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A groupoid generalisation of Leavitt path algebras

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Abstract
Let G be a locally compact, Hausdorff, étale groupoid whose unit space is totally disconnected.We show that the collection A(G) of locally-constant, compactly supported complex-valued functions on G is a dense ∗-subalgebra of Cc(G) and that it is universal for algebraic representations of the collection of compact open bisections of G. We also show that if G is the groupoid associated to a row-finite graph or k-graph with no sources, then A(G) is isomorphic to the associated Leavitt path algebra or Kumjian–Pask algebra. We prove versions of the Cuntz–Krieger and graded uniqueness theorems for A(G).

A GROUPOID GENERALISATION OF LEAVITT PATH ALGEBRAS

LISA ORLOFF CLARK, CYNTHIA FARTHING, AIDAN SIMS, AND MARK TOMFORDE

Abstract. Let G be a locally compact Hausdorff étale groupoid whose unit space is totally disconnected. We show that the collection $A(G)$ of locally-constant, compactly supported complex-valued functions on G is a dense $*$-subalgebra of $C_c(G)$ and that it is universal for algebraic representations of the collection of compact open bisections of G. We also show that if G is the groupoid associated to a row-finite graph or k-graph with no sources, then $A(G)$ is isomorphic to the associated Leavitt path algebra or Kumjian-Pask algebra. We prove versions of the Cuntz-Krieger and graded uniqueness theorems for $A(G)$.

1. Introduction

A ring R is said to have invariant basis number if any two bases (i.e., R-linearly independent spanning sets) of a free left R-module have the same number of elements. Many familiar rings (e.g., fields, commutative rings, left-Noetherian rings) have invariant basis number, but there are many examples of noncommutative rings that do not. A ring R without invariant basis number is said to have module type (m, n) if $m < n$ are natural numbers chosen minimally with $R^m \cong R^n$ as left R-modules. In the 1940's, Leavitt constructed algebras $L_{m,n}$ with module type (m, n) for all pairs of natural numbers with $m < n$ [14, 15]. The $L_{m,n}$ are now known as the *Leavitt algebras*, and when $m = 1$, the Leavitt algebra $L_{1,n}$ is the unique nontrivial unital complex algebra generated by elements $x_1 \dots x_n$ and $y_1, \dots, y_n$ such that $\sum_{i=1}^n x_i y_i = 1$ and $y_i x_j = \delta_{i,j} 1$ for all $i, j \leq n$. In the 1970's, independent of Leavitt's work and motivated by the search for C^*-algebraic analogues of Type III factors, Cuntz defined a class of C^*-algebras $\mathcal{O}_n$, one for each integer $n \geq 2$, which are generated by elements $s_1, \dots, s_n$ satisfying $\sum_{i=1}^n s_i s_i^* = 1$ and $s_i^* s_i = 1$ for all i (it follows that $s_i^* s_j = \delta_{i,j} 1$ for all $i, j \leq n$). A consequence of the uniqueness of $L_{1,n}$ is that it is isomorphic to the dense $*$-subalgebra of $\mathcal{O}_n$ generated by $s_1, \dots, s_n$ via an isomorphism that carries each x_i to s_i and each y_i to s_i^*.

Shortly after Cuntz's work, Cuntz and Krieger generalised Cuntz's results to describe a class of C^*-algebras $\mathcal{O}_A$ associated to binary-valued matrices A [7]. At about the same time, Enomoto and Watatani provided a very elegant description of these Cuntz-Krieger algebras in terms of the directed graphs encoded by the matrices. Nearly twenty years later, Kumjian, Pask, Raeburn, and Renault developed the class of C^*-algebras now known as graph C^*-algebras [13], as a far-reaching generalisation of the Cuntz-Krieger algebras patterned on Enomoto and Watatani's approach. Each graph C^*-algebra is described in terms of generators associated to the vertices and edges in the graph subject to relations encoded by connectivity in the graph. The Cuntz algebra $\mathcal{O}_n$ corresponds to the graph with one vertex and n edges. A remarkable assortment of important C^*-algebraic properties of a graph C^*-algebra can be characterised in terms of the structure of the graph

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(see [18] for a good overview). Shortly afterwards, Kumjian and Pask introduced a sort of higher-dimensional graph [11], now known as a k-graph, and an associated class of C^*-algebras, as a flexible visual model for the higher-rank Cuntz-Krieger algebras discovered by Robertson and Steger [22]. When $k = 1$, a k-graph is essentially a directed graph, and Kumjian and Pask's C^*-algebras coincide with the graph C^*-algebras of [13].

