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Keywords

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The use of Maple platform for the study of geodesic motion on curved spacetimes

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Abstract

The article illustrates the graphical study of geodesic motion on curved space-times (mainly exact solutions of Einstein equations) using the symbolic, numerical and graphical computation facilities of Maple platform. The example of null geodesics on Schwarzschild solution is completely processed. The geodesic curves are plotted directly using DEtools package in Schwarzschild coordinates.

1. Introduction

This article is dedicated to the study of the geodesic motion on curved spacetimes solutions of the Einstein equations (Einstein Spaces - [2], [3],[4]) using the graphical facilities of Maple [6]. The study of geodesics on Einstein spaces plays an important role in general relativity for pointing out several properties as causality, dynamic behavior around black-holes and for long time in the study of movements of cosmic objects (as planets and satellites) in the solar system [4]. The literature concentrates mainly in approximate methods or special analytic methods (which at an end are again approximate methods) for solving the geodesic equations. The use of Maple in general relativity and in the study of geodesic motion has a long standing history (see for example [5], [7], [9], [10],[11] and at [1] for a list of articles using Maple and GrTensorII package)

In this article we developed new analytical methods for treating these equations in their entire complexity using symbolic computation with Maple + GrTensorII package. The main goal of the article was to

bring the geodesic equations in an appropriate form to graphically represent them for visualizing their properties in a more easy and striking way. We done this using directly the DEtools package without explicitly solving analytically the equations. We shall illustrate here this study for the Schwarzschild spacetime, modeling the environment around a black-hole for the null geodesics (describing the motion of light particles). After a short introduction on the geodesic equations on Riemannian manifolds (in section 2) we shall concentrate (in section 3) to the preparing the set of differential equations in Schwarzschild spherical coordinates using Maple analytic facilities (with GrTensorII package for Riemannian differential geometrical manipulations). Finally, in section 4, we shall graphically represent the null geodesic curves using DEtools package included in Maple for direct plotting the solutions of the obtained differential equations.

2 Geodesics

A geodesic on a Riemannian manifold (M, g) is defined as a smooth curve $\gamma : (a, b) \rightarrow M$ satisfying

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \quad (1)$$

where ∇ is the Levi-Civita connexion.

Using local coordinates (x^1, \dots, x^m) for which the connexion Christoffel symbols are Γ_{ij}^k with $i, j, k = \overline{1, m}$ and the geodesic curve is $\gamma(t) = (x^1(t), \dots, x^m(t))$, the above equations are a system of nonlinear ordinary differential equations.

Considering the notations $\frac{da(t)}{dt} = \dot{a}(t)$ and $\frac{\partial}{\partial x^i} = \partial_i$ with $i = \overline{1, m}$ we obviously have $\dot{\gamma}(t) = \dot{x}^i \partial_i$. Thus:

$$\begin{aligned} \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) &= \nabla_{\dot{\gamma}(t)} \dot{x}^i \partial_i = \\ \dot{x}^i \nabla_{\dot{\gamma}(t)} \partial_i + \frac{d}{dt} (\dot{x}^i) \partial_i &= \dot{x}^i \nabla_{\dot{x}^j \partial_j} \partial_i + \ddot{x}^i \partial_i = \end{aligned}$$

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$$\ddot{x}^i \dot{x}^j \nabla_{\partial_j} \partial_i + \ddot{x}^i \partial_i$$

But knowing that $\nabla_{\partial_j} \partial_i = \Gamma_{ij}^k \partial_k$ with $i, j, k = \overline{1, m}$ and choosing a torsionless connexion (i.e. is symmetric : $\Gamma_{ij}^k = \Gamma_{ji}^k$) finally the geodesic equations became [2]

$$\ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0 \quad \forall k = \overline{1, m} \quad (2)$$

Because the coefficients $\Gamma_{ij}^k = \Gamma_{ij}^k(x)$ depend smoothly on x we can use the classical existence Banach-Picard theorem for initial values to deduce the next proposition on the local existence [8]:

