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Classification theorems for the C*-algebras of graphs with sinks

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Abstract

We consider graphs E which have been obtained by adding one or more sinks to a fixed directed graph G. We classify the C^*-algebra of E up to a very strong equivalence relation, which insists, loosely speaking, that $C^*(G)$ is kept fixed. The main invariants are vectors W_E: $G^0 \to$ which describe how the sinks are attached to G; more precisely, the invariants are the classes of the W_E in the cokernel of the map $A - I$, where A is the adjacency matrix of the graph G.

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CLASSIFICATION THEOREMS FOR THE C^*-ALGEBRAS OF GRAPHS WITH SINKS

IAIN RAEBURN, MARK TOMFORDE AND DANA P. WILLIAMS

We consider graphs E which have been obtained by adding one or more sinks to a fixed directed graph G. We classify the C^*-algebra of E up to a very strong equivalence relation, which insists, loosely speaking, that $C^*(G)$ is kept fixed. The main invariants are vectors $W_E : G^0 \to \mathbb{N}$ which describe how the sinks are attached to G; more precisely, the invariants are the classes of the W_E in the cokernel of the map $A - I$, where A is the adjacency matrix of the graph G.

The Cuntz-Krieger algebras $\mathcal{O}_A$ are generated by families of partial isometries satisfying relations determined by a finite matrix A with entries in $\{0,1\}$ and no zero rows [**2**]. One can view $\mathcal{O}_A$ as the C^*-algebra of the finite directed graph E with vertex adjacency matrix A [**12**]; note that E has no sinks because A has no zero rows. In recent years there has been a flurry of interest in analogues of these algebras for infinite graphs and matrices (see [**5**] and [**3**], for example).

It was shown in [**10**] that the graph algebras of [**5**] and the Exel-Laca algebras of [**3**] can be realised as direct limits of C^*-algebras of finite graphs with sinks. Since sinks were specifically excluded in the original papers, it is now of some interest to investigate the effect of sinks on the structure of the graph algebra and its K-theory. The results of [**1**] and [**10**] show that this effect can be substantial, depending on how the sink is attached to the rest of the graph. Here we shall prove some classification theorems for graphs with sinks which describe the effect of adding sinks to a given graph.

Suppose E is a row-finite graph with one sink v. The set $\{v\}$ is hereditary, and therefore gives rise to an ideal $I(v)$ in the C^*-algebra $C^*(E)$ of E. According to general theory, the quotient $C^*(E)/I(v)$ can be identified with the graph algebra $C^*(G)$ of the graph G obtained, loosely speaking, by deleting v and all edges which head only into v (see [**1**, Theorem 4.1]). We consider primarily graphs E with one sink for which this quotient is a fixed row-finite graph G; we call such graphs *1-sink extensions* of G (see Definition 1.1). The results in [**10**] suggest that the appropriate invariant should be the *Wojciech vector* of the extension, which is the element W_E of $\prod_{G^0}\mathbb{Z}$ whose wth entry is the number of paths in $E^1 \setminus G^1$ from w to the sink.

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We now state our main theorem as it applies to the finite graphs which give simple Cuntz-Krieger algebras. We denote by A_G the vertex matrix of a graph G, in which $A_G(w_1, w_2)$ is the number of edges in G from w_1 to w_2. For any row-finite graph G, A_G is a well-defined map on the direct product $\prod_{G^0}\mathbb{Z}$ and A_G^t is well-defined on $\bigoplus_{G^0}\mathbb{Z}$.

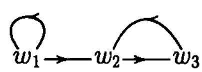
THEOREM *Suppose that E_1 and E_2 are 1-sink extensions of a finite transitive graph G.*

1. *If $W_{E_1} - W_{E_2} \in \operatorname{im}(A_G - I)$, then there exist a 1-sink extension F of G and embeddings $\phi_i : C^*(F) \to C^*(E_i)$ onto full corners of $C^*(E_i)$ such that the following diagram commutes*

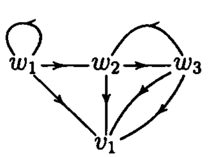
$$\begin{tikzcd} C^*(F) \arrow[rr, "\phi_i"] \arrow[dr, "\pi_F"'] & & C^*(E_i) \arrow[dl, "\pi_{E_i}"] \\ & C^*(G). & \end{tikzcd}$$

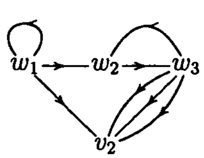
2. *If there exist F and ϕ_i as above, and if $\ker(A_G^t - I) = \{0\}$, then $W_{E_1} - W_{E_2} \in \operatorname{im}(A_G - I)$.*

While the invariants we are dealing with are K-theoretic in nature, and the proof of part (2) uses K-theory, we give constructive proofs of part (1) and of the other main theorems. Thus we can actually find the graph F. For example, if G is given by

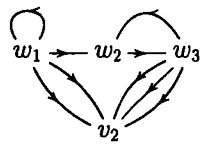


and E_1 and E_2 are the 1-sink extensions





then we can take for F the graph



The concrete nature of these constructions is very helpful when we want to apply them to graphs with more than one sink, as we do in Section 4. It also means that our classification is quite different in nature from the K-theoretic classifications of the algebras of finite graphs without sinks [**11**, **4**]. It would be an interesting and possibly very hard problem to combine our theorems with those of [**11, 4**] to say something about 1-sink extensions of different graphs.

We begin in Section 1 by establishing conventions and notation. We give careful definitions of 1-sink and n-sink extensions, and describe the basic constructions which we use throughout. In Section 2, we consider a class of extensions which we call *essential*; these are the 1-sink extensions E for which the ideal $I(v)$ is an essential ideal in $C^*(E)$. For essential 1-sink extensions of row-finite graphs we have a very satisfactory classification (Theorem 2.3), which includes part (1) of the above theorem. We show by example that we cannot completely discard the essentiality, but in Section 3 we extend the analysis to cover non-essential extensions E_1 and E_2 for which the primitive ideal spaces $\operatorname{Prim} C^*(E_1)$ and $\operatorname{Prim} C^*(E_2)$ are appropriately homeomorphic. This extra generality is crucial in Section 4, where we use our earlier results to prove a classification theorem for extensions with n sinks (Theorem 4.1). In our last section, we investigate the necessity of our hypothesis on the Wojciech vectors. In particular, part (2) of the above theorem follows from Corollary 5.4.

1. Sink extensions and the basic constructions

A directed graph $G = (G^0, G^1, r, s)$ consists of a countable set G^0 of vertices, a countable set G^1 of edges, and maps $r, s : G^1 \to G^0$ which identify the range and source of an edge. A vertex $v \in G^0$ is a *sink* if $s^{-1}(v) = \emptyset$, or a *source* if $r^{-1}(v) = \emptyset$; G is *row-finite* if each vertex emits at most finitely many edges. All graphs in this paper are row-finite and directed, and unless we say otherwise, G will stand for a generic row-finite graph. In general, our notation should be consistent with that of [**1**] and [**5**].

If G is a row-finite graph, a *Cuntz-Krieger G-family* in a C^*-algebra consists of mutually orthogonal projections $\{p_v : v \in G^0\}$ and partial isometries $\{s_e : e \in G^1\}$ which satisfy the *Cuntz-Krieger relations*

$$s_e^* s_e = p_{r(e)} \text{ for } e \in G^1 \quad\text{and}\quad p_v = \sum_{\{e:s(e)=v\}} s_e s_e^* \text{ whenever } v \in G^0 \text{ is not a sink.}$$

We denote by $C^*(G) = C^*(s_e, p_v)$ the C^*-algebra of the graph G, which is generated by a universal Cuntz–Krieger G-family $\{s_e, p_v\}$ (see [**5**, Theorem 1.2]).

