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Abstract

To model cohesionless granular flow using continuum theory, the usual approach is to assume the cohesionless Coulomb-Mohr yield condition. However, this yield condition assumes that the angle of internal friction is constant, when according to experimental evidence for most powders the angle of internal friction is not constant along the yield locus, but decreases for decreasing normal stress component σ from a maximum value of $\pi/2$. For this reason, we consider here the more general yield function which applies for shear-index granular materials, where the angle of internal friction varies with σ . In this case, failure due to frictional slip between particles occurs when the shear and normal components of stress τ and σ satisfy the so-called Warren Spring equation $(|\tau|/c)^n = 1 - (\sigma/t)$, where c , t and n are positive constants which are referred to as the cohesion, tensile strength and shear-index respectively, and experimental evidence indicates for many materials that the value of the shear-index n lies between 1 and 2. For many materials, the cohesion is close to zero and therefore the notion of a cohesionless shear-index granular material arises. For such materials, a continuum theory applying for shear-index cohesionless granular materials is physically plausible as a limiting ideal theory, and any analytical solutions might provide important benchmarks for numerical schemes. Here, we examine the cohesionless shear-index theory for the problem of gravity flow of granular materials through two-dimensional wedge-shaped hoppers, and we attempt to determine analytical solutions. Although some analytical solutions are found, these do not correspond to the actual hopper problem, but may serve as benchmarks for purely numerical schemes. The special analytical solutions obtained are illustrated graphically, assuming only a symmetrical stress distribution.

Keywords

cohesionless, index, limiting, shear, ideal, granular, materials, theory

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THE LIMITING IDEAL THEORY FOR SHEAR-INDEX COHESIONLESS GRANULAR MATERIALS

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Abstract

To model cohesionless granular flow using continuum theory, the usual approach is to assume the cohesionless Coulomb-Mohr yield condition. However, this yield condition assumes that the angle of internal friction is constant, when according to experimental evidence for most powders the angle of internal friction is not constant along the yield locus, but decreases for decreasing normal stress component σ from a maximum value of $\pi/2$. For this reason, we consider here the more general yield function which applies for shear-index granular materials, where the angle of internal friction varies with σ . In this case, failure due to frictional slip between particles occurs when the shear and normal components of stress τ and σ satisfy the so-called Warren Spring equation $(|\tau|/c)^n = 1 - (\sigma/t)$, where c , t and n are positive constants which are referred to as the cohesion, tensile strength and shear-index respectively, and experimental evidence indicates for many materials that the value of the shear-index n lies between 1 and 2. For many materials, the cohesion is close to zero and therefore the notion of a cohesionless shear-index granular material arises. For such materials, a continuum theory applying for shear-index cohesionless granular materials is physically plausible as a limiting ideal theory, and any analytical solutions might provide important benchmarks for numerical schemes. Here, we examine the cohesionless shear-index theory for the problem of gravity flow of granular materials through two-dimensional wedge-shaped hoppers, and we attempt to determine analytical solutions. Although some analytical solutions are found, these do not correspond to the actual hopper problem, but may serve as benchmarks for purely numerical schemes. The special analytical solutions obtained are illustrated graphically, assuming only a symmetrical stress distribution.

1. Introduction

Many industrial processes throughout the world make use of granular materials at various stages. These granular materials are frequently stored in silos or hoppers, where the material can be reclaimed at a later date. For dry powder-like materials that

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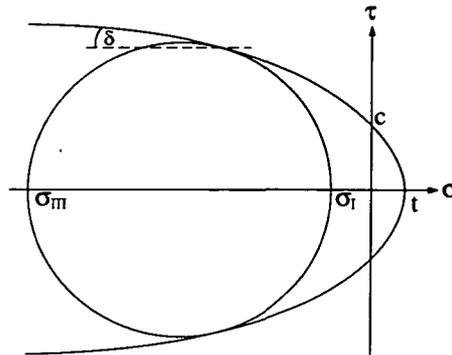


FIGURE 1. General yield locus.

flow freely from a hopper or silo, the cohesion of the material is often effectively zero. For such cohesionless granular materials, the usual continuum approach to modelling the flow is to assume that the material yields according to the Coulomb-Mohr theory, namely

$$|\tau| = c - \sigma \tan \delta, \quad (1.1)$$

where the cohesion c is set to zero. We note that for this yield condition, both the cohesion c and the angle of internal friction δ are assumed constant, and that σ and τ denote the normal and tangential components of compressive traction which are assumed to be positive in tension, namely the usual convention in continuum mechanics is adopted that positive forces are assumed to produce positive extensions. In this paper, we examine the so-called shear-index yield condition for the situation of vanishingly small cohesion.

Experimental evidence (see Williams *et al.* [12], Stainforth *et al.* [10], Eelkman Rooda [1], Eelkman Rooda and Haaker [2] and Farley and Valentin [3]) indicates that the angle of internal friction for most granular materials is not constant along the yield locus, but decreases for decreasing σ from a maximum value of $\pi/2$ at the vertex $(t, 0)$, as indicated in Figure 1. In general therefore, the angle of internal friction is a stress-dependent function $\delta(\sigma)$ which is defined incrementally from the equation

$$d\tau = -d\sigma \tan \delta(\sigma). \quad (1.2)$$

Clearly, the Coulomb-Mohr yield condition (1.1) satisfies (1.2) when the angle of internal friction δ is constant, but for a yield condition that satisfies (1.2) when δ varies with σ , we assume the shear-index yield condition, sometimes referred to as

TABLE 1. Typical values of n , t , c and ρ , when the cohesion is close to zero (Farley and Valentin [3], g denotes 981 dynes).

Granular material (Particle size)	Shear-index n	Tensile strength t (g/cm ²)	Cohesion c (g/cm ²)	Density ρ (gm/cm ³)
CaCO₃				
(5-7 μ)	1.35	0.0066	0.051	0.389
(16-20 μ)	1.29	0.00011	0.00443	0.406
Calcite				
(40-44 μ)	1.12	0.047	0.06	0.445
(+44 μ)	1.12	0.0012	0.0026	0.457
Zinc Dust				
(coarse)	1.27	0.283	0.65	0.498
Alumina				
(+37 μ)	1.19	0.179	0.290	0.250

the Warren Spring equation, namely

$$\left(\frac{|\tau|}{c}\right)^n = 1 - \frac{\sigma}{t}, \quad (1.3)$$

where t and n are positive constants which are referred to as the tensile strength and shear index respectively. We note that the Warren Spring equation (1.3) has been validated experimentally by a number of authors (see Williams and Birks [11] and Stainforth *et al.* [10]). Using the Jenike shear tests, Farley and Valentin [3] suggest that the cohesion c is usually of the order of twice the tensile strength t and that the shear index n for a particular powder is independent of the bulk density of the compact, and can therefore be used to classify powders according to their flow properties. In addition, Farley and Valentin [3] give simple expressions relating n to the ratio of volume to surface mean diameter and t to the bulk density. The known experimental values of shear index n , such as those cited in Farley and Valentin [3], all lie between 1 and 2, and Table 1 gives the typical values of n , t , c and ρ as determined by Farley and Valentin [3] when the cohesion c is close to zero. From Table 1 we see that there exist shear-index granular materials for which the cohesion is close to zero, and as such, a continuum theory applying for shear-index cohesionless granular materials is physically plausible as a limiting ideal theory.