Proposition : *Let (M, g) a Riemannian manifold. For any compact sub-manifold K of the tangent space TM of the manifold there exists $\epsilon > 0$ so that $\forall (x, X) \in K$ exists an unique geodesic $\gamma = \gamma_{x, X} : (-\epsilon, \epsilon) \rightarrow M$ verifying $\gamma(0) = x$ and $\dot{\gamma}(0) = X$.*

3 Maple+GrTensorII programs for geodesic equations

This section is dedicated to the description of commands and short programs in Maple we used for processing the geodesic equations on Schwarzschild space-time [4]. We preferred to calculate the Christoffel symbols and the geodesic equations from the metric with the help of GrTensorII package [1] as being the most easy-to-use tensor manipulation package. Thus at the very beginning we have the next sequence of Maple commands:

```
> restart; grtw(); qload(schw);
> grcalc(Chr(dn, dn, dn));
> grdisplay(Chr(dn, dn, dn));
```

initializing the GrTensorII package, loading the metric, calculating and displaying the Christoffel symbols. The Schwarzschild metric line element has the form:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2(d\theta^2 + \sin(\theta)^2 d\phi^2)$$

in spherical coordinates (t, r, θ, ϕ) . Next we shall define the 4-velocity and 4-acceleration, namely

$$v^i := \frac{dx^i}{d\tau} \quad ; \quad a^i := \frac{dv^i}{d\tau}$$

where τ is the proper time measured in the proper reference frame of the particle. We shall use an intermediate coordinate set **te(tau)**, **er(tau)**, **ph(tau)**, **th(tau)** instead of the coordinates (t, r, θ, ϕ) fixed by the GrTensorII, till we shall come back to Maple, where there is no possibility of confusion. We done this as a series of GrTensorII definitions:

```
> grdef('v{^i}:=
    [diff(er(tau), tau), diff(th(tau), tau),
      diff(ph(tau), tau), diff(te(tau), tau)]');
> grcalc(v(up)); grdisplay(v(up));
> grdef('accel{ ^i }:= [diff(er(tau),
    tau, tau), diff(th(tau), tau, tau), diff(ph
    (tau), tau, tau), diff(te(tau), tau, tau)]');
> grcalc(accel(up)); grdisplay(accel(up));
> grdef('geo{ ^i }:=
    accel{ ^i } +
    Chr{ j k ^i }*v{ ^j }*v{ ^k }');
> grcalc(geo(up)); grdisplay(geo(up));
```

In this way we defined and computed a GrTensorII 4-vector, **geo(dn)** containing as components the four geodesic equations. Next we shall not use anymore GrTensorII so for our purposes we shall “extract” one by one the components of **geo(dn)** as pure Maple objects:

```
> four:=grcomponent(geo(up), [t]);
> one:=grcomponent(geo(up), [r]);
> two:=grcomponent(geo(up), [theta]);
> three:=grcomponent(geo(up), [phi]);
```

A close inspection of the above equations shows off the fact that we have mixed coordinates, namely t, r, θ, ϕ (through the components of the Christoffel symbols as calculated by GrTensorII) and $t(\tau), er(\tau), te(\tau), ph(\tau)$ coming from the components of the 4-acceleration. To avoid this confusion and because we are no more “inside” GrTensorII, we proceed now through several substitutions, as the partially we illustrate below:

```
> one:=subs(r=r(tau), one);
> two:=subs(r=r(tau), two);
.....
> one:=subs(
    diff(er(tau), tau)=diff(r(tau), tau),
    diff(th(tau), tau)=diff(theta(tau), tau),
    diff(ph(tau), tau)=diff(phi(tau), tau),
    diff(te(tau), tau)=diff(t(tau), tau), one);
.....
```

