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The two-prime analogue of the Hecke C*-algebra of Bost and Connes

Nadia Larsen
*University of Copenhagen*

Ian Putnam
*University of Victoria*

Iain Raeburn
*University of Wollongong, raeburn@uow.edu.au*

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Abstract
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The Two-prime Analogue of the Hecke C*-algebra of Bost and Connes

NADIA S. LARSEN, IAN F. PUTNAM & IAIN RAEBURN

ABSTRACT. Let p and q be distinct odd primes. We analyse a semigroup crossed product $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ similar to the semigroup crossed product which models the Hecke C*-algebra of Bost and Connes. We describe a composition series of ideals in $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$, and show that the structure of one of the subquotients reflects interesting number-theoretic information about the multiplicative orders of q in the rings $\mathbb{Z}/p^\ell \mathbb{Z}$.

In [3], Bost and Connes introduced and studied a Hecke C*-algebra $C_\Omega$ which has many fascinating connections with number theory. It was shown in [11] that $C_\Omega$ can be realised as a crossed product $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_\alpha \mathbb{N}^*$ by an endomorphism action $\alpha$ of the multiplicative semigroup $\mathbb{N}^*$ of positive integers, and this realisation gives a great deal of insight into the Bost-Connes analysis (see [9]). Here we fix two odd primes $p$ and $q$, and analyse the semigroup crossed product $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ associated to the subgroup $G_{p,q} := \{ n/p^k q^\ell \mid n \in \mathbb{Z} \}/\mathbb{Z}$ of $\mathbb{Q}/\mathbb{Z}$ and the restriction of $\alpha$ to the subsemigroup $\{ p^k q^\ell \} \subset \mathbb{N}^*$, which is isomorphic to the additive semigroup $\mathbb{N}^2$. This crossed product still exhibits rich connections with number theory, though of a somewhat different nature: it has a subquotient, for example, whose ideal structure encodes the multiplicative orders of $q$ in the rings $\mathbb{Z}/p^\ell \mathbb{Z}$.

We begin our analysis by passing to the Fourier transform of our dynamical system, which involves the algebras of continuous functions on the spaces of $p$-adic and $q$-adic integers. We describe our dynamical system $(C^*(G_{p,q}), \mathbb{N}^2, \alpha)$ and its Fourier transform in Section 1. Next we construct a composition series for $C^*(G_{p,q}) \rtimes_\alpha \mathbb{N}^2$ using general results about invariant ideals and tensor products of semigroup crossed products which have been worked out in [13]. Our main structure theorem is Theorem 2.2, which is proved in Section 2 and Section 3. Theorem 3.1, which gives a detailed description of an ordinary crossed product $C(U(\mathbb{Z}_p)) \rtimes \mathbb{Z}$ arising in our analysis, is interesting in its own right: it shows, for

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example, that $\mathbb{C}(\mathcal{U}(\mathbb{Z}_p)) \times \mathbb{Z}$ is simple if and only if $q$ is a primitive root modulo $p^\ell$ for all $\ell$, which happens if and only if it is primitive modulo $p^\ell$ for any single $\ell > 1$ (see Remark 3.8). In the last section, we describe the topology on the primitive ideal space of $\mathbb{C}^*(G_{p,q}) \times_{\alpha} \mathbb{N}^2$, thus completely determining its ideal structure.

1. The Dynamical System and Its Fourier Transform

Let $p$ and $q$ be distinct odd primes. We consider the additive group

$$\mathbb{Z}[p^{-1}, q^{-1}] = \{rp^{-k}q^{-\ell} \mid r, k, \ell \in \mathbb{Z}\}$$

and its quotient $G_{p,q} := \mathbb{Z}[p^{-1}, q^{-1}]/\mathbb{Z}$. We write $\alpha$ for the action of $\mathbb{N}^2$ by endomorphisms of the group $\mathbb{C}^*$-algebra $\mathbb{C}^*(G_{p,q})$ which is characterised on the canonical generating unitaries $\{\delta_r \mid r \in G_{p,q}\}$ by

$$\alpha_{m,n}(\delta_r) = \frac{1}{p^mq^n} \sum_{s \in G_{p,q} \text{ s.t. } p^mq^ns = r} \delta_s;$$

we can see that there is such an action either by modifying [11, Proposition 2.1] or by applying the general method of [14, Section 1] to the action of $\mathbb{N}^2$ on $\mathbb{Z}$ defined by $\eta_{m,n}(k) = p^mq^nk$ (see [14, Example 1.2]). As in [10, Proposition 2.1], the action satisfies

$$\alpha_{k,\ell}(1)\alpha_{m,n}(1) = \alpha_{km,\ell \cdot n}(1).$$

A covariant representation of the dynamical system $(\mathbb{C}^*(G_{p,q}), \mathbb{N}^2, \alpha)$ consists of a nondegenerate representation $\pi$ of $\mathbb{C}^*(G_{p,q})$ and a representation $V$ of $\mathbb{N}^2$ by isometries on the same space such that

$$\pi(\alpha_{m,n}(a)) = V_{m,n}\pi(a)V_{m,n}^*, \quad \text{for } a \in \mathbb{C}^*(G_{p,q}) \text{ and } (m,n) \in \mathbb{N}^2;$$

the relation (1.2) then implies that the isometric representation $V$ is Nica covariant, in the sense that $V_{k,\ell}^*V_{k,\ell}V_{m,n}V_{m,n}^* = V_{km,\ell \cdot n}V_{km,\ell \cdot n}^*$. One can see that the system has nontrivial covariant representations by modifying the constructions in [11], or by applying [14, Lemma 1.7]. Thus there is a crossed product $(\mathbb{C}^*(G_{p,q}) \times_{\alpha} \mathbb{N}^2, i_A, i_S)$, which is a universal $\mathbb{C}^*$-algebra for covariant representations of the system (see [10, Proposition 2.1]). (To avoid complicated notation, we always write $i_A$ and $i_S$ for the algebra and semigroup components of the universal covariant representation.) This crossed product carries a dual action $\hat{\alpha}$ of $\mathbb{F}^2$ which leaves $i_A(\mathbb{C}^*(G_{p,q}))$ invariant and satisfies $\hat{\alpha}_{w,z}(i_S(m,n)) = w^mz^n i_S(m,n)$.

To compute the Fourier transform of the system, we need a description of the dual group $\hat{G}_{p,q}$. Note that with $G_p := \mathbb{Z}[p^{-1}]/\mathbb{Z}$, the map $(r, s) \mapsto r + s$ is an isomorphism of $G_p \times G_q$ onto $G_{p,q}$, and, dually, we have $\hat{G}_{p,q} \cong \hat{G}_p \times \hat{G}_q$. To
describe \( \hat{G}_p \), note that \( \mathbb{Z}[p^{-1}] = \bigcup_{\ell=1}^{\infty} p^{-\ell}\mathbb{Z} \), so \( G_p = (\bigcup_p p^{-\ell}\mathbb{Z})/\mathbb{Z} \) has a natural description as a direct limit \( \lim_{\ell \to \infty} p^{-\ell}\mathbb{Z}/\mathbb{Z} \), and \( \hat{G}_p \) is an inverse limit \( \lim_{\ell \to \infty} p^{-\ell}\mathbb{Z}/\mathbb{Z} \) of finite groups. The usual pairing \( (t, n) = \exp 2\pi i tn \) of \( \mathbb{Z} \) with \( \mathbb{R}/\mathbb{Z} \) induces an isomorphism of \( \mathbb{Z}/p^\ell\mathbb{Z} \) onto \( (p^{-\ell}\mathbb{Z}/\mathbb{Z})^* \), and it is easy to check that the dual of the inclusion \( p^{-\ell}\mathbb{Z}/\mathbb{Z} \to p^{-(\ell+1)}\mathbb{Z}/\mathbb{Z} \) is the map of \( \mathbb{Z}/p^{\ell+1}\mathbb{Z} \) onto \( \mathbb{Z}/p^\ell\mathbb{Z} \) given by reduction mod \( p^\ell \). Thus \( \hat{G}_p \) is naturally identified as a compact group with the inverse limit \( \lim_{\ell \to \infty} \mathbb{Z}/p^\ell\mathbb{Z} \).

