Regularity of induced representations and a theorem of Quigg and Speilberg

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Abstract
Mackey's imprimitivity theorem characterizes the unitary representations of a locally compact group G which have been induced from representations of a closed subgroup K; Rieffel's influential reformulation says that the group C^*-algebra $C^*(K)$ is Morita equivalent to the crossed product $C_0(G/K) \times G$ [14]. There have since been many important generalizations of this theorem, especially by Rieffel [15, 16] and by Green [3, 4]. These are all special cases of the symmetric imprimitivity theorem of [11], which gives a Morita equivalence between two crossed products of induced C^*-algebras.

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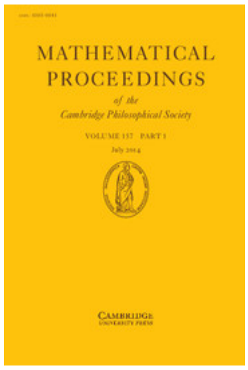

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Regularity of induced representations and a theorem of Quigg and Spielberg

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Mackey's imprimitivity theorem characterizes the unitary representations of a locally compact group G which have been induced from representations of a closed subgroup K; Rieffel's influential reformulation says that the group C^*-algebra $C^*(K)$ is Morita equivalent to the crossed product $C_0(G/K) \times G$ [**14**]. There have since been many important generalizations of this theorem, especially by Rieffel [**15**, **16**] and by Green [**3**, **4**]. These are all special cases of the symmetric imprimitivity theorem of [**11**], which gives a Morita equivalence between two crossed products of induced C^*-algebras.

Quigg and Spielberg proved in [**10**], by ingenious but indirect methods, that the symmetric imprimitivity theorem, and hence all the other generalizations of Rieffel's imprimitivity theorem, have analogues for reduced crossed products. A different Morita equivalence between the same reduced crossed products was obtained by Kasparov [**7**, theorem 3.15].

Here we identify the representations which induce to regular representations under the Morita equivalence of the symmetric imprimitivity theorem (see Theorem 1 and Corollary 6), and thus obtain a direct proof of the theorem of Quigg and Spielberg (see Corollary 3). We discovered Theorem 1 while trying to understand why Rieffel's theory of proper actions in [**17**] gives an equivalence involving reduced crossed products rather than full ones.

Theorem 1 has several other interesting applications. We can use it to see, albeit somewhat indirectly, that regular representations themselves nearly always induce to regular representations (see Corollary 7), and it gives a new proof of the main theorem of [**5**, section 4] which avoids a complicated argument involving a composition of crossed-product Morita equivalences (see Corollary 11). It also sheds light on constructions in [**2**] and [**8**], which, in various special cases of the symmetric imprimitivity theorem, yield pairs of regular representations which induce to each other (see Remarks 10).

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Notation. We shall use left Haar measure, because we always do, and the right-regular representation ρ, because this is what seems to come naturally out of our constructions; this means that modular functions intrude, so that $\rho = \rho^G \colon G \to U(L^2(G))$ is defined by $(\rho_t\xi)(s) = \xi(st)\Delta_G(t)^{1/2}$. We also write ρ for the representation on $L^2(G, \mathscr{H})$ defined by the same formula. With our conventions, the regular representation of $A \times_\alpha G$ induced from a representation $\pi \colon A \to B(\mathscr{H})$ is the integrated form $\widetilde{\pi} \times \rho$ of the covariant representation $(\widetilde{\pi}, \rho)$ of (A, G, α) on $L^2(G, \mathscr{H})$ in which $(\widetilde{\pi}(a)\xi)(s) = \pi(\alpha_s(a))(\xi(s))$. Given a non-degenerate representation μ of A we denote by $\bar{\mu}$ its extension to the multiplier algebra $M(A)$ of A. We shall denote by lt and rt any actions of groups by, respectively, left and right translation; thus if H acts on the left of a locally compact space P and K acts on the right, we have $\mathrm{lt}_s(f)(p) = f(s^{-1}p)$ and $\mathrm{rt}_t(f)(p) = f(pt)$ for $f \in C_0(P)$.

1. *The main theorem*

We start with commuting free and proper actions of two locally compact groups H and K on the left and right of a locally compact space P, and commuting actions $\tau \colon H \to \operatorname{Aut} C$ and $\sigma \colon K \to \operatorname{Aut} C$ on a C^*-algebra C. The *induced C^*-algebra* $\mathrm{Ind}(C, \sigma)$ is the C^*-subalgebra of $C_b(P, C)$ consisting of the functions f such that $f(pt) = \sigma_t^{-1}(f(p))$ for all $t \in K$ and $p \in P$, and such that the function $pK \mapsto \|f(p)\|$ vanishes at infinity on P/K. The diagonal action $\mathrm{lt} \otimes \tau$ on $C_b(P, C) \subset M(C_0(P, C))$ restricts to a well-defined strongly continuous action of H on $\mathrm{Ind}(C, \sigma)$, which is characterized by $(\mathrm{lt} \otimes \tau)_s(f)(p) = \tau_s(f(s^{-1}p))$. (The continuity of this action was established in [**5**, lemma 5.1].) Likewise, $\mathrm{Ind}(C, \tau)$ consists of the bounded continuous functions $f \colon P \to C$ such that $f(sp) = \tau_s(f(p))$ for $s \in H$ and $Hp \mapsto \|f(p)\|$ vanishes at infinity on $H \backslash P$, and we have a natural action $\mathrm{rt} \otimes \sigma$ of K on $\mathrm{Ind}(C, \tau)$ given by $(\mathrm{rt} \otimes \sigma)_t(f)(p) = \sigma_t(f(pt))$.

