ for all $s \in H$. Then $f \mapsto f(e)$ is an isomorphism of $\text{Ind}(C, \text{id})$ onto $C$, and the symmetric imprimitivity theorem says that $\text{Ind}(C, \sigma) \otimes G$ is Morita equivalent to $C \times_{\sigma} K$ [11, section 1.5]. If $\pi: C \to B(\mathcal{H})$ is a non-degenerate representation, then it is proved in [8, proposition 3.2] that $X\text{-Ind}(\pi \times \lambda^K)$ is equivalent to a regular representation1. To see that $X\text{-Ind}(\pi \times \lambda^K)$ is regular using Theorem 1, we define

$$\langle \mu(f)\xi(t) = \pi(\sigma_t(f)(\xi))\rangle (t).$$

Then $(\mu, \rho^K)$ is a covariant representation of $(C(G), C), K, \text{rt} \otimes \sigma$, and Theorem 1 implies that the representation $X\text{-Ind}(\rho|_{\text{Ind}(C, \text{id})} \times \rho^K)$ is regular. The extension $\bar{\rho}$ is given on $C(G, C)$ by the same formula (6), and hence the isomorphism $\text{Ind}(C, \text{id}) \cong C$ carries $\bar{\rho}|_{\text{Ind}(C, \text{id})}$ into $\bar{\pi}$. Thus $X\text{-Ind}(\pi \times \lambda^K)$ is regular, and so is the equivalent representation $X\text{-Ind}(\pi \times \lambda^K)$. (As it stands, though, Theorem 1 does not give the twisted version of this result given in [8].)

In [2, section 1] Echterhoff and Raeburn consider the special case where $H$ and $K$ are subgroups of the same locally compact group $G$ and $P = G$. They construct a pair of regular representations of the induced systems $(\text{Ind}(C, \sigma), H)$ and $(\text{Ind}(C, \tau), K)$, and show that these induce to each other via the equivalence of the symmetric imprimitivity theorem [2, theorem 1.4]. We can use our Theorem 1 to see directly that the induced representations are regular: in the notation of [2], just define $\mu: C_0(G, D) \to B(\text{Ind}^G_H(L^2(K \times \mathcal{H}, \mathcal{H})))$ by $\langle \mu(g)\xi(s)(k, h) = \rho_1(\beta_h(g(sh)))\xi(s, k, h)\rangle$, and then $\mu$ is a representation of $C_0(G, D)$ such that the restriction of $\bar{\rho}$ to the induced algebra is $\text{Ind}^G_H(\rho_1 \otimes 1 \otimes 1)$.

2.3. Amenability of actions on $C_0(P)$-algebras

We shall now show that the main theorem of [5, section 4], which is a generalization of Corollary 3 to actions on $C_0(P)$-algebras, can be deduced from Theorem 1. We recall the set-up of [5] and take $\rho_P$ as usual, but instead of an arbitrary $C^*$-algebra $C$, we fix a $C_0(P)$-algebra $A$; this means that there is a non-degenerate injection $\iota_A$ of $C_0(P)$ into $ZM(A)$ (we write $f \cdot a$ for $\iota_A(f)a$). We insist that the actions $\tau: H \to \text{Aut} A$ and $\sigma: K \to \text{Aut} A$ commute and satisfy

$$\tau_s(f \cdot a) = \iota_L(f) \cdot \tau_s(a) \quad \text{and} \quad \sigma_t(f \cdot a) = \iota_R(f) \cdot \sigma_t(a).$$

1 We warn that $\bar{\pi}$ has different meaning in the presence of $\lambda$ and $\rho$. 

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It is proved in [5, section 3] that both \( \tau \) and \( \sigma \) are proper in the sense of [17], that there are strongly continuous actions \( \tau: H \to \text{Aut} \ A^\tau \) and \( \sigma: K \to \text{Aut} \ A^\sigma \) on the generalized fixed-point algebras of [17], and that \( A^\sigma \times_\tau H \) is Morita equivalent to \( A^\tau \times_\sigma K \).

