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Abstract

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APERIODICITY AND PRIMITIVE IDEALS OF ROW-FINITE K -GRAPHS

SOORAN KANG AND DAVID PASK

ABSTRACT. We describe the primitive ideal space of the C^* -algebra of a row-finite k -graph with no sources when every ideal is gauge invariant. We characterize which spectral spaces can occur, and compute the primitive ideal space of two examples. In order to do this we prove some new results on aperiodicity. Our computations indicate that when every ideal is gauge invariant, the primitive ideal space only depends on the 1-skeleton of the k -graph in question.

1. INTRODUCTION

A k -graph (or higher rank graph) is a higher dimensional analog of a directed graph. The notion of a k -graph Λ and its associated C^* -algebra $C^*(\Lambda)$ was introduced in [22] to provide a graphical approach to the higher dimensional Cuntz-Krieger algebras introduced by Robertson and Steger in [37]. Since then k -graph algebras have been studied by many authors and have provided concrete examples of many classifiable C^* -algebras whose fine structure and invariants may readily be computed (see [14, 29, 33, 34, 35, 36, 41, 42], amongst others). They also provide a fertile class of examples for researchers in non-selfadjoint algebras (see [10, 11, 21, 32, 44] amongst others) and crossed products (see [6, 15, 16, 28]).

One of the reasons for the interest in k -graph algebras is that they provide an important testing ground for more complicated mathematical structures such as topological graphs as in [19], topological quivers as in [27], topological k -graphs as in [45] and Cuntz-Pimsner algebras themselves (see [7, 17, 20, 31] amongst others).

The purpose of this paper is to describe the primitive ideal space of a row-finite k -graph with no sources. This was done for row-finite 1-graphs with no sources in [3]. Row-finite k -graphs with no sources have proved to be the most tractable class of k -graphs to study, and we restrict ourselves to this class to avoid the sort of technicalities which occur when analyzing the primitive ideal space of general 1-graphs (see [2, 4, 18]). It is our hope that our techniques will shed light on similar computations for more complex C^* -algebras.

The ideal structure of the C^* -algebra of a row-finite k -graph Λ with no sources is best understood under the hypothesis of aperiodicity on Λ , introduced in [22] but significantly improved later in [35]. Under the aperiodicity hypothesis (see Definition 2.8) the structure of the gauge invariant ideals of $C^*(\Lambda)$ is completely understood (see [33,

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Theorem 5.2]): The lattice of gauge invariant ideals is isomorphic to the lattice of saturated hereditary sets of vertices of Λ . To begin our analysis, we first seek a condition under which all ideals are gauge invariant. This was first done in [40, Theorem 7.2] using a condition called (D), which we prefer to call strong aperiodicity (see Definition 3.1). In Definition 3.4 we introduce the notion of an aperiodic quartet at a vertex. We then prove new results about aperiodicity in order to give necessary conditions on a 2-graph to be strongly aperiodic in Proposition 3.9.

Following [2, 3, 4, 18] the primitive ideals of a 1-graph algebra are described in terms of a collection of vertices called a maximal tail. In Definition 3.10 we adapt the definition of a maximal tail into the context of row-finite k -graphs with no sources. Theorem 3.12 gives necessary and sufficient conditions on a saturated hereditary set of vertices to give rise to a primitive gauge invariant ideal of $C^*(\Lambda)$ when Λ is strongly aperiodic. Furthermore, in Theorem 3.15 we describe the topology of the primitive ideal space of $C^*(\Lambda)$.

We then continue the analysis of [2] and describe in Lemma 3.16 an equivalent topology of the primitive ideal space of $C^*(\Lambda)$. Moreover, in Theorem 4.2 we give a characterization of those topological spaces which can occur as the primitive ideal space of a row-finite k -graph with no sources: a spectral space in which the compact open sets form a countable base. Indeed, in Corollary 4.7 we show that given a row-finite k -graph Λ with no sources there is an AF algebra with the same primitive ideal space. Unlike in [2, §5] we have been unable to find an algorithm for producing such an AF algebra from the 1-skeleton of Λ .

Finally, we give a detailed analysis of the primitive ideal space of two 2-graphs using the main results in this paper. In Example 5.7 we study the first graph Λ , a skew product graph, which we show, is strongly aperiodic using Proposition 3.9. Then using Theorems 3.12 and 3.15, we describe its primitive ideal space. During the example we show that Λ cannot be realized as a cartesian product graph. In Example 5.8 we consider a second 2-graph $\Omega \times \Omega$, the cartesian product of an aperiodic 1-graph Ω with itself. By Theorem 5.3 (ii), the 2-graph $\Omega \times \Omega$ is also strongly aperiodic. Since $C^*(\Omega \times \Omega) \cong C^*(\Omega) \otimes C^*(\Omega)$, the primitive ideal space of $C^*(\Omega \times \Omega)$ is the cartesian product of primitive ideal space of $C^*(\Omega)$ with itself with the product topology. It turns out that $C^*(\Omega \times \Omega)$ and $C^*(\Lambda)$ have homeomorphic primitive ideal spaces, even though Λ is not a cartesian product.

2. PRELIMINARIES

Let \mathbb{N}^k denote the monoid of k -tuples of natural numbers under addition, and denote the canonical generators by e_1, \dots, e_k . We write n_i for the i th coordinate $n \in \mathbb{N}^k$. For $m, n \in \mathbb{N}^k$, we say that $m \leq n$ if $m_i \leq n_i$ for each i . We write $m \vee n$ for the coordinate-wise maximum of m and n and $m \wedge n$ for the coordinate-wise minimum of m and n .

Definition 2.1. [22, Definitions 1.1]. For $k \geq 1$, a k -graph Λ consists of a countable category Λ together with a functor $d : \Lambda \rightarrow \mathbb{N}^k$ satisfying the factorization property :

for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$, $d(\nu) = n$.

Examples 2.2. (a) The path category of a directed graph is a 1-graph, and vice versa.

In particular for $n \geq 1$ we denote by B_n the path category of the directed graph consisting of a single vertex and n edges.

- (b) Let $\text{Mor } \Omega_k = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$, and $\text{Obj } \Omega_k = \mathbb{N}^k$ then, when it is gifted with the structure maps $r(m, n) = m$, $s(m, n) = n$, composition $(m, n)(n, p) = (m, p)$ and degree map $d(m, n) = n - m$, one checks that (Ω_k, d) is a row-finite k -graph with no sources. We identify $\text{Obj } \Omega_k$ with $\{(m, m) : m \in \mathbb{N}^k\} \subset \text{Mor } \Omega_k$.
- (c) For $n \geq 1$ let $\underline{n} = \{1, \dots, n\}$. For $m, n \geq 1$ let $\theta : \underline{m} \times \underline{n} \rightarrow \underline{m} \times \underline{n}$ be a bijection. Let \mathbb{F}_θ^2 be the 2-graph with single vertex v and edges $f_1, \dots, f_m, g_1, \dots, g_n$, with $d(f_i) = e_1$ for $i \in \underline{m}$, $d(g_j) = e_2$ for $j \in \underline{n}$ and factorization rules $f_i g_j = g_{j'} f_{i'}$ where $\theta(i, j) = (i', j')$ for $(i, j) \in \underline{m} \times \underline{n}$ (see [10, 11, 32, 44]).

For $E, F \subseteq \Lambda$ and $\lambda, \nu \in \Lambda$, we define $\lambda E := \{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\}$ and $F\nu := \{\mu\nu : \mu \in F, s(\mu) = r(\nu)\}$, $\lambda E\nu = \{\lambda\mu\nu : \mu \in E, s(\lambda) = r(\mu), s(\mu) = r(\nu)\}$.

For $n \in \mathbb{N}^k$ let $\Lambda^n = d^{-1}(n)$, then by the factorization property we may identify $\Lambda^0 = d^{-1}(0)$ with the objects of Λ , and for this reason we call elements of Λ^0 vertices. In particular, for $v \in \Lambda^0$ and $n \in \mathbb{N}^k$,

$$v\Lambda^n = \{\lambda \in \Lambda : r(\lambda) = v \text{ and } d(\lambda) = n\}.$$

The k -graph Λ is *row-finite* if the set $v\Lambda^m$ is finite for each $m \in \mathbb{N}^k$ and $v \in \Lambda^0$. Also, Λ has *no sources* if $v\Lambda^{e_i} \neq \emptyset$ for all $v \in \Lambda^0$ and $i \in \{1, \dots, k\}$.

A morphism between two k -graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \rightarrow \Lambda_2$ satisfying $d_2(f(\lambda)) = d_1(\lambda)$ for all $\lambda \in \Lambda_1$.

Notation 2.3. (see [34, §2]) For $0 \leq m \leq n \leq d(\lambda)$, by the factorization property we have $\lambda = \lambda(0, m)\lambda(m, n)\lambda(n, d(\lambda))$ where $d(\lambda(0, m)) = m$, $d(\lambda(m, n)) = n - m$ and $d(\lambda(n, d(\lambda))) = d(\lambda) - n$.

Though originally called a $*$ -representation of Λ in [22, Definitions 1.4], nowadays we call the relations satisfied by the generators of $C^*(\Lambda)$ the Cuntz-Krieger relations.

Definition 2.4. Let Λ be a row-finite k -graph with no sources. A *Cuntz-Krieger* Λ -family in a C^* -algebra B consists of a family of partial isometries $\{t_\lambda : \lambda \in \Lambda\}$ satisfying the *Cuntz-Krieger relations*:

- (a) $\{t_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections,
 (b) $t_{\lambda\mu} = t_\lambda t_\mu$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$,
 (c) $t_\lambda^* t_\lambda = t_{s(\lambda)}$,
 (d) $t_v = \sum_{\lambda \in v\Lambda^m} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

Remark 2.5. As mentioned in [22, 33], given a row-finite k -graph Λ with no sources, there is a C^* -algebra $C^*(\Lambda)$ generated by a universal Cuntz-Krieger Λ -family $\{t_\lambda : \lambda \in \Lambda\}$ with $t_\lambda \neq 0$ for all $\lambda \in \Lambda$ on a separable Hilbert space.

By the universal property of $C^*(\Lambda)$, there is a strongly continuous action of the k -torus \mathbb{T}^k , called the gauge action, $\gamma : \mathbb{T}^k \rightarrow \text{Aut } C^*(\Lambda)$ defined for $z = (z_1, \dots, z_k) \in \mathbb{T}^k$ and $s_\lambda \in C^*(\Lambda)$ by $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$, where $z^m = z_1^{m_1} \dots z_k^{m_k}$ for $m = (m_1, \dots, m_k) \in \mathbb{N}^k$.