The symmetric imprimitivity theorem of [**11**, theorem 1.1] shows how to make $X_0 := C_c(P, C)$ into a pre-imprimitivity bimodule whose completion X implements a Morita equivalence between $\mathrm{Ind}(C, \sigma) \times_{\mathrm{lt} \otimes \tau} H$ and $\mathrm{Ind}(C, \tau) \times_{\mathrm{rt} \otimes \sigma} K$. Since a shortage of Greek letters and left-right ambivalence have previously led to conflicts of notation, it is worthwhile to record the formulas we use:

$$\begin{aligned}
a \cdot x(p) &= \int_H a(s, p)\tau_s(x(s^{-1}p))\, \Delta_H(s)^{1/2}\, ds \\
x \cdot d(p) &= \int_K \sigma_t\big(x(pt)d(t^{-1}, pt)\big)\, \Delta_K(t)^{-1/2}\, dt \\
{}_{\mathrm{Ind}(C,\sigma)\times H}\langle x, y\rangle(s, p) &= \int_K \sigma_t\big(x(pt)\tau_s(y(s^{-1}pt)^*)\big)\, dt\, \Delta_H(s)^{-1/2} \\
\langle x, y\rangle_{\mathrm{Ind}(C,\tau)\times K}(t, p) &= \int_H \tau_s\big(x(s^{-1}p)^*\sigma_t(y(s^{-1}pt))\big)\, ds\, \Delta_K(t)^{-1/2},
\end{aligned}$$

where $a \in C_c(H, \mathrm{Ind}(C, \sigma)) \subset \mathrm{Ind}(C, \sigma) \times H$, $d \in C_c(K, \mathrm{Ind}(C, \tau)) \subset \mathrm{Ind}(C, \tau) \times K$, and x, y belong to $X_0 = C_c(P, C)$.

We shall also need the one-sided version of this bimodule which is based on the same space $Z_0 := C_c(P, C)$ but omits all mention of the group H; this bimodule is the dual of the bimodule first considered in [**12**]. Thus we denote by Z the completion of $Z_0 = C_c(P, C)$ as an $\mathrm{Ind}(C, \sigma)$–$(C_0(P, C) \times_{\mathrm{rt} \otimes \sigma} K)$ imprimitivity bimodule. We use exactly the same conventions as above, so that, for example,

$$\langle x, y\rangle_{C_0(P,C)\times K}(t,p) = x(p)^*\sigma_t(y(pt))\Delta_K(t)^{-1/2}.$$

THEOREM 1. *Suppose (μ, U) is a covariant representation of $\big(C_0(P,C), K, \mathrm{rt}\otimes\sigma\big)$ on $\mathscr{H}$. Then the representation*

$$X\text{-}\mathrm{Ind}_{\mathrm{Ind}(C,\tau)\times K}^{\mathrm{Ind}(C,\sigma)\times H}(\bar\mu|_{\mathrm{Ind}(C,\tau)}\times U)\colon \mathrm{Ind}(C,\sigma)\times_{\mathrm{lt}\otimes\tau} H \to B(X\otimes_{\mathrm{Ind}(C,\tau)\times K}\mathscr{H})$$

is unitarily equivalent to the right-regular representation

$$\big(Z\text{-}\mathrm{Ind}_{C_0(P,C)\times K}^{\mathrm{Ind}(C,\sigma)}(\mu\times U)\big)^{\sim}\times\rho\colon \mathrm{Ind}(C,\sigma)\times_{\mathrm{lt}\otimes\tau} H \to B\big(L^2(H, Z\otimes_{C_0(P,C)\times K}\mathscr{H})\big).$$

Proof. We shall write $D := \mathrm{Ind}(C,\tau)\times_{\mathrm{rt}\otimes\sigma} K$ and $B := C_0(P,C)\times_{\mathrm{rt}\otimes\sigma} K$; we are going to view $X_0 := C_c(P,C)$ as the pre-Hilbert $C_c(K, \mathrm{Ind}(C,\tau))$-module described above, and also as a dense subspace of the Hilbert B-module Z.

For $x\otimes_D h \in X_0\odot\mathscr{H}$, we define $W(x\otimes_D h)\colon H\to Z\otimes_B\mathscr{H}$ by

$$W(x\otimes_D h)(s) := (\mathrm{lt}\otimes\tau)_s(x)\otimes_B h.$$

We shall prove that W extends to a unitary operator of $X\otimes_D\mathscr{H}$ onto $L^2(H, Z\otimes_B\mathscr{H})$ which intertwines the given representations.

We first prove that W is well-defined and isometric: for both, it suffices to show that

$$\big(W(x\otimes_D h)\,\big|\,W(y\otimes_D\bar\mu(f)k)\big) = \big(x\otimes_D h\,\big|\,y\otimes_D\bar\mu(f)k\big) \tag{1}$$

for $x, y\in X_0$, $h,k\in\mathscr{H}$ and $f\in C_c(P)\subset M(C_0(P,C))$. (Inserting the function f allows us to deduce from the properness of the actions that the integrands in the following calculations have compact support.) To prove (1), we note that

$$\begin{aligned}\big(x\otimes_B h\,\big|\,y\otimes_B\bar\mu(f)k\big) &= \big(\mu\times U(\langle y,x\rangle_B)h\,\big|\,\bar\mu(f)k\big)\\ &= \int_K \big(\mu(\langle y,x\rangle_B(t))U_t h\,|\,\bar\mu(f)k\big)\,dt\\ &= \int_K \big(\mu\big(p\mapsto \overline{f(p)}y(p)^*\sigma_t(x(pt))\Delta_K(t)^{-1/2}\big)U_t h\,|\,k\big)\,dt.\end{aligned}$$

From this and an application of Fubini's Theorem, we deduce that

$$\begin{aligned}\big(W(x\otimes_D h)\,\big|\,W(y\otimes_D\bar\mu(f)k)\big) &= \int_H \big((\mathrm{lt}\otimes\tau)_s(x)\otimes_B h\,\big|\,(\mathrm{lt}\otimes\tau)_s(y)\otimes_B\bar\mu(f)k\big)\,ds\\ &= \int_H\int_K \Big(\mu\big(p\mapsto\overline{f(p)}\tau_s(y(s^{-1}p)^*\sigma_t(x(s^{-1}pt)))\\ &\qquad\times\Delta_K(t)^{-1/2}\big)U_t h\,\Big|\,k\Big)\,dt\,ds\\ &= \int_K \big(\bar\mu(\overline{f}\langle y,x\rangle_D(t))U_t h\,\big|\,k\big)\,dt\\ &= \big(x\otimes_D h\,\big|\,y\otimes_D\bar\mu(f)k\big),\end{aligned}$$

as desired.

