Exel's crossed product and relative Cuntz-Pimsner algebras

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crossed, exel, pimsner, cuntz, product, relative, algebras

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Exel’s crossed product and relative Cuntz–Pimsner algebras

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Abstract

We consider Exel’s new construction of a crossed product of a $C^*$-algebra $A$ by an endomorphism $\alpha$. We prove that this crossed product is universal for an appropriate family of covariant representations, and we show that it can be realised as a relative Cuntz–Pimsner algebra. We describe a necessary and sufficient condition for the canonical map from $A$ into the crossed product to be injective, and present several examples to demonstrate the scope of this result. We also prove a gauge-invariant uniqueness theorem for the crossed product.

1. Introduction

If $\alpha$ is an endomorphism of a $C^*$-algebra $A$, we can form a new $C^*$-algebra called the crossed product of $A$ by $\alpha$. This was first done by Cuntz [2], and there are now several general theories [13, 14, 17], which have been applied in a number of settings [1, 9, 10].

In [3], Exel proposed a new definition for the crossed product of a unital $C^*$-algebra $A$ by an endomorphism $\alpha$. Exel’s crossed product depends not only on $A$ and $\alpha$, but also on the choice of a transfer operator, which is a positive continuous linear map $L: A \to A$ such that $L(\alpha(a)b) = aL(b)$ for $a, b \in A$. This new theory generalises previous constructions where the endomorphism is injective and has hereditary range [13], and has applications in the study of classical irreversible dynamical systems [5].

In this paper, we re-examine Exel’s crossed product, denoted $A \rtimes_{\alpha, L} \mathbb{N}$, and identify a family of representations for which $A \rtimes_{\alpha, L} \mathbb{N}$ is universal. We then show that $A \rtimes_{\alpha, L} \mathbb{N}$ can be realised as a relative Cuntz–Pimsner algebra as in [6, 11], and use known results for relative Cuntz–Pimsner algebras to study $A \rtimes_{\alpha, L} \mathbb{N}$. In particular, we identify conditions which ensure that the canonical map $A \to A \rtimes_{\alpha, L} \mathbb{N}$ is injective, thus answering a question raised by Exel in [3], and partially answered by him in [4].

We begin with a brief discussion of relative Cuntz–Pimsner algebras, and we state a lemma which we will use when considering the map $A \to A \rtimes_{\alpha, L} \mathbb{N}$. In Section 3 we

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discuss representations of Exel’s crossed product. The main result in this section is the realization of \( A\rtimes_{\alpha,L} \mathbb{N} \) as a relative Cuntz–Pimsner algebra.

In Section 4 we describe a necessary and sufficient condition on the transfer operator \( L \) for \( A \to A\rtimes_{\alpha,L} \mathbb{N} \) to be injective. We also show that this condition simplifies when \( A \) is a commutative \( C^* \)-algebra, and give examples to illustrate that our results do significantly improve those of Exel. In Section 5 we use our realisation of \( A\rtimes_{\alpha,L} \mathbb{N} \) as a relative Cuntz–Pimsner algebra and results of Katsura [8] and Muhly–Tomforde [12] to prove a gauge-invariant uniqueness theorem for \( A\rtimes_{\alpha,L} \mathbb{N} \), which generalises the one of Exel and Vershik in [5].

2. Relative Cuntz–Pimsner algebras

Suppose that \( A \) is a \( C^* \)-algebra and \( X \) is a Hilbert bimodule over \( A \), where the left action \( a \cdot x \) is given by a homomorphism \( \phi: A \to \mathcal{L}(X) \), so that \( a \cdot x = \phi(a)x \). A Toeplitz representation \( (\psi, \pi) \) of \( X \) in a \( C^* \)-algebra \( B \) is a pair consisting of a linear map \( \psi: X \to B \) and a homomorphism \( \pi: A \to B \) such that

\[
\psi(x \cdot a) = \psi(x)\pi(a), \quad \psi(x)^*\psi(y) = \pi(\langle x, y \rangle_A) \quad \text{and} \quad \psi(\phi(a)x) = \pi(a)\psi(x)
\]

for \( x, y \in X \) and \( a \in A \). Given such a representation, [7, proposition 1.6] says there is a homomorphism \( (\psi, \pi) : \mathcal{K}(X) \to B \) which satisfies

\[
(\psi, \pi)(\Theta_{x,y}) = \psi(x)\psi(y)^* \quad \text{for} \quad x, y \in X
\]

and

\[
(\psi, \pi)(T)\psi(x) = \psi(Tx) \quad \text{for} \quad T \in \mathcal{K}(X) \quad \text{and} \quad x \in X. \tag{2.1}
\]

Remark 2.1. If \( (\psi, \pi) \) is a Toeplitz representation of \( X \) and \( \pi \) is injective, then both \( \psi \) and \( (\psi, \pi) \) are injective. To see this, suppose \( \psi(x) = 0 \). Then

\[
0 = \|\psi(x)\|^2 = \|\psi(x)^*\psi(x)\| = \|\pi(\langle x, x \rangle_A)\| = \|\langle x, x \rangle_A\| = \|x\|^2
\]

and hence \( \psi \) is injective.

Now suppose \( T \in \mathcal{K}(X) \) and \( (\psi, \pi)(T) = 0 \). Then (2.1) gives

\[
0 = (\psi, \pi)(T)\psi(x) = \psi(Tx) \quad \text{for all} \quad x \in X,
\]

so \( Tx = 0 \) for all \( x \in X \), and \( T = 0 \). Hence \( (\psi, \pi) \) is injective.

