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Abstract

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IDEALS OF STEINBERG ALGEBRAS OF STRONGLY EFFECTIVE GROUPOIDS, WITH APPLICATIONS TO LEAVITT PATH ALGEBRAS

LISA ORLOFF CLARK, CAIN EDIE-MICHELL, ASTRID AN HUEF, AND AIDAN SIMS

Abstract. We consider the ideal structure of Steinberg algebras over a commutative ring with identity. We focus on Hausdorff groupoids that are strongly effective in the sense that their reductions to closed subspaces of their unit spaces are all effective. For such a groupoid, we completely describe the ideal lattice of the associated Steinberg algebra over any commutative ring with identity. Our results are new even for the special case of Leavitt path algebras; so we describe explicitly what they say in this context, and give two concrete examples.

1. Introduction

Leavitt path algebras over a field have been studied intensively since their independent introduction, around 2005, by Abrams–Aranda-Pino in [3] and Ara–Moreno–Pardo in [5]. One of the earliest questions asked about these algebras was what the ideals look like. The lattice of ideals is now completely understood, see, for example, [2, Theorem 2.8.10] or [1, Theorem 11]. Work on the ideal structure of the Leavitt path algebra and its irreducible representations continues. See for example the recent papers on the generators of ideals [26], prime and primitive ideals [8, 25, 17], two-sided chain conditions [4], and on irreducible representations [11, 6, 18].

In 2011, Tomforde went on to consider Leavitt path algebras over commutative rings R with identity in [33], and again considered the ideal structure. Things are more complicated in this setting because the ideal structure of the ring R has an effect on the ideal structure of the Leavitt path algebra. Tomforde sidestepped this issue by considering only the "basic" ideals which are, roughly speaking, the ideals that contain a scalar multiple of a generator if and only if they contain the generator itself, and are therefore insensitive to the ideal structure of R. The structure of the basic ideals in Leavitt path algebra has recently been reconsidered by Larki in [21] for more general graphs than were allowed in [33]. Larki also studies the prime and primitive ideals, and this involves non-basic ideals.

In this paper, we investigate the basic and non-basic ideal structure of a large class of Steinberg algebras. The Steinberg algebras, introduced independently in [31] and in [13], are associated to ample groupoids. They include the Kumjian–Pask algebras of

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higher-rank graphs introduced in [7], which in turn include the Leavitt path algebras. The advantage of working with the more general Steinberg algebras is that this brings into play, in the algebraic setting, powerful techniques from Renault's theory of groupoid C^*-algebras. Indeed, Renault's theory previously played a fundamental role in the development of the theory of graph C^*-algebras and their analogues.

Groupoid models in C^*-algebra theory are particularly well-suited to answering questions about ideal structure [28]. We focus on groupoids G which are strongly effective in the sense that in every reduction of G to a closed invariant subspace of its unit space $G^{(0)}$, the interior of the isotropy consists only of units. This reduces to Condition (K) for graphs and to "strong aperiodicity" for higher-rank graphs. (This is folklore, but we provide a proof in Corollary 6.5.) Our results provide a complete description of the lattice of ideals in the Steinberg algebra of such a groupoid. Since these results are new even for Leavitt path algebras, and hence also for Kumjian–Pask algebras, we give an explicit account of what our main theorem says in these special cases.

We start in Section 3 by analysing the basic ideals of the Steinberg algebra of a strongly effective groupoid . We find that the ideals are indexed by the open invariant subsets of the unit space as expected. When R is a field, every ideal is a basic ideal. Thus we can draw some conclusions about the ideals in Steinberg algebras over fields, and the relationship of these to the ideals of the corresponding groupoid C^*-algebra, at least when the groupoid G is amenable.

In Section 4 we build on our analysis of basic ideals to describe all the ideals in the Steinberg algebra. The extra ideals arising from the ideals of the ring R are encoded by functions π, satisfying a consistency condition relating nesting of ideals in R to nesting of subsets of the unit space, from the collection of open invariant subsets of $G^{(0)}$ to the set $\mathcal{L}(R)$ of ideals of R.

Containment of ideals in the Steinberg algebra is encoded by a very natural partial order on the functions π described in the preceding paragraph. So in principle the lattice structure on the set of ideals is explicitly described in terms of the functions π. However, it is difficult to describe the join operation on functions π that corresponds to addition of ideals of the Steinberg algebra. In Section 5, we introduce an alternative characterisation of the ideals in the Steinberg algebra in terms of functions $\rho : G^{(0)} \to \mathcal{L}(R)$ that are continuous with respect to a suitable topology on $\mathcal{L}(R)$. This allows us to describe the join and meet operations quite naturally.

Finally, in Section 6, we translate our results into the language of Leavitt path algebras and Kumjian–Pask algebras. Here, the ideals are parameterised by functions from the collection of saturated hereditary subsets of the vertex set of the graph into the set of ideals of R, again satisfying a suitable consistency condition; or alternatively by continuous functions from the infinite-path space of the graph to the ideal space $\mathcal{L}(R)$. We detail the content of our theorems for two concrete examples of graphs, each emphasising the advantages of one of these two parameterisations.

2. Preliminaries

We use the groupoid conventions of [12]. Let G be a groupoid. A subset U of the unit space $G^{(0)}$ of G is *invariant* if $s(\gamma) \in U$ implies $r(\gamma) \in U$; equivalently,

$$r(s^{-1}(U)) = U = s(r^{-1}(U)).$$

Given $V \subseteq G^{(0)}$, we write $[V]$ for the smallest invariant subset of $G^{(0)}$ containing V. Thus

$$[V] = r(s^{-1}(V)) = s(r^{-1}(V)).$$

We use the standard notation from [27, page 6] where $G_u = \{\gamma \in G : s(\gamma) = u\}$, $G^u = \{\gamma \in G : r(\gamma) = u\}$ and $G_u^u = G_u \cap G^u$ for each unit $u \in G^{(0)}$. The *isotropy groupoid* of G is

$$\mathrm{Iso}(G) := \{g \in G : s(\gamma) = r(\gamma)\} = \bigcup_{u \in G^{(0)}} G_u^u.$$

Let U be an invariant subset of $G^{(0)}$. We write $G_U := s^{-1}(U)$, and then G_U coincides with the *restriction*

$$G|_U := \{\gamma \in G : s(\gamma), r(\gamma) \in U\}.$$

of G to U. This G_U is a groupoid with unit space U.

For subsets $W, V \subseteq G$, we define $WV := \{\gamma\eta : \gamma \in W, \eta \in V, s(\gamma) = r(\eta)\}$.

Now let G be a topological groupoid. A subset B of G is a *bisection* if the source and range maps restrict to homeomorphisms on B; for an open set to be a bisection we require the source and range maps to restrict to homeomorphisms onto open subsets of $G^{(0)}$. Then G is called *ample* if G has a basis of compact open bisections. In this paper, we only consider ample Hausdorff groupoids.

An ample Hausdorff groupoid G is *effective* if the interior of $\mathrm{Iso}(G)$ is just $G^{(0)}$. It follows that when G is effective, if B is a nonempty compact open bisection such that $B \subseteq G \setminus G^{(0)}$, then $B \setminus \mathrm{Iso}(G) \neq \emptyset$.

When G is second countable, G is effective if and only if it is topologically principal in the sense that $\{u \in G^{(0)} : G_u^u = \{u\}\}$ is dense in $G^{(0)}$ (see [29, Proposition 3.6]). Our results apply to groupoids G that are not second countable, so for us the two conditions are, in general, different.

Definition 2.1. A groupoid G is *strongly effective* if for every nonempty closed invariant subset V of $G^{(0)}$, the groupoid G_V is effective.

If G is strongly effective, then it is effective because $G^{(0)}$ is a closed invariant set. If G is second countable, then so is G_V for every invariant subset V of $G^{(0)}$, and so G is strongly effective if and only if it is essentially principal in the sense of [27, Chapter 2, Definition 4.3].

Let G be an ample Hausdorff groupoid and R a commutative ring with identity. We write $A_R(G)$ for the *Steinberg algebra* of all locally constant, compactly supported functions $f : G \to R$, equipped with the convolution product. As a set, $A_R(G)$ is the R-linear span

$$\mathrm{span}_R\{1_B : B \text{ is a compact open bisection}\}.$$

For $f \in A_R(G)$, the set $\{\gamma \in G : f(\gamma) \neq 0\}$ is a finite union of compact open sets, and so is itself compact and open. Since compact subsets of a Hausdorff space are closed, we have

$$\mathrm{supp}(f) := \overline{\{\gamma \in G : f(\gamma) \neq 0\}} = \{\gamma \in G : f(\gamma) \neq 0\}.$$

Under the convolution product on $C_c(G)$, for $f, g \in A_R(G)$ we have $\mathrm{supp}(f * g) \subseteq \mathrm{supp}(f)\,\mathrm{supp}(g)$.

3. Basic ideal structure

Throughout, G is an ample Hausdorff groupoid and R is a commutative ring with identity.

When the coefficient ring R is not a field, the ideal structure of $A_R(G)$ depends on the ideal structure of R. For example, if $G = \{e\}$ is the trivial group, then the Steinberg algebra $A_R(G)$ is isomorphic to R as an R-algebra, and then the ideals of $A_R(G)$ are precisely the ideals of R. An ideal I of $A_R(G)$ is a *basic ideal* if

$$K \text{ a compact open subset of } G^{(0)}, 0 \neq r \in R \text{ and } r1_K \in I \quad \implies \quad 1_K \in I.$$

When $G = \{e\}$, the only nonzero basic ideal is R itself. In general, the basic ideals are the ones that reflect the structure of G alone, and do not reflect the structure of R; we expect the basic-ideal structure to be independent of R. Basic ideals of $A_R(G)$ were introduced by the first two authors in [12], and they generalise the basic ideals of a Leavitt path algebra studied by Tomforde in [33].

The first step in studying the ideal structure of $A_R(G)$ is to study the basic ideals. By [12, Theorem 4.1], if G is an ample Hausdorff groupoid, then $A_R(G)$ has no proper basic ideals if and only if G is effective and minimal. In this paper we consider groupoids that are strongly effective (hence effective) but not minimal. The main result of this section is the following.

Theorem 3.1. *Let G be an ample Hausdorff groupoid, and let R be a commutative ring with identity. Then G is strongly effective if and only if*

$$U \mapsto I_U := \{f \in A_R(G) : \operatorname{supp} f \subseteq G_U\}$$

is a lattice isomorphism from the open invariant subsets of $G^{(0)}$ onto the basic ideals of $A_R(G)$.

Before proving Theorem 3.1, we need to establish some helper results.

Lemma 3.2. *Let G be an ample Hausdorff groupoid, let R be a commutative ring with identity and let $U \subseteq G^{(0)}$ be an open invariant subset. Then I_U is a basic ideal in $A_R(G)$.*

Proof. The set I_U is closed under addition and scalar multiplication. To see that I_U is an ideal, fix $f \in I_U$ and $g \in A_R(G)$. Let $\alpha \notin G_U$ and $\beta \in G^{r(\alpha)}$. Then $s(\beta^{-1}\alpha) = s(\alpha) \notin U$, and hence $s(\beta) \notin U$ because U is invariant. Hence $f(\beta) = 0$. Thus

$$(f * g)(\alpha) = \sum_{\beta \in G^{r(\alpha)}} f(\beta) g(\beta^{-1}\alpha) = 0.$$

So $f * g \in I$. A similar argument gives $g * f \in I$. That I_U is basic follows immediately from its definition. $\square$

Proposition 3.3. *Let G be an ample Hausdorff groupoid and let R be a commutative ring with identity. Then $U \mapsto I_U$ is an injective lattice morphism from the open invariant subsets of $G^{(0)}$ to the basic ideals of $A_R(G)$.*

Proof. We first prove that for open invariant subsets U, V of $G^{(0)}$, we have $I_U \subseteq I_V$ if and only if $U \subseteq V$. Suppose that $I_U \subseteq I_V$, and fix $u \in U$. Choose a compact open neighbourhood K of u such that $K \subseteq U$. Then $1_K \in I_U \subseteq I_V$, giving $u \in K \subseteq V$. Hence $U \subseteq V$. Conversely, if $U \subseteq V$, then $G_U \subseteq G_V$, and hence $I_U \subseteq I_V$.

It follows immediately that $I_U = I_V$ implies $U = V$, so $U \mapsto I_U$ is injective.

We will show that

$$I_{U\cap V} = I_U \cap I_V \quad \text{and} \quad I_{U\cup V} = I_U + I_V$$

Since the set of open invariant subsets of $G^{(0)}$ (with set inclusion, intersection and union) forms a lattice, it will then follow that $\{I_U : U \text{ is an open invariant subset of } G^{(0)}\}$ is a lattice (with set inclusion, intersection and $+$), and that $U \mapsto I_U$ is a lattice morphism.