In the following section we state the basic equations for a cohesive shear-index material. In Section 3, we examine the resulting equations applying to shear-index materials for which the cohesion is zero. The equilibrium equations for a cohesionless shear-index material are then determined in Section 4 for general n , and some simple solutions to these equations are presented in Section 5. In Section 6, for a cohesion-

less shear-index material, we attempt to determine the solution corresponding to the solution for gravity flow through a two-dimensional converging wedge-shaped hopper which is known to exist for the Coulomb-Mohr yield condition.

2. Cohesive shear-index materials

In this section, we state the basic equations for the cohesive shear-index yield condition. However, we first note from Hill and Wu [5] that a general yield function can be expressed parametrically, in terms of principal stress components, as

$$\begin{aligned}(\sigma_I - \sigma_{III}) \cos \delta &= 2f [(\sigma_I + \sigma_{III})/2 + (\sigma_I - \sigma_{III})(\sin \delta)/2], \\ \tan \delta &= -\frac{d}{d\sigma} f [(\sigma_I + \sigma_{III})/2 + (\sigma_I - \sigma_{III})(\sin \delta)/2],\end{aligned}\quad (2.1)$$

where the stress-dependent angle of internal friction $\delta = \delta(\sigma)$ is the parameter, and σ_I and σ_{III} denote the maximum and minimum principal stresses respectively. Thus, for example, if the angle of internal friction is constant and $\tau = f(\sigma)$ is the linear yield condition (1.1), then (2.1) gives the well-known Coulomb-Mohr yield condition

$$\sigma_I - \sigma_{III} = 2c \cos \delta - (\sigma_I + \sigma_{III}) \sin \delta. \quad (2.2)$$

In the case of the Warren Spring equation (1.3), we find $f(\sigma) = c(1 - \sigma/t)^{1/n}$, and so the Warren Spring yield condition (2.1) has the parametric form

$$\frac{\sigma_I}{t} = 1 + \frac{\beta c}{t} (\sec \delta - \tan \delta) - \beta^n, \quad \frac{\sigma_{III}}{t} = 1 - \frac{\beta c}{t} (\sec \delta + \tan \delta) - \beta^n, \quad (2.3)$$

where β is a function of δ which is defined by

$$\beta = \left(\frac{nt}{c} \tan \delta \right)^{1/(1-n)}. \quad (2.4)$$

We note that (2.3) and (2.4) are only valid provided $n \neq 1$. If $n = 1$ then the angle of internal friction is constant and the Coulomb-Mohr yield condition (2.2) arises. We also note for the special cases of $n = 1$ and $n = 2$, that we obtain the explicit yield conditions

$$\begin{aligned}(1 - \sigma_I/t)^{1/2} &= \left\{ (1 + c^2/t^2)^{1/2} - c/t \right\} (1 - \sigma_{III}/t)^{1/2}, & (n = 1) \\ (1 - \sigma_I/t)^{1/2} &= (1 - \sigma_{III}/t)^{1/2} - c/t, & (n = 2)\end{aligned}\quad (2.5)$$

where the special case of $n = 1$ corresponds to the Coulomb-Mohr yield condition (2.2), noting that $c/t = \tan \delta$ when $n = 1$. The above relations become more

transparent by expressing the parametric solution (2.3) in the form

$$1 - \frac{\sigma_I}{t} = \frac{1}{\beta^n} \left\{ \left(\left[\beta^{2n} + \left(\frac{\beta c}{nt} \right)^2 \right]^{1/2} - \frac{\beta c}{2t} \right)^2 - \left(\frac{\beta c}{t} \right)^2 \left(\frac{1}{n} - \frac{1}{2} \right)^2 \right\},$$

$$1 - \frac{\sigma_{III}}{t} = \frac{1}{\beta^n} \left\{ \left(\left[\beta^{2n} + \left(\frac{\beta c}{nt} \right)^2 \right]^{1/2} + \frac{\beta c}{2t} \right)^2 - \left(\frac{\beta c}{t} \right)^2 \left(\frac{1}{n} - \frac{1}{2} \right)^2 \right\},$$

from which the special roles of $n = 1$ and $n = 2$ are apparent and (2.5) can be easily deduced. It appears that $n = 1$ and $n = 2$ are the only values of n giving rise to simple analytical yield functions. However, other special values of n such as $n = 3/2, 4/3$ and $8/5$ permit further analytical investigation, but the final results are still complicated (see the Appendix of Hill and Wu [5]). For an explicit form of the Warren Spring yield condition, we see from Hill and Wu [5] that the parameter β in (2.3) may be eliminated to give

$$\left[\frac{c^2 n^2}{2t^2(n-1)} \right]^{n/(2-n)} \left\{ B - \left[B^2 - \frac{(n-1)}{n^2 t^2} A \right]^{1/2} \right\}^{n/(2-n)} + \frac{n}{2(n-1)} \left\{ B - \left[B^2 - \frac{(n-1)}{n^2 t^2} A \right]^{1/2} \right\} - B = 0,$$

where $A = (\sigma_{rr} - \sigma_{\theta\theta})^2 + \sigma_{r\theta}^2$ and $B = 1 - (\sigma_{rr} + \sigma_{\theta\theta})/(2t)$.

3. Cohesionless shear-index materials

In this section, we deduce the governing equations corresponding to free-flowing (cohesionless) shear-index granular materials. To do this, we examine the cohesive shear-index yield condition (1.3) in the form

$$|\tau|^n = c^n - c^n \sigma/t, \tag{3.1}$$

from which it is clear that for (3.1) to be meaningful when the cohesion is set to zero, we must assume

$$t = \gamma c^n, \tag{3.2}$$

for some finite positive constant γ . Namely, we assume the tensile strength t and cohesion c are such that

$$\lim_{c \rightarrow 0} \frac{t}{c^n} = \gamma. \tag{3.3}$$

In this event, the cohesionless Warren Spring yield condition becomes

$$|\tau|^n = -\sigma/\gamma, \quad (3.4)$$

and we note from (1.1) for the special case of $n = 1$, that $\gamma = \cot \delta$. We also note that the assumption of (3.2) is more than just an analytical necessity, as from Table 1 we see that there do exist shear-index granular materials whose cohesion is effectively zero, but which must still yield according to (3.1), which clearly can only happen provided (3.2) is satisfied.

