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A DICHOTOMY FOR GROUPOID C*-ALGEBRAS

TIMOTHY RAINONE AND AIDAN SIMS

Abstract. We study the finite versus infinite nature of C*-algebras arising from étale groupoids. For an ample groupoid G, we relate infiniteness of the reduced C*-algebra $C^*_r(G)$ to notions of paradoxicality of a K-theoretic flavor. We construct a pre-ordered abelian monoid $S(G)$ which generalizes the type semigroup introduced by Rørdam and Sierakowski for totally disconnected discrete transformation groups. This monoid characterizes the finite/infinite nature of the reduced groupoid C*-algebra of G in the sense that if G is ample, minimal, topologically principal, and $S(G)$ is almost unperforated, we obtain a dichotomy between the stably finite and the purely infinite for $C^*_r(G)$. A type semigroup for totally disconnected topological graphs is also introduced and we prove a similar dichotomy for these graph C*-algebras as well.

1. Introduction

The groupoid C*-algebra construction has been a very fruitful and unifying notion in the theory of operator algebras since Renault's pioneering monograph [40]. Groupoid C*-algebras include all group C*-algebras, all crossed products of commutative C*-algebras by actions and partial actions of groups, inverse-semigroup C*-algebras, AF algebras, and the various Cuntz–Krieger constructions. Even in the seemingly restrictive case of ample groupoids it is known that every Kirchberg algebra that satisfies the Universal Coefficient Theorem is Morita equivalent to the C*-algebra associated to a Hausdorff ample groupoid (see [45]). Groupoids and associated operator algebras have also been used by Connes and others as models for noncommutative topological spaces.

In this piece we study, for a large class of étale groupoids, notions of finiteness, infiniteness, and proper infiniteness, the latter expressed in terms of paradoxical decompositions. Tarski's alternative theorem establishes, for discrete groups, the dichotomy between amenability and paradoxicality. This divide carries over to geometric operator algebras. For example, Rørdam and Sierakowski showed [44] that if a discrete group Γ acts on itself by left-translation, the Roe algebra $C(\beta\Gamma) \rtimes_\lambda \Gamma$ is properly infinite if and only if Γ is paradoxical, which is equivalent to the non-amenability of Γ [44]. In the W*-setting, we know that all projections in a II_1 factor are finite and that the ordering of

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Murray-von-Neumann subequivalence is determined by a unique faithful normal tracial state. By contrast, type III factors admit no traces since all non-zero projections therein are properly infinite. The analogous dichotomy fails for C*-algebras in general as shown by Rørdam's construction [42] of a unital, simple, separable and nuclear algebra which is neither stably finite or purely infinite (the C*-algebraic analog of type III). It is still not known whether the stably finite/purely infinite dichotomy holds for C*-algebras generated by projections. Here we give a partial answer to this question for C*-algebras associated to ample groupoids.

The motivation for such a dichotomy comes from Elliott's program of classification of simple, separable, nuclear C*-algebras by K-theoretic invariants. In the stably finite case the Elliott program has seen stunning advances in recent years [14, 18, 43, 48, 50]. In the purely infinite setting, complete classification was achieved in the groundbreaking results of Kirchberg [25] and Phillips [34]: all Kirchberg algebras (unital, simple, separable, nuclear, and purely infinite) satisfying the Universal Coefficient Theorem (UCT) are classified by their K-theory. For this reason C*-theorists have sought to determine when various C*-constructions yield purely infinite simple algebras. For instance, strong and local boundary actions or certain filling properties displayed in dynamical systems give rise to purely infinite crossed products [2, 23, 29, 46]. For groupoids, there is a locally contracting property, introduced by Anantharaman-Delaroche [1] that guaranteed pure infiniteness of the associated C*-algebra. This has recently been used, for example, to obtain sufficient conditions for pure infiniteness of topological-graph C*-algebras [30].

The notion of paradoxicality, which undergirds Tarki's theorem [49], is paramount in distinguishing the finite from the infinite. This notion was cast in a C*-algebraic framework by Kerr and Nowak [24], Rørdam and Sierakowski [44], and by the first author in [37]. The authors of [44] built a type semigroup $S(X, \Gamma)$ from an action of a discrete group on a compact zero-dimensional space and subsequently tied pure infiniteness of the resulting reduced crossed product to the absence of states on this semigroup.They prove that if a countable, discrete, and exact group Γ acts continuously and freely on the Cantor set X, and the pre-ordered semigroup $S(X, \Gamma)$ is almost unperforated, then the following are equivalent: (i) The reduced crossed product $C(X) \rtimes_\lambda \Gamma$ is purely infinite; (ii) $C(X) \rtimes_\lambda \Gamma$ is traceless; and (iii) $S(X, \Gamma)$ is purely infinite (that is $2\theta \leq \theta$ for every $\theta \in S(X, \Gamma)$). Inspired by their work, the first author extended these results to noncommutative C*-dynamical systems (A, Γ) by constructing an analogous noncommutative type semigroup $S(A, \Gamma)$ [37]. In this paper, we generalise this work in a different direction, to the setting of ample groupoids. That is, we associate to every ample groupoid G a pre-ordered abelian monoid $S(G)$, constructed as an appropriate quotient of the additive monoid of compactly supported locally constant integer valued functions on the unit space. We prove that this monoid is an invariant for equivalence of groupoids, and also that it is isomorphic to the type semigroup $S(X, \Gamma)$ when G is the transformation groupoid $G = X \rtimes \Gamma$.

Our main theorems explore the relationship between the nature of the type semigroup $S(G)$ of a groupoid G and the structure of its reduced C*-algebra $C^*_r(G)$. For instance, when G is ample and minimal, Theorem 6.5 characterizes stable finiteness of $C^*_r(G)$ by means of the non-paradoxical nature of $S(G)$ and also by a coboundary condition. In the amenable case we recover quasidiagonality. That a certain coboundary condition is equivalent to stable finiteness is reminiscent of the work of Pimsner [35] and Brown [10] and also appears in noncommutative C*-systems [12, 36, 39]. In Theorem 7.3 we establish that, again for minimal groupoids with totally disconnected unit space, if every element of $S(G)$ is properly infinite ($2\theta \leq \theta$ for all θ in $S(G)$), then $C^*(G)$ is purely infinite; moreover these conditions are equivalent and coincide with tracelessness provided that $S(G)$ is almost unperforated. We then are able to deduce a stably finite/purely infinite dichotomy for a large class of ample groupoids (see Theorem 7.4). When G is amenable we obtain a quasidiagonal/purely infinite dichotomy. This is the case for infinite-path groupoids that are constructed from topological graphs (see Corollary 8.5).

As we were preparing the final version of this article, the article [5] was posted on the arXiv. In Sections 4 and 5 of that paper, Bönicke and Li have independently obtained a overlapping results, albeit for groupoids with compact unit spaces. Throughout, we have included, where relevant, remarks discussing the relationships between our results and theirs.

The paper is organized as follows. We begin in Section 2 by reviewing the relevant concepts, definitions, and basic results surrounding the theory of groupoids and their algebras. In Section 3 we study notions of paradoxicality displayed in ample groupoids. We construct infinite reduced groupoid C*-algebras and show that stable finiteness is a natural obstruction to paradoxical behavior. Section 4 deals with minimal groupoids, and provides a K-theoretic description of minimality for ample groupoids. In Section 5 we associate to every ample groupoid G a type semigroup $S(G)$ and show that it reflects any paradoxical phenomena present in the groupoid. We show that both isomorphism and equivalence of ample groupoids preserves the type semigroup. In Section 6 we establish our characterization (Theorem 6.5) of stable finiteness for the reduced C*-algebras of minimal ample groupoids with compact unit space, and extend this to non-compact unit spaces using the invariance of the type semigroup under groupoid equivalence. In Section 7 we explore the purely infinite situation. We first pin down a necessary algebraic condition on G for pure infiniteness of $C^*_r(G)$, and thereafter establish that if an ample groupoid G is minimal, topologically principal and has an almost unperforated type semigroup, then the purely infinite nature of its type semigroup characterizes tracelessness as well as pure infiniteness of the C*-algebra (Theorem 7.3). We consequently obtain a dichotomy between the stably finite and the purely infinite for those C*-algebras. As an application we recover the result that principal n-filling groupoids admit purely infinite algebras. Section 8 is divided up into three subsections. First we establish a type semigroup for zero-dimensional topological graphs and obtain a dichotomy between the purely infinite and the quasidiagonal for their associated C*-algebras. Next, we reconcile our definition of the type semigroup $S(G)$ with previous constructions for C*-dynamical systems and for higher-rank graphs. As an application we show that the type semigroup of a dynamical system is invariant under topological orbit equivalence.

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2. Preliminaries and Notation

In this section we provide a brief introduction to the theory of groupoids, topological groupoids, and their reduced C*-algebras. We shall be mainly interested in the étale case. Experts are welcome to move ahead to the next section. There are a variety of good resources on the subject, for example Patterson's book [33], or the work of Reneault [40].

2.1. **Groupoids.** A *groupoid* is a non-empty set G satisfying the following:

G1. There is a distinguished subset $G^{(0)} \subseteq G$, called the unit space. Elements of $G^{(0)}$ are called units.

G2. There are maps $r, s : G \to G^{(0)}$ satisfying $r(u) = s(u) = u$ for all $u \in G^{(0)}$. These maps are called the range and source maps respectively.

G3. Setting $G^{(2)} = \{(\alpha, \beta) \mid \alpha, \beta \in G,\ s(\alpha) = r(\beta)\}$, there is a 'law of composition'

$$m : G^{(2)} \longrightarrow G, \quad m(\alpha, \beta) = \alpha\beta$$

that satisfies

(i) $r(\alpha\beta) = r(\alpha)$, $s(\alpha\beta) = s(\beta)$, for all $(\alpha, \beta) \in G^{(2)}$.

(ii) If (α, β) and (β, γ) belong to $G^{(2)}$, then $(\alpha, \beta\gamma)$ and $(\alpha\beta, \gamma)$ also are in $G^{(2)}$ and $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

(iii) For every $\alpha \in G$, $r(\alpha)\alpha = \alpha = \alpha s(\alpha)$ (note that $(r(\alpha), \alpha)$ and $(\alpha, s(\alpha))$ are in $G^{(2)}$).

G4. For every $\alpha \in G$ there is an 'inverse' $\alpha^{-1} \in G$ (necessarily unique) such that (α, α^{-1}) and (α^{-1}, α) are in $G^{(2)}$ and such that $\alpha\alpha^{-1} = r(\alpha)$, $\alpha^{-1}\alpha = s(\alpha)$.

It follows from definitions that $r(\alpha^{-1}) = s(\alpha)$, $s(\alpha^{-1}) = r(\alpha)$, and that the map $\iota(\alpha) = \alpha^{-1}$ is an involutive bijection of G.

The following notation is common in the groupoid literature. If $A, B \subseteq G$, then

$$AB = \{\alpha\beta \mid \alpha \in A, \beta \in B, r(\beta) = s(\alpha)\} = m\left((A \times B) \cap G^{(2)}\right).$$

Some special cases arise frequently; for instance, if $E \subseteq G$ and $U \subseteq G^{(0)}$ then

$$\begin{aligned} EU =& \{\alpha\beta \mid \alpha \in E, \beta \in U, s(\alpha) = r(\beta)\} = \{\alpha\beta \mid \alpha \in E, \beta \in U, s(\alpha) = \beta\} \\ =& \{\alpha s(\alpha) \mid \alpha \in E, s(\alpha) \in U\} = \{\alpha \mid \alpha \in E, s(\alpha) \in U\} = E \cap s^{-1}(U), \end{aligned}$$

and in this context we will write

$$U_E := r(EU) = \{r(\alpha) \mid \alpha \in E, s(\alpha) \in U\} \subseteq G^{(0)}.$$

As a special case, if u is a unit in G, then

$$G_u := Gu = \{\alpha \in G \mid s(\alpha) = u\}, \quad G^u := uG = \{\alpha \in G \mid r(\alpha) = u\}, \quad G_u^u := G_u \cap G^u.$$

The *isotropy subgroupoid* is defined by $\mathrm{Iso}(G) := \bigcup_{u \in G^{(0)}} G_u^u$.

A map $\varphi : G \to H$ between groupoids is a homomorphism of groupoids if

$$(\alpha, \beta) \in G^{(2)} \Longrightarrow (\varphi(\alpha), \varphi(\beta)) \in H^{(2)} \quad \text{and} \quad \varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta).$$

It is not difficult to show that such a homomorphism satisfies

$$\varphi(G^{(0)}) \subseteq H^{(0)}, \quad \varphi \circ r_G = r_H \circ \varphi, \quad \varphi \circ s_G = s_H \circ \varphi, \quad \varphi \circ \iota_G = \iota_H \circ \varphi.$$

If φ is bijective, then the inverse map $\varphi^{-1} : H \to G$ is easily seen to be a groupoid homomorphism, and such a φ defines a groupoid isomorphism. If φ is an isomorphism, then $\varphi(G^{(0)}) = H^{(0)}$.

We are mainly interested in groupoids that admit a topology. A *topological groupoid* is a groupoid G with a topology for which $G^{(2)} \subseteq G \times G$ is closed (automatic when G is Hausdorff), $m : G^{(2)} \to G$ is continuous, and $\iota : G \to G$ is continuous. It follows that the range and source maps are again continuous, and if G is Hausdorff, it follows that $G^{(0)}$ is closed. An isomorphism of topological groupoids is an isomorphism of groupoids that is also a homeomorphism.

A topological groupoid G is called *minimal* if $\overline{r(G_u)} = G^{(0)}$ for every unit $u \in G^{(0)}$, and is said to be *topologically principal* or *effective* if $\mathrm{Iso}(G)^\circ = G^{(0)}$.

A topological groupoid G is called *étale* if the maps $r, s : G \to G$ are local homeomorphisms. In this setting every $\alpha \in G$ has an open neighborhood $E \subseteq G$ for which $r(E), s(E)$ are open subsets of G, and $r|_E : E \to r(E)$, $s|_E : E \to s(E)$ are homeomorphisms. Such a set E is called an *open bisection*. One shows that the unit space $G^{(0)}$ of an étale groupoid is open and closed in G. If G is a locally compact Hausdorff groupoid, then G is étale if and only if there is a basis for the topology on G consisting of open bisections with compact closure. A topological groupoid is called *ample* if G has a basis of compact open bisections (so every ample groupoid is in particular étale). If G is locally compact, Hausdorff, and étale, then G is ample if and only if $G^{(0)}$ is totally disconnected (see Proposition 4.1 in [15]).

When G is étale we will denote by $\mathcal{B}$ the collection of open bisections, and write $\mathcal{C} \subseteq \mathcal{B}$ for the subcollection of all compact open bisections. It can be shown that both $\mathcal{B}$ and $\mathcal{C}$ are closed under inversion, multiplication, and taking intersections. Note that any open (compact open) $U \subseteq G^{(0)}$ belongs to $\mathcal{B}$ ($\mathcal{C}$). The following facts will surface regularly throughout our work. If E is an open bisection in G, then using the fact that r and s are injective on E, we get $E^{-1}E = s(E)$ and $EE^{-1} = r(E)$. Moreover, if E and F are disjoint open bisections in G, then $E^{-1}F \cap G^{(0)} = \emptyset = EF^{-1} \cap G^{(0)}$. This holds because for any $\alpha, \beta \in G$, $\alpha^{-1}\beta \in G^{(0)}$ implies that $\alpha = \beta$. Finally we note that if E, U belong to $\mathcal{B}$ ($\mathcal{C}$) with $U \subseteq G^{(0)}$, then $EU = (s|_E)^{-1}(U)$ and $U_E = r(EU)$ also belong to $\mathcal{B}$ ($\mathcal{C}$), because $s|_E$ is a homeomorphism and r is an open map.

For convenience we shall henceforth assume that *all topological groupoids are locally compact, second countable, and Hausdorff.*

The theory of topological groupoids offers a unification of several constructions. One motivating special case is that of a topological dynamical system. A *transformation group* is a pair (X, Γ) where X is a locally compact Hausdorff space, Γ is a locally compact Hausdorff group, and $\Gamma \curvearrowright X$ acts continuously by homeomorphisms. Endowed with the product topology, $G = \Gamma \times X$ becomes a locally compact Hausdorff groupoid where the unit space $G^{(0)} := \{(e, x) \mid x \in X\} \cong X$ can be identified with X, and for $t, t' \in \Gamma$, $x, y \in X$, $s(t, x) = x$, $r(t, x) = t.x$, and $m((t', y), (t, x)) = (t't, x)$ provided $t.x = y$. This groupoid is often called the *transformation groupoid* and is denoted by $G = X \rtimes \Gamma$. If Γ is discrete then $X \rtimes \Gamma$ is étale, and if, moreover, X is totally disconnected then $X \rtimes \Gamma$ is ample. In the étale setting one can show that $X \rtimes \Gamma$ is minimal if and only if the action $\Gamma \curvearrowright X$ is minimal (X admits no non-trivial Γ-invariant closed subspaces), and $X \rtimes \Gamma$ is topologically principal if and only if the action $\Gamma \curvearrowright X$ is topologically free (every point in X has nowhere-dense stabilizer). In this dynamical setting Γ naturally acts on functions defined on X through the formula $t.f(x) = f(t^{-1}.x)$, where $f : X \to \mathbb{C}$ is any function. Generalizing this to the groupoid setting we arrive at the following definition.

Definition 2.1. Let G be an étale groupoid and E an open bisection. If $f : G^{(0)} \to \mathbb{C}$ is a function, define $Ef : G^{(0)} \to \mathbb{C}$ by

$$Ef(u) = \begin{cases} f(s(\alpha)) & \text{if } \exists \alpha \in E \text{ with } r(\alpha) = u, \\ 0 & \text{otherwise.} \end{cases}$$

Note that Ef is well defined because $r|_E : E \to G^{(0)}$ is injective. We list a few facts that will surface periodically in our work below. We are always assuming that G is étale. The first easily follows from definitions.

Fact 2.2. *If E is an open bisection, $f, g : G^{(0)} \to \mathbb{C}$ functions, and $z \in \mathbb{C}$, then*

$$E(zf + g) = zEf + Eg.$$

Fact 2.3. *If $E \subseteq G$ is an open bisection and $U \subseteq G^{(0)}$, then $E\mathbb{1}_U = \mathbb{1}_{r(EU)} = \mathbb{1}_{U_E}$.*

Proof. Definitions imply $U_E = r(EU) = \{r(\alpha) \mid \alpha \in E, s(\alpha) \in U\}$, so

$$\mathbb{1}_{U_E}(u) = \begin{cases} 1 & \text{if } \exists \alpha \in E \text{ with } s(\alpha) \in U,\ r(\alpha) = u, \\ 0 & \text{otherwise.} \end{cases}$$

whereas

$$E\mathbb{1}_U(u) = \begin{cases} \mathbb{1}_U(s(\alpha)) & \text{if } \exists \alpha \in E \text{ with } r(\alpha) = u, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1 & \text{if } \exists \alpha \in E \text{ with } s(\alpha) \in U,\ r(\alpha) = u, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathbb{1}_{U_E}(u) = E\mathbb{1}_U(u)$ for every $u \in G^{(0)}$. □

Fact 2.4. *If $f : G^{(0)} \to \mathbb{C}$ is continuous and E is a clopen bisection, then $Ef : G^{(0)} \to \mathbb{C}$ is continuous.*

Proof. Let $u, (u_n)_n \in G^{(0)}$ with $(u_n)_n \longrightarrow u$. Suppose first that there is an $\alpha \in E$ with $r(\alpha) = u$. Then $u \in r(E)$ which is open, so u_n belongs to $r(E)$ for large enough n. Say $r(\alpha_n) = u_n$, for some $\alpha_n \in E$. Now $r|_E$ is a homeomorphism, so $(\alpha_n)_n \longrightarrow \alpha$ which implies $(s(\alpha_n))_n \longrightarrow s(\alpha)$. Therefore

$$(Ef(u_n))_n = (f(s(\alpha_n)))_n \longrightarrow f(s(\alpha)) = Ef(u).$$

Next suppose that $u \notin r(E)$, which means that $Ef(u) = 0$. Then $u \in G^{(0)} \setminus r(E)$ which is again open, so that $u_n \notin r(E)$ for all large n. In this case $Ef(u_n) = 0$ for all large n. □

2.2. **The reduced C*-algebra $C^*_r(G)$.** We briefly describe the construction of the reduced C*-algebra of a locally compact, Hausdorff, étale groupoid G. Fix such a G and consider the complex linear space $C_c(G)$ of compactly supported complex-valued functions on G. Convolution and involution defined by

$$f \cdot g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta), \quad \gamma \in G$$
$$f^*(\alpha) = \overline{f(\alpha^{-1})}, \quad \alpha \in G$$

give $C_c(G)$ the structure of a complex $*$-algebra. Is is important to note that there is a natural inclusion $C_c(G^{(0)}) \hookrightarrow C_c(G)$ of $*$-algebras. Also, if E and F are compact open bisections, then the characteristic functions $\mathbb{1}_E, \mathbb{1}_F \in C_c(G)$ and

$$\mathbb{1}_E^* = \mathbb{1}_{E^{-1}} \quad \text{and} \quad \mathbb{1}_E \mathbb{1}_F = \mathbb{1}_{EF}.$$

The $*$-algebra $C_c(G)$ is then represented on Hilbert spaces as follows: for a unit $u \in G^{(0)}$ define

$$\pi_u : C_c(G) \to \mathbb{B}(\ell^2(G_u)) \quad \text{as} \quad \pi_u(f)(\xi)(\gamma) = f \cdot \xi(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)\xi(\beta).$$

It is verified that π_u is a representation of the $*$-algebra $C_c(G)$, and that the direct sum $\pi_r := \bigoplus_{u\in G^{(0)}} \pi_u$ is faithful. It follows that $\|f\|_r := \|\pi_r(f)\|$ is a C*-norm on $C_c(G)$ and we may define the reduced C*-algebra of the groupoid G as

$$C^*_r(G) := \overline{C_c(G)}^{\|\cdot\|_r}.$$

The inclusion $C_c(G^{(0)}) \hookrightarrow C^*_r(G)$ has a partial 'inverse', namely the conditional expectation. More precisely, the restriction map $C_c(G) \to C_c(G^{(0)})$, $f \mapsto f|_{G^{(0)}}$ extends continuously to a faithful conditional expectation $\mathbb{E} : C^*_r(G) \to C_0(G^{(0)})$.