Definition 1.1: An *n-sink extension* of G is a row-finite graph E which contains G as a subgraph and satisfies:

(1) $H := E^0 \setminus G^0$ is finite, contains no sources, and contains exactly n sinks.
(2) There are no loops in E whose vertices lie in H.
(3) If $e \in E^1 \setminus G^1$, then $r(e) \in H$.
(4) If w is a sink in G, then w is a sink in E.

When we say (E, v_i) is an n-sink extension of G, we mean that $v_1, \dots v_n$ are the n sinks outside G^0. We consistently write H for $E^0 \setminus G^0$ and S for the set of sinks $\{v_1, \dots, v_n\}$ lying in H.

If $w \in H$, then there are at most finitely many paths from w to a given sink v_i. If there is one sink v_1 and exactly one path from every $w \in H$ to v_1, we call (E, v_1) a *1-sink tree extension* of G. Equivalently, $\big(H, s^{-1}(H)\big)$ is a tree.

If we start with a graph E with n sinks, these ideas should apply as follows. Let H be the saturation of the set S of sinks in the sense of [**1**], and take $G := E \setminus H := \big(E^0 \setminus H, E^1 \setminus r^{-1}(H)\big)$. Then E satisfies all the above properties with respect to G except possibly (1); if, however, E is finite and has no sources, this is automatic too. So the situation of Definition 1.1 is quite general. Property (4) ensures that the saturation of S does not extend into G; it also implies that an m-sink extension of an n-sink extension of G is an $(m+n)$-sink extension of G, which is important for an induction argument in Section 4.

Lemma 1.2. *Let (E, v_i) be an n-sink extension of G. Then $H := E^0 \setminus G^0$ is a saturated hereditary subset of E^0. Indeed, H is the saturation $\overline{S}$ of $S := \{v_1, \dots, v_n\}$.*

Proof: Property (3) of Definition 1.1 implies that H is hereditary, and property (4) that H is saturated. Because $\overline{S}$ is the smallest saturated set containing S, it now suffices to prove that $H \subset \overline{S}$. Suppose that $w \notin \overline{S}$. Then either there is a path γ from w to a sink $r(\gamma) \notin S$, or there is an infinite path which begins at w. In the first case, w cannot be in H because $r(\gamma) \notin H$ and H is hereditary. In the second case, w cannot be in H because otherwise we would have an infinite path going round the finite set H, and there would have to be a loop in H. Either way, therefore, $w \notin H$, and we have proved $H \subset \overline{S}$. ∎

Corollary 1.3. *Suppose that (E, v_i) is an n-sink extension of G, and $I(S)$ is the ideal in $C^*(E) = C^*(s_e, p_v)$ generated by the projections p_{v_i} associated to the sinks $v_i \in S$. Then there is a surjection π_E of $C^*(E)$ onto $C^*(G) = C^*(t_f, q_w)$ such that $\pi_E(s_e) = t_e$ for $e \in G^1$ and $\pi_E(p_v) = q_v$ for $v \in G^0$, and $\ker \pi_E = I(S)$.*

Proof: From Lemma 1.2 and [**1**, Lemma 4.3], we see that $I(S) = I(H)$, and the result follows from [**1**, Theorem 4.1]. ∎

Definition 1.4: An n-sink extension (E, v_i) of G is *simple* if $E^0 \setminus G^0 = \{v_i, \dots, v_n\}$.

We want to associate to each n-sink extension (E, v_i) a simple extension by collapsing paths which end at one of the v_i. For the precise definition, we need some notation. An edge e with $r(e) \in H$ and $s(e) \in G^0$ is called a *boundary edge*; the sources of these edges are called *boundary vertices*. We write B^1_E and B^0_E for the sets of boundary edges and vertices. If $w \in G^0$ and $1 \leqslant i \leqslant n$, we denote by $Z(w, v_i)$ the set of paths α from w to v_i which leave G immediately in the sense that $r(\alpha_1) \in H$. The *Wojciech vector* of the sink v_i is the element $W_{(E;v_i)}$ of $\prod_{G^0}\mathbb{N}$ given by

$$W_{(E;v_i)}(w) := \#Z(w,v_i) \text{ for } w \in G^0;$$

notice that $W_{(E;v_i)}(w) = 0$ unless w is a boundary vertex. If E has just one sink, we denote its only Wojciech vector by W_E.

The *simplification* of (E,v_i) is the graph SE with $(SE)^0 := G^0 \cup \{v_1,\dots,v_n\}$,

$$(SE)^1 := G^1 \cup \{e^{(w,\alpha)} : w \in B_E^0 \text{ and } \alpha \in Z(w,v_i) \text{ for some } i\},$$
$$s|_{G^1} = s_E, \quad s(e^{(w,\alpha)}) = w, \quad r|_{G^1} = r_E, \quad \text{and} \quad r(e^{(w,\alpha)}) = r(\alpha).$$

The simplification of (E,v_i) is a simple n-sink extension of G with the same Wojciech vectors as E. We now describe how the graph algebras are related:

PROPOSITION 1.5. *Let (E,v_i) be an n-sink extension of G, and let $\{s_e,p_v\}$, $\{t_f,q_w\}$ denote the canonical Cuntz-Krieger families in $C^*(SE)$ and $C^*(E)$. Then there is an embedding ϕ^{SE} of $C^*(SE)$ onto the full corner in $C^*(E)$ determined by the projection $\sum_{i=1}^{n} q_{v_i} + \sum\{q_w : w \in G^0\}$, which satisfies $\phi^{SE}(p_v) = q_v$ for all $v \in G^0 \cup \{v_i\}$, and for which we have a commutative diagram involving the maps π_E of Corollary 1.3:*

$$\begin{tikzcd} C^*(SE) \arrow[rr, "\phi^{SE}"] \arrow[dr, "\pi_{SE}"'] & & C^*(E) \arrow[dl, "\pi_E"] \\ & C^*(G) & \end{tikzcd}$$

PROOF: The elements

$$P_v := q_v \quad \text{and} \quad S_e := \begin{cases} t_e & \text{if } e \in G^1 \\ t_\alpha & \text{if } e = e^{(w,\alpha)} \end{cases}$$

form a Cuntz-Krieger (SE)-family in $C^*(E)$, so there is a homomorphism $\phi^{SE} := \pi_{S,P} : C^*(SE) \to C^*(E)$ with $\phi^{SE}(p_v) = P_v$ and $\phi^{SE}(s_e) = S_e$. We trivially have $\phi^{SE}(p_v) = q_v$ for $v \in G^0 \cup S$.

To see that ϕ^{SE} is injective, we use the universal property of $C^*(E)$ to build an action $\beta : \mathbb{T} \to \operatorname{Aut} C^*(E)$ such that

$$\beta_z(q_w) = q_w \quad \text{and} \quad \beta_z(t_f) = \begin{cases} zt_f & \text{if } s(f) \in G^0 \\ t_f & \text{otherwise,} \end{cases}$$

note that ϕ^{SE} converts the gauge action on $C^*(SE)$ to β, and apply the gauge-invariant uniqueness theorem [**1**, Theorem 2.1].