In the early 2000's, the algebraic community became interested in the similarity between the constructions of Leavitt and Cuntz and the potential for the graph C^*-algebra template to provide a broad class of interesting new algebras. Following the lead of [13], Abrams and Aranda Pino associated *Leavitt path algebras* to a broad class of directed graphs. The Leavitt path algebra of a directed graph is the universal algebra whose presentation in terms of generators and relations is essentially the same as that of the graph C^*-algebra. Moreover, the graded uniqueness theorem for Leavitt path algebras implies that the C^*-algebra of a directed graph is a norm completion of its Leavitt path algebra [4], [25]. Further generalising Leavitt path algebras, Aranda Pino, (J.) Clark, an Huef, and Raeburn recently constructed a class of algebras associated to k-graphs, which they call Kumjian-Pask algebras [3].

A very powerful framework for constructing C^*-algebras is the notion of a groupoid C^*-algebra. Renault's structure theory for groupoid C^*-algebras [21] is exploited in [13] where structural properties of the graph C^*-algebra are deduced by showing that the graph C^*-algebra is isomorphic to a groupoid C^*-algebra and then tapping into Renault's results [21]. The same approach was taken in [11] to establish important structural properties of k-graph C^*-algebras: the C^*-algebra of a k-graph is defined in terms of generators and relations, but its structure is analysed by identifying it with a groupoid C^*-algebra.

In this paper, from a sufficiently well-behaved groupoid G, we construct a complex algebra $A(G)$ with the following properties:

(1) $A(G)$ has a natural description as a universal algebra (Theorem 3.10);
(2) $A(G)$ is isomorphic to a dense subalgebra of the groupoid C^*-algebra $C^*(G)$ (Proposition 4.2); and
(3) given a k-graph Λ, if $G = G_\Lambda$ is the groupoid corresponding to Λ as in [11] (Proposition 4.3), then $A(G)$ is isomorphic to the Kumjian-Pask algebra $\mathrm{KP}_{\mathbb{C}}(\Lambda)$. In particular, if E is a directed graph and $G = G_E$ is the graph groupoid associated to E, then $A(G)$ is isomorphic to the Leavitt path algebra $L_{\mathbb{C}}(E)$.

In [24], Steinberg defines a groupoid algebra KG for an arbitrary commutative ring K with unit and shows that KG is a quotient of an associated inverse semigroup algebra. We show that the algebra $A(G)$ is identical to KG for $K = \mathbb{C}$ (the complex numbers).[1] Our approach is different from that of [24] and our universal property and uniqueness theorems (see below) provide new tools for studying KG and the inverse semigroup algebras associated to them in the case where $K = \mathbb{C}$; it would be interesting to investigate versions of these theorems for general K.

The Cuntz-Krieger uniqueness theorem and gauge-invariant uniqueness theorem are important tools in the study of graph C^*-algebras. Versions of these theorems have been established for many generalisations of Cuntz-Krieger algebras [6, 9, 11, 12, 13, 19, 20]. For Leavitt path algebras, the graded uniqueness theorem is the analogue of the gauge-invariant uniqueness theorem. The first version of this graded uniqueness theorem was

[1]We would like to thank Steinberg who brought this to our attention after reading an earlier version of this paper.

a corollary to Ara, Moreno, and Pardo's characterisation [2, Theorem 4.3] of the graded ideals in a Leavitt path algebra. It was first stated explicitly by Raeburn who proved both the graded uniqueness theorem and Cuntz-Krieger uniqueness theorem for Leavitt path algebras of row-finite graphs with no sinks and over fields equipped with a positive definite $*$-operation [4, Theorem 1.3.2 and Theorem 1.3.4]. Tomforde extended these results to Leavitt path algebras of arbitrary graphs over arbitrary fields in [25, Theorem 4.8 and Theorem 6.8], and later proved the two uniqueness theorems for Leavitt path algebras of arbitrary graphs over a ring [26, Theorem 5.3 and Theorem 6.5]. Aranda Pino, (J.) Clark, an Huef, and Raeburn subsequently proved versions of these theorems for Kumjian-Pask algebras [3]. In Section 5 we prove versions of the Cuntz-Krieger uniqueness theorem (Theorem 5.1) and the graded uniqueness theorem (Theorem 5.4) for $A(G)$. We also give an example of a groupoid satisfying our hypothesis that is not necessarily the groupoid of a k-graph.