Given a discrete transformation group (X, Γ), the C*-algebra obtained from the resulting transformation groupoid $X \rtimes \Gamma$ is a familiar construction; the reduced C*-crossed product. In fact,

$$C^*_r(X \rtimes \Gamma) \cong C_0(X) \rtimes_r \Gamma.$$

2.3. **K-theory and traces.** Recall that if X is the Cantor set, $K_0(C(X))$ is order isomorphic to $C(X, \mathbb{Z})$ via the dimension map $\dim : K_0(C(X)) \to C(X, \mathbb{Z})$ given by $\dim([p]_0)(x) = \mathrm{Tr}(p(x))$, where p is a projection over the matrices of $C(X)$; $M_n(C(X)) \cong C(X; \mathbb{M}_n)$, and Tr denotes the standard (non-normalized) trace on $\mathbb{M}_n$. It is clear that under this isomorphism $\dim([\mathbb{1}_E]_0) = \mathbb{1}_E$ where $E \subseteq X$ is a compact open subset.

If G is an étale groupoid with unit space $G^{(0)}$ homeomorphic to a Cantor set, the inclusion $\iota : C(G^{(0)}) \hookrightarrow C^*_r(G)$ induces a positive group homomorphism

$$K_0(\iota) : K_0(C(G^{(0)})) \longrightarrow K_0(C^*_r(G))$$

that satisfies $K_0(\iota)([\mathbb{1}_{s(E)}]_0) = K_0(\iota)([\mathbb{1}_{r(E)}]_0)$ for any compact open bisection E. This holds because the projections $\iota(\mathbb{1}_{s(E)})$ and $\iota(\mathbb{1}_{r(E)})$ are Murray-von-Neumann equivalent in $C^*_r(G)$. Indeed, $\mathbb{1}_E \in C_c(G) \subseteq C^*_r(G)$ and

$$\mathbb{1}_E^* \mathbb{1}_E = \mathbb{1}_{E^{-1}} \mathbb{1}_E = \mathbb{1}_{E^{-1}E} = \mathbb{1}_{s(E)},$$
$$\mathbb{1}_E \mathbb{1}_E^{-1} = \mathbb{1}_E \mathbb{1}_{E^{-1}} = \mathbb{1}_{EE^{-1}} = \mathbb{1}_{r(E)}.$$

The following elementary fact will be useful. Let A and B be C*-algebras with B stably finite, and suppose $\varphi : A \hookrightarrow B$ is an embedding. The induced map on K-theory $\mathrm{K}_0(A) \to \mathrm{K}_0(B)$ is faithful. For if p is a projection in $\mathbb{M}_n(A)$ for some n, then

$$\mathrm{K}_0(\varphi)([p]_0) = 0 \Longrightarrow [\varphi(p)]_0 = 0 \Longrightarrow \varphi(p) = 0 \Longrightarrow p = 0,$$

where the second implication follows from the fact that B is stably finite.

Traces and tracial weights on $\mathrm{C}^*_r(G)$ are obtained via invariant measures on the unit space. We briefly describe these.

Definition 2.5. Let G be an étale groupoid. A Borel measure μ on $G^{(0)}$ is said to be G-invariant (or simply invariant) if, $\mu(s(E)) = \mu(r(E))$ for every open bisection E.

If G is étale with compact unit space $G^{(0)}$, and μ is an invariant Borel probability measure on $G^{(0)}$, then we obtain a tracial state τ_μ on $\mathrm{C}^*_r(G)$ by composing the expectation $\mathbb{E} : \mathrm{C}^*_r(G) \to C(G^{(0)})$ with integration against μ; $I_\mu : C(G^{(0)}) \to \mathbb{C}$, $f \mapsto \int_{G^{(0)}} f d\mu$:

$$\tau_\mu := I_\mu \circ \mathbb{E} : \mathrm{C}^*_r(G) \longrightarrow \mathbb{C}, \qquad \tau_\mu(a) = \int_{G^{(0)}} \mathbb{E}(a) d\mu.$$

Since the expectation is faithful, the above trace τ_μ is faithful provided the measure μ has full support.

If the unit space $G^{(0)}$ is not necessarily compact, then an invariant measure μ on $G^{(0)}$ produces a lower-semicontinuous (tracial) weight $I_\mu : C_0(G^{(0)})^+ \to [0,\infty]$, $f \mapsto \int_{G^{(0)}} f d\mu$, which induces a lower-semicontinuous tracial weight $I_\mu \circ \mathbb{E} : \mathrm{C}^*_r(G)^+ \to [0,\infty]$.

3. Paradoxical Groupoids

We begin by studying notions of paradoxicality and paradoxical decompositions in ample groupoids which give rise to infinite reduced groupoid C*-algebras. Recall that a projection p in a C*-algebra A is infinite if there is a partial isometry $v \in A$ with $v^*v = p$ and $vv^* < p$, that is, p is Murray-von Neumann equivalent to a proper subprojection of itself. A C*-algebra A is infinite if it admits an infinite projection.

Definition 3.1. Let G be an ample groupoid, and suppose $A \subseteq G^{(0)}$ is a non-empty compact open subset of the unit space. We say that A is *paradoxical* if there are compact open bisections $E_1, \ldots, E_n$ in $\mathcal{C}$ satisfying

$$\mathbb{1}_A \leq \sum_{i=1}^n \mathbb{1}_{s(E_i)}, \quad \text{and} \quad \sum_{i=1}^n \mathbb{1}_{r(E_i)} < \mathbb{1}_A. \tag{1}$$

We will say that G is *paradoxical* if there is some paradoxical $A \subseteq G^{(0)}$.

A remark is in order. The notation $f < g$, for functions $f, g \in C_c(G^{(0)})$, means that $f \leq g$ and $f \neq g$.

Proposition 3.2. *Let G be an ample groupoid. If G is paradoxical, then the reduced groupoid* C**-algebra* $\mathrm{C}^*_r(G)$ *is infinite.*

Proof. Suppose the compact open $\emptyset \neq A \subseteq G^{(0)}$ is paradoxical. Let $E_1, \ldots E_n$ be the compact open bisections that satisfy 1. The inequality $\sum_{i=1}^n \mathbb{1}_{r(E_i)} < \mathbb{1}_A$ implies that $r(E_i) \cap r(E_j) = \emptyset$ for $i \neq j$. It follows that the E_i are also pairwise disjoint. Moreover, since $\mathbb{1}_A \leq \sum_{i=1}^n \mathbb{1}_{s(E_i)}$ we know that $A \subseteq \bigcup_{i=1}^n s(E_i)$. There are compact open subsets $A_1, \ldots, A_n \subseteq A$ that satisfy $A_i \subseteq s(E_i)$ and $\bigsqcup_{i=1}^n A_i = A$. This is seen by inductively defining

$$A_1 := A \cap s(E_1),\ A_2 := (A \cap s(E_2)) \setminus A_1, \ldots, A_n := (A \cap s(E_n)) \setminus \left(\bigcup_{i=1}^{n-1} A_i \right).$$

Now let $F_i := (s|_{E_i})^{-1}(A_i) \subseteq E_i$ for $i = 1, \ldots, n$. Note that the F_i are again disjoint compact open bisections and that $s(F_i) = A_i$. Observe

$$F_i^{-1}F_j = \{\alpha^{-1}\beta \mid \alpha \in F_i, \beta \in F_j, r(\alpha) = s(\alpha^{-1}) = r(\beta)\} = \begin{cases} s(F_i) = A_i & \text{if } i = j, \\ \emptyset & \text{if } i \neq j. \end{cases}$$

$$F_iF_j^{-1} = \{\alpha\beta^{-1} \mid \alpha \in F_i, \beta \in F_j, s(\alpha) = r(\beta^{-1}) = s(\beta)\} = \begin{cases} r(F_i) & \text{if } i = j, \\ \emptyset & \text{if } i \neq j. \end{cases}.$$

Therefore if we set $v := \sum_{i=1}^n \mathbb{1}_{F_i}$ in $C_c(G) \subseteq \mathrm{C}_r^*(G)$ we obtain

$$\begin{aligned} v^*v &= \left(\textstyle\sum_{i=1}^n \mathbb{1}_{F_i} \right)^* \left(\textstyle\sum_{j=1}^n \mathbb{1}_{F_j} \right) = \textstyle\sum_{i,j=1}^n \mathbb{1}_{F_i}^* \mathbb{1}_{F_j} \\ &= \textstyle\sum_{i,j=1}^n \mathbb{1}_{F_i^{-1}F_j} = \textstyle\sum_{i=1}^n \mathbb{1}_{s(F_i)} = \textstyle\sum_{i=1}^n \mathbb{1}_{A_i} = \mathbb{1}_A. \end{aligned}$$

whereas

$$vv^* = \left(\textstyle\sum_{i=1}^n \mathbb{1}_{F_i} \right) \left(\textstyle\sum_{j=1}^n \mathbb{1}_{F_j} \right)^* = \textstyle\sum_{i,j=1}^n \mathbb{1}_{F_iF_j^{-1}} = \textstyle\sum_{i=1}^n \mathbb{1}_{r(F_i)} \leq \textstyle\sum_{i=1}^n \mathbb{1}_{r(E_i)} < \mathbb{1}_A.$$

The projection $\mathbb{1}_A$ is therefore infinite whence $\mathrm{C}_r^*(G)$ is infinite. □

Parodoxicality carries the connotation of 'duplication of sets', so we revisit the ideas explored in [24] and [37] and define, in the groupoid setting, a notion of paradoxical decomposition with a covering multiplicity.

Definition 3.3. Let G be an ample groupoid and suppose $A \subseteq G^{(0)}$ is a non-empty compact open subset of the unit space. With $k > l > 0$ positive integers, we say that A is *(k, l)-paradoxical* if there are compact open bisections $E_1, \ldots, E_n$ in $\mathcal{C}$ satisfying

$$k\mathbb{1}_A \leq \sum_{i=1}^n \mathbb{1}_{s(E_i)}, \quad \text{and} \quad \sum_{i=1}^n \mathbb{1}_{r(E_i)} \leq l\mathbb{1}_A. \tag{2}$$

We call A *properly paradoxical* if it is $(2, 1)$-paradoxical.

If A fails to be (k, l)-paradoxical for all integers $k > l > 0$ then we say that A is *completely non-paradoxical.*

If every nonempty compact open $A \subseteq G^{(0)}$ is completely non-paradoxical we say that G is completely non-paradoxical.

Remark 3.4. A related definition of paradoxicality appears as [5, Definition 4.4]; our definitions is slightly more flexible in that we do not require that the sets $s(E_i)$ in (2) cover A, and we do not insist on any orthogonality amongst the sets $r(E_i)$. It is clear that if G is $(\mathbb{E}, k, l)$-paradoxical in the sense of Bönicke and Li for some $\mathbb{E}, k, l$, then it is (k, l)-paradoxical in our sense.

It is not surprising that stable finiteness is an obstruction to paradoxicality. The following result is related to [5, Proposition 4.8], modulo the difference in our definitions of paradoxicality.

Proposition 3.5. *Let G be an ample groupoid. If $\mathrm{C}^*_r(G)$ is stably finite then G is completely non-paradoxical.*

Proof. Suppose $A \subseteq G^{(0)}$ is compact open and also (k,l)-paradoxical. Let $E_1, \ldots E_n \in \mathcal{C}$ be the compact open bisections that satisfy 2. Moreover, let $\iota : C(G^{(0)}) \hookrightarrow \mathrm{C}^*_r(G)$ denote the canonical inclusion. Since $\mathrm{C}^*_r(G)$ is stably finite, the induced map on K-theory

$$\mathrm{K}_0(\iota) : \mathrm{K}_0(C(G^{(0)})) \cong C(G^{(0)}, \mathbb{Z}) \longrightarrow \mathrm{K}_0(\mathrm{C}^*_r(G))$$

is faithful (see section 2.3). We compute

$$\begin{aligned} k\mathrm{K}_0(\iota)(\mathbb{1}_A) = \mathrm{K}_0(\iota)(k\mathbb{1}_A) &\leq \mathrm{K}_0(\iota)\Big(\sum_{i=1}^n \mathbb{1}_{s(E_i)}\Big) = \sum_{i=1}^n \mathrm{K}_0(\iota)(\mathbb{1}_{s(E_i)}) \\ &= \sum_{i=1}^n \mathrm{K}_0(\iota)(\mathbb{1}_{r(E_i)}) = \mathrm{K}_0(\iota)\Big(\sum_{i=1}^n \mathbb{1}_{r(E_i)}\Big) \leq \mathrm{K}_0(\iota)(l\mathbb{1}_A) = l\mathrm{K}_0(\iota)(\mathbb{1}_A). \end{aligned}$$

Writing $x = \mathrm{K}_0(\iota)(\mathbb{1}_A)$ we get that $(l-k)x$ belongs to $\mathrm{K}_0(\mathrm{C}^*_r(G))^+ \cap -(\mathrm{K}_0(\mathrm{C}^*_r(G))^+)$, which is trivial by stable finiteness. Thus $(l-k)x = 0$. Again using the fact that $\mathrm{C}^*_r(G)$ is stably finite we get $x = 0$, and so $\mathbb{1}_A = 0$ since $\mathrm{K}_0(\iota)$ is faithful. This contradicts the assumption that A is non-empty. □

One of the principal goals of this paper is to characterize stably finite C*-algebras that arise from étale groupoids. In this vein we aim to establish a converse to Proposition 3.5. What is needed is a technique to pass from complete non-paradoxicality of a groupoid G to constructing faithful tracial states on $\mathrm{C}^*_r(G)$. We will achieve this in the minimal setting (Theorem 6.5). The next section is devoted to expressing the notion of minimality in a K-theoretic framework which will best suit this purpose and enable us to construct faithful traces.

4. Minimal Groupoids

Recall that a C*-dynamical system (A, Γ, α) is said to be minimal if A admits no non-trivial Γ-invariant ideals. If $A = C_0(X)$, where X is a locally compact space, and α is induced by a continuous action $\Gamma \curvearrowright X$, then α is minimal if and only if X admits no non-trivial Γ-invariant closed subsets. It is not difficult to show that the action is minimal if and only if the orbit $\mathrm{Orb}(x) := \{t.x \mid t \in \Gamma\}$ of every $x \in X$ is dense in X. Analogously, an étale groupoid G is said to be *minimal* if the orbit $r(G_u)$ of every unit u is dense in the unit space; that is $\overline{r(G_u)} = G^{(0)}$ for every $u \in G^{(0)}$.

If (X, Γ) is a discrete transformation group with X compact, it is not difficult to show (see Proposition 3.1 in [38]) that the action is minimal if and only if the following holds: for every non-empty open set $U \subseteq X$, there are group elements $t_1, \ldots, t_n \in \Gamma$ such that $X = \cup_{j=1}^n t_j.U$. The following result is similar, where group elements are replaced by open bisections.

Recall from section 2.1 that if E is an open bisection in G and $U \subseteq G^{(0)}$ is open, then both

$$EU = \{\alpha \mid \alpha \in E, s(\alpha) \in U\}, \quad \text{and} \quad U_E = r(EU) = \{r(\alpha) \mid \alpha \in E, s(\alpha) \in U\}$$

are also open.

Proposition 4.1. *Let G be an étale groupoid with compact unit space $G^{(0)}$. The following are equivalent*

(i) G is minimal.

(ii) For every non-empty open $U \subseteq G^{(0)}$, there are open bisections $E_1, \ldots, E_n$ such that

$$\bigcup_{j=1}^{n} U_{E_j} = G^{(0)}.$$

Proof. *(i)* $\Rightarrow$ *(ii)*: Let $U \subseteq G^{(0)}$ be open. For each finite collection of open bisections $\mathcal{E} = \{E_1, \ldots, E_m\} \subseteq \mathcal{B}$ we set

$$U_{\mathcal{E}} = \bigcup_{E \in \mathcal{E}} U_E.$$

We claim that $\bigcup_{\mathcal{E}} U_{\mathcal{E}} = G^{(0)}$ where the union runs over all finite collections $\mathcal{E}$ of open bisections. Suppose the claim is false, then there is a unit $u \in G^{(0)} \setminus \cup_{\mathcal{E}} U_{\mathcal{E}}$. Since $r(G_u)$ is dense and $\cup_{\mathcal{E}} U_{\mathcal{E}}$ is open we know that $r(G_u) \cap (\cup_{\mathcal{E}} U_{\mathcal{E}}) \neq \emptyset$. Let $\alpha \in G_u$ with $r(\alpha) \in U_{\mathcal{F}}$ for some finite collection $\mathcal{F} \subseteq \mathcal{B}$. So there exists $F \in \mathcal{F}$ with $r(\alpha) \in E_F$. This means that there is a $\beta \in F$ with $s(\beta) \in U$ and $r(\alpha) = r(\beta)$. Now set $\gamma = \alpha^{-1}\beta$. It follows that

$$s(\gamma) = s(\beta) \in U, \quad \text{and} \quad r(\gamma) = r(\alpha^{-1}) = s(\alpha) = u.$$

Now let E be an open bisection containing α^{-1}. Then $H = EF$ is an open bisection containing γ, and $u \in U_H$ since $\gamma \in H$ and $r(\gamma) = u$ with $s(\gamma) \in U$. This contradicts the fact that $u \notin \cup_{\mathcal{E}} U_{\mathcal{E}}$. The claim is thus proved.

Compactness of $G^{(0)}$ implies that $G^{(0)} = \cup_{j=1}^{J} U_{\mathcal{E}_j}$ where the $\mathcal{E}_j$ are finite collections of open bisections. Let $\mathcal{E} = \cup_{j=1}^{J} \mathcal{E}_j$, which is again a finite collection. Therefore,

$$G^{(0)} = U_{\mathcal{E}} = \bigcup_{E \in \mathcal{E}} U_E.$$

(ii) $\Rightarrow$ *(i)*: Let $u \in G^{(0)}$, we want to show that $\overline{r(G_u)} = G^{(0)}$. If not, set $U = G^{(0)} \setminus \overline{r(G_u)}$, a non-empty open set in $G^{(0)}$. By our assumption there are open bisections $E_1, \ldots, E_n$ such that

$$\bigcup_{j=1}^{n} U_{E_j} = G^{(0)}.$$

Let i be such that $u \in U_{E_i}$. Then there is an α in E_i with $s(\alpha) \in U$ and $r(\alpha) = u$. This implies that $\alpha^{-1} \in G_u$ since $s(\alpha^{-1}) = r(\alpha) = u$. However,

$$\overline{r(G_u)} \supseteq r(G_u) \ni r(\alpha^{-1}) = s(\alpha) \in U$$

which contradicts the fact that $U \cap \overline{r(G_u)} = \emptyset$. $\square$

We now turn our attention to the ample case. Let G be an étale groupoid with compact and totally disconnected unit space $G^{(0)}$. If $f : G^{(0)} \to \mathbb{Z}$ is a continuous integer-valued function, its range is a finite subset of $\mathbb{Z}$ and we can therefore express f as a finite sum

$$f = \sum_{k=1}^{l} m_k \mathbb{1}_{U_k}$$

where $m_k \in \mathbb{Z}$ and the U_k are closed and open subsets of $G^{(0)}$. If E is a compact open bisection in G, then combining Facts 2.2, 2.3, and 2.4 we get

$$Ef = E\left(\sum_{k=1}^{l} m_k \mathbb{1}_{U_k}\right) = \sum_{k=1}^{l} m_k \mathbb{1}_{r(EU_k)}$$

which is again a continuous integer-valued function on $G^{(0)}$.