Definition 2.6. [22, Definitions 2.1]. Let Λ be a row-finite k -graph with no sources. The set $\Lambda^\infty = \{x : \Omega_k \rightarrow \Lambda : x \text{ is a } k\text{-graph morphism}\}$ is called the *infinite path space* of Λ . For $x \in \Lambda^\infty$ and $v \in \Lambda^0$, we put $v\Lambda^\infty = \{x \in \Lambda^\infty : x(0, 0) = v\}$.

Remark 2.7. For $\lambda \in \Lambda$, let $Z(\lambda) = \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\}$. Then $\{Z(\lambda) : \lambda \in \Lambda\}$ forms a basis of compact and open sets for a topology on Λ^∞ . For $p \in \mathbb{N}^k$, the shift map $\sigma^p : \Lambda^\infty \rightarrow \Lambda^\infty$ defined by $\sigma^p x(m, n) = x(m + p, n + p)$ for $(m, n) \in \Omega_k$ is a local homeomorphism (for more details see [22, Remark 2.5, Lemma 2.6]).

Since we will be dealing with gauge invariant ideals in $C^*(\Lambda)$, we must work with k -graphs which satisfy certain aperiodicity conditions, such as the one given below:

Definition 2.8. [35, Lemma 3.2 (iv)]. Let Λ be a row-finite k -graph with no sources. We say that Λ has *no local periodicity* at $v \in \Lambda^0$ if for each pair $m \neq n \in \mathbb{N}^k$, there is a path $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$(1) \quad \lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

The k -graph Λ is said to be *aperiodic* (or satisfies the *aperiodicity condition*) if every vertex has no local periodicity.

Recall that a loop is a path $\mu \in \Lambda$ such that $s(\mu) = r(\mu)$. In [25, §3] a directed graph is said to satisfy condition (L) if every loop has an exit, which is the analogue of condition (I) in [9]. So the analogue of condition (L) for 1-graphs is that every loop has an entrance: given $\mu \in v\Lambda^p v$ for some $v \in \Lambda^0$ there is $\kappa \in v\Lambda^p$ with $\mu \neq \kappa$. The following result is common knowledge but we have not been able to find a proof.

Lemma 2.9. *For row-finite 1-graphs with no sources, the aperiodicity condition is equivalent to condition (L).*

Proof. Suppose that a Λ is an aperiodic 1-graph with no sources and that $\mu \in v\Lambda^p v$ for some $v \in \Lambda^0$. Let $m = 0$ and $n = p$. Since there is no local periodicity at v and there are no sources, we may find $\lambda \in v\Lambda^{tp}$ for some $t > 1$ such that

$$\lambda(0, t(p-1)) = \lambda(0, tp-p) \neq \lambda(p, p+tp-p) = \lambda(p, tp)$$

which means that μ^t cannot be the only path with range v of length tp and hence μ must have an entrance.

Suppose that every loop in Λ has an entrance. Choose $v \in \Lambda^0$ and $m \not\leq n \in \mathbb{N}$. If v is not connected to a loop then (1) holds for every $\lambda \in v\Lambda$. Hence without loss of generality we may assume that v is connected to a loop; moreover also that v lies on the loop, $\mu \in v\Lambda^p v$. Suppose that $p = 1$, then since the loop has an entrance $e \neq \mu(0, 1)$ the path $\lambda = \mu^n e$ satisfies (1), so we may assume $p > 1$. We may also, without loss of generality, assume that the vertices $\mu(q, q)$ are distinct for all $0 \leq q \leq p-1$. If $n-m$ is not a multiple of p then $\lambda = \mu^t$ will satisfy (1) where $tp > n$ since $\lambda(m, m) \neq \lambda(n, n)$.

Finally, suppose that $n = m + tp$ where $t \geq 1$ and let q be such that $qp \leq n < (q+1)p$. Since μ has an exit there must be a path $\kappa \in v\Lambda^p$ such that $\kappa \neq \mu$. Then one checks that $\lambda = \mu^q \kappa$ satisfies (1). \square

Examples 2.10. For $n \geq 2$ the 1-graphs B_n are aperiodic. Since it has no loops it follows that Ω_1 is aperiodic.

Remark 2.11. An aperiodicity condition for k -graphs was first introduced in [22, Definition 4.3] in terms of infinite paths: A row-finite k -graph Λ satisfies condition (A) if for every vertex $v \in \Lambda^0$, there is an infinite path $x \in v\Lambda^\infty$ such that $\sigma^m(x) \neq \sigma^n(x)$ for all $m \neq n \in \mathbb{N}^k$. There are several other definitions for aperiodicity in the literature such as condition (B) of [33, Theorem 4.3] and the condition described in [35, Definition 1]. It is shown in [35, Lemma 3.2] that all these definitions are equivalent. The benefit of Definition 2.8 is that it is given in terms of Λ directly which makes it a lot easier to work with.

Definition 2.12. Let Λ be a row-finite k -graph with no sources. Define a relation \leq on Λ^0 by $v \leq w$ if and only if $v\Lambda w \neq \emptyset$.

- (a) We say that a subset H of Λ^0 is *hereditary* if $v \in H$ and $v \leq w$ imply that $w \in H$.
- (b) We say that a set $H \subseteq \Lambda^0$ is *saturated* if for all $v \in \Lambda^0$

$$r^{-1}(v) \neq \emptyset \text{ and } \{s(\lambda) : \lambda \in v\Lambda^{e_i}\} \subset H \text{ for some } i \in \{1, \dots, k\} \implies v \in H.$$

The *saturation* of a set H is the smallest saturated subset \overline{H} of Λ^0 containing H .

Remark 2.13. The notion of a saturated hereditary set was first given in [8] and adapted for directed graphs in [24, §6]. The definition of a saturated set for a k -graph was first introduced in [33, §5], but the relation \leq is defined differently. Our definition is adapted from the one given in [40, Definition 3.1] which was given in a more general setting.

The following technical lemma is used in the proof of our main Theorem (Theorem 3.12). We use it to identify when a vertex lies in the saturation of a hereditary set of vertices. Its proof is very similar to the one for a directed graph given in [3, Lemma 6.2]. We include its proof as we need an additional argument because of new saturation condition (b) of Definition 2.12 for a k -graph.

Lemma 2.14. *Suppose that Λ is a row-finite k -graph with no sources and $v \in \Lambda^0$. If $y \in \overline{\{x \in \Lambda^0 : v \leq x\}}$, then there exists $z \in \Lambda^0$ such that $v \leq z$ and $y \leq z$.*

Proof. Let $L_v := \{x \in \Lambda^0 : v \leq x\}$. First note that L_v is hereditary since if $y \in L_v$ and $y \leq z$ then $v \leq y$, which implies that $v \leq z$. Thus, $z \in L_v$. So its saturation is by definition the smallest saturated set containing L_v . Now suppose that K is any saturated set containing L_v . Then $K_1 := \{w \in K : w \leq x \text{ for some } x \in L_v\}$ contains L_v . To show that K_1 is saturated, we suppose that $z \in \Lambda^0$ and $\{s(\lambda) : \lambda \in z\Lambda^{e_i}\} \subset K_1$ for some $i \in \{1, \dots, k\}$. Then $K_1 \subset K$ by definition, so $z \in K$ since K is saturated. We know that there exists λ such that $r(\lambda) = z$ and $s(\lambda) \in K_1$, so $s(\lambda) \leq x$ for some $x \in L_v$. Also, $z = r(\lambda) \leq s(\lambda)$ implies that $z \leq x$ for some $x \in L_v$. Hence, $z \in K_1$

and K_1 is saturated. Thus, if K is the smallest saturated set containing L_v , then $K = \{w \in K : w \leq x \text{ for some } x \in L_v\}$.

If $y \in L_v$, then $v \leq y$. So any $z \in \Lambda^0$ satisfying $y \leq z$ will give the desired property. If $y \in \overline{L_v} \setminus L_v$, then $\{s(\lambda) : \lambda \in y\Lambda^{e_i}\} \subset L_v$ for some $i \in \{1, \dots, k\}$. So there exists $z \in L_v$, i.e. $v \leq z$ such that $y \leq z$, which proves the statement. \square

Remark 2.15. Let $C^*(\Lambda)$ be the C^* -algebra for a row-finite k -graph Λ with no sources. For an ideal I in $C^*(\Lambda)$, we let $H_I = \{v \in \Lambda^0 : t_v \in I\}$. Also, for each subset H of Λ^0 , let I_H denote the ideal in $C^*(\Lambda)$ generated by $\{t_v : v \in H\}$. Then, following Lemma 4.3 in [3], it can be shown that for a saturated hereditary subset H of Λ ,

$$I_H = \overline{\text{span}}\{t_\alpha t_\beta^* : \alpha, \beta \in \Lambda \text{ and } s(\alpha) = s(\beta) \in H\}.$$

In particular, I_H is gauge invariant in the sense that $\gamma_z(a) = a$ for all $a \in I_H$ and $z \in \mathbb{T}^k$. Moreover, every gauge-invariant ideal is of this form.

The following Theorem gives a complete description of the gauge invariant ideals of $C^*(\Lambda)$.

Theorem 2.16. [40, Theorem 5.5], [33, Theorem 5.2]. *Let Λ be a row-finite k -graph with no sources. Let H be a subset of Λ^0 and I be an ideal in $C^*(\Lambda)$. Let I_H and H_I be as defined above.*

- (a) *If I is non-zero gauge invariant ideal of $C^*(\Lambda)$, then H_I is non-empty saturated hereditary subset and $I = I_{H_I}$.*
- (b) *If H is a saturated and hereditary subset of Λ^0 and I_H is the associated ideal, then $H = H_{I_H}$.*
- (c) *The map $H \mapsto I_H$ is an isomorphism of the lattice of saturated hereditary subsets of Λ^0 onto the lattice of closed gauge-invariant ideals of $C^*(\Lambda)$.*
- (d) *Suppose $H \neq \Lambda^0$ is saturated and hereditary. Then*

$$\Gamma(\Lambda \setminus H) := (\Lambda^0 \setminus H, \{\lambda \in \Lambda : s(\lambda) \notin H\}, r, s)$$

is a row-finite k -graph with no sources, and $C^(\Lambda)/I_H$ is canonically isomorphic to $C^*(\Gamma(\Lambda \setminus H))$.*

Proof. The proofs for the generalized case of (a) and (b) are given in [40, Theorem 5.5]. Also, (c) and (d) are shown in [33, Theorem 5.2] for locally convex row-finite k -graphs, so they are certainly true for row-finite k -graphs with no sources. \square

Remark 2.17. Note that Λ^0, \emptyset are always saturated hereditary subsets of Λ^0 , corresponding to the trivial ideals $C^*(\Lambda), \{0\}$ of $C^*(\Lambda)$ respectively. This observation allows us to complete the analysis of Theorem 2.16 (d) for the case $H = \Lambda^0$.