If \( \rho: B \to C \) is a homomorphism of \( C^* \)-algebras, then \( (\rho \circ \psi, \rho \circ \pi) \) is a Toeplitz representation of \( X \), and we have

\[
(\rho \circ \psi, \rho \circ \pi)(\Theta_{x,y}) = \rho \circ \psi(x)\rho \circ \psi(y)^* = \rho \circ (\psi, \pi)(\Theta_{x,y}) \quad \text{for all} \quad x, y \in X.
\]

It follows from linearity and continuity that we have

\[
(\rho \circ \psi, \rho \circ \pi) = \rho \circ (\psi, \pi). \tag{2.2}
\]

We define

\[
J(X) := \phi^{-1}(\mathcal{K}(X)),
\]

which is a closed two-sided ideal in \( A \). Let \( K \) be an ideal contained in \( J(X) \). Following Muhly and Solel, we say that a Toeplitz representation \( (\psi, \pi) \) of \( X \) is \textit{coisometric on} \( K \) if

\[
(\psi, \pi)(\phi(a)) = \pi(a) \quad \text{for all} \quad a \in K.
\]
Proposition 2.2 ([6, Proposition 1-3]). Let $X$ be a Hilbert bimodule over $A$, and let $K$ be an ideal in $J(X)$. Then there are a $C^*$-algebra $\mathcal{O}(K, X)$ and a Toeplitz representation $(k_X, k_A): X \rightarrow \mathcal{O}(K, X)$ which is coisometric on $K$ and satisfies:

(i) for every Toeplitz representation $(\psi, \pi)$ of $X$ which is coisometric on $K$, there is a homomorphism $\psi \circ k_X \pi$ of $\mathcal{O}(K, X)$ such that $(\psi \circ k_X \pi) k_X = \psi$ and $(\psi \circ k_X \pi) k_A = \pi$; and

(ii) $\mathcal{O}(K, X)$ is generated as a $C^*$-algebra by $k_X(X) \cup k_A(A)$.

The triple $(\mathcal{O}(K, X), k_X, k_A)$ is unique: if $(B, k'_X, k'_A)$ has similar properties, there is an isomorphism $\theta: \mathcal{O}(K, X) \rightarrow B$ such that $\theta \circ k_X = k'_X$ and $\theta \circ k_A = k'_A$. There is a strongly continuous gauge action $\gamma: \mathbb{T} \rightarrow \text{Aut}\mathcal{O}(K, X)$ which satisfies $\gamma_z(k_A(a)) = k_A(a)$ and $\gamma_z(k_X(x)) = z k_X(x)$ for $a \in A, x \in X$.

The algebra $\mathcal{O}(K, X)$ is called the relative Cuntz–Pimsner algebra determined by $K$, and was first studied by Muhly and Solel in [11]. The algebra $\mathcal{O}([0], X)$ is the Toeplitz algebra $T(X)$ (see [7, Proposition 1-4]), and $\mathcal{O}(J(X), X)$ is the Cuntz–Pimsner algebra $\mathcal{O}(X)$ [15]. The following lemma tells us when $k_A: A \rightarrow \mathcal{O}(K, X)$ is injective.

Lemma 2.3. Let $X$ be a Hilbert bimodule over $A$ and let $(\mathcal{O}(K, X), k_A, k_X)$ be a relative Cuntz–Pimsner algebra associated to $X$. Then $k_A$ is injective if and only if $\phi|_K: K \rightarrow \mathcal{L}(X)$ is injective.

Proof. If $\phi|_K$ is injective, then [11, Proposition 2-21] implies that $k_A$ is injective. Conversely, suppose $k_A$ is injective and $a \in K$ satisfies $\phi|_K(a) = 0$. Then $k_A(a) = (k_X, k_A)^{(1)}[\phi|_K(a)] = 0$, and since $k_A$ is injective, this implies $a = 0$. Thus $\phi|_K: K \rightarrow \mathcal{L}(X)$ is injective.

3. Exel's crossed product

Let $A$ be a unital $C^*$-algebra and $\alpha$ an endomorphism of $A$; we do not assume that $\alpha$ is unital or injective. In [3], Exel defined a transfer operator $L$ for $(A, \alpha)$ to be a continuous linear map $L: A \rightarrow A$ such that:

(i) $L$ is positive in the sense that $a \geq 0 \Rightarrow L(a) \geq 0$; and

(ii) $L(\alpha(a)b) = \alpha L(b)$, for all $a, b \in A$.

He then defined $T(A, \alpha, L)$ to be the universal unital $C^*$-algebra generated by a copy of $A$ and an element $S$ satisfying the relations $Sa = \alpha(a)S$ and $S^*aS = L(a)$ for $a \in A$, so that $T(A, \alpha, L)$ is by definition universal for the following representations.

Definition 3-1. A pair $(\rho, V)$, consisting of a unital homomorphism $\rho$ of $A$ into a $C^*$-algebra $B$ and an element $T \in B$, is a Toeplitz-covariant representation of $(A, \alpha, L)$ in $B$ if, for every $a \in A$:

(TC1) $V\rho(a) = \rho(\alpha(a))V$; and

(TC2) $V^*\rho(a)V = \rho(L(a))$.

We denote by $(i_A, S)$, the universal Toeplitz-covariant representation of $(A, \alpha, L)$ in $T(A, \alpha, L)$. If $(\rho, V)$ is a Toeplitz-covariant representation of $(A, \alpha, L)$, we denote by $\rho \times V$ the representation of $T(A, \alpha, L)$ such that $(\rho \times V) \circ i_A = \rho$ and $(\rho \times V)(S) = V$.