Since $G_U \cap G_V = s^{-1}(U) \cap s^{-1}(V) = s^{-1}(U \cap V) = G_{U\cap V}$, we have

$$I_U \cap I_V = \{f : \operatorname{supp}(f) \in G_U \cap G_V\} = \{f : \operatorname{supp}(f) \in G_{U\cap V}\} = I_{U\cap V}.$$

If $f \in I_U + I_V$, say $f = f_U + f_V$, then

$$\operatorname{supp}(f) \subseteq \operatorname{supp}(f_U) \cup \operatorname{supp}(f_V) \subseteq G_U \cup G_V = s^{-1}(U) \cup s^{-1}(V) = s^{-1}(U \cup V) = G_{U\cup V}.$$

This gives $I_U + I_V \subseteq I_{U\cup V}$.

For the reverse containment, suppose that $f \in I_{U\cup V}$. So $\operatorname{supp}(f) \subseteq s^{-1}(U \cup V)$. The set $K_U := \operatorname{supp}(f) \setminus s^{-1}(V) \subseteq s^{-1}(U)$ is compact because it is a closed subset of $\operatorname{supp}(f)$, and similarly $K_V := \operatorname{supp}(f) \setminus s^{-1}(U)$ is a compact subset of $s^{-1}(V)$. Let $u \in K_U$. Since $\operatorname{supp}(f) \cap s^{-1}(U)$ is open, and since G is ample, we can find a compact open neighbourhood N_u of $u \in N_u \subseteq \operatorname{supp}(f) \cap s^{-1}(U)$. By taking the union of a finite subcover of the cover $\{N_u : u \in K_U\}$ of K_U, we obtain a compact open subset K'_U of $\operatorname{supp}(f)$ such that $K_U \subseteq K'_U \subseteq s^{-1}(U)$. Let $K'_V := \operatorname{supp}(f) \setminus K'_U$. Then $\operatorname{supp}(f) = K'_U \sqcup K'_V$ with $K'_U \subseteq s^{-1}(U)$ and $K'_V \subseteq s^{-1}(V)$. Since K'_U and K'_V are compact and open, we obtain locally constant functions f_U and f_V by setting $f_U(\gamma) := 1_{K'_U}(\gamma) f(\gamma)$ and $f_V(\gamma) := 1_{K'_V}(\gamma) f(\gamma)$. By construction, $f_U \in I_U$ and $f_V \in I_V$, and so $f = f_U + f_V \in I_U + I_V$. Thus $I_U + I_V = I_{U\cup V}$. As discussed above, this proves the proposition. □

Lemma 3.4. *Let G be an ample Hausdorff groupoid and let R be a commutative ring with identity. Suppose that G is not effective. Then there is a nonzero basic ideal I of $A_R(G)$ such that $I \cap A_R(G^{(0)}) = \{0\}$.*

Proof. Let $\mathbb{F}_R(G^{(0)})$ denote the free R-module generated by a copy of $G^{(0)}$; to reduce confusion, we shall write δ_u for the spanning element of $\mathbb{F}_R(G^{(0)})$ corresponding to $u \in G^{(0)}$. Let $\operatorname{End}(\mathbb{F}_R(G^{(0)}))$ denote the R-algebra of endomorphisms of $\mathbb{F}_R(G^{(0)})$. By applying [12, Proposition 4.2(2)] to the G-invariant set $G^{(0)}$ there is a homomorphism $\pi : A_R(G) \to \operatorname{End}(\mathbb{F}_R(G^{(0)}))$ such that

$$\pi(f)\delta_u = \sum_{\gamma \in G_u} f(\gamma)\delta_{r(\gamma)}.$$

Since G is not effective, $\operatorname{Int}(\operatorname{Iso}(G)) \setminus G^{(0)}$ is nonempty. Since $\operatorname{Int}(\operatorname{Iso}(G)) \setminus G^{(0)}$ is open, there exists a compact open bisection $B \subseteq \operatorname{Int}(\operatorname{Iso}(G)) \setminus G^{(0)}$. If $u \in s(B)$ then $\pi(1_B)\delta_u = \delta_u = \pi(1_{s(B)})\delta_u$, and both are 0 otherwise. Now $0 \neq 1_B - 1_{s(B)} \in \ker \pi$. Thus $\ker \pi$ is a nonzero ideal, and it is basic by [12, Lemma 4.5].

We will show that $\ker \pi \cap A_R(G^{(0)}) = \{0\}$, and this proves the lemma. Let $f \in A_R(G^{(0)}) \setminus \{0\}$. Fix $u \in G^{(0)}$ such that $f(u) \neq 0$. Then $0 \neq f(u)\delta_u = \pi(f)\delta_u$, and so $f \notin \ker \pi$. Thus $\ker \pi \cap A_R(G^{(0)}) = \{0\}$. □

If G is an effective ample Hausdorff groupoid, then every nonzero ideal I of $A_R(G)$ has nonzero intersection with $A_R(G^{(0)})$ by [32, Proposition 3.3]. Combining this with Lemma 3.4 gives the following corollary.

Corollary 3.5. *Let G be an ample Hausdorff groupoid and let R be a commutative ring with identity. Then G is effective if and only if every nonzero ideal I of $A_R(G)$ has nonzero intersection with $A_R(G^{(0)})$.*

Suppose that U is an open invariant subset of $G^{(0)}$ and let $D := G^{(0)} \setminus U$. Since U is open, there is a function $i_U : A_R(G_U) \to A_R(G)$ such that

$$i_U(f)(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in U \\ 0 & \text{otherwise.} \end{cases}$$

Likewise, since D, and hence G_D, is closed, restriction of functions gives a function $q_U : A_R(G) \to A_R(G_D)$.

Lemma 3.6. *Let G be an ample Hausdorff groupoid and let R be a commutative ring with identity. Let U be an open invariant subset of $G^{(0)}$, and $D := G^{(0)} \setminus U$. The functions $i_U : A_R(G_U) \to A_R(G)$ and $q_U : A_R(G) \to A_R(G_D)$ are *-homomorphisms, and the sequence*

$$0 \longrightarrow A_R(G_U) \xrightarrow{i_U} A_R(G) \xrightarrow{q_U} A_R(G_D) \longrightarrow 0$$

is exact. Further, $I_U = i_U(A_R(G_U))$, and

$$I_U = \operatorname{span}_R\{1_B : B \subseteq G \textit{ is a compact open bisection with } s(B) \subseteq U\}.$$

Proof. Since U is invariant, $i_U : A_R(G_U) \to A_R(G)$ is a homomorphism, and since D is invariant, q_U is also a homomorphism. It is clear that i_U is injective. To see that q_U is surjective, fix a compact open subset K of G_D. Since K is also compact in G, and G is ample, we can find a finite cover $\bigcup_{L \in F} L$ of K by mutually disjoint compact open subsets of G. Then $1_K = q_U(\sum_{B \in F} 1_L)$. Since $A_R(G_V)$ is spanned by the 1_K it follows that q_U is surjective. By definition of i_U and q_U it is clear that $\operatorname{im} i_U \subseteq \ker q_U$. For the reverse containment, take $f \in \ker q_U$. Write $f = \sum_{B \in F} r_B 1_B$ where F is a collection of mutually disjoint bisections of G and the r_B are all nonzero. Since $q_U(f) = 0$, each $B \in F$ is contained in G_U, and so is a compact open subset of G_U. So we can define $f_0 \in A_R(G_U)$ by $f_0 = \sum_{B \in F} r_B 1_B$, and we have $i_U(f_0) = f$ by construction. Finally, we have

$$\begin{aligned} I_U = i_U(A_R(G_U)) &= \operatorname{span}_R\{i_U(1_B) : B \text{ is a compact open bisection of } G_U\} \\ &= \operatorname{span}_R\{1_B : B \subseteq G \text{ is a compact open bisection with } s(B) \subseteq U\}. \end{aligned}$$ □

Proof of Theorem 3.1. By Proposition 3.3, $U \mapsto I_U$ is an injective lattice morphism. So it remains to prove that G is strongly effective if and only if $U \mapsto I_U$ is surjective.

Suppose that G is not strongly effective. There exists a nonempty closed invariant subset V of $G^{(0)}$ such that G_V is not effective. By Lemma 3.4, there is a nonzero basic ideal I of $A_R(G_V)$ which has zero intersection with $A_R(G_V^{(0)})$. Let

$$J := \{f \in A_R(G) : f|_{G_V} \in I\}.$$

If $V = G^{(0)}$, then $J = I$. If $V \neq G^{(0)}$, then $I_{G^{(0)} \setminus V}$ is a nonzero ideal of $A_R(G)$ contained in J. In either case, J is a nonzero ideal of $A_R(G)$.

To see that J is a basic ideal, suppose that $K \subseteq G^{(0)}$ is a compact open subset of $G^{(0)}$ and $0 \neq r \in R$ with $r1_K \in J$. Then $r1_K|_{G_V} = r1_{K \cap V} \in I$. Since I is basic, $1_K|_{G_V} = 1_{K \cap V} \in I$, and hence $1_K \in J$. Thus J is basic.

To see that J is not of the form I_U, fix a nonempty open invariant $U \subseteq G^{(0)}$. First suppose that $U \cap V = \emptyset$. Fix a nonzero element $g \in I$. Lemma 3.6 shows that q_V :

$A_R(G) \to A_R(G_V)$ is surjective, so there exists $f \in A_R(G)$ such that $f|_{G_V} = g$, and so $f \in J$. Since $U \cap V = \emptyset$, we have $h|_{G_V} = 0$ for all $h \in I_U$, and we conclude that $f \in J \setminus I_U$. On the other hand, if $U \cap V \neq \emptyset$, then any nonempty compact open subset $K \subseteq U \cap V$ satisfies $1_K \in I_U \cap A_R(G_V^{(0)})$. Since $J \cap A_R(G_V^{(0)}) = \{0\}$, this implies $J \neq I_U$. Thus $U \mapsto I_U$ is not surjective.

Conversely, suppose that G is strongly effective. We just have to show that $U \mapsto I_U$ is surjective. Let I be a nonzero basic ideal in $A_R(G)$. Define

$$D := \bigcap_{f \in I, \operatorname{supp} f \subseteq G^{(0)}} f^{-1}(0) \quad \text{and} \quad U := G^{(0)} \setminus D = \bigcup_{f \in I, \operatorname{supp} f \subseteq G^{(0)}} f^{-1}(R \setminus \{0\}).$$

We will show that U is an open invariant subset of $G^{(0)}$ and that $I = I_U$.

To see that U is invariant, let $u \in U$ and choose γ such that $s(\gamma) = u$. We must show that $r(\gamma) \in U$. Fix $f \in I$ such that $\operatorname{supp} f \subseteq G^{(0)}$ and $f(s(\gamma)) \neq 0$. Let B be a compact open bisection containing γ. A calculation shows that

$$\begin{aligned} 1_B * f * 1_{B^{-1}}(\zeta) &= \sum_{\{\eta \in B : r(\eta) = r(\zeta)\}} f(s(\gamma)) 1_{B^{-1}}(\eta^{-1}\zeta) \\ &= \begin{cases} f(s(\eta)) & \text{if there exists unique } \eta \in B \text{ such that } \zeta = r(\eta) \\ 0 & \text{else.} \end{cases} \end{aligned}$$

In particular, $1_B * f * 1_{B^{-1}}(r(\gamma)) = f(s(\gamma)) \neq 0$ and $\operatorname{supp}(1_B * f * 1_{B^{-1}}) \subseteq G^{(0)}$. Thus $r(\gamma) \notin D$, and hence $r(\gamma) \in U$. Thus U is invariant, and it is open because it is a union of open sets $f^{-1}(R \setminus \{0\})$.

Now we will show that $I = I_U$. For the $\subseteq$ direction, recall from Lemma 3.6 that $I_U = \ker q_U$, and so q_U induces an isomorphism $\tilde{q}_U : A_R(G)/I_U \to A_R(G_D)$. By definition of I_U, we have $I \cap A_R(G^{(0)}) = \{f \in A_R(G^{(0)}) : \operatorname{supp}(f) \subseteq U\} = I_U \cap A_R(G^{(0)})$. Thus

$$\begin{aligned} \tilde{q}_U(I + I_U) \cap A_R((G_D)^{(0)}) &= \tilde{q}_U\big((I \cap A_R(G^{(0)})) + I_U\big) \\ &= q_U(I \cap A_R(G^{(0)})) \\ &= q_U(I_U \cap A_R(G^{(0)})) = \{0\}. \end{aligned}$$

Since G is strongly effective, G_D is effective, and [32, Proposition 3.3] gives $\tilde{q}_U(I + I_U) = \{0\}$. Thus $I \subseteq I_U$.

For the $\supseteq$ direction, we first claim that if B is a compact open bisection and $s(B) \subseteq U$, then $1_{s(B)} \in I$. To see this, observe that $s(B)$ is a compact open subset of U. By definition of U, for each $u \in U$ there is an element $f_u \in I$ such that $\operatorname{supp}(f_u) \subseteq G^{(0)}$ and $f_u(u) \neq 0$. This f_u is locally constant, so $V_u := f^{-1}(f(u))$ is a compact open neighbourhood of u in $G^{(0)}$ and $f|_{V_u} = f(u)1_{V_u}$. Since I is an ideal, we deduce that $f(u)1_{V_u} = f * 1_{V_u}$ belongs to I. Since I is a basic ideal, we deduce that $1_{V_u} \in I$. Now the V_u cover $s(B)$, which is compact, so we can write $s(B)$ as a finite union $s(B) = V_{u_1} \cup \cdots \cup V_{u_n}$. Putting $W_i := V_{u_i} \setminus \bigcup_{j=1}^{i-1} V_{u_j}$ we obtain pairwise disjoint compact open sets that cover $s(B)$, and each $1_{W_i} = 1_{W_i} * 1_{V_{u_i}} \in I$ because I is an ideal. Thus $1_{s(B)} = \sum 1_{W_i} \in I$ as claimed. So the final statement of Lemma 3.6 implies that $I_U \subseteq I$. So $I = I_U$ and hence the map $U \mapsto I_U$ is surjective. □

In the situation where $R = \mathbb{F}$ is a field, all ideals are basic and the following is immediate.