Thus we obtained the next four geodesic equations:

$$\frac{d^2 r}{d\tau^2} - \frac{m}{r(r-2m)} \left(\frac{dr}{d\tau}\right)^2 - (r-2m) \left(\frac{d\theta}{d\tau}\right)^2 - \sin(\theta)^2 (r-2m) \left(\frac{d\phi}{d\tau}\right)^2 - \frac{m}{r^3} (r-2m) \left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\frac{d^2 \theta}{d\tau^2} + \frac{2}{r} \frac{dr}{d\tau} \frac{d\theta}{d\tau} - \sin(\theta) \cos(\theta) \left(\frac{d\phi}{d\tau}\right)^2 = 0$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r} \frac{dr}{d\tau} \frac{d\phi}{d\tau} + 2 \operatorname{ctg}(\theta) \frac{d\theta}{d\tau} \frac{d\phi}{d\tau} = 0$$

$$\frac{d^2t}{d\tau^2} + \frac{2m}{r(r-2m)} \frac{dr}{d\tau} \frac{dt}{d\tau} = 0$$

The last of the above equation can be written as:

$$\frac{d}{d\tau} \left(\left(1 - \frac{2m}{r}\right) \frac{dt}{d\tau} \right) = 0$$

thus we can solve it using the next Maple commands lines:

```
> bau:=diff((1-2*m/r(tau))*
             diff(t(tau),tau),tau);
> expand(simplify(four*(2*m-r(tau))/
                 r(tau)+bau));
> four:=expand(simplify(four*(2*m-r(tau))/
                       r(tau)));
> ecufour:=(1-2*m/r(tau))*diff(t(tau),tau)-C1;
> diffte:=solve(ecufour,diff(t(tau),tau));
> ecufour:=expand(simplify(subs(diff(t(tau),
                                tau)=diffte,ecufour)));
> four:=expand(simplify(subs(diff(t(tau),
                                tau)=diffte,four)));
> four:=expand(simplify(four));
```

obtaining for **diffte** (with C_1 as a constant):

$$\frac{dt}{d\tau} = \frac{C_1 r}{r-2m}$$

Now comes a series of alternate **simplify** and **expand** commands for arranging the terms in the rest of the above equations - with substitution, of course, in everyone of the **diffte** expression [6]. Finally we will solve the **three** equation as:

```
> bau2:=diff(r(tau)^2*sin(theta(tau))^2*
             diff(phi(tau),tau),tau);
> expand(simplify(subs(sin(theta(tau))=
                     sin(theta),cos(theta(tau))=
                     cos(theta),bau2/r(tau)^2/
                     sin(theta(tau))^2)-three)));
> ecuthree:=r(tau)^2*sin(theta)^2*
             diff(phi(tau),tau)-C2;
> diffph:=solve(ecuthree,diff(phi(tau),tau));
> ecuthree:=expand(simplify(subs(diff(
                                phi(tau),tau)=diffph,ecuthree)));
> three:=expand(simplify(subs(diff(phi(tau),
                                tau)=subs(theta=theta(tau),
                                diffph),subs(cos(theta)=cos(theta(tau)),
                                sin(theta)=sin(theta(tau)),three))));
> one:=subs(diff(phi(tau),tau)=diffph,one);
> two:=subs(diff(phi(tau),tau)=diffph,two);
```

obtaining for **diffph** the expression

$$\frac{d\phi}{d\tau} = \frac{C_2}{\sin(\theta)^2 r^2}$$

where C_2 it's another constant. To solve the last two remaining equations (**one** and **two** and after substituting the above expression for **diffph**) is necessary to fix the coordinate $\theta = \pi/2$, the movement being restricted to the equatorial plane. Thus we have:

```
> bau3:=diff(r(tau)^2*diff(theta(tau),tau),
             tau)-r(tau)^2*sin(theta)*cos(
             theta)*diffph^2;expand(simplify(bau3/
             r(tau)^2-two,trigsin));
> one;two; theta(tau):=Pi/2;
> one:=eval(subs(theta=Pi/2,eval(one)));
> two:=eval(subs(theta=Pi/2,eval(two)));
```

