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Guiyuan Ma ^{*} Song-Ping Zhu [†] Wenting Chen [‡]

Abstract

This paper studies European call option pricing problem under a hard-to-borrow stock model where stock price and buy-in rate are fully coupled. Avellaneda and Lipkin (2009) proposed a simplified solution approach with an independence assumption, and then derived a semi-explicit pricing formula. However, such an approach has limited its application to more general cases. In this paper, we propose a partial differential equation(PDE) approach for pricing European call options, regardless of the independence assumption. A two-dimensional PDE is derived first with a set of appropriate boundary conditions. Then, two numerical schemes are provided with different treatments of the jump term.

Through our numerical results, we find that the semi-explicit formula is a good approximate solution when the coupling parameter is small. However, when the stock price and the buy-in rate are significantly coupled, the PDE approach is preferred to solve the option pricing problem under the full hard-to-borrow model.

Keywords. Hard-to-borrow stock; Buy-in rate; Regulation SHO; Option pricing; ADI scheme.

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1 Introduction

Option pricing is one of the most important topics in mathematical finance. The pioneering work was attributed to Black and Scholes (1973), who first proposed an analytical pricing formula, which has laid the solid foundation for modern option pricing theory. One of the assumptions in the Black-Scholes model is that short-selling is permitted without transaction cost, while the market regulations suggest otherwise. In most stock markets, naked short-selling is forbidden for it may lead to too much settlement risk. However, short-selling with some restrictions is still allowed in some markets. If an investor wants to short a stock, he could borrow from others in advance. The availability of stocks for borrowing depends on market conditions. While some are easily borrowed, others may be in short supply. A *hard-to-borrow list* is an inventory record used by brokerages to indicate what securities are difficult to borrow for short-selling transactions. Those listed stocks are referred to as *hard-to-borrow* stocks.

On January 3, 2005, the Securities and Exchange Commission (SEC) implemented Regulation SHO¹ to modify rules for short selling in U.S. equity market. In order to cover shortfalls in delivery of stocks, Rule 203 of Regulation SHO requires that the short-seller should be “forced” to repurchase the stock when the associated settlement risk increases to a certain level. Such a regulation is called a *buy-in* mechanism. The *short interest*, a ratio of shares sold short to shares outstanding, is an indicator of market sentiment that tells whether investors expect a stock’s price is likely to fall. As the short interest goes up, the settlement risk accumulates gradually in the market. When the risk is out of control, a buy-in is triggered. Once a buy-in occurs, the settlement risk then is reduced as the short interest falls down. After that, the risk accumulates again with time going by and the short interest goes up again. When it comes to a certain level again, another buy-in follows. Investors should take the possible buy-ins into account. Mathematically, these buy-ins are considered as stochastic dividend yields or convenience yields, because

¹To learn more about Regulation SHO, readers are referred to the website: www.sec.gov.

the holders of hard-to-borrow stocks can obtain lending fees by lending their stocks to the investors who wish to maintain short positions and not risk buy-ins (Avellaneda and Lipkin, 2009).

In the literature, there is a considerable amount of research about short-selling. Diamond and Verrecchia (1987) considered constraints on short-selling and asset price adjustment to private information. Nielsen (1989) studied the asset market equilibrium with short-selling. Duffie et al. (2002) presented a model of asset valuation in which short-selling is achieved by searching for security lenders and bargaining over the terms of lending fee. Jones and Lamont (2002) pointed out that, from market data, stocks are overpriced when short-sale constraints are imposed. Evans et al. (2009) mainly focused on how options market and short-selling interact with each other. Boehmer et al. (2013) studied the impact of short-selling ban on stock price and found that there would be a pronounced but temporary increase.

To characterize the buy-in mechanism associated with hard-to-borrow stocks, Avellaneda and Lipkin (2009) proposed a dynamic model (hereafter, it is referred to as the A&L model) by introducing a stochastic *buy-in rate*. Such a rate represents the frequency at which buy-ins take place. Since then, the A&L model has attracted much attention from different aspects, such as Avellaneda and Zhang (2010); Li et al. (2014); Jensen and Pedersen (2016); Ma and Zhu (2018).

Mathematically, the A&L model is a fully coupled system. In order to derive a pricing formula for European call options, Avellaneda and Lipkin (2009) proposed a simplified solution approach by assuming that the buy-in rate is independent of the Brownian motion that drives the stock price, which is referred to as *the independence assumption* hereafter. However, the A&L's formula is semi-explicit because a series of unknown weight functions are still involved. In order to calculate these weight functions, Monte Carlo simulations need to be carried out, which is absent in Avellaneda and Lipkin (2009). On the other hand, we note that the independence assumption is significantly important because it

simplifies the fully coupled hard-to-borrow stock model to a decoupled one. Although such an assumption has really facilitated the derivation of the pricing formula, it has also limited its application to general cases.

The contribution of this paper is that we propose a partial differential equation (PDE) approach to price European call options and our approach works not only for the fully coupled hard-to-borrow stock model but also for the simplified model with the independence assumption. To obtain the numerical results, two schemes are carefully chosen based on different approaches to the jump term. Both of them have adopted the ADI scheme to improve the computational efficiency. In addition, we also present how to implement Monte Carlo simulations for the dynamics when a doubly stochastic Poisson process is involved. From our numerical experiments, we find that when the coupling parameter is small, the A&L's pricing formula is an efficient tool and a good approximate solution to calculate option price. However, as the value of the coupling parameter increases, the simplified model departs from the fully coupled hard-to-borrow model more and more significantly. Consequently, the PDE solution approach is preferred for solving European option pricing problem when the coupling relation is pronounced.

The paper is organized as follows. In Section 2, the A&L's model is reviewed briefly with a semi-explicit pricing formula. In Section 3, we derive a two-dimensional PDE to price European call options with appropriate boundary conditions. In Section 4, two numerical methods are presented to solve the PDE system based on the different treatments for the jump term. In addition, Monte Carlo simulations are also provided to calculate the unknown weight functions. In Section 5, the numerical results are provided and some discussions are presented. Conclusions are given in the last section.

2 The A&L model for hard-to-borrow stocks

Avellaneda and Lipkin (2009) proposed a fully coupled hard-to-borrow stock model by

characterizing buy-ins with a doubly stochastic Poisson process $N_{\lambda_t}(t)$. The buy-in rate λ_t represents the frequency at which buy-ins take place. The stock price S_t and the buy-in rate λ_t satisfy the following stochastic differential equations (SDEs) under the physical measure $\mathbb{P}$:

$$\begin{cases} \frac{dS_t}{S_t} = \sigma dW_t + \gamma \lambda_t dt - \gamma dN_{\lambda_t}(t) \\ dx_t = \kappa dZ_t + \alpha(\bar{x} - x_t)dt + \beta \frac{dS_t}{S_t}, \quad x_t = \ln\left(\frac{\lambda_t}{\lambda_0}\right), \end{cases} \quad (2.1)$$

where dN_{λ_t} denotes the increment of a doubly stochastic Poisson process with intensity λ over the interval $(t, t + dt)$. σ represents constant volatility and γ is price elasticity of demand due to buy-ins; W_t and Z_t are two Brownian motions with covariance $cov(dW, dZ) = \rho dt$.² For convenience, x_t is also called buy-in rate hereafter. The second equation in (2.1) describes the evolution of the buy-in rate with its volatility κ , long-time equilibrium value $\bar{x}$, speed of mean-reversion α and coupling parameter β that couples the change in price with the change in the buy-in rate.

In order to introduce a positive feedback between buy-in rate and stock price, the coupling parameter β is assumed to be positive by Avellaneda and Lipkin (2009). When a buy-in occurs, the stock price drops down and the buy-in rate also falls to a low level simultaneously because of the coupling term $\beta \frac{dS_t}{S_t}$. Another buy-in is unlikely to occur immediately following the previous one because the previous one has reduced some settlement risk. With time moving on, the risk accumulates gradually and the buy-in rate goes up again. Another buy-in occurs in the future once market risk goes up to reach a certain level again. This demonstrates how the hard-to-borrow stock model (2.1) characterizes the buy-in mechanism imposed by Regulation SHO.