Corollary 3.7. *Let G be an ample Hausdorff groupoid and let $\mathbb{F}$ be a field. Suppose that G is strongly effective. Then*

$$U \mapsto I_U := \{f \in A_F(G) : \operatorname{supp} f \subseteq G_U\}$$

is an isomorphism from the lattice of open invariant subsets of $G^{(0)}$ onto the lattice of ideals of $A_{\mathbb{F}}(G)$.

If in addition the groupoid G is second countable and amenable, then [10, Corollary 5.9] shows that there is a lattice isomorphism between the open invariant subsets of $G^{(0)}$ and the closed ideals of the C^*-algebra $C^*(G) = C^*_r(G)$. Combining this with Corollary 3.7 gives the following.

Corollary 3.8. *Let G be an ample Hausdorff groupoid. Suppose that G is second countable, amenable and strongly effective. If we regard $A_{\mathbb{C}}(G)$ as a *-subalgebra of $C^*(G)$, then the closure operation is a lattice isomorphism from the ideals of $A_{\mathbb{C}}(G)$ to the closed ideals of $C^*(G)$.*

4. Nonbasic ideal structure

Proposition 4.1. *Let G be an ample Hausdorff groupoid and let R be a commutative ring with identity. Suppose that G is strongly effective, and let I be an ideal in $A_R(G)$. Then*

$$I = \operatorname{span}\{r1_B : B \textit{ is a compact open bisection and } r1_{s(B)} \in I\}.$$

The proof of this proposition uses two technical lemmas. The first is about compact open bisections in the complement of the unit space of a strongly effective groupoid. The second shows that restriction of functions to the unit space in such groupoids respects ideal structure.

Lemma 4.2. *Let G be an ample Hausdorff groupoid which is strongly effective. Suppose that $U \subseteq G^{(0)}$ is a compact open set and $B \subseteq G\backslash G^{(0)}$ is a compact open bisection such that $s(B), r(B) \subseteq U$. Then there is a finite collection of compact open bisections $\{N_1, \dots, N_m\}$ such that*

(a) *for each i, $r(N_i) \subseteq U$;*
(b) *for each i, $N_i^{-1}BN_i = \emptyset$;*
(c) *$s(N_i) \cap s(N_j) = \emptyset$ for $i \neq j$; and*
(d) *$s(B) = \bigsqcup_i s(N_i)$.*

Proof. Since G is strongly effective it is effective, and hence $B \setminus \operatorname{Iso}(G) \neq \emptyset$. For each $\gamma \in B\backslash\operatorname{Iso}(G)$, we can apply [10, Claim 3.2] to obtain a compact open bisection $V_\gamma \subseteq s(B)$ such that $\gamma \in BV_\gamma$ and $V_\gamma BV_\gamma = \emptyset$. Let

$$C := G^{(0)} \setminus \bigcup_{\gamma\in B\setminus\operatorname{Iso}(G)} [V_\gamma].$$

Since each $[V_\gamma]$ is an open invariant set, C is closed and invariant. We have

$$B \setminus \operatorname{Iso}(G) \subseteq \bigcup_{\gamma\in B\setminus\operatorname{Iso}(G)} BV_\gamma \subseteq G \setminus G_C,$$

and hence $B \cap G_C \subseteq \operatorname{Iso}(G)$. We claim that $B \cap G_C = \emptyset$. Suppose, aiming for a contradiction, that $B \cap G_C \neq \emptyset$. Then $B \cap G_C$ is a nonempty open compact bisection of G_C. Since G is strongly effective, G_C is effective. Thus $(B \cap G_C) \setminus \mathrm{Iso}(G_C) \neq \emptyset$, contradicting that $B \cap G_C \subseteq \mathrm{Iso}(G_C)$. Thus $B \cap G_C = \emptyset$. Now

$$B = B \setminus G_C = \bigcup_{\gamma \in B \setminus \mathrm{Iso}(G)} B[V_\gamma],$$

and in particular, $s(B) \subseteq \bigcup_{\gamma \in B}[V_\gamma]$.

Fix $x \in s(B)$. We will construct a compact open bisection M_x such that $x \in s(M_x)$, $r(M_x) \subseteq U$, and and $M_x^{-1}BM_x = \emptyset$. Choose γ with $x \in [V_\gamma]$ and choose $\eta \in V_\gamma Gx$. First suppose that $r(\eta) \neq x$. Since G is ample and Hausdorff, there exist compact open neighbourhoods U_x of x and $U_{r(\eta)}$ of $r(\eta)$ such that $U_x \cap U_{r(\eta)} = \emptyset$. We have $\eta \in s^{-1}(U_x) \cap r^{-1}(U_{r(\eta)})$. Thus by intersecting an open compact bisection containing η with the closed set $s^{-1}(U_x) \cap r^{-1}(U_{r(\eta)})$ we obtain a compact open bisection M_x containing η such that $r(M_x) \cap s(M_x) = \emptyset$. Since $r(\eta) \in V_\gamma$, we can replace M_x with $V_\gamma M_x$ to obtain $r(M_x) \subseteq V_\gamma \subseteq U$ (the replacement makes the range smaller, so we still have $r(M_x) \cap s(M_x) = \emptyset$). We then have

$$M_x^{-1}BM_x = M_x^{-1}V_\gamma BV_\gamma M_x = \emptyset.$$

Now suppose that $r(\eta) = x$. Then $x = r(\eta) \in V_\gamma$ and so $M_x := BV_\gamma$ satisfies $x \in s(M_x)$, $r(M_x) \subseteq r(B) \subseteq U$, and $r(M_x) \cap s(M_x) = \emptyset$. Furthermore,

$$M_x^{-1}BM_x = V_\gamma B^{-1}BBV_\gamma = V_\gamma s(B)BV_\gamma = V_\gamma BV_\gamma = \emptyset.$$

We have $s(B) \subseteq \bigcup_{x \in s(B)} s(M_x)$, and since $s(B)$ is compact, there exist $M_1, \dots, M_n \in \{M_x : x \in s(B)\}$ such that $s(B) \subset \bigcup_{i=1}^n s(M_i)$. For $1 \leq i \leq n$ let

$$N_i := M_i \setminus \big(\bigcup_{j<i} M_i s(M_j)\big).$$

These N_i satisfy (a)–(c) because the M_i do. Since each $s(N_i) = s(M_i) \setminus \bigcup_{j<i} s(M_j)$, the $s(N_i)$ are mutually disjoint and $\bigcup_i s(N_i) = \bigcup_i s(M_i) = s(B)$, giving (d). ☐

Lemma 4.3. *Let G be an ample Hausdorff groupoid which is strongly effective, and let R be a commutative ring with identity. Let I be an ideal in $A_R(G)$. If $f \in I$, then $f|_{G^{(0)}} \in I$.*

Proof. Fix $f \in I$. Since $G^{(0)}$ is closed and open, we can write $f = \sum_{V \in F_0} r_V 1_V + \sum_{C \in F_1} r_C 1_C$, where F_0 and F_1 are finite collections of mutually disjoint compact open bisections in $G^{(0)}$ and $G \setminus G^{(0)}$ respectively, and the r_U and r_V are all nonzero in R. It suffices to show that $r_V 1_V \in I$ for all $V \in F_0$. Fix $U \in F_0$. We have

$$1_U f 1_U = r_U 1_U + \sum_{B \in F_1} r_B 1_{UBU} \in I,$$

and $\{UBU : B \in F_1\}$ is a set of mutually disjoint compact open bisections contained in $G \setminus G^{(0)}$. We will show that $r_U 1_U \in I$; we will do this by induction.

Let $n \geq 0$. Our inductive hypothesis is: if $r1_U + \sum_{B \in F} r_B 1_B \in I$ where $U \subseteq G^{(0)}$ is compact open and F is a set of n mutually disjoint compact open bisections in $UGU \setminus G^{(0)}$, then $r1_U \in I$. When $n = 0$ the induction hypothesis holds trivially.

Now let $g \in I$ be of the form

$$g = r1_U + \sum_{B \in H} r_B 1_B$$

where $U \subseteq G^{(0)}$ is compact open and H is a collection of $n+1$ mutually disjoint compact open bisections in $UGU \setminus G^{(0)}$.

Fix $B_0 \in H$. We will first show that $a1_{s(B_0)} \in I$. Since G is strongly effective, we apply Lemma 4.2 to U and B_0 to obtain compact open bisections $\{N_1, \dots, N_m\}$ satisfying properties (a)–(d) of the lemma. For $0 \le i \le m$, we have $s(N_i) = N_i^{-1}UN_i$ and $N_i^{-1}B_0N_i = \emptyset$ by properties (a) and (b) of Lemma 4.2, respectively. Hence

$$h_i := 1_{N_i^{-1}}g1_{N_i} = r1_{s(N_i)} + \sum_{B\in H, B\neq B_0} r_B 1_{N_i^{-1}BN_i} \in I.$$

By property (d) of Lemma 4.2 we have $\sum_{i=1}^m r1_{s(N_i)} = r1_{s(B_0)}$, and thus

$$\sum_{i=1}^m h_i = r1_{s(B_0)} + \sum_{i=1}^m \sum_{B\in H, B\neq B_0} r_B 1_{N_i^{-1}BN_i} \in I. \tag{4.1}$$

For $i \neq j$, by property (c) of Lemma 4.2 we have

$$s(N_i^{-1}BN_i) \cap s(N_j^{-1}BN_j) \subseteq s(N_i) \cap s(N_j) = \emptyset,$$
$$r(N_i^{-1}BN_i) \cap r(N_j^{-1}BN_j) \subseteq s(N_i) \cap s(N_j) = \emptyset.$$

Hence $N_i^{-1}BN_i \cap N_j^{-1}BN_j = \emptyset$. For $B \in H$ with $B \neq B_0$ set

$$D_B := \bigsqcup_{i=1}^n N_i^{-1}BN_i.$$

Then each D_B is a compact open bisection contained in UGU because the source and range of the N_i are contained in U by properties (a) and (d) of Lemma 4.2. Hence (4.1) is

$$r1_{s(B_0)} + \sum_{B\in H\setminus\{B_0\}} r_B 1_{D_B} \in I. \tag{4.2}$$

To apply the inductive hypothesis, we must verify that each $D_B \cap G^{(0)} = \emptyset$. Fix $\gamma \in D_B$. Then $\gamma \in N_i^{-1}BN_i$ for some i. If $r(\gamma) \neq s(\gamma)$, then $\gamma \notin G^{(0)}$. So suppose that $r(\gamma) = s(\gamma)$. Since N_i is a bisection, there is a unique element $\alpha \in N_i$ such that $s(\alpha) = s(\gamma)$ and $\gamma = \alpha^{-1}\beta\alpha$ where β is the unique element of B with $s(\beta) = r(\alpha)$. Since $B \cap G^{(0)} = \emptyset$, $\beta \notin G^{(0)}$, and so $\gamma \notin G^{(0)}$. Thus $D_B \cap G^{(0)} = \emptyset$. Now the inductive hypothesis applies to (4.2), giving $r1_{s(B_0)} \in I$.

Since our choice of B_0 was arbitrary, we obtain $r1_{s(B)} \in I$ for every $B \in H$. We may also assume the collection $\{s(B)\}_{B\in H}$ is disjoint (by disjointification). So

$$V := \bigcup_{B\in H} s(B) \subseteq U$$

satisfies

$$r1_V = \sum_{B\in H} r1_{s(B)} \in I.$$

Since $s(B) \cap U \setminus V = \emptyset$ for $B \in H$ we have $1_{U\setminus V}g1_{U\setminus V} = r1_{U\setminus V} \in I$. Thus $r1_U = r1_V + r1_{U\setminus V} \in I$ as well. □

Proof of Proposition 4.1: It suffices to show that

$$I \subseteq \operatorname{span}\{r1_B : B \text{ is a compact open bisection and } r1_{s(B)} \in I\}.$$

Fix $f \in I$. Since G is strongly effective, Lemma 4.3 implies that $f|_{G^{(0)}} \in I$, and hence $f - f|_{G^{(0)}} \in I$. Since f is locally constant, $A := f(G^{(0)}) \subseteq R$ is finite. The sets $B_r := f^{-1}(r) \cap G^{(0)}$, $r \in A$ are mutually disjoint compact open subsets of $G^{(0)}$, and we have $f|_{G^{(0)}} = \sum_{r\in A} r1_{B_r}$. Each $r1_{B_r} = 1_{B_r} f|_{G^{(0)}} \in I$ and we deduce that $f|_{G^{(0)}} \in \operatorname{span}\{r1_B : r1_{s(B)} \in I\}$.