Each \( \mathbb{Z}/p^\ell\mathbb{Z} \) is a ring, and the reduction maps are ring homomorphisms, so \( \lim_{\ell \to \infty} \mathbb{Z}/p^\ell\mathbb{Z} \) is a compact topological ring \( \mathbb{Z}_p \), which is called the ring of \( p \)-adic integers; in the previous paragraph, we identified \( \hat{G}_p \) with the additive group of \( \mathbb{Z}_p \). However, the multiplicative structure of \( \mathbb{Z}_p \) plays a crucial role in our analysis, for two reasons. First, we can use it to describe the action \( \alpha \): the reduction maps \( \mathbb{Z} \to \mathbb{Z}/p^\ell\mathbb{Z} \) induce an embedding of \( \mathbb{Z} \) in \( \mathbb{Z}_p \), and \( \alpha_{m,n} \), loosely speaking, division by \( p^m q^n \) (see Lemma 1.1 below). Second, the group \( \mathcal{U}(\mathbb{Z}_p) \) of units in \( \mathbb{Z}_p \) (the multiplicatively invertible elements) appears in our theorems. We need to know that there is a natural identification of \( \mathcal{U}(\mathbb{Z}_p) \) with \( \lim_{\ell \to \infty} \mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z}) \), and that an integer \( m \) is a unit in \( \mathbb{Z}_p \) precisely when \( m \) is coprime to \( p \). For these and other properties of \( \mathbb{Z}_p \), we refer to [16, Chapter II].

We are now ready to describe the Fourier-transform system. The dual of \( G_{p,q} \) is \( \mathbb{Z}_p \times \mathbb{Z}_q \); if \( \pi_\ell \) denotes the canonical map of \( \mathbb{Z}_p \) onto \( \mathbb{Z}/p^\ell\mathbb{Z} \), then the pairing is given by

\[
(1.4) \quad \langle r + s, (x, y) \rangle = \exp 2\pi i (r \pi_\ell(x) + s \pi_\ell(y)) \quad \text{for } r \in \mathbb{Z}[p^{-1}],
\]
\[
s \in \mathbb{Z}[q^{-1}], \text{ and } \ell \text{ large}.
\]

**Lemma 1.1.** The Fourier transform \( C^*(G_{p,q}) \equiv C(\mathbb{Z}_p \times \mathbb{Z}_q) \) carries the action defined by (1.1) into the action given by

\[
(1.5) \quad \alpha_{m,n}(f)(x, y) = \begin{cases} 
  f(p^{-m}q^{-n}x, p^{-m}q^{-n}y) & \text{if } x \in p^m q^n \mathbb{Z}_p \text{ and } y \in p^m q^n \mathbb{Z}_q, \\
  0 & \text{otherwise}.
\end{cases}
\]

**Proof.** We aim to apply [13, Proposition 4.5]. To do this, note that \( \alpha_{m,n} \) is defined by averaging over the solutions \( s \) of \( \beta_{m,n}(s) = r \), where \( \beta_{m,n} \) is the endomorphism of \( G_{p,q} \) defined by \( \beta_{m,n}(s) = p^m q^n s \). From the pairing (1.4), we see that the endomorphism \( \beta_{m,n} \) of \( \mathbb{Z}_p \times \mathbb{Z}_q \) is given in terms of the ring structure by \( \beta_{m,n}(x, y) = (p^m q^n x, p^m q^n y) \). Thus the Lemma follows directly from [13, Proposition 4.5].
2. The structure theorem

Our main theorem describes the structure of $C^*(G, d) \rtimes_\alpha \mathbb{N}^2$—or, equivalently, of the crossed product $C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes_\alpha \mathbb{N}^2$ of the Fourier-transform system described in Lemma 1.1. To state it, we need a number-theoretic lemma. If $k$ and $m$ are coprime integers, so that $m$ is a unit in $\mathbb{Z}/k\mathbb{Z}$, we write $o_k(m)$ for the order of $m$ in $\mathbb{Z}/k\mathbb{Z}$.

**Lemma 2.1.** Let $p$ and $q$ be distinct odd primes. Then there is a positive integer $L = L_p(q)$ such that

\[
o_p(q) = \begin{cases} o_p(q) & \text{if } 1 \leq \ell \leq L, \\ p^{\ell - L} o_p(q) & \text{if } \ell > L. \end{cases}
\]

This lemma is presumably well-known; certainly some of its immediate consequences are (see Remark 3.8). We are not going to prove it now, because we shall prove a slightly more general result in Theorem 3.1. However, we want to use the integers $L_p(q)$ from this lemma in the statement of our main theorem.

**Theorem 2.2.** Let $p$ and $q$ be distinct odd primes. Then there are $\alpha$-invariant ideals $I_1$ and $I_2$ in $C^*(G, d) \rtimes_\alpha \mathbb{N}^2$ such that $I_1 \subset I_2$,

\[
I_1 \cong \mathcal{K}(\ell^2(\mathbb{N}^2)) \otimes C(\mathbb{N}_p) \rtimes \mathcal{U}(\mathbb{Z}_q),
\]

\[
I_2/I_1 \cong (\mathcal{K}(\ell^2(\mathbb{N})) \otimes C) \oplus (\mathcal{K}(\ell^2(\mathbb{N})) \otimes D),
\]

\[
(C^*(G, d) \rtimes_\alpha \mathbb{N}^2)/I_2 \cong C(\mathbb{T}^2),
\]

where $C$ is the direct sum of $(p - 1)p^{l_p(q) - 1}/o_p(q)$ Bunce-Deddens algebras with supernatural number $o_p(q)p^\infty$ and $D$ is the direct sum of $(q - 1)q^{l_q(p) - 1}/o_q(p)$ Bunce-Deddens algebras with supernatural number $o_q(p)q^\infty$.

The algebra $C^*(G, d) \cong C(\mathbb{Z}_p \times \mathbb{Z}_q)$ decomposes as a tensor product $C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)$, and the action $\alpha$ given by (1.5) decomposes as a tensor product of two actions of $\mathbb{N}^2$. At this point, we cannot separate the actions of the two copies of $\mathbb{N}$ (as Bost and Connes say, the two primes interact), but there is a large invariant ideal $C_0(\mathbb{Z}_p \setminus \{0\})$ in $C(\mathbb{Z}_p)$ where the action does split as a tensor product of two actions of $\mathbb{N}$. The ideals $I_1$ and $I_2$ will be crossed products of different invariant ideals in $C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)$ built from $C_0(\mathbb{Z}_p \setminus \{0\})$ and its twin.

For ordinary crossed products $A \rtimes G$ by group actions, invariant ideals in $A$ give rise to short exact sequences

\[
0 \rightarrow I \rtimes G \rightarrow A \rtimes G \rightarrow (A/I) \rtimes G \rightarrow 0.
\]

For semigroup crossed products $A \rtimes_\alpha S$, one has to know that the ideal $I$ is **extendibly invariant**, in the sense that each endomorphism $\alpha_s$ extends to endomorphisms of $M(I)$ and $M(A)$ in such a way that $\alpha_s(1_M(I)) = \alpha_s(1_M(A))$ as multipliers of $I$ (see [1, 13]). Since the endomorphism $x \mapsto p^m q^n x$ of $\mathbb{Z}_p$ leaves
both \( \mathbb{Z}_p \setminus \{0\} \) and \( \{0\} \) invariant, it follows from Lemma 1.1 and [13, Theorem 4.3] that \( I := C_0(\mathbb{Z}_p \setminus \{0\}) \) and \( J := C_0(\mathbb{Z}_q \setminus \{0\}) \) are extendibly invariant ideals in \( A := C(\mathbb{Z}_p) \) and \( B := C(\mathbb{Z}_q) \). We can therefore apply [13, Theorem 3.1] to deduce that the ideals \( I_1 := (I \otimes J) \times \mathbb{N}^2 \) and \( I_2 := (I \otimes B + A \otimes J) \times \mathbb{N}^2 \) form a composition series in which

\[
(2.5) \quad I_1 \cong (I \otimes J) \times_{\alpha} \mathbb{N}^2, \\
(2.6) \quad I_2/I_1 \cong ((A/I) \otimes J) \times \mathbb{N}^2 \oplus (I \otimes (B/J)) \times \mathbb{N}^2, \quad \text{and} \\
(2.7) \quad (A \otimes B) \times_{\alpha} \mathbb{N}^2/I_2 \cong ((A/I) \otimes (B/J)) \times \mathbb{N}^2.
\]

Notice that because the ideals are crossed products, they are \( \alpha \)-invariant. To prove Theorem 2.2, therefore, we have to identify the subquotients.