Our aim in defining and initiating the analysis of $A(G)$ is twofold: (1) to provide a broad framework for future generalisations of Leavitt path algebras from other combinatorial structures; and (2) to make available the powerful toolkit of groupoid analysis to study these algebras. In addition, we hope this will provide a new and useful perspective on the interplay between algebra and analysis at the interface between Leavitt path algebras and graph C^*-algebras.

2. Preliminaries

A groupoid is a small category with inverses. We write $G^{(2)} \subseteq G \times G$ for the set of composable pairs in G; we write $G^{(0)}$ for the unit space of G, and we denote by r and s the range and source maps $r, s : G \to G^{(0)}$. So $(\alpha, \beta) \in G^{(2)}$ if $s(\alpha) = r(\beta)$. For $U, V \subseteq G$, we define

$$UV := \{\alpha\beta : \alpha \in U,\ \beta \in V,\ \text{and } r(\beta) = s(\alpha)\}. \tag{2.1}$$

A topological groupoid is a groupoid endowed with a topology under which r and s are continuous, the inverse map is continuous, and such that composition is continuous with respect to the relative topology on $G^{(2)}$ inherited from $G \times G$.

Recall that if G is a groupoid, then an *open bisection* of G is an open subset $U \subseteq G$ such that $r|_U$ and $s|_U$ are homeomorphisms. We will work exclusively with locally compact, Hausdorff groupoids which are étale in the sense that the source map $s : G \to G^{(0)}$ is a local homeomorphism. The range map is then a local homeomorphism as well. If G is étale then $G^{(0)}$ is open in G and G admits a Haar system consisting of counting measures. The following also appears as [8, Proposition 4.1].

Lemma 2.1. *Let G be a locally compact, Hausdorff, étale groupoid. Suppose that $G^{(0)}$ is totally disconnected. Then the topology on G has a basis of clopen bisections. Moreover, if G is locally compact and Hausdorff, then G has a basis of compact open bisections.*

Proof. Proposition 2.8 of [21] implies that G has a basis of open bisections. For each $\gamma \in G$, let U be an open bisection containing γ. Since r is an open map there exists a basic clopen neighbourhood X of $r(\gamma)$ such that $X \subseteq r(U)$. Then $XU = \{h \in U : r(h) \in X\} = U \cap r^{-1}(X)$ is homeomorphic to X by choice of U and in particular is a clopen bisection containing γ. If G is also locally compact, then U may be chosen to be precompact. Hence the clopen subset XU is a compact open bisection. □

Notation 2.2. For the remainder of this paper, Γ will denote a discrete group, G will denote a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and c will denote a continuous cocycle from G to Γ (that is, c carries composition in G to the group operation in Γ).

By Lemma 2.1, with Γ, G and c as above, G has a basis of compact open bisections. Since G is Hausdorff, compact subsets of G are closed. We will use this fact frequently and without further comment.

Remark 2.3. These hypotheses might sound very restrictive, but, for instance, every k-graph groupoid satisfies them (see, for example, [9]).

Remark 2.4. Let U be a compact open subset of a topological space X. Let F be a finite cover of U by compact open subsets of U. For each nonempty $H \subseteq F$, let $V_H := \big(\bigcap H\big) \setminus \big(\bigcup (F \setminus H)\big)$. Since each $V \in F$ is compact and open, so is each V_H. In particular, since F is finite, so is $K := \{H \subseteq F : H \neq \emptyset, V_H \neq \emptyset\}$, and

$$U = \bigsqcup_{H \in K} V_H$$

is an expression for U as a finite disjoint union of nonempty compact open sets such that for each $W \in K$ we have $W \subseteq V$ for at least one $V \in F$, and such that whenever $W \in K$ and $V \in F$ satisfy $W \not\subseteq V$, we have $W \cap V = \emptyset$. We refer to this as the *disjointification* of the cover F of U.

Throughout this paper, unless stated otherwise, all algebras are taken to be complex *-algebras, and all representations are assumed to preserve adjoints.

3. The algebra $A(G)$

Definition 3.1. Let X be a topological space. A function $f : X \to Y$ is *locally constant* if for every $x \in X$ there exists a neighbourhood U of x such that $f|_U$ is constant.

Observe that if $f : X \to \mathbb{C}$ is locally constant then it is automatically continuous, and the support of f is clopen in X.