We next prove that W is surjective. We begin by observing that, with the pointwise action $w \cdot b(s) \coloneqq w(s) \cdot b$ and the inner product

$$\langle v, w\rangle_B \coloneqq \int_H \langle v(s), w(s)\rangle_B \, ds,$$

$C_c(H, Z_0)$ is a pre-Hilbert B-module; we denote its completion by $L^2(H, Z)$. (Although we shall not need this, it might help to observe that $L^2(H, Z)$ is naturally isomorphic to the external Hilbert-module tensor product $L^2(H)_{\mathbb{C}} \otimes Z_B$ of, for example, [**13**, section 3.4].) Writing out the formulas shows that the map $w \otimes_B h \mapsto (s \mapsto w(s) \otimes_B h)$ is inner-product preserving from $L^2(H, Z) \otimes_B \mathscr{H}$ to $L^2(H, Z \otimes_B \mathscr{H})$, and we can see by considering elementary tensors $w = \xi \otimes z \in C_c(H) \odot Z_0$ that it has dense range. So we can view $L^2(H, Z \otimes_B \mathscr{H})$ as $L^2(H, Z) \otimes_B \mathscr{H}$. Functions $f \in C_c(P) \subset C_0(P)$ act as multipliers on $C_0(P, C)$ and hence also on $B = C_0(P, C) \times K$ and X_0, where the action is given by $(x \cdot f)(t, p) = x(t, p)f(p)$. Since

$$W(x \otimes_D \bar{\mu}(f)h)(s) \coloneqq (\mathrm{lt} \otimes \tau)_s(x) \otimes_B \bar{\mu}(f)h = ((\mathrm{lt} \otimes \tau)_s(x) \cdot f) \otimes_B h,$$

and since an isometry with dense range is surjective, it suffices for us to show that

$$L_0 \coloneqq \operatorname{span}\{(s, p) \mapsto \tau_s(x(s^{-1}p))f(p) \colon x \in X_0, f \in C_c(P)\}$$

is dense in $L^2(H, Z_B)$. If $v \in C_c(H \times P, C) \subset C_c(H, X_0)$ has $\operatorname{supp} v \subset M \times N$, then using the L^1-norm to estimate the C^*-norm gives

$$\begin{aligned}
\|v\|_B^2 &= \int_H \|\langle v(s, \cdot), v(s, \cdot)\rangle_B\| \, ds \\
&\leqslant \int_H \Big(\int_K \sup_{p \in P} \|v(s, p)^* \sigma_t(v(s, pt))\| \Delta_K(t)^{-1/2} \, dt \Big) \, ds \\
&\leqslant \mu_H(M) \int_{\{t \in K : N \cap N t^{-1} \neq \varnothing\}} \|v\|_\infty^2 \Delta_K(t)^{-1/2} \, dt.
\end{aligned}$$

Thus it suffices to show that we can approximate $v \in C_c(H \times P, C)$ uniformly on a compact neighbourhood of $\operatorname{supp} v$ by functions in L_0 with support in that neighbourhood.

Because H acts freely and properly on P, the map $(s, p) \mapsto (sp, p)$ is a homeomorphism of $H \times P$ onto $P \times_{H\backslash P} P \coloneqq \{(q, p) \in P \times P \colon Hq = Hp\}$. (The inverse is given by $(q, p) \mapsto (\mathrm{tr}(q, p), p)$, where $\mathrm{tr}\colon P \times_{H\backslash P} P \to H$ is the translation function characterized by $q = \mathrm{tr}(q, p)p$; a routine compactness argument shows that tr is continuous.) Thus the map $\Phi\colon C_c(P \times_{H\backslash P} P, C) \to C_c(H \times P, C)$ defined by $\Phi(w)(s, p) = \tau_s(w(s^{-1}p, p))$ is a linear isomorphism which preserves the kind of approximation we require. For $v \in C_c(H \times P, C)$, we choose an extension w of $\Phi^{-1}(v)$ to a function of compact support on $P \times P$, and now standard arguments show that we can approximate w in $C_c(P \times P, C)$ by functions $\sum x_i \otimes f_i$ in $C_c(P, C) \odot C_c(P)$; then $\Phi(\sum x_i \otimes f_i)$ is the required approximation to v. Thus W is surjective.