²As suggested by an anonymous referee, in this paper, we consider a general hard-to-borrow model where two Brownian motions are correlated. Obviously, when $\rho = 0$, it degenerates to the original A&L model.

2.1 The risk-neutral measure

It should be pointed out that the A&L model operates in an incomplete market since an additional source of uncertainty has been introduced through the buy-in rate, which is not a tradable quantity (Tankov, 2003). Consequently, it is impossible to perfectly hedge a portfolio composed of hard-to-borrow stocks and there does not exist a unique risk-neutral measure. For pricing a derivative, a risk-neutral measure needs to be defined for the processes S_t and x_t first. What Avellaneda and Lipkin (2009) did was to introduce an arbitrage-free pricing measure, which is equivalent to changing the drift of the Brownian motion associated with the underlying stock. Mathematically, to conduct measure transform, two new processes are defined as

$$\tilde{W}_t = W_t + \int_0^t \frac{\gamma\lambda_l - r}{\sigma} dl, \quad (2.2)$$

$$\tilde{Z}_t = Z_t + \int_0^t \frac{\alpha z(l, x_l, S_l)}{\kappa} dl, \quad (2.3)$$

where $z(t, x, S)$ is an arbitrary function. By Girsanov's theorem, $\tilde{W}$ and $\tilde{Z}$ are two new Brownian motions under the risk-neutral measure $\mathbb{Q}$ defined by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_t = \exp \left\{ - \int_0^t \left[\frac{\gamma\lambda_l - r}{\sigma} + \frac{\alpha z(l, x_l, S_l)}{\kappa} \right] dW_l - \frac{1}{2} \int_0^t \left[\frac{(\gamma\lambda_l - r)^2}{\sigma^2} + \frac{\alpha^2 z^2(l, x_l, S_l)}{\kappa^2} \right] dl \right\},$$

which is the so-called Radon-Nikodym derivative. Under this risk-neutral measure, the dynamics of the A&L model become

$$\frac{dS_t}{S_t} = \sigma d\tilde{W}_t + r dt - \gamma dN_{\lambda_t}(t), \quad (2.4)$$

$$dx_t = \kappa d\tilde{Z}_t + [\alpha(\bar{x}^* - x_t)]dt + \beta \frac{dS_t}{S_t}, \quad (2.5)$$

where $\bar{x}^* = \bar{x} - z$.

Financially, any source of uncertainty needs to be compensated by the associated market

price of risk or risk premium. In the classic Black-Scholes model, the market price of risk for the underlying is $\frac{\mu - r}{\sigma}$ (Wilmott et al., 1995). On the other hand, in the Heston model, an additional source uncertainty is introduced by the stochastic volatility and an additional market price of volatility risk is defined through an arbitrary function, i.e., $\lambda(t, S, v)$ in Heston (1993), which may appear in a more general form for other stochastic volatility models discussed in Fouque et al. (2000). In the A&L model, the new buy-in process also brings in an additional source of uncertainty and the corresponding market price of buy-in risk is represented by the function $z(t, x, S)$ in Equation (2.3). Furthermore, it should be remarked that the market price of risk for the stock in the A&L model becomes $\frac{\gamma\lambda - r}{\sigma}$, which is different from its counterpart $\frac{\mu - r}{\sigma}$ in the Black-Scholes model.

When market is complete, the market price of risk for stock is unique, such as the term $\frac{\mu - r}{\sigma}$ in the Black-Scholes model. When a market is incomplete, the market price of risk is specified after a risk-neutral measure is chosen, or in a vice versa way in financial practice that a market price of risk is extracted from market data first, which then implicitly dictates the risk-neutral measure to be used in pricing a derivative. Therefore, the market price of buy-in risk in the A&L model should be determined by market data, just as the market price of volatility risk in the Heston model needs to be calibrated from market data (Bollerslev et al., 2011). For simplicity, Avellaneda and Lipkin (2009) effectively set $z(t, x, S)$ to be zero, which is a standard treatment in the Heston model as well (Rouah, 2013).

2.2 The A&L's pricing formula

In order to simplify the fully coupled model, Avellaneda and Lipkin (2009) made an approximation assumption that λ_t is independent of $\tilde{W}_t$, which results in

$$dx_t = [\alpha(\bar{x} - x_t) + \beta r]dt + \kappa d\tilde{Z}_t - \beta\gamma dN_{\lambda_t}. \quad (2.6)$$

Such an approximate assumption holds when β is small. Financially, it means that the coupling relation between the stock price and the buy-in rate is not very significant. Under the simplified model consisting of (2.4) and (2.6), Avellaneda and Lipkin (2009) derived a pricing formula for European call options

$$C(S, K, T, r) = \sum_{n=0}^{\infty} \Pi(n, T) C^{BS}(S(1 - \gamma)^n, K, T, r, \sigma), \quad (2.7)$$

where $C^{BS}(S, K, T, r, \sigma)$ represents the Black-Scholes value of a European call option with stock price S , time to maturity T , and strike price K . Weight functions are defined

$$\Pi(n, T) = P\left(\int_0^T dN_{\lambda_t} = n\right). \quad (2.8)$$

Obviously, the pricing formula (2.7) still involves a series of unknown functions $\Pi(n, T)$ which are unnecessary for computing option price from a completely explicit formula, such as the famous Black-Scholes formula. It is for this reason that pricing formula (2.7) is semi-explicit.

In addition, the original model consisting of (2.4) and (2.5) is fully coupled because S_t and x_t are twisted together; while the independence assumption makes the fully coupled model to a simplified or decoupled one. In (2.6), the buy-in rate x_t no longer depends on the stock price S_t . This independence assumption has really facilitated the derivation of A&L's semi-explicit pricing formula for European call options under the simplified model. However, at the same time, such an assumption has limited the application of the full hard-to-borrow stock model.

3 The PDE approach for European call options

In this section, we derive a two-dimensional PDE to price European call options under the full hard-to-borrow stock model. To obtain a properly-closed PDE system, some boundary

conditions are presented from mathematical and financial aspects. A similar PDE is also derived for the simplified model with the independence assumption.

Theorem 3.1. Under the hard-to-borrow stock model consisting of (2.4) and (2.5), the value of European call options written on the hard-to-borrow stock at time t is defined as

$$u(x, S, t) = \mathbf{E}_t[e^{-r(T-t)}h(S_T)],$$

where $\mathbf{E}_t$ is conditional expectation on $S_t = S, x_t = x$ under the risk-neutral measure $\mathbb{Q}$ and the payoff function is $h(S) = (S - K)^+$. Then $u(x, S, t)$ satisfies the PDE system

$$\begin{cases} -\frac{\partial u}{\partial t} = (\mathcal{L}_1 + \mathcal{L}_2)u, \\ u(x, S, T) = (S - K)^+, \end{cases} \quad (x, S, t) \in \mathbf{R} \times [0, \infty) \times [0, T], \quad (3.1)$$

where the diffusion operator $\mathcal{L}_1$ and the jump operator $\mathcal{L}_2$ are defined as

$$\mathcal{L}_1 u = \frac{\kappa^2 + \beta^2 \sigma^2 + 2\kappa\beta\sigma\rho}{2} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + (\beta\sigma^2 + \rho\kappa\sigma) S \frac{\partial^2 u}{\partial x \partial S} + [\alpha(\bar{x} - x) + \beta r] \frac{\partial u}{\partial x} + rS \frac{\partial u}{\partial S} - ru, \quad (3.2)$$

$$\mathcal{L}_2 u = e^x [u(x - \beta\gamma, S(1 - \gamma), t) - u(x, S, t)]. \quad (3.3)$$

The details of the proof are left in Appendix A.

Remark 3.1. The operator $\mathcal{L}_2$ would be in a form of integration if γ was a random variable. In that case, (3.1) would become an integro-PDE. However, in the A&L model, γ is a deterministic constant and (3.1) is still a two-dimensional standard PDE, instead of an integro-PDE.