We begin by noting that the maps \( f \mapsto f(0) \) induce isomorphisms \( A/I \cong \mathbb{C} \) and \( B/J \cong \mathbb{C} \), so \( (A/I) \otimes (B/J) \cong \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \). Thus (2.7) is \( \mathbb{C} \times_{\mathrm{id}} \mathbb{N}^2 \). When the action is unital, as the identity action id certainly is, the covariance relation (1.3) implies that the isometries are all unitary; thus \( \mathbb{C} \times_{\mathrm{id}} \mathbb{N}^2 \) is the universal \( C^* \)-algebra generated by a unitary representation of \( \mathbb{Z}^2 \). In other words, \( \mathbb{C} \times_{\mathrm{id}} \mathbb{N}^2 = C^*(\mathbb{Z}^2) \cong C(T^2) \), and we have proved (2.4).

For the other two parts, we need the promised decomposition of the action of \( \mathbb{N}^2 \) on \( I = C_0(\mathbb{Z}_p \setminus \{0\}) \).

**Lemma 2.3.** The map \((n, x) \mapsto p^n x\) is a homeomorphism of \( \mathbb{N} \times \mathcal{U}(\mathbb{Z}_p) \) onto \( \mathbb{Z}_p \setminus \{0\} \).

**Proof.** Since every nonzero \( p \)-adic number can be uniquely written as a power of \( p \) times a unit (by Proposition 2 of [16, Chapter II], for example), the map is a bijection. It is a homeomorphism because it carries the basic open sets \([n] \times V\) for the topology on \( \mathbb{N} \times \mathcal{U}(\mathbb{Z}_p) \) into the basic open sets \( p^n V \) for the topology on \( \mathbb{Z}_p \setminus \{0\} \).

The lemma implies that \( I = C_0(\mathbb{Z}_p \setminus \{0\}) \cong c_0(\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p)) \). To describe what happens to the action \( \alpha \) under this isomorphism, we need some notation. We let \( \tau \) denote the action of \( \mathbb{N} \) on \( c_0(\mathbb{N}) \) by forward shifts; if we think of elements of \( c_0(\mathbb{N}) \) as functions on \( \mathbb{N} \), then

\[
\tau_m (f)(k) = \begin{cases} 
  f(k - m) & \text{if } k \geq m \\
  0 & \text{if } k < m.
\end{cases}
\]

Since \((q, p) = 1\), \( q \) is a unit in \( \mathbb{Z}_p \), and division by powers of \( q \) defines an action \( \sigma = \sigma_{p,q} \) of \( \mathbb{Z} \) by automorphisms of \( C(\mathcal{U}(\mathbb{Z}_p)) \): \( \sigma_n (f)(x) = f(q^{x-n} x) \). We now have the following immediate corollary of Lemma 2.3:

**Corollary 2.4.** The isomorphism \( C_0(\mathbb{Z}_p \setminus \{0\}) \cong c_0(\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p)) \) induced by the homeomorphism of Lemma 2.3 carries \( \alpha \) into the tensor product action \( \tau \otimes \sigma : (m, n) \mapsto \tau_m \otimes \sigma_n \).
Lemma 2.5. There is an isomorphism

\[(2.8) \quad I_2/I_1 \cong (\mathcal{K}(\ell^2(\mathbb{N}))) \otimes (C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,d}} \mathbb{Z}))
\oplus (\mathcal{K}(\ell^2(\mathbb{N}))) \otimes (C(\mathcal{U}(\mathbb{Z}_q)) \rtimes_{\sigma^{q,d}} \mathbb{Z})).\]

Proof. First, recall that \( A/I \cong \mathbb{C} \) and \( B/J \cong \mathbb{C} \), so from (2.6) we have

\[(2.9) \quad I_2/I_1 \cong (I \rtimes_{\alpha} \mathbb{N}^2) \oplus (J \rtimes_{\alpha} \mathbb{N}^2).\]

Next, we use the decomposition of Corollary 2.4 and [13, Theorem 2.6] (which applies because our action satisfies (1.2)), to see that

\[(2.10) \quad I \rtimes_{\alpha} \mathbb{N}^2 \cong (c_0(\mathbb{N}) \rtimes_{\tau} \mathbb{N}) \otimes (C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma^{p,d}} \mathbb{N}).\]

Because \( \sigma^{p,d} \) consists of automorphisms, the isometries in any covariant representation of \((C(\mathcal{U}(\mathbb{Z}_p)), \mathbb{N}, \sigma)\) are unitary, and \( C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma} \mathbb{N} \) is the usual crossed product \( C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma} \mathbb{Z} \).

To handle the other factor in (2.10), recall that \( \mathcal{C} \rtimes_{\tau} \mathbb{N} = B_{\mathbb{N}} \rtimes_{\tau} \mathbb{N} \) is the Toeplitz algebra, and \( c_0(\mathbb{N}) \rtimes_{\tau} \mathbb{N} \) is the ideal of compact operators. More precisely, let \( M \) denote the representation of \( c \) by multiplication operators on \( \ell^2(\mathbb{N}) \), and let \( S \) be the unilateral shift on \( \ell^2(\mathbb{N}) \). Then \( (M, S) \) is a covariant representation of \((C, \mathbb{N}, \tau)\) such that \( M \times S \) is an isomorphism of \( c \rtimes_{\tau} \mathbb{N} \) onto the \( C^* \)-algebra generated by \( S \). (This formulation of Coburn's Theorem is described in [2], for example.) It is easy to check that \( M \times S \) carries the ideal \( c_0 \rtimes_{\tau} \mathbb{N} \) onto \( \mathcal{K}(\ell^2(\mathbb{N})) \).

Thus (2.10) implies that \( I \rtimes_{\alpha} \mathbb{N}^2 \cong \mathcal{K} \otimes (C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\sigma} \mathbb{Z}) \). Swapping \( p \) and \( q \) gives an analogous description of \( J \rtimes_{\alpha} \mathbb{N}^2 \), and the Lemma follows from (2.9). \( \Box \)

The description of \( I_2/I_1 \) in (2.3) will follow from this lemma and Theorem 3.1.