So it suffices to show that $g := f - f|_{G^{(0)}} \in \operatorname{span}\{r1_B : r1_{s(B)} \in I\}$. Express $g = \sum_{B\in F} r_B 1_B$ where F is a finite set of mutually disjoint compact open bisections in $G\backslash G^{(0)}$. Fix $C \in F$; we just have to establish that $r_C 1_{s(C)} \in I$. We have

$$1_{C^{-1}} g = r_C 1_{s(C)} + \sum_{B\in F\backslash\{C\}} r_B 1_{C^{-1}B} \in I.$$

We claim that for each $B \in F \setminus \{C\}$ we have $C^{-1}B \subseteq G \setminus G^{(0)}$. Fix $B \in F \setminus \{C\}$ and $\gamma \in C^{-1}B$. Then $\gamma = \alpha^{-1}\beta$ for some $\alpha \in C$ and $\beta \in B$. Since $C \cap B = \emptyset$, $\alpha \neq \beta$, and so $\gamma \notin G^{(0)}$. Thus $(1_{C^{-1}} g)|_{G^{(0)}} = r_C 1_{s(C)}$. Since G is strongly effective, Lemma 4.3 gives $r_C 1_{s(C)} \in I$ as needed. □

For any ring R, we write $\mathcal{L}(R) := \{I : I \text{ is a two-sided ideal of } R\}$ for the set of ideals of R. We now state our main theorem.

Theorem 4.4. *Let G be an ample Hausdorff groupoid which is strongly effective, and let R be a commutative ring with identity. Let $\mathcal{O}$ be the set of all nonempty open invariant subsets of $G^{(0)}$, and let $\mathcal{F}$ be the set of all functions $\pi : \mathcal{O} \to \mathcal{L}(R)$ such that for all $\mathcal{A} \subseteq \mathcal{O}$*

$$\pi\big(\textstyle\bigcup_{U\in\mathcal{A}} U\big) = \textstyle\bigcap_{U\in\mathcal{A}} \pi(U). \tag{4.3}$$

There is a bijection $\Gamma : \mathcal{F} \to \mathcal{L}(A_R(G))$ such that

$$\Gamma(\pi) = \operatorname{span}_R\Big(\bigcup_{U\in\mathcal{O}} \{rf : r \in \pi(U), f \in A_R(G), \operatorname{supp}(f) \subseteq G_U\}\Big).$$

For each $U \in \mathcal{O}$, we have

$$\pi(U) = \{r \in R : r1_B \in \Gamma(\pi) \text{ for all compact open } B \subseteq U\}.$$

The following observation will be useful a couple of times: suppose that $V, W \in \mathcal{O}$ with $V \subseteq W$. Then $\mathcal{A} := \{V, W\}$ satisfies $\bigcup_{U\in\mathcal{A}} U = W$. So if π satisfies (4.3), we have $\pi(V) \cap \pi(W) = \pi(W)$. Hence (4.3) implies that

$$V \subseteq W \Longrightarrow \pi(W) \subseteq \pi(V) \quad \text{for all } V, W \in \mathcal{O}. \tag{4.4}$$

Before proving the theorem, we establish a lemma.

Lemma 4.5. *Resume the notation of Theorem 4.4. Let $\pi_1, \pi_2 \in \mathcal{F}$. Then*

$$\Gamma(\pi_1) \subseteq \Gamma(\pi_2) \text{ if and only if } \pi_1(U) \subseteq \pi_2(U) \text{ for all } U \in \mathcal{O}.$$

Proof. Suppose that $\Gamma(\pi_1) \subseteq \Gamma(\pi_2)$. Fix $U \in \mathcal{O}$. Fix $r \in \pi_1(U)$. Let $u \in U$ and let K be a compact open subset of U with $u \in K$. Then $r1_K \in \Gamma(\pi_1) \subseteq \Gamma(\pi_2)$. By definition of $\Gamma(\pi_2)$,

$$r1_K = \sum_{V\in\mathcal{O}} r_V f_V,$$

where $r_V \in \pi_2(V)$ and $\operatorname{supp}(f_V) \subseteq G_V \cap K = V \cap K$. Let F be the finite collection of $V \in \mathcal{O}$ such that $f_V(u) \neq 0$. Then $r = \sum_{V \in F} r_V f_V(u)$. Let

$$W_u := \bigcap_{V \in F} V \cap U.$$

For $V \in F$, we have $W_u \subseteq V$, and so (4.4) gives $r_V \in \pi_2(V) \subseteq \pi_2(W_u)$. Since $\pi_2(W_u)$ is an ideal, it follows that $r = \sum_{V \in F} r_V f_V(u) \in \pi_2(W_u)$. Now (4.3) gives

$$\pi_2(U) = \pi_2\big(\bigcup_{u \in U} W_u\big) = \bigcap_{u \in U} \pi_2(W_u).$$

Thus $r \in \pi_2(U)$, and hence $\pi_1(U) \subseteq \pi_2(U)$.

If $\pi_1(U) \subseteq \pi_2(U)$ for all $U \in \mathcal{O}$, then it is immediate from the definition of Γ that $\Gamma(\pi_1) \subseteq \Gamma(\pi_2)$. $\square$

Proof of Theorem 4.4. Since each $\pi(U)$ is an ideal of R and each I_U is an ideal of $A_R(G)$, it follows that $\Gamma(\pi)$ is an ideal in $A_R(G)$.

To see that Γ is injective, suppose that $\Gamma(\pi_1) = \Gamma(\pi_2)$. Then two applications of Lemma 4.5 show that $\pi_1(U) = \pi_2(U)$ for every U, and so $\pi_1 = \pi_2$. Hence Γ is injective.

To see that Γ is surjective, let I be an ideal in $A_R(G)$. Let $U \in \mathcal{O}$. Set

$$\pi(U) = \{r \in R : r1_B \in I \text{ for all compact open } B \subseteq U\}. \tag{4.5}$$

Then $\pi(U) \in \mathcal{L}(R)$, and we claim that $\pi \in \mathcal{F}$, that is, π satisfies (4.3).

Let $\mathcal{A} \subseteq \mathcal{O}$. Since π reverses set inclusion, we have $\pi(\bigcup_{U \in \mathcal{A}} U) \subseteq \pi(U)$ for all $U \in \mathcal{A}$. Thus $\pi(\bigcup_{U \in \mathcal{A}} U) \subseteq \bigcap_{U \in \mathcal{A}} \pi(U)$.

For the reverse containment, fix $r \in \bigcap_{U \in \mathcal{A}} \pi(U)$. Let B be a compact open subset of $\bigcup_{U \in \mathcal{A}} U$. We need to show $r1_B \in I$. For each $b \in B$, there exists $U_b \in \mathcal{A}$ such that $b \in U_b$. Because $G^{(0)}$ has a basis of compact open sets, there exists compact open $K_b \subseteq U_b$ such that $b \in K_b$. Since $r \in \pi(U_b)$ we have $r1_{K_b} \in I$. Since B is compact, there is a finite set $C \subseteq \{K_b : b \in B\}$ that covers B. Since I is an ideal, $r1_{K_b} \in I$ implies $r1_K \in I$ for any compact open $K \subseteq K_b$. So we may disjointify C to obtain a finite cover, still called C, of B by compact open sets satisfying $r1_K \in I$ for all $K \in C$. So

$$r1_B = \sum_{K \in C} r1_K \in I.$$

Thus $r \in \pi(\bigcup_{U \in \mathcal{A}} U)$. Thus π satisfies (4.3), and $\pi \in \mathcal{F}$ as claimed.

Finally, we show that $\Gamma(\pi) = I$. To see that $\Gamma(\pi) \subseteq I$, we take $U \in \mathcal{O}$, $r \in \pi(U)$ and $f \in A_R(G)$ with $\operatorname{supp}(f) \subseteq G_U$, so that rf is a typical spanning element of $\Gamma(\pi)$. Write $rf = \sum_{B \in F} r_B 1_B$ where F is a finite set of mutually disjoint compact open bisections and $0 \neq r_B$ for $B \in F$. Fix $L \in F$ and take $\gamma \in L$. Then $(rf)(\gamma) = r_L \in \pi(U) \setminus \{0\}$. Since $\operatorname{supp}(rf) \subseteq \operatorname{supp} f \subseteq G_U$ we must have $s(L) \subseteq U$ and hence $r_L 1_{s(L)} \in I$ by definition of $r_L \in \pi(U)$. Thus $r_L 1_L = 1_L * (r_L 1_{s(L)}) \in I$. Thus $rf \in I$, and hence $\Gamma(\pi) \subseteq I$.

Conversely, fix $f \in I$. Because G is strongly effective, by Proposition 4.1 we have $f = \sum_{B \in F} r_B 1_B$ where each $r_B 1_{s(B)} \in I$. Fix $L \in F$. Recall that $[s(L)]$ is the smallest invariant subset of $G^{(0)}$ containing $s(L)$. We claim that $r_L 1_K \in I$ for every compact open $K \subseteq [s(L)]$; this implies $r_L \in \pi([s(L)])$ and hence $r_L 1_L \in \pi([s(L)]) I_{s([L])} \subseteq \Gamma(\pi)$. It then follows that $f \in \Gamma(\pi)$.

To prove the claim, fix $K \subseteq [s(L)]$. For each $k \in K$, there exists γ_k such that $s(\gamma_k) \in s(L)$ and $r(\gamma_k) = k$. Let B_k be a compact open bisection containing γ_k. We can assume that $s(B_k) \subseteq s(L)$ and $r(B_k) \subseteq K$ (by taking intersections). Now $\{r(B_k) : k \in K\}$ is an open cover of K. By taking a finite subcover and disjointifying, we get a collection of compact open bisections $\{B_1, \dots, B_n\}$ whose ranges form a disjoint cover of K. For $1 \le i \le n$ we have $r_L 1_{B_i} = 1_{B_i} * r_L 1_{s(L)} \in I$ and hence $r_L 1_{r(B_i)} = r_L 1_{B_i} * 1_{B_i^{-1}} \in I$. Now $r_L 1_K = \sum_{i=1}^n r_L 1_{r(B_i)} \in I$ as claimed. Thus $I \subseteq \Gamma(\pi)$. Now $I = \Gamma(\pi)$ and we have shown that Γ is surjective. By definition of π—see (4.5)—this also establishes the final statement of the theorem. □

5. The lattice isomorphism

In this section, we study the lattice structure of the set $\mathcal{L}(A_R(G))$ of ideals of $A_R(G)$. We have established in Theorem 4.4 a bijection from

$$\mathcal{F} := \{\pi : \mathcal{O} \to \mathcal{L}(R) : \pi \text{ satisfies (4.3)}\}$$

onto $\mathcal{L}(A_R(G))$. Since $(\mathcal{L}(A_R(G)), \subseteq, +, \cap)$ is a lattice, Γ induces a lattice structure $(\mathcal{F}, \preccurlyeq, \vee, \wedge)$ on $\mathcal{F}$ via

$$\pi_1 \preccurlyeq \pi_2 \iff \Gamma(\pi_1) \subseteq \Gamma(\pi_2).$$

However, it seems difficult to explicitly describe the element $\pi_1 \vee \pi_2 \in \mathcal{F}$ such that $\Gamma(\pi_1 \vee \pi_2) = \Gamma(\pi_1) + \Gamma(\pi_2)$.

Here we start by explaining the difficulties with $(\mathcal{F}, \preccurlyeq, \vee, \wedge)$, and then present a new parameterisation $\mathcal{F}'$ of the ideals of $A_R(G)$ which is better suited to describing the lattice structure. We will also see that $\mathcal{F}'$ has the additional advantage that it does not require a computation of the lattice $\mathcal{O}$ of open invariant subsets of $G^{(0)}$.

Let $\pi_1, \pi_2 \in \mathcal{F}$. By Lemma 4.5 we have

$$\pi_1 \preccurlyeq \pi_2 \iff \pi_1(U) \subseteq \pi_2(U) \text{ for all } U \in \mathcal{O}.$$

It is then easy to verify that the function $U \mapsto \pi_1(U) \cap \pi_2(U)$ belongs to $\mathcal{F}$, and is the meet $\pi_1 \wedge \pi_2$ of π_1 and π_2. The join $\pi_1 \vee \pi_2$ in $\mathcal{F}$ is more complicated. One might guess that $\pi_1 \vee \pi_2$ is the function g defined by $g(U) = \pi_1(U) + \pi_2(U)$ for $U \in \mathcal{O}$. But the next example shows that g may not even belong to $\mathcal{F}$.