followed again by an appropriate sequence of **simplify** and **expand** commands, for arranging the terms. Observe that the second equation (**two**) is now canceled by the fixing of θ coordinate. Finally we have the equation (from **one** Maple object):

$$\frac{d^2r}{d\tau^2} - \frac{m}{r(r-2m)} \left(\frac{dr}{d\tau} \right)^2 - \frac{C_2^2(r-2m)}{r^4} + \frac{C_1^2 m}{r(r-2m)} = 0$$

coupled with the above expressions for **diffte** and **diffph**.

Our purpose is from now one to split the above equation in two different ones, expressing the derivatives for r and t in terms of the angular coordinate ϕ , thus eliminating the proper time τ . Physically this means we shall study the geodesic movement of a particle in Schwarzschild coordinates from the point of view of an inertial observer situated far away from the two singularities ($r = 2m$ or $r = 0$) where the reference frame can be considered a Minkowski one (flat spacetime). This can be done directly through appropriate manipulating of the equations above. Finally we will split the **one** remaining equation in two (Maple defined objects **one1** and **one2**) as:

```
> aba1:=C2*diff(r(phi),phi)/r(phi)^2;
> aba2:=C2^2/r(phi)^4*(diff(r(phi),phi,phi)-
                        diff(r(phi),phi)^2/r(phi));
> one1:=subs(diff(r(tau),tau,tau)=aba2,
             diff(r(tau),tau)=aba1,r(tau)=r(phi),one);
> one1:=expand(simplify(one1));
> one2:=diff(t(phi),phi) -
             C1*r(phi)^3/C2/(r(phi)-2*m);
```

Thus we obtained the next two differential equations:

$$\frac{C_2^2}{r^4} \frac{d^2 r}{d\phi^2} - \frac{C_2^2(r-m)}{r^5(r-2m)} \left(\frac{dr}{d\phi} \right)^2 - \frac{C_2^2(r-2m)}{r^4} + \frac{C_1^2 m}{r(r-2m)} = 0 \quad (3)$$

$$\frac{dt}{d\phi} - \frac{C_1 r^3}{C_2(r-2m)} = 0 \quad (4)$$

The first of the above equations is, obviously, the generalized Binet equation (with extra terms coming from the gravitational interaction) for a central field movement [4] - this fact is easy to prove when we are changing as usual the variable $r \rightarrow 1/u$. The standard procedure here is to solve it approximatively (as is done for the study of the movement in the solar system) or to use elliptic functions (see for example in [4], cap. 15). Instead we preferred here the direct solving of this system of differential equations, as we shall explain in the next section.

4 Graphical results

Maple has the DEtools package, which permits, among other facilities the numerical integration (and the direct visualization through appropriate DEplot commands) of certain systems of differential equations - see [6]. An entire class of numerical integration methods are at the user disposal (as for example several Runge-Kutta methods, Taylor series, Rosenbrock method, and so on). A problem here could be the appropriate choosing of the integration methods as well as the dimension the integration steps (`stepsize` option). In our particular case the main problem will be to avoid the singularity of the Schwarzschild metric at $r = 2m$ (in this coordinate frame).

We used the Maple commands sequence below for numerical integration of the system of equations (3,4) obtained in the previous section. Of course we specified before certain values for the constants, in our examples we used $m = 1, C_2 = \pi, C_1 = 1$.

```
> with(DEtools);
> one1graph:=subs(C2=Pi,C1=1,m=1,one1);
> ini1:=r(0)=3,D(r)(0)=1;
> one2graph:=subs(m=1,C2=Pi,C1=1,one2);
> ini2:=t(0)=0;
> DEplot(one1graph,r(phi),phi=0..Pi,[[ini1]],
        method=classical,axes=BOXED,
        thickness=5,stepsize=0.001);
> DEplot3d([one1graph,one2graph],[r(phi),
        t(phi)],-Pi/4..Pi,[[ini1,ini2]],
```