The homomorphism $i_A: A \rightarrow T(A, \alpha, L)$ is injective; to see this, we need an example of a Toeplitz-covariant representation $(\rho, V)$ with $\rho$ injective, and one such example is given in [3].
Given the triple $\( (A, \alpha, L) \)$, we recall from [3] the construction of the Hilbert $A$-bimodule $M_L$. We let $A_L$ be a copy of the underlying vector space of $A$. We define a right action of $A$ on $A_L$ by

$$m \cdot a = m\alpha(a) \text{ for } m \in A_L \text{ and } a \in A,$$

and an $A$-valued map $\langle \cdot, \cdot \rangle_L$ on $A_L$ by

$$\langle m, n \rangle_L = L(m^*n) \text{ for } m, n \in A_L.$$ 

We define $N := \{ a \in A_L : \langle a, a \rangle_L = 0 \}$; it follows from the Cauchy–Schwarz inequality that $N$ is a subspace of $A_L$, and we can form the quotient space $A_L/N$. We denote the quotient map by $q : A_L \to A_L/N$, and then $A_L/N$ is a right $A$-module with inner-product $\langle q(a), q(b) \rangle_L = L(a^*b)$. By completing $A_L/N$ we get a right Hilbert $A$-module which we denote by $M_L$. For $a \in A$ and $m \in A_L$ we have

$$\|\langle am, am \rangle_L\| = \|L(m^*a^*am)\| \leq \|a\|^2\|L(m^*m)\| = \|a\|^2\|\langle m, m \rangle_L\|,$$

and it follows that left multiplication by $a$ on $A_L$ extends to a bounded adjointable operator on $M_L$. This defines a homomorphism $\phi : A \to \mathcal{L}(M_L)$, and writing $\phi(a)m := a \cdot m$ makes $M_L$ a Hilbert bimodule over $A$. Note that $q(A_L)$ is dense in $M_L$.

In the following lemma we see that there is a one-to-one correspondence between Toeplitz-covariant representations of $\( (A, \alpha, L) \)$ and Toeplitz representations of $M_L$.

**Lemma 3.2.** Given a Toeplitz-covariant representation $\( \rho, V \)$ of $\( (A, \alpha, L) \)$ in a $C^*$-algebra $B$, there exists a linear map $\psi_V : M_L \to B$ such that $\psi_V(q(a)) = \rho(a)V$ and the pair $\( \psi_V, \rho \)$ is a Toeplitz representation of $M_L$ in $B$. Conversely, if $\( \psi, \pi \)$ is a Toeplitz representation of $M_L$ in $B$ and $\pi$ is unital, then the pair $\( \pi, \psi(q(1)) \)$ is a Toeplitz-covariant representation of $\( (A, \alpha, L) \)$, and $\psi(q(1)) = \psi$.

**Proof.** We define $\theta : A_L \to B$ by $\theta(a) = \rho(a)V$. Then $\theta$ is linear, and for $a \in A$ we have

$$\|\theta(a)\|^2 = \|\rho(a)V\|^2 = \|\rho(a)V^*\rho(a)V\| = \|V^*\rho(a)V\| = \|\rho(L(a^*a))\|$$

$$\leq \|L(a^*a)\| = \|\langle a, a \rangle_L\|,$$

so $\theta$ is bounded for the semi-norm on $A_L$. Thus $\theta$ induces a bounded map $\psi_V : M_L \to B$ satisfying $\psi_V(q(a)) = \rho(a)V$ for $a \in A$. For $a, b, c \in A$ we have

$$\psi_V(q(b) \cdot a) = \psi_V(q(ba(a))) = \rho(ba(a))V = \rho(b)V\rho(a) = \psi_V(q(b))\rho(a),$$

$$\psi_V(q(b)^*q(c)) = (\rho(b)V)^*\rho(c)V = V^*\rho(b^*c)V = \rho(L(b^*c)) = \rho(\langle q(b), q(c) \rangle_L)$$

and

$$\psi_V(a \cdot q(b)) = \psi_V(q(ab)) = \rho(ab)V = \rho(a)\rho(b)V = \rho(a)\psi_V(q(b)).$$

Thus $\( \psi_V, \rho \)$ is a Toeplitz representation of $M_L$ in $B$.

Now let $\( \psi, \pi \) : M_L \to B$ be a Toeplitz representation of $M_L$ in a $C^*$-algebra $B$ with $\pi$ unital. Then for $a \in A$ we have

$$\psi(q(1))\pi(a) = \psi(q(1) \cdot a) = \psi(q(\alpha(a))) = \psi(\alpha(a) \cdot q(1)) = \pi(\alpha(a))\psi(q(1)),$$

and

$$\psi(q(1))^*\pi(a)\psi(q(1)) = \psi(q(1))^*\psi(a \cdot q(1)) = \psi(q(1))^*\psi(q(a))$$

$$= \pi(\langle q(1), q(a) \rangle_L) = \pi(L(1^*a)) = \pi(L(a)),$$
so \((\pi, \psi(q(1)))\) is a Toeplitz-covariant representation of \((A, \alpha, L)\). Finally, for \(a \in A\) we have
\[
\psi_{\psi(q(1))}(q(a)) = \pi(a)\psi(q(1)) = \psi(a \cdot q(1)) = \psi(q(a)),
\]
which implies that \(\psi_{\psi(q(1))} = \psi\).

**Corollary 3.3.** The \(C^*\)-algebra \(T(A, \alpha, L)\) is isomorphic to the Toeplitz algebra \(T(M_L)\).

**Proof.** We prove that \(T(A, \alpha, L)\) has the universal property which characterises \(T(M_L)\). Applying the lemma to the pair \((i_A, S)\) gives a Toeplitz representation \((\psi_S, i_A)\) of \(M_L\) in \(T(A, \alpha, L)\), which generates \(T(A, \alpha, L)\) because \(i_A\) and \(S\) do. Now suppose \((\psi, \pi)\) is a Toeplitz representation of \(M_L\). Note that \(M_L\) is essential as a left \(A\)-module, in the sense that \(A \cdot M_L = M_L\). This implies that the essential subspace \(\pi(1)H\) is reducing for \((\psi, \pi)\), so we can apply the lemma to the restriction of \((\psi, \pi)\) to \(\pi(1)H\); this gives a Toeplitz-covariant representation \((\pi|_H, \psi|_{\pi(H)})\) on \(\pi(1)H\). Now the representation \(\mu := (\pi|_H \times \psi|_{\pi(H)}) \oplus 0\) has \(\mu \circ i_A = \pi|_H \oplus 0 = \pi\), and for \(a \in A\) we have
\[
\mu \circ \psi_S(q(a)) = \mu(i_A(a)) = \mu(i_A(a)) \mu(S) = (\pi(a)\psi|_{\pi(H)}(1)) \oplus 0
= \pi(a)\psi(q(1)) = \psi_{\psi(q(1))}(q(a)) = \psi(q(a)),
\]
which implies that \(\mu \circ \psi_S = \psi\).