Example 5.1. Consider the groupoid G that consists of two units x and y with the discrete topology. That is, $G = G^{(0)} = \{x, y\}$. Then $A_{\mathbb{Z}}(G) = \mathbb{Z} \oplus \mathbb{Z}$. The set of nonempty open invariant subsets of $G^{(0)}$ is

$$\mathcal{O} = \{\{x\}, \{y\}, G^{(0)}\}.$$

Define $\pi_1, \pi_2 : \mathcal{O} \to \mathcal{L}(\mathbb{Z})$ by

$$\pi_1(\{x\}) = 2\mathbb{Z},\ \pi_1(\{y\}) = 3\mathbb{Z},\ \pi_1(G^{(0)}) = 6\mathbb{Z},$$
$$\pi_2(\{x\}) = 3\mathbb{Z},\ \pi_2(\{y\}) = 5\mathbb{Z},\ \pi_2(G^{(0)}) = 15\mathbb{Z}.$$

Then π_1 and π_2 satisfy (4.3). Also $\Gamma(\pi_1) = 2\mathbb{Z} \oplus 3\mathbb{Z}$ and $\Gamma(\pi_2) = 3\mathbb{Z} \oplus 5\mathbb{Z}$. Hence $\Gamma(\pi_1) + \Gamma(\pi_2) = (2\mathbb{Z} \oplus 3\mathbb{Z}) + (3\mathbb{Z} \oplus 5\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, and it follows that $\pi_1 \vee \pi_2$ is given by

$$(\pi_1 \vee \pi_2)(\{x\}) = (\pi_1 \vee \pi_2)(\{y\}) = (\pi_1 \vee \pi_2)(\{x, y\}) = \mathbb{Z}.$$

Since $\pi_1(G^{(0)}) + \pi_2(G^{(0)}) = 6\mathbb{Z} + 15\mathbb{Z} = 3\mathbb{Z} \ne \mathbb{Z}$, we see that $\pi_1 \vee \pi_2$ is not given by pointwise addition of ideals. Indeed, since $G^{(0)} = \{x\} \cup \{y\}$ but $\pi_1(G^{(0)}) + \pi_2(G^{(0)}) = 3\mathbb{Z} \ne \mathbb{Z} = \big(\pi_1(\{x\}) + \pi_2(\{x\})\big) \cap \big(\pi_1(\{x\}) + \pi_2(\{x\})\big)$, we see that $U \mapsto \pi_1(U) + \pi_2(U)$ does not satisfy (4.3).

To overcome this problem, we will reparameterise $\mathcal{F}$ in terms of a set $\mathcal{F}'$ of functions from $G^{(0)}$ to $\mathcal{L}(R)$ that are continuous with respect to a suitable topology, and are suitably G-invariant. We will show in Theorem 5.4 below that the order relation and the meet and join operations on $(\mathcal{F}', \preccurlyeq)$ translate to pointwise containment, intersection and addition of functions, giving a natural description of the lattice structure on $\mathcal{L}(A_R(G))$.

Define a topology on $\mathcal{L}(R)$ as follows: Given a finite set $F \subseteq R$, define

$$Z(F) := \{I \in \mathcal{L}(R) : F \subseteq I\}.$$

Then $Z(F_1) \cap Z(F_2) = Z(F_1 \cup F_2)$ for finite $F_1, F_2 \subseteq R$, and hence the collection of all such $Z(F)$ forms a basis for a topology on $\mathcal{L}(R)$. We equip $\mathcal{L}(R)$ with this topology. It is a fairly weak topology: it is T_0 because if $I, J \in \mathcal{L}(R)$ and $r \in I \setminus J$, then $Z(\{r\})$ contains I but not J. However it is not a T_1 topology: if $J \subseteq I$ then every open set containing J contains I.

The following lemma is straightforward to prove.

Lemma 5.2. *Let $\rho : G^{(0)} \to \mathcal{L}(R)$. Then ρ is continuous at $u \in G^{(0)}$ if and only if for all $a \in \rho(u)$ there exists an open neighbourhood W of u such that $a \in \rho(w)$ for every $w \in W$.*

Proof. Fix $u \in G^{(0)}$. The sets $\{Z(\{a\}) : a \in \rho(u)\}$ form a neighbourhood subbasis at $\rho(u)$ for the topology on $\mathcal{L}(R)$. Thus ρ is continuous at u if and only if for each $a \in \rho(u)$ there is an open neighbourhood W of u such that $\rho(W) \subseteq Z(\{a\})$. □

We say $\rho : G^{(0)} \to \mathcal{L}(R)$ is *G-invariant* if $\rho(s(\gamma)) = \rho(r(\gamma))$ for all $\gamma \in G$. We set

$$\mathcal{F}' := \{\rho : G^{(0)} \to \mathcal{L}(R) : \rho \text{ is } G\text{-invariant and continuous}\}.$$

Lemma 5.3. *Let G be an ample Hausdorff groupoid and let R a commutative ring with identity.*

(a) *For any function $\rho : G^{(0)} \to \mathcal{L}(R)$, the function $\pi_\rho : \mathcal{O} \to \mathcal{L}(R)$ given by*

$$\pi_\rho(U) = \bigcap_{u \in U} \rho(u) \tag{5.1}$$

satisfies (4.3).

(b) *For any function $\pi : \mathcal{O} \to \mathcal{L}(R)$, the formula*

$$\rho_\pi(u) = \bigcup_{U \, open, u \in U} \pi([U]) \tag{5.2}$$

defines a g-invariant continuous function $\rho_\pi : G^{(0)} \to \mathcal{L}(R)$.

(c) *We have $\pi_{\rho_\pi} = \pi$ for $\pi \in \mathcal{F}$ and $\rho_{\pi_\rho} = \rho$ for $\rho \in \mathcal{F}'$. In particular, $\rho \mapsto \pi_\rho$ is a bijection from $\mathcal{F}'$ to $\mathcal{F}$.*

Proof. (a) Given $\rho : G^{(0)} \to \mathcal{L}(R)$ and $\mathcal{A} \subseteq \mathcal{O}$, we have

$$\pi_\rho\Big(\bigcup_{U \in \mathcal{A}} U\Big) = \bigcap_{u \in \bigcup_{U \in \mathcal{A}} U} \rho(u) = \bigcap_{U \in \mathcal{A}} \Big(\bigcap_{u \in U} \rho(u)\Big) = \bigcap_{U \in \mathcal{A}} \pi_\rho(U).$$

(b) Fix $\pi \in \mathcal{F}$. We start by showing that $\rho_\pi(u) \in \mathcal{L}(R)$. Fix $u \in G^{(0)}$. Let $r \in \rho_\pi(u)$ and $s \in R$. By definition of ρ_π there exists an open neighbourhood U of u such that $r \in \pi([U])$. Then $rs, sr \in \pi([U]) \subseteq \rho_\pi(u)$ because $\pi([U])$ is an ideal. Also, if $r, s \in \rho_\pi(u)$, there exist open neighbourhoods U_r and U_s of u such that $r \in \pi([U_r])$ and $s \in \pi([U_s])$.

Now (4.4) gives $r, s \in \pi([U_r \cap U_s])$, and hence $r + s \in \pi([U_r \cap U_s]) \subseteq \rho_\pi(u)$. Thus $\rho_\pi(u)$ is an ideal.

To see that ρ_π is continuous, we use Lemma 5.2. Fix $u \in G^{(0)}$ and fix $a \in \rho_\pi(u)$. By definition of ρ_π, there exists an open neighbourhood W of u such that $a \in \pi([W])$. Let $v \in W$. Then

$$a \in \pi([W]) \subseteq \bigcup_{V \text{ open}, v \in V} \pi([V]) = \rho_\pi(v).$$

It follows that ρ_π is continuous. It is G-equivariant because $\{[U] : r(\gamma) \in U, U \text{ open}\} = \{[U] : s(\gamma) \in U, U \text{ open}\}$ for every $\gamma \in G$.

(c) First fix $\pi \in \mathcal{F}$. We must show that $\pi_{\rho_\pi}(U) = \pi(U)$ for all $U \in \mathcal{O}$. Fix $U \in \mathcal{O}$. First suppose that $a \in \pi(U)$. Then $a \in \pi([W])$ for every open $W \subseteq U$. Since $[W] \subseteq U$ implies $\pi(U) \subseteq \pi([W])$, we get

$$a \in \bigcap_{u \in U} \Big(\bigcup_{W \text{ open}, \, u \in W} \pi([W]) \Big) = \bigcap_{u \in U} \rho_\pi(u) = \pi_{\rho_\pi}(U).$$

Now suppose that $a \in \pi_{\rho_\pi}(U)$. Then for each $u \in U$, there exists an open neighbourhood $W_u \subseteq U$ of u such that $a \in \pi([W_u])$. Since π satisfies (4.3), we obtain

$$a \in \bigcap_{u \in U} \pi([W_u]) = \pi\Big(\bigcup_{u \in U} [W_u] \Big) = \pi(U).$$

Now fix $\rho \in \mathcal{F}'$. We must show that $\rho_{\pi_\rho} = \rho$. Fix $u \in G^{(0)}$. Using that ρ is G-invariant for the final equality, we calculate:

$$\begin{aligned} \rho_{\pi_\rho}(u) &= \bigcup_{W \text{ open}, \, u \in W} \pi_\rho([W]) \\ &= \bigcup_{W \text{ open}, \, u \in W} \Big(\bigcap_{v \in [W]} \rho(v) \Big) \\ &= \bigcup_{W \text{ open}, \, u \in W} \Big(\bigcap_{v \in W} \rho(v) \Big). \end{aligned} \tag{5.3}$$

To see that this is equal to $\rho(u)$, first fix $a \in \rho_{\pi_\rho}(u)$. Then there is a neighbourhood U of u such that $a \in \rho(v)$ for every $v \in [U]$. In particular, $a \in \rho(u)$. Now fix $a \in \rho(u)$. Then there exists an open neighbourhood $W \subseteq G^{(0)}$ of u such that $\rho(v) \in Z(\{a\})$ for all $v \in W$. That is $a \in \rho(v)$ for all $v \in W$. So (5.3) gives

$$a \in \bigcap_{v \in W} \rho(v) \subseteq \rho_{\pi_\rho}(u). \qquad \square$$

Theorem 5.4. *Let G be an ample Hausdorff groupoid which is strongly effective, and let R be a commutative ring with identity. Let $\mathcal{F}'$ be the set of continuous G-invariant functions $\rho : G^{(0)} \to \mathcal{L}(R)$. There is a bijection $\Gamma' : \mathcal{F}' \to \mathcal{L}(A_R(G))$ such that*

$$\Gamma'(\rho) = \operatorname{span}_R \Big\{ r1_B : B \textit{ is a compact open bisection and } r \in \bigcap_{u \in [s(B)]} \rho(u) \Big\}.$$

Define a relation $\preccurlyeq$ on $\mathcal{F}'$ by

$$\rho_1 \preccurlyeq \rho_2 \textit{ if and only if } \rho_1(u) \subseteq \rho_2(u) \textit{ for all } u \in G^{(0)}.$$

Then $(\mathcal{F}', \preccurlyeq)$ is a lattice with join and meet operations given by

$$\rho_1 \vee \rho_2(u) = \rho_1(u) + \rho_2(u) \text{ and} \tag{5.4}$$

$$\rho_1 \wedge \rho_2(u) = \rho_1(u) \cap \rho_2(u), \tag{5.5}$$

and $\Gamma' : (\mathcal{F}', \preccurlyeq) \to (\mathcal{L}(A_R(G)), \subseteq)$ is a lattice isomorphism.

Proof. To see that $\Gamma'(\rho) = \Gamma(\pi_\rho)$, we start by unravelling $\Gamma(\pi_\rho)$:

$$\Gamma(\pi_\rho) = \operatorname{span}_R\Big(\bigcup_{U \in \mathcal{O}} \{rf : r \in \pi(U), f \in A_R(G), \operatorname{supp}(f) \subseteq G_U\}\Big).$$

Take $rf \in \Gamma(\pi_\rho)$. Then $rf = \sum_{B \in F} r_B 1_B$ where F is a set of mutually disjoint compact open bisections contained in G_U for some $U \in \mathcal{O}$. Fix $L \in F$ and let $\gamma \in L$. Then $r_L = rf(\gamma) \in \bigcap_{u \in U} \rho(u) \subseteq \bigcap_{u \in [s(L)]} \rho(u)$ since $[s(L)] \subseteq U$. Thus each $r_B 1_B \in \Gamma'(\rho)$, and hence $rf \in \Gamma'(\rho)$. Now $\Gamma(\pi_\rho) \subseteq \Gamma'(\rho)$. The reverse set inclusion is immediate. Thus $\Gamma'(\rho) = \Gamma(\pi_\rho)$.

Now Γ' is the composition of the bijections $\rho \mapsto \pi_\rho$ and $\pi \mapsto \Gamma(\pi)$ of Lemma 5.3 and Theorem 4.4, respectively. Hence Γ' is a bijection. To see that it is a lattice isomorphism, we must show that $\Gamma'(\rho_1) \subseteq \Gamma'(\rho_2)$ if and only if $\rho_1 \preccurlyeq \rho_2$.

First suppose that $\Gamma'(\rho_1) \subseteq \Gamma'(\rho_2)$. Then $\Gamma(\pi_{\rho_1}) \subseteq \Gamma(\pi_{\rho_2})$, forcing $\pi_{\rho_1}(U) \subseteq \pi_{\rho_2}(U)$ for all U. Fix $u \in G^{(0)}$ and $a \in \rho_1(u)$. We show that $a \in \rho_2(u)$. We have

$$a \in \rho_1(u) = \rho_{\pi_{\rho_1}}(u) = \bigcup_{W \text{ open}, \, u \in W} \pi_{\rho_1}([W]).$$

Hence there is an open neighbourhood $W \subseteq G^{(0)}$ of u such that $a \in \pi_{\rho_1}([W])$. Let K be a compact open subset of W. Then

$$a1_K \in \Gamma(\pi_{\rho_1}) = \Gamma'(\rho_1) \subseteq \Gamma'(\rho_2),$$

forcing

$$a \in \bigcap_{v \in [K]} \rho_2(v),$$

and in particular $a \in \rho_2(u)$. Thus $\rho_1 \preccurlyeq \rho_2$.