3. THE PRIMITIVE IDEAL SPACE OF $C^*(\Lambda)$

In this section, we describe and completely characterize the primitive ideal space of the C^* -algebra of a k -graph Λ all of whose ideals are gauge invariant. Since $C^*(\Lambda)$ is separable (see Remark 2.5) it follows from [30, Proposition 3.13.10 and Proposition

4.3.6] that every primitive ideal is prime and vice versa. We shall therefore use the terms primitive and prime interchangeably.

We begin in Proposition 3.3 where we give a condition on Λ which ensures that all ideals are gauge invariant. Then, in Proposition 3.9 we show how it may be possible to check this condition. Next in Theorem 3.12 we give necessary and sufficient conditions on a gauge invariant ideal to be prime when all ideals are gauge invariant. Finally, in Theorem 3.15 we describe the topology of $\text{Prim } C^*(\Lambda)$.

Definition 3.1. Let Λ be a row-finite k -graph with no sources. We say that Λ is *strongly aperiodic* if $\Gamma(\Lambda \setminus H)$ satisfies aperiodicity condition for all saturated hereditary subsets $H \subsetneq \Lambda^0$.

Note that by taking $H = \emptyset$ we see that if a row-finite k -graph Λ with no sources is strongly aperiodic, then it is automatically aperiodic.

A loop μ in a 1-graph Λ is said to be *simple* if the vertices $\{\mu(i, i) : 0 \leq i \leq d(\mu) - 1\}$ are distinct. In [24, §6], a directed graph is said to satisfy condition (K) if every vertex is the range of either two distinct simple loops or none; that is for all $v \in \Lambda^0$ either $v\Lambda v = \emptyset$ or there are simple loops $\mu, \nu \in v\Lambda v$ such that $\mu \neq \nu$. When considering 1-graphs, the analogue of condition (K) is the same. Parts of the proof of the following result can be found in [4] for directed graphs (see also [24, Remark 6.11]). However, the result has not been stated in this form, so we give the proof for completeness and to complement the proof of Lemma 2.9.

Lemma 3.2. *For row-finite 1-graphs with no sources, strong aperiodicity is equivalent to condition (K).*

Proof. Let Λ be a strongly aperiodic 1-graph. Suppose, for contradiction, that Λ does not satisfy condition (K). Then there is a vertex $v \in \Lambda^0$ which has only one simple loop based at $v \in \Lambda^0$. Let $\mu \in v\Lambda^p v$ for some $p > 0$. Let $X = \{s(\nu) \mid \nu \neq \mu, \nu \in v\Lambda^q\}$. It is clear that $v \notin X$ since v only has one simple loop μ . Let $L_X = \{u \in \Lambda^0 \mid x \leq u \text{ for some } x \in X\}$, then L_X is hereditary. If $x\Lambda v \neq \emptyset$ for some $x \in L_X$, then v is the range of two distinct loops and so Λ satisfies condition (K), so $x\Lambda v = \emptyset$ for all $x \in L_X$. Since v does not connect to any vertex in L_X , it follows that $v \notin \overline{L_X}$. Moreover, $\Gamma(\Lambda \setminus \overline{L_X})$ contains v and μ . But $v\Gamma(\Lambda \setminus \overline{L_X})^q = \{\mu\}$ by construction. Hence by Lemma 2.9, the 1-graph $\Gamma(\Lambda \setminus \overline{L_X})$ does not satisfy condition (L) and so is not aperiodic. This contradicts Λ being strongly aperiodic. Thus strong aperiodicity implies condition (K).

Let Λ be a 1-graph which satisfies condition (K). Suppose for contradiction, that Λ is not strongly aperiodic. Then there is a saturated hereditary subset $H \subsetneq \Lambda^0$ such that $\Gamma(\Lambda \setminus H)$ is not aperiodic. By Lemma 2.9 there is $v \in \Gamma(\Lambda \setminus H)^0$ and $\mu \in v\Gamma(\Lambda \setminus H)^p v$ such that $|v\Gamma(\Lambda \setminus H)^q| = 1$ for all $q \geq 1$. Replacing μ by a sub-path if necessary we may, without loss of generality, assume that μ is a simple loop. Since Λ itself satisfies condition (K) there is a simple loop $\nu \in v\Lambda^q v$ for some $q \geq 1$ such that $\mu \neq \nu$. By definition of $\Gamma(\Lambda \setminus H)$ it follows that $\nu \in \Gamma(\Lambda \setminus H)$ since $v = s(\nu) \notin H$. Suppose that $q < p$ then since $|\Gamma(\Lambda \setminus H)^q| = 1$ it follows that $\mu(0, q) = \nu(0, q)$. Since

$r(\nu) = \mu(0, q) = v$ it follows that μ is not simple and a contradiction. We obtain a similar contradiction if $p > q$. If $p = q$ then since $|v\Gamma(\Lambda \setminus H)^q| = 1$ it follows that $\mu = \nu$ which contradicts the assumption that $\mu \neq \nu$. Hence in all cases, the assumption must have been false, and so Λ is strongly aperiodic. \square

Proposition 3.3. *Let Λ be a row-finite strongly aperiodic k -graph with no sources, then all ideals of $C^*(\Lambda)$ are gauge invariant.*

Proof. If Λ is strongly aperiodic, then one can check that Λ satisfies condition (D) of [40, Definition 7.1]. The result follows from [40, Theorem 7.2]. \square

We now look for conditions on a 2-graph which ensure that it is strongly aperiodic. First we need a definition.

Definition 3.4. Let Λ be a 2-graph and $u \in \Lambda^0$, suppose that there are distinct $\alpha_i \in u\Lambda^{ae_1}u$ and distinct $\beta_i \in u\Lambda^{be_2}u$ for $i = 1, 2$, where $a, b \in \mathbb{Z}^+$ such that

$$\beta_2\alpha_1 = \alpha_1\beta_2, \beta_2\alpha_2 = \alpha_2\beta_2, \beta_1\alpha_1 = \alpha_2\beta_1, \beta_1\alpha_2 = \alpha_1\beta_1.$$

Then $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is called an (a, b) -*aperiodic quartet* at u .

Theorem 3.5. *Let Λ be a row-finite 2-graph with no sources and $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ be an $(1, 1)$ -aperiodic quartet at $u \in \Lambda^0$, then there is no local periodicity at u .*

Proof. Fix $u \in \Lambda^0$, and let $m \neq n \in \mathbb{N}^2$. Suppose that $m \vee n \neq m$ or n . Let $q_1, q_2 \geq 1$ be such that $q_1 \geq (m \vee n)_1$ and $q_2 \geq (m \vee n)_2$ and define $\lambda \in u\Lambda^q$ by $\lambda = \alpha_1^{q_1}\beta_2^{q_2}$. Without loss of generality, suppose that $n_1 < m_1$ and choose $\mu = \lambda\alpha_2\beta_2$. Because of the factorisation property $\beta_2\alpha_1 = \alpha_1\beta_2$, we have $\mu = \alpha_1^{q_1}\beta_2^{q_2}\alpha_2\beta_2 = \alpha_1^{m_1}\beta_2^{m_2}\alpha_1^{q_1-m_1}\beta_2^{q_2-m_2}\alpha_2\beta_2 = \alpha_1^{n_1}\beta_2^{n_2}\alpha_1^{q_1-n_1}\beta_2^{q_2-n_2}\alpha_2\beta_2$. Since $q_1 - m_1 < q_1 - n_1$ we have

$$\mu(m, m + (q_1 - m_1 + 1)e_1) = \alpha_2 \neq \alpha_1 = \mu(n, n + (q_1 - m_1 + 1)e_1),$$

and so there is no local periodicity at u .

Now suppose that $m \vee n = m$ or n . Without loss of generality suppose that $m \vee n = m$. Let $q_1, q_2 \geq 1$ be such that $q_1 \geq m_1$ and $q_2 \geq m_2$ and define $\lambda \in u\Lambda^q$ by $\lambda = \alpha_1^{q_1}\beta_2^{q_2}$. If $m > n$, then put $\mu = \lambda(\beta_1\alpha_2)^{m_2-n_2}$, then

$$\mu(m, m + (q_2 - m_2 + 1)e_2) = \beta_1 \neq \beta_2 = \mu(n, n + (q_2 - m_2 + 1)e_2),$$

and so there is no local periodicity at u . If $m_1 = n_1$ but $m_2 > n_2$ let $\lambda \in u\Lambda^q$ be defined by $\lambda = \alpha_1^{q_1}\beta_2^{q_2}$. Let $\mu = \lambda(\beta_1\alpha_2)^{m_2-n_2}$, then

$$\mu(n, n + (q_2 - m_2 + 1)e_2) = \beta_2 \neq \beta_1 = \mu(m, m + (q_2 - m_2 + 1)e_2),$$

and so there is no local periodicity at u . If $m_2 = n_2$ but $m_1 > n_1$ then similar argument applies using $\mu = \lambda(\beta_1\alpha_1)^{m_1-n_1}$, then the proof is complete. \square

Once we have checked no local aperiodicity at a vertex, we are able to deduce that many other vertices have no local aperiodicity.

Lemma 3.6. *Let Λ be a row-finite k -graph with no sources. Suppose that there is no local periodicity at $u \in \Lambda^0$ and that $v \leq u$, then there is no local periodicity at v .*

Proof. Let $\kappa \in v\Lambda u$, and $m \neq n \in \mathbb{N}^k$. Let $p = m - m \wedge n$ and $q = n - m \wedge n$, then $p \neq q \in \mathbb{N}^k$. Since u has no local periodicity, there is $\mu \in u\Lambda^t$, where $t > p \vee q$ such that

$$(2) \quad \mu(p, p + d(\mu) - p \vee q) \neq \mu(q, q + d(\mu) - p \vee q).$$

Let $\lambda = \kappa\mu$ then we have

$$\begin{aligned} \lambda(d(\kappa) + m - m \wedge n, d(\kappa) + m - m \wedge n + d(\mu) - p \vee q) \\ &= \mu(p, p + d(\mu) - p \vee q) \\ &\neq \mu(q, q + d(\mu) - p \vee q) \\ &= \lambda(d(\kappa) + n - m \wedge n, d(\kappa) + n - m \wedge n + d(\mu) - p \vee q). \end{aligned}$$

So we have

$$\lambda(m, m + d(\lambda) - m \vee n) \neq \lambda(n, n + d(\lambda) - m \vee n),$$

since

$$0 \leq d(\kappa) - m \wedge n \leq d(\kappa) - m \wedge n + d(\mu) - p \vee q \leq d(\lambda) - m \vee n$$

and the result follows. \square

Pulling the previous two results together we have the following two Propositions.