Corollary 3.3 has been obtained independently by Nadia Larsen.

**Remark 3.4.** The Toeplitz representation \((\psi_S, i_A)\) induces a homomorphism \((\psi_S, i_A)^{(1)}\) of \(\mathcal{K}(M_L)\) into \(T(A, \alpha, L)\), and \((\psi_S, i_A)^{(1)}\) is injective because \(i\) is injective (see Remark 2.1). The range of any homomorphism of \(C^*\)-algebras is closed, and since \((\psi_S, i_A)^{(1)}(\mathcal{K}(M_L))\) is dense in \(\psi_S(M_L)\psi_S(M_L)^*\), it follows that \((\psi_S, i_A)^{(1)}\) is an isomorphism of \(\mathcal{K}(M_L)\) onto the \(C^*\)-algebra \(\psi_S(M_L)\psi_S(M_L)^* = i_A(A)SS^*i_A(A)\).

We will now discuss Exel’s notion of a redundancy. Define \(M := \overline{i_A(A)S} = \psi_S(M_L)\). Conditions (TC1) and (TC2) imply that \(i_A(A)M \subseteq M, Mi_A(A) \subseteq M\) and \(M^*M \subseteq i_A(A)\), so \(M\) is a Hilbert bimodule over \(i_A(A)\). It follows that left multiplication by elements of \(i_A(A)\) on \(M\) could coincide with left multiplication by elements in \(\overline{i_A(A)SS^*i_A(A)}\). In [3], Exel defines a redundancy to be a pair \((i_A(a), k)\) such that \(a \in A, k \in i_A(A)SS^*i_A(A)\) and
\[
i_A(a)i_A(b)S = k_i_A(b)S \text{ for all } b \in A.
\]
The next lemma provides a useful identification of the redundancies.

**Lemma 3.5.** Let \(a \in A\) and let \(k \in T(A, \alpha, L)\). Then \((i_A(a), k)\) is a redundancy if and only if \(a \in J(M_L) := \phi^{-1}(\mathcal{K}(M_L))\) and \(k = (\psi_S, i_A)^{(1)}(\phi(a))\).

**Proof.** First suppose that \(a \in J(M_L)\) and \(k = (\psi_S, i_A)^{(1)}(\phi(a))\). Then \(k\) belongs to the image \(i_A(A)SS^*i_A(A)\) of \((\psi_S, i_A)^{(1)}\), and for \(b \in A\) we have
\[
i_A(a)i_A(b)S = i_A(a)\psi_S(q(b)) = \psi_S(\phi(a)q(b))
= (\psi_S, i_A)^{(1)}(\phi(a))\psi_S(q(b))
= (\psi_S, i_A)^{(1)}(\phi(a))i_A(b)S
\]
where the second last equality follows from (2.1). Thus \((i_A(a), k)\) is a redundancy.
Now suppose that \((i_A(a), k)\) is a redundancy. It follows from Remark 3.4 that there exists a unique \(t \in \mathcal{K}(M_L)\) such that \((\psi_S, i_A)^{(1)}(t) = k\). Then for \(b \in A\) we have
\[
\psi_S(\phi(a)(q(b))) = \psi_S(q(ab)) = i_A(ab)S = i_A(a)i_A(b)S
\]
\[
= ki_A(b)S = (\psi_S, i_A)^{(1)}(t)\psi_S(q(b)) = \psi_S(t(q(b))).
\]
The map \(\psi_S\) is injective because \(i\) is injective, and it follows that \(\phi(a)(m) = t(m)\) for all \(m \in M_L\). Hence \(\phi(a) = t\), and the result follows.

Exel defined the crossed product of \((A, \alpha, L)\) to be the quotient of \(T(A, \alpha, L)\) by the ideal generated by the set
\[
\{i_A(a) - k: (i_A(a), k) \text{ is a redundancy with } a \in \overline{A\alpha(A)A}\}.
\]
We denote the quotient map by \(Q: T(A, \alpha, L) \to A\times_{\alpha, L} N\). The next corollary follows immediately from Lemma 3.5.

**Corollary 3.6.** Let \(K_\alpha := \overline{A\alpha(A)A} \cap J(M_L)\) and denote by \(I(A, \alpha, L)\) the ideal in \(T(A, \alpha, L)\) generated by
\[
\{i_A(a) - (\psi_S, i_A)^{(1)}(\phi(a)): a \in K_\alpha\}.
\]
Then \(A\times_{\alpha, L} N = T(A, \alpha, L)/I(A, \alpha, L)\).

To describe \(A\times_{\alpha, L} N\) as a universal object, we need to identify the Toeplitz-covariant representations that vanish on the ideal \(I(A, \alpha, L)\). We need a lemma:

**Lemma 3.7.** Suppose \((\rho, V)\) is a covariant representation of \((A, \alpha, L)\). Then we have:
\[
(\rho \times V) \circ (\psi_S, i_A)^{(1)} = (\psi_V, \rho)^{(1)}. \quad (3.1)
\]

**Proof.** We know from (2.2) that
\[
(\rho \times V) \circ (\psi_S, i_A)^{(1)} = ((\rho \times V) \circ \psi_S, (\rho \times V) \circ i_A)^{(1)}.
\]
Since \((i_A, S)\) is the universal Toeplitz-covariant representation, we have \((\rho \times V) \circ i_A = \rho\), and \((\rho \times V)(S) = V\). So for \(a \in A\) we have
\[
(\rho \times V) \circ \psi_S(q(a)) = \rho \times V(i_A(a)S) = \rho(a)V = \psi_V(q(a)),
\]
and hence we also have \((\rho \times V) \circ \psi_S = \psi_V\).