The next corollary is [5, theorem 4.5]. There the one-sided case was proved first, by realizing the bimodule of [17] for the reduced crossed product as a quotient of the one from [12]; the general case was then deduced from several applications of the one-sided case and a theorem of Combes [1]. The proof of Corollary 11, on the other hand, uses only the general results of [11, section 1–2] and Theorem 1: the techniques, and in particular the indirect applications of Theorem 1, may be of independent interest.

**Corollary 11.** In the above set-up, the system \( (A^\tau, H, \tilde{\tau}) \) is amenable if and only if \( (A^\tau, K, \sigma) \) is amenable.

As in [5, section 3], we view \( A^\tau \) as a quotient of \( \text{Ind}(A, \tau) \) – indeed, for our purposes we could define \( A^\tau \) this way, and avoid all reference to proper actions. To see how this works, recall that the non-degenerate action of \( C_0(P) \) on \( A \) induces a continuous surjection \( q_A: \text{Prim} \ A \to P \), which is characterized by

\[
q_A(I) = p \iff \{ f \cdot a \in C_0(P) \cdot A : f(p) = 0 \} \subset I;
\]

the hypothesis (7) implies that \( q_A \) is \( H \)- and \( K \)-equivariant. It is proved in [5, section 3, proposition 3.6] and the end of the proof of [5, theorem 3.1] that the map \( f \otimes a \mapsto f \cdot a \) extends to \( C_0(P, A) \subset M(C_0(P) \otimes A) \), and induces an equivariant isomorphism of \( (\text{Ind}(A, \tau)/I(\tau), K, \text{rt} \otimes \sigma) \) onto \( (A^\tau, K, \sigma) \), where

\[
I(\tau) = \{ b \in \text{Ind}(A, \tau) : b(q_A(I)) \in I \text{ for all } I \in \text{Prim} \ A \}.
\]

This ideal is \( \text{rt} \otimes \sigma \)-invariant, and hence by [4, proposition 12] the crossed product \( I(\tau) \times_{\text{rt} \otimes \sigma} K \) embeds naturally as an ideal in \( \text{Ind}(A, \tau) \times_{\text{rt} \otimes \sigma} K \) with quotient

\[
(\text{Ind}(A, \tau)/I(\tau)) \times_{\text{rt} \otimes \sigma} K \cong A^\tau \times_\sigma K.
\]

To apply our theorem, we need a representation \( \nu \) of \( C_0(P, A) \) such that \( \tilde{\nu}|_{\text{Ind}(A, \tau)} \) has kernel \( I(\tau) \). We choose a non-degenerate representation \( \nu \) of \( C_0(P, A) \) with

\[
\ker \nu = I_\Delta \doteq \{ b \in C_0(P, A) : b(q_A(I)) \in I \text{ for all } I \in \text{Prim} \ A \}.
\]

Then for \( b \in \text{Ind}(A, \tau) \), we have

\[
\nu(b) = 0 \iff \nu(bc) = 0 \text{ for all } c \in C_0(P, A)
\]

\[
\iff b(q_A(I))c(q_A(I)) \in I \text{ for all } I \in \text{Prim} A, c \in C_0(P, A)
\]

\[
\iff b(q_A(I))a \in I \text{ for all } I \in \text{Prim} A, a \in A
\]

\[
\iff b(q_A(I)) \in I \text{ for all } I \in \text{Prim} A,
\]

so \( \ker \nu|_{\text{Ind}(A, \tau)} = I(\tau) \), as we wanted.

The representation \( \tilde{\nu}|_{\text{Ind}(A, \tau)} \) is the restriction of (the extension to \( M(C_0(P, A)) \) of) the representation \( \tilde{\nu} \) of \( C_0(P, A) \); we aim to apply Theorem 1 to the covariant representation \( \tilde{\nu} \otimes \rho \) of \( (C_0(P, A), K, \text{rt} \otimes \sigma) \). For this to be useful, we need to know that \( \ker (\tilde{\nu} \times \rho) \) is the ideal in \( C_0(P, A) \times K \) corresponding to the diagonal ideal \( I_\Delta \) in \( C_0(P, A) \).
Lemma 12. With the above notation, \( \ker(\tilde{\nu} \times \rho) = I_\Lambda \times_{\lt \otimes \sigma} K \).