Second, suppose that $\rho_1 \preccurlyeq \rho_2$. Then $\rho_1(u) \subseteq \rho_2(u)$ for all $u \in G^{(0)}$, and take $\Gamma'(\rho_1) \subseteq \Gamma'(\rho_2)$ by definition of Γ'.

It remains only to show that $\rho_1 \vee \rho_2$ and $\rho_1 \wedge \rho_2$ are given by the formulas (5.4) and (5.5). For this, define $\tau_\vee, \tau_\wedge : G^{(0)} \to \mathcal{L}(R)$ by

$$\tau_\vee(u) = \rho_1(u) + \rho_2(u) \quad \text{and} \quad \tau_\wedge(u) = \rho_1(u) \cap \rho_2(u) \quad \text{for all } u \in G^{(0)}.$$

We first check that $\tau_\vee \in \mathcal{F}'$. To see that $\tau_\vee$ is continuous, we use Lemma 5.2. Fix $u \in G^{(0)}$ and $a \in \tau_\vee(u)$. Write $a = a_1 + a_2$ where $a_1 \in \rho_1(u)$ and $a_2 \in \rho_(u)$. Since ρ_1 and ρ_2 are continuous, there exist open neighbourhoods W_1 and W_2 of u such that $a_i \in \bigcap_{v \in W_i} \rho_i(v)$ for $i = 1, 2$. Hence $W := W_1 \cap W_2$ is an open neighbourhood of v such that

$$a = a_1 + a_2 \in \tau_\vee(v) \text{ for all } v \in W.$$

It follows that $\tau_\vee$ is continuous. It is G-equivariant because ρ_1 and ρ_2 are. Thus $\tau_\vee \in \mathcal{F}'$. A similar argument shows that $\tau_\wedge \in \mathcal{F}'$ as well.

We have $\tau_\wedge(u) = \rho_1(u) \cap \rho_2(u) \subseteq \rho_1(u), \rho_2(u)$ for all $u \in G^{(0)}$, and so $\tau_\wedge \preccurlyeq \rho_1, \rho_2$. The maximality of $\rho_1 \wedge \rho_2$ gives $\tau_\wedge \preccurlyeq \rho_1 \wedge \rho_2$. On the other hand, we have $\rho_1 \wedge \rho_2 \preccurlyeq \rho_1, \rho_2$, so for all $u \in G^{(0)}$ we have $(\rho_1 \wedge \rho_2)(u) \subseteq \rho_1(u), \rho_2(u)$, forcing $(\rho_1 \wedge \rho_2)(u) \subseteq \rho_1(u) \cap \rho_2(u) = \tau_\wedge(u)$, and so $\rho_1 \wedge \rho_2 \preccurlyeq \tau_\wedge$. Since $\preccurlyeq$ is a partial order, we deduce that $\tau_\wedge = \rho_1 \wedge \rho_2$. A similar argument gives $\tau_\vee = \rho_1 \vee \rho_2$. $\square$

Remark 5.5. We chose to first present the description of $\mathcal{L}(A_R(G))$ in terms of

$$\mathcal{F} := \{\pi : \mathcal{O} \to \mathcal{L}(R) : \pi \text{ satisfies (4.3)}\}$$

of Theorem 4.4 rather than the description in terms of

$$\mathcal{F}' := \{\rho : G^{(0)} \to \mathcal{L}(R) : \rho \text{ is } G\text{-invariant and continuous}\}$$

of Theorem 5.4. We did this because $\mathcal{F}$ is closer in spirit to the description, in terms of open invariant sets, of the ideals in a groupoid C^*-algebra in [28, Corollary 4.9] or the basic ideals of a Steinberg algebra in Section 3. In the context of graph groupoids, $\mathcal{F}$ is also much more closely related to the Bates–Pask–Raeburn–Szymański catalogue of ideals of $C^*(E)$ in [9, §4], Tomforde's catalogue of basic ideals of $L_R(E)$ in terms of saturated hereditary sets in [33, Theorem 7.9], and the analogous theorems for the Kumjian–Pask algebras of higher-rank graphs [7, Theorem 5.1] and [14, Theorem 9.4]. (see also Theorems 6.1 and 6.3 below). Nevertheless, we believe that the description in terms of $\mathcal{F}'$ is very natural, and at least in some cases much easier to compute with (as is the case in Example 6.7 below). The main advantage of the description in terms of $\mathcal{F}$ is that it is easy to decide which elements $r1_B$ belong to an ideal of the form $\Gamma(\pi)$; the principal advantages of the description in terms of $\mathcal{F}'$ are that there is no need to compute the collection of all open invariant sets, and that it makes the join operation easier to compute.

6. Leavitt path algebras and Kumjian–Pask algebras

In this section we explain what Theorem 4.4, and its crucial ingredient Proposition 4.1, say about a Leavitt path algebra of a directed graph and about a Kumjian–Pask algebra of a higher-rank graph. Since a Leavitt path algebra is a Kumjian–Pask algebra of a 1-graph, we will deduce Theorem 6.1 about the Leavitt path algebra from the analogous theorem about the Kumjian–Pask algebra. We start by gathering background needed to state Theorem 6.1.

Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sources. A subset $H \subseteq E^0$ is *hereditary* if $r(e) \in H$ implies $s(e) \in H$ for all $e \in E^1$, and is *saturated* if $s(vE^1) \subseteq H$ implies $v \in H$ for all $v \in E^0$. We write $\mathcal{H}_E$ for the collection of all saturated hereditary subsets of E^0. Given $\mathcal{A} \subseteq \mathcal{H}_E$, we write $\bigvee \mathcal{A}$ for the smallest saturated hereditary set containing every $H \in \mathcal{A}$; that is,

$$\bigvee \mathcal{A} = \bigcap_{H \in \mathcal{H}_E,\ K \subseteq H \text{ for all } K \in \mathcal{A}} H.$$

A graph E is satisfies *Condition (L)* if every cycle has an entry. Further, E satisfies *Condition (K)* if for every $v \in E^0$, either there is no cycle based at v, or there are at least two distinct return paths based at v. A graph satisfies Condition (K) if and only if for every saturated hereditary subset $H \neq E^0$ of E^0, the subgraph $E \backslash H = (E^0 \backslash H, s^{-1}(E^0 \backslash H), r, s)$ satisfies Condition (L) [24, Lemma 4.7]. It follows from Corollary 6.5 below that E satisfies Condition (K) if and only if the graph groupoid of E is strongly effective.

We refer to [33, §2] for the definition of the Leavitt path algebra $L_R(E)$. We write (p, s) for the universal generating Leavitt family in $L_R(E)$. Let $H \in \mathcal{H}_E$. Then the ideal I_H of $L_R(E)$ generated by $\{p_v : v \in H\}$ is a basic ideal by [33, Proposition 7.7]. When E

satisfies Condition (K), the map $H \mapsto I_H$ is an isomorphism from the lattice of hereditary saturated subsets of E^0 onto the lattice of basic ideals of $L_R(E)$ by [33, Corollary 7.18]. Theorem 6.1 addresses the non-basic ideal structure of $L_R(E)$ when E satisfies (K).

Theorem 6.1. *Let E be a row-finite directed graph with no sources and let R a commutative ring with identity. Suppose that E satisfies Condition (K).*

(a) *Suppose that I is an ideal in $L_R(E)$. Then*
$$I = \operatorname{span}_R\{rs_\lambda s_{\mu^*} : rp_{s(\mu)} \in I\}.$$

(b) *Let $\mathcal{H}_E$ be the set of all saturated hereditary subsets of E^0, let $\mathcal{L}(R)$ be the set of ideals of R and let $\mathcal{F}$ be the set of all functions $\pi : \mathcal{H}_E \to \mathcal{L}(R)$ such that*
$$\pi\Big(\bigvee_{H\in\mathcal{A}} H\Big) = \bigcap_{H\in\mathcal{A}} \pi(H) \quad \textit{for all } \mathcal{A} \subseteq \mathcal{H}_E.$$
Then the map $\Gamma : \mathcal{F} \to \mathcal{L}(L_R(E))$ given by
$$\Gamma(\pi) = \operatorname{span}_R\{rs_\mu s_{\nu^*} : \textit{ there exists } H \in \mathcal{H}_E \textit{ such that } r \in \pi(H) \textit{ and } s_\mu s_{\nu^*} \in I_H\}$$
is a bijection.

(c) *Let $\pi_1, \pi_2 \in \mathcal{F}$. Then $\Gamma(\pi_1) \subseteq \Gamma(\pi_2)$ if and only if $\pi_1(H) \subseteq \pi_2(H)$ for all $H \in \mathcal{H}_E$.*

Roughly, part (a) of Theorem 6.1 comes from Proposition 4.1, (b) comes from Theorem 4.4, and (c) comes from Lemma 4.5 used in the proof of Theorem 4.4. As we said above, the proof of Theorem 6.1 follows from the analogous Theorem 6.3 for Kumjian–Pask algebras, which we state and prove below. We now outline the background needed to state Theorem 6.3.

For a positive integer k, the additive semigroup $\mathbb{N}^k$ can be viewed as a category with one object. Following Kumjian and Pask's [20, Definitions 1.1], a *graph of rank k* or *k-graph* is a countable category $\Lambda = (\Lambda^0, \Lambda, r, s)$ together with a functor $d : \Lambda \to \mathbb{N}^k$, called *the degree map*, satisfying the following *factorisation property*: if $\lambda \in \Lambda$ and $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then there are unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$.

Let Λ be a k-graph. We use the notational convention whereby the juxtaposition UV of subsets $U, V \subseteq \Lambda$ means $\{\mu\nu : \mu \in U, \nu \in V, s(\mu) = r(\nu)\}$. If one of U, V is a singleton, we typically drop the braces from our notation; so for $v \in \Lambda^0$, the expression $v\Lambda$ means the same as $\{v\}\Lambda$, namely $\{\lambda \in \Lambda : r(\lambda) = v\}$.

Following [20], Λ is *row-finite* if $v\Lambda^n$ is finite for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$; Λ *has no sources* if $v\Lambda^n$ is nonempty for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. In this paper we are only interested in row-finite k-graphs with no sources.

Example 6.2. Let Ω_k be the category with objects $\mathbb{N}^k$, morphisms $\{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q\}$, domain and codomain maps given by $s(p, q) = q$ and $r(p, q) = p$ respectively, and composition given by $(p, q)(q, r) = (p, r)$. Define $d : \Omega_k \to \mathbb{N}^k$ by $d(p, q) := q - p$. With this structure, Ω_k is a k-graph (where we identify $\Omega_k^0 = \{(p, p) : p \in \mathbb{N}^k\}$ with $\mathbb{N}^k$ via $(p, p) \mapsto p$).

Following [23], a subset H of Λ^0 is *hereditary* if $s(H\Lambda) \subseteq H$ and is *saturated* if $v \in H$ whenever $s(v\Lambda^n) \subseteq H$. Analogously to the definitions for directed graphs E above, define
$$\mathcal{H}_\Lambda := \{H \subseteq \Lambda^0 : H \text{ is saturated and hereditary}\}.$$

Given a subset $\mathcal{A}$ of $\mathcal{H}_\Lambda$, let $\bigvee \mathcal{A}$ denote the smallest element of $\mathcal{H}_\Lambda$ containing $\bigcup_{K\in\mathcal{A}} K$. So

$$\bigvee \mathcal{A} := \bigcap_{H\in\mathcal{H}_\Lambda,\ K\subseteq H \text{ for all } K\in\mathcal{A}} H.$$

(We describe this $\bigvee$ operation explicitly in Lemma 6.4.)

The following is from [20, §2]. Let Ω_k be the k-graph of Example 6.2. An *infinite path* in Λ is a k-graph morphism $x : \Omega_k \to \Lambda$; the set of infinite paths is denoted by Λ^∞. We write $x(m)$ for the vertex $x(m, m)$. Then the range of an infinite path x is the vertex $r(x) := x(0)$, and we write $v\Lambda^\infty := r^{-1}(v)$.

For $\lambda \in \Lambda$, set $Z(\lambda) = \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\}$. Then $\{Z(\lambda) : \lambda \in \Lambda\}$ is a basis for a topology, and we equip Λ^∞ with this topology. Then Λ^∞ is a totally disconnected, locally compact Hausdorff space, and each $Z(\lambda)$ is compact and open. For $p \in \mathbb{N}^k$ define $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m, n) = x(m + p, n + p)$.