Proposition 4.2. *Let G be an ample groupoid with compact unit space $G^{(0)}$. The following are equivalent.*

(i) *G is minimal.*

(ii) *For every non-empty clopen $U \subseteq G^{(0)}$, there are clopen bisections $E_1, \ldots, E_n$ such that*

$$\bigcup_{j=1}^{n} U_{E_j} = G^{(0)}.$$

(iii) *For every non-zero $f \in C(G^{(0)}, \mathbb{Z})^+$, there are clopen bisections $E_1, \ldots, E_n$ such that*

$$\sum_{j=1}^{n} E_j f \geq \mathbb{1}_{G^{(0)}}.$$

Proof. *(i)* $\Rightarrow$ *(ii)*: This direction follows the same lines as Proposition 4.1 except we choose compact open (as opposed to merely open) bisections every time.

(ii) $\Rightarrow$ *(i)*: If $u \in G^{(0)}$ and $\overline{r(G_u)} \neq G^{(0)}$, we choose $U \subseteq G^{(0)} \setminus \overline{r(G_u)}$, a non-empty close and open subset (this is possible because these form a base for the topology of $G^{(0)}$. The rest of the proof is identical to that of Proposition 4.1.

(ii) $\Rightarrow$ *(iii)*: If $f \in C(G^{(0)}, \mathbb{Z})^+$ is non-zero, we may write $f = \sum_{k=1}^{l} m_k \mathbb{1}_{U_k}$ where the m_k are strictly positive integers and $U_k \subseteq G^{(0)}$ are closed and open and non-empty. Setting $U = U_1$, there are compact open bisections $E_1, \ldots, E_n$ with

$$\bigcup_{j=1}^{n} U_{E_j} = G^{(0)}.$$

Note that

$$E_j f = E_j\left(\sum_{k=1}^{l} m_k \mathbb{1}_{U_k}\right) = \sum_{k=1}^{l} m_k \mathbb{1}_{r(E_j U_k)} \geq m_1 \mathbb{1}_{r(E_j U_1)} \geq \mathbb{1}_{r(E_j U)}.$$

Therefore

$$\mathbb{1}_{G^{(0)}} = \mathbb{1}_{\cup_{j=1}^{n} U_{E_j}} \leq \sum_{j=1}^{n} \mathbb{1}_{U_{E_j}} = \sum_{j=1}^{n} \mathbb{1}_{r(E_j U)} \leq \sum_{j=1}^{n} E_j f.$$

(iii) $\Rightarrow$ *(ii)*: Let $\emptyset \neq U \subseteq G^{(0)}$ be closed and open, and set $f = \mathbb{1}_U$. By our assumption there are compact open bisections $E_1, \ldots, E_n \in \mathcal{C}$ with $\sum_{j=1}^n E_j f \geq \mathbb{1}_{G^{(0)}}$. Then

$$\mathbb{1}_{G^{(0)}} \leq \sum_{j=1}^n E_j f = \sum_{j=1}^n E_j \mathbb{1}_U = \sum_{j=1}^n \mathbb{1}_{r(E_j U)} = \sum_{j=1}^n \mathbb{1}_{U_{E_j}},$$

which shows that $G^{(0)} \subseteq \cup_{j=1}^n U_{E_j}$. □

Proposition 4.2 part (iii) will be instrumental in constructing faithful traces on completely non-paradoxical groupoid C*-algebras in the following section.

5. The type semigroup of an ample groupoid

Given a transformation group (X, Γ) where Γ is discrete and X is the Cantor set, Rørdam and Sierakowski [44] construct a type semigroup $S(X, \Gamma)$ that witnesses the pure infiniteness of the reduced C*-crossed product $C(X) \rtimes_\lambda \Gamma$. This construction is much in the spirit of the classical type semigroup for arbitrary actions studied in [49]. Given a (not necessarily commutative) C*-dynamical system (A, Γ, α) with A stably finite and $\mathrm{K}_0(A)$ admitting the Riesz refinement property, the first author constructs in [37] a type semigroup $S(A, \Gamma)$ that distinguishes stable finitness versus pure infiniteness of the reduced crossed product $A \rtimes_\lambda \Gamma$. In this section we construct the analogue in the setting of ample groupoids. We restrict our attention to ample groupoids so that the unit space $G^{(0)}$ is totally disconnected. It then follows that the set of compactly-supported, integer-valued, continuous functions on the unit space $C_c(G^{(0)}, \mathbb{Z})$ is a dimension group with positive cone $C_c(G^{(0)}, \mathbb{Z})^+$. We will show in Section 8 that our type semigroup $S(G)$ for a transformation groupoid coincides with Rørdam and Sierakowski's semigroup $S(X, \Gamma)$.

Definition 5.1. Let G be an ample groupoid. Define a relation on $C_c(G^{(0)}, \mathbb{Z})^+$ as follows: for $f, g \in C_c(G^{(0)}, \mathbb{Z})^+$, set $f \sim_G g$ if there are compact open bisections $E_1, \ldots, E_n \in \mathcal{C}$ with

$$f = \sum_{i=1}^n \mathbb{1}_{s(E_i)}, \quad \text{and} \quad g = \sum_{i=1}^n \mathbb{1}_{r(E_i)}. \tag{3}$$

Proposition 5.2. *Let G be an ample groupoid. Then the relation on $C_c(G^{(0)}, \mathbb{Z})^+$ described in Definition 5.1 is an equivalence relation.*

Proof. Reflexivity is clear because compact open subsets of $G^{(0)}$ are bisections, and symmetry is clear because if E is a compact open bisection, then so is E^{-1}.

To prove transitivity, we suppose that $f \sim g \sim h$ via

$$f = \sum_{i=1}^m \mathbb{1}_{s(E_i)}, \quad g = \sum_{i=1}^m \mathbb{1}_{r(E_i)}, \quad g = \sum_{j=1}^n \mathbb{1}_{s(F_j)}, \quad h = \sum_{j=1}^n \mathbb{1}_{r(F_j)}, \tag{4}$$

where the E_i and F_j are compact open bisections.

Claim 5.3. *In the equations 4, we may assume that $m = n$ and that $r(E_i) = s(F_i)$ for $i = 1, \ldots, m$.*

By applying the refinement property to the two expressions for g we can find compact open subsets $\{Z_{i,j}\}_{i,j}$ of $G^{(0)}$ satisfying

$$\mathbb{1}_{r(E_i)} = \sum_{j=1}^{n} \mathbb{1}_{Z_{i,j}}, \quad \mathbb{1}_{s(F_j)} = \sum_{i=1}^{m} \mathbb{1}_{Z_{i,j}}$$

for every i and j. Note that $\bigsqcup_{j=1}^{n} Z_{i,j} = r(E_i)$ and $\bigsqcup_{i=1}^{m} Z_{i,j} = s(F_j)$. For each pair (i,j) consider the compact open bisections

$$G_{i,j} := E_i \cap r^{-1}(Z_{i,j}) \subseteq E_i, \quad \text{and} \quad H_{i,j} := F_j \cap s^{-1}(Z_{i,j}) \subseteq F_j.$$

We then have $E_i = \bigsqcup_{j=1}^{n} G_{i,j}$ and $F_j = \bigsqcup_{i=1}^{m} H_{i,j}$. Thus

$$f = \sum_{i=1}^{m} \mathbb{1}_{s(E_i)} = \sum_{i,j} \mathbb{1}_{s(G_{i,j})}, \quad g = \sum_{i=1}^{m} \mathbb{1}_{r(E_i)} = \sum_{i,j} \mathbb{1}_{r(G_{i,j})},$$

$$g = \sum_{j=1}^{n} \mathbb{1}_{s(F_j)} = \sum_{i,j} \mathbb{1}_{s(H_{i,j})}, \quad h = \sum_{j=1}^{n} \mathbb{1}_{r(F_j)} = \sum_{i,j} \mathbb{1}_{r(H_{i,j})}.$$

Note that $r(G_{i,j}) = Z_{i,j} = s(H_{i,j})$. This proves the Claim.

Now look at the compact open bisections $B_i = F_i E_i$, for $1 \leq i \leq m$. Note that $s(B_i) = s(E_i)$ and $r(B_i) = r(F_i)$. We end up with $f \sim h$ since

$$f = \sum_{i=1}^{m} \mathbb{1}_{s(E_i)} = \sum_{i=1}^{m} \mathbb{1}_{s(B_i)}, \quad h = \sum_{i=1}^{m} \mathbb{1}_{r(F_i)} = \sum_{i=1}^{m} \mathbb{1}_{r(B_i)},$$

and this finishes the proof.

□

We can now make the following definition.

Definition 5.4. Let G be an ample groupoid. We define the *type semigroup of G* as $S(G) := C_c(G^{(0)}, \mathbb{Z})^+ / \sim_G$, and write $[f]_G$ for the equivalence class with representative $f \in C_c(G^{(0)}, \mathbb{Z})^+$.

Remark 5.5. Our definition of the type semigroup of G is related to the definition given by Bönicke and Li in [5, Definition 5.3], though our definition is somewhat more algebraic and does not involve amplifying and passing to levels of $G^{(0)} \times \mathbb{N}$. The apparent discrepancy can be resolved using Proposition 5.7 below: Write $\mathcal{R} := \mathbb{N} \times \mathbb{N}$ regarded as a discrete principal groupoid, and let $\mathcal{K}G$ denote the groupoid $G \times \mathcal{R}$. Identifying $\mathcal{R}^{(0)}$ with $\mathbb{N}$ in the obvious way, it is straightforward to see that the type semigroup of G as defined in [5, Definition 5.3] is precisely the type semigroup of $\mathcal{K}G$ as defined by Definition 5.4. Proposition 5.7 shows that the type semigroups, in the sense of our definition, of $\mathcal{K}G$ and of G coincide, so we see that our definition of the type semigroup of G agrees with [5, Definition 5.3] up to canonical isomorphism.

We can define addition of classes simply by $[f]_G + [g]_\alpha := [f+g]_G$ for f, g in $C_c(G^{(0)}, \mathbb{Z})^+$. It is routine to check that this operation is well defined; indeed if $f \sim_G f'$ and $g \sim_G g'$ via

$$f = \sum_{i=1}^{n} \mathbb{1}_{s(E_i)}, \quad f' = \sum_{i=1}^{n} \mathbb{1}_{r(E_i)}, \quad g = \sum_{j=1}^{m} \mathbb{1}_{s(F_j)}, \quad g' = \sum_{j=1}^{m} \mathbb{1}_{r(F_j)},$$

then

$$\begin{aligned}[f]_G + [g]_G = [f+g]_G &= \left[\sum_{i=1}^{n} \mathbb{1}_{s(E_i)} + \sum_{j=1}^{m} \mathbb{1}_{s(F_j)}\right]_G = \left[\sum_{i=1}^{n} \mathbb{1}_{r(E_i)} + \sum_{j=1}^{m} \mathbb{1}_{r(F_j)}\right]_G \\ &= [f'+g']_G = [f']_G + [g']_G.\end{aligned}$$

We make a few elementary observations about $S(G)$ for such an ample G. Firstly, $S(G)$ is not only a semigroup but an abelian monoid as $[0]_G$ is clearly the neutral additive element. Impose the algebraic ordering on $S(G)$, that is, set $[f]_G \leq [g]_G$ if there is an $h \in C_c(G^{(0)}, \mathbb{Z})^+$ with $[f]_G + [h]_G = [g]_G$. This gives $S(G)$ the structure of a pre-ordered abelian monoid. Note that if $f, g \in C_c(G^{(0)}, \mathbb{Z})^+$ with $f \leq g$ (in the ordering of $C_c(G^{(0)}, \mathbb{Z})$) then $[f]_G \leq [g]_G$ in $S(G)$. To see this, $f \leq g$ implies $g - f := h \in C_c(G^{(0)}, \mathbb{Z})^+$, so $[g]_G = [f+h]_G = [f]_G + [h]_\alpha$ which gives $[f]_G \leq [g]_G$. Next, we observe that if $[f]_G = [0]_G$, for some f in $C_c(G^{(0)}, \mathbb{Z})^+$, then in fact $f = 0$ in $C_c(G^{(0)}, \mathbb{Z})$. Indeed, if $f = \sum_i \mathbb{1}_{s(E_i)}$, and $\sum_i \mathbb{1}_{r(E_i)} = 0$ for some compact open bisections $E_1, \ldots, E_n \in \mathcal{C}$, then $r(E_i) = \emptyset$ for every i which implies $E_i = \emptyset$ and $s(E_i) = \emptyset$ for all i. All together, there is a surjective, order preserving, faithful, monoid homomorphism

$$\pi : C_c(G^{(0)}, \mathbb{Z})^+ \longrightarrow S(G) \quad \text{given by} \quad \pi(f) = [f]_G.$$

In the remainder of this section we show that the type semigroup of an ample groupoid is an invariant for groupoid equivalence in the sense that equivalent groupoids have isomorphic type semigroups. This will also enable us to extend our characterization of stable finiteness for reduced C*-algebras of minimal ample groupoids from the situation of groupoids with compact unit space to the general case.

We first show that the type semigroup is an isomorphism invariant.

Proposition 5.6. *Let $\varphi : G \to H$ be an isomorphism of ample groupoids. The induced map $\overline{\varphi} : C_c(H^{(0)}, \mathbb{Z})^+ \longrightarrow C_c(G^{(0)}, \mathbb{Z})^+$ given by $\overline{\varphi}(f) = f \circ \varphi|_{G^{(0)}}$ drops to an isomorphism of monoids $\phi : S(H) \to S(G)$.*

Proof. This is clear: If $f \sim g$ in $C_c(H^{(0)}, \mathbb{Z})^+$ with $f = \sum_i \mathbb{1}_{s(E_i)}$ and $g = \sum_{i=1}^n \mathbb{1}_{r(E_i)}$, then $\overline{\varphi}(f) \sim \overline{\varphi}(g)$ with $\overline{\varphi}(f) = \sum_i \mathbb{1}_{s(\varphi^{-1}(E_i))}$ and $\overline{\varphi}(g) = \sum_i \mathbb{1}_{r(\varphi^{-1}(E_i))}$. So $\overline{\varphi}(f)$ drops to a homomorphism $\phi : S(H) \to S(G)$, and $\overline{\varphi^{-1}}$ drops to an inverse for ϕ. □

To see that equivalent ample groupoids have isomorphic type semigroups, we will appeal to the groupoid analogue [11] of the Brown–Green–Rieffel theorem [9]. We first need to know that the type semigroup is invariant under stabilization of groupoids.

Recall from Remark 5.5 that $\mathcal{R}$ is the discrete groupoid $\mathbb{N} \times \mathbb{N}$ and $\mathcal{R}^{(0)}$ is identified with $\mathbb{N}$. The stabilization of a groupoid G is the product $\mathcal{K}G := G \times \mathcal{R}$ with its natural groupoid structure. Note that $\mathcal{K}G^{(0)} = G^{(0)} \times \mathcal{R}^{(0)}$. If G is ample, one notes that $E \times \{(i,j)\} \subseteq \mathcal{K}G$ is a compact open bisection if and only if $E \subseteq G$ is a compact open bisection. Moreover, a brief compactness argument shows that every compact open bisection $F \subseteq \mathcal{K}G$ can be written as a disjoint union $F = \bigsqcup_{k=1}^K E_k \times \{(i_k, j_k)\}$ where E_k are compact open bisections in G and $(i_k, j_k) \in \mathcal{R}$. Likewise, any compact open $A \subseteq \mathcal{K}G^{(0)}$ can be written as a disjoint union $A = \bigsqcup_{k=1}^K A_k \times \{(i_k, i_k)\}$ where the A_k are compact open subsets of $G^{(0)}$ and $i_k \in \mathbb{N}$.

We will write $\mathbb{1}_{(m,n)}$ for the point-mass functions in $C_c(\mathcal{R})$. Given functions $f \in C_c(G)$ and $g \in C_c(\mathcal{R})$, we denote by $f \times g : G \times \mathcal{R} \to \mathbb{C}$ the function defined by sending $(\alpha, (m,n)) \mapsto f(\alpha)g((m,n))$, where $\alpha \in G$, $(m,n) \in \mathcal{R}$. Note that the association $(f,g) \mapsto f \times g$ is a well-defined bilinear map $C_c(G) \times C_c(\mathcal{R}) \to C_c(\mathcal{K}G)$. Also, if $A \subseteq G$ and $B \subseteq \mathcal{R}$ are compact open, then $\mathbb{1}_A \times \mathbb{1}_B = \mathbb{1}_{A \times B}$.

Proposition 5.7. *Let G be an ample Hausdorff groupoid.*

(i) For any $n \in \mathbb{N}$ and $f \in C_c(G^{(0)}, \mathbb{Z})^+$, we have

$$[f \times \mathbb{1}_{(0,0)}]_{\mathcal{K}G} = [f \times \mathbb{1}_{(n,n)}]_{\mathcal{K}G} \quad \text{in} \quad S(\mathcal{K}G).$$

(ii) There is an isomorphism of monoids $\varphi : S(G) \to S(\mathcal{K}G)$ satisfying

$$\varphi([f]_G) = [f \times \mathbb{1}_{(0,0)}]_{\mathcal{K}G}, \quad f \in C_c(G^{(0)}, \mathbb{Z})^+.$$

Proof. Let $n \in \mathbb{N}$ and $f \in C_c(G^{(0)}, \mathbb{Z})^+$. We can write f as a finite sum $f = \sum_k m_k \mathbb{1}_{A_k}$ with $m_k \in \mathbb{Z}^+$ and $A_k \subseteq G^{(0)}$ are compact open subsets. We then see that

$$\begin{aligned} f \times \mathbb{1}_{(0,0)} &= \Big(\sum_k m_k \mathbb{1}_{A_k}\Big) \times \mathbb{1}_{(0,0)} = \sum_k m_k (\mathbb{1}_{A_k} \times \mathbb{1}_{(0,0)}) = \sum_k m_k \mathbb{1}_{A_k \times \{(0,0)\}} \\ &= \sum_k m_k \mathbb{1}_{s(A_k \times \{(n,0)\})} \sim \sum_k m_k \mathbb{1}_{r(A_k \times \{(n,0)\})} = \sum_k m_k \mathbb{1}_{A_k \times \{(n,n)\}} = f \times \mathbb{1}_{(n,n)}, \end{aligned}$$

which proves (i).

The map $C_c(G^{(0)}, \mathbb{Z})^+ \to S(\mathcal{K}G)$ which sends $f \mapsto [f \times \mathbb{1}_{(0,0)}]_{\mathcal{K}G}$ is clearly well-defined and additive. If $f \sim_G g$ in $C_c(G, \mathbb{Z})^+$, then there exist compact open bisections $E_1, \dots, E_n$ in G such that $f = \sum_{i=1}^n \mathbb{1}_{s(E_i)}$ and $g = \sum \mathbb{1}_{r(E_i)}$. Hence

$$\begin{aligned} f \times \mathbb{1}_{(0,0)} &= \Big(\sum_{i=1}^n \mathbb{1}_{s(E_i)}\Big) \times \mathbb{1}_{(0,0)} = \sum_{i=1}^n \mathbb{1}_{s(E_i)} \times \mathbb{1}_{(0,0)} = \sum_{i=1}^n \mathbb{1}_{s(E_i) \times \{(0,0)\}} \\ &= \sum_{i=1}^n \mathbb{1}_{s(E_i \times \{(0,0)\})} \sim \sum_{i=1}^n \mathbb{1}_{r(E_i \times \{(0,0)\})} = g \times \mathbb{1}_{(0,0)}. \end{aligned}$$

There is, therefore, a well-defined monoid homomorphism $\varphi : S(G) \to S(\mathcal{K}G)$ satisfying the description in (ii).