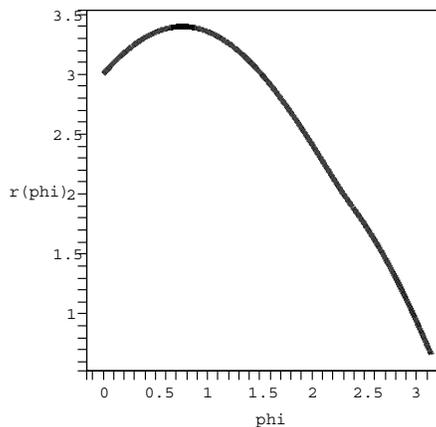


Figure 1.

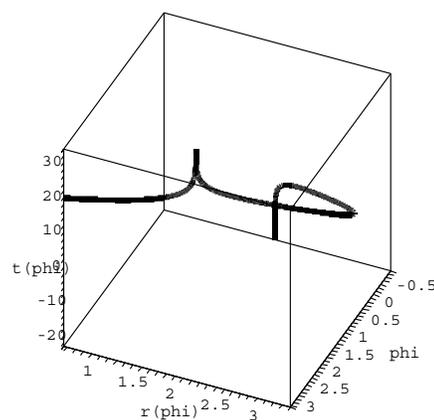


Figure 2.

```
method=classical,stepsize=0.001,axes=
        BOXED,thickness=5);
```

The first DEplot command above is producing the bidimensional graphical representation of the first equation (3) - (**one1**) practically being the function $r(\phi)$. We used Runge-Kutta method (“classical” option). The result is the figure no. 1.

Much more interesting is the graphical output obtained through the second DEplot3d command above. Here are numerically integrated (and graphically plotted) the system of the two equations (eqs. 3 and 4) through the Maple objects **one1** and **one2**). We obtained a 3-d plot, being represented both functions $r(\phi), t(\phi)$ in terms of the radial variable ϕ . At an appropriate resolution Maple can deal with the singularity at $r = 2m$. This plot is presented here in Figure 2

The 3-dimensional plot we have here can also out-

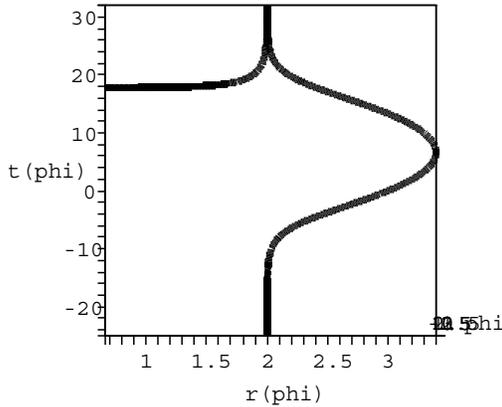


Figure 3.

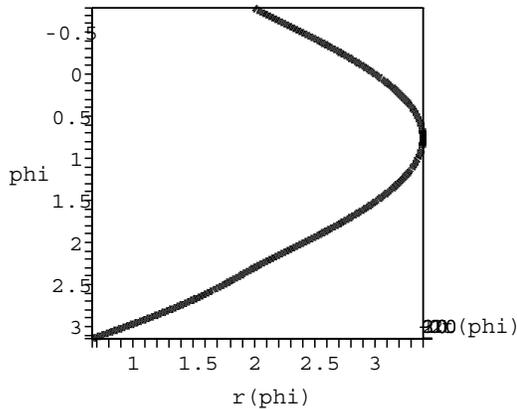


Figure 4.