It remains to check that W intertwines the given representations as claimed. Let $a \in C_c(H, \mathrm{Ind}(C, \sigma))$, and for this calculation denote the action $\mathrm{lt} \otimes \tau$ of H on $\mathrm{Ind}(C, \sigma)$ by α. Then for $x \otimes_D h \in X_0 \odot \mathscr{H}$, we have

$$W\big(X\text{-}\mathrm{Ind}(\bar{\mu} \times U)(a)(x \otimes_D h)\big)(s) = W((a \cdot x) \otimes_D h)(s) = \alpha_s(a \cdot x) \otimes_B h. \qquad (2)$$

On the other hand,

$$\begin{aligned}
\big(Z\text{-Ind}(\mu \times U)\tilde{} \times \rho(a)\big)\big(W(x \otimes_D h)\big)(s) \\
&= \int_H Z\text{-Ind}(\mu \times U)(\alpha_s(a(r)))\big(\rho_r(W(x \otimes_D h))(s)\big)\, dr \\
&= \int_H Z\text{-Ind}(\mu \times U)(\alpha_s(a(r)))\big(W(x \otimes_D h)(sr)\Delta_H(r)^{1/2}\big)\, dr \\
&= \int_H \big((\alpha_s(a(r)) \cdot \alpha_{sr}(x)) \otimes_B h\big)\Delta_H(r)^{1/2}\, dr. \qquad (3)
\end{aligned}$$

Since we have

$$\begin{aligned}
\alpha_s(a \cdot x)(p) &= \tau_s(a \cdot x(s^{-1}p)) \\
&= \tau_s\Big(\int_H a(r, s^{-1}p)\tau_r(x(r^{-1}s^{-1}p))\Delta_H(r)^{1/2}\, dr\Big) \\
&= \int_H \big(\alpha_s(a(r)) \cdot \alpha_{sr}(x)\big)(p)\Delta_H(r)^{1/2}\, dr,
\end{aligned}$$

the only difference between (2) and (3) is the location of the integral with respect to $\otimes_B h$. But the integrands in both formulas are continuous and compactly supported, so there is no difficulty verifying that they have the same inner product with every vector of the form $y \otimes_B k \in Z_0 \odot \mathscr{H}$, and hence must be equal. We deduce that

$$W\big(X\text{-Ind}(\bar{\mu} \times U)(a)\big) = \big(Z\text{-Ind}(\mu \times U)\tilde{} \times \rho\big)(a)W,$$

and we have proved the theorem.

Remark 2. The referee pointed out that Theorem 1 suggests that there is, and would follow from, an isomorphism

$$X \otimes_{\text{Ind}(C,\tau)\times K} (C_0(P, C) \times K) \cong Y \otimes_{\text{Ind}(C,\sigma)} Z \qquad (4)$$

of right-Hilbert $(\text{Ind}(C, \sigma) \times H)$–$(C_0(P, C) \times K)$ bimodules, where Y is the Hilbert bimodule of Green which induces representations of $\text{Ind}(C, \sigma)$ to regular representations of $\text{Ind}(C, \sigma) \times H$. Such isomorphisms have proved to be a powerful tool for studying the duality between induction and restriction of representations [**6**]. If applications arise which require functorial properties of the equivalence in Theorem 1, then establishing such an isomorphism might be an efficient way to proceed; for our present applications, Theorem 1 suffices.

2. *Applications*

2·1. *The theorem of Quigg and Spielberg*

We retain the notation of the preceding section. We shall say that a dynamical system (A, G, α) is *amenable* if the regular representation induced from a faithful representation of A is faithful on $A \times_\alpha G$.

Corollary 3 ([**10**]). *Denote by I and J the kernels of the quotient maps of* $\text{Ind}(C, \tau) \times_{\text{rt}\otimes\sigma} K$ *and* $\text{Ind}(C, \sigma) \times_{\text{lt}\otimes\tau} H$ *onto the reduced crossed products. Then* $X\text{-Ind}\, I = J$. *In particular, this implies that* $\text{Ind}(C, \tau) \times_{\text{rt}\otimes\sigma,\, r} K$ *is Morita equivalent to* $\text{Ind}(C, \sigma) \times_{\text{lt}\otimes\tau,\, r} H$, *and that the system* $\big(\text{Ind}(C, \sigma), H, \text{lt} \otimes \tau\big)$ *is amenable if and only if* $\big(\text{Ind}(C, \tau), K, \text{rt} \otimes \sigma\big)$ *is amenable.*

Proof. Every regular representation $\widetilde{\pi} \times \rho$ induced from a faithful representation π of $\mathrm{Ind}(C, \tau)$ has the same kernel I, and X-$\mathrm{Ind}(\widetilde{\pi} \times \rho)$ has kernel X-$\mathrm{Ind}\, I$. We *choose* π to be the restriction $\bar{\nu}|_{\mathrm{Ind}(C,\tau)}$ of a faithful nondegenerate representation of $C_0(P, C)$. Then $(\widetilde{\pi}, \rho)$ is the restriction of the regular representation $(\widetilde{\nu}, \rho)$ of $\big(C_0(P, C), K, \mathrm{rt} \otimes \sigma\big)$, and we can apply Theorem 1 with $(\mu, U) = (\widetilde{\nu}, \rho)$. We deduce that, for this π, X-$\mathrm{Ind}(\widetilde{\pi} \times \rho)$ is equivalent to the right-regular representation of $\mathrm{Ind}(C, \sigma) \times_{\mathrm{lt} \otimes \tau} H$ induced from the representation Z-$\mathrm{Ind}(\widetilde{\nu} \times \rho)$ of $\mathrm{Ind}(C, \sigma)$. Thus

$$X\text{-}\mathrm{Ind}\, I = \ker\big(X\text{-}\mathrm{Ind}(\widetilde{\pi} \times \rho)\big) = \ker\big(Z\text{-}\mathrm{Ind}(\widetilde{\nu} \times \rho)^{\sim} \times \rho\big) \subset J. \tag{5}$$

(If we knew that Z-$\mathrm{Ind}(\widetilde{\nu} \times \rho)$ is faithful then we would have equality in 5 above; instead we show that equality holds and deduce that Z-$\mathrm{Ind}(\widetilde{\nu} \times \rho)$ is faithful.) By symmetry $\widetilde{X}$-$\mathrm{Ind}\, J \subset I$, and now applying X-$\mathrm{Ind}\, I$ we see that $J \subset X$-$\mathrm{Ind}\, I$.