To see that φ is surjective, fix $h \in C_c(\mathcal{K}G^{(0)}, \mathbb{Z})^+$ and write $h = \sum_k m_k \mathbb{1}_{A_k}$ for a finite list of positive integers m_k and compact open subsets $A_k \subseteq \mathcal{K}G^{(0)}$. By our discussion before the statement of the Proposition we may assume that each $A_k = B_k \times \{(i_k, i_k)\}$ with $B_k \subseteq G^{(0)}$ compact and open. Setting $f = \sum_k m_k \mathbb{1}_{B_k}$ in $C_c(G^{(0)}, \mathbb{Z})^+$ we get

$$\begin{aligned} [h]_{\mathcal{K}G} &= \Big[\sum_k m_k \mathbb{1}_{B_k \times \{(i_k, i_k)\}}\Big] = \sum_k m_k [\mathbb{1}_{B_k} \times \mathbb{1}_{(i_k, i_k)}] \overset{(i)}{=} \sum_k m_k [\mathbb{1}_{B_k} \times \mathbb{1}_{(0,0)}] \\ &= \sum_k m_k \varphi([\mathbb{1}_{B_k}]) = \varphi\Big(\sum_k m_k [\mathbb{1}_{B_k}]\Big) = \varphi([f]_G). \end{aligned}$$

We show that φ is injective by constructing a left inverse. The map $\rho : C_c(\mathcal{K}G^{(0)},\mathbb{Z})^+ \to C_c(G^{(0)},\mathbb{Z})^+$ given by

$$\rho(h)(u) := \sum_{n\in\mathbb{N}} h(u,n), \quad u \in G^{(0)}$$

is clearly well-defined and additive. Now if $F = E \times \{(i,j)\}$ is a compact open bisection in $\mathcal{K}G$, we see that for all $u \in G^{(0)}$

$$\begin{aligned}\rho(\mathbb{1}_{s(F)})(u) &= \sum_n \mathbb{1}_{s(F)}(u,n) = \sum_n \mathbb{1}_{s(E)\times\{(j,j)\}}(u,n)\\ &= \sum_n (\mathbb{1}_{s(E)} \times \mathbb{1}_{(j,j)})(u,n) = \mathbb{1}_{s(E)}(u),\end{aligned}$$

whence $\rho(\mathbb{1}_{s(F)}) = \mathbb{1}_{s(E)}$. Similarly, $\rho(\mathbb{1}_{r(F)}) = \mathbb{1}_{r(E)}$. Given any compact open bisection $F \subseteq \mathcal{K}G$, we can write $F = \bigsqcup_{k=1}^K F_k$ with $F_k = E_k \times \{(i_k, j_k)\}$, and we get

$$\begin{aligned}\rho(\mathbb{1}_{s(F)}) &= \rho\Big(\sum_k \mathbb{1}_{s(F_k)}\Big) = \sum_k \rho(\mathbb{1}_{s(F_k)}) = \sum_k \mathbb{1}_{s(E_k)} \sim \sum_k \mathbb{1}_{r(E_k)}\\ &= \sum_k \rho(\mathbb{1}_{r(F_k)}) = \rho\Big(\sum_k \mathbb{1}_{r(F_k)}\Big) = \rho(\mathbb{1}_{r(F)}).\end{aligned}$$

Since ρ is additive we conclude that if $h \sim g$ in $C_c(\mathcal{K}G^{(0)},\mathbb{Z})^+$ then $\rho(h) \sim \rho(g)$ in $C_c(G^{(0)},\mathbb{Z})^+$. Thus ρ descends to a monoid homomorphism $\tilde{\rho} : S(\mathcal{K}G) \to S(G)$ given by $\tilde{\rho}([h]_{\mathcal{K}G}) = [\rho(h)]_G$. It is quite clear that $\rho(f \times \mathbb{1}_{(0,0)}) = f$, so $\tilde{\rho}$ is the desired inverse of φ. □

Corollary 5.8. *Let G and H be ample groupoids with σ-compact unit spaces. If G and H are groupoid equivalent, then $S(G) \cong S(H)$.*

Proof. We know that $S(G) \cong S(\mathcal{K}G)$ and $S(H) \cong S(\mathcal{K}H)$ by Proposition 5.7. Since G and H are equivalent, [11, Theorem 2.1] shows that $\mathcal{K}G \cong \mathcal{K}H$, so the result follows from Proposition 5.6. □

6. Stably finite groupoid C*-algebras

This section is concerned with characterizing stably finite C*-algebras that arise from ample groupoids by the non-paradoxical nature of its type semigroup and by a K-theoretic coboundary property analogous to that seen in the work of N. Brown [10]. Our unified statements (Theorem 6.5 and Corollary 6.6) are the main results of this section and include, as promised, a converse to Proposition 3.5.

We first look at how the type semigroup $S(G)$ of an ample groupoid G reflects the notion of paradoxicality introduced in Definition 3.1.

Lemma 6.1. *Let G be an ample groupoid, and let $A \subseteq G^{(0)}$ be a non-empty compact open subset. Writing $\theta = [\mathbb{1}_A]_G$ in $S(G)$, A is (k,l)-paradoxical if and only if $k\theta \leq l\theta$ in $S(G)$.*

Proof. Suppose A is (k,l)-paradoxical and that $E_1,\dots,E_n \in \mathcal{C}$ satisfy 3. Then

$$k\theta = k[\mathbb{1}_A]_G = [k\mathbb{1}_A]_G \leq \Big[\sum_{i=1}^n \mathbb{1}_{s(E_i)}\Big]_G = \Big[\sum_{i=1}^n \mathbb{1}_{r(E_i)}\Big]_G \leq [l\mathbb{1}_A]_G = l[\mathbb{1}_A]_G = l\theta.$$

Conversely, suppose $k\theta \leq l\theta$. Then there is an $f \in C(G^{(0)}, \mathbb{Z})^+$ such that

$$[k\mathbb{1}_A + f]_G = [k\mathbb{1}_A]_G + [f]_G = k[\mathbb{1}_A]_G + [f]_G = l[\mathbb{1}_A]_G = [l\mathbb{1}_A]_G.$$

This means that there are compact open bisections $E_1, \dots, E_n$ with

$$k\mathbb{1}_A + f = \sum_{i=1}^{n} \mathbb{1}_{s(E_i)}, \quad \text{and} \quad \sum_{i=1}^{n} \mathbb{1}_{r(E_i)} = l\mathbb{1}_A.$$

It follows that A is (k, l)-paradoxical since $k\mathbb{1}_A \leq k\mathbb{1}_A + f$ and so 2 is satisfied. □

Before moving forward we recall some terminology for pre-ordered abelian monoids $(S, +)$. For positive integers $k > l > 0$, we say that a nonzero element $\theta \in S$ is (k, l)-*paradoxical* provided that $k\theta \leq l\theta$. If $\theta \neq 0$ fails to be (k, l)-paradoxical for all pairs of integers $k > l > 0$, call θ *completely non-paradoxical*. Note that θ is completely non-paradoxical if and only if $(n + 1)\theta \not\leq n\theta$ for all $n \in \mathbb{N}$. If every nonzero element in S is completely non-paradoxical we say that S is completely non-paradoxical. The above lemma basically states that in its setting, an subset $A \subseteq G^{(0)}$ is completely non-paradoxical precisely when $[\mathbb{1}_A]_G$ is completely non-paradoxical in the pre-ordered abelian monoid $S(G)$, and G is completely non-paradoxical if and only if $S(G)$ is completely non-paradoxical. A *state* on S is a map $\nu : S \to [0, \infty]$ which is additive, respects the pre-ordering $\leq$, and satisfies $\nu(0) = 0$. If a state assumes a value other than 0 or ∞, it said to be *non-trivial*.

The following result is a main ingredient in the proof of Tarski's theorem. It is a Hahn-Banach type extension result and is essential in establishing a converse to Proposition 3.5. A proof can be found in [49].

Theorem 6.2. *Let $(S, +)$ be an abelian monoid equipped with the algebraic ordering, and let θ be an element of S. Then the following are equivalent:*

(i) $(n + 1)\theta \not\leq n\theta$ for all $n \in \mathbb{N}$, that is θ is completely non-paradoxical.
(ii) There is a non-trivial state $\nu : S \to [0, \infty]$ with $\nu(\theta) = 1$.

For the next lemma we need the notion of an invariant state. If G is an étale, ample groupoid, we will call a state $\beta : C_c(G^{(0)}, \mathbb{Z}) \to \mathbb{R}$ *invariant* if $\beta(\mathbb{1}_{s(E)}) = \beta(\mathbb{1}_{r(E)})$ for any compact open bisection E.

Lemma 6.3. *Let G be an ample groupoid with compact unit space $G^{(0)}$. Consider the following properties.*

(i) For every non-empty closed and open $U \subseteq G^{(0)}$, there is a faithful invariant positive group homomorphism $\beta : C(G^{(0)}, \mathbb{Z}) \to \mathbb{R}$ with $\beta(\mathbb{1}_U) = 1$.
(ii) There is a faithful invariant state β on the dimension group

$$(C(G^{(0)}, \mathbb{Z}), C(G^{(0)}, \mathbb{Z})^+, \mathbb{1}_{G^{(0)}}).$$

(iii) G is completely non-paradoxical.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) always hold. If G is minimal then (iii) $\Rightarrow$ (i) whence all conditions are equivalent.

Proof. $(i) \Rightarrow (ii)$: Simply take $U = G^{(0)}$.

$(ii) \Rightarrow (iii)$: Suppose a compact open subset $A \subseteq G^{(0)}$ is (k,l)-paradoxical for a pair of positive integers $k > l > 0$. There are then $E_1, \dots, E_n \in \mathcal{C}$ with

$$k\mathbb{1}_A \leq \sum_{j=1}^n \mathbb{1}_{s(E_j)}, \quad \text{and} \quad \sum_{j=1}^n \mathbb{1}_{r(E_j)} \leq l\mathbb{1}_A.$$

Applying β one gets

$$\begin{aligned} k\beta(\mathbb{1}_A) = \beta(k\mathbb{1}_A) \leq \beta\Big(\sum_{j=1}^n \mathbb{1}_{s(E_j)}\Big) &= \sum_{j=1}^n \beta(\mathbb{1}_{s(E_j)}) = \sum_{j=1}^n \beta(\mathbb{1}_{r(E_j)}) = \beta\Big(\sum_{j=1}^n \mathbb{1}_{r(E_j)}\Big) \\ \leq \beta(l\mathbb{1}_A) = l\beta(\mathbb{1}_A). \end{aligned}$$

If A is non-empty then $\mathbb{1}_A \neq 0$ and since β is faithful we may divide by $\beta(\mathbb{1}_A) > 0$ to yield $k \leq l$, which is contradictory. Therefore $A = \emptyset$ and G is completely non-paradoxical.

Assuming G is minimal we prove $(iii) \Rightarrow (i)$. Let $U \subseteq G^{(0)}$ be a non-empty compact open set, and consider $\theta = [\mathbb{1}_U]_G$ in $S(G)$. Since G is completely non-paradoxical θ is completely non-paradoxical and therefore Theorem 6.2 provides a non-trivial state $\nu : S(G) \to [0,\infty]$ with $\nu(\theta) = 1$. Composing this state with the quotient mapping $\pi : C(G^{(0)},\mathbb{Z})^+ \to S(G)$ yields

$$\beta := \nu\pi : C(G^{(0)},\mathbb{Z})^+ \longrightarrow [0,\infty],$$

an additive, order-preserving map with $\beta(0) = 0$ and $\beta(\mathbb{1}_U) = 1$. By construction β is invariant; indeed if E is a compact open bisection in G then

$$\beta(\mathbb{1}_{s(E)}) = \nu\pi(\mathbb{1}_{s(E)}) = \nu([\mathbb{1}_{s(E)}]_G) = \nu([\mathbb{1}_{r(E)}]_G) = \nu\pi(\mathbb{1}_{r(E)}) = \beta(\mathbb{1}_{r(E)}).$$

We claim that β is in fact finite. Since G is minimal, there are compact open bisections $E_1, \dots, E_n$ with

$$\mathbb{1}_{G^{(0)}} \leq \sum_{j=1}^n E_j\mathbb{1}_U = \sum_{j=1}^n \mathbb{1}_{r(E_jU)}.$$

Note that for a compact open bisection $E \in \mathcal{C}$ we have $s(EU) \subseteq U$ and so $\mathbb{1}_{s(EU)} \leq \mathbb{1}_U$. Using this fact and applying β to the above inequality gives

$$\beta(\mathbb{1}_{G^{(0)}}) \leq \beta\Big(\sum_{j=1}^n \mathbb{1}_{r(E_jU)}\Big) = \sum_{j=1}^n \beta(\mathbb{1}_{r(E_jU)}) = \sum_{j=1}^n \beta(\mathbb{1}_{s(E_jU)}) \leq \sum_{j=1}^n \beta(\mathbb{1}_U) = n < \infty,$$

where we have used the invariance and order-preserving properties of β. If $f \in C(G^{(0)},\mathbb{Z})^+$, write $f = \sum_{k=1}^K m_k\mathbb{1}_{A_k}$, where $m_k \in \mathbb{Z}^+$ and the A_k are closed and open subsets of $G^{(0)}$. Since $\mathbb{1}_{A_k} \leq \mathbb{1}_{G^{(0)}}$ we have

$$\beta(f) = \beta\Big(\sum_{k=1}^K m_k\mathbb{1}_{A_k}\Big) = \sum_{k=1}^K m_k\beta(\mathbb{1}_{A_k}) \leq \beta(\mathbb{1}_{G^{(0)}})\sum_{k=1}^K m_k$$

which is finite.

Now we show that β is faithful. Suppose that $A \subseteq G^{(0)}$ is a non-empty closed and open subset with $\beta(\mathbb{1}_A) = 0$. Again we use minimality to find compact open bisections $E_1, \dots, E_n$ satisfying

$$\mathbb{1}_{G^{(0)}} \leq \sum_{j=1}^{n} E_j \mathbb{1}_A = \sum_{j=1}^{n} \mathbb{1}_{r(E_j A)}.$$

Applying β and using its properties we get

$$\begin{aligned} 1 = \beta(\mathbb{1}_U) \leq \beta(\mathbb{1}_{G^{(0)}}) \leq \beta\Big(\sum_{j=1}^{n} \mathbb{1}_{r(E_j A)}\Big) &= \sum_{j=1}^{n} \beta(\mathbb{1}_{r(E_j A)}) = \sum_{j=1}^{n} \beta(\mathbb{1}_{s(E_j A)}) \\ &\leq \sum_{j=1}^{n} \beta(\mathbb{1}_A) = 0 \end{aligned}$$

which is absurd. Therefore $\beta(\mathbb{1}_A) > 0$ for every non-empty closed and open $A \subseteq G^{(0)}$. Now suppose $f \in C(G^{(0)}, \mathbb{Z})^+$ is non-zero. We may write $f = \sum_{k=1}^{K} m_k \mathbb{1}_{A_k}$ where each $A_k \subseteq G^{(0)}$ is non-empty, closed and open, and every m_k is a strictly positive integer. Since $\beta(\mathbb{1}_{A_k}) > 0$ for every k, it easily follows that $\beta(f) > 0$ as well.

To complete the proof we need only extend β to all of $C(G^{(0)}, \mathbb{Z})$; but this is easily done by expressing $f \in C(G^{(0)}, \mathbb{Z})$ as the difference of its positive and negative parts $f = f^+ - f^-$, $f^+, f^- \in C(G^{(0)}, \mathbb{Z})^+$ and applying β additively. □

With lemma 6.3 at our disposal we can characterize stably finite C*-algebras that arise from ample groupoids. Before we delve into the details, it is natural to tie stable finiteness with the idea of a coboundary subgroup.

Definition 6.4. Let G be an ample groupoid and let $\mathcal{C}$ denote the family of all compact open bisections in G. We define the *coboundary subgroup of* G as the subgroup of $C_c(G^{(0)}, \mathbb{Z})$ generated by differences $\mathbb{1}_{s(E)} - \mathbb{1}_{r(E)}$, $E \in \mathcal{C}$, that is

$$H_G := \big\langle \mathbb{1}_{s(E)} - \mathbb{1}_{r(E)} \mid E \in \mathcal{C} \big\rangle.$$

We say that G satisfies the *coboundary condition* if $H_G \cap C_c(G^{(0)}, \mathbb{Z})^+ = \{0\}$.

Here is the main result of this section.

Theorem 6.5. *Let G be an ample groupoid with compact unit space $G^{(0)}$. Consider the following properties.*

(i) The C*-*algebra* $C^*_r(G)$ *admits a faithful tracial state.*
(ii) The C*-*algebra* $C^*_r(G)$ *is stably finite.*
(iii) G *satisfies the coboundary condition.*
(iv) G *is completely non-paradoxical.*

The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ *always hold. If G is minimal, then* $(iv) \Rightarrow (i)$ *and all properties are equivalent.*

If G is minimal and amenable, then properties (i) *through* (iv) *are all equivalent to*

(v) The C*-*algebra* $C^*_r(G)$ *is quasidiagonal.*

Proof. *(i)* $\Rightarrow$ *(ii)*: Any unital C*-algebra that admits a faithful tracial state is stably finite.

(ii) $\Rightarrow$ *(iii)*: Applying the K_0-functor to the canonical inclusion $\iota : C(G^{(0)}) \hookrightarrow \mathrm{C}^*_r(G)$ gives a positive group homomorphism

$$\mathrm{K}_0(\iota) : C(G^{(0)}, \mathbb{Z}) \cong \mathrm{K}_0(C(G^{(0)})) \longrightarrow \mathrm{K}_0(\mathrm{C}^*_r(G)).$$

The fact that $\mathrm{C}^*_r(G)$ is stably finite ensures that $\mathrm{K}_0(\iota)$ is faithful (see section 2.3), that is, its kernel contains no non-zero positive elements. However, we know that $\mathrm{K}_0(\iota)(\mathbb{1}_{s(E)}) = \mathrm{K}_0(\iota)(\mathbb{1}_{r(E)})$ for every compact open bisection E (once more see section 2.3), therefore $H_G \subseteq \ker(\mathrm{K}_0(\iota))$. It now follows that $H_G \cap C(G^{(0)}, \mathbb{Z})^+ = \{0\}$.

(iii) $\Rightarrow$ *(iv)*: Assume, by way of contradiction, that G is not completely non-paradoxical so that we can find a non-empty closed and open $A \subseteq G^{(0)}$, positive integers $k > l > 0$ and compact open bisections $E_1, \dots, E_n \in \mathcal{C}$ satisfying

$$k\mathbb{1}_A \leq \sum_{i=1}^{n} \mathbb{1}_{s(E_i)}, \quad \text{and} \quad \sum_{i=1}^{n} \mathbb{1}_{r(E_i)} \leq l\mathbb{1}_A.$$

Then

$$0 < (k-l)\mathbb{1}_A = k\mathbb{1}_A - l\mathbb{1}_A \leq \sum_{i=1}^{n} \mathbb{1}_{s(E_i)} - \sum_{i=1}^{n} \mathbb{1}_{r(E_i)} = \sum_{i=1}^{n} \left(\mathbb{1}_{s(E_i)} - \mathbb{1}_{r(E_i)}\right)$$

which certainly belongs to $H_G \cap C(G^{(0)}, \mathbb{Z})^+$, a contradiction. Thus G is completely non-paradoxical.

Assuming G is minimal and completely non-paradoxical we prove *(iv)* $\Rightarrow$ *(i)*. By Lemma 6.3 there is an invariant faithful state $\beta : C(G^{(0)}, \mathbb{Z}) \to \mathbb{R}$. Composing with the isomorphism of dimension groups $\dim : \mathrm{K}_0(C(G^{(0)})) \cong C(G^{(0)}, \mathbb{Z})$ gives us a state on the K_0-group

$$\tilde{\beta} := \beta \circ \dim : \mathrm{K}_0(C(G^{(0)})) \longrightarrow \mathbb{R}.$$

As $C(G^{(0)})$ is a commutative AF algebra, states on $\mathrm{K}_0(C(G^{(0)}))$ arise from traces ([4], [19]), so let τ be the tracial state on $C(G^{(0)})$ such that $\mathrm{K}_0(\tau) = \tilde{\beta}$. Moreover, τ is simply integration against a regular, Borel, probability measure μ on $G^{(0)}$. All together, if $A \subseteq G^{(0)}$ is closed and open we get

$$\mu(A) = \int_{G^{(0)}} \mathbb{1}_A d\mu = \tau(\mathbb{1}_A) = \mathrm{K}_0(\tau)([\mathbb{1}_A]_0) = \tilde{\beta}([\mathbb{1}_A]_0) = \beta \circ \dim([\mathbb{1}_A]_0) = \beta(\mathbb{1}_A).$$

The G-invariance of μ now follows from the invariance of β, indeed, if E is a compact open bisection, then

$$\mu(s(E)) = \beta(\mathbb{1}_{s(E)}) = \beta(\mathbb{1}_{r(E)}) = \mu(r(E)).$$

We also claim that τ is faithful. To see this we note that the measure μ has full support. For if $\mu(U) = 0$ for a non-empty open subset $U \subseteq G^{(0)}$, we can find a non-empty closed and open $A \subseteq U$ with $\mu(A) = 0$. It then follows that $\beta(\mathbb{1}_A) = 0$ which contradicts the fact that β is faithful. Thus no such U exists and so μ has full support whence τ is faithful. Finally, composing τ with the faithful conditional expectation $\mathbb{E} : \mathrm{C}^*_r(G) \to C(G^{(0)})$ gives a faithful tracial state $\tau \circ \mathbb{E} : \mathrm{C}^*_r(G) \to \mathbb{C}$ as desired.