Now, upon substituting (3.4) into the general parametric representation of the yield function (2.1), we find

$$\frac{\sigma_1}{\gamma} = \frac{\xi}{\gamma} (\sec \delta - \tan \delta) - \xi^n, \quad \frac{\sigma_{III}}{\gamma} = -\frac{\xi}{\gamma} (\sec \delta + \tan \delta) - \xi^n, \quad (3.5)$$

where the parameter ξ is defined by

$$\xi = (n\gamma \tan \delta)^{1/(1-n)}, \quad (3.6)$$

and again we note that $n \neq 1$. If $n = 1$ then the yield condition (3.4) becomes the usual cohesionless Coulomb-Mohr yield condition and ξ becomes constant since δ is constant. It should be noted that (3.5) and (3.6) can be derived directly from (2.3) using (3.2), where $\xi = c\beta$. Thus (3.5) is the parametric representation of the cohesionless shear-index yield condition.

To determine an explicit form of the cohesionless Warren Spring yield equation, we first need to introduce the positive quantities p and q given by

$$p = -(\sigma_1 + \sigma_{III})/2, \quad q = (\sigma_1 - \sigma_{III})/2, \quad (3.7)$$

so that from (3.5), we get

$$p = \frac{\xi^n}{n\gamma} (n\gamma^2 + \xi^{2-2n}), \quad q = \frac{\xi}{n\gamma} (n^2\gamma^2 + \xi^{2-2n})^{1/2}. \quad (3.8)$$

Next, we introduce $P = ((\sigma_1 - \sigma_{III})/2\gamma)^2$ and $Q = -(\sigma_1 + \sigma_{III})/2\gamma$, so that from (3.7) and (3.8) we find

$$P = \frac{\xi^2}{\gamma^2} + \frac{\xi^{4-2n}}{n^2\gamma^4}, \quad Q = \xi^n + \frac{\xi^{2-n}}{n\gamma^2}. \quad (3.9)$$

Following the derivation of the explicit cohesive shear-index yield condition, as given by Hill and Wu [5], we introduce $\chi = \xi^{2-n}$, such that

$$\frac{\xi^2}{\xi^n} = \chi = \frac{\gamma^2 P - \chi^2/n^2\gamma^2}{Q - \chi/n\gamma^2},$$

which leads to a quadratic equation for χ , namely

$$\frac{(n-1)}{n^2\gamma^2}\chi^2 - Q\chi + \gamma^2P = 0. \quad (3.10)$$

Solving (3.10) for χ gives

$$\chi = \frac{n^2\gamma^2}{2(n-1)} \left\{ Q \pm \left[Q^2 - \frac{4(n-1)}{n^2}P \right]^{1/2} \right\}, \quad (3.11)$$

where from (3.9)₂, we note that as $Q - \chi/n\gamma^2 = \xi^n > 0$, which means that $\chi < n\gamma^2Q$, then we must take the minus sign in (3.11), namely

$$\chi = \frac{n^2\gamma^2}{2(n-1)} \left\{ Q - \left[Q^2 - \frac{4(n-1)}{n^2}P \right]^{1/2} \right\}. \quad (3.12)$$

Thus the yield condition for a cohesionless shear-index granular material can be determined from either (3.9)₁ or (3.9)₂ together with (3.12). From (3.9)₂ we have

$$\chi^{n/(2-n)} + \frac{\chi}{n\gamma^2} - Q = 0,$$

and using (3.12) we find

$$\begin{aligned} & \left[\frac{n^2\gamma^2}{2(n-1)} \right]^{n/(2-n)} \left\{ Q - \left[Q^2 - \frac{4(n-1)}{n^2}P \right]^{1/2} \right\}^{n/(2-n)} \\ & + \frac{n}{2(n-1)} \left\{ Q - \left[Q^2 - \frac{4(n-1)}{n^2}P \right]^{1/2} \right\} - Q = 0, \end{aligned} \quad (3.13)$$

where P and Q are defined by (3.9). Thus (3.13) is the explicit cohesionless shear-index yield condition.

4. General equilibrium equations for gravity flow

In this section, we utilise the cohesionless shear-index yield condition to determine the resulting equilibrium equations for quasi-static gravity flow from a two-dimensional converging hopper. In terms of the usual cylindrical polar coordinates (r, θ, z) as defined by Figure 2, the stress components for quasi-static plane strain flow in hoppers satisfy the equilibrium equations

$$\begin{aligned} \frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= \rho g \cos\theta, \\ \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{2\sigma_{r\theta}}{r} &= -\rho g \sin\theta, \end{aligned} \quad (4.1)$$

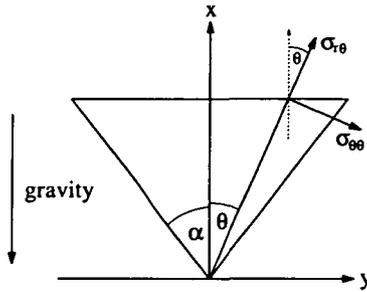


FIGURE 2. Coordinates for a two-dimensional wedge-shaped hopper.

where ρ is the bulk density, g is acceleration due to gravity, and σ_{rr} , $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ denote the in-plane physical stress components, which are assumed to be positive in tension. Namely, we adopt the usual convention in continuum mechanics that positive forces are assumed to produce positive extensions. We note that we have followed the notation of Spencer and Bradley [9], including adopting the unusual convention of the x axis being vertical. Thus, following Spencer and Bradley [9], these components can be expressed in the standard form

$$\sigma_{rr} = -p + q \cos 2\psi, \quad \sigma_{\theta\theta} = -p - q \cos 2\psi, \quad \sigma_{r\theta} = q \sin 2\psi, \quad (4.2)$$

where the positive quantities p and q are defined by (3.7). We note that an equivalent definition of p and q is

$$p = -\frac{1}{2} (\sigma_{rr} + \sigma_{\theta\theta}), \quad q = \frac{1}{2} \{(\sigma_{rr} - \sigma_{\theta\theta})^2 + 4\sigma_{r\theta}^2\}^{1/2},$$

while the stress angle ψ is given by $\tan 2\psi = 2\sigma_{r\theta}/(\sigma_{rr} - \sigma_{\theta\theta})$, where physically speaking, ψ is the angle between the maximum principal stress axis and the x direction, in the direction of increasing θ .