Definition 3.2. Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. We define $A(G)$ to be the compex vector space

$$A(G) = \{f \in C_c(G) : f \text{ is locally constant}\}$$

with pointwise addition and scalar multiplication.

The following lemma shows that $A(G)$ is precisely the algebra $\mathbb{C}G$ of [24, Definition 4.1]. (In fact, Definition 3.2 agrees precisely with the definition of $\mathbb{C}G$ given in the preprint version of [24] — see [23, Definition 3.1] — and then the following Lemma is [23, Proposition 3.3].)

Lemma 3.3. *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. If $\mathcal{U}$ is the basis of all compact open subsets of G, we have $A(G) = \operatorname{span}\{1_U : U \in \mathcal{U}\}$.*

Proof. For any $U \in \mathcal{U}$, the function 1_U is locally constant, and hence $\operatorname{span}\{1_U : U \in \mathcal{U}\} \subseteq A(G)$. We must show that $A(G) \subseteq \operatorname{span}\{1_U : U \in \mathcal{U}\}$. Fix $f \in A(G)$. Since f is locally constant and $\mathcal{U}$ is a basis, for each $\alpha \in \operatorname{supp}(f)$, there is a neighbourhood $U_\alpha \in \mathcal{U}$ of α such that $f|_{U_\alpha}$ is constant. Since $\operatorname{supp}(f)$ is clopen, we may assume that

$U_\alpha \subseteq \operatorname{supp}(f)$. Since $\operatorname{supp}(f)$ is compact there is a finite subset $F \subseteq \{U_\alpha\}_{\alpha \in \operatorname{supp} f}$ such that $\operatorname{supp}(f) = \bigcup F$. Let K be the disjointification of F discussed in Remark 2.4. Since f is constant on each $V \in F$ and each $W \in K$ is a subset of some $V \in F$, the function f is constant on each $W \in K$. Hence, writing $f(W)$ for the unique value taken by f on $W \in K$, we have $f = \sum_{W \in K} f(W) 1_W$. □

Definition 3.4. Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. For $n \in \Gamma$ we write $G_n := c^{-1}(n)$. We write $A_n(G)$ for the subset of $A(G)$ consisting of functions whose support is contained in G_n. We say that a subset S of G is *graded* if the cocycle c is constant on S. If $S \subseteq G_n$, we say that S is *n-graded*. For each $n \in \Gamma$ we write $B_n^{\mathrm{co}}(G)$ for the collection of all n-graded compact open bisections of G. We write $B_*^{\mathrm{co}}(G)$ for $\bigcup_{n \in \Gamma} B_n^{\mathrm{co}}(G)$.

Lemma 3.5. *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. We have $A(G) = \operatorname{span}\{A_n : n \in \Gamma\} \subseteq C_c(G)$. Every $f \in A(G)$ can be expressed as $f = \sum_{U \in F} a_U 1_U$ where F is a finite subset of $B_*^{\mathrm{co}}(G)$ whose elements are mutually disjoint and $a : U \mapsto a_U$ is a function from F to $\mathbb{C}$.*

Proof. We have $A(G) \supseteq \operatorname{span}\{A_n : n \in \Gamma\}$ because each A_n consists of locally constant functions. For the reverse inclusion, fix $f \in A(G)$. Since Γ is discrete and c is continuous, each G_n is clopen. Since $\operatorname{supp}(f)$ is compact, there is a finite collection $N \subseteq G$ such that $\operatorname{supp}(f) \subseteq \bigcup_{n \in N} G_n$. For $n \in N$ let f_n denote the pointwise product $f 1_{G_n}$. Then f_n is locally constant and continuous because 1_{G_n} and f are. We then have $f = \sum_{n \in N} f_n \in \operatorname{span}\{A_n : n \in \Gamma\}$.

Let $f \in A(G)$. By Lemma 3.3, there is a finite set K_0 of compact open sets and an assignment $W \mapsto d_W$ of scalars to the elements of K_0 such that $f = \sum_{W \in K_0} d_W 1_W$. Let

$$K := \{W \cap G_n : W \in K_0, n \in \Gamma, W \cap G_n \neq \emptyset\}.$$

Since Γ is discrete and c is continuous, each G_n is open. Since each $W \in K_0$ is compact, K is finite. Each $V \in K$ is graded; we write $c(V)$ for the unique value taken by c on V. For each $V \in K$, let

$$b_V = \sum_{W \in K_0, W \cap G_{c(V)} = V} d_W$$

Then $f = \sum_{V \in K} b_V 1_V$.