The boundary conditions along the stock price direction are easy to impose. They are similar to the counterparts in the Black-Scholes model. The stock price stays at zero once it hits zero. Therefore, European call options become worthless even if there is still a long time to expiry. Consequently, we set $u(x, 0, t) = 0$. On the other hand, as the stock price

becomes large, it is more likely that the European call options will be exercised at the expiry. The corresponding boundary condition is imposed as

$$\lim_{S \rightarrow \infty} \frac{u(x, S, t)}{S} = 1.$$

Then we turn to the boundary conditions along the buy-in rate direction. When x tends to $-\infty$ (i.e. $\lambda \rightarrow 0$), there is no jump in the stock price. In this case, the model is equivalent to the Black-Scholes model. Therefore, we set the value of European call options as the counterpart in the Black-Scholes model, i.e.

$$\lim_{x \rightarrow -\infty} u(x, S, t) = \lim_{\lambda \rightarrow 0} u(\ln \lambda, S, t) = C^{BS}(S, K, T - t, r, \sigma).$$

Finally, we come to the boundary condition on $x \rightarrow \infty$. One needs to understand how the buy-in rate affects the option price first. Roughly speaking, the buy-in rate is a measure of the frequency at which buy-ins occur. When the buy-in rate increases, the buy-ins occur more often, resulting in higher lending fees. On the other hand, the holder of European call options needs to hedge the risk by shorting stocks. If the lending fees are very large, hedging the call options would become very expensive. Fewer and fewer investors are willing to buy the corresponding European call options as the buy-in rate increases. When the buy-in rate $x = \ln(\lambda)$ becomes large enough, it has few effect on the option value. In other words, the option value would be expected to be insensitive to the buy-in rate change when x has been very large, which is similar to the boundary condition imposed by Clarke and Parrott (1999). Accordingly, a Neumann boundary condition can be given as

$$\lim_{x \rightarrow \infty} \frac{\partial u}{\partial x}(x, S, t) = 0.$$

In a brief summary, the properly-closed PDE system can be written as

$$-\frac{\partial u}{\partial t} = (\mathcal{L}_1 + \mathcal{L}_2)u, \quad (x, S, t) \in \mathbf{R} \times [0, \infty) \times [0, T] \quad (3.4)$$

where the operator $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined in (3.2)-(3.3) and boundary conditions are

$$\left\{ \begin{array}{l} u(x, S, T) = (S - K)^+, \\ u(x, 0, t) = 0, \\ \lim_{S \rightarrow \infty} \frac{u(x, S, t)}{S} = 1, \\ \lim_{x \rightarrow -\infty} u(x, S, t) = C^{BS}(S, K, T - t, r, \sigma), \\ \lim_{x \rightarrow \infty} \frac{\partial u}{\partial x}(x, S, t) = 0, \end{array} \right. \quad (3.5)$$

In order to compare our PDE approach with A&L's semi-explicit pricing formula, we then derive a PDE for the simplified model consisting of (2.4) and (2.6) with the independence assumption. Similar to Theorem 3.1, a simplified PDE is derived as

$$-\frac{\partial u}{\partial t} = (\tilde{\mathcal{L}}_1 + \mathcal{L}_2)u, \quad (3.6)$$

where

$$\tilde{\mathcal{L}}_1 u = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + \frac{\kappa^2}{2} \frac{\partial^2 u}{\partial x^2} + \rho \kappa \sigma S \frac{\partial^2 u}{\partial x \partial S} + [\alpha(\bar{x} - x) + \beta r] \frac{\partial u}{\partial x} + r S \frac{\partial u}{\partial S} - r u, \quad (3.7)$$

with the same boundary conditions (3.5). The results calculated from the PDE (3.6) now can be compared with those calculated from A&L's pricing formula since both of them are under the simplified model with independence assumption.

By comparing the PDE (3.4) for the full model and the PDE (3.6) for the simplified model with the independence assumption, we note that the only difference between them lies in the coefficients of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial x \partial S}$. The coefficient of $\frac{\partial^2 u}{\partial x^2}$ changes from $\frac{\kappa^2 + \beta^2 \sigma^2 + 2\kappa\beta\sigma\rho}{2}$ to $\frac{\kappa^2}{2}$ and the coefficient of term $\frac{\partial^2 u}{\partial x \partial S}$ changes from $(\rho\kappa\sigma + \beta\sigma^2)S$ to $\rho\kappa\sigma S$ when the

independence assumption is imposed. Obviously, the PDE (3.6) is a good approximation to PDE (3.4) when β is small, which indicates that the buy-in rate is not strongly affected by the change of stock price. However, when the coupling parameter β is sufficiently large, the independence assumption is unreasonable and the PDE (3.4) needs to be solved directly instead of the simplified model.

Unlike the A&L's pricing formula (2.7), which depends heavily on the independence assumption, our PDE approach can deal with both the full model and the simplified model, regardless of the independence assumption. In addition, our PDE approach can also be extended to deal with the American case³, while it may be a quite difficult task to extend the semi-explicit pricing formula to the American case.

4 Numerical schemes

In this section, two numerical schemes are presented with different treatments of the jump term. We just take the PDE (3.4) for the full model as an example when demonstrating the details of numerical schemes. Of course, these schemes can also be applied to solve the PDE (3.6) for the simplified model.

4.1 Numerical scheme for the PDE system

For the convenience of numerical implementation, we introduce transforms $\tau = T - t$ and $y = \ln(\frac{S}{K})$. Then the PDE (3.4) is rewritten as

$$\frac{\partial u}{\partial \tau} = \overline{\mathcal{L}}_1 u + \overline{\mathcal{L}}_2 u, \quad (x, y, \tau) \in \mathbf{R} \times \mathbf{R} \times [0, T], \quad (4.1)$$

³When this paper is under review, our another paper (Ma and Zhu, 2018), which applies the PDE approach proposed in this paper to deal with the American call option pricing problem, has been published.

where

$$\begin{aligned}\overline{\mathcal{L}}_1 u &= \frac{\kappa^2 + \beta^2 \sigma^2 + 2\kappa\beta\sigma\rho}{2} \frac{\partial^2 u}{\partial x^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial y^2} + (\beta\sigma^2 + \rho\kappa\sigma) \frac{\partial^2 u}{\partial x \partial y} + [\alpha(\bar{x} - x) + \beta r] \frac{\partial u}{\partial x} + (r - \frac{1}{2}\sigma^2) \frac{\partial u}{\partial y} - ru \\ \overline{\mathcal{L}}_2 u &= e^x [u(x - \beta\gamma, y + \ln(K(1 - \gamma)), \tau) - u]\end{aligned}$$

with boundary conditions

$$\left\{ \begin{array}{l} u(x, y, 0) = K(e^y - 1)^+, \\ \lim_{y \rightarrow -\infty} u(x, y, \tau) = 0, \\ \lim_{y \rightarrow \infty} \frac{u(x, y, \tau)}{K e^y} = 1, \\ \lim_{x \rightarrow -\infty} u(x, y, \tau) = C^{BS}(K e^y, K, \tau, r, \sigma), \\ \lim_{x \rightarrow \infty} \frac{\partial u}{\partial x}(x, y, \tau) = 0, \end{array} \right. \quad (4.2)$$

Before discretizing the PDE, we truncate the domain as

$$(x, y, \tau) \in [X_{\min}, X_{\max}] \times [Y_{\min}, Y_{\max}] \times [0, T]. \quad (4.3)$$

Theoretically, to eliminate the boundary effect, $X_{\max}(Y_{\max})$ should be sufficiently large and $X_{\min}(Y_{\min})$ should be sufficiently small. According to Wilmott et al. (1995), the upper bound of stock price is always three or four times the strike price. Therefore we set $Y_{\max} = \ln(5K)$ and $Y_{\min} = -Y_{\max}$ so that $S_{\min} = \frac{1}{5K} \approx 0$. As for the buy-in rate, we set $\lambda_{\max} = 252$, which means that buy-ins occur every day at most. Therefore, $X_{\min} = -X_{\max} = -\ln(252)$ so that $\lambda_{\min} = \frac{1}{252} \approx 0$. The space (x, y, τ) is divided into a uniform grid with

$$\begin{aligned}x_i &= X_{\min} + (i - 1) \cdot \Delta x, i = 1, \dots, N_x; \\ y_j &= Y_{\min} + (j - 1) \cdot \Delta y, j = 1, \dots, N_y; \\ \tau_l &= (l - 1) \cdot \Delta \tau, l = 1, \dots, N_T;\end{aligned}$$

where N_x, N_y, N_T are the number of grid points in the x, y, τ , respectively. The option values at the grid points thus are $u_{i,j}^l = u(x_i, y_j, \tau_l)$.