For an amenable G, we have that $C^*_r(G)$ is separable, nuclear, and satisfies the UCT. If, moreover, $C^*_r(G)$ admits a faithful trace, then [48, Theorem A] and the subsequent remarks ensure that $C^*_r(G)$ is quasidiagonal. □

Using Proposition 5.7, we can extend the key statement of Theorem 6.5 to the situation of groupoids with non-compact unit space.

Corollary 6.6. *Let G be a minimal, ample groupoid. The following are equivalent:*

(i) $C^*_r(G)$ admits a faithful semifinite trace τ such that $0 < \tau(\mathbb{1}_K) < \infty$ for every compact open $K \subseteq G^{(0)}$;
(ii) $C^*_r(G)$ is stably finite;
(iii) G satisfies the coboundary condition; and
(iv) G is completely non-paradoxical.

If G is also amenable, then properties (i) through (iv) are all equivalent to

(v) The C^*-algebra $C^*_r(G)$ is quasidiagonal.

Proof. If $C^*_r(G)$ admits a nontrivial faithful semifinite trace as in (i) then it is stably finite because the collection $\{\mathbb{1}_K \mid K \subseteq G^{(0)}$ compact open $\}$ forms an approximate identity for $C^*_r(G)$.

Fix a compact open set $K \subseteq G^{(0)}$ and set $H := KGK = \{\gamma \in G : r(\gamma), s(\gamma) \in K\}$. Since G is minimal, we have $G^{(0)} = \{r(\gamma) : s(\gamma) \in K\}$. Hence GK is a G–H equivalence. This gives $\mathcal{K}G \cong \mathcal{K}H$ by [11, Theorem 2.1]. Also $C^*_r(G)$ and $C^*_r(H)$ are stably isomorphic, and we deduce that $C^*_r(G)$ is stably finite if and only if $C^*_r(H)$ is. By the proof of Proposition 5.7 we see that G satisfies the coboundary condition if and only its stabilization $\mathcal{K}G$ does. Using that G and H are stably isomorphic we learn that G satisfies the coboundary condition if and only if H does which in turn occurs if and only if $C^*_r(H)$ is stably finite by Theorem 6.5. Thus $C^*_r(G)$ is stably finite if and only if G satisfies the coboundary condition.

Moreover, if $C^*_r(G)$ is stably finite, then so is $C^*_r(H)$, so there is a faithful trace τ on $C^*_r(H)$. From this we obtain a faithful semifinite trace $\tilde{\tau}$ on $C^*_r(\mathcal{K}H)$ satisfying $\tilde{\tau}(f) = \sum_{n\in\mathbb{N}} \tau(f|_{H^{(0)}\times\{(n,n)\}})$ for $f \in C_c(\mathcal{K}H)$. Since $\mathcal{K}G \cong \mathcal{K}H$, the map $\tilde{\tau}$ induces a semifinite trace on $C^*_r(\mathcal{K}G)$ which is nonzero and finite on the indicator function of any compact open set of units. This semifinite trace then restricts to a faithful semifinite trace on the corner $C^*_r(G)$ of $C^*_r(\mathcal{K}G)$.

By Proposition 3.5, if $C^*_r(G)$ is stably finite, then G is completely non-paradoxical. Conversely, if G is completely non-paradoxical, then since compact open bisections of H are compact open bisections of G, we see that H is completely non-paradoxical, and then (iv) $\implies$ (ii) of Theorem 6.5 shows that $C^*_r(H)$ is stably finite. We saw above that $C^*_r(G)$ is stably isomorphic to $C^*_r(G)$, so we deduce that $C^*_r(G)$ is stably finite.

If G is amenable and $C^*_r(G)$ is stably finite, then $C^*_r(H)$ is unital, separable, nuclear, has a faithful trace and satisfies the UCT. Qasidiagonality of $C^*_r(H)$ again follows from the main result in [48]. Since $C^*_r(G)$ and $C^*_r(H)$ are stably isomorphic we conclude that $C^*_r(G)$ is quasidiagonal too. □

Thus stable finiteness for C^*-algebras arising from minimal, ample groupoids G is characterized by the 'non-infinite' nature of the type semigroup G. More precisely, if we call an element $\theta \in S(G)$ *infinite* provided $(n+1)\theta \leq n\theta$ for some $n \in \mathbb{N}$, then Theorem 6.5 says that $C^*_r(G)$ is stably finite provided that $S(G)$ contains no infinite elements. In the next section we shall look at the diametrically opposite setting where every element in $S(G)$ is not only infinite, but 'properly infinite'.

7. Purely Infinite Groupoid C*-algebras

We now wish to characterize purely infinite reduced groupoid C*-algebras by the 'properly infinite' nature of the corresponding type semigroup constructed above. This coincides in spirit with the work of Rørdam and Sierakowski in [44] and of the first author in [37].

Recall that a projection p in a C*-algebra A is properly infinite if there are two subprojections $q, r \leq p$ with $qr = 0$ and $q \sim p \sim r$. A unital C*-algebra A is properly infinite if its unit 1_A is properly infinite. Purely infinite C*-algebras were introduced by J. Cuntz in [13]; an algebra A is called purely infinite if every hereditary C*-subalgebra of A contains a properly infinite projection. It was a longstanding open question whether all unital, separable, simple, and nuclear C*-algebra satisfied the stably finite/ purely infinite dichotomy until M. Rørdam answered this query negatively in [42]. He constructed a unital, simple, nuclear, and separable C*-algebra D containing a finite projection p and an infinite projection q. It follows that $A = qDq$ is unital, separable, nuclear, simple, and properly infinite, but not purely infinite. It seems natural to ask if there is a smaller class of algebras for which a stably finite/purely infinite dichotomy holds. Theorem 7.4 below gives a partial answer in this direction.

We first present a necessary condition for a groupoid C*-algebra to be purely infinite.

Proposition 7.1. *Let G be an étale groupoid. If $C^*_r(G)$ is purely infinite, then for any non-empty compact open $U \subseteq G^{(0)}$, there is an open bisection $E \in \mathcal{B}$ with $E \cap G^{(0)} = \emptyset$ such that*

$$r(EU) \cap U \neq \emptyset.$$

In particular, for every non-empty compact open $U \subseteq G^{(0)}$, there is an $\alpha \notin G^{(0)}$ with $r(\alpha), s(\alpha) \in U$.

Proof. Since $C^*_r(G)$ is purely infinite the projection $p = \mathbb{1}_U$ is properly infinite (Theorem 4.16 in [26]). So there are $x, y \in C^*_r(G)$ that satisfy

$$x^*x = p = y^*y, \quad xx^* \perp yy^*, \quad xx^*, yy^* \leq p.$$

The $*$-algebra $C_c(G)$ is norm-dense in $C^*_r(G)$, so we may find sequences $(a_n)_n, (b_n)_n$ in $C_c(G)$ converging to x and y respectively. We now compress by setting $x_n := pa_np$ and $y_n := pb_np$ and note the following:

$$(x_n)_n \longrightarrow x \quad \text{since} \quad x_n = pa_np \longrightarrow pxp = pxx^*x = xx^*x = x,$$

$$(y_n)_n \longrightarrow y \quad \text{by a similar argument,}$$

$$(x_n^*x_n)_n \longrightarrow p \quad \text{since} \quad x_n^*x_n \longrightarrow x^*x = p,$$

$$\text{similarly} \quad (y_n^*y_n)_n \longrightarrow p, \quad \text{and}$$

$$(x_n^*y_n)_n \longrightarrow 0 \quad \text{since} \quad x_n^*y_n \longrightarrow x^*y = x^*xx^*yy^*y = 0.$$

Moreover, the x_n cannot be normal for all large n. To see why, suppose $x_n^*x_n = x_nx_n^*$ for n large, then we would have

$$p = p^2 = \Big(\lim_n x_n^*x_n\Big)\, p = \Big(\lim_n x_nx_n^*\Big)\, p = xx^*p = xx^*$$

which contradicts the fact that p is infinite. By passing to a subsequence we may assume that all the x_n are non-normal.

Since G is étale we can write $x_n = \sum_{k=1}^{K_n} f_{n,k}$ such that, for a fixed n, the $f_{n,k} \in C_c(G)$ are non-zero and supported on distinct open bisections, say $E_{n,k}$, with $E_{n,1} \subseteq G^{(0)}$ and $E_{n,k} \cap G^{(0)} = \emptyset$ for $2 \le k \le K_n$. Since $px_np = x_n$ we get

$$x_n = \sum_{k=1}^{K_n} f_{n,k} = \sum_{k=1}^{K_n} \mathbb{1}_U f_{n,k} \mathbb{1}_U.$$

If $\mathbb{1}_U f_{n,k}\mathbb{1}_U = 0$ for $k = 2, \ldots, K_n$, then $x_n = \mathbb{1}_U f_{n,1}\mathbb{1}_U$ is normal, contradicting our assumption. Therefore, there is an open bisection, say E, with $E \cap G^{(0)} = \emptyset$, and a non-zero $f \in C_c(G)$ supported in E such that $\mathbb{1}_U f \mathbb{1}_U \neq 0$. But

$$\emptyset \neq \operatorname{supp}(\mathbb{1}_U f \mathbb{1}_U) \subseteq UEU = \{\alpha \mid \alpha \in E, s(\alpha), r(\alpha) \in U\}.$$

It now follows that $r(EU) \cap U \neq \emptyset$. □

As alluded to above, it is the properly infinite nature of the type semigroup $S(G)$ that will generate properly infinite projections in $C_r^*(G)$. This is seen in Lemma 7.2 below. For this reason we introduce some terminology. An element θ in a pre-ordered abelian group S is said to *properly infinite* if $2\theta \le \theta$, that is, if it is $(2,1)$-paradoxical, or equivalently it is $(k,1)$-paradoxical for any $k \ge 2$. If every member of S is properly infinite then S is said to be *purely infinite*. A pre-ordered monoid S is said to be *almost unperforated* if, whenever $\theta, \eta \in S$, and $n, m \in \mathbb{N}$ are such that $n\theta \le m\eta$ and $n > m$, then $\theta \le \eta$.

Lemma 7.2. *Let G be an ample groupoid, and suppose $A \subseteq G^{(0)}$ is a compact open subset. Let $\theta = [\mathbb{1}_A]_G$ denote the class of $\mathbb{1}_A$ in $S(G)$. The following are equivalent.*

(i) There are mutually disjoint compact open bisections

$$F_1, \ldots, F_n, H_1, \ldots, H_n \in \mathcal{C}$$

satisfying the following: writing $x = \sum_{i=1}^n \mathbb{1}_{F_i}$ and $y = \sum_{i=1}^n \mathbb{1}_{H_i}$ in $C_r^*(G)$,

$$x^*x = \mathbb{1}_A, \quad y^*y = \mathbb{1}_A, \quad xx^* + yy^* \le \mathbb{1}_A.$$

(ii) A is $(k,1)$-paradoxical for some $k \ge 2$.

(iii) θ is properly infinite in $S(G)$.

Proof. (i) $\Rightarrow$ (ii): By assumption we have

$$\mathbb{1}_A = x^*x = \Big(\sum_{i=1}^n \mathbb{1}_{F_i}\Big)^* \Big(\sum_{i=1}^n \mathbb{1}_{F_i}\Big) = \sum_{i,j} \mathbb{1}_{F_i}^* \mathbb{1}_{F_j} = \sum_{i,j} \mathbb{1}_{F_i^{-1}} \mathbb{1}_{F_j} = \sum_{i,j} \mathbb{1}_{F_i^{-1}F_j}.$$

Applying the conditional expectation $\mathbb{E}$ on both sides we get

$$\mathbb{1}_A = \mathbb{E}(\mathbb{1}_A) = \mathbb{E}\Big(\sum_{i,j} \mathbb{1}_{F_i^{-1}F_j}\Big) = \sum_{i,j} \mathbb{E}(\mathbb{1}_{F_i^{-1}F_j}) = \sum_{i,j} \mathbb{1}_{F_i^{-1}F_j \cap G^{(0)}} = \sum_{i=1}^n \mathbb{1}_{s(F_i)},$$

where we have used the fact that the F_i are mutually disjoint to ensure that for $i \neq j$, $F_i^{-1}F_j \cap G^{(0)} = \emptyset$, and $F_i^{-1}F_i = s(F_i)$. By a similar calculation we also have

$$\mathbb{1}_A = \sum_{i=1}^{n} \mathbb{1}_{s(H_i)}.$$

Moreover, applying the conditional expectation to

$$\begin{aligned}\mathbb{1}_A \geq xx^* + yy^* &= \Big(\sum_{i=1}^{n} \mathbb{1}_{F_i}\Big)\Big(\sum_{i=1}^{n} \mathbb{1}_{F_i}\Big)^* + \Big(\sum_{i=1}^{n} \mathbb{1}_{H_i}\Big)\Big(\sum_{i=1}^{n} \mathbb{1}_{H_i}\Big)^* \\ &= \sum_{i,j} \mathbb{1}_{F_iF_j^{-1}} + \sum_{i=i,j} \mathbb{1}_{H_iH_j^{-1}}\end{aligned}$$

gives

$$\begin{aligned}\mathbb{1}_A = \mathbb{E}(\mathbb{1}_A) \geq \mathbb{E}\Big(\sum_{i,j} \mathbb{1}_{F_iF_j^{-1}} + \sum_{i,j} \mathbb{1}_{H_iH_j^{-1}}\Big) &= \sum_{i,j} \mathbb{1}_{F_iF_j^{-1}\cap G^{(0)}} + \sum_{i=i,j} \mathbb{1}_{H_iH_j^{-1}\cap G^{(0)}} \\ = \sum_{i=1}^{n} \mathbb{1}_{F_iF_i^{-1}} + \sum_{i=1}^{n} \mathbb{1}_{H_iH_i^{-1}} = \sum_{i=1}^{n} \mathbb{1}_{r(F_i)} + \sum_{i=1}^{n} \mathbb{1}_{r(H_i)}.\end{aligned}$$

Again here we use disjointness so that $F_iF_j^{-1} \cap G^{(0)} = \emptyset = H_iH_j^{-1} \cap G^{(0)}$ for $i \neq j$. All together,

$$2\mathbb{1}_A = \sum_{i=1}^{n} \mathbb{1}_{s(F_i)} + \sum_{i=1}^{n} \mathbb{1}_{s(H_i)}, \quad \text{and} \quad \sum_{i=1}^{n} \mathbb{1}_{r(F_i)} + \sum_{i=1}^{n} \mathbb{1}_{r(H_i)} \leq \mathbb{1}_A$$

which says that $\mathbb{1}_A$ is $(2,1)$-paradoxical.

$(ii) \Rightarrow (i)$: We may suppose that $k = 2$ since $2\mathbb{1}_A \leq k\mathbb{1}_A$. There are, therefore, compact open bisections $E_1, \ldots, E_n$ in G with

$$2\mathbb{1}_A \leq \sum_{i=1}^{n} \mathbb{1}_{s(E_i)}, \quad \text{and} \quad \sum_{i=1}^{n} \mathbb{1}_{r(E_i)} \leq \mathbb{1}_A.$$

This condition implies that the $r(E_i)$ are mutually disjoint and therefore the bisections E_i are themselves mutually disjoint.

The inequality $2\mathbb{1}_A \leq \sum_{i=1}^{n} \mathbb{1}_{s(E_i)}$ implies that by partitioning copies of A, we can find compact open sets $\{A_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq 2\}$ such that

$$\bigsqcup_{i=1}^{n} A_{i,1} = A, \quad \bigsqcup_{i=1}^{n} A_{i,2} = A, \quad A_{i,1} \sqcup A_{i,2} \subseteq s(E_i) \quad \forall i \in \{1, \ldots, n\}.$$

Now for each $i = 1, \ldots, n$ set

$$F_i := (s|_{E_i})^{-1}(A_{i,1}) \quad \text{and} \quad H_i := (s|_{E_i})^{-1}(A_{i,2}).$$

Note the following:

$$\begin{aligned}F_i \subseteq E_i, \quad s(F_i) = A_{i,1}, \quad i \neq j \Rightarrow r(F_i) \cap r(F_j) = \emptyset, \\ H_i \subseteq E_i, \quad s(H_i) = A_{i,2}, \quad i \neq j \Rightarrow r(F_i) \cap r(F_j) = \emptyset.\end{aligned}$$

Moreover, $F_i \cap H_i = \emptyset$ for every i, and since the E_i are disjoint, it follows that the compact open bisections $F_1, \ldots, F_n, H_1, \ldots, H_n$ are mutually disjoint.

Using these facts and writing x and y as in the statement of the lemma we get

$$x^*x = \left(\sum_{i=1}^n \mathbb{1}_{F_i}\right)^* \left(\sum_{i=1}^n \mathbb{1}_{F_i}\right) = \sum_{i,j} \mathbb{1}_{F_i^{-1}F_j} = \sum_{i=1}^n \mathbb{1}_{s(F_i)} = \sum_{i=1}^n \mathbb{1}_{A_{i,1}} = \mathbb{1}_A,$$

$$y^*y = \left(\sum_{i=1}^n \mathbb{1}_{H_i}\right)^* \left(\sum_{i=1}^n \mathbb{1}_{H_i}\right) = \sum_{i,j} \mathbb{1}_{H_i^{-1}H_j} = \sum_{i=1}^n \mathbb{1}_{s(H_i)} = \sum_{i=1}^n \mathbb{1}_{A_{i,2}} = \mathbb{1}_A,$$

where we use the fact that $F_i^{-1}F_j = \emptyset = H_i^{-1}H_j = \emptyset$ for $i \neq j$ since these have disjoint ranges. Now using the fact that $s(F_i) \cap s(F_j) = \emptyset = s(H_i) \cap s(H_j)$ for $i \neq j$, and that E_i are compact open bisections we get

$$\begin{aligned} xx^* + yy^* &= \sum_{i,j} \mathbb{1}_{F_iF_j^{-1}} + \sum_{i,j} \mathbb{1}_{H_iH_j^{-1}} = \sum_{i=1}^n \mathbb{1}_{r(F_i)} + \sum_{i=1}^n \mathbb{1}_{r(H_i)} \\ &= \sum_{i=1}^n (\mathbb{1}_{r(F_i)} + \mathbb{1}_{r(H_i)}) \leq \sum_{i=1}^n \mathbb{1}_{r(E_i)} \leq \mathbb{1}_A, \end{aligned}$$

thus (i) holds.

The equivalence $(ii) \Leftrightarrow (iii)$ follows directly from Lemma 6.1. □

We now come to the main result of this section. We use the constructed type semigroup $S(G)$ to characterize purely infinite and traceless C*-algebras that arise from ample groupoids. Recall from [27] that a C*-algebra A is said to be traceless if it admits no non-zero lower semi-continuous (possibly unbounded) 2-quasitrace defined on an algebraic ideal of A. The following theorem is in the spirit of Theorem 5.4 of [44] and Theorem 5.6 of [37].

Theorem 7.3. *Let G be an ample groupoid which is minimal and topologically principal. Consider the following properties:*

(i) The semigroup $S(G)$ is purely infinite.
(ii) Every non-empty compact open $A \subseteq G^{(0)}$ is properly paradoxical.
(iii) The C-algebra $C_r^*(G)$ is purely infinite.*
(iv) The C-algebra $C_r^*(G)$ is traceless.*
(v) The semigroup $S(G)$ admits no non-trivial state.

The following implications always hold: $(i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$. If the semigroup $S(G)$ is almost unperforated then $(v) \Rightarrow (i)$ and all properties are equivalent.

Proof. $(i) \Rightarrow (ii)$: Let $A \subseteq G^{(0)}$ be compact and open. Since we are assuming that $\theta := [\mathbb{1}_A]_G$ is properly infinite, we have $2\theta \leq \theta$. Lemma 6.1 now says that A is $(2,1)$-paradoxical whence properly paradoxical.