It follows from standard properties of the Rieffel correspondence [**13**, proposition 3.25] that the reduced crossed products are Morita equivalent. Finally, if $\big(\mathrm{Ind}(C, \tau), K, \mathrm{rt} \otimes \sigma\big)$ is amenable, then $I = \{0\}$, so $J = \{0\}$ and the system $\big(\mathrm{Ind}(C, \sigma), H, \mathrm{lt} \otimes \tau\big)$ is amenable; the last part follows by symmetry.

One special case is worth mentioning because the possibility of such a result was specifically mooted in [**17**, page **171**], and because the proof we have given is more direct than others.

COROLLARY 4. *Suppose H acts freely and properly on P and τ is an action of H on a C^*-algebra C. Then*

$$C_0(P, C) \times_{\mathrm{lt} \otimes \tau} H = C_0(P, C) \times_{\mathrm{lt} \otimes \tau,\, r} H.$$

Proof. This is the special case of Corollary 3 in which $K = \{e\}$; the dynamical system $\big(\mathrm{Ind}(C, \tau), K, \mathrm{rt} \otimes \sigma\big)$ is then trivially amenable.

2·2. *Inducing regular representations*

Because the symmetric imprimitivity theorem passes to reduced crossed products, it is natural to ask whether the symmetric imprimitivity theorem matches up the regular representations themselves. More precisely, if $\tilde{\pi} \times \rho$ is a regular representation of $\mathrm{Ind}(C, \tau) \times_{\mathrm{rt} \otimes \sigma} K$, is the induced representation X-$\mathrm{Ind}(\tilde{\pi} \times \rho)$ of $\mathrm{Ind}(C, \sigma) \times_{\mathrm{lt} \otimes \tau} H$ regular? We can settle this question, though in a rather roundabout fashion (see Corollary 7 below).

We begin our analysis of this question by observing that Theorem 1 can be used to characterize the representations of $\mathrm{Ind}(C, \tau) \times_{\mathrm{rt} \otimes \sigma} K$ which induce to regular representations of $\mathrm{Ind}(C, \sigma) \times_{\mathrm{lt} \otimes \tau} H$.

COROLLARY 5. *Let (ν, V) be a covariant representation of $(\mathrm{Ind}(C, \tau), K, \mathrm{rt} \otimes \sigma)$ on $\mathscr{H}$. Then the representation X-$\mathrm{Ind}(\nu \times V)$ is regular if and only if there is a covariant representation (μ, U) of $(C_0(P, C), K, \mathrm{rt} \otimes \sigma)$ on $\mathscr{H}$ such that (ν, V) is equivalent to $(\bar{\mu}|_{\mathrm{Ind}(C,\tau)}, U)$.*

Proof. Theorem 1 immediately gives the 'if' direction. So suppose X-$\mathrm{Ind}(\nu \times V)$ is equivalent to the regular representation $\tilde{\pi} \times \rho$ for some representation π of $\mathrm{Ind}(C, \sigma)$. Let $\mu \times U := \widetilde{Z}$-$\mathrm{Ind}_{\mathrm{Ind}(C,\sigma)}^{C_0(P,C) \times K} \pi$, and note that Z-$\mathrm{Ind}(\mu \times U)$ is equivalent to π. Theorem 1 implies that X-$\mathrm{Ind}(\bar{\mu}|_{\mathrm{Ind}(C,\tau)} \times U)$ is equivalent to $\tilde{\pi} \times \rho$, and applying $\widetilde{X}$-Ind shows that (ν, V) is equivalent to $(\bar{\mu}|_{\mathrm{Ind}(C,\tau)}, U)$.

COROLLARY 6. *Let π be a non-degenerate representation of* $\mathrm{Ind}(C,\tau)$. *Then there is a covariant representation* (μ, U) *of* $(C_0(P,C), H, \mathrm{lt}\otimes\tau)$ *such that*

$$X\text{-}\mathrm{Ind}_{\mathrm{Ind}(C,\tau)\times K}^{\mathrm{Ind}(C,\sigma)\times H}(\widetilde{\pi}\times\rho) = \bar{\mu}|_{\mathrm{Ind}(C,\sigma)}\times U.$$

Proof. This follows from Corollary 5 by taking advantage of the symmetry of the situation. Suppose that π is a representation of $\mathrm{Ind}(C,\sigma)$ instead. Let $(\nu, V) = \widetilde{X}\text{-}\mathrm{Ind}(\tilde{\pi}\times\rho)$ and note that $X\text{-}\mathrm{Ind}(\nu\times V)$ is regular because it is equivalent to $\tilde{\pi}\times\rho$. By Corollary 5 (ν, V) is the restriction of a covariant representation (μ, U) of $(C_0(P,C), K, \mathrm{rt}\otimes\sigma)$.

COROLLARY 7. *Suppose, in addition to our standard assumptions on* ${}_HP_K$, *that P is a locally trivial principal H-bundle. Then for every non-degenerate representation π of* $\mathrm{Ind}(C,\tau)$, *the induced representation* $X\text{-}\mathrm{Ind}(\widetilde{\pi}\times\rho)$ *of* $\mathrm{Ind}(C,\sigma)\times_{\mathrm{lt}\otimes\tau} H$ *is regular.*

By Green's imprimitivity theorem [**4**, theorem 6], it suffices to construct a non-degenerate representation ν of $C_0(H)$ on the Hilbert space $X\otimes_D \mathscr{H}_\pi$ of $X\text{-}\mathrm{Ind}(\widetilde{\pi}\times\rho)$ which commutes with the action of $\mathrm{Ind}(C,\sigma)$ and, together with the unitary part of $X\text{-}\mathrm{Ind}(\widetilde{\pi}\times\rho)$, is covariant for the action lt: $H\to \mathrm{Aut}\, C_0(H)$: the imprimitivity theorem then implies that $X\text{-}\mathrm{Ind}(\widetilde{\pi}\times\rho)$ is induced from the subgroup $\{e\}$ of H, and hence is regular. Let (μ, U) be the covariant representation of $(C_0(P,C), H, \mathrm{lt}\otimes\tau)$ from Corollary 6. We shall use μ and the copies of H inside the principal bundle P to construct the required representation ν of $C_0(H)$. We make this precise in the following lemma:

LEMMA 8. *Suppose P is a locally trivial principal H-bundle and (μ, U) is a covariant representation of* $(C_0(P,C), H, \mathrm{lt}\otimes\tau)$ *on $\mathscr{H}$. Then there is a (μ, U)-invariant subspace $\mathscr{H}_1$ of $\mathscr{H}$ and a representation $\nu_1\colon C_0(H)\to B(\mathscr{H}_1)$ such that $(\nu_1, U|_{\mathscr{H}_1})$ is a covariant representation of* $(C_0(H), H, \mathrm{lt})$ *and each $\nu_1(f)$ commutes with each $\mu(g)|_{\mathscr{H}_1}$.*

Proof. We can use a partition of unity on $H\backslash P$ to write every function in the dense subalgebra $C_c(P,C)$ as a sum of functions supported on H-saturated open subsets of P which are trivial as H-bundles. Since μ is non-degenerate and in particular non-zero, μ must be non-zero on one of these sets. More formally, there is an H-saturated open set N such that there is a bundle isomorphism ϕ: $N\to (H\backslash N)\times H$, and such that μ is not identically zero on $I_N := \{g\in C_0(P,C)\colon g(p)=0 \text{ for } p\notin N\}$. Because N is H-saturated, I_N is invariant, and thus $\mathscr{H}_1 := \overline{\mathrm{span}}\{\mu(g)h\colon g\in I_N, h\in\mathscr{H}\}$ is a non-zero (μ, U)-invariant subspace of $\mathscr{H}$.

We now let ϕ_2: $N\to (H\backslash N)\times H\to H$ denote the composition of ϕ with the projection onto the second factor H, and define ι: $C_0(H)\to ZM(I_N)$ by

$$(\iota(f)g)(p) = \begin{cases} f(\phi_2(p))g(p) & \text{for } p\in N \\ 0 & \text{for } p\notin N, \end{cases}$$

where $g\in I_N$. Because $\mu|_{\mathscr{H}_1}$ is non-degenerate on I_N it extends to a representation $\bar{\mu}_1$ of $M(I_N)$. Since ι is non-degenerate as a homomorphism into $M(I_N)$ it follows that $\nu_1 := \bar{\mu}_1\circ\iota$ is a non-degenerate representation of $C_0(H)$ on $\mathscr{H}_1$ whose range commutes with every $\mu(g)|_{\mathscr{H}_1}$. Since ϕ is H-equivariant, ι is equivariant for the action lt of H on $C_0(H)$ and the action $\mathrm{lt}\otimes\tau$ of H on $I_N\subset C_0(P,C)$; thus the covariance of (μ, U) implies that $(\nu_1, U|_{\mathscr{H}_1})$ is a covariant representation of $(C_0(H), H, \mathrm{lt})$.

Proof of Corollary 7. From Lemma 8 and a Zorn's Lemma argument, we obtain a decomposition $\mathscr{H} = \bigoplus \mathscr{H}_i$ into (μ, U)-invariant subspaces, each of which admits a suitable non-degenerate representation ν_i: $C_0(H) \to B(\mathscr{H}_i)$. Now we just take $\nu := \bigoplus \nu_i$, and apply Green's imprimitivity theorem as described above.

Remark 9. The local triviality hypothesis in Corollary 7 is a minor one, and is automatic if H is a Lie group, for example. Indeed, because the action of H is free and proper, P is locally trivial if and only if the orbit map q: $P \to H\backslash P$ admits local continuous cross-sections [**13**, proposition 4.65], and a theorem of Palais says that q always admits such sections when H is a Lie group [**9**, section 4.1].

Remarks 10. One situation in which the induced representation is naturally regular is that considered by Kirchberg and Wassermann in [**8**]. Suppose K is a closed subgroup of a locally compact group G, $P = G$, $H = G$ and $\tau_s = \mathrm{id}$ for all $s \in H$. Then $f \mapsto f(e)$ is an isomorphism of $\mathrm{Ind}(C, \mathrm{id})$ onto C, and the symmetric imprimitivity theorem says that $\mathrm{Ind}(C, \sigma) \times_{\mathrm{lt}\otimes\mathrm{id}} G$ is Morita equivalent to $C \times_\sigma K$ [**11**, section 1.5]. If π: $C \to B(\mathscr{H})$ is a non-degenerate representation, then it is proved in [**8**, proposition 3.2] that $X\text{-}\mathrm{Ind}(\widetilde{\pi} \times \lambda^K)$ is equivalent to a regular representation.[1]

To see that $X\text{-}\mathrm{Ind}(\widetilde{\pi} \times \lambda^K)$ is regular using Theorem 1, we define

$$(\mu(f)\xi)(t) = \pi(\sigma_t(f(t)))(\xi(t)). \tag{6}$$

Then (μ, ρ^K) is a covariant representation of $(C_0(G, C), K, \mathrm{rt} \otimes \sigma)$, and Theorem 1 implies that the representation $X\text{-}\mathrm{Ind}(\bar{\mu}|_{\mathrm{Ind}(C,\mathrm{id})} \times \rho^K)$ is regular. The extension $\bar{\mu}$ is given on $C_b(G, C)$ by the same formula (6), and hence the isomorphism $\mathrm{Ind}(C, \mathrm{id}) \cong C$ carries $\bar{\mu}|_{\mathrm{Ind}(C,\mathrm{id})}$ into $\widetilde{\pi}$. Thus $X\text{-}\mathrm{Ind}(\widetilde{\pi} \times \rho^K)$ is regular, and so is the equivalent representation $X\text{-}\mathrm{Ind}(\widetilde{\pi} \times \lambda^K)$. (As it stands, though, Theorem 1 does not give the twisted version of this result given in [**8**].)