After meshing the computational domain, we provide two different treatments for the jump term $\overline{\mathcal{L}_2}u$ in PDE (4.1). They are marked as Method 1 and Method 2, respectively.

4.1.1 Method 1

Upon discretizing the computational domain, the discretization associated with the jump term requires values of the unknown functions being evaluated off grid points. In Method 1, we apply bilinear interpolation to approximate $u(x - \beta\gamma, y + \ln(K(1 - \gamma)), \tau)$. For convenience of notation, the value on point P is denoted as $u_P(x, y)$, while the grid points around point P are denoted as $Q_{1,1} = (x_1, y_1), Q_{1,2} = (x_1, y_2), Q_{2,1} = (x_2, y_1), Q_{2,2} = (x_2, y_2)$ as shown in Figure 1. According to the bilinear interpolation method, the value on point P can be approximated as

$$u_P(x, y) \approx \frac{1}{(x_2 - x_1)(y_2 - y_1)}(x_2 - x, x - x_1) \begin{pmatrix} u_{Q_{1,1}} & u_{Q_{1,2}} \\ u_{Q_{2,1}} & u_{Q_{2,2}} \end{pmatrix} \begin{pmatrix} y_2 - y \\ y - y_1 \end{pmatrix}, \quad (4.4)$$

where $u_{Q_{i,j}}$ are the values on the grid points $Q_{i,j}, i, j = 1, 2$. According to the interpolation

Figure 1: Interpolation grid

formula, the jump term in the n -th time step is approximated as

$$\overline{\mathcal{L}_2} u^n \approx e^x [u_P^n(x - \beta\gamma, y + \ln(K(1 - \gamma)) - u^n].$$

The Alternative Direction Implicit (ADI) scheme is applied to the operator $\overline{\mathcal{L}_1}$ and an explicit scheme to the operator $\overline{\mathcal{L}_2}$. When discretizing the operator $\overline{\mathcal{L}_1}$ in first step, only the derivatives with respect to x are evaluated in terms of the unknown values of u^{2n+1} , while the other derivatives are replaced in terms of known values of u^{2n} . The difference equation obtained in first step is implicit in the x -direction and explicit in y -direction. This procedure is then repeated at next step with the difference equation implicit in the y -direction and explicit in the x -direction. Both operator $\overline{\mathcal{L}_2}$ and cross-derivative $\frac{\partial u^2}{\partial x \partial y}$ are approximated explicitly. Thus, we obtain two difference equations,

$$\begin{aligned} \frac{u_{i,j}^{2n+1} - u_{i,j}^{2n}}{\Delta\tau} = & \hat{a} \frac{u_{i+1,j}^{2n+1} - 2u_{i,j}^{2n+1} + u_{i-1,j}^{2n+1}}{\Delta x^2} + \hat{c}_i \frac{u_{i+1,j}^{2n+1} - u_{i-1,j}^{2n+1}}{2\Delta x} - \frac{r}{2} u_{i,j}^{2n+1} \\ & + \hat{b} \frac{u_{i,j+1}^{2n} - 2u_{i,j}^{2n} + u_{i,j-1}^{2n}}{\Delta y^2} + \hat{d} \frac{u_{i,j+1}^{2n} - u_{i,j-1}^{2n}}{2\Delta y} - \frac{r}{2} u_{i,j}^{2n} \\ & + \hat{\varrho} \frac{u_{i+1,j+1}^{2n} - u_{i-1,j+1}^{2n} - u_{i+1,j-1}^{2n} + u_{i-1,j-1}^{2n}}{4\Delta x \Delta y} + \overline{\mathcal{L}_2} u_{i,j}^{2n}, \end{aligned} \quad (4.5)$$

$$\begin{aligned} \frac{u_{i,j}^{2n+2} - u_{i,j}^{2n+1}}{\Delta\tau} = & \hat{b} \frac{u_{i,j+1}^{2n+2} - 2u_{i,j}^{2n+2} + u_{i,j-1}^{2n+2}}{\Delta y^2} + \hat{d} \frac{u_{i,j+1}^{2n+2} - u_{i,j-1}^{2n+2}}{2\Delta y} - \frac{r}{2} u_{i,j}^{2n+2} \\ & + \hat{a} \frac{u_{i+1,j}^{2n+1} - 2u_{i,j}^{2n+1} + u_{i-1,j}^{2n+1}}{\Delta x^2} + \hat{c}_i \frac{u_{i+1,j}^{2n+1} - u_{i-1,j}^{2n+1}}{2\Delta x} - \frac{r}{2} u_{i,j}^{2n+1} \\ & + \hat{\varrho} \frac{u_{i+1,j+1}^{2n+1} - u_{i-1,j+1}^{2n+1} - u_{i+1,j-1}^{2n+1} + u_{i-1,j-1}^{2n+1}}{4\Delta x \Delta y} + \overline{\mathcal{L}_2} u_{i,j}^{2n+1}, \end{aligned} \quad (4.6)$$

where $\hat{a} = \frac{\kappa^2 + \beta^2 \sigma^2 + 2\kappa\beta\sigma\rho}{2}$, $\hat{b} = \frac{\sigma^2}{2}$, $\hat{c}_i = \alpha(\bar{x} - x_i) + \beta r$, $\hat{d} = r - \frac{\sigma^2}{2}$ and $\hat{\varrho} = \beta\sigma^2 + \rho\kappa\sigma$. The corresponding matrix form for the above equations can be simply written as

$$\mathbf{H}_1 u_j^{2n+1} = P_j^{2n} + \mathbf{x} \mathbf{B} \mathbf{n} d_j^1, \quad (4.7)$$

$$\mathbf{H}_2 u_i^{2n+2} = Q_i^{2n+1} + \mathbf{y} \mathbf{B} \mathbf{n} d_i^1, \quad (4.8)$$

where the matrices $\mathbf{H}_1$ and $\mathbf{H}_1$ are both tridiagonal⁴. Therefore, the Thomas algorithm can be adopted to accelerate the computational speed (Strikwerda, 2004).

4.1.2 Method 2

The other method to deal with the jump term $\overline{\mathcal{L}}_2 u$ is a second-order Taylor expansion. After applying Taylor expansion, we have

$$\begin{aligned}\overline{\mathcal{L}}_2 u &= e^x [u(x - \beta\gamma, y + \ln(K(1 - \gamma)), t) - u(x, y, t)] \\ &= e^x [-\beta\gamma \frac{\partial u}{\partial x} + \ln(K(1 - \gamma)) \frac{\partial u}{\partial y} + \frac{\beta^2 \gamma^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\ln^2(K(1 - \gamma))}{2} \frac{\partial^2 u}{\partial y^2} - \beta\gamma \ln(K(1 - \gamma)) \frac{\partial^2 u}{\partial x \partial y}] \\ &\quad + o(e^x (\beta^2 \gamma^2 + \ln^2(K(1 - \gamma)))) .\end{aligned}$$

With the high-order terms being dropped out, the PDE (4.1) becomes

$$\frac{\partial u}{\partial \tau} = \bar{a} \frac{\partial^2 u}{\partial x^2} + \bar{b} \frac{\partial^2 u}{\partial y^2} + \bar{c} \frac{\partial u}{\partial x} + \bar{d} \frac{\partial u}{\partial y} + \bar{\varrho} \frac{\partial^2 u}{\partial x \partial y} - ru, \quad (4.9)$$

with new coefficients $\bar{a} = \frac{\kappa^2 + \beta^2 \sigma^2 + 2\kappa\beta\sigma\rho + \beta^2 \gamma^2 e^x}{2}$, $\bar{b} = \frac{\sigma^2 + \ln^2(K(1 - \gamma))e^x}{2}$, $\bar{c} = \alpha(\bar{x} - x) + \beta r - e^x \beta \gamma$, $\bar{d} = r - \frac{\sigma^2}{2} + e^x \ln(K(1 - \gamma))$, and $\bar{\varrho} = \beta \sigma^2 + \rho \kappa \sigma - \ln(K(1 - \gamma)) \beta \gamma e^x$. Since the PDE (4.9) does not include the jump term any longer, the ADI scheme can be directly applied. The finite difference equation for the PDE (4.9) is of the form

$$(I - \theta A_1)(I - \theta A_2)u^{n+1} = [I + A_0 + (1 - \theta)A_1 + (1 - \theta)A_2 + \theta^2 A_1 A_2]u^n. \quad (4.10)$$

The details of the derivation for (4.10) are left in Appendix B.