Equation (3.1) motivates the following definition.

**Definition 3.8.** Consider the triple \((A, \alpha, L)\), and let \((\rho, V)\) be a Toeplitz-covariant representation in a \(C^*\)-algebra \(B\). We say that \((\rho, V)\) is a covariant representation of \((A, \alpha, L)\) if in addition we have
\[
(C3) \quad \rho(a) = (\psi_V, \rho)^{(1)}(\phi(a)) \text{ for all } a \in K_\alpha.
\]

The following Proposition says that \(A\times_{\alpha, L} N\) is universal for covariant representations of \((A, \alpha, L)\).

**Proposition 3.9.** Let \(\alpha\) be an endomorphism of a unital \(C^*\)-algebra \(A\), and let \(L\) be a transfer operator for \((A, \alpha)\). The pair \((j_A, T) := (Q \circ i_A, Q(S))\) is a covariant representation of \((A, \alpha, L)\) in \(A\times_{\alpha, L} N\), and for every covariant representation \((\rho, V)\) of \((A, \alpha, L)\), there is a representation \(\tau_{\rho, V}\) of \(A\times_{\alpha, L} N\) such that \(\tau_{\rho, V} \circ j_A = \rho\) and \(\tau_{\rho, V}(T) = V\).
Proof. The pair \((Q \circ i_A, Q(S))\) is Toeplitz-covariant because \((i_A, S)\) is, and its integrated form \((Q \circ i_A) \times Q(S)\) is precisely \(Q\). By Lemma 3.2, we get a Toeplitz representation \((\psi_{Q(S)}, Q \circ i_A) : M_L \rightarrow A \times_{\alpha, L} \mathbb{N}\), and for \(a \in K_{\alpha}\) we have

\[
(Q \circ i_A)(a) = Q(i_A(a)) = Q((\psi_S, i_A)(\phi(a))) \\
= ((Q \circ i_A) \times Q(S)((\psi_S, i_A)(\phi(a))) \\
= (\psi_{Q(S)}, Q \circ i_A)(\phi(a)),
\]

using Lemma 3.7. So the pair \((Q \circ i_A, Q(S))\) is covariant.

Now suppose \((\rho, V)\) is a covariant representation of \((A, \alpha, L)\). The Toeplitz-covariant representation \((\rho, V)\) gives us a representation \(\rho \times V\) of \(T(A, \alpha, L)\), and condition (C3) says that \(\rho \times V\) vanishes on the generators of the ideal \(I(A, \alpha, L)\). Hence Corollary 3.6 implies that \(\rho \times V\) factors through a representation \(\tau_{\rho, V}\) of \(A \times_{\alpha, L} \mathbb{N}\). Then

\[
\tau_{\rho, V} \circ j_A = \tau_{\rho, V} \circ Q \circ i_A = (\rho \times V) \circ i_A = \rho \quad \text{and} \quad \tau_{\rho, V}(T) = \tau_{\rho, V}(Q(S)) = (\rho \times V)(S) = V,
\]

so \(\tau_{\rho, V}\) has the required properties.

We now realize \(A \times_{\alpha, L} \mathbb{N}\) as a relative Cuntz–Pimsner algebra.

**Proposition 3.10.** Suppose \(\alpha\) is an endomorphism of a unital \(C^*\)-algebra \(A\) and \(L\) is a transfer operator for \((A, \alpha)\). Then there is an isomorphism \(\theta : \mathcal{O}(K_{\alpha}, M_L) \rightarrow A \times_{\alpha, L} \mathbb{N}\) such that \(\theta \circ k_A = j_A\) and \(\theta(k_{M_L}(q(1))) = T\).

**Proof.** Consider the triple \((A \times_{\alpha, L} \mathbb{N}, \psi_T, j_A)\), where \((\psi_T, j_A)\) is the Toeplitz representation of \(M_L\) induced by the pair \((j_A, T)\), as in Lemma 3.2. We will prove that \((A \times_{\alpha, L} \mathbb{N}, \psi_T, j_A)\) satisfies the conditions of Proposition 2.2.

Since \((j_A, T)\) is covariant, it satisfies (C3), which says precisely that \((\psi_T, j_A)\) is coisometric on \(K_{\alpha}\). Let \((\psi, \pi)\) be a Toeplitz representation of \(M_L\) which is coisometric on \(K_{\alpha}\); since \(M_L\) is essential, we suppose by throwing away a trivial representation that \(\pi\) is unital (see the proof of Corollary 3.3). Then Lemma 3.2 gives a Toeplitz-covariant representation \((\tau, \psi(q(1)))\). Since \(\psi_{\psi(q(1))} = \psi\) and \((\psi, \pi)\) is coisometric on \(K_{\alpha}\); \((\pi, \psi(q(1)))\) is covariant. Now Proposition 3.9 gives a representation \(\tau_{\pi, \psi(q(1))}\) of \(A \times_{\alpha, L} \mathbb{N}\) such that \(\tau_{\pi, \psi(q(1))} \circ j_A = \pi\) and \(\tau_{\pi, \psi(q(1))}(T) = \psi(q(1))\). For \(a \in A\) we have

\[
\tau_{\pi, \psi(q(1))}(\psi_T(q(a))) = \tau_{\pi, \psi(q(1))}(j_A(a)T) = \pi(a)\psi(q(1)) = \psi(q(a)),
\]

and it follows that \(\tau_{\pi, \psi(q(1))} \circ \psi_T = \psi\). So \(\psi \times_{K_{\alpha}} \pi = \tau_{\pi, \psi(q(1))}\) satisfies condition (i) of Proposition 2.2. Since \(\psi_T(M_L) \cup j_A(A)\) generates \(A \times_{\alpha, L} \mathbb{N}\), condition (ii) is also satisfied, and applying Proposition 2.2 gives the result.

Notice that when \(\alpha(1) = 1\), we have \(K_{\alpha} = J(M_L)\), and the crossed product \(A \times_{\alpha, L} \mathbb{N}\) is the Cuntz–Pimsner algebra \(\mathcal{O}(M_L)\).