$(ii) \Rightarrow (i)$: Let $\theta = [f]_G$ be a non-zero element in $S(G)$. We may write $f = \sum_k \mathbb{1}_{A_k}$ where $A_k \subseteq G^{(0)}$ are nonempty compact open subsets. Setting $\theta_k := [\mathbb{1}_{A_k}]_G$, again

Lemma 6.1 implies that $2\theta_k \leq \theta_k$ for each k. The quotient map is additive so $\sum_k \theta_k = \theta$. It easily follows that θ is properly paradoxical since

$$2\theta = 2\sum_k \theta_k = \sum_k 2\theta_k \leq \sum_k \theta_k = \theta.$$

$(ii) \Rightarrow (iii)$: Since G is minimal and topologically principal, by the main result in [7] it suffices to check that every non-empty compact open set $A \subseteq G^{(0)}$ defines a properly infinite projection $\mathbb{1}_A$ in $\mathrm{C}^*_r(G)$. By our assumption every such A is $(2,1)$-paradoxical, and Lemma 7.2 implies that $\mathbb{1}_A$ is properly infinite.

$(iii) \Rightarrow (iv)$: Purely infinite algebras are always traceless [27].

$(iv) \Rightarrow (v)$: Suppose $\nu : S(G) \to [0,\infty]$ is a non-trivial state, so that $0 < \nu([\mathbb{1}_U]_G) < \infty$ for some non-empty compact open subset $U \subseteq G^{(0)}$. Composing with the quotient map $\pi : C_c(G^{(0)},\mathbb{Z})^+ \to S(G)$ we get an order preserving monoid homomorphism $\beta = \nu \circ \pi : C_c(G^{(0)},\mathbb{Z})^+ \to [0,\infty]$ with $0 < \beta(\mathbb{1}_U) < \infty$. If $\mathcal{C}_0$ denotes the algebra of all of all compact open subsets of $G^{(0)}$, then the map

$$\mu_0 : \mathcal{C}_0 \to [0,\infty] \qquad \mu_0(A) = \beta(\mathbb{1}_A)$$

defines a pre-measure in the sense of [16] (page 30) satisfying $\mu_0(U) \in (0,\infty)$, and $\mu_0(r(E)) = \mu_0(s(E))$ for every compact open bisection E. The σ-algebra generated by $\mathcal{C}_0$ is the Borel σ-algebra $\mathcal{B}_{G^{(0)}}$, so μ_0 extends to an invariant Borel measure μ. As in the last paragraph of section 2 we obtain a lower-semicontinuous tracial weight $\tilde{\tau}_\mu := I_\mu \circ \mathbb{E} : \mathrm{C}^*_r(G)^+ \to [0,\infty]$ with $\tilde{\tau}(\mathbb{1}_U) = \mu(U) \in (0,\infty)$. By standard techniques we restrict (and extend) $\tilde{\tau}_\mu$ to a linear, unbounded, lower-semicontinuous trace τ_μ defined on the algebraic ideal generated by the domain $D := \{a \in \mathrm{C}^*_r(G)^+ \mid \tilde{\tau}_\mu(a) < \infty\}$. Since $\tilde{\tau}_\mu(\mathbb{1}_U) < \infty$, the element $\mathbb{1}_U$ belongs to D and so $\tau_\mu(\mathbb{1}_U) = \tilde{\tau}_\mu(\mathbb{1}_U) > 0$, whence τ_μ is non-zero. This contradicts the assumption that $\mathrm{C}^*_r(G)$ is traceless.

Now we assume that $S(G)$ is almost unperforated and prove $(v) \Rightarrow (i)$. Let θ be a non-zero element in $S(G)$. If θ is completely non-paradoxical then by Theorem 6.2 $S(G)$ admits a non-trivial state. So, assuming (v), we must have $(k+1)\theta \leq k\theta$ for some $k \in \mathbb{N}$. Thus

$$(k+2)\theta = (k+1)\theta + \theta \leq k\theta + \theta = (k+1)\theta \leq k\theta.$$

Repeating we obtain $(k+1)2\theta \leq k\theta$. Since $S(G)$ is almost unperforated we may conclude that $2\theta \leq \theta$, hence θ is properly infinite. □

As promised, we combine Theorem 6.5, Corollary 6.6 and Theorem 7.3 to obtain the long-desired dichotomy.

Theorem 7.4. *Let G be a minimal, ample, and topologically principal groupoid with an almost unperforated type semigroup $S(G)$.*

(i) The C^*-algebra $\mathrm{C}^*_r(G)$ is simple and is either purely infinite or stably finite.

(ii) If G is also amenable, then $\mathrm{C}^*_r(G)$ is simple and is either purely infinite or quasidiagonal.

Remark 7.5. Since, by Remark 5.5, our type semigroup coincides with that of Bönicke and Li [5, Definition 5.3], our Theorems 7.3 and 7.4 recover [5, Theorem 5.11], and extend it to groupoids with non-compact unit spaces. In particular, our Theorem 7.4 extends [5, Corollary 5.13] to groupoids with not-necessarily-compact unit spaces.

Due to the deep classification achievements of Kirchberg [25] and Phillips [34], several authors have sought to express dynamical conditions for systems (A,Γ) that yield purely infinite crossed products $A \rtimes_r \Gamma$. For example, a continuous action $\Gamma \curvearrowright X$ of a discrete group on a locally compact Hausdorff space is called a *local boundary action* if for every non-empty open $U \subseteq X$ there is an open $V \subseteq U$ and $t \in \Gamma$ such that $t.\overline{V} \subsetneq V$. Laca and Spielberg showed in [29] that if $\Gamma \curvearrowright X$ is a topologically free local boundary action the reduced crossed product $C_0(X) \rtimes_r \Gamma$ is purely infinite and simple. Jolissaint and Robertson [20] generalized this notion to noncommutative C*-systems (A,Γ) by defining a property called *n-filling*. When $A = C(X)$ (X compact and infinite) this filling property is equivalent to a weak version of hyperbolicity expressed as follows: for every collection $U_1, \dots, U_n \subseteq X$ of nonempty open sets we can find group elements $t_1, \dots, t_n$ in Γ such that $\bigcup_{i=1}^n t_j.U_j = X$. This n-filling property naturally generalizes to groupoids as expressed by Suzuki [47], who proved that n-filling topologically principal groupoids have purely infinite reduced C*-algebras. Here we demonstrate how this n-filling property lends itself naturally to our notions of paradoxicality explored in this section, and in the process we recover Suzuki's results.

The following definition looks slightly different from that given by Suzuki in [47, Definition 3.2], but we show below that they are equivalent in Remark 7.8.

Definition 7.6. Let G be an ample groupoid with compact unit space, and let $n \in \mathbb{N}$. We say that G has the n-filling property if for every collection $U_1, \dots, U_n \subseteq G^{(0)}$ of non-empty open sets we can find compact open bisections $E_1, \dots, E_n$ such that

$$\bigcup_{i=1}^n r(E_j U_j) = G^{(0)}.$$

Proposition 7.7. *Let G be an ample groupoid whose unit space $G^{(0)}$ is compact and has no isolated points. If G is n-filling, then G is minimal and every non-empty compact open $A \subseteq G^{(0)}$ is properly paradoxical.*

If, moreover, G is topologically principal, then $C^*_r(G)$ is purely infinite.

Proof. Minimality is clear by Proposition 4.2. Suppose $A \subseteq G^{(0)}$ is a non-empty compact open subset. Since $G^{(0)}$ is totally disconnected without isolated points we can find $2n$ disjoint clopen sets $U_1, \dots, U_{2n} \subseteq A$. By the filling property we can find compact open bisections $E_1, \dots, E_{2n}$ such that

$$\bigcup_{j=1}^n r(E_j U_j) = G^{(0)}, \quad \text{and} \quad \bigcup_{j=n+1}^{2n} r(E_j U_j) = G^{(0)}.$$

Consider the compact open bisections $F_j = (E_j U_j)^{-1}$ for $j = 1, \dots, 2n$. We note that

$$\sum_{j=1}^{2n} \mathbb{1}_{s(F_j)} = \sum_{j=1}^{2n} \mathbb{1}_{r(E_j U_j)} \geq \mathbb{1}_{\bigcup_{j=1}^n r(E_j U_j)} + \mathbb{1}_{\bigcup_{j=n+1}^{2n} r(E_j U_j)} \geq \mathbb{1}_{G^{(0)}} + \mathbb{1}_{G^{(0)}} = 2\mathbb{1}_{G^{(0)}}.$$

Since $r(F_j) = s(E_j U_j) \subseteq U_j$ for all j, and these are disjoint, we have

$$\sum_{j=1}^{2n} \mathbb{1}_{r(F_j)} \leq \sum_{j=1}^{2n} \mathbb{1}_{U_j} = \mathbb{1}_{\bigcup_{j=1}^{2n} U_j} \leq \mathbb{1}_A,$$

thus A is properly paradoxical.

The final assertion follows from Theorem 7.3. □

It would be reasonable to suspect that this notion of filling is related to the locally contracting property of C. Anantharaman-Delaroche found in [1].

Remark 7.8. To see how that our Proposition 7.7 recovers the pure-infiniteness statement in [47, Proposition 3.9], we show that our notion of n-filling is equivalent to Suzuki's notion of n-filling. Suzuki defines G to be n-filling if for every nonempty open W there are open bisections $E_1, \ldots, E_n$ such that $\bigcup_i r(E_i W) = G^{(0)}$ (see [47, Definition 3.2]). If G is n-filling in our sense, then it is also n-filling in Suzuki's sense: given W, apply our definition with $U_1 = U_2 = \cdots = U_n = W$. The argument for the converse is essentially that of [20]. If G is n-filling in Suzuki's sense, then Proposition 4.2 (or [47, Proposition 3.9] and [6, Proposition 5.7]) shows that G is minimal. So we can find $\gamma_{1,2} \in U_1 G U_2$ and then a compact open bisection $F_{1,2} \subseteq U_1 G U_2$ containing $\gamma_{1,2}$. We then repeat this argument with U_1 replaced by $r(F_{1,2})$ and U_2 replaced by U_3 to find a compact open bisection $F_{1,3} \in r(F_{1,2}) G U_3$. Iteratively, we obtain compact open bisections $F_{1,i} \subseteq \bigcap_{j<i} r(F_{1,j}) G U_i$. Apply Suzuki's n-filling property with $W = \bigcap_{i=1}^n r(F_{1,i})$ to obtain compact open bisections F_i' with $\bigcup_i r(F_i' W) = G^{(0)}$. Now putting $E_i = F_i' F_{1,i}$, we obtain $\bigcup_i r(E_i U_i) = \bigcup_i r(F_i' r(F_{1,i})) \supseteq \bigcup_i r(F_i' W) = G^{(0)}$.

8. Graphs and dynamical systems

The C*-algebras that arise from groupoids form an extensive class. In this last section of our work we study three prominent cases; topological graphs, transformation groups, and higher-rank graphs.

Topological graphs were defined and studied by Katsura in [21]. Confining our focus to the zero-dimensional case, we show (see Definition 8.3) that a regular totally disconnected graph E admits a natural semigroup $S(E)$ which agrees with the groupoid type semigroup $S(G_E)$ (Proposition 8.4), where G_E is the infinite-path groupoid as defined by Yeend [51]. As topological graph C*-algebras are nuclear and satisfy the UCT, we obtain Corollary 8.5; a quasidiagonal/purely infinite dichotomy result.

Given a discrete dynamical system $\Gamma \curvearrowright X$, one can form the transformation groupoid $X \rtimes \Gamma$. The reduced groupoid C*-algebra $C_r^*(X \rtimes \Gamma)$ is isomorphic to the reduced crossed product algebra $C_0(X) \rtimes_r \Gamma$. When X is compact and zero-dimensional, we would expect our type semigroup $S(X \rtimes \Gamma)$ to coincide with the type semigroup $S(X, \Gamma)$ of the system $\Gamma \curvearrowright X$ as defined and studied in [44, 37]. We indeed establish this assertion in Proposition 8.8. We then apply our results to the theory of rigidity of dynamical systems and show that the type semigroup of a Cantor system is a continuous orbit-equivalence invariant.

Finally, if Λ is a row-finite k-graph with no sources, its C*-algebra is Morita equivalent to a crossed product of an AF algebra by $\mathbb{Z}^k$. The type semigroup $S(\Lambda)$ of this action can be computed directly in terms of the adjacency matrices of the k-graph [32]. Since k-graph C*-algebras can also be modeled as the C*-algebras of ample groupoids, it is natural to compare the semigroup $S(\Lambda)$ with the type semigroup of the associated groupoid. We prove in Proposition 8.11 that the two are isomorphic.

8.1. **Zero-dimensional topological graphs.** Recall from [21] that a topological graph is a quadruple $E = (E^0, E^1, r, s)$ where E^0 and E^1 are locally compact Hausdorff spaces, $r : E^1 \to E^0$ (the range map) is continuous, and $s : E^1 \to E^0$ (the source map) is a local homeomorphism. For any $n = 1, \dots, \infty$ we have the path spaces

$$E^n = \big\{(\lambda_k)_{k=1}^n \mid \lambda_k \in E^1, s(\lambda_k) = r(\lambda_{k+1})\big\} \subseteq \prod_{k=1}^n E^1, \qquad E^* = \bigsqcup_{n\geq 0} E^n$$

endowed with the obvious topologies. For any $n \in \mathbb{N}$ the range and source maps can be naturally extended (with the same properties—r is continuous and s is a local homeomorphism) to $r, s : E^n \to E^0$. Moreover, the range map extends to infinite paths $r : E^\infty \to E^0$ via $r(\lambda_k)_{k=1}^\infty = r(\lambda_1)$. A subset $U \subseteq E^n$ for which $s|_U : U \to s(U)$ is a homeomorphism is called an s-section. Note that for any $n \in \mathbb{N}$ and vertex $v \in E^0$, $s^{-1}(v) \subseteq E^n$ is discrete.

A topological graph E is said to be totally disconnected if E^0 is totally disconnected. For the remainder of this section we restrict our attention to totally disconnected topological graphs whose range map is proper and surjective. In this setting the infinite path space E^∞ is also totally disconnected. Here is a brief justification. For $n \in \mathbb{N}$ and $U \subseteq E^n$, we have the cylinder sets

$$Z(U) := \big\{\lambda \in E^\infty \mid \lambda = \alpha\beta, \alpha \in U, \beta \in E^\infty, s(\alpha) = r(\beta)\big\} \subseteq E^\infty.$$

If $U \subseteq E^n$ is compact and open, one verifies that $Z(U)$ is compact open too, and a routine argument shows that the collection

$$\big\{Z(U) \mid U \subseteq E^n \text{ is a compact open } s\text{-section for some } n \geq 0\big\}$$

forms a basis for the topology on E^∞. One can also show that every compact open $A \subseteq E^\infty$ can be written as a finite disjoint union $A = \sqcup_j Z(A_j)$ where the $A_j \subseteq E^{p_i}$ are compact open s-sections. These facts will prove useful in our work below.

The topological-graph C*-algebra $C^*(E)$ is defined to be the Cuntz-Pimsner algebra of a C*-correspondence constructed from the topological graph E. However, it can also be realized as the reduced C*-algebra of a Deaconu-Reneault infinite-path groupoid G_E. We briefly recall the construction of G_E. Recall from Sections 2.3 and 2.4 of [41] that a boundary path of E is either an infinite path or else a finite path $\lambda \in E^*$ such that $s(\lambda)E^1$ is empty or has no compact neighborhood in E^1. Since $r : E^1 \to E^0$ is proper and surjective by hypothesis, we deduce that the boundary-path space of E is precisely E^∞. The groupoid G_E is defined as follows. The underlying set is

$$G_E := \left\{\big(\alpha\lambda, |\alpha| - |\beta|, \beta\lambda\big) \;\middle|\; \alpha, \beta \in E^*, \lambda \in E^\infty, s(\alpha) = s(\beta) = r(\lambda)\right\} \subseteq E^\infty \times \mathbb{Z} \times E^\infty,$$

endowed with the subspace topology. The unit space is $G_E^{(0)} = \{(\lambda, 0, \lambda) : \lambda \in E^\infty\}$ identified with E^∞. The range and source maps are defined via

$$r, s : G_E \to G_E^{(0)} \qquad r(\mu, n, \nu) = \mu, \quad s(\mu, n, \nu) = \nu,$$

while the law of composition and inverse operation are given by

$$(\mu, m, \nu)(\nu, n, \lambda) = (\mu, m + n, \lambda), \quad \text{and} \quad (\mu, n, \nu)^{-1} = (\nu, -n, \mu).$$

The topology on G_E has basic compact open bisections

$$Z(U,V) := \Big\{(\alpha\lambda, |\alpha| - |\beta|, \beta\lambda) \mid \alpha \in U, \beta \in V, \lambda \in E^\infty, r(\lambda) = s(\alpha) = s(\beta)\Big\}.$$

where $U \subseteq E^n, V \subseteq E^m$ are compact open s-sections for some $m,n \in \mathbb{N}$. Note that in this context $r(Z(U,V)) = Z(U)$ and $s(Z(U,V)) = Z(V)$. Yeend proves in [51] that $C^*(E) \cong C^*(G_E)$.

We now define a semigroup $S(E)$ associated to a totally disconnected topological graph E. Again, we are restricting our attention to totally disconnected topological graphs whose whose range map is proper and surjective. If $K \subseteq E^0$ is compact and $v \in E^0$, then $r^{-1}(K) \cap s^{-1}(v)$ is the intersection of a compact set and a discrete set, and is therefore finite. For any $f \in C_c(E^0)$ and any integer $n \geq 0$ we define $\Theta^n(f) : E^0 \to \mathbb{Z}$ by

$$\Theta^n(f)(v) = \sum_{\lambda \in E^n v} f(r(\lambda)). \tag{5}$$

Note that the sum runs over all $\lambda \in r^{-1}(\mathrm{supp}(f)) \cap s^{-1}(v)$ which is a finite set. The support of $\Theta^n(f)$ is compactly supported since $\mathrm{supp}(\Theta^n(f)) \subseteq s(r^{-1}(\mathrm{supp}(f)))$ which is compact. We therefore have an operator $\Theta^n : C_c(E^0,\mathbb{Z}) \to C_c(E^0,\mathbb{Z})$ which is a positive group homomorphism and satisfies $\Theta^{n+m} = \Theta^n \circ \Theta^m$. Observe that if E is a discrete directed graph, then Θ^n is just multiplication by the transpose $(A_E^n)^t$ of the nth power of the adjacency matrix of E.

It is useful to see how Θ^n operates on characteristic functions. To that end, suppose $U \subseteq E^0$ is a compact open subset. As discussed above, $s : E^n \to E^0$ is a local homeomorphism, so E^n has a basis of compact open s-sections. Since $r : E^n \to E^0$ is proper, UE^n is compact, and it follows that there are finitely many mutually disjoint compact open s-sections $U_j \subseteq E^n$ such that

$$UE^n := \{\lambda \in E^n \mid r(\lambda) \in U\} = \bigsqcup_j^k U_j.$$

One checks that $\Theta^n(\mathbb{1}_U) = \sum_{j=1}^k \mathbb{1}_{s(U_j)}$.

Definition 8.1. Let E be a totally disconnected topological graph whose range map is proper and surjective.

(i) We define a relation $\sim$ on $C_c(E^0,\mathbb{Z})^+$ as follows: $f \sim g$ if there exist $p,q \in \mathbb{N}$ such that $\Theta^p(f) = \Theta^q(g)$.

(ii) Define the relation $\approx$ on $C_c(E^0,\mathbb{Z})^+$ by $f \approx g$ if there exist finitely many pairs $(f_i,g_i)_{i=1}^n$ in $C_c(E^0,\mathbb{Z})^+ \times C_c(E^0,\mathbb{Z})^+$ satisfying

$$f \sim \sum_{i=1}^n f_i, \quad g \sim \sum_{i=1}^n g_i \quad \text{and} \quad f_i \sim g_i \quad \text{for each } i.$$

Lemma 8.2. *Let E be a totally disconnected topological graph whose range map is proper and surjective. The relations $\sim$ and $\approx$ are equivalence relations. If $f \approx g$ then there exist finitely many compact open sets $U_i \subseteq E^0$, and natural numbers p,q and q_i such that $\Theta^p(f) = \sum_i \Theta^{q_i}(\mathbb{1}_{U_i})$ and $\Theta^q(g) = \sum_i \mathbb{1}_{U_i}$.*

Proof. It is clear that both $\sim$ and $\approx$ are symmetric and reflexive. To see that $\sim$ is transitive, if $f \sim g \sim h$, say $\Theta^m(f) = \Theta^n(g)$ and $\Theta^p(g) = \Theta^q(h)$, then $\Theta^{m+p}(f) = \Theta^{n+p}(g) = \Theta^{n+q}(h)$, giving $f \sim h$.