Proposition 3.7. *Let Λ be a row-finite 2-graph with no sources. Suppose that every vertex is connected to a vertex with an aperiodic quartet, then Λ is aperiodic.*

Proof. Follows by Theorem 3.5 and Lemma 3.6. \square

Example 3.8. As seen in Example 2.2 (c), let $m, n \geq 1$ and $\theta : \underline{m} \times \underline{n}$ be a bijection such that there are $i \neq i' \in \underline{m}$ and $j \neq j' \in \underline{n}$ such that $\theta(i, j) = (i', j)$, $\theta(i', j) = (i, j)$, $\theta(i, j') = (i, j')$ and $\theta(i', j') = (i', j')$ then $f_i, f_{i'}, g_j, g_{j'}$ is a $(1, 1)$ -aperiodic quartet and so \mathbb{F}_θ^2 is aperiodic by Proposition 3.7. More generally, for $a, b \geq 1$ the bijection θ induces a bijection $\tilde{\theta} : \underline{m}^a \times \underline{n}^b \rightarrow \underline{m}^a \times \underline{n}^b$. If there are $i \neq i' \in \underline{m}^a$ and $j \neq j' \in \underline{n}^b$ such that $\theta(i, j) = (i', j)$, $\theta(i', j) = (i, j)$, $\theta(i, j') = (i, j')$ and $\theta(i', j') = (i', j')$ then $f_i, f_{i'}, g_j, g_{j'}^1$ is an (a, b) -aperiodic quartet and so \mathbb{F}_θ^2 is aperiodic by Proposition 3.7. These results are compatible with [11, Theorem 3.4]. Indeed, when \mathbb{F}_θ^2 is aperiodic we have $C^*(\mathbb{F}_\theta^2) \cong \mathcal{O}_m \otimes \mathcal{O}_n$ (cf. [44, §5]).

Proposition 3.9. *Let Λ be a row-finite 2-graph with no sources. Suppose that every vertex has an aperiodic quartet, then Λ is strongly aperiodic.*

Proof. Let H be a hereditary subset of Λ^0 . Since every vertex in Λ^0 has an aperiodic quartet, there is an aperiodic quartet $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ at any $v \in \Lambda^0 \setminus H$. Since the source and range of α_i and β_i for $i = 1, 2$ are the vertex $v \in \Lambda^0 \setminus H$, $\alpha_i, \beta_i \in \Gamma(\Lambda \setminus H)$ for $i = 1, 2$. Thus, by Proposition 3.7 $\Lambda^0 \setminus H$ is aperiodic, which proves that Λ is strongly aperiodic. \square

¹where for $i \in \underline{m}^a$ we have $f_i = f_{i_1} \dots f_{i_a}$ and similarly for $j \in \underline{n}^b$ we have $g_j = g_{j_1} \dots g_{j_b}$

Now we turn our attention to the description of all the prime ideals in $C^*(\Lambda)$. we update the definition of a maximal tail given in [3, Proposition 6.1] into the context of row-finite k -graphs with no sources.

Definition 3.10. Let Λ be a row-finite k -graph with no sources. A nonempty subset γ of Λ^0 is called *maximal tail* if

- (a) for every $v_1, v_2 \in \gamma$ there is $w \in \gamma$ such that $v_1\Lambda w \neq \emptyset$ and $v_2\Lambda w \neq \emptyset$,
- (b) for every $v \in \gamma$ and $1 \leq i \leq k$ there is $e \in v\Lambda^{e_i}$ such that $s(e) \in \gamma$, and
- (c) for $w \in \gamma$ and $v \in \Lambda^0$ with $v\Lambda w \neq \emptyset$ we have $v \in \gamma$.

Remark 3.11. Versions of condition (a) in Definition 3.10 can be traced back to [2, Theorem 3.8 (iii)] and [12, Lemma 3.1 (iii)]. The word “tail” in the above definition is meant to convey the sense of conditions (a) and (b), and “maximal” to convey that of condition (c). Also the notion of maximal tail for a k -graph was introduced in Sims’ thesis [39, Proposition 5.5.3], but the condition (b) in the above definition is stronger than the condition (MT2) in [39] since not every finite exhaustive set contains edges $e \in v\Lambda^{e_i}$ for all $1 \leq i \leq k$.

Now we state the main theorem of this paper, which is a generalization of [3, Proposition 6.1]. We give complete proof because of the new condition (b) in the definition of the maximal tail of a k -graph.

Theorem 3.12. *Let Λ be a row-finite strongly aperiodic k -graph with no sources. Let $H \subsetneq \Lambda^0$ be a saturated hereditary subset. Then I_H is primitive if and only if $\gamma := \Lambda^0 \setminus H$ is a maximal tail.*

Proof. First suppose that $\gamma \in \Lambda^0$ is a maximal tail. Let $H = \Lambda^0 \setminus \gamma$, so $H \neq \Lambda^0$. To see that H is hereditary: let $v \in H$ and suppose $v \leq w$. If $w \in \gamma$, then $v \in \gamma$ by (c). Thus, $w \in H = \Lambda^0 \setminus \gamma$. To see that H is saturated: let $v \in \Lambda^0$ such that $\{s(\lambda) : \lambda \in v\Lambda^{e_i}\} \subset H$ for some $i \in \{1, \dots, k\}$. If $v \in \gamma$, then there are $\lambda \in v\Lambda^{e_i}$ for all i such that $s(\lambda) \in \gamma$ by (b). But this contradicts $\{s(\lambda) : \lambda \in v\Lambda^{e_i}\} \subset H$ for some $i \in \{1, \dots, k\}$. Thus, $v \in H = \Lambda^0 \setminus \gamma$. Now to show that I_H is prime, suppose I_1 and I_2 are ideals of $C^*(\Lambda)$ and $I_1 \cap I_2 \subset I_H$. Since Λ is strongly aperiodic, every ideal is gauge invariant by Proposition 3.3. Then by Theorem 2.16 (a) and (c), there exist corresponding saturated hereditary subsets H_1 and H_2 such that $I_1 = I_{H_1}$, $I_2 = I_{H_2}$ and $I_{H_1 \cap H_2} = I_{H_1} \cap I_{H_2}$. Thus, $I_1 \cap I_2 \subset I_H$ implies that $H_1 \cap H_2 \subset H$. We claim that $H_1 \subset H$ or $H_2 \subset H$. Suppose that $H_1 \not\subset H$ and $H_2 \not\subset H$, then there exist $v_1 \in H_1 \setminus H$, $v_2 \in H_2 \setminus H$. i.e. $v_1 \in \gamma$ and $v_2 \in \gamma$. By (a), there is $v \in \gamma$ such that $v_1 \leq v$ and $v_2 \leq v$. Since H_1 and H_2 are hereditary, $v \in H_1$ and $v \in H_2$. So $v \in H_1 \cap H_2 \subset H = \Lambda^0 \setminus \gamma$ which contradicts $v \in \gamma$. Thus, $H_1 \subset H$ or $H_2 \subset H$ that implies that $I_1 = I_{H_1} \subset I_H$ or $I_2 = I_{H_2} \subset I_H$. Hence I_H is prime.

Now suppose H is saturated and hereditary and I_H is primitive. Let $\gamma = \Lambda^0 \setminus H$, then γ satisfies (c) : suppose not then $v \leq w$ and $w \in \gamma$. If $v \notin \gamma$, i.e. $v \in H$, then $w \in H$ since H is hereditary, which contradicts $w \in \gamma$. So $v \in \gamma$. To show that γ satisfies (b), let $v \in \gamma$. Since Λ has no sources, $v\Lambda^{e_i} \neq \emptyset$ for all $i \in \{1, \dots, k\}$.

So there are $\lambda \in v\Lambda^{e_i}$ for all $i \in \{1, \dots, k\}$. Suppose $\{s(\lambda) : v\Lambda^{e_i}\} \subset H$ for some $i \in \{1, \dots, k\}$, then $v \in H$ since H is saturated, which contradicts $v \in \gamma$. Thus, there are $\lambda \in v\Lambda^{e_i}$ such that $s(\lambda) \in \gamma$ for all $i \in \{1, \dots, k\}$. To prove (a), recall that for a hereditary saturated set H , $C(\Lambda)/I_H$ is isomorphic to $C^*(\Gamma(\Lambda \setminus H))$ by Theorem 2.16 (d). Because I_H is primitive in $C^*(\Lambda)$, $\{0\}$ is primitive in $C^*(\Gamma(\Lambda \setminus H))$. Suppose that $v_1, v_2 \in \Lambda^0 \setminus H$. Then $H_i = \{x \in \Lambda^0 \setminus H : v_i \leq x\}$ are non-empty hereditary subsets of $\Lambda^0 \setminus H = \Gamma(\Lambda \setminus H)^0$. Since $\{0\}$ is prime in $C^*(\Gamma(\Lambda \setminus H))$, we must have $I_{\overline{H_1}} \cap I_{\overline{H_2}} \neq \{0\}$. If $I_{\overline{H_1}} \cap I_{\overline{H_2}} = \{0\}$, then $I_{\overline{H_1}} \cap I_{\overline{H_2}} \subseteq \{0\}$ and the fact that $\{0\}$ is prime implies that $I_{\overline{H_1}} \subseteq \{0\}$ or $I_{\overline{H_2}} \subseteq \{0\}$. Hence, $I_{\overline{H_1}} = \{0\}$ or $I_{\overline{H_2}} = \{0\}$, but $I_{H_1} \neq \{0\}$ since $H_1 \neq \emptyset$. Thus, $I_{\overline{H_1}} \cap I_{\overline{H_2}} \neq \{0\}$ implies $\overline{H_1} \cap \overline{H_2} \neq \emptyset$ by Theorem 2.16 (c). Say $y \in \overline{H_1} \cap \overline{H_2}$, then $y \in \overline{H_1} = \{x \in \Lambda^0 \setminus H : v_1 \leq x\}$. By Lemma 2.14, there is $x \in \Lambda^0 \setminus H$ such that $y \leq x$ and $v_1 \leq x$. Since $y \in \overline{H_2}$ and $\overline{H_2}$ is hereditary, we have $x \in \overline{H_2} = \{x \in \Lambda^0 \setminus H : v_2 \leq x\}$. Applying Lemma 2.14 again on x and $\overline{H_2}$, we have $z \in \Lambda^0 \setminus H$ such that $x \leq z$ and $v_2 \leq z$. So we have $v_1 \leq x \leq z$ and $v_2 \leq z$ in $\Gamma(\Lambda^0 \setminus H)$. Thus $\gamma = \Lambda^0 \setminus H$ satisfies (a). \square

Remark 3.13. Observe that the strongly aperiodic condition on Λ was not used in the second half of the proof of Theorem 3.12. Hence, for any row finite k -graph with no sources if the ideal I_H is primitive, then $\gamma = \Lambda^0 \setminus H$ is a maximal tail. This is proved for arbitrary finitely aligned k -graph in [39, Proposition 5.5.3].