4. **Injectivity of \(j_A : A \rightarrow A \times_{\alpha, L} \mathbb{N}\)**

**Definition 4.1.** Suppose that \(A\) is a unital \(C^*\)-algebra, \(\alpha\) is an endomorphism of \(A\) and \(L\) is a transfer operator for \((A, \alpha)\). We say that \(L\) is faithful on an ideal \(I\) of \(A\) if

\[
a \in I \quad \text{and} \quad L(a^* a) = 0 \implies a = 0;
\]
we say that $L$ is almost faithful on $I$ if
\[ a \in I \text{ and } L((ab)^*ab) = 0 \text{ for all } b \in A \implies a = 0. \]

**Theorem 4.2.** Let $\alpha$ be an endomorphism of a unital $C^*$-algebra $A$, and let $L$ be a transfer operator for $(A, \alpha)$. Then the map $j_A: A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective if and only if $L$ is almost faithful on $K_\alpha = A\alpha(A)A \cap J(M_L)$.

**Proof.** It follows from Proposition 3.10 that the map $j_A$ is injective if and only if $k_A: A \to \mathcal{O}(K_\alpha, M_L)$ is injective. By Lemma 2.3 this is true if and only if $\phi|_{K_\alpha}: K_\alpha \to \mathcal{L}(M_L)$ is injective, and so it suffices to prove that the transfer operator $L$ is almost faithful on $K_\alpha$ if and only if $\phi|_{K_\alpha}: K_\alpha \to \mathcal{L}(M_L)$ is injective. But for $a \in K_\alpha$ and $b \in A$, we have
\[
\|L((ab)^*ab)\| = \|\langle q(ab), q(ab) \rangle_L\| = \|q(ab)\|^2 = \|a \cdot q(b)\|^2 = \|\phi(a)(q(b))\|^2 = \|\phi|_{K_\alpha}(a)(q(b))\|^2,
\]
and this implies the desired equivalence.

**Corollary 4.3.** Let $\alpha$ be an endomorphism of a unital commutative $C^*$-algebra $A$, and let $L$ be a transfer operator for $(A, \alpha)$. Then the map $j_A: A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective if and only if $L$ is faithful on $K_\alpha$.

**Proof.** If $L$ is faithful on $K_\alpha$ then it follows from Theorem 4.2 that $j_A: A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective. Conversely, suppose $j_A: A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective. By Theorem 4.2, this implies that $L$ is almost faithful on $K_\alpha$. Suppose $a \in K_\alpha$ satisfies $L(a^*a) = 0$. Then for every $b \in A$ we have
\[
\|L((ab)^*ab)\| = \|L((ba)^*ba)\| = \|L(a^*b^*ba)\| \leq \|b\|^2\|L(a^*a)\| = 0.
\]
Thus $L((ab)^*ab) = 0$ for every $b \in A$, which implies $a = 0$, and we have shown that $L$ is faithful on $K_\alpha$.

In [4], Exel assumed that $\alpha$ is a unital injective endomorphism and $L = \alpha^{-1} \circ E$, where $E$ is a conditional expectation of $A$ onto $\alpha(A)$ satisfying $E(a^*a) = 0 \implies a = 0$ (Exel says $E$ is non-degenerate). Under these conditions he proves that $j_A: A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective [4, theorem 4.12]. Notice that such $L$ are almost faithful, and so [4, theorem 4.12] follows from Theorem 4.2. The following examples show that our theorem is stronger in several different ways.

**Example 4.4.** In this example, the endomorphism is not unital. Let $A = c$, the space of convergent sequences under the sup norm, and let $\alpha$ be the forward shift $\tau_f$. Then the backward shift $L = \tau_b$ is a transfer operator for $(c, \tau_f)$ and we have
\[
M_{\tau_b} = c/\mathbb{C}c_0, \ J(M_{\tau_b}) = c, \text{ and } K_{\tau_f} = \{f \in c: f(0) = 0\};
\]
notice that $L = \tau_b$ is faithful on $K_{\tau_f}$, but not on all of $c$. It follows from Corollary 4.3 that the map $j_c: c \to c \rtimes_{\tau_f,\tau_b} \mathbb{N}$ is injective.

**Example 4.5.** In this example, the endomorphism is not injective. Again the algebra $A$ is $c$, but now we view the backward shift $\tau_b$ as the endomorphism, and take for $L$ the forward shift $\tau_f$. Then we have $M_{\tau_f} = A_L, \ J(M_{\tau_f}) = c$, and $K_{\tau_b} = c$. In this case, $L = \tau_f$ is faithful on $K_{\tau_b}$, so Corollary 4.3 shows that $j_c: c \to c \rtimes_{\tau_b,\tau_f} \mathbb{N}$ is injective.
Example 4-6. In this example, the transfer operator is almost faithful but is not faithful. We take $A$ to be the UHF algebra $\text{UHF}(n^\infty)$, viewed as the direct limit $\lim_{\rightarrow} (A_N, i_N)$ with $A_N = \bigotimes_{k=1}^{N} M_n(\mathbb{C})$ and

$$i_N(a_1 \otimes \ldots \otimes a_N) := a_1 \otimes \ldots \otimes a_N \otimes 1;$$

we denote the canonical embeddings by $i^N: A_N \to A$. The maps $\alpha_N: A_N \to A_{N+1}$ defined by

$$\alpha_N(a_1 \otimes \ldots \otimes a_N) = e_{11} \otimes a_1 \otimes \ldots \otimes a_N,$$

induce an injective endomorphism $\alpha: A \to A$ such that $\alpha(i^N(a)) = i^{N+1}(\alpha_N(a))$ for $a \in A_N$. Since range $\alpha$ is closed, it follows that range $\alpha = i^1(e_{11})A i^1(e_{11})$. We can then define $L: A \to A$ by