It follows from [**1**, Lemma 1.1] that $\sum_{i=1}^{n} q_{v_i} + \sum\{q_w : w \in G^0\}$ converges strictly to a projection $q \in M(C^*(E))$ such that

$$qt_\alpha t_\beta^* = \begin{cases} t_\alpha t_\beta^* & \text{if } s(\alpha) \in G^0 \cup S \\ 0 & \text{otherwise.} \end{cases}$$

Thus $qC^*(E)q$ is spanned by the elements $t_\alpha t_\beta^*$ with $s(\alpha) = s(\beta) \in G^0 \cup S$, and by applying the Cuntz-Krieger relations we may assume $r(\alpha) = r(\beta) \in G^0 \cup S$ also, so that the range of ϕ is the corner $qC^*(E)q$. To see that this corner is full, suppose I is an ideal containing $qC^*(E)q$. Then [1, Lemma 4.2] implies that $K := \{v : q_v \in I\}$ is a saturated hereditary subset of E^0; since K certainly contains $G^0 \cup S$, we deduce that $K = E^0$. But then $I = C^*(E)$ by [**1**, Theorem 4.1]. Finally, to see that the diagram commutes, we just need to check that π_{SE} and $\pi_E \circ \phi^{SE}$ agree on generators. ∎

It is convenient to have a name for the situation described in this proposition:

DEFINITION 1.6: Suppose (E, v_i) and (F, w_i) are n-sink extensions of G. We say that $C^*(F)$ is *$C^*(G)$-embeddable in $C^*(E)$* if there is an isomorphism ϕ of $C^*(F) = C^*(s_e, p_v)$ onto a full corner in $C^*(E) = C^*(t_f, q_w)$ such that $\phi(p_{w_i}) = q_{v_i}$ for all i and $\pi_E \circ \phi = \pi_F : C^*(F) \to C^*(G)$. If ϕ is an isomorphism onto $C^*(E)$, we say that $C^*(F)$ is *$C^*(G)$-isomorphic to $C^*(E)$*.

Notice that if $C^*(F)$ is $C^*(G)$-embeddable in $C^*(E)$, then $C^*(F)$ is Morita equivalent to $C^*(E)$ in a way which respects the common quotient $C^*(G)$.

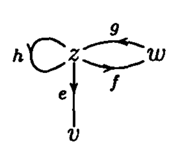
We now describe the basic construction by which we manipulate the Wojciech vectors of graphs.

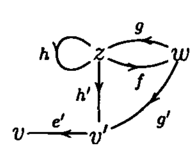
DEFINITION 1.7: Let (E, v_i) be an n-sink extension of G, and let e be a boundary edge such that $s(e)$ is not a source of G. The *outsplitting* of E by e is the graph $E(e)$ defined by

$$E(e)^0 := E^0 \cup \{v'\}; \quad E(e)^1 := (E^1 \setminus \{e\}) \cup \{e'\} \cup \{f' : f \in E^1 \text{ and } r(f) = s(e)\}$$
$$(r, s)|_{E^1 \setminus \{e\}} := (r_E, s_E); \quad r(e') := r_E(e),\ s(e') := v'; \quad r(f') := v',\ s(f') := s_E(f).$$

In general, we call $E(e)$ a *boundary outsplitting* of E.

The following example might help fix the ideas:

$E :$ 

$E(e) :$ 

If (E, v_i) is an n-sink extension of G, then every boundary outsplitting $(E(e), v_i)$ is also an n-sink extension of G; if (E, v_0) is a 1-sink tree extension, so is $(E(e), v_0)$. We need to assume that $s(e)$ is not a source of G to ensure that $E(e)$ is an n-sink extension, and we make this assumption implicitly whenever we talk about boundary outsplittings. As the name suggests, boundary outsplittings are special cases of the outsplittings discussed in [**7**, Section 2.4].

PROPOSITION 1.8. *Suppose $(E(e), v_i)$ is a boundary outsplitting of an n-sink extension (E, v_i) of G. Then $C^*(E(e))$ is $C^*(G)$-isomorphic to $C^*(E)$. If E is a 1-sink tree extension, then the Wojciech vector of $E(e)$ is given in terms of the vertex matrix A_G of G by*

$$W_{E(e)} = W_E + (A_G - I)\,\delta_{s(e)}. \tag{1.1}$$

PROOF: Let $C^*(E) = C^*(t_h, q_w)$. Then

$$P_v := \begin{cases} q_v & \text{if } v \neq s(e) \text{ and } v \neq v' \\ t_e t_e^* & \text{if } v = v' \\ q_{s(e)} - t_e t_e^* & \text{if } v = s(e) \end{cases}$$
$$S_g := \begin{cases} t_e & \text{if } g = e' \\ t_g(q_{s(e)} - t_e t_e^*) & \text{if } g \neq e' \text{ and } r(g) = s(e) \\ t_f t_e t_e^* & \text{if } g = f' \text{ for some } f \in E^1 \text{ with } r(f) = s(e) \\ t_g & \text{otherwise} \end{cases}$$

is a Cuntz-Krieger $E(e)$-family which generates $C^*(E)$. The universal property of $C^*(E(e)) = C^*(s_g, p_v)$ gives a homomorphism $\phi = \pi_{S,P} : C^*(E(e)) \to C^*(E)$ such that $\phi(s_g) = S_g$ and $\phi(p_v) = P_v$, which is an isomorphism by the gauge-invariant uniqueness theorem [**1**, Theorem 2.1]. It is easy to check on generators that ϕ is a $C^*(G)$-isomorphism.

When H is a tree with one sink v_0, there is precisely one path γ in E from $r(e)$ to v_0, and hence all the new paths from a vertex v to v_0 have the form $f'\gamma$. Thus if $v \neq s(e)$,

$$\begin{aligned} W_{E(e)}(v) &= W_E(v) + \#\{f' \in E(e)^1 : s(f') = v \text{ and } f' \notin E^1\} \\ &= W_E(v) + \#\{f \in G^0 : s(f) = v \text{ and } r(f) = s(e)\} \\ &= W_E(v) + A_G(v, s(e)). \end{aligned}$$

On the other hand, if $v = s(e)$, then

$$\begin{aligned} W_{E(e)}(s(e)) &= W_E(s(e)) + \#\{f' \in E(e)^1 : s(f') = s(e) \text{ and } f' \notin E^1\} - 1 \\ &= W_E(s(e)) + \#\{f \in G^0 : s(f) = s(e) = r(f)\} - 1 \\ &= W_E(v) + A_G(s(e), s(e)) - 1. \end{aligned}$$

Together these calculations give (1.1). ∎

Suppose that $\alpha = \alpha_1\alpha_2\cdots\alpha_n$ is a path in G and there is a boundary edge e with $s(e) = r(\alpha)$. Then $E(e)$ will have a boundary edge α_n' at $r(\alpha_{n-1})$, and therefore we can outsplit again to get $E(e)(\alpha_n')$. This graph has a boundary edge α_{n-1}' at $r(\alpha_{n-2})$, and we can outsplit again. Continuing this process gives an extension $E(e,\alpha)$ in which $s(\alpha)$ is a boundary vertex. We shall refer to this process as *performing outsplittings along the path* α. From Proposition 1.8 we can calculate the Wojciech vector of $E(e,\alpha)$:

COROLLARY 1.9. *Suppose E is a 1-sink tree extension of G and α is a path in G for which $r(\alpha)$ is a boundary vertex. Then for any boundary edge e with $s(e) = r(\alpha)$, we have*

$$W_{E(e,\alpha)} = W_E + \sum_{i=1}^{|\alpha|}(A_G - I)\delta_{r(\alpha_i)}.$$

2. A CLASSIFICATION FOR ESSENTIAL 1-SINK EXTENSIONS

We now ask to what extent the Wojciech vector determines a 1-sink extension. Suppose that E_1 and E_2 are 1-sink extensions of G. Our main results say, loosely speaking, that if the Wojciech vectors W_{E_i} determine the same class in $\operatorname{coker}(A_G - I)$, then there will be a simple extension F such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$ and $C^*(E_2)$. However, we shall need some hypotheses on the way the sinks are attached to G; the hypotheses in this section are satisfied if, for example, G is one of the finite transitive graphs for which $C^*(G)$ is a simple Cuntz-Krieger algebra. We begin by describing the class of extensions which we consider in this section.

Recall that if v, w are vertices in G, then $v \geqslant w$ means there is a finite path γ with $s(\gamma) = v$ and $r(\gamma) = w$. For $K, L \subset G^0$, $K \geqslant L$ means that for each $v \in K$ there exists $w \in L$ such that $v \geqslant w$. If γ is a loop, we write $\gamma \geqslant L$ when $\{r(\gamma_i)\} \geqslant L$.