To describe \( I[1] := (I \otimes J) \rtimes_{\alpha} \mathbb{N}^2 \), we use two applications of Corollary 2.4 to get an isomorphism

\[ I \otimes J = C_0(\mathbb{Z}_p \setminus \{0\}) \otimes C_0(\mathbb{Z}_q \setminus \{0\}) \cong C_0(\mathbb{N} \times \mathbb{N} \times \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)) \]

which carries the endomorphism \( \alpha_{m,n} \) into \( \tau_m \otimes \tau_n \otimes \sigma_{m,n}^{p,q} \). We now borrow another idea from the theory of ordinary crossed products: recall that \((C_0(\mathbb{G}) \otimes A) \rtimes_{\tau^A} G \cong (C_0(\mathbb{G}) \rtimes_{\tau} G) \otimes A \) for any action \( \beta \). Because \( q \in \mathcal{U}(\mathbb{Z}_p) \)

and \( p \in \mathcal{U}(\mathbb{Z}_q) \), the endomorphism \( \varphi \) of \( C_0(\mathbb{N} \times \mathbb{N} \times \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)) \) defined by \( \varphi(f)(k, \ell, x, y) = f(k, \ell, q^k x, p^\ell y) \) is an automorphism. A quick calculation shows that \( \varphi \circ (\tau_m \otimes \tau_n \otimes \sigma_{m,n}^{p,q} \otimes \sigma_{m,n}^{q,p}) = \tau_m \otimes \tau_n \otimes \text{id} \otimes \text{id} \), so \( \varphi \) induces an isomorphism

\[(I \otimes J) \rtimes_{\alpha} \mathbb{N}^2 \cong (C_0(\mathbb{N} \times \mathbb{N}) \rtimes_{\tau \otimes \tau} (\mathbb{N} \times \mathbb{N})) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)).\]
To finish off the proof of (2.2), either note that
\[
C_0(\mathbb{N}^2) \rtimes_{\alpha} \mathbb{N}^2 \cong (C_0 \rtimes_{\alpha} \mathbb{N}) \otimes (C_0 \rtimes_{\alpha} \mathbb{N}) \\
\cong \mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathcal{K}(\ell^2(\mathbb{N})) = \mathcal{K}(\ell^2(\mathbb{N}^2)),
\]
or check directly that the natural covariant representation of $B_{\mathbb{N}^2} \rtimes_{\alpha} \mathbb{N}^2$ on $\ell^2(\mathbb{N}^2)$ restricts to an isomorphism of $C_0(\mathbb{N}^2) \rtimes \mathbb{N}^2$ onto $\mathcal{K}(\ell^2(\mathbb{N}^2))$.

To prove Theorem 2.2, therefore, it remains to prove Lemma 2.1 and to identify $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\alpha} \mathbb{Z}$ with the appropriate number of Bunce-Deddens algebras. We do this in Theorem 3.1.

3. The crossed products $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\alpha} \mathbb{Z}$

Our analysis of $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\alpha} \mathbb{Z}$ does not require that $q$ is prime, only that it is coprime to $p$. We therefore fix an odd prime $p$ and an integer $m$ coprime to $p$, and consider the action $\sigma = p^m \cdot \sigma$ of $\mathbb{Z}$ on $C(\mathcal{U}(\mathbb{Z}_p))$ defined by

\[(3.1) \quad \sigma_n^{p^m}(f)(x) = f(m^{-n}x).\]

**Theorem 3.1.** Suppose that $p$ is an odd prime and $(m,p) = 1$, and denote by $o_{p^\ell}(m)$ the order of $m$ in $\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$. Then there is a positive integer $L$ such that

\[(3.2) \quad o_{p^\ell}(m) = \begin{cases} o_p(m) & \text{if } 1 \leq \ell \leq L \\ p^{\ell-1}o_p(m) & \text{if } \ell > L, \end{cases}\]

and $C(\mathcal{U}(\mathbb{Z}_p)) \rtimes_{\alpha} \mathbb{Z}$ is the direct sum of $p^{L-1}(p - 1)/o_p(m)$ Bunce-Deddens algebras with supernatural number $o_p(m)p^m$.

We begin by establishing the number-theoretic statements. Because $\mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z})$ is cyclic of order $(p - 1)p^{\ell-1}$ (see Theorem 2 of [8, Chapter 4], for example), we can apply the following elementary lemma about cyclic groups.

**Lemma 3.2.** Suppose that $(n,p) = 1$ and $G, H$ are cyclic groups of orders $p^\ell n$, $p^{\ell-1} n$, respectively. If $\pi : G \to H$ is a surjective homomorphism and $g$ is a generator of $G$, then the order of $\pi(g)$ is given by

\[o(\pi(g^r)) = \begin{cases} |G|/(r,|G|) & \text{if } p^\ell \text{ divides } r \\ |G|/p(r,|G|) & \text{if } p^\ell \text{ does not divide } r \end{cases}\]

**Proof.** Since $\pi(g)$ is a generator of $H$, we have

\[o(\pi(g^r)) = o(\pi(g))^r = \frac{|H|}{(r,|H|)} = \frac{|G|}{p(r,|H|)}.\]
If \( p^\ell \) divides \( r \), say \( r = sp^\ell \), then
\[
p(r, |H|) = p(p^\ell s, p^{\ell-1}n) = p^\ell(ps, n) = p^\ell(s, n) = (r, p^\ell n) = (r, |G|),
\]
as claimed. If \( p^\ell \) does not divide \( r \), then \( (r, |G|) = (r, p^\ell n) = (r, p^{\ell-1}n) = (r, |H|) \).

**Corollary 3.3.** Suppose \( p \) is prime and \( (p, m) = 1 \). Then
\[
o_{p^\ell}(m) = \begin{cases} o_{p^{\ell+1}}(m) & \text{if } p \text{ does not divide } o_{p^{\ell+1}}(m) \\ o_{p^{\ell+1}}(m)/p & \text{if } p \text{ does divide } o_{p^{\ell+1}}(m). \end{cases}
\]

**Proof.** Since a number is coprime to \( p^\ell \) iff it is coprime to \( p^{\ell+1} \), the reduction map \( \pi \) is a homomorphism of \( \mathcal{U}(\mathbb{Z}/p^{\ell+1}\mathbb{Z}) \) onto \( \mathcal{U}(\mathbb{Z}/p^\ell\mathbb{Z}) \), and Lemma 3.2 applies. Indeed, there is a generator \( g \) such that \( m = g^r \) where \( r := (p-1)p^\ell/o_{p^{\ell+1}}(m) \). Then
\[
o_{p^\ell}(m) = o(\pi(g^r)) = \begin{cases} o_{p^{\ell+1}}(m) & \text{if } p^\ell \text{ divides } (p-1)p^\ell/o_{p^{\ell+1}}(m), \\ o_{p^{\ell+1}}(m)/p & \text{if } p^\ell \text{ does not divide } (p-1)p^\ell/o_{p^{\ell+1}}(m), \end{cases}
\]
which translates into what we want.

**Corollary 3.4.** There is a positive integer \( L \) such that (3.2) holds.

**Proof.** We first note that the sequence \( \{o_{p^\ell}(m) \mid \ell \in \mathbb{N}\} \) must be unbounded: for fixed \( N \), \( m^N \) is eventually less than \( p^\ell \), and then \( o_{p^\ell}(m) > N \). In particular, \( \{o_{p^\ell}(m)\} \) is certainly not constant. Let \( L \) be the first integer such that \( o_{p^L}(m) < o_{p^{L+1}}(m) \). Then \( o_{p^L}(m) = o_p(m) \) for \( 1 \leq \ell \leq L \), and by Corollary 3.3, we have \( o_{p^{L+1}}(m) = p o_p(m) \), and \( p \) divides \( o_{p^{L+1}}(m) \). Since \( o_{p^{L+1}}(m) \) divides \( o_{p^L}(m) \) for all \( \ell > L \), it follows that \( p \) divides \( o_{p^L}(m) \) for all \( \ell > L \), and \( \ell - L \) applications of Corollary 3.3 show that \( o_{p^L}(m) = p^{L-1}o_{p^L}(m) = p^{L-L}o_p(m) \).

**Remark 3.5.** The referee has pointed out that one can also deduce Corollary 3.4 from the isomorphism of \( \mathcal{U}(\mathbb{Z}/p\mathbb{Z}) \times p\mathbb{Z}_p^+ \) onto \( \mathcal{U}^+(\mathbb{Z}_p) \) provided by sending elements of \( \mathcal{U}(\mathbb{Z}/p\mathbb{Z}) \) to their Teichmüller representatives and the exponential isomorphism of the additive group \( p\mathbb{Z}_p^+ \) onto \( 1 + p\mathbb{Z}_p \) (see [7, Corollary 4.5.10], for example). This isomorphism is compatible with the inverse limit decompositions of \( \mathcal{U}^+(\mathbb{Z}_p) \) and \( p\mathbb{Z}_p^+ \), and hence it suffices to prove the analogous properties of additive orders in \( p\mathbb{Z}_p^+ \).