$$L(a) = \alpha^{-1}(i^1(e_{11})a i^1(e_{11})).$$

Then $L$ is positive, continuous and linear. To see that $L$ is a transfer operator, let $a = \bigotimes a_i \in A_N$, $b = \bigotimes b_i \in A_{N+1}$, and compute:

$$L(\alpha(i^N(a))i^{N+1}(b)) = L(i^{N+1}(e_{11}b_1 \otimes a_1b_2 \otimes \ldots \otimes a_Nb_{N+1}))$$

$$= \alpha^{-1}(i^1(e_{11}))i^{N+1}(e_{11}b_1 \otimes a_1b_2 \otimes \ldots \otimes a_Nb_{N+1})i^1(e_{11})$$

$$= (b_1)_{11} \alpha^{-1}(i^{N+1}(e_{11} \otimes a_1b_2 \otimes \ldots \otimes a_Nb_{N+1}))$$

$$= (b_1)_{11} \alpha^{-1}(i^{N+1}(e_{11} \otimes a_1 \otimes \ldots \otimes a_N))$$

$$= i^N(a)(b_1)_{11} \alpha^{-1}(i^{N+1}(e_{11} \otimes b_2 \otimes \ldots \otimes b_{N+1}))$$

$$= i^N(a)\alpha^{-1}(i^1(e_{11})i^{N+1}(b)i^1(e_{11}))$$

$$= i^N(a)L(i^{N+1}(b)).$$

It follows from linearity and continuity of $L$ and $\alpha$ that $L(\alpha(a)b) = aL(b)$ for all $a, b \in A$, and hence $L$ is a transfer operator for $(A, \alpha)$.

For $j \in \{1, \ldots, n\}$ define $b_j := i^1(e_{jj})$. Suppose $a \in A$ satisfies $L((ab)^*ab) = 0$ for all $b \in A$. Then $0 = L((ab_j)^*ab_j) = \alpha^{-1}(i^1(e_{11})b_j^*a^*ab_ji^1(e_{11}))$ for all $j$, and this implies that $ab_j = 0$ for all $j$. Thus

$$0 = \sum_{j=1}^{n} ab_jb_j^* = a i^1(\sum_{j=1}^{n} e_{jj}) = a i^1(1) = a,$$

and hence $L$ is almost faithful on $A$. To see that $L$ is not faithful we let $a_0 \in M_n(\mathbb{C})$ be a non-zero matrix whose first column is zero. Then $(a_0^*a_0)_{11} = 0$ and

$$L(i^1(a_0)^*i^1(a_0)) = \alpha^{-1}(i^1(e_{11}a_0^*a_0e_{11})) = \alpha^{-1}((a_0^*a_0)_{11}i^1(e_{11})) = \alpha^{-1}(0) = 0,$$

whereas $i^1(a_0) \neq 0$ because $i^1$ is injective.

The endomorphism $\alpha$ is injective and has hereditary range. Under these assumptions, Exel proved in [3, theorem 4-7] that $A \rtimes_{\alpha,L} \mathbb{N}$ is isomorphic to the Stacey crossed product $A \rtimes \mathbb{N}$. This crossed product was first considered by Cuntz, who showed in [2] that $\text{UHF}(n^\infty) \rtimes \mathbb{N}$ is isomorphic to the Cuntz algebra $O_n$.

Example 4-7. This is an example of a commutative $C^*$-algebra with a transfer operator $L$ which is not faithful on $K_\alpha$, so that $A$ does not embed in Exel’s crossed product.
product. Let $A := C([0, 2])$, and define $\alpha: C([0, 2]) \to C([0, 2])$ by

$$\alpha(f)(x) := \begin{cases} f(2x) & \text{if } x \in [0, 1] \\ f(4 - 2x) & \text{if } x \in [1, 2] \end{cases}$$

Then the map $L: C([0, 2]) \to C([0, 2])$ defined by $L(f)(x) = f(x/2)$ is a transfer operator for $(A, \alpha)$. We have $A_L = C([0, 2])$ as a vector space, and

$$N := \{ f \in C([0, 2]): L(f) = 0 \} = \{ f \in C([0, 2]): f(x) = 0 \text{ for all } x \in [0, 1] \}.$$  

Thus the restriction map $r: f \mapsto f|_{[0, 1]}$ induces a vector-space isomorphism of $A_L/N$ onto $C([0, 1])$, which converts the bimodule structure into

$$\langle g, h \rangle_L(x) = (g(x/2))h(x/2), \quad g \cdot f(x) = g(x)f(2x), \quad f \cdot g(x) = f(x)g(x)$$

for $g, h \in C([0, 1])$ and $f \in A = C([0, 2])$; it follows from the first formula that $r$ is isometric for the sup-norm on $C([0, 1])$, so $A_L/N$ is complete and $M_L = A_L/N$. Now for $f \in A$ and $x \in [0, 1]$, we have

$$\Theta_{r(f)}(x) = r(f)(x)\langle 1, g \rangle_L(2x) = f(x)g(x) = (\phi(f)g)(x),$$

so $f \in J(M_L)$. Thus $J(M_L) = A$, which implies $K_{\alpha} = A$ because $\alpha(1) = 1$. The transfer function $L$ is not faithful on $C([0, 2])$: any nonzero function $f \in C([0, 2])$ with $f|_{[0, 1]} = 0$ will satisfy $L(f) = 0$. Hence it follows from Corollary 4.3 that the canonical map $C([0, 2]) \to C([0, 2]) \times_{\alpha, L} N$ is not injective.

5. Gauge invariant uniqueness theorem

Using the isomorphism $\theta: \mathcal{O}(K_{\alpha}, M_L) \to A \times_{\alpha, L} N$ of Proposition 3.10, we can see that there is a natural gauge action $\delta: T \to \text{Aut}(A \times_{\alpha, L} N)$ such that $\delta_z(j_A(a)) = j_A(a)$, $\delta_z(T) = zT$ and $\theta \circ \gamma_z = \delta_z \circ \theta$.