DEFINITION 2.1: A 1-sink extension (E, v_0) of a graph G is an *essential extension* if $G^0 \geqslant v_0$.

We can see immediately that simplifications of essential extensions are essential, and consideration of a few cases shows that boundary outsplittings of essential extensions are essential. To see why we chose the name, recall that an ideal I in a C^*-algebra A is essential if $I \cap J \neq 0$ for all nonzero ideals J in A, or equivalently, if $aI = 0$ implies $a = 0$. Then we have:

LEMMA 2.2. *Let (E, v_0) be a 1-sink extension of G. Then (E, v_0) is an essential extension of G if and only if the ideal $I(v_0)$ generated by p_{v_0} is an essential ideal in $C^*(E) = C^*(s_e, p_v)$.*

PROOF: Suppose that there exists $w \in G^0$ such that $w \not\geqslant v_0$. Then since

$$I(v_0) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) = v_0\}$$

(see [**1**, Lemma 4.3]), we have $p_w I(v_0) = 0$, and $I(v_0)$ is not essential.

Conversely, suppose that $G^0 \geqslant v_0$. To show that $I(v_0)$ is an essential ideal it suffices to prove that if $\pi : C^*(E) \to B(\mathcal{H})$ is a representation with $\ker \pi \cap I(v_0) = \{0\}$, then π is faithful. So suppose $\ker \pi \cap I(v_0) = \{0\}$. In particular, we have $\pi(p_{v_0}) \neq 0$. For every $v \in G^0$ there is a path α in E such that $s(\alpha) = v$ and $r(\alpha) = v_0$. Then $\pi(s_\alpha^* s_\alpha) = \pi(p_{v_0}) \neq 0$, and hence $\pi(p_v) \geqslant \pi(s_\alpha s_\alpha^*) \neq 0$. Since every loop in a 1-sink extension E must lie entirely in G, every loop in G has an exit in E; thus we can apply [**1**, Theorem 3.1] to deduce that π is faithful, as required. $\square$

We can now state our classification theorem for essential extensions.

Theorem 2.3. *Let G be a row-finite graph with no sources, and suppose that (E_1, v_1) and (E_2, v_2) are essential 1-sink extensions of G with finitely many boundary vertices. If there exists $n \in \bigoplus_{G^0} \mathbb{Z}$ such that the Wojciech vectors satisfy $W_{E_1} - W_{E_2} = (A_G - I)n$, then there is a simple 1-sink extension F of G such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$ and $C^*(E_2)$.*

We begin by observing that, since a full corner in a full corner of a C^*-algebra A is a full corner in A, the composition of two $C^*(G)$-embeddings is another $C^*(G)$-embedding. Thus it suffices by Proposition 1.5 to prove the theorem for the simplifications SE_1 and SE_2. However, since we are going to perform boundary outsplittings and these do not preserve simplicity, we assume merely that E_1 and E_2 are 1-sink tree extensions. The following lemma is the key to many of our constructions:

Lemma 2.4. *Let (E_1, v_1) and (E_2, v_2) be 1-sink tree extensions of G with finitely many boundary vertices, and suppose that $B^0_{E_1} \geqslant B^0_{E_2} \geqslant B^0_{E_1}$. If γ is a loop in G such that $\gamma \geqslant B^0_{E_1}$, then for any $a \in \mathbb{Z}$ there are 1-sink tree extensions E_1' and E_2' which are formed by performing a finite number of boundary outsplittings to E_1 and E_2, respectively, and for which*

$$W_{E_1'} - W_{E_2'} = W_{E_1} - W_{E_2} + a\bigg(\sum_{j=1}^{|\gamma|}(A_G - I)\delta_{r(\gamma_j)}\bigg).$$

Proof: Since the statement is symmetric in E_1 and E_2, it suffices to prove this for $a > 0$. Choose a path α in G such that $s(\alpha) = r(\gamma)$ and $r(\alpha) \in B^0_{E_1}$. Since $B^0_{E_1}$ is finite, going along paths from $r(\alpha)$ to $B^0_{E_2}$ and then to and fro between $B^0_{E_2}$ and $B^0_{E_1}$ must eventually give either (a) a loop μ which visits both $B^0_{E_1}$ and $B^0_{E_2}$, and a path β with $s(\beta) = r(\alpha)$ and $r(\beta) = s(\mu) \in B^0_{E_1}$, or (b) a vertex $v \in B^0_{E_1} \cap B^0_{E_2}$ and a path β with $s(\beta) = r(\alpha)$ and $r(\beta) = v$.

We deal with case (a) first. Since there are boundary edges $e_1 \in B^1_{E_1}$ and $e_2 \in B^1_{E_2}$ with $s(e_i)$ on μ, we can perform outsplittings along μ to get new tree extensions $E_i(e_i, \mu^i)$, where μ^i is the loop μ relabelled so that it ends at $s(e_i)$. Because μ^1 and μ^2 have the same vertices as μ in a different order, Corollary 1.9 gives

$$W_{E_i(e_i,\mu^i)} = W_{E_i} + \sum_{j=1}^{|\mu|}(A_G - I)\delta_{r(\mu_j)},$$

so we have $W_{E_1(e_1,\mu^1)} - W_{E_2(e_2,\mu^2)} = W_{E_1} - W_{E_2}$. Since $r(\beta_{|\beta|}) = s(\mu)$, and in forming both $E_i(e_i,\mu^i)$ we have performed an outsplitting at $s(\mu)$, $s(\beta_{|\beta|})$ is a boundary vertex in both $E_i(e_i,\mu^i)$; say $f_i \in B^1_{E_i}$ has $s(f_i) = s(\beta_{|\beta|})$. Write $\beta = \beta'\beta_{|\beta|}$, γ^a for the path obtained by going a times around γ, and define

$$E_1' := E_1(e_1,\mu^1)(f_1,\gamma^a\alpha\beta') \quad\text{and}\quad E_2' := E_2(e_2,\mu^2)(f_2,\alpha\beta').$$

We now compute the Wojciech vectors using Corollary 1.9: for example,

$$W_{E_1'} = W_{E_1(e_1,\mu^1)} + (A_G - I)\Bigg(\sum_{j=1}^{|\beta|-1}\delta_{r(\beta_j)} + \sum_{j=1}^{|\alpha|}\delta_{r(\alpha_j)} + \sum_{j=1}^{|\gamma|}a\delta_{r(\gamma_j)}\Bigg).$$

The formula for $W_{E_2'}$ is the same except for the last term, so

$$\begin{aligned} W_{E_1'} - W_{E_2'} &= W_{E_1(e_1,\mu^1)} - W_{E_2(e_2,\mu^2)} + \sum_{j=1}^{|\gamma|}a(A_G - I)\delta_{r(\gamma_j)} \\ &= W_{E_1} - W_{E_2} + \sum_{j=1}^{|\gamma|}a(A_G - I)\delta_{r(\gamma_j)}, \end{aligned}$$

as required.