Let \( H \) be the closed subgroup of \( \mathcal{U}(\mathbb{Z}_p) \) generated by \( m \). Then \( H \) is invariant under multiplication by powers of \( m \), and the formula (3.1) also defines an action \( \sigma \) of \( \mathbb{Z} \) on \( C(H) \). This is where the Bunce-Deddens algebras come from:
Proposition 3.6. The crossed product \( C(H) \rtimes_{\sigma} \mathbb{Z} \) is a Bunce-Deddens algebra with supernatural number \( o_p(m) p^\infty \).

The Bunce-Deddens algebras were originally defined to be the \( C^* \)-algebras generated by certain weighted shifts on \( l^2 \) [5, Section V.3], but we shall recognize them as crossed products associated to odometer actions. Let \( \{n_k\} \) be a sequence of integers each of which is at least 2, and let \( X_k = \{0, 1, \ldots, n_k - 1\} \). The odometer action \( \tau \) of \( \mathbb{Z} \) on \( \prod_{k \geq 1} X_k \) is given by addition with carry over: let \( N_1 = 1 \), \( N_k := \prod_{i < k} n_i \) for \( k > 1 \), and then

\[
\tau_n(\{a_k\}) = \{b_k\}, \quad \text{where} \quad \sum_{k \geq 1} b_k N_k = n + \sum_{k \geq 1} a_k N_k \pmod{N_{\ell+1}}.
\]

The crossed product \( C(\prod_{k \geq 1} X_k) \rtimes_{\tau} \mathbb{Z} \) is then a Bunce Deddens algebra with supernatural number \( \prod_{k \geq 1} n_k \) [5, Theorem VIII.4.1]. In general, Bunce-Deddens algebras are simple [5, Theorem V.3.3], and are determined up to isomorphism by their supernatural number [5, Theorem V.3.5].

Proof. Write \( d \) for \( o_p(m) \), and let

\[
\mathcal{O} := \{0, 1, \ldots, d - 1\} \times \{0, 1, \ldots, p - 1\}^\mathbb{N}.
\]

For \( \ell > L \), we define \( h_\ell : \mathcal{O} \to \mathcal{U}(\mathbb{Z}/p^\ell \mathbb{Z}) \) by

\[
h_\ell(\{a_n\}) = m^{a_0 + da_1 + d^2 a_2 + \cdots + d^{\ell-1} a_{\ell-1}} + (\text{mod } p^\ell \mathbb{Z});
\]

because the order of \( m \) in \( \mathcal{U}(\mathbb{Z}/p^\ell \mathbb{Z}) \) is \( dp^{\ell-L} \), the maps \( h_\ell \) satisfy \( h_{\ell+1}(\{a_n\}) = h_\ell(\{a_n\}) \pmod{p^\ell \mathbb{Z}} \). Since the \( h_\ell \) are continuous by definition of the product topology, they induce a continuous map \( h : \mathcal{O} \to \mathcal{U}(\mathbb{Z}_p) = \lim \mathcal{U}(\mathbb{Z}/p^\ell \mathbb{Z}) \), which is an injection because \( h_\ell(\{a_n\}) \) determines \( a_0, \ldots, a_{\ell-1} \) uniquely. The range of \( h \) is a compact subgroup, and contains the positive powers of \( m \), which are the images of the sequences in \( \mathcal{O} \) which are eventually zero; since such sequences are dense in \( \mathcal{O} \), their images generate the range. In other words, \( h \) is a continuous injection of \( \mathcal{O} \) onto \( H \), and is therefore a homeomorphism. Since \( h(\tau^1 \{a_n\}) = mh(\{a_n\}) \) for all \( \{a_n\} \), we deduce that the Bunce-Deddens algebra \( C(\mathcal{O}) \rtimes_{\tau} \mathbb{Z} \) is isomorphic to \( C(H) \rtimes_{\sigma} \mathbb{Z} \).

To finish the proof of our theorem, we need to decompose the dynamical system \( (C(\mathcal{U}(\mathbb{Z}_p)), \mathbb{Z}, \sigma) \) as a sum of copies of \( (C(H), \mathbb{Z}, \sigma) \). This needs a simple group-theoretic lemma.

Lemma 3.7. Suppose \( G = \lim_{\leftarrow} G_n \) is a compact group which is the inverse limit of finite groups \( G_n \), and suppose that the canonical maps \( \pi_n : G \to G_n \) are surjective. If \( H \) is a closed subgroup of \( G \) and there is an integer \( k \) such that \( |G_n/\pi_n(H)| = k \) for all \( n \), then \( |G/H| = k \).
Proof. Certainly \(|G/H| \geq |\pi_n(G)/\pi_n(H)| = k\). Suppose \(g_1 H, \ldots, g_{k+1} H\) are cosets in \(G/H\); we shall prove that two must be the same. The hypothesis implies that for each \(n\), two of \(\pi_n(g_1 H)\) coincide. Since there are only finitely many possibilities, we can assume by passing to a subsequence that the same two coincide in each \(G_n/\pi_n(H)\); say \(\pi_n(g_1 H) = \pi_n(g_2 H)\) for all \(n\). Then \(\pi_n(g_1 g_2^{-1}) \in \pi_n(H)\); say \(\pi_n(g_1 g_2^{-1}) = \pi_n(h_n)\). By definition of the topology on the inverse limit, we have \(h_n \rightarrow g_1 g_2^{-1}\) in \(G\), so that \(g_1 g_2^{-1} \in H\) and \(g_1 H = g_2 H\).

End of the proof of Theorem 3.1. Since \(\pi_\ell(H)\) is the subgroup of \(U(\mathbb{Z}/p^\ell \mathbb{Z})\) generated by \(m\), we have

\[
\frac{|U(\mathbb{Z}/p^\ell \mathbb{Z})/\pi_\ell(H)|}{(p - 1)p^{\ell-1}/o_p(m)} = (p - 1)p^{\ell-1}/o_p(m) \quad \text{for all } \ell \geq L.
\]

We can therefore apply Lemma 3.7 to \(U(\mathbb{Z}_p) = \lim_\leftarrow (U(\mathbb{Z}/p^\ell \mathbb{Z}))\), \(\ell \geq L\) to deduce that \(H\) has index \(N := (p - 1)p^{L-1}/o_p(m)\) in \(U(\mathbb{Z}_p)\).

Next, note that because \(H\) is a closed subgroup of finite index, it is also open: its complement is the finite union of cosets of \(H\), and hence closed. Since \(H\) is by definition invariant under multiplication by powers of \(m\), it follows that \(U(\mathbb{Z}_p)\) is the disjoint union of \(N\) open and closed invariant sets of the form \(xH\), and \(C(U(\mathbb{Z}_p))\) is the direct sum of \(\sigma\)-invariant ideals of the form \(C(xH)\). The dynamical systems \((C(xH), \mathbb{Z}, \sigma)\) are all conjugate to \((C(H), \mathbb{Z}, \sigma)\). Thus the Theorem follows from Proposition 3.6.

Remark 3.8. An integer \(m\) which generates \(U(\mathbb{Z}/p^\ell \mathbb{Z})\) is called a primitive root modulo \(p^\ell\). If \(m\) is a primitive root modulo \(p^\ell\) for one \(\ell > 1\), then (3.2) implies that \(L_p(m) = 1\) and \(o_p(m) = p - 1\), and hence that \(m\) is a primitive root modulo \(p^k\) for all \(k\). (This is known; see [6, Section 17, Exercise VI.4], for example.) Theorem 3.1 gives a curious \(*\)-algebraic characterisation of primitive roots: \(m\) is primitive modulo \(p^\ell\) for all \(\ell\) if and only if \(C(U(\mathbb{Z}_p)) \rtimes_\sigma \mathbb{Z}\) is simple. More generally, the cardinality of \(\text{Prim } C(U(\mathbb{Z}_p)) \rtimes \mathbb{Z}\) determines the orders \(o_{p^\ell}(m)\) of \(m\) in \(U(\mathbb{Z}/p^\ell \mathbb{Z})\).