In [**2**, section 1] Echterhoff and Raeburn consider the special case where H and K are subgroups of the same locally compact group G and $P = G$. They construct a pair of regular representations of the induced systems $(\mathrm{Ind}(C, \sigma), H)$ and $(\mathrm{Ind}(C, \tau), K)$, and show that these induce to each other via the equivalence of the symmetric imprimitivity theorem [**2**, theorem 1.4]. We can use our Theorem 1 to see directly that the induced representations are regular: in the notation of [**2**], just define μ: $C_0(G, D) \to B\big(\mathrm{Ind}_H^G(L^2(K \times H, \mathscr{H}_{\rho_1}))\big)$ by $(\mu(g)\xi)(s)(k, h) = \rho_1(\beta_h(g(sh)))(\xi(s, k, h))$, and then μ is a representation of $C_0(G, D)$ such that the restriction of $\bar{\mu}$ to the induced algebra is $\mathrm{Ind}_H^G(\rho_1 \otimes 1 \otimes 1)$.

2·3. *Amenability of actions on $C_0(P)$-algebras*

We shall now show that the main theorem of [**5**, section 4], which is a generalization of Corollary 3 to actions on $C_0(P)$-algebras, can be deduced from Theorem 1. We recall the set-up of [**5**] and take ${}_HP_K$ as usual, but instead of an arbitrary C^*-algebra C, we fix a $C_0(P)$-algebra A; this means that there is a non-degenerate injection ι_A of $C_0(P)$ into $ZM(A)$ (we write $f \cdot a$ for $\iota_A(f)a$). We insist that the actions τ: $H \to \mathrm{Aut}\, A$ and σ: $K \to \mathrm{Aut}\, A$ commute and satisfy

$$\tau_s(f \cdot a) = \mathrm{lt}_s(f) \cdot \tau_s(a) \quad \text{and} \quad \sigma_t(f \cdot a) = \mathrm{rt}_t(f) \cdot \sigma_t(a). \tag{7}$$

[1] We warn that $\widetilde{\pi}$ has different meaning in the presence of λ and ρ.

It is proved in [**5**, section 3] that both τ and σ are proper in the sense of [**17**], that there are strongly continuous actions $\bar{\tau}$: $H \to \operatorname{Aut} A^\sigma$ and $\bar{\sigma}$: $K \to \operatorname{Aut} A^\tau$ on the generalized fixed-point algebras of [**17**], and that $A^\sigma \times_{\bar{\tau}} H$ is Morita equivalent to $A^\tau \times_{\bar{\sigma}} K$.

The next corollary is [**5**, theorem 4.5]. There the one-sided case was proved first, by realizing the bimodule of [**17**] for the reduced crossed product as a quotient of the one from [**12**]; the general case was then deduced from several applications of the one-sided case and a theorem of Combes [**1**]. The proof of Corollary 11, on the other hand, uses only the general results of [**11**, section 1–2] and Theorem 1; the techniques, and in particular the indirect applications of Theorem 1, may be of independent interest.

Corollary 11. *In the above set-up, the system $(A^\sigma, H, \bar{\tau})$ is amenable if and only if $(A^\tau, K, \bar{\sigma})$ is amenable.*

As in [**5**, section 3], we view A^τ as a quotient of $\operatorname{Ind}(A, \tau)$ – indeed, for our purposes we could define A^τ this way, and avoid all reference to proper actions. To see how this works, recall that the non-degenerate action of $C_0(P)$ on A induces a continuous surjection q_A: $\operatorname{Prim} A \to P$, which is characterized by

$$q_A(I) = p \iff \{f \cdot a \in C_0(P) \cdot A\colon f(p) = 0\} \subset I;$$

the hypothesis (7) implies that q_A is H- and K-equivariant. It is proved in [**5**, section 3, proposition 3.6] and the end of the proof of [**5**, theorem 3.1] that the map $f \otimes a \mapsto f \cdot a$ extends to $C_b(P, A) \subset M(C_0(P) \otimes A)$, and induces an equivariant isomorphism of $\big(\operatorname{Ind}(A, \tau)/I(\tau), K, \mathrm{rt} \otimes \sigma\big)$ onto $(A^\tau, K, \bar{\sigma})$, where

$$I(\tau) := \{b \in \operatorname{Ind}(A, \tau)\colon b(q_A(I)) \in I \text{ for all } I \in \operatorname{Prim} A\}.$$

This ideal is $(\mathrm{rt} \otimes \sigma)$-invariant, and hence by [**4**, proposition 12] the crossed product $I(\tau) \times_{\mathrm{rt}\otimes\sigma} K$ embeds naturally as an ideal in $\operatorname{Ind}(A, \tau) \times_{\mathrm{rt}\otimes\sigma} K$ with quotient

$$(\operatorname{Ind}(A, \tau)/I(\tau)) \times_{\mathrm{rt}\otimes\sigma} K \cong A^\tau \times_{\bar{\sigma}} K.$$

To apply our theorem, we need a representation ν of $C_0(P, A)$ such that $\bar{\nu}|_{\operatorname{Ind}(A,\tau)}$ has kernel $I(\tau)$. We choose a non-degenerate representation ν of $C_0(P, A)$ with

$$\ker \nu = I_\Delta := \{b \in C_0(P, A)\colon b(q_A(I)) \in I \text{ for all } I \in \operatorname{Prim} A\}.$$

Then for $b \in \operatorname{Ind}(A, \tau)$, we have

$$\begin{aligned} \bar{\nu}(b) = 0 &\iff \nu(bc) = 0 \text{ for all } c \in C_0(P, A) \\ &\iff b(q_A(I))c(q_A(I)) \in I \text{ for all } I \in \operatorname{Prim} A,\ c \in C_0(P, A) \\ &\iff b(q_A(I))a \in I \text{ for all } I \in \operatorname{Prim} A,\ a \in A \\ &\iff b(q_A(I)) \in I \text{ for all } I \in \operatorname{Prim} A, \end{aligned}$$

so $\ker \bar{\nu}|_{\operatorname{Ind}(A,\tau)} = I(\tau)$, as we wanted.