In case (b), we can dispense with the first step in the preceding argument: we choose boundary edges $f_i \in B^1_{E_i}$ with $s(f_i) = v$, and then

$$E_1' := E_1(f_1,\gamma^a\alpha\beta) \quad\text{and}\quad E_2' := E_2(f_2,\alpha\beta)$$

have the required properties. $\square$

Proof of Theorem 2.3: As we indicated earlier, it suffices to prove the theorem when E_1 and E_2 are tree extensions. It also suffices to prove that we can perform boundary outsplittings on E_1 and E_2 to achieve extensions F_1 and F_2 with the same Wojciech vector; Propositions 1.5 and 1.8 then imply that we can take for F the common simplification of F_1 and F_2. We can write $n = \sum_{k=1}^{m} a_k\delta_{w_k}$ for some finite set $\{w_1, w_2, \dots, w_m\} \subset G^0$. We shall prove by induction on m that we can perform the required outsplittings. If $m = 0$, then $W_{E_1} = W_{E_2}$, and there is nothing to prove. So we suppose that we can perform the outsplittings whenever n has the form $\sum_{k=1}^{m} a_k\delta_{w_k}$, and that $n = \sum_{k=1}^{m+1} a_k\delta_{w_k}$. Let D be the subgraph of G with vertices $D^0 := \{w_1, w_2, \dots, w_{m+1}\}$ and edges $D^1 := \big\{e \in G^1 : s(e), r(e) \in D^0\big\}$. Since D is a finite graph it contains either a sink or a loop.

If D contains a sink, then by relabelling we can assume the sink is w_{m+1}. Since $A_G(w_{m+1}, w_j) = 0$ for all j, we have

$$W_{E_1}(w_{m+1}) = W_{E_2}(w_{m+1}) - a_{m+1}.$$

Thus either E_1 or E_2 has at least $|a_{m+1}|$ boundary edges leaving w_{m+1}: we may as well assume that $a_{m+1} > 0$, so that $W_{E_2}(w_{m+1}) \geqslant a_{m+1}$. We can then perform a_{m+1} boundary outsplittings on E_2 at w_{m+1} to get a new extension E_2'. From Proposition 1.8, we have $W_{E_2'} = W_{E_2} + a_{m+1}(A_G - I)\delta_{w_{m+1}}$, and therefore

$$W_{E_1} = W_{E_2'} + (A_G - I)\Big(\sum_{k=1}^{m} a_k \delta_{w_k}\Big).$$

Since E_2' is formed by performing boundary outsplittings to the essential tree extension E_2, it is also an essential tree extension, and the inductive hypothesis implies that we can perform boundary outsplittings on E_1 and E_2' to arrive at extensions with the same Wojciech vector.

If D does not have a sink, it must contain a loop γ. If necessary, we can shrink γ so that its vertices are distinct, and by relabelling, we may assume that w_{m+1} lies on γ. Because the extensions are essential, we have $G^0 \geqslant B^0_{E_1}$ and $G^0 \geqslant B^0_{E_2}$, so we can apply Lemma 2.4. Thus there are 1-sink tree extensions E_1' and E_2' formed by performing boundary outsplittings to E_1 and E_2, and for which

$$W_{E_1'} - W_{E_2'} = W_{E_1} - W_{E_2} - a_{m+1}\sum_{j=1}^{|\gamma|}(A_G - I)\delta_{r(\gamma_j)}.$$

But because $W_{E_1} = W_{E_2} + (A_G - I)n$ this implies that

$$W_{E_1'} = W_{E_2'} + (A_G - I)\Big(\sum_{j=1}^{m} b_j \delta_{w_j}\Big),$$

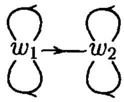
where $b_j = a_j - a_{m+1}$ if w_j lies on γ, and $b_j = a_j$ otherwise. We can now invoke the inductive hypothesis to see that we can perform boundary outsplittings to E_1' and E_2' to arrive at extensions with the same Wojciech vector.

This completes the proof of the inductive step, and the result follows. $\square$

REMARK 2.5. The graph F in Theorem 2.3 has actually been constructed in a very specific way, and it will be important in Section 4 that we can keep track of the procedures used. We shall say that one simple extension F has been obtained from another E *by a standard construction* if it is the simplification of a graph obtained by performing a sequence of boundary outsplittings to E. The graph F in Theorem 2.3 has been obtained from both SE_1 and SE_2 by a standard construction.

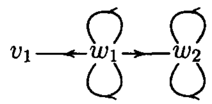
The next example shows that the hypothesis of essentiality in Theorem 2.3 cannot be completely dropped.

EXAMPLE 2.6. Consider the following graph G

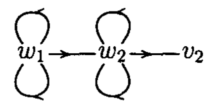


and its extensions E_1 and E_2;

E_1 :



E_2 :



Note that E_2 is essential but E_1 is not. On one hand, we have $A_G = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $W_{E_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $W_{E_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so

$$W_{E_1} - W_{E_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = (A_G - I) \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

On the other hand, we claim that $C^*(E_1)$ is not Morita equivalent to $C^*(E_2)$, so that they cannot have a common full corner. To see this, recall from [**1**, Theorem 4.4] that the ideals in $C^*(E_i)$ are in one-to-one correspondence with the saturated hereditary subsets of E_i^0. The saturated hereditary subsets of E_1^0 are $\{v_1\}$, $\{v_1, w_2\}$, $\{v_1, w_1, w_2\}$ and $\{w_2\}$, and those of E_2^0 are $\{v_2\}$, $\{v_2, w_2\}$ and $\{v_2, w_1, w_2\}$. Thus $C^*(E_1)$ has more ideals than $C^*(E_2)$. But if they were Morita equivalent, the Rieffel correspondence would set up a bijection between their ideal spaces.

This example shows that the way the sinks v_i are attached to G can affect how the ideal $I(v_0)$ lies in the ideal space of $C^*(E)$. In the next section, we give a simple condition on the way v_i are attached which ensures that the primitive ideal spaces of $C^*(E_i)$ are homeomorphic, and show that under this condition there is a good analogue of Theorem 2.3. However, there is one situation in which essentiality is not needed: when $C^*(G)$ is an AF-algebra.

COROLLARY 2.7. *Let G be a graph with no sources for which $C^*(G)$ is an AF-algebra, and let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. If there exists $n \in \bigoplus_{G^0} \mathbb{Z}$ such that $W_{E_1} = W_{E_2} + (A_G - I)n$, then then there is a simple 1-sink extension F of G such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$ and $C^*(E_2)$.*

PROOF: We first recall from [**5**, Theorem 2.4] that $C^*(G)$ is AF if and only if G has no loops. Now we proceed as in the proof of Theorem 2.3. Everything goes the same until we come to consider the finite subgraph D associated to the support of the vector $n_\cdot$. Since there are no loops in G, D must have a sink, and the argument in the second paragraph of the proof of Theorem 2.3 suffices; this does not use essentiality. $\square$

3. A classification for non-essential 1-sink extensions

Recall from [**1**, Section 6] that a *maximal tail* in a graph E is a nonempty subset of E^0 which is cofinal under $\geqslant$, is backwards hereditary ($v \geqslant w$ and $w \in \gamma$ imply $v \in \gamma$), and contains no sinks (for each $w \in \gamma$, there exists $e \in E^1$ with $s(e) = w$ and $r(e) \in \gamma$).

Definition 3.1: Let (E, v_0) be a 1-sink extension of G. The *closure* of the sink v_0 is the set

$$\overline{v_0} := \cup\{\gamma : \gamma \text{ is a maximal tail in } G \text{ and } \gamma \geqslant v_0\}.$$

To motivate this definition, we notice first that the extension is essential if and only if $\overline{v_0} = G^0$. More generally (although it is not logically necessary for our results), we explain how this notion of closure is related to the closure of sets in $\operatorname{Prim} C^*(E)$, as described in [**1**, Section 6]. For each sink v, let $\lambda_v := \{w \in E^0 : w \geqslant v\}$, and let

$$\Lambda_E := \{\text{maximal tails in } E\} \cup \{\lambda_v : v \text{ is a sink in } E\}.$$

The set Λ_E has a topology in which the closure of a subset S is $\{\lambda : \lambda \geqslant \bigcup_{\chi \in S} \chi\}$, and it is proved in [**1**, Corollary 6.5] that when E satisfies Condition (K) of [**6**], $\lambda \mapsto I(E^0 \setminus \lambda)$ is a homeomorphism of Λ_E onto $\operatorname{Prim} C^*(E)$. If (E, v_0) is a 1-sink extension of G, then the only loops in E are those in G, so E satisfies Condition (K) whenever G does. A subset of G^0 is a maximal tail in E if and only if it is a maximal tail in G, and because every sink in G is a sink in E, we deduce that $\Lambda_E = \Lambda_G \cup \{\lambda_{v_0}\}$.