Notation 3.14. Let Λ be a row-finite k -graph with no sources, then we denote the set of maximal tails of Λ by χ_Λ .

For nonempty subsets K, L of Λ^0 , we write $K \leq L$ to mean that for each $v \in K$, there exists $w \in L$ such that $v \leq w$. Thus condition (c) of Definition 3.10 says that “ $\{v\} \leq \gamma$ implies $v \in \gamma$ ”. Also, we can describe the saturated hereditary set H_γ corresponding to $\gamma \in \chi_\Lambda$ as either $H_\gamma = \Lambda^0 \setminus \gamma$ or $H_\gamma = \{v : \{v\} \not\leq \gamma\}$, (as in [1, Proposition 4.1]).

The following description of the Jacobson topology of the primitive ideal space of $C^*(\Lambda)$ is a generalization of [3, Theorem 6.3] which has an identical proof, so we omit it (note that this topology is T_0 (see [30, 4.1.4])).

Theorem 3.15. *Let Λ be a row-finite strongly aperiodic k -graph with no sources. Then there is a topology \mathcal{T}_0 on the set χ_Λ of maximal tails in Λ such that*

$$\overline{S} = \{\delta \in \chi_\Lambda : \delta \subseteq \bigcup_{\gamma \in S} \gamma\}$$

for $S \subset \chi_\Lambda$, and then $\gamma \mapsto I_{H_\gamma}$ is a homeomorphism of χ_Λ onto $\text{Prim } C^*(\Lambda)$.

Following [2], we can describe an equivalent topology on χ_Λ which will be useful in the next section.

Lemma 3.16. *Let Λ be a row-finite strongly aperiodic k -graph with no sources. For $v \in \Lambda^0$, let*

$$S(v) := \{\chi \in \chi_\Lambda : v \in \chi\}.$$

Then, $\{S(v) : v \in \Lambda^0\}$ form a countable base of open sets for a topology \mathcal{T}_1 on χ_Λ . Moreover, the topologies \mathcal{T}_0 and \mathcal{T}_1 on χ_Λ are equal.

4. CHARACTERIZING $\text{Prim } C^*(\Lambda)$ FOR A STRONGLY APERIODIC k -GRAPH Λ

Definition 4.1. [5]. A closed subset C of a topological space is *irreducible* if it cannot be written as the union of two proper closed subsets of itself. A *spectral space* is a T_0 space in which every irreducible set is the closure of a point.

The main theorem in this section describes precisely which topological spaces can occur as the primitive ideal space of the C^* -algebra of a strongly aperiodic row-finite k -graph with no sources, and generalizes [2, Theorem 4.2].

Theorem 4.2. *Let X be a topological space. Then X is homeomorphic to $\text{Prim } C^*(\Lambda)$ for a row-finite strongly aperiodic k -graph with no sources if and only if X is a spectral space in which the compact open sets form a countable base.*

We first characterize the irreducible subsets of χ_Λ . The proof of following result is the same as that of [2, Lemma 4.3].

Lemma 4.3. *Let Λ be a row-finite strongly aperiodic k -graph with no sources and let $C \subseteq \chi_\Lambda$ be a closed set. Write $\chi_C = \cup_{\chi \in C} \chi$. Then the following statements are equivalent:*

- (a) *the set C is irreducible.*
- (b) *the set χ_C is a maximal tail.*

If these conditions are satisfied, then for all $v \in \Lambda^0$, $\chi_C \in S(v)$ if and only if $S(v) \cap C \neq \emptyset$.

The proof of the following Proposition is the same as that of [2, Proposition 4.4].

Proposition 4.4. *Let Λ be a row-finite strongly aperiodic k -graph with no sources. Then χ_Λ has the property that every irreducible set is the closure of a point in χ_Λ .*

Proposition 4.4 shows that χ_Λ is a spectral space. We now turn our attention to finding a base of compact open sets using the topology described in Lemma 3.16.

We prove a technical lemma first. It shows that every vertex in a maximal tail can be reached by a path of strictly positive degree from some other vertex in the same maximal tail.

Lemma 4.5. *Let Λ be a row-finite k -graph with no sources. Let $\chi \in \chi_\Lambda$ be a maximal tail, then for all $v \in \chi$ there is $\mu \in v\Lambda$ such that $d(\mu) > 0$ and $s(\mu) \in \chi$. (i.e. $d(\mu)_i > 0$ for $i = 1, \dots, k$).*

Proof. Fix $v \in \chi$. Since χ is a maximal tail, there is $\lambda \in v\Lambda^{e_i}$ such that $s(\lambda) \in \chi$ for all $i = 1, \dots, k$. So let $\lambda_1 \in v\Lambda^{e_1}$ such that $s(\lambda_1) \in \chi$. Let $u_1 = s(\lambda_1)$. Then, let $\lambda_2 \in u_1\Lambda^{e_2}$ such that $s(\lambda_2) \in \chi$. Continuing this process, we find $\lambda_i \in \Lambda^{e_i}$ such that $s(\lambda_i) \in \chi$ and $r(\lambda_{i+1}) = s(\lambda_i)$ for all $i = 1, \dots, k$. Let $\mu = \lambda_1 \dots \lambda_k \in v\Lambda$. Then the factorization property implies $d(\mu)_i > 0$ for all $i = 1, \dots, k$. \square

The following Proposition is a generalization of [2, Lemma 4.5], whose proof is much more intricate due to the complex topology of Λ^∞ and the new condition (b) in the definition of the maximal tail of a k -graph.

Proposition 4.6. *Let Λ be a row-finite strongly aperiodic k -graph with no sources. Define a map $\beta : \Lambda^\infty \rightarrow \Lambda^0$ by*

$$\beta(x) = \{v \in \Lambda^0 : v\Lambda x(n, n) \neq \emptyset \text{ for some } n \in \mathbb{N}^k\}.$$

Then

(A) $\beta(x)$ is a maximal tail.

(B) β is continuous open surjection and χ_Λ is a quotient space of Λ^∞ .

(C) The open sets $\{S(v) : v \in \Lambda^0\}$ are compact subsets of χ_Λ .

Proof. (A): To check the condition (a) of maximal tail, for $v_1, v_2 \in \beta(x)$, we want to find $z \in \beta(x)$ such that $v_1 \leq z$ and $v_2 \leq z$. Since $v_1 \in \beta(x)$, there is $n_1 \in \mathbb{N}^k$ such that $v_1 \leq x(n_1, n_1)$. Similarly, there is $n_2 \in \mathbb{N}^k$ such that $v_2 \leq x(n_2, n_2)$. Let $N = \max(n_1, n_2)$, and let $z = x(N, N)$. Then, $v_1 \leq x(n_1, n_1) \leq x(N, N) = z$ and $v_2 \leq x(n_2, n_2) \leq x(N, N) = z$. Thus, the condition (a) is satisfied. Now fix $v \in \beta(x)$, then there is $n \in \mathbb{N}^k$ such that $v \leq x(n, n)$. Let $\mu \in v\Lambda x(n, n)$. Since $x \in \Lambda^\infty$, there is $x(n, n + e_i) \in x(n, n)\Lambda x(n + e_i, n + e_i)$ for all $1 \leq i \leq k$. Then the factorization property implies that there is $\lambda_i \in v\Lambda^{e_i}$ such that $\mu x(n, n + e_i) = \lambda_i \mu'$. Since $s(\mu') = s(x(n, n + e_i)) = x(n + e_i, n + e_i)$ and $s(\lambda_i) \leq s(\mu')$, $s(\lambda_i) \in \beta(x)$. Thus, the condition (b) is satisfied. To show that the condition (c) is satisfied, suppose $v \leq w$ and $w \in \beta(x)$. Then, there is $n \in \mathbb{N}^k$ such that $w \leq x(n, n)$. So $v \leq w \leq x(n, n)$ implies that $v \in \beta(x)$. Hence, $\beta(x)$ is a maximal tail.

(B): To show that β is continuous, we claim

$$\begin{aligned} \beta^{-1}(S(v)) &= \{x \in \Lambda^\infty : v \leq x(n, n) \text{ for some } n \in \mathbb{N}^k\} \\ (3) \quad &= \bigcup_{v \leq s(\mu)} Z(\mu). \end{aligned}$$

To show the first equality, let $x \in \Lambda^\infty$ be such that $v \leq x(n, n)$ for some $n \in \mathbb{N}^k$. Then $v \in \beta(x)$. Since $\beta(x)$ is a maximal tail, we have $\beta(x) \in S(v)$. On the other hand, let $x \in \Lambda^\infty$ such that $\beta(x) \in S(v)$. So $v \in \beta(x)$ and it gives $n \in \mathbb{N}^k$ such that $v \leq x(n, n)$. To show the second equality, take $x \in Z(\mu)$ such that $v \leq s(\mu)$. Then $x = \mu t$ for $t \in \Lambda^\infty$. Clearly $s(\mu) = x(n, n)$ for some $n \in \mathbb{N}^k$. Thus, $v \leq x(n, n)$. So x belongs to the set. Now take $x \in \Lambda^\infty$ such that $v \leq x(n, n)$ for some $n \in \mathbb{N}^k$. Let $\mu = x(0, n)$. Then $x \in Z(\mu)$ and $v \leq x(n, n) = s(\mu)$. Thus, $x \in \bigcup_{v \leq s(\mu)} Z(\mu)$, which proves the above equalities. Since $Z(\mu)$ is open set, $\beta^{-1}(S(v))$ is open. Thus, β is continuous.