Now, we find from (3.8) and (4.2) that the stresses become

$$\begin{aligned} \sigma_{rr} &= -\frac{\xi^n}{n\gamma} (n\gamma^2 + \xi^{2-2n}) + \frac{\xi}{n\gamma} (n^2\gamma^2 + \xi^{2-2n})^{1/2} \cos 2\psi, \\ \sigma_{\theta\theta} &= -\frac{\xi^n}{n\gamma} (n\gamma^2 + \xi^{2-2n}) - \frac{\xi}{n\gamma} (n^2\gamma^2 + \xi^{2-2n})^{1/2} \cos 2\psi, \\ \sigma_{r\theta} &= \frac{\xi}{n\gamma} (n^2\gamma^2 + \xi^{2-2n})^{1/2} \sin 2\psi, \end{aligned} \quad (4.3)$$

where $\psi = \psi(r, \theta)$ and $\xi = \xi(r, \theta)$. Substitution of these expressions directly into

the equilibrium equations (4.1), and simplifying, gives

$$\begin{aligned} \frac{\partial \psi}{\partial r} - \frac{1}{2r} \frac{\partial}{\partial \theta} \{\ln F\} &= \frac{1}{2\xi^{2-n}} \left[\frac{\cos 2\psi}{r} \frac{\partial F}{\partial \theta} - \sin 2\psi \frac{\partial F}{\partial r} \right] - \frac{\rho g n \gamma \sin(\theta + 2\psi)}{2F}, \\ \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \frac{1}{r} + \frac{1}{2} \frac{\partial}{\partial r} \{\ln F\} &= \frac{1}{2\xi^{2-n}} \left[\frac{\sin 2\psi}{r} \frac{\partial F}{\partial \theta} + \cos 2\psi \frac{\partial F}{\partial r} \right] + \frac{\rho g n \gamma \cos(\theta + 2\psi)}{2F}, \end{aligned} \quad (4.4)$$

which are the general equilibrium equations for a cohesionless shear-index granular material, where F is defined by

$$F(r, \theta) = \xi [n^2 \gamma^2 + \xi^{2-2n}]^{1/2}, \quad (4.5)$$

and noting we have used the fact that

$$\frac{\partial}{\partial \xi} \left\{ \xi [n^2 \gamma^2 + \xi^{2-2n}]^{1/2} \right\} = \frac{[n^2 \gamma^2 + (2-n)\xi^{2-2n}]}{[n^2 \gamma^2 + \xi^{2-2n}]^{1/2}}.$$

Now, we need to determine the appropriate boundary conditions for gravity flow through a hopper. To do this, we assume that due to the geometry of the hopper, the stress distribution is symmetrical around the vertical axis. As a result, we observe from the equilibrium equations (4.1) that σ_{rr} and $\sigma_{\theta\theta}$ must be even functions of θ , while $\sigma_{r\theta}$ must be an odd function or skew-symmetric. Thus, to ensure continuity, $\sigma_{r\theta}$ must vanish at the origin giving rise to the boundary condition

$$\psi(0) = 0. \quad (4.6)$$

To determine the second stress boundary condition, following Spencer and Bradley [9] we assume a Coulomb friction condition at the wall of the hopper at $\theta = \alpha$, such that

$$\sigma_{r\theta} = -\sigma_{\theta\theta} \tan \mu, \quad \text{at } \theta = \alpha, \quad (4.7)$$

where μ is the angle of wall friction and α denotes the semi-vertex angle as indicated in Figure 2. Thus, from (4.2) and (4.3) we find that (4.7) becomes

$$\frac{\sin(2\psi - \mu)}{\sin \mu} = \frac{\xi^{n-1}(n\gamma^2 + \xi^{2-2n})}{(n^2\gamma^2 + \xi^{2-2n})^{1/2}}, \quad \text{at } \theta = \gamma.$$

5. Some simple solutions of (4.4)

In this section, we examine the general equilibrium equations (4.4) for a cohesionless shear-index granular material in order to determine some simple solutions. We first look for a simple solution of the form

$$\psi = -\theta + \psi_0, \quad (5.1)$$

for some constant ψ_0 , which is a well-known solution for the Coulomb-Mohr yield condition (Sokolovsky [8]) and corresponds to the Cartesian stresses being linear in both x and y . From (4.4) and (5.1), we find

$$\left[\frac{F \cos[2(\psi_0 - \theta)]}{\xi^{2-n}} + 1 \right] \frac{1}{r} \frac{\partial F}{\partial \theta} - \frac{F \sin[2(\psi_0 - \theta)]}{\xi^{2-n}} \frac{\partial F}{\partial r} = k \sin(2\psi_0 - \theta), \quad (5.2)$$

$$\frac{F \sin[2(\psi_0 - \theta)]}{\xi^{2-n}} \frac{1}{r} \frac{\partial F}{\partial \theta} + \left[\frac{F \cos[2(\psi_0 - \theta)]}{\xi^{2-n}} - 1 \right] \frac{\partial F}{\partial r} = -k \cos(2\psi_0 - \theta),$$

where $k = \rho g n \gamma$. On solving (5.2) for $\partial F/\partial r$ and $\partial F/\partial \theta$, we obtain the expressions

$$\frac{\partial F}{\partial r} = -k \frac{[G \cos \theta + \cos(2\psi_0 - \theta)]}{[G^2 - 1]}, \quad (5.3)$$

$$\frac{\partial F}{\partial \theta} = kr \frac{[G \sin \theta - \sin(2\psi_0 - \theta)]}{[G^2 - 1]},$$

where G is defined by

$$G = F/\xi^{2-n}. \quad (5.4)$$

Now, upon checking the consistency of (5.3), we obtain

$$(G^2 \sin \theta + 2G \sin(\theta - 2\psi_0) + \sin \theta) r \frac{\partial G}{\partial r} + (G^2 \cos \theta + 2G \cos(\theta - 2\psi_0) + \cos \theta) \frac{\partial G}{\partial \theta} = 0, \quad (5.5)$$

which is a first-order partial differential equation and is readily solved by the method of characteristics to yield

$$(G^2 + 2G \cos 2\psi_0 + 1) r \cos \theta + 2G \sin 2\psi_0 r \sin \theta = \Phi(G), \quad (5.6)$$