Lemma 3.2. *Suppose G satisfies Condition* (K), *and (E_1, v_1), (E_2, v_2) are 1-sink extensions of G. Then $\overline{v_1} = \overline{v_2}$ if and only if there is a homeomorphism h of* $\operatorname{Prim} C^*(E_1)$ *onto* $\operatorname{Prim} C^*(E_2)$ *such that*

$$h\big(I(E_1^0 \setminus \lambda)\big) = I(E_2^0 \setminus \lambda) \text{ for } \lambda \in \Lambda_G, \text{ and } h\big(I(E_1^0 \setminus \lambda_{v_1})\big) = I(E_2^0 \setminus \lambda_{v_2}). \tag{3.1}$$

Proof: For any 1-sink extension (E, v_0), the map $J \mapsto \pi_E^{-1}(J)$ is a homeomorphism of $\operatorname{Prim} C^*(G)$ onto the closed subset $\{J \in \operatorname{Prim} C^*(E) : J \supset I(v_0)\}$. If $\lambda \in \Lambda_G \subset \Lambda_E$, then $\pi_E^{-1}\big(I(G^0 \setminus \lambda)\big) = I(E^0 \setminus \lambda)$, and hence h is always a homeomorphism of the closed set $\{I(E_1^0 \setminus \lambda) : \lambda \in \Lambda_G\}$ in $\operatorname{Prim} C^*(E_1)$ onto the corresponding subset of $\operatorname{Prim} C^*(E_2)$. So the only issue is whether the closures of the sets $I(E_1^0 \setminus \lambda_{v_1})$ and $I(E_2^0 \setminus \lambda_{v_2})$ match up. But

$$\overline{I(E_i^0 \setminus \lambda_{v_i})} = \{I(E_i^0 \setminus \lambda) : \lambda \geqslant \lambda_{v_i}\} = \{I(E_i^0 \setminus \lambda) : \lambda \geqslant v_i\}.$$

Since other sets λ_v associated to sinks are never $\geqslant v_i$, the ideals on the right-hand side are those associated to the maximal tails lying in $\overline{v_i}$, and the result follows. $\square$

We now return to the problem of proving analogues of Theorem 2.3 for non-essential extensions. Notice that the closure is a subset of G^0 rather than E^0: we have defined it this way because we want to compare the closures in different extensions.

PROPOSITION 3.3. *Suppose that (E_1, v_1) and (E_2, v_2) are 1-sink extensions of G with finitely many boundary vertices, and suppose that $\overline{v_1} = \overline{v_2} = C$, say. If $W_{E_1} - W_{E_2}$ has the form $(A_G - I)n$ for some $n \in \bigoplus_C \mathbb{Z}$, then there is a simple 1-sink extension F of G such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$ and $C^*(E_2)$.*

We aim to follow the proof of Theorem 2.3, so we need to check that the operations used there will not affect the hypotheses in Proposition 3.3. It is obvious that the closure is unaffected by simplifications. It is true but not so obvious that it is unaffected by boundary outsplittings:

LEMMA 3.4. *Suppose (E, v_0) is a 1-sink extension of a graph G, and e is a boundary edge in E. Then the closures of v_0 in E and $E(e)$ are the same.*

PROOF: Suppose γ is a maximal tail such that $\gamma \geqslant v_0$ in $E(e)$ and $z \in \gamma$; we want to prove $z \geqslant B_E^0$. We know $z \geqslant w$ for some $w \in B_{E(e)}^0$. If $w \in B_E^0$, there is no problem. If $w \notin B_E^0$, then $w = s(f)$ for some $f \in G^1$ with $r(f) = s(e)$, so $z \geqslant w \geqslant s(e) \in B_E^0$.

Now suppose $\gamma \geqslant v_0$ in E and $z \in \gamma$; we want to prove that $z \geqslant B_{E(e)}^0$. We know that there is a path α with $s(\alpha) = z$ and $r(\alpha) \in B_E^0$. If $r(\alpha) \neq s(e)$, we have $z \geqslant r(\alpha) \in B_{E(e)}^0$. If $r(\alpha) = s(e)$ and $|\alpha| \geqslant 1$, we have $z \geqslant r(\alpha_{|\alpha|-1}) \in B_{E(e)}^0$. The one remaining possibility is that $z = s(e)$ and there is no path of length at least 1 from $s(e)$ to $s(e)$. Because γ is a tail, there exists $f \in G^1$ such that $s(f) = s(e)$ and $r(f) \in \gamma$. Now we use $\gamma \geqslant v_0$ to get a path β with $s(\beta) = r(f)$ and $r(\beta) \in B_{E^0}^0 \setminus \{s(e)\}$, and we are back in the first case with $\alpha = f\beta$. ∎

PROOF OF PROPOSITION 3.3: Since the closures $\overline{v_1}$ and $\overline{v_2}$ are unaffected by simplification and boundary outsplitting, we can run the argument of Theorem 2.3. In doing so, we never have to leave the common closure C: by hypothesis, $n = \sum_{k=1}^{m} a_k \delta_{w_k}$ for some $w_k \in C$, so all the vertices on the subgraph D used in the inductive step lie in C. When D has a sink, the argument goes over verbatim. When D has a loop γ, all the vertices on γ lie in C, and the hypothesis $\overline{v_1} = C = \overline{v_2}$ implies that $\gamma \geqslant B_{E_1}^0 \geqslant B_{E_2}^0 \geqslant B_{E_1}^0$, so we can still apply Lemma 2.4. The rest of the argument carries over. ∎

The catch in Proposition 3.3 is that the vector n is required to have support in the common closure C. For our applications to n-sink extensions in the next section, this is just what we need. However, if we are only interested in 1-sink extensions, this requirement might seem a little unnatural. So it is interesting that we can often remove it:

LEMMA 3.5. *Suppose that (E_1, v_1) and (E_2, v_2) are 1-sink extensions of G, and suppose that $\overline{v_1} = \overline{v_2} = C$, say. Suppose that 1 is not an eigenvalue of the $(G^0 \setminus C)$*

$\times (G^0 \setminus C)$ corner of A_G. Then if $W_{E_1} - W_{E_2}$ has the form $(A_G - I)n$ for some $n \in \bigoplus_{G^0} \mathbb{Z}$, we have $n \in \bigoplus_C \mathbb{Z}$.

PROOF: Since the maximal tails comprising C are backwards hereditary, there are no paths from $G^0 \setminus C$ to C. Thus A_G decomposes with respect to the decomposition $G^0 = (G^0 \setminus C) \cup C$ as $A_G = \left(\begin{smallmatrix} B & 0 \\ C & D \end{smallmatrix}\right)$, and $A_G - I = \left(\begin{smallmatrix} B-I & 0 \\ C & D-I \end{smallmatrix}\right)$. Writing n as $\left(\begin{smallmatrix} k \\ m \end{smallmatrix}\right)$ and noting that $W_{E_1} - W_{E_2}$ has support in C shows that $(B-I)k = 0$, which by the hypothesis on A_G implies $k = 0$. But this says exactly what we want. ∎

4. A CLASSIFICATION FOR n-SINK EXTENSIONS

We say that an n-sink extension is *essential* if $G^0 \geqslant v_i$ for $1 \leqslant i \leqslant n$.