The relations (3.2) are the only restrictions on the possible values of \(o_p(m)\). Indeed, given an odd prime \(p\), a divisor \(d\) of \(p - 1\), and an integer \(L \geq 1\), there are infinitely many primes \(q\) with \(o_p(q) = d\) and \(L_p(q) = L\). To see this, choose \(k\) such that \(o_p(k) = p d\). Then every integer \(q\) in the arithmetic progression \(k + np L + 1\) has \(o_p(q) = d\) and \(o_p(q) = p d\), and it follows from (3.2) that \(o_{p^\ell}(q) = p^{\ell-1}d\) for all \(\ell > L\). Now our assertion follows from Dirichlet's Theorem: every arithmetic progression \(k + nr\) with \((k, r) = 1\) contains infinitely many primes [8, Section 16.1].
4. THE PRIMITIVE IDEAL SPACE

Since \( \text{Prim} C(X, \mathcal{K}) \) is homeomorphic to \( X \) [15, Example A.24] and Bunce-Deddens algebras are simple [5, Theorem V.3.3], Theorem 2.2 gives us a setwise description of the primitive ideal space of the algebra \( C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_\alpha \mathbb{N}^2 \). It consists of a copy \( \{I_{x,y}\} \) of \( U(\mathbb{Z}_p) \times U(\mathbb{Z}_q) \) embedded as an open subset, a copy \( \{L_{w,z}\} \) of \( \mathbb{T}^2 \) embedded as a closed subset, and two finite sets \( \{J_{xH_p}\}, \{K_{yH_q}\} \) parametrised by the quotients \( U(\mathbb{Z}_p)/H_p = U(\mathbb{Z}_p)/q^2 \) and \( U(\mathbb{Z}_q)/H_q = U(\mathbb{Z}_q)/p^2 \) whose cardinalities determine the number of Bunce-Deddens algebras in the subquotients. The topology on \( \text{Prim}(C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_\alpha \mathbb{N}^2) \) is then given by:

**Theorem 4.1.** The maps \((x, y) \mapsto I_{x,y}, xH_p \mapsto J_{xH_p}, yH_q \mapsto J_{yH_q}\) and \((w, z) \mapsto L_{w,z}\) combine to give a bijection of the disjoint union

\[
(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)) \sqcup \mathcal{U}(\mathbb{Z}_p)/q^2 \sqcup \mathcal{U}(\mathbb{Z}_q)/p^2 \sqcup \mathbb{T}^2
\]

onto \( \text{Prim}(C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_\alpha \mathbb{N}^2) \). Write \( \pi_p \) for the map \( U(\mathbb{Z}_p) \times U(\mathbb{Z}_q) \to U(\mathbb{Z}_p) \to U(\mathbb{Z}_p)/q^2 \). Then the hull-kernel closure of a nonempty subset \( F \) of (4.1) is

(a) the usual closure of \( F \) in \( \mathbb{T}^2 \) if \( F \subset \mathbb{T}^2 \);

(b) \( F \cup \mathbb{T}^2 \) if \( F \subset U(\mathbb{Z}_p)/q^2 \sqcup U(\mathbb{Z}_q)/p^2 \);

(c) the usual closure of \( F \) in \( U(\mathbb{Z}_p) \times U(\mathbb{Z}_q) \) together with \( \pi_p(F) \cup \pi_q(F) \cup \mathbb{T}^2 \) if \( F \subset U(\mathbb{Z}_p) \times U(\mathbb{Z}_q) \).

We prove this by finding irreducible representations of \( C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_\alpha \mathbb{N}^2 \) realising each of these primitive ideals, identifying their kernels as crossed products of invariant ideals in \( C(\mathbb{Z}_p \times \mathbb{Z}_q) \) using results from [13], and then reading off the topology from standard properties of the topology on \( \mathbb{Z}_p \times \mathbb{Z}_q \).

The ideals \( L_{w,z} \) are lifted from the quotient \( (C(\mathbb{Z}_p \times \mathbb{Z}_q) \times \mathbb{N}^2)/I_2 = C \rtimes \text{id} \mathbb{N}^2 \), and are the kernels of the characters \( \gamma_{w,z} : (m, n) \mapsto w^m z^n \); more precisely, \( L_{w,z} = \ker(\varepsilon_{0,0} \times \gamma_{w,z}) \), where \( \varepsilon_{0,0}(f) := f(0,0) \). Because \( \text{Prim}(C \rtimes \text{id} \mathbb{N}^2) \) is a closed subset of \( \text{Prim}(C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{N}^2) \), this also proves part (a) of Theorem 4.1.

The ideals \( J_{xH_p} \) are inverse images under the map \((\text{id} \otimes \varepsilon_0)^* \) of \( C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{N}^2 \) onto \( C(\mathbb{Z}_p) \rtimes \mathbb{N}^2 \) induced by \( \text{id} \otimes \varepsilon_0 : C(\mathbb{Z}_p \times \mathbb{Z}_q) \to C(\mathbb{Z}_p) \), and are determined by the intersections of their images with the ideal \( C_0(\mathbb{Z}_p \setminus \{0\}) \rtimes \mathbb{N}^2 \). Recall that the homeomorphism \( h_p : (k, x) \mapsto p^k x \) induces an isomorphism

\[
(4.2) \quad h_p^* : C_0(\mathbb{Z}_p \setminus \{0\}) \rtimes \mathbb{N}^2 \cong (\mathcal{U}(\mathbb{Z}_p), C_0(\mathbb{N}) \rtimes_\tau \mathbb{N}) \rtimes_\sigma \text{id Z}.
\]

Because \( M \times T \) is an isomorphism of \( C_0(\mathbb{N}) \rtimes_\tau \mathbb{N} \) onto \( \mathcal{K}(\ell^2(\mathbb{N})) \) and \( Z \) acts freely on \( U(\mathbb{Z}_p) = \text{Prim} C(U(\mathbb{Z}_p), \mathcal{K}) \), the primitive ideals of the right-hand side of (4.2) are induced from the ideals \( \ker(M \times T) \circ \varepsilon_x \). In particular, we have

\[
J_{xH_p} \cap (C_0(\mathbb{Z}_p \setminus \{0\}) \rtimes \mathbb{N}^2) = \ker((\text{Ind}_{\{0\}}^{\mathbb{Z}} (M \times T) \circ \varepsilon_x) \circ h_p^* \circ (\text{id} \otimes \varepsilon_0)^*).
\]
We can now use the standard form \( \overline{\pi} \times \lambda \) of the induced representation to see that the ideal \( I_{X T} \) is the kernel of the representation \( \rho_x \times (T \otimes \lambda) \) of \( C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \alpha \mathbb{N}^2 \) on \( \ell^2(\mathbb{N} \times \mathbb{N}) \), where \( (\rho_x(f) \xi)(k, \ell) := f(p \ell q^k x, 0) \xi(k, \ell) \). Similarly, with \( \sigma_y : C(\mathbb{Z}_p \times \mathbb{Z}_q) \to B(\ell^2(\mathbb{Z} \times \mathbb{N})) \) defined by

\[
(\sigma_y(f) \xi)(k, \ell) = f(0, p \ell q^k y) \xi(k, \ell),
\]
we have \( \ker(\sigma_y \times (\lambda \otimes T)) = K_{Y T} \).

The ideals \( I_{X, Y} \) are determined by their intersection with \( I_1 \), and \( I_{X, Y} \cap I_1 \) is pulled back under the isomorphism \( \epsilon_{x, y} : C(\mathbb{U}(\mathbb{Z}_p) \times \mathbb{U}(\mathbb{Z}_q), \mathcal{K}) \to \mathcal{K} \). This isomorphism is induced by the homeomorphism \( h : (\ell, k, x, y) \to p \ell q^k x, y \) of \( \mathbb{N}^2 \times \mathbb{U}(\mathbb{Z}_p) \times \mathbb{U}(\mathbb{Z}_q) \) onto \( (\mathbb{Z}_p \setminus \{0\}) \times (\mathbb{Z}_q \setminus \{0\}) \), and the Toeplitz representation \( M \times T \) of \( c_0(\mathbb{N}^2) \rtimes \mathbb{T} \mathbb{N}^2 \onto \mathcal{K}(\ell^2(\mathbb{N}^2)) \). The representation \( (\pi_{x, y}(f) \xi)(k, \ell) = f(p \ell q^k x, p \ell q^k y) \xi(k, \ell) \) satisfies \( \pi_{x, y}(C(\mathbb{Z}_p \times \mathbb{Z}_q, \mathbb{N}^2, \alpha)) \) with \( I_{x, y} = \ker(\pi_{x, y} \times T) \).