The representation $\big(\bar{\nu}|_{\operatorname{Ind}(A,\tau)}\big)^{\sim}$ is the restriction of (the extension to $M(C_0(P, A))$ of) the representation $\widetilde{\nu}$ of $C_0(P, A)$; we aim to apply Theorem 1 to the covariant representation $(\widetilde{\nu}, \rho)$ of $\big(C_0(P, A), K, \mathrm{rt} \otimes \sigma\big)$. For this to be useful, we need to know that $\ker(\widetilde{\nu} \times \rho)$ is the ideal in $C_0(P, A) \times K$ corresponding to the diagonal ideal I_Δ in $C_0(P, A)$.

LEMMA 12. *With the above notation,* $\ker(\widetilde{\nu} \times \rho) = I_\Delta \times_{\mathrm{rt}\otimes\sigma} K$.

Proof. To avoid having to write out the opposite version of Theorem 1, we instead prove the equivalent assertion that the regular representation $(\widetilde{\nu}, \rho)$ of the system $(C_0(P, A), H, \mathrm{lt} \otimes \tau)$ satisfies $\ker(\widetilde{\nu} \times \rho) = I_\Delta \times_{\mathrm{lt}\otimes\tau} H$. To achieve this, we apply Theorem 1 with K absent. Then X_0 is the bimodule $Y_0 := {}_{C_0(P,A)\times H}C_c(P, A)_{\mathrm{Ind}(A,\tau)}$ of [**12**, theorem 2.2], Z_0 is the trivial bimodule ${}_{C_0(P,A)}C_c(P, A)_{C_0(P,A)}$, and Theorem 1 says that

$$Y\text{-Ind}(\bar{\nu}|_{\mathrm{Ind}(A,\tau)}) \sim (Z\text{-Ind}\,\nu)^{\sim} \times \rho = \widetilde{\nu} \times \rho.$$

Since $K = \{e\}$, we have $I(\sigma) = I_\Delta$, and [**11**, corollary 2.1] implies that $Y\text{-Ind}\,I(\tau) = I_\Delta \times H$. Thus our choice of ν implies that $\ker\bar{\nu}|_{\mathrm{Ind}(A,\tau)} = I(\tau)$, and

$$\ker(\widetilde{\nu} \times \rho) = \ker\big(Y\text{-Ind}(\bar{\nu}|_{\mathrm{Ind}(A,\tau)})\big) = Y\text{-Ind}\,I(\tau) = I_\Delta \times_{\mathrm{lt}\otimes\tau} H,$$

as required.

Proof of Corollary 11. Suppose that $(A^\tau, K, \bar{\sigma})$ is amenable. Applying Theorem 1 to $(\widetilde{\nu}, \rho)$ shows that

$$\ker\big(X\text{-Ind}((\bar{\nu}|_{\mathrm{Ind}(A,\tau)})^{\sim} \times \rho)\big) = \ker\big((Z\text{-Ind}(\widetilde{\nu} \times \rho))^{\sim} \times \rho\big). \tag{8}$$

Since $\ker\bar{\nu}|_{\mathrm{Ind}(A,\tau)} = I(\tau)$, $\bar{\nu}|_{\mathrm{Ind}(A,\tau)}$ factors through a faithful representation κ_1 of $A^\tau \cong \mathrm{Ind}(A, \tau)/I(\tau)$; the amenability of $(A^\tau, K, \bar{\sigma})$ implies that $\widetilde{\kappa}_1 \times \rho$ is faithful, or, equivalently, that $\ker\big((\bar{\nu}|_{\mathrm{Ind}(A,\tau)})^{\sim} \times \rho\big) = I(\tau) \times K$. [**11**, corollary 2.1] says that the Rieffel correspondence X-Ind carries $I(\tau) \times K$ to $I(\sigma) \times H$. Thus

$$I(\sigma) \times H = X\text{-Ind}(I(\tau) \times K) = \ker\big(X\text{-Ind}((\bar{\nu}|_{\mathrm{Ind}(A,\tau)})^{\sim} \times \rho)\big). \tag{9}$$

On the other hand, another application of [**11**, corollary 2.1], with H missing this time, shows that the Rieffel correspondence Z-Ind takes $I_\Delta \times K$ to $I(\sigma)$. Thus Lemma 12 gives

$$\ker\big(Z\text{-Ind}(\widetilde{\nu} \times \rho)\big) = Z\text{-Ind}(I_\Delta \times K) = I(\sigma).$$

So $Z\text{-Ind}(\widetilde{\nu} \times \rho)$ factors through a faithful representation κ_2 of $A^\sigma \cong \mathrm{Ind}(A, \sigma)/I(\sigma)$, and the representation $\big(Z\text{-Ind}(\widetilde{\nu}\times\rho)\big)^{\sim}\times\rho$ appearing in (8) factors through the regular representation $\widetilde{\kappa}_2 \times \rho$. Since we know from (8) and (9) that $\big(Z\text{-Ind}(\widetilde{\nu} \times \rho)\big)^{\sim} \times \rho$ has kernel $I(\sigma) \times H$, $\widetilde{\kappa}_2 \times \rho$ must be faithful on $A^\sigma \times H \cong (\mathrm{Ind}(A, \sigma) \times H)/(I(\sigma) \times H)$. Thus $(A^\sigma, H, \bar{\tau})$ is amenable.

The result follows by symmetry.

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