THEOREM 4.1. *let (E, v_i) and (F, w_i) be essential n-sink extensions of G with finitely many boundary vertices. Suppose that the Wojciech vectors satisfy*

$$W_{(E;v_i)} - W_{(F;w_i)} \in (A_G - I)\left(\textstyle\bigoplus_{G^0}\mathbb{Z}\right) \text{ for } 1 \leqslant i \leqslant n. \tag{4.1}$$

Then there is a simple n-sink extension D of G such that $C^*(D)$ is $C^*(G)$-embeddable in both $C^*(E)$ and $C^*(F)$.

We shall prove this theorem by induction on n. At a key point we need to convert $(n-1)$-sink extensions to n-sink extensions. If $m \in \prod_{G^0} \mathbb{N}$ and (E, v_i) is an $(n-1)$-sink extension, we denote by $(E * m, v_i)$ the n-sink extension of G obtained by adding an extra vertex v_n and $m(w)$ edges from each vertex $w \in G^0$ to v_n. Note that $E * m$ has one new Wojciech vector $W_{(E*m;v_n)} = m$, and the other Wojciech vectors are unchanged. If E is a simple extension, then so is $E * m$. Conversely, if (F, w_i) is a simple n-sink extension, then $F \setminus w_n := \left(F^0 \setminus \{w_n\}, F^1 \setminus r^{-1}(w_n)\right)$ is a simple $(n-1)$-sink extension for which $(F \setminus w_n) * W_{(F;w_n)}$ can be naturally identified with F.

We need to know how the operation $E \mapsto E * m$ interacts with our other constructions:

LEMMA 4.2. *If e is a boundary edge for E, then e is a boundary edge for $E * m$, and the boundary outsplittings satisfy $E(e) * m = (E * m)(e)$. The simplification construction $E \mapsto SE$ satisfies $S(E * m) = (SE) * m$.*

PROOF: The only edges which are affected in forming $E(e)$ are e and the edges f with $r(f) = s(e)$. Since none of the new edges in $E * m$ have range in E, they are not affected by the outsplitting. Simplifying collapses paths which end at one of the sinks v_i, and forming $E * m$ adds only paths of length 1 ending at v_n, so there is nothing extra to collapse in simplifying $E * m$. ∎

PROOF OF THEOREM 4.1: As in the 1-sink case, it suffices by Proposition 1.5 to prove the result when E and F are simple. So we assume this. Our proof is by induction on n, but we have to be careful to get the right inductive hypothesis. So we shall prove that by performing n standard constructions on both E and F, we can arrive at simple n-sink extensions of G with all their Wojciech vectors equal; these graphs are then isomorphic, and we can take D to be either of them. Theorem 2.3 says that this is true for $n = 1$ (see Remark 2.5).

So we suppose that our inductive hypothesis holds for all simple $(n-1)$-sink extensions satisfying the hypotheses of Theorem 4.1. Then $E\backslash v_n$ and $F\backslash w_n$ are simple $(n-1)$-sink extensions of G with Wojciech vectors $W_{(E\backslash v_n;v_i)} = W_{(E;v_i)}$ and $W_{(F\backslash w_n;w_i)} = W_{(F;w_i)}$ for $i \leqslant n-1$. So the Wojciech vectors of $E\setminus v_n$ and $F\setminus w_n$ satisfy the hypothesis (4.1). Since $G^0 \geqslant v_i$ in E, and we have not deleted any edges except those ending at v_n and w_n, we still have $G^0 \geqslant v_i$ in $E\setminus v_n$ for $i \leqslant n-1$, and similarly $G^0 \geqslant w_i$ in $F\setminus w_n$. By the inductive hypothesis, therefore, we can perform $(n-1)$ standard constructions on each of E and F to arrive at the same simple $(n-1)$-sink extension (D, u_i) of G.

By Lemma 4.2, $D * W_{(E;v_n)}$ and $D * W_{(F;w_n)}$ are obtained from $E = (E\setminus v_n) * W_{(E;v_n)}$ and $F = (F\setminus w_n) * W_{(F;w_n)}$ by $(n-1)$ standard constructions. We now view (D^E, v_n) $:= D * W_{(E;v_n)}$ and $(D^F, w_n) := D * W_{(F;w_n)}$ as two simple 1-sink extensions of the graph D. Since the standard constructions have not affected the path structure of G inside D, and we assumed $G^0 \geqslant v_n$ in E, we still have $G^0 \geqslant v_n$ in D^E, and similarly $G^0 \geqslant w_n$ in D^F. Because any sink in G has to be a sink in E, the hypothesis $G^0 \geqslant v_n$ in E implies that G has no sinks; thus every vertex in G lies on an infinite path x, and hence in the maximal tail $\gamma := \{v : v \geqslant x\}$. Thus $G^0 \geqslant v_n$ says precisely that G^0 is the closure of v_n in D^E. Of course the same is true of w_n in D^F. We can therefore apply Proposition 3.3 to deduce that we can by one more standard construction on each of D^E and D^F reach the same 1-sink extension (C, u_n) of D; since all the boundary vertices of D lie in G, this standard construction for extensions of D is a also standard for extensions of G, and hence C can also be obtained by performing n standard constructions to each of E and F.

This completes the proof of the inductive hypothesis, and hence of the theorem. $\square$

5. K-THEORY OF 1-SINK EXTENSIONS

PROPOSITION 5.1. *Suppose that G is a row-finite graph with no sinks, and (E, v_0) is a 1-sink extension of G such that $W_E \perp \ker(A_G^t - I)$. If (F, w_0) is a 1-sink extension of G and $\phi : C^*(F) \to C^*(E)$ is a $C^*(G)$-embedding, then there exists $k \in \prod_{G^0} \mathbb{Z}$ such that $W_E - W_F = (A_G - I)k$.*

For the proof, we need to know the K-theory of the C^*-algebras of graphs with sinks, which was was calculated in [**10**, Section 3]. We summarise some results from [**10**] in a convenient form:

LEMMA 5.2. *Suppose G has no sinks and (E, v_0) is a 1-sink extension of G with graph algebra $C^*(E) = C^*(s_e, p_v)$. Let ψ^E be the homomorphism of $\left(\bigoplus_{G^0} \mathbb{Z}\right) \oplus \mathbb{Z}$ into*

$K_0(C^*(E))$ which is determined on the standard basis elements by $\psi^E(\delta_v, 0) := [p_v]$ for $v \in G^0$ and $\psi^E(0,1) = [p_{v_0}]$. Then ψ^E induces an isomorphism of the cokernel of $((A_G^t - I) \oplus W_E^t) : \bigoplus_{G^0} \mathbb{Z} \to (\bigoplus_{G^0} \mathbb{Z}) \oplus \mathbb{Z}$ onto $K_0(C^*(E))$.

PROOF: We first suppose that (E, v_0) is simple. Then $(\bigoplus_{G^0} \mathbb{Z}) \oplus \mathbb{Z}$ is the group $\mathbb{Z}^{G^0} \oplus \mathbb{Z}^W$ considered in [**10**, Section 3], and it suffices to show that ψ^E is the homomorphism $\overline{\phi}$ considered there. To do this, we need to check that the map S of $K_0(C^*(E) \times_\gamma \mathbb{T})$ onto $K_0(C^*(E))$ in [**10**, (3.3)] satisfies $S([p_v\chi_1]) = [p_v]$. The map S is built up from the homomorphisms induced by the embedding of $C^*(E) \times_\gamma \mathbb{T}$ in the dual crossed product $(C^*(E) \times_\gamma \mathbb{T}) \times_{\hat{\gamma}} \mathbb{Z}$, the Takesaki-Takai duality isomorphism $(C^*(E) \times_\gamma \mathbb{T}) \times_{\hat{\gamma}} \mathbb{Z} \cong C^*(E) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$, and the map $a \mapsto a \otimes p$ of $C^*(E)$ into $C^*(E) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ determined by a rank-one projection p. The formulas at the start of the proof of [**9**, Theorem 6] show that, because p_v is fixed under γ, the duality isomorphism carries $p_v\chi_1 \in C^*(E) \times_\gamma \mathbb{T} \subset (C^*(E) \times_\gamma \mathbb{T}) \times_{\hat{\gamma}} \mathbb{Z}$ into $p_v \otimes M(\chi_1)$, where $M(\chi_1)$ is the projection onto the subspace spanned by the basis element e_1. Thus S has the required property, and the result for simple extensions now follows from [**10**, Theorem 3.2].