By [20, Definition 4.3], a k-graph is *aperiodic* if for every $v \in \Lambda^0$ there exists $x \in Z(v) = v\Lambda^\infty$ such that

$$\sigma^m(x) \neq \sigma^n(x) \text{ for all distinct } m, n \in \mathbb{N}^k. \tag{6.1}$$

We say Λ is *strongly aperiodic* if for every saturated hereditary subset $H \neq \Lambda^0$ of Λ, the k-graph $\Lambda \setminus H$ is aperiodic. This is the analogue for k-graphs of Condition (K) for directed graphs. (The terminology "strongly aperiodic" was coined in [19, Definition 3.1], but the condition itself appeared earlier, for example, in [30, Proposition 4.5].) We prove in Corollary 6.5 below that Λ is strongly aperiodic if and only if the graph groupoid of Λ is strongly effective.

We refer to [7, §3] for the definition of the Kumjian–Pask algebra path $\mathrm{KP}_R(\Lambda)$. We write (p, s) for the universal generating Kumjian–Pask family in $\mathrm{KP}_R(\Lambda)$. For $H \in \mathcal{H}_\Lambda$, the ideal I_H of $\mathrm{KP}_R(E)$ generated by $\{p_v : v \in H\}$ is a basic ideal by [7, Lemma 5.4]. When Λ is strongly aperiodic, the map $H \mapsto I_H$ is an isomorphism from the lattice of saturated hereditary subsets of Λ^0 onto the lattice of basic ideals of $\mathrm{KP}_R(\Lambda)$ by [7, Corollary 5.7]. We can now state our theorem for Kumjian–Pask algebras — it looks very similar to Theorem 6.1.

Theorem 6.3. *Let Λ be a row-finite k-graph with no sources and let R a commutative ring with identity. Suppose that Λ is strongly aperiodic.*

(a) *Suppose that I is an ideal in $\mathrm{KP}_R(\Lambda)$. Then*

$$I = \mathrm{span}_R\{r s_\lambda s_{\mu^*} : r p_{s(\mu)} \in I\}.$$

(b) *Let $\mathcal{H}_\Lambda$ be the set of all saturated hereditary subsets of Λ^0, let $\mathcal{L}(R)$ be the set of ideals of R and let $\mathcal{F}$ be the set of all functions $\pi : \mathcal{H}_\Lambda \to \mathcal{L}(R)$ such that*

$$\pi\Big(\bigvee_{H\in\mathcal{A}} H\Big) = \bigcap_{H\in\mathcal{A}} \pi(H) \quad \textit{for all } \mathcal{A} \subseteq \mathcal{H}_\Lambda. \tag{6.2}$$

Then the map $\Gamma : \mathcal{F} \to \mathcal{L}(\mathrm{KP}_R(\Lambda))$ given by

$$\Gamma(\pi) = \mathrm{span}_R\{r s_\mu s_{\nu^*} : \textit{ there exists } H \in \mathcal{H}_\Lambda \textit{ such that } r \in \pi(H) \textit{ and } s_\mu s_{\nu^*} \in I_H\}$$

is a bijection.

(c) *Let $\pi_1, \pi_2 \in \mathcal{F}$. Then $\Gamma(\pi_1) \subseteq \Gamma(\pi_2)$ if and only if $\pi_1(H) \subseteq \pi_2(H)$ for all $H \in \mathcal{H}_\Lambda$.*

To recover Theorem 6.1 from Theorem 6.3, recall that for a row-finite graph E with no sources, the Leavitt path algebra $L_R(E)$ is canonically isomorphic to the Kumjian–Pask algebra $\mathrm{KP}_R(E^*)$ where E^* is the path-category of E as in [20, Example 1.3].

Before starting the proof of Theorem 6.3, we need to introduce the graph groupoid G_Λ from [20, Definition 2.7]. Let Λ be a row-finite k-graph Λ with no sources. Define

$$G_\Lambda := \{(x, l, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : \exists m, n \in \mathbb{N}^k \text{ such that } l = m - n \text{ and } \sigma^m(x) = \sigma^n(y)\}.$$

Then G_Λ is a groupoid with composition and inverse given by

$$(x, l, y)(y, m, z) = (x, l + m, z) \quad \text{and} \quad (x, l, y)^{-1} = (y, -l, x).$$

For $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ set

$$Z(\mu, \nu) := \{(\mu z, d(\mu) - d(\nu), \nu z) : z \in Z(s(\mu))\}$$

Then $\{Z(\mu, \nu) : \mu, \nu \in \Lambda, s(\mu) = s(\nu)\}$ is a basis for a topology on G_Λ; we equip G_Λ with this topology. Then G_Λ is an ample Hausdorff groupoid (see [20, Proposition 2.8]). The unit space $G_\Lambda^{(0)}$ is $\{(x, 0, x) : x \in \Lambda^\infty\}$, which we identify with Λ^∞; the identification takes $Z(\mu, \mu)$ to $Z(\mu)$.

Let R be any commutative ring with identity. The Kumjian–Pask algebra $\mathrm{KP}_R(\Lambda)$ is canonically isomorphic to the Steinberg algebra of $A_R(G_\Lambda)$. This is proved in [13, Proposition 4.3] when $R = \mathbb{C}$, for a directed graph in [16, Example 3.2], and, most generally, for a finitely aligned k-graph in [15, Proposition 5.4]. (A row-finite k-graph with no sources is finitely aligned.)

The next lemma establishes the relationship between saturated hereditary subsets of Λ^0 and open invariant subsets of $G_\Lambda^{(0)}$; we write $\mathcal{O}_\Lambda$ for the latter. This lemma is known but doesn't seem to be recorded in the literature.

Lemma 6.4. *Let Λ be a row-finite k-graph with no sources. With notation as above, for $\mathcal{A} \subseteq \mathcal{H}_\Lambda$, we have*

$$\bigvee_{H \in \mathcal{A}} H = \{v \in \Lambda^0 : \text{ there exists } n \in \mathbb{N}^k \text{ such that } s(v\Lambda^n) \subseteq \textstyle\bigcup_{H \in \mathcal{A}} H\}. \tag{6.3}$$

The map $H \mapsto U_H$ from $\mathcal{H}_\Lambda$ to $\mathcal{O}_\Lambda$ given by

$$U_H := \{x \in \Lambda^\infty : x(n) \in H \text{ for large } n \in \mathbb{N}^k\}$$

is a bijection. For $H \in \mathcal{H}_\Lambda$, we have $H = \{v \in E^0 : Z(v) \subseteq U_H\}$, and $U_{\bigvee_{H \in \mathcal{A}} H} = \bigcup_{H \in \mathcal{A}} U_H$ for all $\mathcal{A} \subseteq \mathcal{H}_\Lambda$.

Proof. To establish (6.3), first take $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ such that $s(v\Lambda^n) \subseteq \bigcup_{H \in \mathcal{A}} H$. So v belongs to every saturated hereditary set containing $\bigcup_{H \in \mathcal{A}} H$, and therefore to $\bigvee_{H \in \mathcal{A}} H$. This establishes "$\supseteq$" in (6.3).

For the reverse containment, we show that any v that does not belong to the right-hand side of (6.3) also does not belong to the left-hand side. Fix $v \in \Lambda^0$ such that $s(v\Lambda^n) \not\subseteq \bigcup_{H \in \mathcal{A}} H$ for all $n \in \mathbb{N}^k$. For each $i \in \mathbb{N}$, let $n_i := (i, i, \ldots, i) \in \mathbb{N}^k$. For each i, choose $\lambda_i \in v\Lambda^{n_i}$ such that $s(\lambda_i) \notin \bigcup_{H \in \mathcal{A}} H$. Since Λ has no sources, we can extend each λ_i to an infinite path $x_i \in v\Lambda^\infty$ such that $x_i(n_i) \notin \bigcup_{H \in \mathcal{A}} H$. Since $v\Lambda^\infty = Z(v)$ is compact, there is a subsequence of $\{x_i\}$ converging to some $x \in Z(v)$. Since the $H \in \mathcal{A}$ are all hereditary, each $x_i(n_i) \notin \bigcup_{H \in \mathcal{A}} H$ implies $x_i(n) \notin \bigcup_{H \in \mathcal{A}} H$ for $n \leq n_i$. Since $\{n_i\}$ is a cofinal sequence, it follows that $x(n) \notin \bigcup_{H \in \mathcal{A}} H$ for all n. Set

$$K := \{w \in \Lambda^0 : w\Lambda x(n) = \emptyset \text{ for all } n \in \mathbb{N}^k\}.$$

Then K is hereditary.

To see that K is saturated, fix $w \in \Lambda^0$ and $n \in \mathbb{N}^k$ such that $s(w\Lambda^n) \subseteq K$. Fix $\lambda \in w\Lambda$. For any $\alpha \in s(\lambda)\Lambda^n$, by the factorisation property there exist $\mu \in w\Lambda^n$ and $\beta \in \Lambda$ such that $\lambda\alpha = \mu\beta$. Now $s(\mu) \in s(w\Lambda^n) \subseteq K$. Thus $s(\beta) \neq x(m)$ for all $m \in \mathbb{N}$. In particular, $\alpha \neq x(p, p+n)$ for any p. Thus $s(\lambda) \neq x(p)$ for all p, and hence $w \in K$.

We have $v \notin K$ by construction of K. Fix $u \in H \in \mathcal{A}$. If $u\Lambda x(n)$ were nonempty for some n, we would have $x(n) \in H$ because H is hereditary, and this is impossible by construction of x. So $u \in K$. That is, K is a saturated hereditary set containing $\bigcup_{H\in\mathcal{A}} H$ and not containing v, and it follows from the definition of $\bigvee_{H\in\mathcal{A}} H$ that $v \notin \bigvee_{H\in\mathcal{A}} H$ as required. This gives (6.3).

To see that $H \mapsto U_H$ is injective, suppose that $U_{H_1} = U_{H_2}$. By symmetry, we just have to show that $H_1 \subseteq H_2$. Let $v \in H_1$. Then $Z(v) \subseteq U_{H_1} = U_{H_2}$. So for each $x \in Z(v)$, there exists $n_x \in \mathbb{N}^k$ such that $x(n_x) \in H_2$. The sets $Z(x(0, n_x))$ cover $Z(v)$, and so there is a finite $F \subseteq Z(v)$ such that $\{Z(x(0, n_x)) : x \in F\}$ covers $Z(v)$. Take $N := \bigvee_{x\in F} n_x$. Let $\lambda \in v\Lambda^N$. Then $\lambda \in Z(v)$ implies $\lambda \in Z(x(0, n_x))$ for some $x \in F$. Since $x(n_x) \in H_2$ and H_2 is hereditary, we have $s(\lambda) \in H_2$. Thus $s(v\Lambda^N) \subseteq H_2$. Since H_2 is saturated as well, we have $v \in H_2$.

To see that $H \mapsto U_H$ is surjective, fix an open invariant $U \subseteq G_\Lambda^{(0)}$. Put

$$H(U) := \{v \in \Lambda^0 : Z(v) \subseteq U\}.$$

We claim that $H(U)$ is saturated and hereditary, and that $U_{H(U)} = U$. To see that $H(U)$ is hereditary, let $w \in H(U)$ and $\lambda \in w\Lambda$. Let $x \in Z(s(\lambda))$. Then $(\lambda x, d(\lambda), x) \in G_\Lambda$ and $\lambda x = r(\lambda x, d(\lambda), x) \in Z(w) \subseteq U$. Since U is invariant, $x = s(\lambda x, d(\lambda), x) \in U$ as well. Thus $Z(s(\lambda)) \subseteq U$, and hence $s(\lambda) \in H(U)$. So $H(U)$ is hereditary. To see that $H(U)$ is saturated, let $n \in \mathbb{N}^k$ and $w \in \Lambda^0$, and suppose that $s(w\Lambda^n) \subseteq H(U)$. If $\lambda \in w\Lambda^n$, then $s(\lambda) \in H(U)$ implies $\lambda x \in U$ by invariance of U. Thus

$$Z(w) = \bigcup_{\lambda\in w\Lambda^n} Z(\lambda) = \bigcup_{\lambda\in w\Lambda^n} \{r(\lambda x, d(\lambda), x) : x \in Z(s(\lambda))\} \subseteq U.$$

So $w \in H(U)$. Thus $H(U)$ is saturated.

To see that $U_{H(U)} = U$, first let $x \in U$. Since U is open there exists $\lambda \in \Lambda$ such that $x \in Z(\lambda) \subseteq U$. Since U is invariant and $(x, d(\lambda), \sigma^{d(\lambda)}(x)) \in G_\Lambda$ we see that $Z(s(\lambda)) \subseteq U$ as well. Thus $s(\lambda) \in H(U)$. Since $H(U)$ is hereditary, $x(n) \in H(U)$ for all $n \geq d(\lambda)$, and hence $x \in U_{H(U)}$.

Second, let $x \in U_{H(U)}$. Then there exists n such that $x(n) \in H(U)$. Then $Z(x(n)) \subseteq U$, and hence $\sigma^n(x) \in U$. Since U is invariant, it follows that $x = r(x, n, \sigma^n(x)) \in U$ as well. Thus $U_{H(U)} = U$, and $H \mapsto U_H$ is surjective.

That $H = \{v \in \Lambda^0 : Z(v) \subseteq U_H\}$ follows quickly: given H, we have $U_{H(U_H)} = U_H$, and since $H \mapsto U_H$ is injective, we deduce that $H = H(U_H)$, which is $\{v \in E^0 : Z(v) \subseteq U_H\}$ by definition.