To show that β is surjective, we need to consider two cases : $|\chi| < \infty$ and $|\chi| = \infty$. If $|\chi| < \infty$, then let $\chi = \{u_1, \dots, u_n\}$. In the case that $n = 1$, we apply Lemma 4.5 to obtain $\lambda \in u_1\Lambda$ such that $d(\lambda) > 0$ and $s(\lambda) \in \chi$. So $s(\lambda) = u_1$. Let x be an infinite path of the form $x = \mu\mu \dots$, then $\beta(x) = \chi$. In the case that $n > 1$, let $p_1 = u_1, u_2 \in \chi$. By applying the maximal tail condition (a) we have $p_2 \in \chi$ such that $u_1\Lambda p_2 \neq \emptyset$ and

$u_2\Lambda p_2 \neq \emptyset$. Let $\mu_1 \in u_1\Lambda p_2 = p_1\Lambda p_2$. By similar argument, we can find $p_i \in \chi$ and $\mu_i \in p_i\Lambda p_{i+1}$ for $i = 1, \dots, n-1$ such that $u_i \leq p_i$ for $i = 1, \dots, n$. Let $\delta_i \in u_i\Lambda p_i$. Then apply the maximal tail condition (b) on p_n , we find $\lambda \in p_n\Lambda^{e_i}$ for all $i = 1, \dots, k$ such that $s(\lambda) \in \chi$. Since χ is finite, $s(\lambda) = u_l$ for some $1 \leq l \leq n$. So $\phi = \lambda\delta_{l+1}\mu_{l+1} \dots \mu_n$ is a loop based on p_n since $s(\phi) = s(\mu_n) = p_n$ and $r(\phi) = r(\lambda) = p_n$. Let $x = (\phi)^\infty$. Then we certainly have $\beta(x) = \chi$.

If $|\chi| = \infty$, we write $\chi = \{v_i\}_{i=1}^\infty$. Let $p_1 = v_1$, the the maximal tail condition (a) gives $p_2 \in \chi$ such that $v_1\Lambda p_2 \neq \emptyset$ and $v_2\Lambda p_2 \neq \emptyset$. Let $\mu_1 \in v_1\Lambda p_2 = p_1\Lambda p_2$. Then by Lemma 4.5, there is $\rho_1 \in p_2\Lambda$ such that $d(\rho_1) > 0$ and $s(\rho_1) \in \chi$. Let $q_1 = s(\rho_1)$ and $\lambda_1 = \mu_1\rho_1$. Then certainly $d(\lambda_1) > 0$. Apply the similar argument to v_3 and q_1 to obtain $\lambda_2 \in s(\lambda_1)\Lambda$ such that $d(\lambda_2) > 0$ and $s(\lambda_2) \in \chi$. Then $\{v_1, v_2, v_3\} \leq (\lambda_1\lambda_2)^0$, where $(\lambda_1\lambda_2)^0$ is the set of vertices on $\lambda_1\lambda_2$. So inductively we can find $\lambda_1, \dots, \lambda_n$ such that $\{v_1, \dots, v_{n+1}\} \leq (\lambda_1 \dots \lambda_n)^0$. Let $x = \lambda_1\lambda_2 \dots$, then $\beta(x) = \chi$. Hence, β is surjective.²

To see that β is an open map, we claim the following.

$$(4) \quad \beta(Z(\mu)) = \{\chi : s(\mu) \in \chi\}.$$

Let $x = \mu t \in Z(\mu)$. Then, $s(\mu) \in \beta(x)$. Since $\beta(x)$ is a maximal tail, $\beta(x) \in \{\chi : s(\mu) \in \chi\}$. Now take $\delta \in \{\chi : s(\mu) \in \chi\}$. Then by the condition (c) of maximal tail, for $s(\mu) \in \delta$ we have $r(\mu) \in \delta$. As shown in the proof of the surjectivity of β , for $s(\mu) \in \delta$, there is $x \in s(\mu)\Lambda^\infty$ such that $\beta(x) = \delta$. Then $\mu x \in Z(\mu)$ and $\beta(\mu x) = \delta$, which proves the claim. Thus, we have $\beta(Z(\mu)) = \{\chi : s(\mu) \in \chi\} = S(s(\mu))$, which is open in χ_Λ . Therefore, β is an open map.

(C): For $v \in \Lambda^0$, the subset $Z(v) \subset \Lambda^\infty$ is compact by Lemma 2.6 in [22]. By (4) above, we have

$$S(v) = \{\chi \in \chi_\Lambda : v \in \chi\} = \beta(Z(v)).$$

Since β is continuous, $\beta(Z(v))$ is compact, which implies that $S(v)$ is compact. \square

Proof of Theorem 4.2. Let X be a spectral space in which the compact open sets form a countable base. Then by [5, Theorem 5], there is an AF-algebra \mathcal{A} such that $\text{Prim } \mathcal{A} \simeq X$. Moreover by [13, Theorem 1], \mathcal{A} is Morita equivalent to $C^*(\mathcal{D}(\mathcal{A}))$ where $\mathcal{D}(\mathcal{A})$ is a Bratteli diagram for \mathcal{A} . Hence, by [38, Corollary 3.33], we have $\text{Prim } C^*(\mathcal{D}(\mathcal{A})) \simeq \text{Prim } \mathcal{A}$. Moreover, a Bratteli diagram satisfies condition (K), hence it is strongly aperiodic by Lemma 3.2.

To prove the converse, let Λ be a row-finite strongly aperiodic k -graph with no sources. Then by Theorem 3.15 we have $\text{Prim } C^*(\Lambda) \simeq \chi_\Lambda$. It follows from Proposition 4.4 that χ_Λ is a spectral space, and by Lemma 4.6, the set $\{S(v) : v \in \Lambda^0\}$ forms a countable base of compact open sets for the topology on χ_Λ . \square

As in [2, Corollary 4.6], we may apply [5, Theorem 5] to Theorem 4.2 to get:

² The above proof suggests that when $|\chi| = \infty$, for given $v \in \chi$ we can construct an infinite path x such that $\beta(x) = \chi$ and $r(x) = v$.

Corollary 4.7. *If Λ is a row-finite strongly aperiodic k -graph with no sources, then there is an AF-algebra \mathcal{A} such that $\text{Prim } \mathcal{A} \simeq \text{Prim } C^*(\Lambda)$.*

Remark 4.8. In [2, §5], Bates gives an algorithm to construct from a directed graph E an auxiliary directed graph \tilde{E} whose C^* -algebra is AF and has the same primitive ideal space $C^*(E)$. Since we are dealing with k -graph Λ which has k -colored 1-skeleton, Bates' construction does not work as expected. The main idea of her construction in [2] is that there is one-to-one correspondence between maximal tails in E and maximal tails in the auxiliary graph \tilde{E} . When we apply the same construction of Bates' auxiliary graph to k -graph, we can only show that the map from Λ to $\tilde{\Lambda}$ takes the maximal tail of k -graph Λ to the maximal tail of 1-graph $\tilde{\Lambda}$ but not the other way. If we start with a maximal tail of 1-graph to construct a maximal tail of k -graph, the corresponding set of vertices no longer satisfy the conditions of maximal tail, in particular the condition (b). Thus, we need a different construction of auxiliary graph for k -graph that represents AF-algebra, and currently we do not know any construction that works.

5. CARTESIAN PRODUCT AND SKEW PRODUCT EXAMPLES

In this section we describe two strongly aperiodic 2-graphs with no sources, one a skew product graph which is not a cartesian product graph and the other a cartesian product graph. We compute the primitive ideal spaces of their associated C^* -algebras and show that they are homeomorphic even though the 2-graphs are not isomorphic. The common feature that the examples have is that they have the same 1-skeleton, which we now describe.

A k -graph Λ can be visualized by its 1-skeleton: This is a directed graph E_Λ with vertices Λ^0 and edges $\cup_{i=1}^k \Lambda^{e_i}$ which have range and source in E_Λ determined by their range and source in Λ . Each edge in E_Λ with degree e_i is assigned the same color c_i , so E_Λ is a colored graph. It is common to call edges with degree e_1 in Λ blue edges in E_Λ and draw them with solid lines; edges with degree e_2 in Λ are then called red edges and are drawn as dashed lines. Different k -graphs can determine the same 1-skeleton since the skeleton does not encode the factorization property of the k -graph. In practice, along with the 1-skeleton we give a collection of factorization rules which relate the edges of E_Λ that occur in the factorization of morphisms of degree $e_i + e_j$ ($i \neq j$) in Λ . For more information about 1-skeletons and their relationship with k -graphs we refer the reader to [33].

Before we give an example, we introduce the cartesian product graphs and skew-product graphs.

Proposition 5.1. [22, Proposition 1.8]. *Let (Λ_i, d_i) be k_i -graphs for $i = 1, 2$, then the cartesian product graph $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$ is a $(k_1 + k_2)$ -graph where $\Lambda_1 \times \Lambda_2$ is the product category and $d_1 \times d_2 : \Lambda_1 \times \Lambda_2 \rightarrow \mathbb{N}^{k_1+k_2}$ is given by $d_1 \times d_2(\lambda_1, \lambda_2) = (d_1(\lambda_1), d_2(\lambda_2)) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ for $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$.*

When we identify $\mathbb{N}^{k_1+k_2}$ with $\mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ in the above Proposition we may suppose that $\mathbb{N}^{k_1+k_2}$ has a basis $\{e_1, \dots, e_{k_1}, f_1, \dots, f_{k_2}\}$ where $\{e_1, \dots, e_{k_1}\}$ and $\{f_1, \dots, f_{k_2}\}$

are the canonical bases for \mathbb{N}^{k_1} and \mathbb{N}^{k_2} respectively. Then the factorization rule for elements of degree $(e_i + f_j)$ in $\Lambda_1 \times \Lambda_2$ are of the form

$$(5) \quad (r(a), b)(a, s(b)) = (a, r(b))(s(a), b) \text{ where } b \in \Lambda_1^{e_i}, a \in \Lambda_2^{f_j}.$$

Remarks 5.2. (a) Suppose that a 2-graph Λ has edges $b \in \Lambda^{e_1}, a \in \Lambda^{e_2}$ such that $r(a) = s(a) = r(b) = s(b)$. If Λ is a cartesian product of 1-graphs then by (5) we must have $ab = ba$.

(b) By (a) it follows that for $m, n \geq 1$ the 2-graph \mathbb{F}_θ^2 described in Examples 2.2 (3) is the cartesian product $B_m \times B_n$ if and only if θ is the identity function.

Theorem 5.3. *Let Λ_i be k_i -graphs for $i = 1, 2$. Then*

(a) $\Lambda_1 \times \Lambda_2$ is a row-finite $(k_1 + k_2)$ -graph with no sources if and only if Λ_i is a row-finite k_i -graph with no sources for $i = 1, 2$;

(b) $\Lambda_1 \times \Lambda_2$ is aperiodic if and only if Λ_1, Λ_2 are aperiodic; and

(c) $\Lambda_1 \times \Lambda_2$ is strongly aperiodic if and only if Λ_1, Λ_2 are strongly aperiodic.