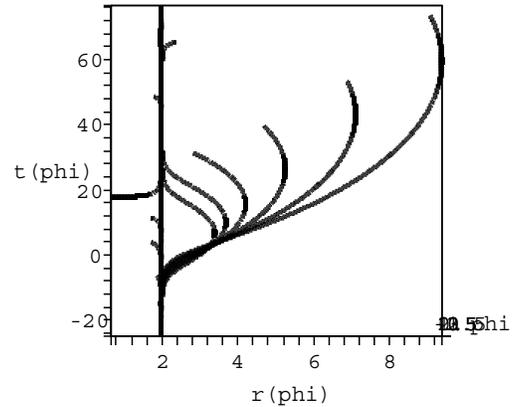


Figure 5.

put several projections on different planes of the corresponding image by rotating the image obtained in the Maple graphical interface. These results are illustrated in the next two figures (3) and (4). The first one, namely 3 represents a projection on the plane $\phi = const$ plotting the function $r(t)$. It is very striking here the situation around the singularity at $r = 2m$ (here $r = 2m$ being unity). Because our equations are the null geodesic (i.e. photons trajectories) this type of plotting can be used for the study of causality on the space-time represented by the Schwarzschild metric. It is obvious the different behavior of these geodesics outside and inside the Schwarzschild horizon at $r = 2m$ (see [4]).

The image plotted in Figure 4 represents the projection of the 3-dimensional image we obtained on the plane $t = const$ being actually the image of the function $r(\phi)$.

It is also possible to represent, on the same plot multiple geodesics, as emerging from a single point with different initial data (angles) as we illustrate in Figure 5 using the same plot command.

5. Conclusions

We illustrated here the possibility of the study of geodesic motion using the graphical facilities of an integrated (algebraic+graphic) computer platform as Maple. We applied the DEtools package included in several versions of Maple (we mainly used Maple 9 but

this package is present even in older versions) for direct numerical integrations and numerical output plotting of geodesic equations. For this purpose it was necessary to bring the equations in an appropriate form which imposed a special treatment by algebraic computing methods (including tensorial manipulations done in GrTensorII package). This treatment was somehow different of the classical analytical methods used to study the geodesic equations in the literature. For the case of null geodesics on Schwarzschild spacetime (which was our working example) we obtained the geodesic equations in Schwarzschild coordinates which we considered more appropriate for visualizing the trajectories described by the geodesic equations.

The Maple programs and commands sequences we used can be easily adapted for other examples. DeSitter, anti-DeSitter and Robertson-Walker metrics are in our view, as future development of this study.

The main conclusion is that, at least Maple (equipped with appropriate packages as we mentioned : GrTensorII and DEtools) can deal through its graphical and analytic facilities, with the study of geodesic equations on curved space-times, as part of the more general topic of numerical relativity (computer solving of Einstein equations)

References

- [1] <http://grtensor.org>.
- [2] A. L. Besse. *Einstein Manifolds*. Springer Verlag, Berlin Heidelberg, 1987.
- [3] I. Chavel. *Riemannian Geometry - a Modern Introduction*. Cambridge Univ. Press, Cambridge, 1993.
- [4] R. d'Inverno. *Introducing Einstein's Relativity*. Clarendon Press, Oxford, 1992.
- [5] G. F. R. Ellis and H. van Elst. gr-qc/9709060, 1997.
- [6] K. H. Heal, M. L. Hansen, and K. M. Richard. *Maple 6 - Learning guide*. Waterloo Maple Inc., 2000.
- [7] V. L. Kalashnikov. gr-qc/0103023, 2001.
- [8] W. Kuhnel. *Differential Geometry: Curves - Surfaces - Manifolds*, volume 16 of *Student Math. Library*. AMS, Providence, Rhode Island, 2002.
- [9] P. Norbury and J. H. Rubinstein. math.GT/0309159, 2003.
- [10] M. Vasudevan and K. A. Stevens. *Phys.Rev.*, D72:124008, 2005.
- [11] D. N. Vulcanov. gr-qc/0010085, 2000.