It remains to check compatibility of $\bigvee$ with $\bigcup$. Fix $\mathcal{A} \subseteq \mathcal{H}_\Lambda$. First suppose that $x \in U_{\bigvee_{H\in\mathcal{A}} H}$. Then there exists n such that $x(n) \in \bigvee_{H\in\mathcal{A}} H$. Equation (6.3) shows that there exists m such that $x(n+m) \in \bigcup_{H\in\mathcal{A}} H$, so we may fix $H \in \mathcal{A}$ with $x(n+m) \in H$. Since H is hereditary, we have $x(p) \in H$ for large p, giving $x \in U_H \subseteq \bigcup_{H\in\mathcal{A}} U_H$. Second, suppose that $x \in \bigcup_{H\in\mathcal{A}} U_H$. Then $x \in U_H$ for some $H \in \mathcal{A}$, and since $U_H \subseteq U_{\bigvee_{H\in\mathcal{A}} H}$, we deduce that $x \in U_{\bigvee_{H\in\mathcal{A}} H}$. Thus $U_{\bigvee_{H\in\mathcal{A}} H} = \bigcup_{H\in\mathcal{A}} U_H$. □

Corollary 6.5. (a) *Let Λ be a row-finite k-graph with no sources. Then Λ is strongly aperiodic if and only if the k-graph groupoid G_Λ is strongly effective.*
(b) *Let E be a row-finite directed graph with no sources. Then E satisfies Condition (K) if and only if the graph groupoid G_E is strongly effective.*

Proof. The equivalence [22, Corollary 3.9 (i) $\iff$ (iii)] says that Λ is strongly aperiodic if and only if G_Λ is essentially principal. So the comment immediately following Definition 2.1 proves (a).

For (b), let F be a row-finite directed graph with no sources and let F^* be the path category as in [20, Example 1.3]. Then F^* is a 1-graph. Applying [7, Lemma 4.6] to the graph $F \setminus HF$ for each saturated hereditary $H \subseteq F^0$ shows that F^* is strongly aperiodic if and only if every $F \setminus HF$ satisfies Condition (L). So, by part (a) above, it suffices to show that F satisfies (K) if and only if each $F \setminus HF$ satisfies (L). Remark 4.5 of [9] proves the "only if" implication. The "if" implication is certainly folklore, but we couldn't find it explicitly in the literature. To prove it, suppose that F fails (K). Then there exists $v \in F^0$ lying on exactly one cycle μ. Then $H := \{v \in F^0 : vF^*r(\mu) = \emptyset\}$ is a saturated and hereditary set and μ is a cycle with no entrance in $F \setminus HF$. So $F \setminus HF$ fails (L). $\square$

Proof of Theorem 6.3. Since Λ is strongly aperiodic, G_Λ is strongly effective by Corollary 6.5. Thus Proposition 4.1 and Theorem 4.4 apply to G_Λ.

(a) Every compact open bisection in G_Λ is a finite union of basic compact open bisections $Z(\lambda, \mu)$. So in $A_R(G_\Lambda)$, we have

$$\begin{aligned} I &= \operatorname{span}_R\{r1_B : B \text{ is a compact open bisection with } r1_{s(B)} \in I\} \\ &= \operatorname{span}_R\{r1_{Z(\lambda,\mu)} : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu), r1_{Z(\mu)} \in I\}. \end{aligned}$$

We have

$$r1_{Z(\mu)} = 1_{Z(\mu,s(\mu))}\big(r1_{Z(s(\mu))}\big)1_{Z(s(\mu),\mu)} \quad \text{and} \quad r1_{Z(s(\mu))} = 1_{Z(s(\mu),\mu)}\big(r1_{Z(\mu)}\big)1_{Z(\mu,s(\mu))},$$

and hence $r1_{Z(\mu)} \in I$ if and only if $r1_{Z(s(\mu))} \in I$. Hence

$$\begin{aligned} &\operatorname{span}_R\{r1_B : B \text{ is a compact open bisection with } r1_{s(B)} \in I\} \\ &\qquad = \operatorname{span}_R\{r1_{Z(\lambda,\mu)} : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu), r1_{Z(s(\mu))} \in I\}. \end{aligned}$$

The canonical isomorphism of $A_R(G_\Lambda)$ onto $\mathrm{KP}_R(\Lambda)$ of [15, Proposition 5.4] carries $1_{Z(\lambda,\mu)}$ to $s_\lambda s_{\mu^*}$ and $1_{Z(v)}$ to p_v. Thus $I = \operatorname{span}_R\{rs_\lambda s_{\mu^*} : rp_{s(\mu)} \in I\}$ by Proposition 4.1.

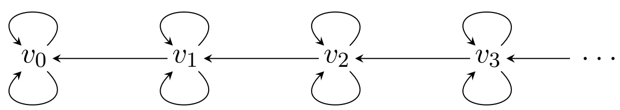
(b) Composition with the bijection $U \mapsto H_U$ from $\mathcal{O}_\Lambda$ to $\mathcal{H}_\Lambda$ of Lemma 6.4 carries the functions $\pi : \mathcal{H}_\Lambda \to \mathcal{L}(R)$ satisfying (6.2) to functions from $\mathcal{O}_\Lambda$ to $\mathcal{L}(R)$ satisfying (4.3) in Theorem 4.4. Thus it follows from Theorem 4.4 that $\Gamma : \mathcal{F} \to \mathcal{L}(R)$ is a bijection.

The argument of part (a) shows that the isomorphism $\mathrm{KP}_R(\Lambda)$ onto $A_R(G_\Lambda)$ carries the ideal I_H generated by the p_v with $v \in H$ to the ideal I_{U_H}, and hence $\Gamma(\pi)$ has the form claimed.

(c) This follows from Lemma 4.5 because $H \mapsto U_H$ preserves containment. $\square$

We conclude by applying our results to two illustrative examples of Leavitt path algebras.

Example 6.6. Consider the directed graph E pictured below.



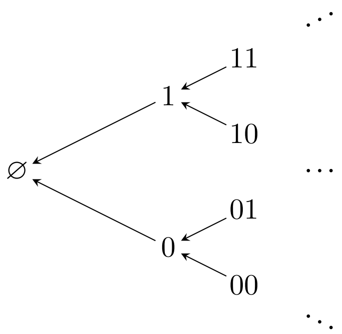
This E satisfies Condition (K) because every vertex has two loops, and it has a linear lattice $\mathcal{H}_E = \{H_n : n \in \mathbb{N}\}$ of saturated hereditary sets $H_n = \{v_n, v_{n+1}, v_{n+2}, \dots\}$.

Consider the ring $R = \mathbb{Z}$, which has nonzero ideals $\{m\mathbb{Z} : m \in \mathbb{N} \setminus \{0\}\}$. As a notational convenience, we write $\infty\mathbb{Z} := \{0\}$, the trivial ideal. So we may identify the set of functions $\pi : \mathcal{H}_E \to \mathcal{L}(R)$ with the set of all functions $\pi : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$. Given a subset $\mathcal{A} \subseteq \mathbb{N}$, we have $\bigvee_{n \in \mathcal{A}} H_n = H_{\min \mathcal{A}}$, and so a given $\pi : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ belongs to $\mathcal{F}$ if $\pi(\min \mathcal{A}) = \operatorname{lcm}\{\pi(n) : n \in \mathcal{A}\}$ for all $\mathcal{A} \subseteq \mathbb{N}$. This is equivalent to the condition that $\pi(n+1) \mid \pi(n)$ for all n (with the convention that $n \mid \infty$ for every $n \in \mathbb{N} \cup \{\infty\}$). So $\mathcal{F}$ consists of functions $\pi : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ such that $\pi(n+1) \mid \pi(n)$ for all n. Given such a function π, the corresponding ideal $\Gamma(\pi)$ of $A_{\mathbb{Z}}(E)$ is

$$\Gamma(\pi) = \operatorname{span}_{\mathbb{Z}}\{r s_\mu s_\nu^* : n \in \mathbb{N}, \mu, \nu \in E^* v_n \text{ and } r \in \pi(n)\mathbb{Z}\}.$$

We have $\Gamma(\pi) \subseteq \Gamma(\pi')$ if and only if $\pi'(n) \mid \pi(n)$ for all n. Theorem 6.1 shows that this completely describes all the ideals of $L_{\mathbb{Z}}(E)$.

Example 6.7. Consider the directed graph E pictured below.



To describe the ideals in $L_{\mathbb{Z}}(E)$ for this example, it is easiest to apply the description given in Theorem 5.4. For this, observe that the infinite paths in E, which are the units of the associated groupoid G_E, can be identified with pairs (ω, x) consisting of a finite word $\omega \in \{0,1\}^*$ and an infinite word $x \in \{0,1\}^\infty$ (when thinking of ω and x as paths, ω corresponds to the unique finite path to the root $\varnothing$ of the tree E from the range of the infinite path x).

The graph groupoid G_E then consists of triples of the form $\big((\omega, x), p - q, (\omega', y)\big)$ such that $p + |\omega| = q + |\omega'|$ and $x_{p+k} = y_{q+k}$ for all k, and from this it is easy to see that every orbit of G_E intersects exactly once with the set $\{\varnothing\} \times \{0,1\}^\infty$ of infinite paths with range $\varnothing$. So the G_E-invariant functions $\rho : G_E^{(0)} \to \mathcal{L}(\mathbb{Z})$ are in bijective correspondence with functions $\rho_0 : \{0,1\}^\infty \to \mathcal{L}(\mathbb{Z})$; specifically, $\rho_0(x) = \rho\big((\varnothing, x)\big)$ and $\rho\big((\omega, x)\big) = \rho_0(\omega x)$. Moreover, ρ is continuous with respect to the topology on $\mathcal{L}(\mathbb{Z})$ described just after Example 5.1 if and only if ρ_0 is continuous with respect to the same topology on $\mathcal{L}(\mathbb{Z})$ and the product topology on $\{0,1\}^\infty$. So the assignment $\rho \mapsto \rho_0$ restricts to a bijection between the set $\mathcal{F}'$ of Theorem 5.4 and the set of continuous functions from $\{0,1\}^\infty$ (under the product topology) to $\mathcal{L}(\mathbb{Z})$.

To describe the topology on $\mathcal{L}(\mathbb{Z})$, observe that for a finite set $F \subseteq \mathbb{Z}$, the corresponding open set $Z(F) \subseteq \mathcal{L}(\mathbb{Z})$ is the set $\{n\mathbb{Z} : n \mid \gcd(F)\}$. Identifying $\mathcal{L}(\mathbb{Z})$ with $\mathbb{N} \cup \{\infty\}$ as in the previous example, we see that the open sets in $\mathbb{N} \cup \{\infty\}$ are the sets $\{n : n \mid N\}$ indexed by $N \in \mathbb{N} \cup \{\infty\}$. So a function $\rho_0 : \{0,1\} \to \mathbb{N} \cup \{\infty\}$ is continuous if and only if whenever $x_j \to x$ in the product topology on $\{0,1\}^\infty$ we have $\rho_0(x_j) \mid \rho_0(x)$ for large j.

For $\nu \in E^*$, write ω_ν for the unique element of $\{0,1\}^\infty$ that corresponds to the path to the root $\varnothing$ from $s(\nu)$. By Theorem 5.4, the ideal corresponding to such a function ρ_0 is

$$\Gamma'(\rho_0) = \operatorname{span}_{\mathbb{Z}}\{n s_\mu s_{\nu^*} : \mu, \nu \in E^* s(\nu), n \in \rho_0(\omega_\nu x) \text{ for all } x \in \{0,1\}^\infty\},$$

and these are all the ideals of $L_{\mathbb{Z}}(E)$. Moreover, $\Gamma'(\rho_0) \subseteq \Gamma'(\tau_0)$ if and only if $\tau_0(x) \mid \rho_0(x)$ for all $x \in \{0,1\}^\infty$.

Remark 6.8. In the preceding example we argued directly to prove that we could reduce the problem of describing $\mathcal{F}'$ to that of describing the collection of continuous functions $\rho_0 : \{0,1\}^\infty \to \mathcal{L}(\mathbb{Z})$; but as an alternative, we could have used the results of [16]. The set $X := \{\varnothing\} \times \{0,1\}^\infty \cong \{0,1\}^\infty$ is compact open and intersects every G_E-orbit. Hence the restriction H of G_E to this subset of the unit space is equivalent, in the sense of Renault, to G_E by [16, Lemma 6.1]. So [16, Theorem 5.1] implies that $A_{\mathbb{Z}}(G_E)$ is Morita equivalent to $A_{\mathbb{Z}}(H)$, and hence the ideals of the former are in bijection with the ideals of the latter. Since X intersects every G_E-orbit exactly once, $H = H^{(0)}$ is just a copy of the topological space X, so H-equivariance of a function $\rho_0 : H^{(0)} \to \mathcal{L}(\mathbb{Z})$ is a vacuous requirement, and we deduce, once again, that the ideals of $L_{\mathbb{Z}}(E)$ are in bijection with the continuous functions from $\{0,1\}^\infty$ to $\mathcal{L}(\mathbb{Z})$.

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