To see that $\approx$ is transitive, we first prove the final statement of the lemma. If $f \approx g$ there are finitely many pairs $(f_i, g_i)_{i=1} \in C_c(E^0, \mathbb{Z})^+ \times C_c(E^0, \mathbb{Z})^+$, and natural numbers a, b, c, d, p_i, r_i in $\mathbb{N}$ satisfying

$$\Theta^a(f) = \Theta^b\big(\sum_i f_i\big), \quad \Theta^c(g) = \Theta^d\big(\sum_i g_i\big), \quad \text{and} \quad \Theta^{p_i}(f_i) = \Theta^{r_i}(g_i) \quad \text{for each } i.$$

Set $P := \max_i p_i$ and $p := a + d + P$. Then

$$\Theta^p(f) = \Theta^{P+d}\bigg(\Theta^b\big(\sum_i f_i\big)\bigg) = \sum_i \Theta^{P-p_i+b+d}(\Theta^{p_i}(f_i)) = \sum_i \Theta^{P-p_i+r_i+b+d}(g_i).$$

So putting $q_i := P - p_i + r_i$, $q = b + c$ and $h_i = \Theta^{b+d}(g_i)$ for each i, we have

$$\Theta^p(f) = \sum_i \Theta^{q_i}(h_i) \quad \text{and} \quad \Theta^q(g) = \Theta^b(\Theta^c(f)) = \Theta^b\bigg(\Theta^d\big(\sum_i g_i\big)\bigg) = \sum_i h_i.$$

Now writing each $h_i = \sum_{j=1}^{m_i} 1\!\!1_{U_j}$, we obtain

$$\Theta^p(f) = \sum_i \sum_{j \le m_i} \Theta^{q_i}(1\!\!1_{U_i}) \quad \text{and} \quad \Theta^q(g) = \sum_i \sum_{j \le m_i} 1\!\!1_{U_i},$$

so reindexing gives the final statement of the lemma.

Now suppose that $f \approx g \approx h$. By the preceding paragraph, we can choose finitely many characteristic functions $g_1, \dots, g_M$ and $h_1, \dots, h_N$ of compact open subsets of E^0, and natural numbers p, q, a, b, q_i and a_j such that $\Theta^p(f) = \sum_i \Theta^{q_i}(g_i)$, $\Theta^q(g) = \sum_i g_i$, $\Theta^b(h) = \sum_j \Theta^{a_j}(h_j)$ and $\Theta^a(g) = \sum_j h_j$.

We therefore have $\sum_i \Theta^a(g_i) = \Theta^{a+q}(g) = \sum_j \Theta^q(h_j)$. So replacing each g_i by $\Theta^a(g_i)$, p by $p + a$, q by $q + a$, each h_j by $\Theta^q(h_j)$ and b by $b + q$, we obtain

$$\Theta^p(f) = \sum_i \Theta^{q_i}(g_i), \quad \Theta^b(h) = \sum_j \Theta^{a_j}(h_j), \quad \text{and} \quad \sum_i g_i = \sum_j h_j.$$

Applying the refinement property to the equality $\sum_i g_i = \sum_j h_j$ we get characteristic functions $\{k_{i,j}\}_{i,j}$ satisfying

$$g_i = \sum_j k_{i,j}, \quad h_j = \sum_i k_{i,j}$$

for every i and j. Consequently

$$\Theta^p(f) = \sum_i \Theta^{q_i}(g_i) = \sum_{i,j} \Theta^{q_i}(k_{i,j}), \qquad \Theta^b(h) = \sum_j \Theta^{a_j}(h_j) = \sum_{i,j} \Theta^{a_j}(k_{i,j}).$$

Now for every pair (i, j) we have $\Theta^{q_i}(k_{i,j}) \sim \Theta^{a_j}(k_{i,j})$, so by definition $f \approx h$.

□

Definition 8.3. Let E be a totally disconnected topological graph whose range map is proper and surjective. We define the *type semigroup of* E as

$$S(E) := C_c(E^0, \mathbb{Z})^+/\approx \tag{6}$$

and write $[f]_E$ for the equivalence class with representative $f \in C_c(E^0, \mathbb{Z})^+$.

Since the relation $\approx$ clearly respects addition in $C_c(E^0, \mathbb{Z})^+$, we see that $S(E)$ is indeed an abelian monoid under the operation $[f]_E + [g]_E = [f + g]_E$. As usual we give $S(E)$ the algebraic ordering.

Our next goal is to prove that this $S(E)$ coincides with the semigroup $S(G_E)$ constructed from the ample groupoid G_E associated to E.

Proposition 8.4. *Let E be a totally disconnected topological graph whose range map is proper and surjective. There is an isomorphism of monoids $\tau : S(E) \to S(G_E)$ such that $\tau([\mathbb{1}_U]_E) = [\mathbb{1}_{Z(U)}]_{G_E}$ for every compact open $U \subseteq E^0$.*

Proof. Since $r : E^\infty \to E^0$ is proper and surjective, the map $\tilde{\tau} : C_c(E^0, \mathbb{Z})^+ \to C_c(G_E^{(0)}, \mathbb{Z})^+$ defined by $\tilde{\tau}(f) = f \circ r$ is a well-defined, injective monoid homomorphism. One immediately observes that $\tilde{\tau}(\mathbb{1}_U) = \mathbb{1}_U \circ r = \mathbb{1}_{Z(U)}$ for every compact open $U \subseteq E^0$.

To see that $\tilde{\tau}$ descends to $\tau : S(E) \to S(G_E)$, we must show that if $f \approx g$, then $\tilde{\tau}(f) \sim_{G_E} \tilde{\tau}(g)$. We first show that $\tilde{\tau}(f) \sim_{G_E} \tilde{\tau}(\Theta^n(f))$ for any $f \in C_c(E^0, \mathbb{Z})^+$ and $n \in \mathbb{N}$. Since Θ^n and $\tilde{\tau}$ are additive, and $\sim_{G_E}$ respects addition, it suffices to consider $f = \mathbb{1}_U$ for a compact open subset $U \subseteq E^0$. We write $UE^n = \bigsqcup_j^k U_j$ where the $U_j \subseteq E^n$ are compact open s-sections. For each j, let F_j denote the open bisection $F_j := Z(U_j, s(U_j)) \subseteq G_E$. We then have

$$\begin{aligned}\tilde{\tau}(\Theta^n(\mathbb{1}_U)) &= \tilde{\tau}\bigg(\sum_{j=1}^k \mathbb{1}_{s(U_j)}\bigg) = \sum_{j=1}^k \tilde{\tau}(\mathbb{1}_{s(U_j)}) = \sum_{j=1}^k \mathbb{1}_{Z(s(U_j))} \\ &= \sum_{j=1}^k \mathbb{1}_{s(F_j)} \sim_{G_E} \sum_{j=1}^k \mathbb{1}_{r(F_j)} = \sum_{j=1}^k \mathbb{1}_{Z(U_j)} = \mathbb{1}_{\sqcup_j^k Z(U_j)} = \mathbb{1}_{Z(U)} = \tilde{\tau}(\mathbb{1}_U)\end{aligned}$$

Since $\sim_{G_E}$ is an equivalence relation on $C_c(G_E^{(0)}, \mathbb{Z})^+$, we deduce that if $f \sim g$ in $C_c(E^0, \mathbb{Z})^+$ then $\tilde{\tau}(f) \sim_{G_E} \tilde{\tau}(g)$. Now suppose that $f \approx g \in C_c(E^0, \mathbb{Z})^+$, then there exist finitely many pairs $(f_i, g_i)_{i=1}^n$ in $C_c(E^0, \mathbb{Z})^+ \times C_c(E^0, \mathbb{Z})^+$ satisfying

$$f \sim \sum_i^n f_i, \quad g \sim \sum_i^n g_i \quad \text{and} \quad f_i \sim g_i \quad \text{for each } i.$$

Then

$$\tilde{\tau}(f) \sim_{G_E} \tilde{\tau}\bigg(\sum_i f_i\bigg) = \sum_i \tilde{\tau}(f_i) \sim_{G_E} \sum_i \tilde{\tau}(g_i) = \tilde{\tau}\bigg(\sum_i g_i\bigg) \sim_{G_E} \tilde{\tau}(g).$$

So $\tilde{\tau}$ induces a homomorphism $\tau : S(E) \to S(G_E)$ satisfying $\tau([\mathbb{1}_U]_E) = [\mathbb{1}_{Z(U)}]_{G_E}$.

To show surjectivity, it suffices to verify that $[\mathbb{1}_A]_{G_E}$ is in the image of τ for any compact open $A \subseteq E^\infty$. Since any such A can be written as a finite disjoint union

$A = \sqcup_j Z(B_j)$ where the $B_j \subseteq E^{p_i}$ are compact open s-sections, we need only show that such $[Z(B)]_{G_E} \in \mathrm{im}(\tau)$ for such an s-section B. To that end

$$[\mathbb{1}_{Z(B)}]_{G_E} = [\mathbb{1}_{r(Z(B,s(B)))}]_{G_E} = [\mathbb{1}_{s(Z(B,s(B)))}]_{G_E} = [\mathbb{1}_{Z(s(B))}]_{G_E} = \tau([\mathbb{1}_{s(B)}]_E),$$

so τ is indeed surjective.

Lastly we show that τ is injective. Suppose $f, g \in C_c(E^0, \mathbb{Z})^+$ and that $\tilde{\tau}(f) \sim_{G_E} \tilde{\tau}(g)$. We need to show that $f \approx g$. Choose compact open bisections $U_i \subset G_E$ such that

$$\tilde{\tau}(f) = \sum_i \mathbb{1}_{r(U_i)}, \quad \text{and} \quad \tilde{\tau}(g) = \sum_i \mathbb{1}_{s(U_i)}.$$

By definition of the topology on G_E, we can decompose each U_i as $U_i = \bigsqcup_{j=1}^{n_i} Z(V_j, W_j)$ where $V_j \subset E^{p_j}$, $W_j \subseteq E^{q_j}$ are compact open s-sections for some $p_j, q_j \in \mathbb{N}$, and $s(V_j) = s(W_j)$. By relabeling, we can assume that each U_i is equal to such a $Z(V_i, W_i)$. By taking $p = \max_i p_i$, covering each $s(V_i)E^{p-p_i}$ with mutually disjoint compact open s-sections $\{Y_{i,j} : j \leq m_i\}$ and then replacing each $Z(V_i, W_i)$ with $\{Z(V_iY_{i,j}, W_iY_{i,j}) : j \leq m_i\}$, we can moreover assume that the p_i are all equal.

Since $\tilde{\tau}(f)(\lambda x) = f(r(\lambda)) = \tilde{\tau}(f)(\lambda' y)$ whenever $\lambda, \lambda' \in E^p$ satisfy $r(\lambda) = r(\lambda')$, we see that for each $\lambda \in \mathrm{supp}(f)E^p$, we have $|\{i : \lambda \in V_i\}| = f(x)$. It follows that $\tilde{\tau}(\mathbb{1}_{s(V_i)}) = \sum_i \mathbb{1}_{Z(s(V_i))} = \tilde{\tau}(\Theta^p(f))$. We have $s(V_i) = s(W_i)$ for all i, and so $\sum_i \mathbb{1}_{Z(s(W_i))} = \tilde{\tau}(\Theta^p(f))$ as well. Each $\mathbb{1}_{Z(s(W_i))} = \tilde{\tau}(\mathbb{1}_{s(W_i)})$, so it suffices to show that $g \approx \sum_i \mathbb{1}_{s(W_i)}$. So, putting $q = \max_i q_i$, it suffices to show that $g \approx \sum_i \Theta^{q-q_i}(\mathbb{1}_{s(W_i)})$.

By covering each $s(W_i)E^{q-q_i}$ by mutually disjoint compact open s-sections $Y_{i,j}$, we can write $\sum_i \tilde{\tau}(\Theta^{q-q_i}(\mathbb{1}_{s(W_i)})) = \sum_{i,j} \mathbb{1}_{s(W_iY_{i,j})}$. Now the compact open bisections $B_{i,j} = Z(W_iY_{i,j}, s(Y_{i,j}))$ satisfy $\tilde{\tau}\Big(\sum_i \Theta^{q-q_i}(\mathbb{1}_{s(W_i)})\Big) = \sum_{i,j} \mathbb{1}_{s(B_{i,j})}$ and $\tilde{\tau}(g) = \sum_{i,j} \mathbb{1}_{r(B_{i,j})}$. Since each $W_iY_{i,j} \subseteq E^q$, the argument of the preceding paragraph shows that

$$\tilde{\tau}\Big(\sum_i \Theta^{q-q_i}(\mathbb{1}_{s(W_i)})\Big) = \sum_{i,j} \mathbb{1}_{Z(s(B_{i,j}))} = \tilde{\tau}(\Theta^q(g)).$$

Since $\tilde{\tau}$ is injective, it follows that $\Theta^q(g) = \sum_i \Theta^{q-q_i}(\mathbb{1}_{s(W_i)})$, and so $g \approx \sum_i \Theta^{q-q_i}(\mathbb{1}_{s(W_i)})$ as required. □

We arrive at the following dichotomy result for zero-dimensional graphs.

Corollary 8.5. *Let E be a totally disconnected topological graph whose range map is proper and surjective. If $\mathrm{C}^*(E)$ is simple and the semigroup $S(E)$ of* (8.3) *is almost unperforated, then $\mathrm{C}^*(E)$ is either purely infinite or quasidiagonal.*

Proof. Since $\mathrm{C}^*(G_E) \cong \mathrm{C}^*(E)$ is simple, [6, Theorem 5.1] shows that G_E is minimal and topologically principal. Proposition 8.4 shows that $S(G_E)$ is almost unperforated. The result now follows from Theorem 7.4. □

We finish with an alternative description of the type semigroup of a regular topological graph in the spirit of a coboundary monoid. Given a totally disconnected topological graph E with proper and surjective range map, the ordered abelian group $C_c(E^\infty, \mathbb{Z})$ can be realized as an inductive limit as follows. There are proper surjective maps

$$p_n : E^{n+1} \longrightarrow E^n;\ \lambda_1 \cdots \lambda_{n+1} \mapsto \lambda_1 \cdots \lambda_n, \text{ and } \pi_n : E^\infty \longrightarrow E^n;\ (\lambda_j)_{j \geq 1} \mapsto \lambda_1 \cdots \lambda_n$$

which induce injective positive group homomorphisms

$$\rho_n : C_c(E^n, \mathbb{Z}) \to C_c(E^{n+1}, \mathbb{Z}), \quad \text{and} \quad \rho_{\infty,n} : C_c(E^n, \mathbb{Z}) \to C_c(E^\infty, \mathbb{Z})$$

given by $\rho_n(f) = f \circ p_n$, and $\rho_{\infty,n}(f) = f \circ \pi_n$ for $f \in C_c(E^n, \mathbb{Z})$. Note that if $V \subseteq E^n$ is compact and open, then $\rho_n(\mathbb{1}_V) = \mathbb{1}_{p_n^{-1}(V)}$ and $\rho_{\infty,n}(\mathbb{1}_V) = \mathbb{1}_{Z(V)}$. It is routine to check that the inductive limit gives

$$\lim_{n\to\infty}(C_c(E^n, \mathbb{Z}), \rho_n) \cong (C_c(E^\infty, \mathbb{Z}), \rho_{\infty,n}).$$

We will write $\mathcal{C}_n \subset E^n$ for the collection of all compact open s-sections in E^n. Note that $\mathcal{C}_0$ consists of all the compact open subsets of E^0. We define the coboundary monoid of E to be the free abelian monoid with generators $\{a_U : U \in \mathcal{C}_0\}$ subject to the relations

$$a_{U\sqcup U'} = a_U + a_{U'} \quad \text{for disjoint } U, U' \in \mathcal{C}_0, \qquad \text{and}$$

$$a_U = \textstyle\sum_i a_{s(V_i)} \quad \text{for any finite cover } UE^1 = \bigsqcup_i V_i \text{ by mutually disjoint } V_i \in \mathcal{C}_i.$$

That is, M_E is the quotient of $\mathbb{N}\mathcal{C}_0$ by the smallest congruence containing the above two relations. The first of these relations simply says that disjoint unions in E^0 correspond to addition in M. The second relation is akin to a coboundary condition, and is related to the equivalence relation $\approx$ in that we already saw that if $UE^1 = \bigsqcup_i V_i$ is a finite cover of UE^1 by mutually disjoint $V_i \in \mathcal{C}_1$, then the operator $\Theta := \Theta^1 : C_c(E^0, \mathbb{Z}) \to C_c(E^0, \mathbb{Z})$ of (5) satisfies $\Theta(\mathbb{1}_U) = \sum_{i=1}^I \mathbb{1}_{s(V_i)}$.

Proposition 8.6. *Let E be a totally disconnected topological graph whose range map is proper and surjective. Then the type semigroup $S(E)$ is isomorphic to M_E.*

Proof. If $U = \bigsqcup_j U_j$, with $U_j \in \mathcal{C}_0$, then

$$[\mathbb{1}_{Z(U)}]_{G_E} = [\mathbb{1}_{Z(\sqcup U_j)}]_{G_E} = [\mathbb{1}_{\sqcup Z(U_j)}]_{G_E} = \Big[\textstyle\sum_j \mathbb{1}_{Z(U_j)}\Big]_{G_E} = \textstyle\sum_j [\mathbb{1}_{Z(U_j)}]_{G_E}.$$

Also, if $U \in \mathcal{C}_0$ with $UE^1 = \bigsqcup_i V_i$, $V_i \subseteq E^1$ compact open s-sections, then

$$\begin{aligned}[\mathbb{1}_{Z(U)}]_{G_E} &= [\mathbb{1}_{Z(\sqcup V_i)}]_{G_E} = [\mathbb{1}_{\sqcup Z(V_i)}]_{G_E} \\ &= \Big[\textstyle\sum_i \mathbb{1}_{Z(V_i)}\Big]_{G_E} = \Big[\textstyle\sum_i \mathbb{1}_{Z(s(V_i))}\Big]_{G_E} = \textstyle\sum_i [\mathbb{1}_{Z(s(V_i))}]_{G_E}\end{aligned}$$

Hence there is a well-defined monoid homomorphism $\Psi : M_E \to S(G_E)$ such that $\Psi(a_U) = [\mathbb{1}_{Z(U)}]_{G_E}$ for all $U \in \mathcal{C}_0$.

To see that Ψ is an isomorphism, we construct an inverse. For each $n \in \mathbb{N}$ define $\phi_n : C_c(E^n, \mathbb{Z})^+ \to M_E$ as follows. Given f in $C_c(E^n, \mathbb{Z})^+$, write f as a finite sum $f = \sum_i \mathbb{1}_{U_i}$ where the U_i are compact open s-sections, and put $\phi_n\big(\sum_i \mathbb{1}_{U_i}\big) = \sum_i a_{s(U_i)}$. A refinement argument shows that ϕ_n is well-defined. We claim that $\phi_{n+1} \circ \rho_n = \phi_n$ for every $n \geq 0$. To see this, consider any compact open subset $V \subseteq E^n$. We may write $s(V)E^1 = \bigsqcup_i V_i$ for finitely many mutually disjoint $V_i \in \mathcal{C}_1$. We then have $s(p_n^{-1}(V)) = \bigsqcup_i s(V_i)$. By the defining relations of M_E we obtain

$$\phi_{n+1} \circ \rho_n(\mathbb{1}_V) = \phi_{n+1}(\mathbb{1}_{p_n^{-1}(V)}) = a_{s(p_n^{-1}(V))} = \sum_i a_{s(V_i)} = a_{s(V)} = \phi_n(\mathbb{1}_V).$$

The universal property of the inductive limit $C_c(E^\infty, \mathbb{Z})^+ = \varinjlim(C_c(E^n, \mathbb{Z})^+, \rho_n)$ now yields a homomorphism $\phi : C_c(E^\infty, \mathbb{Z})^+ \to M_E$ satisfying $\phi \circ \rho_{\infty,n} = \phi_n$.

Now we show that ϕ drops to $S(G_E)$. For this, fix $f, g \in C_c(E^\infty, \mathbb{Z})^+$ with $f \sim_{G_E} g$. Fix finitely many compact open bisections U_i in G_E with

$$f = \sum_i \mathbb{1}_{s(U_i)}, \quad \text{and} \quad g = \sum_i \mathbb{1}_{r(U_i)}.$$

As above we may assume that $U_i = Z(V_i, W_i)$, where $V_i, W_i \subseteq E^n$ are compact open s-sections, and $s(W_i) = s(V_i)$. Then

$$f = \sum_i \mathbb{1}_{s(U_i)} = \sum_i \mathbb{1}_{Z(W_i)} = \sum_i \rho_{\infty,n}(\mathbb{1}_{W_i}), \quad \text{and}$$
$$g = \sum_i \mathbb{1}_{r(U_i)} = \sum_i \mathbb{1}_{Z(V_i)} = \sum_i \rho_{\infty,n}(\mathbb{1}_{V_i}).$$

Therefore

$$\phi(f) = \sum_i \phi(\rho_{\infty,n}(\mathbb{1}_{W_i})) = \sum_i \phi_n(\mathbb{1}_{W_i}) = \sum_i a_{s(W_i)}, \quad \text{and}$$
$$\phi(g) = \sum_i \phi(\rho_{\infty,n}(\mathbb{1}_{V_i})) = \sum_i \phi_n(\mathbb{1}_{V_i}) = \sum_i a_{s(V_i)},$$

so $\phi(f) = \phi(g)$. Thus ϕ drops to a morphism $\Phi : S(G_E) \to M_E$.