Another ADI scheme, Douglas-Rachford (DR) method (Douglas and Rachford, 1956), is applied to solve the finite difference equation (4.10). The DR method involves two steps,

⁴It is standard to rewrite the difference schemes (4.5)-(4.6) as matrix form (4.7)-(4.8). As requested by an anonymous referee, we omit the definitions of $\mathbf{H}_1$, $\mathbf{H}_1$, $\mathbf{xBnd}^1$ and $\mathbf{yBnd}^1$ in (4.7)-(4.8). Similarly, we also omit the definitions of $A, P_j, B, Q_i, \mathbf{xBnd}^2$ and $\mathbf{yBnd}_i^2$ in (4.13)-(4.14). Readers are referred to Zhu and Chen (2011) for the details.

in which the original operator in (4.10) is split into two that are applied in two spatial directions, respectively. First, we calculate an intermediate variable Z from

$$(I - \theta A_1)Z = [I + A_0 + (1 - \theta)A_1 + A_2]u^n, \quad (4.11)$$

with values in the y direction fixed. After obtaining the intermediate variable Z , the second step is to calculate u^{n+1} from

$$(I - \theta A_2)u^{n+1} = Z - \theta A_2 u^n, \quad (4.12)$$

by fixing the value in x direction. The matrix form for (4.11) and (4.12) are represented as

$$AZ_j = P_j + \mathbf{x} \mathbf{B} \mathbf{d}_j^2, \quad (4.13)$$

$$Bu_i = Q_i + \mathbf{y} \mathbf{B} \mathbf{d}_i^2. \quad (4.14)$$

The von Neumann stability analysis is restricted to the PDE with constant coefficients in general. It is extended to the PDE with variable coefficients with the frozen coefficient technique (Zhu and Chen, 2011). By fixing the coefficients at their values attained at each grid point in the computational domain, the variable coefficients problem becomes constant coefficient ones. If each frozen coefficient problem is stable, then the variable coefficient problem is also stable (Strikwerda, 2004). The next proposition demonstrate the stability of schemes (4.11) and (4.12).

Theorem 4.1. When $\theta \geq \frac{1}{2}$, schemes (4.11) and (4.12) for the PDE (4.9) is unconditionally stable in von Neumann sense.

Proof. Following the idea of von Neumann stability analysis (Strikwerda, 2004), $u_{k,m}^n$ in (4.11) and (4.12) is expressed by $g_1^n e^{ik\phi} e^{im\psi}$ and $Z_{k,m}$ by $g_1^n g_2 e^{ik\phi} e^{im\psi}$, where g_1 is the amplification factor of (4.11) and g_2 is the amplification factor of (4.12), with $\phi, \psi \in [-\pi, \pi]$.

Therefore, the Equations (4.11) and (4.12) are transformed to

$$\begin{aligned} g_2(1 - \theta z_1) &= 1 + z_0 + (1 - \theta)z_1 + z_2, \\ g_1(1 - \theta z_2) &= g_2 - \theta z_2. \end{aligned}$$

After simple calculations, we obtain $g_1 = 1 + \frac{z_0 + z_1 + z_2}{(1 - \theta z_2)(1 - \theta z_1)}$, where $z_0 = -\frac{\bar{\varrho}\Delta\tau}{\Delta x\Delta y}\sin\phi\sin\psi$, $z_1 = -\frac{4\bar{a}\Delta\tau}{\Delta x^2}\sin^2\frac{\phi}{2} - \frac{r\Delta\tau}{2} + i\frac{\bar{c}\Delta\tau}{\Delta x}\sin\phi$ and $z_2 = -\frac{4\bar{b}\Delta\tau}{\Delta y^2}\sin^2\frac{\psi}{2} - \frac{r\Delta\tau}{2} + i\frac{\bar{d}\Delta\tau}{\Delta y}\sin\psi$. It is easy to check that the coefficients satisfy

$$\begin{aligned} &4\bar{a}\bar{b} - \bar{\varrho}^2 \\ &= [\kappa^2 + \beta^2\sigma^2 + 2\kappa\beta\sigma\rho + \beta^2\gamma^2e^x][\sigma^2 + \ln^2(K(1 - \gamma))e^x] - [\beta\sigma^2 + \rho\kappa\sigma - \ln(K(1 - \gamma))\beta\gamma e^x]^2 \\ &= (1 - \rho^2)\kappa^2 + e^x\{(1 - \rho^2)\kappa^2 + [\beta\gamma\ln(K(1 - \gamma)) - (\beta\sigma^2 + \rho\kappa\sigma)]^2\} \\ &\geq 0. \end{aligned}$$

Consequently, we come to

$$\begin{aligned} 4\mathcal{R}(z_1)\mathcal{R}(z_2) - |z_0|^2 &\geq \frac{16ab\Delta\tau^2}{\Delta x^2\Delta y^2}\sin^2\frac{\phi}{2}\sin^2\frac{\psi}{2} - \frac{\Delta\tau^2\bar{\varrho}^2}{\Delta x^2\Delta y^2}\sin\phi\sin\psi \\ &= \frac{4\Delta\tau^2}{\Delta x^2\Delta y^2}\sin^2\frac{\phi}{2}\sin^2\frac{\psi}{2}(4\bar{a}\bar{b} - \bar{\varrho}^2\cos^2\frac{\phi}{2}\cos^2\frac{\psi}{2}) \\ &\geq \frac{4\Delta\tau^2}{\Delta x^2\Delta y^2}\sin^2\frac{\phi}{2}\sin^2\frac{\psi}{2}(4\bar{a}\bar{b} - \bar{\varrho}^2) \\ &\geq 0. \end{aligned}$$

Define vectors $\mathbf{v}_i = (\sqrt{-2\mathcal{R}(z_i)}, \frac{|1+\theta z_i|}{\sqrt{2\theta}})^\top$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{|1-\theta z_1||1-\theta z_2|}{2\theta} &= \|\mathbf{v}_1\|\|\mathbf{v}_2\| \geq \mathbf{v}_1 \cdot \mathbf{v}_2 = 2\sqrt{\mathcal{R}(z_1)\mathcal{R}(z_2)} + \frac{(1+\theta z_1)(1+\theta z_2)}{2\theta} \\ &\geq |z_0| + \frac{(1-\theta z_1)(1-\theta z_2)}{2\theta} + z_1 + z_2. \end{aligned}$$

Dividing both sides with $|1 - \theta z_1||1 - \theta z_2|$, we have

$$\frac{1}{2\theta} \geq \frac{|z_0|}{|1 - \theta z_1||1 - \theta z_2|} + \left| \frac{1}{2\theta} + \frac{z_1 + z_2}{|1 - \theta z_1||1 - \theta z_2|} \right| \geq \left| \frac{1}{2\theta} + \frac{z_1 + z_2}{|1 - \theta z_1||1 - \theta z_2|} \right|. \quad (4.15)$$

Finally, we have such an estimate for the amplification factor g_1

$$\begin{aligned}
|g_1| &= \left| 1 + \frac{z_0 + z_1 + z_2}{(1 - \theta z_1)(1 - \theta z_2)} \right| \\
&= \left| 1 - \frac{1}{2\theta} + \frac{1}{2\theta} + \frac{z_0 + z_1 + z_2}{(1 - \theta z_1)(1 - \theta z_2)} \right| \quad (\theta \geq \frac{1}{2}) \\
&\leq 1 - \frac{1}{2\theta} + \left| \frac{1}{2\theta} + \frac{z_0 + z_1 + z_2}{(1 - \theta z_1)(1 - \theta z_2)} \right| \\
&\leq 1.
\end{aligned}$$

Therefore, the scheme of DR method (4.11) and (4.12) is unconditionally stable. $\square$

4.2 The Monte Carlo simulation

Although Avellaneda and Lipkin (2009) obtained an elegant pricing formula (2.7) with the independence assumption, it is hard to compute option price directly because it still involves the calculation of a series of weight functions. In this subsection, we demonstrate how to implement a Monte Carlo scheme to approximate the weight functions. Recall the definition of the weight functions,

$$\Pi(n, T) = P\left(\int_0^T dN_{\lambda_t} = n\right) = \mathbf{E}\left\{e^{-\int_0^T \lambda(t, \omega) dt} \frac{[\int_0^T \lambda(t, \omega) dt]^n}{n!}\right\}. \quad (4.16)$$

If the intensity λ_t of Poisson process is a constant or a deterministic function, the weight functions can be calculated through the above formula directly. When the intensity is described by a stochastic process, we approximate the probability by frequency over a large number of paths. Here, we present a scheme of Monte Carlo simulation for the Poisson process whose intensity follows (2.6).