Let F be the disjointification of K. Each $U \in F$ is graded because F is a refinement of K. For $U \in F$, define

$$a_U = \sum_{V \in K, U \subseteq V} b_V.$$

Then $f = \sum_{U \in F} a_U 1_U$ is the desired expression. □

Recall that given a locally compact, Hausdorff, étale groupoid G such that $s : G \to G^{(0)}$ is a local homeomorphism, and given $f, g \in A(G) \subseteq C_c(G)$, the functions f^* and $f * g$ are given by

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \tag{3.1}$$

$$(f * g)(\gamma) = \sum_{r(\alpha) = r(\gamma)} f(\alpha) g(\alpha^{-1}\gamma). \tag{3.2}$$

Proposition 3.6. *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. Under the operations* (3.1) *and* (3.2)*, $A(G)$ is a Γ-graded $*$-algebra with graded subspaces A_n as described in Definition 3.4.*

Remark 3.7. For us, an involution on a $*$-algebra over $\mathbb{C}$ is always *conjugate* linear.

Remark 3.8. We do not assume that Γ is abelian so we will write the group operation multiplicatively.

Proof. That $A(G)$ is a complex algebra follows from [24, Proposition 4.6]. We must verify that $A(G)$ is a $*$-algebra and that $A(G)$ is graded. The A_n are mutually linearly independent because the G_n are disjoint and restriction of functions gives a vector-space isomorphism of each A_n onto the space of locally constant functions on G_n. Observe that the $*$-operation is a conjugate-linear involution on $A(G)$ and takes A_n to $A_{n^{-1}}$. Next we will show that the multiplication defined on $A(G)$ is a graded multiplication. If $f \in A_m$ and $g \in A_n$, then if $(f * g)(\gamma) \neq 0$ we have $f(\alpha) \neq 0$ and $g(\alpha^{-1}\gamma) \neq 0$ for some α with $r(\alpha) = r(\gamma)$. In particular, $c(\alpha) = m$, and $c(\alpha^{-1}\gamma) = n$ forcing $c(\gamma) = mn$ (because $c(\gamma) = c(\alpha\alpha^{-1}\gamma) = c(\alpha)c(\alpha^{-1}\gamma)$). Hence $\operatorname{supp}(f * g) \subseteq G_{mn}$. □

We finish this section by presenting of $A(G)$ as a universal algebra.

Definition 3.9. Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. Let B be an algebra over $\mathbb{C}$. A *representation of $B^{co}_*(G)$* in B is a family $\{t_U : U \in B^{co}_*(G)\} \subseteq B$ satisfying

- (R1) $t_\emptyset = 0$;
- (R2) $t_U t_V = t_{UV}$ for all $U, V \in B^{co}_*(G)$; and
- (R3) $t_U + t_V = t_{U\cup V}$ whenever U and V are disjoint elements of $B^{co}_n(G)$ for some n such that $U \cup V$ is a bisection.

The following theorem gives an alternative formulation of [23, Theorem 3.11].

Theorem 3.10. *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. Then $\{1_U : U \in B^{co}_*(G)\} \subseteq A(G)$ is a representation of $B^{co}_*(G)$ which spans $A(G)$. Moreover, $A(G)$ is universal for representations of $B^{co}_*(G)$ in the sense that for every representation $\{t_U : U \in B^{co}_*(G)\}$ of $B^{co}_*(G)$ in an algebra B, there is a unique homomorphism $\pi : A(G) \to B$ such that $\pi(1_U) = t_U$ for all $U \in B^{co}_*(G)$.*

Proof. The collection $\{1_U : U \in B^{co}_*(G)\}$ certainly satisfies (R1) and (R3), and it satisfies (R2) by [24, Proposition 4.5 (3)]. That this family spans $A(G)$ follows from Lemma 3.5.

Let B be a complex algebra and let $\{t_U : U \in B^{co}_*(G)\}$ be a representation of $B^{co}_*(G)$ in B. We must show that there is a homomorphism $\pi : A(G) \to B$ satisfying $\pi(1_U) = t_U$ for all $U \in B^{co}_*(G)$; uniqueness follows from the previous paragraph. We begin by showing that

(3.3) $\sum_{U\in F} t_U = t_{\bigcup F}$ for $n \in \Gamma$ and finite $F \subseteq B^{co}_n(G)$ consisting of mutually disjoint bisections such that $\bigcup F \in B^{co}_n(G)$.