where Φ denotes an arbitrary function. Although (5.6) constitutes the general solution of (5.5), its usefulness is limited because $G(r, \theta)$ is not given explicitly. Alternatively, suppose that

$$G = \Psi(r \sin(\theta - \theta_0)), \quad (5.7)$$

for some function Ψ and constant θ_0 , then from (5.5) we have immediately

$$(G^2 + 1) \cos \theta_0 + 2G \cos(2\psi_0 - \theta_0) = 0,$$

so that for non-constant G , we require

$$\theta_0 = (2a + 1)\pi/2, \quad \psi_0 = (a + b + 1)\pi/2, \quad (5.8)$$

for any integers a and b . If instead G is a constant, then we find from (4.5) and (5.4) that both F and ξ must also be a constant. However, from (3.6) we find that ξ is only constant for the special case of $n = 1$, namely for the Coulomb-Mohr yield condition, and so we assume that G is a non-constant. Thus, for a cohesionless shear-index material ($1 < n \leq 2$), non-trivial solutions of the form (5.1) and (5.7) only exist provided θ_0 and ψ_0 are given by (5.8).

Now, for θ_0 and ψ_0 defined by (5.8), we see that $\sin 2\psi_0 = 0$ and so both the general solution (5.6) and (5.7) yield

$$G = G^*(r \cos \theta), \quad (5.9)$$

for some function G^* . From (4.5), (5.4) and (5.9) we also find that

$$F = f(r \cos \theta), \quad \xi = E(r \cos \theta), \quad (5.10)$$

for some functions f and E . Thus (5.2) becomes

$$\begin{aligned} f' [G \sin(2\psi_0 - \theta) + \sin \theta] &= -k \sin(2\psi_0 - \theta), \\ f' [G \cos(2\psi_0 - \theta) - \cos \theta] &= -k \cos(2\psi_0 - \theta), \end{aligned} \quad (5.11)$$

where the prime denotes differentiation with respect to $r \cos \theta$. For θ_0 and ψ_0 given by (5.8), we see that (5.11) becomes

$$f' (1 - \varepsilon_1 f / E^{2-n}) = \varepsilon_2 k, \quad (5.12)$$

where ε_1 and ε_2 are parameters whose values depend on ψ_0 . In particular, $\varepsilon_1 = \varepsilon_2 = 1$ for $\psi_0 = 0$, and $\varepsilon_1 = -1$ and $\varepsilon_2 = \mp 1$ for $\psi_0 = \pm\pi/2$. In order to solve (5.12), we find from (4.5) and (5.10) that $f = E[n^2\gamma^2 + E^{2-2n}]^{1/2}$, and therefore (5.12) can be written in the form

$$\left\{ E [n^2\gamma^2 + E^{2-2n}]^{1/2} \right\}' - \varepsilon_1 n^2 \gamma^2 E^{n-1} E' - (2-n)\varepsilon_1 E^{1-n} E' = \varepsilon_2 k. \quad (5.13)$$

Solving (5.13) yields

$$\varepsilon_1 E^n [n\gamma^2 + E^{2-2n}] - E [n^2\gamma^2 + E^{2-2n}]^{1/2} = -\varepsilon_2 k r \cos \theta + C, \quad (5.14)$$

which is a transcendental equation for $E(r \cos \theta)$, where C is a constant of integration. Thus we have determined an exact solution for a cohesionless shear-index granular material which satisfies the equilibrium equations (4.4), where $\psi(\theta)$ is of the form of (5.1) and $\xi(r, \theta) = E(r \cos \theta)$ satisfies (5.14). Therefore, from (5.1) and (5.14) we can write the stresses (4.3) as

$$\begin{aligned}\sigma_{rr} &= \varepsilon_1 \varepsilon_2 \rho g r \cos \theta + C^* - \frac{2\varepsilon_1 E}{n\gamma} [n^2 \gamma^2 + E^{2-2n}]^{1/2} \sin^2 \theta, \\ \sigma_{\theta\theta} &= \varepsilon_1 \varepsilon_2 \rho g r \cos \theta + C^* - \frac{2\varepsilon_1 E}{n\gamma} [n^2 \gamma^2 + E^{2-2n}]^{1/2} \cos^2 \theta, \\ \sigma_{r\theta} &= -\frac{2\varepsilon_1 E}{n\gamma} [n^2 \gamma^2 + E^{2-2n}]^{1/2} \sin \theta \cos \theta,\end{aligned}\quad (5.15)$$

where $C^* = C/n\gamma$ and $E = E(r \cos \theta)$ satisfies (5.14).

To apply the simple solution (5.15) for gravity flow through a two-dimensional wedge-shaped hopper, we need to apply the boundary conditions (4.6) and (4.7). We know that if $\varepsilon_1 = \varepsilon_2 = 1$, then $\psi_0 = 0$ and (4.6) is satisfied. However, we find that we are unable to satisfy (4.7) unless $n = 1$. This means that while the simple solution (5.15) satisfies the governing equations for gravity flow through a wedge-shaped hopper, we are unable to satisfy the appropriate boundary conditions. Despite this, for the purpose of completeness, in Figure 3 we demonstrate the variation of the stresses (5.15) for three values of the shear-index, where (4.6) is satisfied, for the constant values $C = 25$, $\gamma = 1$, $\rho = 1.018$ and $\alpha = 287\pi/900$. We observe from Figure 3 that the curves given tend to straight lines as the shear-index n approaches unity, a result which might be expected.

In this section, we have looked for a simple solution for ψ which is known to exist for the Coulomb-Mohr yield condition, namely where $\psi = -\theta + \psi_0$ and ψ_0 is a constant. This solution for ψ leads to a corresponding solution for F of the form $F = f(r \cos \theta)$, but which cannot satisfy the appropriate boundary conditions for gravity flow through a wedge-shaped hopper. For this reason, we now look for a more general solution of the form

$$\psi = -\theta + h(y), \quad F = f(y), \quad (5.16)$$

where $y = r \sin(\theta - \theta_0)$ for some constant θ_0 . This means that the equilibrium equations (4.4), after simplifying, become

$$\begin{aligned}\frac{dh}{dy} - \frac{k \cos(2h - \theta_0)}{2f} &= \frac{\sin(2h - 2\theta_0)}{2\xi^{2-n}} \frac{df}{dy}, \\ \frac{1}{f} \frac{df}{dy} - \frac{k \sin(2h - \theta_0)}{f} &= -\frac{\cos(2h - 2\theta_0)}{\xi^{2-n}} \frac{df}{dy},\end{aligned}\quad (5.17)$$