In our scheme, the time axis in the domain $[0, T]$ is first discretized into a set of discrete nodal points, $0 = t_0 < t_1 < \dots < t_L = T$, with $\Delta t = t_{i+1} - t_i = \frac{T}{L}$ and L being the total number of the nodes. All the values of λ_t or x_t on the nodes $\{t_i\}_{i=1}^L$ need to be obtained via Monte Carlo simulations. Let us take the calculation of λ_{t_1} or x_{t_1} as an example to illustrate the simulation process. The pseudocode is presented in Algorithm 1. Once obtaining the

value of x_{t_1} , we can move on to next node x_{t_2} and repeat Algorithm 1. Similarly, we can simulate a path for x_t and obtain the total number of jumps for each path.

Algorithm 1 Monte Carlo simulation for SDE (2.6)

- 1: Set $\lambda_0 = e^{x_0}$, $k = 0$;
 - 2: Generate a random number τ from exponential distribution with parameter λ_0 ;
 - 3: $k = k + 1$ and $\tau_1 = \tau$;
 - 4: **if** $\tau_1 > t_1$ **then**
 - 5: Generate a random number $\tilde{Z}$ from normal distribution $\mathcal{N}(0, t_1)$;
 - 6: Set $x_{t_1} = x_{t_0} + [\alpha(\bar{x} - x_{t_0}) + \beta r](t_1 - t_0) + \kappa \tilde{Z}$;
 - 7: **else if** $\tau_1 < t_1$ **then**
 - 8: **repeat**
 - 9: Generate a random number τ from exponential distribution with parameter λ_{k-1} ;
 - 10: $\tau_k = \tau_{\tau_{k-1}} + \tau$ and $k = k + 1$;
 - 11: Generate a random number $\tilde{Z}$ from normal distribution $\mathcal{N}(0, \tau)$;
 - 12: $x_{\tau_k} = x_{\tau_{k-1}} + [\alpha(\bar{x} - x_{\tau_{k-1}}) + \beta r](\tau_k - \tau_{k-1}) + \kappa \tilde{Z} - \beta \gamma$;
 - 13: **until** for some k , $\tau_k > t_1$
 - 14: Generate a random number $\tilde{Z}$ from normal distribution $\mathcal{N}(0, t_1 - \tau_{k-1})$;
 - 15: Set $x_{t_1} = x_{t_{\tau_{k-1}}} + [\alpha(\bar{x} - x_{\tau_{k-1}}) + \beta r](t_1 - \tau_{k-1}) + \kappa \tilde{Z}$;
 - 16: **end if**
-

Upon simulating paths, we have a summary of the number of jumps for each path. Using these data, we can approximate the weight functions as

$$\Pi(n, T) \approx \frac{\text{Number of paths with } n \text{ jumps}}{\text{Number of total paths}}. \quad (4.17)$$

The remaining calculation is simple and straightforward by substituting the estimated weight functions (4.17) into the formula (2.7) to obtain the option price. When Monte Carlo simulations are implemented, the standard deviation and confidence interval of the option value can also be calculated. The numerical results are reported in next section.

5 Numerical results and discussions

This section consists of three subsections. In first subsection, the Monte Carlo scheme is implemented to estimate the weight functions which are involved in the semi-explicit

pricing formula. In the second subsection, we first demonstrate the convergence of the two different numerical schemes for the PDE approach. Finally, comparisons are carried out between the numerical results from A&L's semi-explicit pricing formula and those from both the PDE (3.4) for the full model and the PDE (3.6) for the simplified model. Unless stated otherwise, the model parameters in this section are $T = 0.5, \sigma = 0.45, \alpha = 2, \kappa = 0.2, \bar{x} = \ln(10), \lambda_0 = 12, \gamma = 0.01, \beta = 1, \rho = 0, K = 10$.

5.1 The implementation of the semi-explicit pricing formula

In order to obtain numerical results from A&L's semi-explicit formula, a series of weight functions $\{\Pi(n, T)\}_{n \geq 0}$ need to be computed first. This is achieved through Monte Carlo simulations. A simulated path for λ_t is shown in Figure 2. When a buy-in occurs, the intensity or buy-in rate drops down to a low level. With time moving on, it accumulates gradually to an average level.

Figure 2: A simulated path for λ_t .

The intensity x_t or λ_t is simulated Q times consecutively starting with a lower $Q = 100$ until the convergency has been achieved when Q reaches 1,000,000. To demonstrate the convergency of the calculated $\Pi(n, T)$, the variation of values as a function of the number of Monte Carlo simulation paths is tabulated in Table 1 for $n = 6, 7, 8, 9, 10$. Clearly, for

all these five functions, an accuracy at the 3rd decimal place has been achieved when the number of Monte Carlo simulations has reached 1,000,000. Therefore, the results obtained when $Q = 1,000,000$ are adopted to carry out the subsequent calculations of the option price using the semi-explicit pricing formula. These weight functions listed in Table 1 are just some examples of the series of weight functions. All of them can be computed with the approximation (4.17). The subsequently calculated option values from the A&L's semi-

Q	$\Pi(6, T)$	$\Pi(7, T)$	$\Pi(8, T)$	$\Pi(9, T)$	$\Pi(10, T)$
100	0.16	0.11	0.09	0.07	0.05
1,000	0.162	0.114	0.081	0.045	0.032
10,000	0.1609	0.1277	0.083	0.043	0.0306
100,000	0.1598	0.1231	0.0829	0.0497	0.0268
1,000,000	0.1593	0.1228	0.0825	0.0490	0.0262

Table 1: Convergence of the weight functions.

explicit pricing formula and the standard derivations for our Monte Carlo Simulations are both tabulated in Table 2.

stock price	$S_0 = 8$	$S_0 = 9$	$S_0 = 10$	$S_0 = 11$	$S_0 = 12$
option value	0.3299	0.6405	1.0767	1.6302	2.2842
standard deviation	0.00045	0.00079	0.00117	0.00157	0.00197

Table 2: The values calculated from A&L's pricing formula .

5.2 Numerical results for the simplified model

In this subsection, we reported numerical results for the simplified model through solving the PDE (3.6). Comparisons are also carried out with those from A&L's pricing formula.

Both Method 1 and Method 2 are implemented to solve the PDE (3.6) on different grids in order to observe the convergence of the computed results as grid space becomes fine. To measure the convergence, the l_2 error between the value on a fine grid and the coarse grid at the previous level is also listed in Table 3. As shown in Table 3, the l_2 error decreases both in Method 1 and Method 2 as the size of grid spacing diminishes, indicating a significant convergence of both methods.