Let $F \subseteq B^{co}_n(G)$ be a finite collection of mutually disjoint bisections such that $\bigcup F$ is a bisection. We claim that $r(U) \cap r(V) = \emptyset$ for distinct $U, V \in F$. To see this, fix $x \in r(U)$.

There exists $\alpha \in U$ such that $r(\alpha) = x$, and this α is the unique element of $\bigcup F$ whose range is x because $\bigcup F$ is a bisection. Since $U \cap V = \emptyset$, we have $\alpha \notin V$ and hence $x \notin r(V)$. So the sets $r(U)$ where $U \in F$ are mutually disjoint as claimed. Thus each $U \in F$ satisfies $U = r(U)(\bigcup F)$. A standard induction extends (R3) to finite collections of mutually disjoint compact open subsets of $G^{(0)}$. Combining this with (R2), we obtain

$$t_{\bigcup F} = t_{r(\bigcup F)} t_{\bigcup F} = \sum_{U \in F} t_{r(U)} t_{\bigcup F} = \sum_{U \in F} t_{r(U)\left(\bigcup F\right)} = \sum_{U \in F} t_U.$$

We show next that the formula $\sum_{U\in F} a_U 1_U \mapsto \sum_{U \in F} a_U t_U$ is well-defined on linear combinations of indicator functions where $F \subseteq B^{\mathrm{co}}_*(G)$ is a finite collection of mutually disjoint bisections. It will follow from Lemma 3.5 that there is a unique linear map $\pi : A(G) \to B$ such that $\pi(1_U) = t_U$ for each $U \in B^{\mathrm{co}}_*(G)$. Fix $f \in A(G)$ and suppose that

$$\textstyle\sum_{U\in F} a_U 1_U = f = \sum_{V \in H} b_V 1_V$$

where each of F and H is a finite set of mutually disjoint elements of $B^{\mathrm{co}}_*(G)$. We must show that

$$\textstyle\sum_{U\in F} a_U t_U = \sum_{V \in H} b_V t_V.$$

Since the G_n are mutually disjoint, for each $n \in \Gamma$ we have

$$\sum_{U \in F \cap B^{\mathrm{co}}_n(G)} a_U 1_U = f|_{G_n} = \sum_{V \in H \cap B^{\mathrm{co}}_n(G)} b_V 1_V,$$

so we may assume that $F, G \subseteq B^{\mathrm{co}}_n(G)$ for some $n \in \Gamma$.

Let $K = \{U \cap V : U \in F, V \in H, U \cap V \neq \emptyset\}$. Then each $W \in K$ belongs to $B^{\mathrm{co}}_n(G)$. Moreover, for $U \in F$ we have $U = \bigsqcup\{W \in K : W \subseteq U\}$. Hence (3.3) gives $t_U = \sum_{W \in K, W \subseteq U} t_W$ for each $U \in F$; a similar decomposition holds for t_V for each $V \in H$. Therefore

$$\sum_{U \in F} a_U t_U = \sum_{U \in F} \sum_{W \in K, W \subseteq U} a_U t_W = \sum_{W \in K} \Big(\sum_{U \in F, W \subseteq U} a_U \Big) t_W,$$

and similarly

$$\sum_{V \in F} b_V t_V = \sum_{W \in K} \Big(\sum_{V \in F, W \subseteq V} b_V \Big) t_W.$$

Fix $W \in K$. It suffices now to show that $\sum_{U \in F, W \subseteq U} a_U = \sum_{V \in F, W \subseteq V} b_V$. By definition of K, the set W is nonempty, so let $\alpha \in W$. Then for $U \in F$, we have $\alpha \in U \implies W \cap U \neq \emptyset \implies W \subseteq U$. Since $\alpha \in W$, this implies that $\alpha \in U \iff W \subseteq U$. Hence

$$f(\alpha) = \sum_{U \in F} a_U 1_U(\alpha) = \sum_{U \in F, \alpha \in U} a_U = \sum_{U \in F, W \subseteq U} a_U.$$

a similar calculation shows that $\sum_{V \in F, W \subseteq V} b_V = f(\alpha)$ as well. So there is a linear map $\pi : A(G) \to B$ such that $\pi(1_U) = t_U$ for all $U \in B^{\mathrm{co}}_*(G)$.