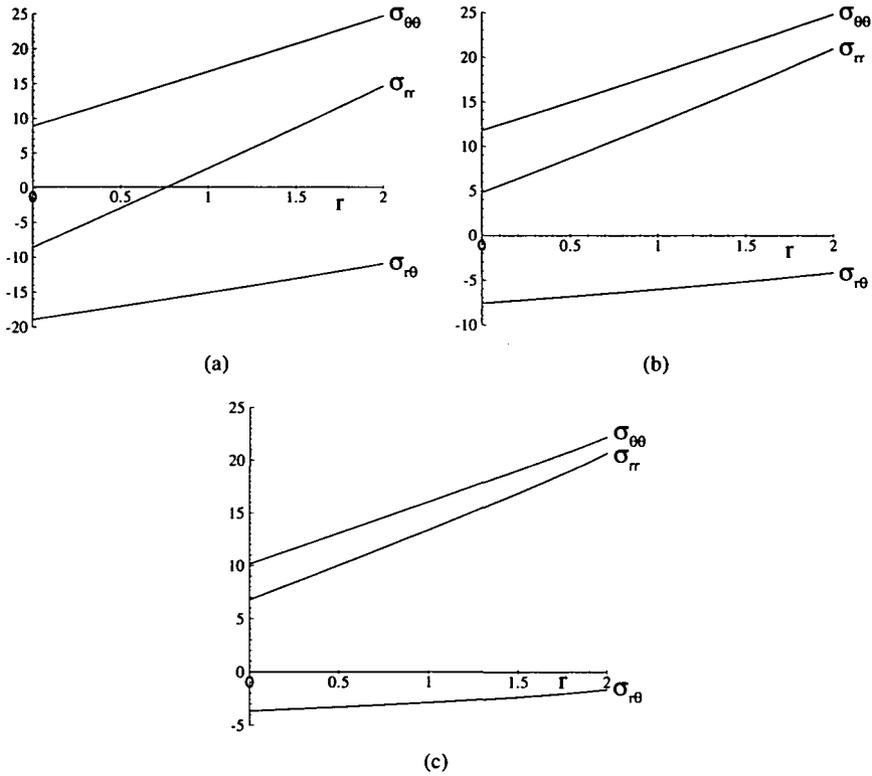


FIGURE 3. Variation of the stresses with r for three values of the shear-index, where ψ is defined by (5.1) with $\psi_0 = 0$, $C = 25$, $\gamma = 1$, $\alpha = 287\pi/900$ and E satisfies (5.14). ((a) $n = 1.19$, (b) $n = 1.5$ and (c) $n = 2$.)

where $k = \rho g n \gamma$. On dividing (5.17)₂ by (5.17)₁, we find

$$\sin(2h - 2\theta_0) \frac{df}{dy} + 2f \cos(2h - 2\theta_0) \frac{dh}{dy} = k \cos \theta_0. \tag{5.18}$$

Clearly, (5.18) can be solved to yield

$$\sin(2h - 2\theta_0) = \frac{ky \cos \theta_0 + c_1}{f}, \tag{5.19}$$

where c_1 is a constant of integration. Thus, upon substituting (5.19) into either (5.17)₁ or (5.17)₂, we obtain

$$\frac{ff' - k \cos \theta_0 (ky \cos \theta_0 + c_1)}{[f^2 - (ky \cos \theta_0 + c_1)^2]^{1/2}} + \frac{ff'}{\xi^{2-n}} = k \sin \theta_0, \tag{5.20}$$

where the prime denotes differentiation with respect to y . In order to solve (5.20) we note from (4.5) and (5.16) that $f = \xi[n^2\gamma^2 + \xi^{2-2n}]^{1/2}$, and therefore (5.20) becomes

$$\left\{ [\xi^2(n^2\gamma^2 + \xi^{2-2n}) - (ky \cos \theta_0 + c_1)^2]^{1/2} \right\}' + n^2\gamma^2\xi^{n-1}\xi' + (2-n)\xi^{1-n}\xi' = k \sin \theta_0,$$

which upon integrating yields

$$\left[\xi^2(n^2\gamma^2 + \xi^{2-2n}) - (ky \cos \theta_0 + c_1)^2 \right]^{1/2} + \xi^n [n\gamma^2 + \xi^{2-2n}] = ky \sin \theta_0 + c_2, \quad (5.21)$$

where c_2 is a second constant of integration. Thus we have determined a solution of the equilibrium equations (4.4) for a cohesionless shear-index granular material, where ψ and F are of the form of (5.16). Note that when $h(y) = \text{constant} = \psi_0$, and in particular is given by (5.8)₂, then we find from (5.19) that $c_1 = 0$ and θ_0 must satisfy (5.8)₁, and hence (5.21) becomes (5.14) where $\xi = E(r \cos \theta)$.

To apply the solution (5.21) for gravity flow through a two-dimensional wedge-shaped hopper, we need to apply the boundary conditions (4.6) and (4.7). Unfortunately, we are unable to determine the constants c_1 and c_2 such that (4.6) and (4.7) are satisfied. Again, for the purpose of completeness, in Figure 4 we demonstrate the variation of the stresses (4.3) where ξ satisfies (5.21), for three values of the shear-index and the constant values $c_1 = 1$, $c_2 = 140$, $\gamma = 1$, $\rho = 1.018$ and $\alpha = 287\pi/900$. We again observe that the curves tend towards linear dependence as n approaches unity.

6. Possible forms for hopper flow solutions of (4.4)

In this section, for cohesionless shear-index materials we attempt to determine the corresponding solution known to exist for the Coulomb-Mohr yield condition and which applies for granular flow through a wedge-shaped hopper under the action of gravity. The Coulomb-Mohr theory assumes a wedge-field solution for the stresses so that all stress components are linear in r , thus

$$\sigma_{rr} = rf_1(\theta), \quad \sigma_{r\theta} = rf_2(\theta), \quad \sigma_{\theta\theta} = rf_3(\theta), \quad (6.1)$$

for certain functions f_1 , f_2 and f_3 . Accordingly, we look for a wedge-field solution for a cohesionless shear-index granular material with $1 < n \leq 2$. From (4.3), on assuming $\psi = \psi(\theta)$, we require that

$$\xi^n (n\gamma^2 + \xi^{2-2n}) = rF^*(\theta), \quad \xi (n^2\gamma^2 + \xi^{2-2n})^{1/2} = rG^*(\theta), \quad (6.2)$$