Method	(N_x, N_y, N_T)	$S_0 = 8$	$S_0 = 9$	$S_0 = 10$	$S_0 = 11$	$S_0 = 12$	l_2 error
Method 1	(20,80,100)	0.3642	0.6829	1.1256	1.6864	2.3471	
	(40,160,200)	0.3425	0.6560	1.0967	1.6518	2.3040	0.0712
	(80,320,300)	0.3342	0.6456	1.0834	1.6372	2.2913	0.0270
	(160,640,400)	0.3304	0.6408	1.0770	1.6303	2.2840	0.0134
	(320,1280,500)	0.3299	0.6403	1.0762	1.6294	2.2831	0.0017
Method 2 $\theta = 0.5$	(20,40,40)	0.3127	0.6490	1.0144	1.6409	2.2583	
	(40,80,50)	0.3384	0.6488	1.0846	1.6401	2.2965	0.0840
	(80,160,60)	0.3313	0.6409	1.0781	1.6311	2.2824	0.0208
	(160,320,70)	0.3301	0.6401	1.0766	1.6298	2.2834	0.0027
	(320,640,80)	0.3298	0.6400	1.0761	1.6293	2.2830	0.0001

Table 3: Comparison of values calculated with different grids.

In order to obtain the order of convergence in one direction, we study the ratios of the consecutive errors with the grid spacing along this direction being successively decreased, while the grid spacing in other directions is fixed. Since there is no analytical solution for the PDE (3.6), we construct a benchmark solution for Method 1 with the number of grids defined as $(N_x, N_y, N_T) = (320, 1280, 500)$ and a benchmark solution for Method 2 with the number of grids defined as $(N_x, N_y, N_T) = (320, 640, 80)$. The Experimental Order of Convergence (EOC) appearing in the $i + 1$ th row in Tables 4 and 5 are defined as

$$\text{EOC}_{i+1} = \frac{\ln \text{error}_i - \ln \text{error}_{i+1}}{\ln N_j^{i+1} - \ln N_j^i}, \quad j = x, y,$$

where error_i and N_j^i are the l_2 error and the number of grid in the i th row, respectively. Similarly, the EOCs in x and y directions for Method 1 and Method 2 are listed in Tables 4 and 5.

(a) EOC in x direction			(b) EOC in y direction		
N_x	l_2 error	EOC	N_y	l_2 error	EOC
30	0.0010		10	1.2481	
40	7.65×10^{-4}	1.05	20	0.4267	1.55
50	6.69×10^{-4}	0.61	30	0.2255	1.57
70	5.59×10^{-4}	0.54	40	0.1496	1.43

Table 4: EOC in each direction for Method 1

(a) EOC in x direction			(b) EOC in y direction		
N_x	l_2 error	EOC	N_y	l_2 error	EOC
20	0.0047		10	0.7282	
40	0.0021	2.02	20	0.2128	1.77
60	9.64×10^{-4}	1.86	40	0.0727	1.55
80	5.04×10^{-5}	2.25	80	0.0161	2.18

Table 5: EOC in each direction for Method 2

From Table 5, the experimental order of convergence for Method 2 are approximately two, which implies that Method 2 is second-order convergent in both x and y directions. Comparing Tables 4 and 5, we can conclude that the results produced with Method 1 converges significantly slower than those with Method 2 due to the fact that the explicit treatment of the jump term in Method 1 really slows down the convergence rate.

On the other hand, Method 1 requires much finer grid than Method 2 and thus more time in order to reach almost the same level of convergence. As far as the total computational time associated with each method is concerned, the total consumed CPU time for a particular run is adopted to measure computational efficiency, which is complemented by a measure of relative error defined as

$$\text{Relative Error I} = \frac{\|V_{\text{PDE}} - V_{\text{A\&L}}\|_2}{\|V_{\text{A\&L}}\|_2} \times 100\%,$$

where V_{PDE} denotes the option value obtained from the PDE (3.6) with Method 1 or Method 2 and $V_{\text{A\&L}}$ represents the value calculated from A\&L's formula. Recorded in Table 6 are the CPU time and Relative Error I.

From Table 6, we find that Relative Error I decreases close to zero as the size of grid is diminished, which indicates that the numerical results from the PDE (3.6) are consistent with those from A\&L's pricing formula because both of them are solution for the simplified model with the independence assumption. In addition, Method 1 is more time-consuming than Method 2, in order to reach almost the same level of accuracy. This is due to the adopted numerical scheme. In fact, Method 1 is a modified version of implicit-explicit

Methods	(N_x, N_y, N_T)	CPU times (s)	Relative Error I(%)
Method 1	(20,80,100)	0.229	3.87
	(40,160,200)	1.386	1.56
	(80,320,300)	9.102	0.69
	(160,640,400)	101.7	0.26
	(320,1280,500)	1130	0.20
Method 2 $\theta = 0.5$	(20,40,40)	0.239	2.18
	(40,80,50)	0.461	0.93
	(80,160,60)	1.53	0.27
	(160,320,70)	8.17	0.22
	(320,640,80)	84.9	0.19
A&L	$Q = 1,000,000$	250.1	

Table 6: Comparison of CPU time and Relative Error I.

(IMEX) scheme. We apply an ADI scheme to the operator $\overline{\mathcal{L}_1}$, and an explicit scheme to the operator $\overline{\mathcal{L}_2}$. Such a scheme with hybrid feature is more efficient than a fully explicit scheme because the ADI scheme can speed up the process of convergence. However, the explicit treatment of the jump term slows down the total speed of convergence. As for the Method 2, an ADI scheme is applied after adopting the second-order Taylor expansion. It has been proved that the ADI scheme is unconditionally stable when $\theta \geq \frac{1}{2}$. Therefore, the results of Method 2 exhibit faster convergence than those of Method 1.

However, in Method 2, a second-order Taylor expansion is adopted to approximate the jump term. Actually, this indicates that we have implicitly assumed that γ is of a small value, so that the high-order terms can be dropped out. This implicit assumption directly affects the accuracy of Method 2. The numerical error of Method 2 includes not only the truncation error led by the finite difference scheme but also the approximation error introduced by adopting the second-order Taylor expansion. The truncation error can be eliminated gradually when the size of grids become small, while the approximation error cannot. Therefore, its accuracy will be significantly affected by the value of γ with large relative error for great γ . As for Method 1, the error source is the truncation error and interpolation error which do not heavily depend on the value of γ . Define the relative error

Figure 3: Relative Error II for different values of γ .

between Methods 1 and 2 as

$$\text{Relative Error II} = \frac{\|V_1 - V_2\|_2}{\|V_2\|_2} \times 100\%,$$

where V_i represents the results with Method $i = 1, 2$. Such a relative error is shown in Figure 3 with different values of γ , the price elasticity of demand due to buy-ins. It is observed that the relative error is still acceptable (less than 0.5%) when γ is smaller than 0.05. As it increases from 0.05 to 0.1, the relative error goes up significantly, which implies that the results for Method 2 depart from those from Method 1 gradually. Although Method 2 shows a good convergence rate in Table 5 and more efficiency in Table 6, Figure 3 shows us that Method 1 is more accuracy when the value of γ is large.

5.3 Comparisons between the full and simplified models

In the subsection, we numerically demonstrate the relations between the full and simplified models through comparing the results from the PDE (3.4) for the full model, the PDE (3.6) for the simplified model, and the A&L pricing formula.

In Figure 4(a), the option price for $S_0 = 9$ are shown with different values of coupling

parameter β with different solution approaches. It is easy to observe that the results from the PDE (3.6) numerically agree with those from the A&L's pricing formula. Theoretically, they are consistent with each other because both of them are solution for the simplified model with the independence assumption, which also explains why both of them do not change too much as β increases. On the other hand, the results from the PDE (3.4)

Figure 4: Comparisons among the PDE (3.4), PDE (3.6) and the A&L's formula (2.7).

are affected by the value of the coupling parameter β . Let V_{full} and V_{simp} represent the results from the PDE (3.4) for the full model and the PDE (3.6) for the simplified model, respectively. The relative error between the PDEs (3.4) and (3.6) is defined as

$$\text{Relative Error III} = \frac{\|V_{\text{full}} - V_{\text{simp}}\|_2}{\|V_{\text{full}}\|_2} \times 100\%,$$

which is depicted in Figure 4(b). As β increases from 0 to 5, we find that the relative error becomes more and more significantly. In other words, when β is large, which means the buy-in rate depends heavily on the change of stock price, it is unreasonable to make such an independence assumption. We should solve the PDE (3.4) for the full model instead of the PDE (3.6) or the A&L's pricing formula for the simplified model with the independence assumption.