Proof. (a): Proposition 5.1 implies that $\Lambda_1 \times \Lambda_2$ is a $(k_1 + k_2)$ -graph. It is straightforward to check that $\Lambda_1 \times \Lambda_2$ is row-finite with no sources, and conversely.

(b): Suppose that Λ_1, Λ_2 are aperiodic. Fix $(v, w) \in (\Lambda_1 \times \Lambda_2)^0$ and $m \neq n \in \mathbb{N}^{k_1 + k_2}$. Let $m' = (m_1, \dots, m_{k_1})$, $m'' = (m_{k_1+1}, \dots, m_{k_1+k_2})$ and $n' = (n_1, \dots, n_{k_1})$, $n'' = (n_{k_1+1}, \dots, n_{k_1+k_2})$. Then $m' \neq n'$, or $m'' \neq n''$ (or both). Assume that $m' \neq n'$. Since Λ_1 is aperiodic, for $v \in \Lambda_1^0$ there exists $\lambda_1 \in v\Lambda_1$ such that $d(\lambda_1) > m' \vee n'$ and

$$\lambda_1(m', m' + d(\lambda_1) - m' \vee n') \neq \lambda_1(n', n' + d(\lambda_1) - m' \vee n').$$

For $w \in \Lambda_2^0$, let $\lambda_2 \in w\Lambda_2$ be any path with $d(\lambda_2) > m'' \vee n''$. We claim that

$$(\lambda_1, \lambda_2)(m, m + d_1 \times d_2(\lambda_1, \lambda_2) - m \vee n) \neq (\lambda_1, \lambda_2)(n, n + d_1 \times d_2(\lambda_1, \lambda_2) - m \vee n).$$

Since the left hand side of the above equation is equal to

$$(\lambda_1(m', m' + d_1(\lambda_1) - m' \vee n'), \lambda_2(m'', m'' + d_2(\lambda_2) - m'' \vee n''))$$

by the definition of degree functor $d_1 \times d_2$ on $\Lambda_1 \times \Lambda_2$, and, similarly the right hand side of the above equation is equal to $(\lambda_1(n', n' + d_1(\lambda_1) - m' \vee n'), \lambda_2(n'', n'' + d_2(\lambda_2) - m'' \vee n''))$. Since $\lambda_1(m', m' + d_1(\lambda_1) - m' \vee n') \neq \lambda_1(n', n' + d_1(\lambda_1) - m' \vee n')$ the claim follows. Therefore, $\Lambda_1 \times \Lambda_2$ is aperiodic. If we assume that $m'' \neq n''$ then a similar argument, using the aperiodicity of Λ_2 shows that $\Lambda_1 \times \Lambda_2$ is aperiodic.

Now suppose that $\Lambda_1 \times \Lambda_2$ is aperiodic. First we identify Λ_1 with $\Lambda_1 \times \{v_2\}$ for some $v_2 \in \Lambda_2^0$. Fix $v_1 \in \Lambda_1^0$, then we claim that Λ_1 has no local periodicity at v_1 . To see this, observe that $\Lambda_1 \times \Lambda_2$ has no local periodicity at (v_1, v_2) . So for $(m, 0) \neq (n, 0) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ there is a path $(\lambda_1, v_2) \in (v_1, v_2)(\Lambda_1 \times \Lambda_2)^{(m', 0)}$ with $m' > m \vee n$ such that (1) in Definition 2.8 holds. Identifying $(\lambda_1, v_2) \in \Lambda_1 \times \{v_2\}$ with $\lambda_1 \in v_1\Lambda_1$ proves the claim. A similar argument applies to Λ_2 .

(c): Suppose that Λ_1, Λ_2 are strongly aperiodic. Then it is straightforward to see that every saturated hereditary subset of $(\Lambda_1 \times \Lambda_2)^0 = \Lambda_1^0 \times \Lambda_2^0$ is of the form $H_1 \times H_2$

where H_i is a saturated hereditary subset of Λ_i^0 for $i = 1, 2$. If $H_i \neq \Lambda_i$ for $i = 1, 2$ then by Theorem 2.16 (4) we have

$$\begin{aligned}
& \Gamma((\Lambda_1 \times \Lambda_2) \setminus (H_1 \times H_2)) \\
&= ((\Lambda_1^0 \times \Lambda_2^0) \setminus (H_1 \times H_2), \{(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2 : s(\lambda_1, \lambda_2) \notin H_1 \times H_2\}, r, s) \\
(6) \quad &= \Gamma(\Lambda_1 \setminus H_1) \times \Gamma(\Lambda_2 \setminus H_2)
\end{aligned}$$

then since Λ_1, Λ_2 are strongly aperiodic it follows that $\Gamma(\Lambda_i \setminus H_i)$ is aperiodic for $i = 1, 2$. Hence by (6) it follows that $\Gamma((\Lambda_1 \times \Lambda_2) \setminus (H_1 \times H_2))$ is aperiodic by part (b). Now suppose $H_2 = \Lambda_2^0$ and $H_1 \neq \Lambda_1^0$ then $H_1 \times \Lambda_2^0 \subsetneq \Lambda_1^0 \times \Lambda_2^0$ and

$$\begin{aligned}
& \Gamma((\Lambda_1 \times \Lambda_2) \setminus (H_1 \times \Lambda_2^0)) = ((\Lambda_1^0 \times \Lambda_2^0) \setminus (H_1 \times \Lambda_2^0), \{(\lambda_1, \lambda_2) : s(\lambda_1, \lambda_2) \notin H_1 \times \Lambda_2^0\}, r, s) \\
(7) \quad &= ((\Lambda_1^0 \setminus H_1) \times \Lambda_2^0, \{(\lambda_1, \lambda_2) : s(\lambda_1) \notin H_1\}, r, s)
\end{aligned}$$

which is isomorphic to $\Gamma(\Lambda_1 \setminus H_1) \times \Lambda_2$. Since Λ_1 is strongly aperiodic it follows that $\Gamma(\Lambda_1 \setminus H_1)$ and hence $\Gamma((\Lambda_1 \times \Lambda_2) \setminus (H_1 \times \Lambda_2^0))$ is aperiodic by part (b). A similar argument applies if $H_1 = \Lambda_1^0$ and $H_2 \neq \Lambda_2^0$. Hence $\Lambda_1 \times \Lambda_2$ is strongly aperiodic.

Finally if we assume that $\Lambda_1 \times \Lambda_2$ is strongly aperiodic and that $H_1 \subsetneq \Lambda_1^0$ is a saturated hereditary subset of Λ_1 , then $H_1 \times \Lambda_2^0 \subsetneq \Lambda_1^0 \times \Lambda_2^0$ is a saturated hereditary subset of $\Lambda_1 \times \Lambda_2$. So by (7) we may identify $\Gamma((\Lambda_1 \times \Lambda_2) \setminus (H_1 \times \Lambda_2^0))$ with $\Gamma(\Lambda_1^0 \setminus H_1) \times \Lambda_2$. By part (b) it follows that $\Gamma(\Lambda_1^0 \setminus H_1)$ is aperiodic and since H_1 was arbitrary it follows that Λ_1 is strongly aperiodic. A similar argument shows that Λ_2 is strongly aperiodic. \square

Example 5.4. Since $\Omega_k \cong \overbrace{\Omega_1 \times \dots \times \Omega_1}^{k\text{-times}}$ it follows from Theorem 5.3 (a) and Examples 2.10 that Ω_k is aperiodic.

Definition 5.5 (Definition 5.1 in [22]). Let G be a discrete group, (Λ, d) a k -graph. Given $c : \Lambda \rightarrow G$ a functor, then define the *skew product* graph $\Lambda \times_c G$ as follows : the objects are identified with $\Lambda^0 \times G$ and the morphisms are identified with $\Lambda \times G$ with the following structure maps $s(\lambda, g) = (s(\lambda), gc(\lambda))$ and $r(\lambda, g) = (r(\lambda), g)$. If $s(\lambda) = r(\mu)$, then (λ, g) and $(\mu, gc(\lambda))$ are composable in $\Lambda \times_c G$ and $(\lambda, g)(\mu, gc(\lambda)) = (\lambda\mu, g)$. The degree map is given by $d(\lambda, g) = d(\lambda)$.

Remark 5.6. It is straightforward to check that if Λ is a row-finite k -graph with no sources, $\Lambda \times_c G$ is a row-finite k -graph with no sources. The factorization rule in $\Lambda \times_c G$ for elements of degree $e_i + e_j$, where $i \neq j$ is induced from the factorization rule in Λ as follows: If $d(\lambda) = e_i + e_j$ then $\lambda = \lambda(0, e_i)\lambda(e_i, d(\lambda)) = \lambda(0, e_j)\lambda(e_j, d(\lambda))$ in Λ and

$$(8) \quad (\lambda(0, e_i), g)(\lambda(e_i, d(\lambda)), gc(\lambda(0, e_i))) = (\lambda(0, e_j), g)(\lambda(e_j, d(\lambda)), gc(\lambda(0, e_j)))$$

in $\Lambda \times_c G$.

Example 5.7. Suppose that Γ is the following 2-graph with the 1-skeleton, shown below



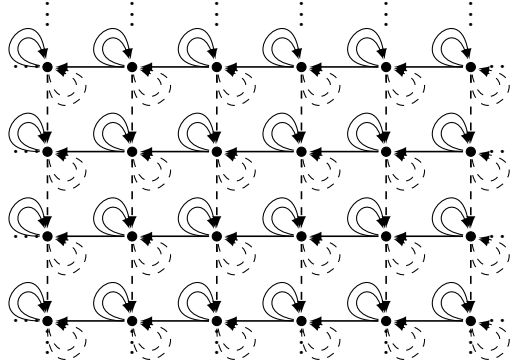
and factorization rules

$$(9) \quad \begin{aligned} g_i f_1 &= f_2 g_j, & g_j f_2 &= f_1 g_j, & g_j f_3 &= f_3 g_j & \text{for } j = 1, 3 \text{ and} \\ g_2 f_i &= f_i g_2 & \text{where } f_i &\in \Lambda^{e_1} \text{ and } g_j &\in \Lambda^{e_2} & \text{for } i, j = 1, 2, 3. \end{aligned}$$

In fact $\Gamma = \mathbb{F}_\theta^2$ where $\theta : \underline{3} \times \underline{3} \rightarrow \underline{3} \times \underline{3}$ is given by

$$\begin{aligned} \theta(2, j) &= (1, j), & \theta(1, j) &= (2, j), & \theta(3, j) &= (3, j) & \text{for } j = 1, 3 \text{ and} \\ \theta(i, 2) &= (i, 2) & \text{for } i = 1, 2, 3. \end{aligned}$$

Define a map $c : \Gamma \rightarrow \mathbb{Z}^2$ by $c(g_3) = (0, 1)$, $c(f_3) = (1, 0)$, $c(f_i) = (0, 0)$, $c(g_i) = (0, 0)$, for $i = 1, 2$ then c preserves the factorization rules in Γ , thus it extends to a functor $c : \Gamma \rightarrow \mathbb{Z}^2$. Then the 1-skeleton of $\Lambda = \Gamma \times_c \mathbb{Z}^2$ is given as follows.