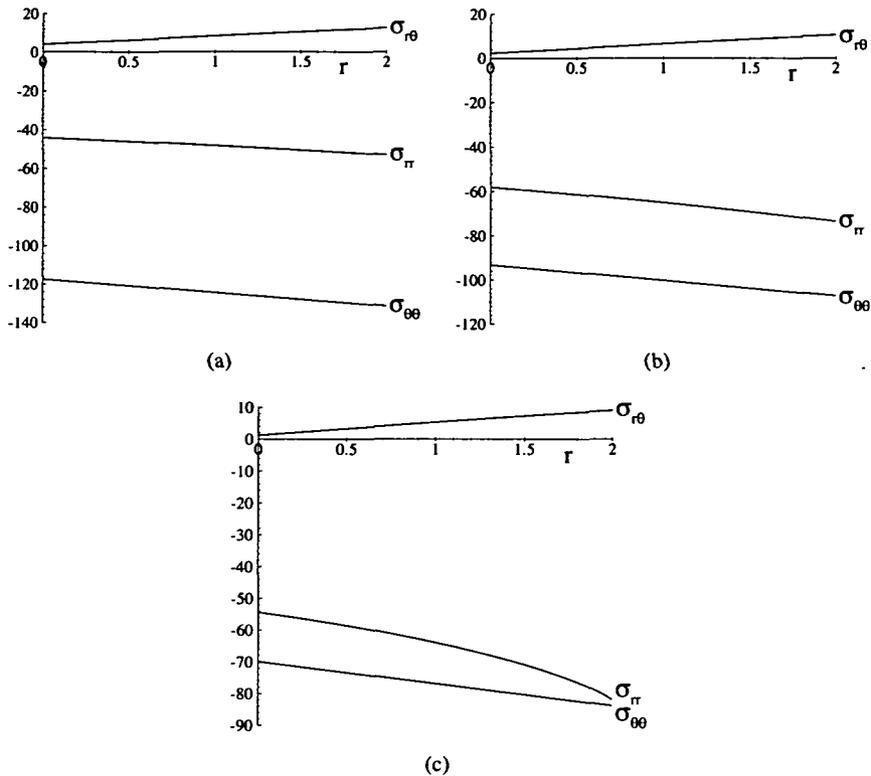


FIGURE 4. Variation of the stresses with r for three values of the shear-index, where ψ is defined by (5.16) with $\psi_0 = 0, \theta_0 = \pi/3, c_1 = 1, c_2 = 140, \gamma = 1, \alpha = 287\pi/900$ and ξ satisfies (5.21). ((a) $n = 1.19$, (b) $n = 1.5$ and (c) $n = 2$.)

for some $F^*(\theta)$ and $G^*(\theta)$ to be determined. Upon rewriting (6.2) in the form

$$n\gamma^2 + \xi^{2-2n} = \frac{rF^*(\theta)}{\xi^n}, \quad n^2\gamma^2 + \xi^{2-2n} = \frac{r^2G^{*2}(\theta)}{\xi^2}, \tag{6.3}$$

we find that subtracting (6.3)₁ from (6.3)₂ gives

$$n(n-1)\gamma^2 = \frac{r^2G^{*2}(\theta)}{\xi^2} - \frac{rF^*(\theta)}{\xi^n}, \tag{6.4}$$

which in principle is a transcendental equation for ξ in terms of F^* and G^* . Alternatively, upon multiplying (6.3)₁ by n and subtracting from (6.3)₂, we determine the following quadratic equation for ξ^{2-n} , namely

$$(n-1)\xi^{4-2n} - nrF^*(\theta)\xi^{2-n} + r^2G^{*2}(\theta) = 0,$$

which upon solving yields

$$\xi^{2-n} = \frac{r}{2(n-1)} \left\{ nF^*(\theta) \pm [n^2F^{*2}(\theta) - 4(n-1)G^{*2}(\theta)]^{1/2} \right\}, \quad (6.5)$$

noting that we need to assume the minus sign in (6.5) to ensure that the special case of $n = 1$ is well-defined. Substituting (6.5) into (6.4) yields a relationship between $F^*(\theta)$ and $G^*(\theta)$ where the r dependence must cancel out in order that the relationship be valid, since F^* and G^* are functions of θ only. However, from these equations this relationship becomes

$$\begin{aligned} n(n-1)\gamma^2 \left[\frac{r}{2(n-1)} \right]^{2/(2-n)} \left[nF^*(\theta) - \{n^2F^{*2}(\theta) - 4(n-1)G^{*2}(\theta)\}^{1/2} \right]^{2/(2-n)} \\ + \frac{r^2F^*(\theta)}{2(n-1)} \left[nF^*(\theta) - \{n^2F^{*2}(\theta) - 4(n-1)G^{*2}(\theta)\}^{1/2} \right] - r^2G^{*2}(\theta) = 0, \end{aligned}$$

from which it is clear to see that the assumed r dependence is consistent only for the special case of $n = 1$, namely only for the Coulomb-Mohr yield condition. Therefore for $1 < n \leq 2$, it is not possible to obtain a wedge-field solution of the form (6.1) to the equilibrium equations. We note that a similar situation also applies even if we neglect gravity. In this case we have equations of the form

$$\xi^n (n\gamma^2 + \xi^{2-2n}) = r^a F^*(\theta), \quad \xi (n^2\gamma^2 + \xi^{2-2n})^{1/2} = r^b G^*(\theta), \quad (6.6)$$

for certain constants a and b and the same conclusion may be deduced.

Thus we need to assume that the dependence on r is more complicated than either (6.2) or (6.6) allows. This means that we need to look for a solution of the form

$$\psi = -\theta + h(\theta), \quad F = F(r, \theta), \quad (6.7)$$

for some function $h(\theta)$. In this event, after solving for $\partial F/\partial r$ and $\partial F/\partial \theta$, the equilibrium equations (4.4) give

$$\begin{aligned} \frac{\partial F}{\partial r} &= \frac{\left[\frac{2F}{r} [G \cos[2(h-\theta)] + 1] \frac{dh}{d\theta} - k [G \cos \theta + \cos(2h-\theta)] \right]}{[G^2 - 1]}, \\ \frac{\partial F}{\partial \theta} &= r \frac{\left[\frac{2F}{r} G \sin[2(h-\theta)] \frac{dh}{d\theta} + k [G \sin \theta - \sin(2h-\theta)] \right]}{[G^2 - 1]}, \end{aligned} \quad (6.8)$$

where $k = \rho g n \gamma$, and G is defined by (5.4). We note if $h(\theta) = \text{constant} = \psi_0$, then (6.8) simplifies to become (5.3). We now assume that gravity may be neglected. This assumption is made to facilitate the analytical results, noting that including gravity further increases the complexity of the results, and if an analytical solution cannot be