To identify the kernels of these representations, we shall use the following analogue of the standard characterisations of faithful representations.

**Lemma 4.2.** Let \( (\eta, T) \) be a covariant representation of a semigroup dynamical system \( (A, \mathbb{N}^k, \alpha) \) with extendible endomorphisms. Suppose that \( \ker \eta \) is an extendibly \( \alpha \)-invariant ideal, and and \( W \) is a unitary representation of \( \mathbb{T}^k \) such that \( (\eta \times T, W) \) is a covariant representation of the dual system \( (A \rtimes \mathbb{N}^k, \mathbb{T}^k, \hat{\alpha}) \). Then

\[
\ker(\eta \times T) = (\ker \eta) \rtimes \mathbb{N}^k
\]

\[
= \overline{\text{span}} \{ i_A(a) i_S(n) \mid m, n \in \mathbb{N}^k, a \in \ker \eta \}.
\]

**Proof.** We know from [13, Theorem 1.8] that \( (\ker \eta) \rtimes \mathbb{N}^k \) is naturally isomorphic to the ideal

\[
\overline{\text{span}} \{ i_A(a) i_S(n) \mid m, n \in \mathbb{N}^k, a \in \ker \eta \} \subset A \rtimes \mathbb{N}^k,
\]
and that the quotient map \( \pi : A \to A/(\ker \eta) \) induces a homomorphism \( \pi \times \text{id} \) of \( A \rtimes \mathbb{N}^k \) onto \( (A/\ker \eta) \rtimes \mathbb{N}^k \) with kernel \( (\ker \eta) \rtimes \mathbb{N}^k \). There is a faithful representation \( \zeta \) of \( A/\ker \eta \) such that \( \eta = \zeta \circ \pi \), and then \( (\zeta \times T, W) \) are covariant. It suffices to prove that \( \zeta \times T \) is faithful, for then \( \eta \times T = (\zeta \times T) \circ (\pi \times \text{id}) \), and

\[
\ker(\eta \times T) = \ker(\zeta \times T) \circ (\pi \times \text{id}) = \ker(\pi \times \text{id}) = (\ker \eta) \rtimes \mathbb{N}^k.
\]

To prove \( \zeta \times T \) faithful, we follow the standard procedure of [4, Lemma 2.2]. Write \( C = A/\ker \eta \), and let \( \theta : C \rtimes \mathbb{N}^k \to C \rtimes \mathbb{N}^k \) be the expectation obtained by averaging over the dual action \( \hat{\alpha} \) on \( C \rtimes \mathbb{N}^k \), which is faithful on positive elements.
The two-prime Hecke C*-algebra

by [10, Remark 3.6]. Because \( S = \mathbb{N}^k \) is abelian, \( C \cong \mathbb{N}^k \) is spanned by the elements \( i_S(m)^*i_C(c)i_S(n) \) [13, Lemma 1.3], and hence \( \theta(C \cong \mathbb{N}^k) \) is spanned by the elements \( i_S(m)^*i_C(c)i_S(m) \); because every finite set of elements in \( \mathbb{N}^k \) has an upper bound, we can imitate the proof of [2, Lemma 1.5] to see that \( \zeta \times T \) is faithful on \( \theta(C \cong \mathbb{N}^k) \). Now we can use the covariance of \( (\zeta \times T, W) \) to get an estimate

\[
\|(\zeta \times T)(\theta(f))\| = \left\| \int_{\mathbb{N}^k} W^*_z (\zeta \times T)(f)W_z \, dz \right\|
\leq \int_{\mathbb{N}^k} \|W^*_z \zeta \times T(f)W_z\| \, dz
= \|\zeta \times T(f)\|,
\]

and follow the argument of [4, Lemma 2.2] to see that \( \zeta \times T \) is faithful. \( \square \)

The ideal \( \ker \pi_{x,y} \) consists of the functions which vanish on the closure of the orbit \( p^nq^n(x,y) \); to check that \( \ker \pi_{x,y} \) is extendibly invariant, we need to know exactly what this closure is.

**Lemma 4.3.** Let \((x, y) \in \mathbb{Z}_p \times \mathbb{Z}_q\). Then \( q^n x \) has the same closure in \( \mathbb{Z}_p \) as \( q^2 x \), and the closure of \( p^n q^n(x, y) \) in \( \mathbb{Z}_p \times \mathbb{Z}_q \) is

\[
(4.3) \quad p^n q^n(x, y) \cup (p^n q^2 x \times \{0\}) \cup (\{0\} \times p^2 q^n y).
\]

**Proof.** Since \( q \in \mathcal{U}(\mathbb{Z}_p) \), multiplication by \( q \) is a homeomorphism of \( \mathcal{U}(\mathbb{Z}_p) \), and defines a free and minimal action of \( \mathbb{Z} \) on \( q^2 x \). The sequence \( \{q^k x \mid k \in \mathbb{N}\} \) has a convergent subsequence, \( q^{k_n} x \to x_0 \), say, and then \( q^2 x_0 = q^2 x \) by minimality. Thus every element of \( q^2 x \) can be approximated first by \( q^n x_0 \), and then by elements \( q^{n+k_n} x \) of \( q^n x \). Thus \( q^n x = q^2 x \). This argument also shows that every element of \( q^2 x \) is the limit of a sequence \( q^{m_n} x \) in which \( m_n \to \infty \).

Since \((0, 0) = \lim_n p^n q^n(x, y)\), it certainly belongs to the orbit closure. Suppose \( p^k q^l x \to s \) and \( s \neq 0 \). Write \( s = p^i s_0 \) for \( s_0 \in \mathcal{U}(\mathbb{Z}_p) \). Then \( p^l \mathcal{U}(\mathbb{Z}_p) \) is an open neighbourhood of \( s_0 \), so \( k_n = i \) for large \( n \), and \( q^l x \to -p^{-i}s \).

As observed above, we may as well suppose \( l_n \to \infty \); then \( q^l y \to 0 \), and \( p^k q^l(x, y) \to (s, 0) \). Thus \( p^n q^2 x \times \{0\} \) is contained in the orbit closure, and, by symmetry, so is \( \{0\} \times p^2 q^n y \).

For the other inclusion, suppose \((w, z) \in \mathbb{Z}_p \times \mathbb{Z}_q \) and \( p^k q^i(x, y) \to (w, z) \). It is obvious that \((w, z)\) belongs to (4.3) if one of \( w \) or \( z \) is 0, so suppose \( w \) and \( z \) are both nonzero. We can write \((w, z) = (p^i w_0, q^j z_0)\) for units \( w_0, z_0 \) and \( i, j \in \mathbb{N} \), and then \( p^l \mathcal{U}(\mathbb{Z}_p) \times q^j \mathcal{U}(\mathbb{Z}_q) \) is a neighbourhood of \((w, z)\). Thus \((k_n, l_n) = (i, j)\) for large \( n \), and \((w, z) = p^n q^l(x, y)\) belongs to \( p^n q^n(x, y)\), as required. \( \square \)

**Lemma 4.4.** Let \((x, y) \in \mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)\). Then
(a) \( J_{xH_p} = \text{span}\{i_S(i, j) \ast i_A(f) i_S(m, n) \mid f \equiv 0 \text{ on } p^N q^N x \times \{0\}\}; \\
(b) \ K_{yH_q} = \text{span}\{i_S(i, j) \ast i_A(f) i_S(m, n) \mid f \equiv 0 \text{ on } \{0\} \times p^2 q^N y\}; \text{ and} \\
(c) \ I_{x, y} = \text{span}\{i_S(i, j) \ast i_A(f) i_S(m, n) \mid f \equiv 0 \text{ on } p^N q^N (x, y)\}.