If $U \subseteq E^0$ is compact and open, then

$$\Phi \circ \Psi(U) = \Phi([\mathbb{1}_{Z(U)}]_{G_E}) = \phi(\mathbb{1}_{Z(U)}) = \phi \circ \rho_{\infty,0}(\mathbb{1}_U) = \phi_0(\mathbb{1}_U) = a_{s(U)} = a_U.$$

Finally, if $V \subseteq E^n$ is any compact open s-section, then

$$\begin{aligned}\Psi \circ \Phi\big([\mathbb{1}_{Z(V)}]_{G_E}\big) &= \Psi \circ \phi \circ \rho_{\infty,n}(\mathbb{1}_V) \\ &= \Psi \circ \phi_n(\mathbb{1}_V) = \Psi(a_{s(V)}) = [\mathbb{1}_{Z(s(V))}]_{G_E} = [\mathbb{1}_{Z(V)}]_{G_E}.\end{aligned}$$

Thus Φ and Ψ are mutually inverse. □

Remark 8.7. If E is a countable row-finite discrete graph, then it is readily seen that M_E coincides with the abelian monoid with presentation

$$\Big\langle v \in E^0 \mid v = \sum_{r(e)=v} s(e) \Big\rangle$$

as studied in [3]. The authors of [3] show that the Murray-von Neumann semigroup $V(\mathrm{C}^*(E))$ is naturally isomorphic to M_E and these are unperforated. It remains an interesting question whether M_E is always almost unperforated for zero-dimensional topological graphs; an affirmative answer would shed considerable light on the quasidiagonal/purely infinite divide for topological graph C*-algebras.

8.2. **Dynamical systems and continuous orbit equivalence.** We now work to reconcile the type semigroup of certain transformation groupoids with previous constructions appearing in the literature, and subsequently apply our results to the theory of continuous orbit equivalence.

Proposition 8.8. *Let $\Gamma \curvearrowright X$ be an action of a discrete group on a compact and totally disconnected space X, and let $G = \Gamma \rtimes X$ denote the resulting ample transformation groupoid. Then $S(G)$ and $S(X, \Gamma)$ (as constructed in* [44]*) are isomorphic as preordered abelian monoids.*

Proof. Proposition 4.4 in [37] shows that $S(X, \Gamma) \cong S(C(X), \Gamma)$, so it suffices to show that $S(G) \cong S(C(X), \Gamma)$. In what follows we may, and will, identify the spaces $G^{(0)}$ and X. We thus consider the identity mapping $C(G^{(0)}, \mathbb{Z})^+ \to C(X, \mathbb{Z})^+$ and prove that $f \sim_G g$ if and only if $f \sim_\alpha g$ (in the sense of definition 4.1 of [37]) where $f, g \in C(X, \mathbb{Z})^+$ and $\alpha : \Gamma \curvearrowright C(X)$ is the induced action.

Let $E \subseteq G$ be a compact open bisection. Writing $\pi_\Gamma : G \to \Gamma$ and $\pi_X : G \to X$ for the canonical projections, we set $E_X = \pi_X(E) = s(E) \subseteq X$ and $E_\Gamma = \pi_\Gamma(E)$. Note that for each $x \in E_X$ there is a unique t_x in E_Γ such that $(t_x, x) \in E$. For $t \in E_\Gamma$, let

$$E_X(t) = \{x \in E_X \mid t_x = t\} = s(E \cap \pi_\Gamma^{-1}(\{t\})).$$

Clearly $E_X(t)$ is compact and open and since $E = \bigsqcup_{t \in E_\Gamma} (E \cap \pi_\Gamma^{-1}(\{t\})$ and s is bijective on E we can see that $\bigsqcup_{t \in E_\Gamma} E_X(t) = E_X$. By compactness there are finitely many $t_j \in E_\Gamma$, $1 \leq j \leq m$ such that

$$\bigsqcup_{j=1}^{m} E_X(t_j) = E_X = s(E).$$

By construction it also follows that

$$\bigsqcup_{j=1}^{m} t_j.E_X(t_j) = r(E).$$

Setting $E_j = E_X(t_j)$ for $j = 1, \dots, m$ we get

$$\mathbb{1}_{s(E)} = \sum_{j=1}^{m} \mathbb{1}_{E_j}, \quad \text{and} \quad \mathbb{1}_{r(E)} = \sum_{j=1}^{m} \mathbb{1}_{t_j.E_j}.$$

Now suppose $f, g \in C(G^{(0)}, \mathbb{Z})^+$ with $f \sim_G g$. Then there are compact open bisections $E_1, \dots, E_n$ with

$$f = \sum_{i=1}^{n} \mathbb{1}_{s(E_i)}, \quad \text{and} \quad g = \sum_{i=1}^{n} \mathbb{1}_{r(E_i)}.$$

By our work above, for each $i = 1, \dots, n$ we can find an m_i, group elements $t_{i,1}, \dots, t_{i,m_i}$, and compact open subsets $E_{i,1}, \dots, E_{i,m_i} \subseteq s(E_i)$ with

$$\mathbb{1}_{s(E_i)} = \sum_{j=1}^{m_i} \mathbb{1}_{E_{i,j}} \quad \text{and} \quad \mathbb{1}_{r(E_i)} = \sum_{j=1}^{m_i} \mathbb{1}_{t_{i,j} E_{i,j}}.$$

It then follows that $f \sim_\alpha g$ because

$$f = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \mathbb{1}_{E_{i,j}} \quad \text{and} \quad g = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \mathbb{1}_{t_{i,j} E_{i,j}}.$$

Conversely, suppose $f, g \in C(X, \mathbb{Z})^+$ with $f \sim_\alpha g$. Then there are compact open subsets $A_1, \dots, A_n \subseteq X$ and group elements $t_1, \dots, t_n \in \Gamma$ such that $f = \sum_{i=1}^{n} \mathbb{1}_{A_i}$ and $g = \sum_{i=1}^{n} \mathbb{1}_{t_i.A_i}$. Simply set $E_i = \{t_i\} \times A_i$, which are clearly compact open bisections in G. Then $s(E_i) = A_i$ and $r(E_i) = t_i.E_i$

$$f = \sum_{i=1}^{n} \mathbb{1}_{A_i} = \sum_{i=1}^{n} \mathbb{1}_{s(E_i)}, \quad \text{and} \quad g = \sum_{i=1}^{n} \mathbb{1}_{t_i.A_i} = \sum_{i=1}^{n} \mathbb{1}_{r(E_i)}$$

which means $f \sim_G g$. □

We recall the notion of continuous orbit equivalence studied by Giordano, Putnam, and Skau in [17], and by X. Li in [31].

Let (X,Γ) and (Y,Λ) be discrete transformation groups and write $X \rtimes \Gamma$ and $Y \rtimes \Lambda$ for the resulting étale transformation groupoids. The actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are said to be *continuous orbit equivalent* (COE) if there are

(i) a homeomorphism $\varphi : X \to Y$ (with inverse $\psi : Y \to X$),
(ii) continuous maps $a : \Gamma \times X \to \Lambda$, and $b : \Lambda \times Y \to \Gamma$

such that for all $x \in X$, $t \in \Gamma$, $y \in Y$, $s \in \Lambda$

$$\varphi(t.x) = a(t,x)\varphi(x), \quad \text{and} \quad \psi(s.y) = b(s,y)\psi(y).$$

X. Li proves the following rigidity property for free systems.

Theorem 8.9 ([31])**.** *Let (X,Γ) and (Y,Λ) be topologically free dynamical systems. The following are equivalent.*

(i) (X,Γ) and (Y,Λ) are continuous orbit equivalent.
(ii) The transformation groupoids $X \rtimes \Gamma$ and $Y \rtimes \Lambda$ are topologically isomorphic.
(iii) There is a C^*-isomorphism $\Phi : C_0(X) \rtimes \Gamma \to C_0(Y) \rtimes \Lambda$ with $\Phi(C_0(X)) = C_0(Y)$.

The proof of *(i)* $\Rightarrow$ *(ii)* relies on topological freeness. Indeed, following the notation in the definition above of COE the maps

$$\begin{aligned} X \rtimes \Gamma &\to Y \rtimes \Lambda, \quad (t,x) \mapsto (a(t,x), \varphi(x)) \\ Y \rtimes \Lambda &\to X \rtimes \Gamma, \quad (s,y) \mapsto (b(s,y), \psi(y)) \end{aligned}$$

are easily seen to be topological groupoid homomorphisms. Topological freeness guarantees that these maps are mutual inverses.

From this result and our work above we can show that the type semigroup is a continuous orbit equivalence invariant under the assumption of freeness. Recall that for a Cantor system (X,Γ), we write $S(X,\Gamma)$ for the type semigroup as defined in [44].

Theorem 8.10. *Let X and Y be totally disconnected spaces and suppose $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are topologically free dynamical systems. If (X,Γ) and (Y,Λ) are continuous orbit equivalent then $S(X,\Gamma) \cong S(Y,\Lambda)$ as abelian monoids.*

Proof. Theorem 8.9 says that $X \rtimes \Gamma \cong Y \rtimes \Lambda$ are topologically isomorphic groupoids. Since the underlying spaces are totally disconnected these groupoids are ample. Now apply Proposition 5.6 to get $S(X \rtimes \Gamma) \cong S(Y \rtimes \Lambda)$. Finally, appealing to Proposition 8.8 gives us the desired result. □

8.3. **Higher-rank graphs.** We end our paper by reconciling our type semigroup with the semigroup of a k-graph constructed in [32]. Given a row-finite k-graph Λ with no sources, the associated semigroup $S(\Lambda)$ is defined [32, Definition 3.5] as follows. For $n \in \mathbb{N}^k$, we write A^n_Λ for the $\Lambda^0 \times \Lambda^0$ integer matrix with entries $A^n_\Lambda(v,w) = |v\Lambda^n w|$. The semigroup $S(\Lambda)$ is defined to be the quotient of $\mathbb{N}\Lambda^0$ by the equivalence relation $\approx$ defined as follows: we first write $x \sim y$ if there exist $p,q \in \mathbb{N}^k$ such that $(A^p_\Lambda)^t x = (A^q_\Lambda)^t y$; and then $x \approx y$ if there exist finitely many pairs (x_i, y_i) in $\mathbb{N}\Lambda^0$ such that

$$\sum_i x_i \sim x, \quad \sum_i y_i \sim y \quad \text{and} \quad x_i \sim y_i \text{ for all } i.$$

In [32, Lemma 3.7] the semigroup $S(\Lambda)$ is related to the type semigroup of an associated C*-dynamical system: each twisted C*-algebra $C^*(\Lambda, c)$ of Λ is realised, up to stable isomorphism, as a crossed product of an AF algebra, and [32, Lemma 3.7] shows that the type semigroup for this dynamical system is isomorphic to $S(\Lambda)$.

Kumjian and Pask [28] showed that every k-graph has an associated infinite-path groupoid G_Λ such that $C^*(\Lambda) \cong C^*(G_\Lambda)$. Here we prove that $S(\Lambda)$ agrees with the type semigroup $S(G_\Lambda)$ of the infinite-path groupoid G_Λ of Λ.

We need to briefly recall the notion of a k-graph and the definition of G_Λ. The following is all taken from [28]. A k-graph is a countable category Λ endowed with a map $d : \Lambda \to \mathbb{N}^k$ that carries composition to addition and has the property, called the *factorisation property* that composition restricts to a bijection from $\{(\mu, \nu) : d(\mu) = m, d(\nu) = n, s(\mu) = r(\nu)\}$ to $d^{-1}(m+n)$. It follows that $d^{-1}(0)$ is precisely the collection of identity morphisms. We write $\Lambda^n := d^{-1}(n)$ for $n \in \mathbb{N}^k$, so that r, s can be regarded as maps from Λ to Λ^0. We say that Λ is row-finite with no sources if $v\Lambda^n$ is finite and nonempty for every $n \in \mathbb{N}^k$ and $v \in \Lambda^0$.

An *infinite path* in a k-graph Λ is a map $x : \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\} \to \Lambda$ such that $x(m, n)x(n, p)$ is defined and equal to $x(m, p)$ whenever $m \leq n \leq p$. The space Λ^∞ of all such infinite paths is a totally disconnected locally compact space under the topology with basic compact open sets $Z(\lambda) := \{x : x(0, d(\lambda)) = \lambda\}$ indexed by $\lambda \in \Lambda$. Given $x \in \Lambda^\infty$ and $n \in \mathbb{N}^k$ we write $\sigma^n(x) \in \Lambda^\infty$ for the element such that $\sigma^n(x)(p, q) = x(n + p, n + q)$. The maps σ^n constitute an action of $\mathbb{N}^k$ on Λ^∞ by local homeomorphisms. The groupoid G_Λ is the set

$$G_\Lambda := \{(x, p - q, y) : x, y \in \Lambda^\infty, p, q \in \mathbb{N}^k, \sigma^p(x) = \sigma^q(y)\},$$

with composable pairs $G_\Lambda^{(2)} = \{((x, m, y), (w, n, z)) : y = w\}$, composition given by $(x, m, y)(y, n, z) = (x, m + n, z)$ and inverses $(x, m, y)^{-1} = (y, -m, x)$. The unit space is $G_\Lambda^{(0)} = \{(x, 0, x) : x \in \Lambda^\infty\}$, and we identify it with Λ^∞ without further comment. Under the topology generated by the sets $Z(\mu, \nu) := \{(x, d(\mu) - d(\nu), y) : x(0, d(\mu)) = \mu$ and $x(0, d(\nu)) = \nu\}$ indexed by pairs $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, the groupoid G_Λ is an ample, amenable groupoid, and the sets $Z(\mu, \nu)$ are compact open bisections.

Proposition 8.11. *Let Λ be a row-finite k-graph with no sources, and let G_Λ be the associated k-graph groupoid. Then there is an isomorphism $\tau : S(\Lambda) \cong S(G_\Lambda)$ such that $\tau([\delta_v]) = [\mathbb{1}_{Z(v)}]$ for all $v \in \Lambda^0$.*

Proof. There is a homomorphism $\tilde{\tau} : \mathbb{N}\Lambda^0 \to C_c(G_\Lambda, \mathbb{Z})_+$ that carries δ_v to $\mathbb{1}_{Z(v)}$ for all v. For $v \in \Lambda^0$ and $p \in \mathbb{N}^k$, we have

$$Z(v) = \bigsqcup_{\lambda \in v\Lambda^p} Z(\lambda) = \bigsqcup_{\lambda \in v\Lambda^p} r(Z(\lambda, s(\lambda)))$$

Hence

$$\begin{aligned}\tilde{\tau}(\delta_v) = \mathbb{1}_{Z(v)} &= \sum_{\lambda \in v\Lambda^p} \mathbb{1}_{r(Z(\lambda, s(\lambda)))} \\ &\sim \sum_{\lambda \in v\Lambda^p} \mathbb{1}_{s(Z(\lambda, s(\lambda)))} = \sum_{\lambda \in v\Lambda^p} \mathbb{1}_{Z(s(\lambda))} = \sum_{\lambda \in v\Lambda^p} \tilde{\tau}(\delta_{s(\lambda)}) = \tilde{\tau}\big((A_\Lambda^p)^t \delta_v\big).\end{aligned}$$

A simple calculation then shows that if $x \approx y$ in $\mathbb{N}\Lambda^0$ then $\tilde{\tau}(x) \sim \tilde{\tau}(y)$ in $C_c(G_\Lambda, \mathbb{Z})_+$ and so $\tilde{\tau}$ descends to a homomorphism from $S(\Lambda)$ to $S(G_\Lambda)$. To see that this τ is surjective, it suffices to show that its range contains $[\mathbb{1}_K]$ for every compact open $K \subseteq G_\Lambda^0 = \Lambda^\infty$. To see this, fix such a compact open K. Since the cylinder sets $\{Z(\lambda) : \lambda \in \Lambda\}$ are a base for the topology on Λ^∞, for each $x \in K$ we can find λ_x such that $x \in Z(\lambda_x) \subseteq K$. By compactness, we there is a finite $F \subseteq \Lambda$ such that $K = \bigcup_{\lambda \in F} Z(\lambda)$. Let $p := \bigvee_{\lambda \in F} d(\lambda)$. Then each $Z(\lambda) = \bigsqcup_{\alpha \in s(\lambda)\Lambda^{p-d(\lambda)}} Z(\lambda\alpha)$. Let $\overline{F} := \{\lambda\alpha : \lambda \in F, \alpha \in s(\lambda)\Lambda^{p-d(\lambda)}\}$. Then $K = \bigcup_{\mu \in \overline{F}} Z(\mu)$. Since the sets $\{Z(\mu) : \mu \in \Lambda^p\}$ are mutually disjoint, we conclude that $K = \bigsqcup_{\mu \in G} Z(\mu)$. Hence

$$[\mathbb{1}_K] = \Big[\sum_{\mu \in \overline{F}} \mathbb{1}_{r(Z(\mu, s(\mu)))}\Big] = \Big[\sum_{\mu \in \overline{F}} \mathbb{1}_{s(Z(\mu), s(\mu))}\Big] = \tau\Big(\sum_{\mu \in \overline{F}} \delta_{s(\mu)}\Big).$$

To see that K is injective, we show that if $\tilde{\tau}(x) \sim \tilde{\tau}(y)$, then $x \approx y$. Since $\tilde{\tau}(x) \sim \tilde{\tau}(y)$, then we can find compact open bisections E_i such that $\sum_v x(v)\mathbb{1}_{Z(v)} = \sum_i \mathbb{1}_{s(E_i)}$ and $\sum_w y(w)\mathbb{1}_{Z(w)} = \sum_i \mathbb{1}_{r(E_i)}$. Recall that the map $c : (x, m, y) \mapsto m$ is a continuous $\mathbb{Z}^k$-valued cocycle on G_Λ and so the sets $\{c^{-1}(p) : p \in \mathbb{Z}^k\}$ are mutually disjoint clopen sets. So by replacing each E_i with the finitely many nonempty intersections $E_i \cap c^{-1}(p)$ of which it is comprised, we can assume that each $E_i \subseteq c^{-1}(p_i)$ for some i. Now an argument very similar to the preceding paragraph shows that we can express each E_i as a finite disjoint union $E_i = \bigcup_{(\mu,\nu) \in F_i} Z(\mu, \nu)$, so we may assume that each E_i has the form $Z(\mu_i, \nu_i)$. By taking $p = \sup_i d(\mu_i)$ and writing each $Z(\mu_i, \nu_i) = \bigsqcup_i \bigsqcup_{\alpha \in s(\mu_i)\Lambda^{p-d(\lambda_i)}} Z(\mu_i\alpha, \nu_i\alpha)$, we can further assume that each $d(\mu_i) = p$. So the sets $r(Z(\mu_i, \nu_i))$ and $r(Z(\mu_j, \nu_j))$ are either equal or disjoint for all pairs i, j. Now for each $v \in \Lambda^0$ such that $x(v) \neq 0$ and each $\lambda \in v\Lambda^p$, we must have $|\{i : \mu_i = \lambda\}| = x(v)$. We deduce that $\sum_i \delta_{s(\mu_i)} = (A_\Lambda^p)^t x \approx x$.

Now let $q = \bigvee_i d(\nu_i)$. For each i, we have

$$\delta_{s(\nu_i)} \approx (A_\Lambda^{q-d(\nu_i)})^t \delta_{s(\nu_i)} = \sum_{\alpha \in s(\nu_i)\Lambda^{q-d(\nu_i)}} \delta_{s(\nu_i\alpha)}.$$

Thus

$$(A_\Lambda^p)^t x \approx \sum_i \sum_{\alpha \in s(\nu_i)\Lambda^{q-d(\nu_i)}} \delta_{s(\nu_i\alpha)} =: z.$$

As above, we have $\sum_i \sum_\alpha \mathbb{1}_{Z(\nu_i\alpha)} = \tilde{\tau}(y)$, and since each $d(\nu_i\alpha) = q$ the sets $Z(\nu_i\alpha)$ are mutually disjoint. So for each $w \in \Lambda^0$ with $y(w) \neq 0$ and each $\eta \in w\Lambda^q$, we have $|\{(i, \alpha) : \nu_i\alpha = \eta\}| = y(w)$. Hence

$$z = (A_\lambda^q)^t y \approx y.$$

Since $\approx$ is transitive, we conclude that $x \approx y$. So τ is injective. □

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