If (E, v_0) is an arbitrary 1-sink extension, we consider its simplification SE and the embedding ϕ^{SE} of $C^*(SE)$ in $C^*(E)$ provided by Proposition 1.5, which by [**8**, Proposition 1.2] induces an isomorphism ϕ_*^{SE} in K-theory. But now it is easy to check that $\phi_*^{SE} \circ \psi^{SE} = \psi^E$, and the result follows. ∎

We now begin the proof of Proposition 5.1. Since the image of ϕ is a full corner in $C^*(E)$, it induces an isomorphism ϕ_* of $K_0(C^*(F))$ onto $K_0(C^*(E))$ (by, for example, [**8**, Proposition 1.2]). The properties of the $C^*(G)$-embedding ϕ imply that $\phi_*([p_{w_0}]) = [p_{v_0}]$ and $(\pi_E)_* \circ \phi_* = (\pi_F)_*$. We need to know how ϕ_* interacts with the descriptions of K-theory provided by Lemma 5.2.

LEMMA 5.3. *The induced homomorphism $\phi_* : K_0(C^*(F)) \to K_0(C^*(E))$ satisfies $\phi_*(\psi^F(0,1)) = \psi^E(0,1)$, and for each $z \in \bigoplus_{G^0} \mathbb{Z}$, there exists $\ell \in \mathbb{Z}$ such that $\phi_*(\psi^F(z,0)) = \psi^E(z,\ell)$.*

PROOF: The first equation is a translation of the condition $\phi_*([p_{w_0}]) = [p_{v_0}]$. For the second, let $\psi^G : \bigoplus_{G^0} \mathbb{Z} \to K_0(C^*(G))$ be the homomorphism such that $\psi^G(\delta_v) = [p_v]$, which induces the usual isomorphism of $\operatorname{coker}(A_G^t - I)$ onto $K_0(C^*(G))$. If $\rho : (\bigoplus_{G^0} \mathbb{Z}) \oplus \mathbb{Z} \to \bigoplus_{G^0} \mathbb{Z}$ is given by $\rho(z, \ell) := z$, then we have $(\pi_E)_* \circ \psi^E = \psi^G \circ \rho$, and similarly for F. Thus

$$(5.1) \qquad (\pi_E)_* \circ \phi_* \circ \psi^F = (\pi_F)_* \circ \psi^F = \psi^G \circ \rho.$$

Now fix $z \in \bigoplus_{G^0} \mathbb{Z}$. Since ψ^E is surjective, there exists $(x, y) \in (\bigoplus_{G^0} \mathbb{Z}) \oplus \mathbb{Z}$ such that $\psi^E(x,y) = \phi_*(\psi^F(z,0))$. From (5.1) we have

$$\psi^G(z) = (\pi_E)_* \circ \phi_* \circ \psi^F(z,0) = (\pi_E)_* \circ \psi^E(x,y) = \psi^G(x),$$

and hence there exists $u \in \bigoplus_{G^0} \mathbb{Z}$ such that $x = z + (A_G^t - I)u$. Now because ψ^E is constant on the image of $(A_G^t - I) \oplus W_E^t$, we have

$$\phi_*\big(\psi^F(z,0)\big) = \psi^E(x,y) = \psi^E(z + (A_G^t - I)u, y) = \psi^E(z, y - W_E^t u),$$

and $\ell := y - W_E^t u$ will do. $\square$

PROOF OF PROPOSITION 5.1: By Lemma 5.3, for each $v \in G^0$ there exists $k_v \in \mathbb{Z}$ such that $\phi_*\big(\psi^F(\delta_v, 0)\big) = \psi^E(\delta_v, k_v)$. We define $k = (k_v) \in \prod_{G^0}\mathbb{Z}$. A calculation shows that for any $(y, \ell) \in \big(\bigoplus_{G^0} \mathbb{Z}\big) \oplus \mathbb{Z}$ we have

$$\begin{aligned} \phi_* \circ \psi^F(y,\ell) &= \sum_v y_v(\phi_* \circ \psi^F)(\delta_v, 0) + \ell(\phi_* \circ \psi^F)(0,1) \\ &= \Big(\sum_v \psi^E(y_v\delta_v, y_v k_v)\Big) + \ell\psi^E(0,1) \\ &= \psi^E(y, k^t y + \ell). \end{aligned} \tag{5.2}$$

Now let $z \in \bigoplus_{G^0} \mathbb{Z}$. On one hand, we have from (5.2) that

$$\phi_* \circ \psi^F\big(((A_G^t - I) \oplus W_F^t)(z)\big) = \psi^E\big((A_G^t - I)z, k^t(A_G^t - I)z + W_F^t z\big). \tag{5.3}$$

On the other hand, since $\psi^F \circ \big((A_G^t - I) \oplus W_F^t\big) = 0$, its composition with ϕ_* is also 0. Thus the class (5.3) must vanish in $K_0\big(C^*(E)\big)$, and there exists $x \in \bigoplus_{G^0} \mathbb{Z}$ such that

$$\big((A_G^t - I)z, k^t(A_G^t - I)z + W_F^t z\big) = \big((A_G^t - I)x, W_E^t x\big). \tag{5.4}$$

Comparing (5.3) and (5.4) shows that $x - z \in \ker(A_G^t - I)$ and

$$k^t(A_G^t - I)z + W_F^t z = W_E^t x = W_E^t z + W_E^t(x - z).$$

Since we are supposing $W_E \perp \ker(A_G^t - I)$, we deduce that $W_E^t(x - z) = 0$. We have now proved that

$$k^t(A_G^t - I)z = (W_E^t - W_F^t)z \quad \text{for all } z \in \textstyle\bigoplus_{G^0}\mathbb{Z},$$

which implies $(A_G - I)k = W_E - W_F$, as required. $\square$

COROLLARY 5.4. *Suppose that G is a row-finite graph with no sinks and with the property that $\ker(A_G^t - I) = \{0\}$. Let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. If there is a 1-sink extension F such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$ and $C^*(E_2)$, then there exists $k \in \prod_{G^0} \mathbb{Z}$ such that $W_{E_1} - W_{E_2} = (A_G - I)k$.*

COROLLARY 5.5. *Suppose that G is a finite graph with no sinks or sources whose vertex matrix A_G satisfies $\ker(A_G^t - I) = 0$. Let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G such that $\overline{v_1} = \overline{v_2}$. Then there is a 1-sink extension F such that $C^*(F)$ is $C^*(G)$-embeddable in both $C^*(E_1)$ and $C^*(E_2)$ if and only if there exists $k \in \bigoplus_{G^0} \mathbb{Z}$ such that $W_{E_1} - W_{E_2} = (A_G - I)k$.*

PROOF: The forward direction follows from the previous corollary. For the converse, we seek to apply Proposition 3.3. To see that n has support in the common closure $C := \overline{v_1} = \overline{v_2}$, recall that A_G decomposes as $A_G = \left(\begin{smallmatrix} B & 0 \\ C & D \end{smallmatrix}\right)$ with respect to $G^0 = (G^0 \setminus C) \cup C$. Thus 1 is an eigenvalue for the $(G^0 \setminus C) \times (G^0 \setminus C)$ corner B of A_G if and only if it is an eigenvalue for A_G, and hence if and only if it is an eigenvalue for A_G^t. So Lemma 3.5 applies, $\operatorname{supp} n$ lies in C, and the result follows from Proposition 3.3. $\square$

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