\textbf{Proof}: For part (a), we want to apply Lemma 4.2 with \( \eta = \rho_x \), and we therefore need to know that \( \ker \rho_x \) is extendibly invariant. We have \( \rho_x(f) = 0 \) iff \( f \equiv 0 \) on \( p^N q^N x \times \{0\} \), which is equivalent by Lemma 4.3 to \( f \equiv 0 \) on \( p^N q^2 x \times \{0\} \). Thus it is enough by [13, Theorem 4.3] to prove that \( p^N q^N x \times \{0\} \) and its complement are invariant under multiplication by \( p^k q^\ell \). This is trivially true for \( p^N q^N x \times \{0\} \). Suppose \((w, z) \notin p^N q^N x \times \{0\}. \text{ If } z \neq 0, \text{ then } p^k q^\ell (w, z) \) is certainly not in \( p^N q^N x \times \{0\}. \text{ So we consider the case } z = 0, \text{ and suppose } p^k q^N x \rightarrow p^k q^\ell w. \) Since \( w \) cannot be 0, we can write \( w = p^i w_0 \) for \( w_0 \in U(Z_p) \). Eventually \( p^k q^N x \in p^k+i U(Z_p) \), so \( k_n = k + i \) for large \( n, \) and \( q^\ell w = \lim p^k q^N x \rightarrow p^i q^\ell x \). Since \( q^2 x = q^2 x \), this implies that \( w \in p^i q^N x \), and hence that \((w, z) \in p^N q^N x \times \{0\}, \) which is a contradiction. So \( p^k q^\ell (w, z) \notin p^N q^N x \times \{0\} \) for all \( k, \ell \in \mathbb{N} \), and we have shown that \( \ker \rho_x \) is extendibly invariant.

Next we observe that \( W_{w, z} \xi(k, \ell) := w^k z^\ell \xi(k, \ell) \) defines a unitary representation \( W \) of \( \mathbb{T}^2 \) on \( L^2(\mathbb{N} \times \mathbb{Z}) \) such that \((\rho_x \times (T \otimes \lambda), W)\) is covariant for the dual action. Thus we can deduce from Lemma 4.2 that \( J_{xH_p} = \ker(\rho_x \times (T \otimes \lambda)) \) has the required form. This gives (a), and of course (b) is exactly the same.

For (c), we apply the same argument to

\( \ker \pi_{x, y} = \{ f \in C(\mathbb{Z}_p \times \mathbb{Z}_q) \mid f \equiv 0 \text{ on } p^N q^N (x, y)\}; \)

as above, the crux is to prove that if \( p^k q^\ell (w, z) \) is in the closure of \( p^N q^N (x, y) \), then so is \((w, z)\). Suppose \((w, z) \in \mathbb{Z}_p \times \mathbb{Z}_q \) and \( p^k q^N (x, y) \rightarrow p^k q^\ell (w, z). \) If \( w \) or \( z \) is 0, we are in the situation covered by the first paragraph. So suppose \( w \) and \( z \) are both nonzero: say \( w = p^i w_0 \) and \( z = q^j z_0 \) for units \( w_0, z_0 \). By Lemma 4.3, we must have \( p^k q^\ell (w, z) = p^m q^n (x, y) \) for some \( m, n \in \mathbb{N} \). Then \( p^k+i q^\ell w_0 = p^m q^n x \) and \( p^k q^\ell + j z_0 = p^m q^n y \). The first of these equations implies that \( k + i = m \), so \( k \leq m \), and the second that \( \ell \leq n \). Thus \((w, z) = p^m q^{n-\ell} (x, y) \) belongs to \( p^N q^N (x, y) \). This proves that \( \ker \pi_{x, y} \) is extendibly invariant. Part (c) follows from an application of Lemma 4.2 with \( W \) given by the same formula as before.

\textbf{Proof of Theorem 4.1}. We have already observed that (a) is easy. For (b), notice that for any \( x \in U(Z_p) \), the spanning elements \( i_S(i, j) \ast i_A(f) i_S(m, n) \) of \( J_{xH_p} \) go to \( f(0, 0) i_S(i, j) \ast i_S(m, n) \) in the quotient \( C \rtimes_{\text{id}} \mathbb{N}^2 \), and hence \( J_{xH_p} \subset L_{w,z} \) for all \((w, z) \in \mathbb{T}^2 \).

For (c), we observe that \( F \cap (U(Z_p) \times U(Z_q)) \) is the usual closure because \((x, y) \rightarrow I_{x, y} \) is a homeomorphism of \( U(Z_p) \times U(Z_q) \) onto the open set \( \text{Prim} I_1 \).
That the closure contains the other points follows from Lemma 4.4: \( f \in \ker \pi_{X,Y} \) implies \( f \in \ker \rho_X \), so all the generators for \( I_{X,Y} \) described in Lemma 4.4 belong to \( J_{X,H_P} \), and \( (x, y) \in F \) implies

\[
J_{X,H_P} \in \tilde{F} = \left\{ P \in \text{Prim}(C(\mathbb{Z}_p \times \mathbb{Z}_q) \rtimes \mathbb{N}^2) \mid \bigcap_{Q \in F} Q \subset P \right\}.
\]

To see that \( J_{X,H_P} \) does not belong to \( \tilde{F} \) when \( x_0 H_P \notin \pi_p(F) \), let \( F_1 \) be the union of the cosets in \( \pi_p(F) \). Choose \( g \in C(\mathbb{U}(\mathbb{Z}_p)) \) such that \( g(x_0) = 1 \) and \( g = 0 \) on \( F_1 \), extend \( g \) to a continuous function on \( \mathbb{Z}_p \) by taking it to be zero outside \( \mathbb{U}(\mathbb{Z}_p) \), and define \( f(x, y) = g(x) \). Now we can see from Lemma 4.3 that \( f \) vanishes on the closure of \( \mathbb{p}q^n(x, y) \) for every \( (x, y) \in F \), and hence \( i_A(f) \) belongs to \( \bigcap \{ I_{x,y} \mid x, y \in F \} \) but not to \( J_{X,H_P} \). Thus \( \tilde{F} \cap \mathbb{U}(\mathbb{Z}_p)/H_P \) is precisely \( \pi_p(F) \), and part (c) follows from (b).

**Remark 4.5.** It is interesting to compare our description of \( \text{Prim} C^*(G_{p,q}) \rtimes \mathbb{N}^2 \) with that obtained for the Bost-Connes algebra \( C_Q \) in [12]. In \( \text{Prim} C_Q \), the finite sets coming from \( \text{Prim} I_2/I_1 \) do not appear; loosely speaking, we believe this happens because \( C_Q \) contains all the primes, and some of these will act minimally on any given \( \mathbb{U}(\mathbb{Z}_p) \) (see Remark 3.8). So the numbers \( o_p(q) \) cannot be recovered from \( \text{Prim} C_Q \). Of course this information is still buried somewhere in \( C_Q \): it follows from [14, Theorem 2.1] that the inclusion of \( G_{p,q} \) in \( \mathbb{Q}/\mathbb{Z} \) induces an isomorphism of \( C^*(G_{p,q}) \rtimes \mathbb{N}^2 \) into \( C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^* = C_Q \).

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**References**


NADIA S. LARSEN:
Department of Mathematics
Institute for Mathematical Sciences
University of Copenhagen
Universitetsparken 5, 2100 Copenhagen
Denmark

E-MAIL: nadia@math.ku.dk

IAN F. PUTNAM:
Department of Mathematics and Statistics
University of Victoria
British Columbia V8W 3P4
Canada

E-MAIL: putnam@math.uvic.ca

IAIN RAEBURN:
Department of Mathematics
University of Newcastle, NSW 2308
Australia

E-MAIL: iain@math.newcastle.edu.au

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