We must check that π is a homomorphism. To see that π is multiplicative, fix $f, g \in A(G)$. Express $f = \sum_{U \in F} a_U 1_U$ and $g = \sum_{V \in H} b_V 1_V$ where F and H are finite subsets of $B^{\mathrm{co}}_*(G)$, and calculate:

$$\pi(fg) = \pi\Big(\Big(\sum_{U \in F} a_U 1_U\Big)\Big(\sum_{V \in H} b_V 1_V\Big)\Big) = \pi\Big(\sum_{U \in F}\sum_{V \in H} a_U b_V 1_U 1_V\Big).$$

Since [24, Proposition 4.5 (3)] gives $1_U 1_V = 1_{UV}$ for all U, V, we then have

$$\pi(fg) = \pi\Big(\sum_{U\in F}\sum_{V\in H} a_U b_V 1_{UV}\Big) = \sum_{U\in F}\sum_{V\in H} a_U b_V t_{UV}.$$

Each $t_{UV} = t_U t_V$ by (R2), so

$$\pi(fg) = \sum_{U\in F}\sum_{V\in H} a_U b_V t_U t_V = \Big(\sum_{U\in F} a_U t_U\Big)\Big(\sum_{V\in H} b_V t_V\Big) = \pi(f)\pi(g)$$

as required. □

4. $A(G)$ is dense in $C^*(G)$

Since our aim is to produce algebras associated to totally disconnected, locally compact, Hausdorff groupoids whose relationship to the groupoid C^*-algebra is analogous to that of Leavitt path algebras to graph C^*-algebras, we show in this section that the subalgebra $A(G)$ of $C_c(G)$ is dense in the full (and hence also the reduced) C^*-algebra of G. We could prove this as in [13, Proposition 4.1] or [24, Proposition 6.7] by using the Stone-Weierstrass theorem, but a direct argument takes about the same amount of effort.

We first prove a technical lemma.

Lemma 4.1. *Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. Fix a compact open bisection U of G and suppose that $f \in C_c(G)$ is supported on U. Fix $\varepsilon > 0$. There exists a finite set $\mathcal{V}$ of nonempty compact open bisections of G such that $U = \bigsqcup \mathcal{V}$ and such that for each $V \in \mathcal{V}$, we have $|f(\alpha) - f(\beta)| \leq \varepsilon$ for all $\alpha, \beta \in V$.*

Proof. For each $\gamma \in U$ let U_γ be a compact open neighbourhood of γ such that $U_\gamma \subseteq U$ and $|f(\alpha) - f(\gamma)| < \varepsilon/2$ for all $\alpha \in U_\gamma$. Since U is compact, there is a finite subset F of U such that $\{U_\gamma : \gamma \in F\}$ covers U. Let $\mathcal{V}$ be the disjointification of the U_γ as in Remark 2.4. Fix $V \in \mathcal{V}$. Then there exists $\gamma \in F$ such that $V \subseteq U_\gamma$, and then for $\alpha, \beta \in V$, we have $|f(\alpha) - f(\beta)| \leq |f(\alpha) - f(\gamma)| + |f(\gamma) - f(\beta)| < \varepsilon$. □

To state the next proposition, we recall from [21] that for a locally compact, Hausdorff, étale groupoid G, the I-norm on $C_c(G)$ is defined as follows. For $f \in C_c(G)$, let

$$\|f\|_{I,r} := \sup_{u\in G^{(0)}} \Big\{ \sum_{r(\alpha)=u} |f(\alpha)| \Big\} \quad \text{and} \quad \|f\|_{I,s} := \sup_{u\in G^{(0)}} \Big\{ \sum_{s(\alpha)=u} |f(\alpha)| \Big\}.$$

Then the I-norm of f is $\|f\|_I := \max\{\|f\|_{I,r}, \|f\|_{I,s}\}$. The I-norm dominates each of the universal norm, the reduced norm, and the uniform norm on $C_c(G)$. (See [21] for further details.)

Proposition 4.2. *Let Γ be a discrete group, G a locally compact, Hausdorff étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. With notation as above, $A(G)$ is dense in $C_c(G)$ under each of the reduced norm, the universal norm, the I-norm, and the uniform norm.*

Proof. Since the I-norm dominates the other three norms, it suffices to prove the result for the I-norm. Fix $f \in C_c(G)$ and $\varepsilon > 0$. Since f has compact support, $\operatorname{supp}(f)$ can be written as a finite union of elements of $B^{\mathrm{co}}_*(G)$. So we can write $f = \sum_{i=1}^n f_i$ where each f_i is supported on an element of $B^{\mathrm{co}}_*(G)$. For each i, apply Lemma 4.1 to $\operatorname{supp}(f_i)$ to obtain a cover $\mathcal{U}_i$ of the support of f_i by disjoint compact open bisections such that for