**Theorem 5.1.** Let $\alpha$ be an endomorphism of a unital $C^*$-algebra $A$, and let $L$ be a transfer operator for $(A, \alpha)$. Suppose $B$ is a $C^*$-algebra and $(\rho, V)$ is a covariant representation of $(A, \alpha, L)$ in $B$ satisfying:

(i) for $a \in A$, $\rho(a) = 0 \implies j_A(a) = 0$;

(ii) if $\rho(a) \in (\psi_V, \rho)^{1/2}(K(M_L))$, then $j_A(a) \in j_A(K)$;

(iii) there exists a strongly continuous action $\beta: T \to \text{Aut}_{\rho, V}(A \times_{\alpha, L} N)$ such that $\beta_z \circ \tau_{\rho, V} = \tau_{\rho, V} \circ \delta_z$ for all $z \in T$.

Then the corresponding representation $\tau_{\rho, V}: A \times_{\alpha, L} N \to B$ is faithful.

The proof of Theorem 5.1 will use the following gauge-invariant uniqueness theorem for relative Cuntz–Pimsner algebras, which is due to Katsura [8, corollary 11.7] and Muhly–Tomforde [12, Section 5].

**Theorem 5.2.** Suppose $X$ is a Hilbert bimodule over $A$ and $K$ is an ideal in $J(M_L)$. If $\mu: \mathcal{O}(K, X) \to B$ is a homomorphism into a $C^*$-algebra $B$ satisfying:

(i) the restriction of $\mu$ to $k_A(A)$ is injective;

(ii) if $\mu(k_A(a)) \in \mu(k_X, k_A)^{1/2}(K(X))$, then $k_A(a) \in k_A(K)$;

(iii) there exists a strongly continuous action $\beta: T \to \text{Aut}_{\mu}(\mathcal{O}(K, X))$ such that $\beta_z \circ \mu = \mu \circ \gamma_z$ for all $z \in T$.

then $\mu$ is injective.
Proof of Theorem 5.1. We will prove that \( \tau_{\rho,V} \circ \theta \) satisfies the conditions of Theorem 5.2. Suppose \( a \in A \) satisfies \((\tau_{\rho,V} \circ \theta)(k_A(a)) = 0\). Then
\[
\rho(a) = \tau_{\rho,V}(j_A(a)) = \tau_{\rho,V}(\theta(a)) = 0,
\]
which by (i) implies that \( j_A(a) = 0 \). Hence \( k_A(a) = \theta^{-1}(j_A(a)) = 0 \), and so \( \tau_{\rho,V} \circ \theta \) is injective on \( k_A(A) \).

Now suppose \( a \in A \) and \((\tau_{\rho,V} \circ \theta)(k_A(a)) \in (\tau_{\rho,V} \circ \theta)((k_{ML}, k_A)^{(1)}(\mathcal{K}(ML)))\). We have \((\tau_{\rho,V} \circ \theta)(k_A(a)) = \rho(a)\), and Lemma 3.7 gives
\[
(\tau_{\rho,V} \circ \theta)((k_{ML}, k_A)^{(1)}(\mathcal{K}(ML))) = \tau_{\rho,V}((\theta \circ k_{ML}, \theta \circ k_A)^{(1)}(\mathcal{K}(ML)))
\]
\[
= \tau_{\rho,V}((\psi_T, j_A)^{(1)}(\mathcal{K}(ML)))
\]
\[
= \tau_{\rho,V} Q((\psi_S, i_A)^{(1)}(\mathcal{K}(ML)))
\]
\[
= (\rho \times V)((\psi_S, i_A)^{(1)}(\mathcal{K}(ML)))
\]
\[
= (\psi_V, \rho)^{(1)}(\mathcal{K}(ML)).
\]
So \( \rho(a) \in (\psi_V, \rho)^{(1)}(\mathcal{K}(ML)) \), and then it follows from (ii) that \( j_A(a) \in j_A(K_\alpha) \). Hence \( k_A(a) \in k_A(K_\alpha) \). By (iii), we have
\[
\beta_z \circ \tau_{\rho,V} \circ \theta = \tau_{\rho,V} \circ \delta_z \circ \theta = \tau_{\rho,V} \circ \theta \circ \gamma_z,
\]
so Theorem 5.2 implies that \( \tau_{\rho,V} \circ \theta \) is injective. Thus \( \tau_{\rho,V} \) is injective.

When the transfer operator \( L \) is almost faithful on \( K_\alpha \), our main theorem says that \( j_A \) is injective. Using [8, corollary 11.8] instead of Theorem 5.2 yields the following gauge-invariant uniqueness theorem which directly generalises [5, theorem 4.2] (because the second condition \((2')\) trivially holds when \( K_\alpha = J(M_L) \), as is the case when \( \alpha(1) = 1 \)).

Corollary 5.3. Let \( \alpha \) be an endomorphism of a unital C*-algebra \( A \), and let \( L \) be a transfer operator for \((A, \alpha)\) which is almost faithful on \( K_\alpha \). Suppose \( B \) is a C*-algebra and \((\rho, V)\) is a covariant representation of \((A, \alpha, L)\) in \( B \) satisfying:

\((1')\) \( \rho \) is faithful;
\((2')\) for \( a \in J(M_L) \), \( \rho(a) = (\psi_V, \rho)^{(1)}(\phi(a)) \) implies \( j_A(a) = (\psi_T, j_A)^{(1)}(\phi(a)) \);
\((3)\) there exists a strongly continuous action \( \beta : \mathbb{T} \to \text{Aut} \tau_{\rho,V}(A \rtimes_{\alpha,L} \mathbb{N}) \) such that
\[
\beta_z \circ \tau_{\rho,V} = \tau_{\rho,V} \circ \gamma_z \quad \text{for all} \quad z \in \mathbb{T}.
\]
Then the corresponding representation \( \tau_{\rho,V} : A \rtimes_{\alpha,L} \mathbb{N} \to B \) is faithful.

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