6 Conclusions

In this paper, we apply the PDE approach to price European call options under the A&L model. Two numerical methods are proposed to solve the PDE system based on different approaches to the jump term. We have also numerically realized A&L's semi-explicit pricing formula via Monte Carlo simulations for the buy-in rate.

The PDE approach is a broader way than A&L's pricing formula for it does not require an independence assumption imposed by Avellaneda and Lipkin (2009). The results from the PDE (3.6) are consistent with those from A&L's formula, which implies that our PDE approach works for the simplified model, as well. Comparisons between the simplified and full models show that the A&L's pricing formula is an efficient tool and a good approximate solution to calculate the option price when the coupling parameter is small. However, when the stock price and the buy-in rate are significantly coupled, the PDE approach is preferred for solving the full hard-to-borrow model as demonstrated in this paper.

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Appendix A Proof of Theorem 3.1

Under the risk-neutral measure, the dynamics are shown in Equation (2.4). According to the definition of European call options, we have

$$u(x, S, t) = \mathbf{E}_t[e^{-r(T-t)}h(S_T)] = \mathbf{E}[e^{-r(T-t)}h(S_T)|\mathcal{G}_t], \quad (\text{A.1})$$

where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, $\mathcal{F}_t$ is the filtration generated by the standard Brownian motion W_t and Z_t and $\mathcal{H}_t$ is the filtration generated by the Poisson process N_{λ_t} . Supposing that $0 \leq s \leq t \leq T$, using Equation (A.1), we obtain

$$\mathbf{E}[e^{-rt}u(x_t, S_t, t)|\mathcal{G}_s] = \mathbf{E}[e^{-rT}h(S_T)|\mathcal{G}_s] = e^{-rs}\mathbf{E}[e^{-r(T-s)}h(S_T)|\mathcal{G}_s] = e^{-rs}u(x_s, S_s, s).$$

Therefore, $e^{-rt}u(x_t, S_t, t)$ is a martingale. By applying Ito's formula to $e^{-rt}u(x_t, S_t, t)$, we have

$$\begin{aligned} d(e^{-rt}u(x_t, S_t, t)) &= e^{-rt}du(x_t, S_t, t) - re^{-rt}u(x_t, S_t, t)dt \\ &= e^{-rt}[u_t dt + u_S dS_t + u_x dx_t + \frac{u_{SS}(dS_t)^2 + u_{xx}(dx_t)^2}{2} + u_{xS}dx_t dS_t \\ &\quad + [u(x_t - \beta\gamma, S_t(1 - \gamma), t) - u(x_t, S_t, t)]dN_{\lambda_t}] - re^{-rt}u(x_t, S_t, t)dt \\ &= e^{-rt}\{u_t + rS_t u_S + [\alpha(\bar{x} - x_t) + \beta r]u_x + \frac{1}{2}\sigma^2 S_t^2 u_{SS} + \frac{\kappa^2 + \sigma^2 \beta^2 + 2\kappa\beta\sigma\rho}{2}u_{xx} + (\beta\sigma^2 + \rho\kappa\sigma)S_t u_{xS} \\ &\quad - ru\}dt + e^{-rt}[(u_S \sigma S + u_x \beta \sigma)dW_t + \kappa u_x dZ_t] + e^{-rt}[u(x_t - \beta\gamma, S_t(1 - \gamma), t) - u(x_t, S_t, t)]dN_{\lambda_t} \end{aligned}$$

where $u_t = \frac{\partial u(x,S,t)}{\partial t}$, $u_S = \frac{\partial u(x,S,t)}{\partial S}$, $u_x = \frac{\partial u(x,S,t)}{\partial x}$, $u_{xx} = \frac{\partial^2 u(x,S,t)}{\partial x^2}$, $u_{SS} = \frac{\partial^2 u(x,S,t)}{\partial S^2}$, and $u_{xS} = \frac{\partial^2 u(x,S,t)}{\partial x \partial S}$. Based on the martingale representation theorem, we set dt term to be zero and derive

$$-\frac{\partial u}{\partial t} = (\mathcal{L}_1 + \mathcal{L}_2)u,$$

where operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined in (3.2)-(3.3).

Appendix B The derivation for Equation (4.10)

The space discretization is performed first. The governing PDE system (4.9) becomes

$$\frac{\partial u_{i,j}^n}{\partial \tau} = a_i \delta_{xx} u_{i,j}^n + b_i \delta_{yy} u_{i,j}^n + c_i \delta_x u_{i,j}^n + d_i \delta_y u_{i,j}^n + \varrho_i \delta_{xy} u_{i,j}^n - r u_{i,j}^n \quad (\text{B.1})$$

where $(\delta_{xy} u)_{i,j} = \frac{u_{i+1,j+1} + u_{i-1,j-1} - u_{i-1,j+1} - u_{i+1,j-1}}{4\Delta x \Delta y}$ and

$$\begin{aligned} (\delta_{xx} u)_{i,j} &= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} & (\delta_{yy} u)_{i,j} &= \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \\ (\delta_x u)_{i,j} &= \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} & (\delta_y u)_{i,j} &= \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}. \end{aligned}$$

The mixed derivative, the spatial derivatives in the x direction and the spatial derivatives in the y direction are denoted as linear operators A_0, A_1, A_2 with

$$A_0 u_{i,j}^n = \Delta \tau \varrho_i \delta_{xy} u_{i,j}^n, A_1 u_{i,j}^n = \Delta \tau (a_i \delta_{xx} u_{i,j}^n + c_i \delta_x u_{i,j}^n - \frac{r}{2} u_{i,j}^n), A_2 u_{i,j}^n = \Delta \tau (b_i \delta_{yy} u_{i,j}^n + d_i \delta_y u_{i,j}^n - \frac{r}{2} u_{i,j}^n).$$

Thus, the weighted average of fully implicit scheme and explicit scheme is represented as

$$[I - \theta(A_0 + A_1 + A_2)]u^{n+1} = [I + (1 - \theta)(A_0 + A_1 + A_2)]u^n + \mathcal{O}((\Delta \tau)^3). \quad (\text{B.2})$$

Note that when $\theta = 0$ or $\theta = 1$, (B.2) becomes fully explicit or fully implicit respectively. When $\theta = 0.5$, it is equivalent to apply Crank-Nicolson scheme to the time derivative $\frac{\partial u}{\partial \tau}$.

After some simple algebraic calculations, we obtain

$$\begin{aligned} [I - \theta(A_0 + A_1 + A_2) + \theta^2 A_1 A_2] u^{n+1} &= [I + (1 - \theta)A_0 + (1 - \theta)A_1 + (1 - \theta)A_2 + \theta^2 A_1 A_2] u^n \\ &+ \theta^2 A_1 A_2 (u^{n+1} - u^n) + \mathcal{O}((\Delta\tau)^3). \end{aligned} \quad (\text{B.3})$$

As $\theta^2 A_1 A_2 (u^{n+1} - u^n) \sim \mathcal{O}((\Delta\tau)^3)$, taking it into the error term leads to

$$(I - \theta A_1)(I - \theta A_2) u^{n+1} - \theta A_0 u^{n+1} = [I + (1 - \theta)A_0 + (1 - \theta)A_1 + (1 - \theta)A_2 + \theta^2 A_1 A_2] u^n + \mathcal{O}((\Delta\tau)^3).$$

Moving the term $\theta A_0 u^{n+1}$ to the right side results in

$$\begin{aligned} (I - \theta A_1)(I - \theta A_2) u^{n+1} &= [I + A_0 + (1 - \theta)A_1 + (1 - \theta)A_2 + \theta^2 A_1 A_2] u^n \\ &+ \theta A_0 (u^{n+1} - u^n) + \mathcal{O}((\Delta\tau)^3) \\ &= [I + A_0 + (1 - \theta)A_1 + (1 - \theta)A_2 + \theta^2 A_1 A_2] u^n + \mathcal{O}((\Delta\tau)^2). \end{aligned}$$

The finite difference equation for PDE system (B.1) is of the form

$$(I - \theta A_1)(I - \theta A_2) u^{n+1} = [I + A_0 + (1 - \theta)A_1 + (1 - \theta)A_2 + \theta^2 A_1 A_2] u^n. \quad (\text{B.4})$$