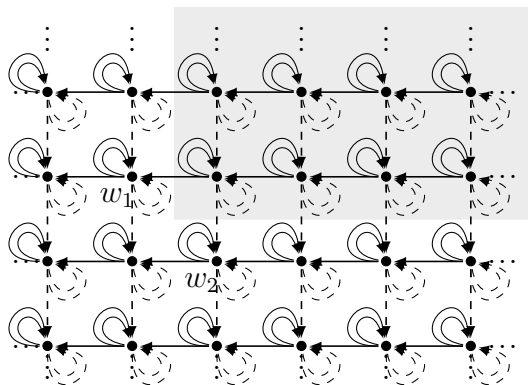


and factorization rules induced from those in Γ given in (9) to $\Lambda \times_c \mathbb{Z}^2$ using (8).

For each $(v, m) \in \Lambda^0$ one checks that $((f_1, m), (f_2, m), (g_1, m), (g_2, m))$ is a $(1, 1)$ -aperiodic quartet, so Proposition 3.9 shows that Λ is strongly aperiodic. Moreover, the graph Λ is not a cartesian product by Remark 5.2 since the edges $(f_i, m) \in \Lambda^{e_1}$ and $(g_i, m) \in \Lambda^{e_2}$, $i = 1, 2$ have source (v, m) , but have the factorization rule $(g_1, m)(f_2, m) = (f_1, m)(g_1, m)$ by (9) and (8) since $c(g_i) = c(f_i) = 0$ for $i = 1, 2$.

Now we describe the ideal structure of $C^*(\Lambda)$. In the following figure, the vertices in the gray shaded area shown form a hereditary collection of vertices in Λ . The region is also saturated since vertices $w_1 = (v, m_1), w_2 = (v, m_2) \in \Lambda^0$ receive both solid and dashed edges from vertices not in the shaded area. However, the complement

of the shaded area is not a maximal tail since it fails to satisfy the condition (a) of Definition 3.10. i.e. there is no common ancestor for w_1 and w_2 .



In the following figure, the vertices in the shaded areas form a hereditary subset of Λ . This region is saturated since vertex $w \in \Lambda^0$ receives both dashed and solid edges from vertices not in the shaded area. Also, the complement of the shaded region satisfies conditions (a), (b) and (c) of Definition 3.10.

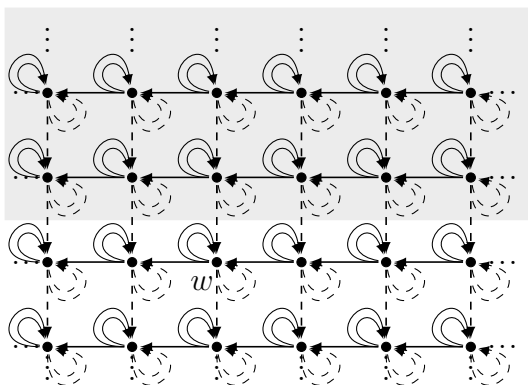


Figure 1

Similarly, the vertices in the shaded area of the following figure form a hereditary and saturated subset in Λ , and the complement is a maximal tail.

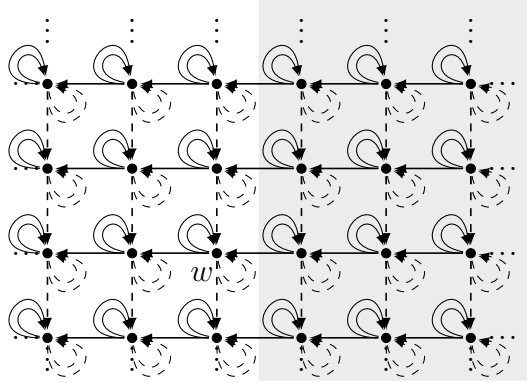


Figure 2

The following figure shows the last type of the hereditary and saturated subset, the shaded area, of Λ , of which the complement is a maximal tail.

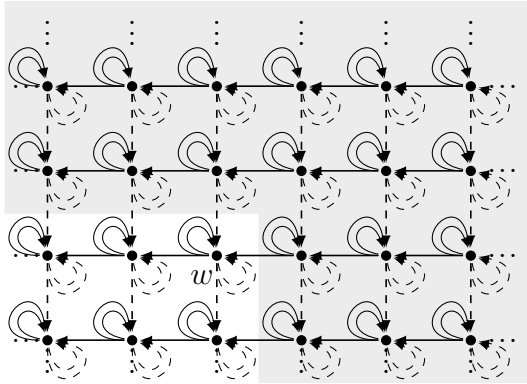


Figure 3

To describe the topology of $\text{Prim } C^*(\Lambda)$ we first identify the vertices of Λ with \mathbb{Z}^2 . It is tedious, but not difficult to show that every maximal tail in χ_Λ is of the form depicted in Figures 1,2,3, plus the tail Λ^0 . In Figure 1, we have a tail of the form

$$H_b = \{(x, y) \in \mathbb{Z}^2 : y \leq b\} \text{ for each } b \in \mathbb{Z}.$$

In Figure 2, we we have a tail of the form

$$V_a = \{(x, y) \in \mathbb{Z}^2 : x \leq a\} \text{ for each } a \in \mathbb{Z}.$$

In Figure 3, we have a tail of the form

$$LQ_{(a,b)} = \{(x, y) \in \mathbb{Z}^2 : x \leq a, y \leq b\} \text{ for each } a, b \in \mathbb{Z}.$$

So

$$\chi_\Lambda = \{V_a : a \in \mathbb{Z}\} \cup \{H_b : b \in \mathbb{Z}\} \cup \{LQ_{(a,b)} : (a, b) \in \mathbb{Z}^2\} \cup \{\Lambda^0\}.$$

Recall that the closed set \overline{S} of χ_Λ given in Theorem 3.15 is $\overline{S} = \{\delta \in \chi_\Lambda : \delta \subseteq \bigcup_{\gamma \in S} \gamma\}$. Hence

$$\begin{aligned} \overline{\{H_b\}} &= \{H_d : d \leq b\}, & \overline{\{LQ_{(a,b)}\}} &= \{LQ_{(c,d)} : (c,d) \leq (a,b)\}, \\ \overline{\{V_c\}} &= \{V_a : c \leq a\}, & \overline{\{\Lambda^0\}} &= \chi_\Lambda, \end{aligned}$$

so Λ^0 is a dense point in χ_Λ . Since

$$\overline{\{LQ_{(a,b)}\}}^c = \overline{\{V_a\}}^c \cup \overline{\{H_b\}}^c = \{V_c : c > a\} \cup \{H_d : d > b\} \cup \{\Lambda^0\},$$

a basis for the topology on χ_Λ is

$$\{\{H_d : d > b\} \cup \{\Lambda^0\} : b \in \mathbb{Z}\}, \{\{V_c : c > a\} \cup \{\Lambda^0\} : c \in \mathbb{Z}\}.$$

Let $X = \mathbb{Z} \cup \{\infty\}$ and extend the order on \mathbb{Z} to X by declaring $x \leq \infty$ for all $x \in \mathbb{Z}$. Then X is totally ordered and has right-order topology with basis $(a, \infty] = \{x \in \mathbb{Z} : x > a\}$. Let $Y = X \times X$ with the product topology. Define a map $\phi : \chi_\Lambda \rightarrow Y$ by

$$\phi(H_b) = (\infty, b), \quad \phi(V_a) = (a, \infty), \quad \phi(LQ_{(a,b)}) = (a, b), \quad \phi(\Lambda^0) = (\infty, \infty),$$

then ϕ is a bijection such that

$$\phi(\{H_d : d > b\} \cup \{\Lambda^0\}) = X \times (b, \infty) \text{ and } \phi(\{V_c : c > a\} \cup \{\Lambda^0\}) = (a, \infty) \times X,$$

and so it is a homeomorphism. Therefore, the topology of $\text{Prim } C^*(\Lambda)$ corresponds to the right-order topology of $(\mathbb{Z} \cup \{\infty\}) \times (\mathbb{Z} \cup \{\infty\})$.

Example 5.8. Let Ω be the 1 graph with the following 1-skeleton.



Then the 1-skeleton of $\Omega \times \Omega$ is the same as the 1-skeleton of Λ in the above example. Since Ω satisfies condition (K), it follows that Ω is strongly aperiodic by Lemma 3.2. Then by Theorem 5.3 (b), $\Omega \times \Omega$ is strongly aperiodic. If we identify the vertices of Ω with \mathbb{Z} , it is straightforward to see that every saturated hereditary subset of Ω^0 is of the form $[n+1, +\infty)$ for some $n \in \mathbb{Z}$, or the empty set \emptyset , or Ω^0 . Similarly every maximal tail is of the form $\chi_n := (-\infty, n]$ for some $n \in \mathbb{Z}$, or Ω^0 (recall that a maximal tail is nonempty). So $\chi_\Omega = \{\chi_n : n \in \mathbb{Z}\} \cup \{\Omega^0\}$ which we may identify (as a set) with $\mathbb{Z} \cup \{\infty\}$. The closure of $\{\chi_n\}$ is $\{\chi_j : j \leq n\}$ and the closure of Ω^0 is χ_Ω . Thus, the nontrivial open sets in χ_Ω are of the form $\{\chi_j : j > n\} \cup \{\Omega^0\}$. When we identify χ_Ω with $\mathbb{Z} \cup \{\infty\}$, this topology coincides with the right-order topology on $\mathbb{Z} \cup \{\infty\}$. By [22, Corollary 3.5 (iv)] we have $C^*(\Omega \times \Omega) \cong C^*(\Omega) \otimes C^*(\Omega)$, so by [43] the primitive ideal space of $C^*(\Omega \times \Omega)$ is the cartesian product of the primitive ideal space of $C^*(\Omega)$ with itself with the product topology.

Remark 5.9. The primitive ideal space of $C^*(\Omega \times \Omega)$ is precisely the same topology as the primitive ideal space of $C^*(\Lambda)$ described in Example 5.7. Even though Λ and Ω are not isomorphic, it is unclear whether $C^*(\Lambda)$ and $C^*(\Omega \times \Omega)$ are isomorphic.

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