obtained when gravity is neglected, then obtaining an analytical solution with gravity included is not likely. Thus (6.8) simplifies to become

$$\begin{aligned} \frac{\partial F}{\partial r} &= \frac{H}{r} [G \cos[2(h - \theta)] + 1] \frac{dh}{d\theta}, \\ \frac{\partial F}{\partial \theta} &= HG \sin[2(h - \theta)] \frac{dh}{d\theta}, \end{aligned} \tag{6.9}$$

where $H = 2F/[G^2 - 1]$. Equations (6.9) can be shown to be consistent provided that

$$\left[\cos[2(h - \theta)] + \frac{1}{G} \right] \frac{dh}{d\theta} = R(r), \tag{6.10}$$

for some function $R(r)$, and hence, from (6.10) we are able to write

$$\sin[2(h - \theta)] = \frac{[G^2 h^2 - (GR - h')^2]^{1/2}}{Gh'}, \tag{6.11}$$

where the prime denotes differentiation with respect to θ . Thus, from (6.10) and (6.11), we find that (6.9) becomes

$$\frac{\partial F}{\partial r} = \frac{HGR}{r}, \quad \frac{\partial F}{\partial \theta} = H [G^2 h^2 - (GR - h')^2]^{1/2}. \tag{6.12}$$

Since we have rewritten (6.9) in the form of (6.12), we again need to confirm the consistency of (6.12), from which we find

$$-\frac{H}{2} \frac{\partial}{\partial r} \left\{ \left[R - \frac{h'}{G} \right]^2 \right\} \left[h^2 - \left(R - \frac{h'}{G} \right)^2 \right]^{-1/2} = 0,$$

and therefore we have

$$R(r) = \frac{h'(\theta)}{G(r, \theta)} + T(\theta), \tag{6.13}$$

for some arbitrary function $T(\theta)$. However, as R is only a function of r , then we see that we need to take $T(\theta) = 0$ and G becomes a separable function. Thus (6.12) becomes

$$\frac{\partial F}{\partial r} = \frac{HGR}{r}, \quad \frac{\partial F}{\partial \theta} = HGh', \tag{6.14}$$

which may be readily verified to be consistent. However, from (6.13) with $T(\theta) = 0$, we can rewrite (6.14) as

$$\frac{\partial F}{\partial r} = \frac{Hh'}{r}, \quad \frac{\partial F}{\partial \theta} = HGh', \tag{6.15}$$

and the consistency of (6.15) now yields the separable equation

$$\frac{h''(\theta)}{h'(\theta)} = -\frac{r}{R^2(r)} \frac{dR}{dr} = \lambda, \tag{6.16}$$

where λ is the constant of separation. Upon solving (6.16), we find

$$h(\theta) = -\frac{1}{\lambda} \ln\left(\frac{\lambda}{\mu_1}\theta + \frac{\mu_0}{\mu_1}\right), \quad R(r) = \frac{1}{\lambda \ln(r/r_0)}, \tag{6.17}$$

where μ_0, μ_1 and r_0 are constants of integration and we must assume that $\lambda \neq 0$. We note that if $\lambda = 0$ then (6.16) yields

$$h(\theta) = \mu_0\theta + \mu_1, \quad R(r) = r_0. \tag{6.18}$$

Thus, from (6.13) and (6.17), we find for $\lambda \neq 0$ that

$$G(r, \theta) = -\frac{\ln(r/r_0)}{\theta + \nu}, \tag{6.19}$$

where $\nu = \mu_0/\lambda$ and from (4.5) and (5.4), we find that (6.19) gives

$$\xi(r, \theta) = \frac{\kappa}{[\eta^2 - 1]^{1/(2-2n)}}, \tag{6.20}$$

where

$$\eta(r, \theta) = \frac{\ln(r/r_0)}{\theta + \nu},$$

and κ is a constant defined by $\kappa = (n\gamma)^{1/(1-n)}$. We further note that we obtain $G = -\eta$. From (4.5) and (6.20), we have $F = f(\eta)$ and therefore (6.14) becomes

$$\frac{df}{d\eta} = -\frac{2f}{\lambda[\eta^2 - 1]}, \tag{6.21}$$

which can be solved to yield

$$f(\eta) = \eta_0 \left(\frac{\eta + 1}{\eta - 1}\right)^{1/\lambda}, \tag{6.22}$$

where η_0 is a constant of integration. Now, from (5.4), (6.19) and (6.20) we may determine an alternative expression for $f(\eta)$, namely

$$f(\eta) = -\kappa^{2-n} \frac{\eta}{(\eta^2 - 1)^{(2-n)/(2-2n)}}. \tag{6.23}$$

From (6.22) and (6.23), the two expressions for $f(\eta)$ are clearly not consistent and by substituting (6.23) into (6.21), we find the following quadratic equation for η , namely

$$\lambda\eta^2 - 2(1 - n)\eta + \lambda(1 - n) = 0,$$

from which it is clear to see that we must have both $n = 1$ and $\lambda = 0$ for (6.22) to be consistent with (6.23). This means that a solution of the form (6.7) only exists when both $n = 1$ and $\lambda = 0$, which corresponds to the Coulomb-Mohr yield condition. This means that we are unable to determine a solution of the form $\psi = \psi(\theta)$ for a shear-index granular material with $1 < n \leq 2$ for flow through a wedge-shaped hopper. For the case when $\lambda = 0$, we note that $h(\theta)$ and $R(r)$ are given by (6.18) and for $\lambda \neq 0$ we still require $n = 1$.

7. Conclusions

In this paper, we have presented the limiting ideal theory for shear-index cohesionless granular materials. We find that the cohesive shear-index yield condition can only apply for cohesionless granular materials provided that in the limit as c tends to zero, $t = \gamma c^n$, where t , c and n are positive constants referred to as the tensile strength, cohesion and shear index respectively, and γ is a positive constant such that (3.3) is satisfied. While the notion of a shear-index yield condition is generally used for cohesive materials, it is clear from Table 1 that there exist shear-index granular materials for which the cohesion is close to zero, and therefore there is a need to formulate the corresponding theory for free flowing or cohesionless materials as a plausible limiting ideal theory. Here, we have attempted to apply the cohesionless shear-index yield condition to the problem of gravity flow in a two-dimensional wedge-shaped hopper. Although we have determined some simple solutions of the appropriate cohesionless equilibrium equations (4.4), we are unable to determine values of the constants of integration which satisfy the boundary conditions (4.6) and (4.7), except for the special case when $n = 1$, namely the Coulomb-Mohr yield condition. Despite this, these simple solutions might be useful as numerical benchmarks for purely numerical schemes.

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