Proof. To avoid having to write out the opposite version of Theorem 1, we instead prove the equivalent assertion that the regular representation \((\tilde{\nu} \times \rho)\) of the system \((C_0(P,A), H, I \otimes \tau)\) satisfies \( \ker(\tilde{\nu} \times \rho) = I_\Lambda \times_{\lt \otimes \tau} H \). To achieve this, we apply Theorem 1 with \( K \) absent. Then \( X_0 \) is the bimodule \( Y_0 := C_c(P,A) \times H C_c(P,A) \) of [12, theorem 2.2]. \( Z_0 \) is the trivial bimodule \( C_0(P,A) \times C_0(P,A) \), and Theorem 1 says that

\[
Y \text{-Ind}(\tilde{\nu}|_{\text{Ind}(A,\tau)}) \sim (Z \text{-Ind} \nu)^- \times \rho = \tilde{\nu} \times \rho.
\]

Since \( K = \{e\} \), we have \( I(\sigma) = I_\Lambda \), and [11, corollary 2.1] implies that \( Y \text{-Ind} I(\tau) = I_\Lambda \times H \). Thus our choice of \( \nu \) implies that \( \ker(\tilde{\nu}|_{\text{Ind}(A,\tau)}) = I(\tau) \), and

\[
\ker(\tilde{\nu} \times \rho) = \ker \left( Y \text{-Ind}(\tilde{\nu}|_{\text{Ind}(A,\tau)}) \right) \times Y \text{-Ind} I(\tau) = I_\Lambda \times_{\lt \otimes \tau} H,
\]

as required.

Proof of Corollary 11. Suppose that \((A^\gamma, K, \sigma)\) is amenable. Applying Theorem 1 to \((\tilde{\nu}, \rho)\) shows that

\[
\ker(\text{X-Ind}((\rho|_{\text{Ind}(A,\tau)})^- \times \rho)) = \ker\left( (Z \text{-Ind}(\tilde{\nu} \times \rho))^\gamma \times \rho \right).
\]

Since \( \ker(\rho|_{\text{Ind}(A,\tau)}) = I(\tau) \), \( \rho|_{\text{Ind}(A,\tau)} \) factors through a faithful representation \( \kappa_1 \) of \( A^\gamma \cong \text{Ind}(A,\tau)/I(\tau) \); the amenability of \((A^\gamma, K, \sigma)\) implies that \( \kappa_1 \times \rho \) is faithful, or, equivalently, that \( \ker((\rho|_{\text{Ind}(A,\tau)})^- \times \rho) = I(\tau) \times K \). [11, corollary 2.1] says that the Rieffel correspondence \( X \text{-Ind} \) carries \( I(\tau) \times K \) to \( I(\sigma) \times H \). Thus

\[
I(\sigma) \times H = X \text{-Ind}(I(\tau) \times K) = \ker(\text{X-Ind}((\rho|_{\text{Ind}(A,\tau)})^- \times \rho)). \tag{9}
\]

On the other hand, another application of [11, corollary 2.1], with \( H \) missing this time, shows that the Rieffel correspondence \( Z \text{-Ind} \) takes \( I_\Lambda \times K \) to \( I(\sigma) \). Thus Lemma 12 gives

\[
\ker(\text{Z-Ind}(\tilde{\nu} \times \rho)) = \text{Z-Ind}(I_\Lambda \times K) = I(\sigma).
\]

So \( Z \text{-Ind}(\tilde{\nu} \times \rho) \) factors through a faithful representation \( \kappa_2 \) of \( A^\sigma \cong \text{Ind}(A,\sigma)/I(\sigma) \), and the representation \( (Z \text{-Ind}(\tilde{\nu} \times \rho))^\gamma \times \rho \) appearing in (8) factors through the regular representation \( \kappa_2 \times \rho \). Since we know from (8) and (9) that \( (Z \text{-Ind}(\tilde{\nu} \times \rho))^\gamma \times \rho \) has kernel \( I(\sigma) \times H \), \( \kappa_2 \times \rho \) must be faithful on \( A^\sigma \times H \cong \text{Ind}(A,\sigma) \times H / (I(\sigma) \times H) \). Thus \( (A^\gamma, H, \tilde{\tau}) \) is amenable.

The result follows by symmetry.

REFERENCES
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