$U \in \mathcal{U}_i$, we have $|f_i(\alpha) - f_i(\beta)| \le \varepsilon/n$ for all $\alpha, \beta \in U_i$. For each $i \le n$ and each $U \in \mathcal{U}_i$, fix $z_{i,U} \in f(U)$, so $|f_i(\alpha) - z_{i,U}| \le \varepsilon/n$ for all $\alpha \in U$. Then let $g_i := \sum_{U \in \mathcal{U}_i} z_{i,U} 1_U$ for all $i \le n$ and define $g := \sum_{i=1}^n g_i \in A(G)$. We have

$$\|f - g\|_I \le \sum_{i=1}^n \|f_i - g_i\|_I.$$

Fix $i \le n$. It suffices to show that $\|f_i - g_i\|_I \le \varepsilon/n$. Fix $u \in G^{(0)}$. Since f_i is supported on a bisection, there is at most one $\alpha \in s^{-1}(u) \cap \operatorname{supp}(f_i)$. If there is no such α, then $\sum_{s(\alpha)=u} |(f_i - g_i)(\alpha)| = 0$ and we are done. So suppose that $\alpha \in s^{-1}(u) \cap \operatorname{supp}(f_i)$. Then there is a unique $U_0 \in \mathcal{U}_i$ such that $\alpha \in U_0$. Therefore $\sum_{s(\alpha)=u} |(f_i - g_i)(\alpha)| = |f_i(\alpha) - z_{i,U_0}| \le \varepsilon/n$. Since $u \in G^{(0)}$ was arbitrary, we conclude that $\|f_i - g_i\|_{I,s} \le \varepsilon/n$. A symmetric argument gives $\|(f_i - g_i)(\alpha)\|_{I,r} \le \varepsilon/n$. Hence $\|f_i - g_i\|_I \le \varepsilon/n$ as required. □

Proposition 4.3. *Suppose that Λ is a row-finite, k-graph with no sources and that G_Λ is the corresponding k-graph groupoid. Then $A(G_\Lambda)$ as constructed above is isomorphic to the Kumjian-Pask algebra* $\mathrm{KP}(\Lambda, \mathbb{C})$.

Proof. By [11, Corollary 3.5], $t_\lambda := 1_{Z(\lambda, s(\lambda))}$ determines a Cuntz-Krieger Λ-family in $C^*(G)$. In particular, there is a Kumjian-Pask family ([3, Definition 3.1]) for Λ determined by $t_\lambda = 1_{Z(\lambda, s(\lambda))}$ and $t_{\lambda^*} = 1_{Z(s(\lambda), \lambda)}$ for all $\lambda \in \Lambda$. It follows from the universal property of $\mathrm{KP}(\Lambda, \mathbb{C})$ that there is a homomorphism $\phi : \mathrm{KP}(\Lambda, \mathbb{C}) \to A(G_\Lambda)$ which carries each s_λ to t_λ and each s_{λ^*} to t_{λ^*}.

By [3, Theorem 3.4] the algebra $\mathrm{KP}(\Lambda, \mathbb{C})$ is spanned by the elements $t_\mu t_{\nu^*}$ where $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, and the $\mathbb{Z}$-grading of $\mathrm{KP}(\Lambda, \mathbb{C})$ carries each $s_\mu s_{\nu^*}$ to $d(\mu) - d(\nu)$. So to see that ϕ is graded, it suffices to show that it preserves the grading of each $s_\mu s_{\nu^*}$, which it does since

$$\phi(s_\mu s_{\nu^*}) = 1_{Z(\mu,\nu)} = 1_{\{(\mu x, d(\mu) - d(\nu), \nu x) : x \in \Lambda^\infty, r(x) = s(\mu)\}} \in A_{d(\mu) - d(\nu)}.$$

Since each $Z(v)$ is nonempty, $\phi(p_v) \neq 0$ for each $v \in E^0$. Thus the graded uniqueness theorem for Kumjian-Pask algebras [3, Theorem 4.1] implies that ϕ is injective.

It remains to show that ϕ is surjective. By Lemma 3.3, $A(G_\Lambda)$ is spanned by the functions 1_U where U ranges over all compact open bisections in G_Λ. Let U be a compact open bisection. Since the grading is continuous and U is compact, we can write 1_U as the finite sum $\sum_{U \cap G_n \neq \emptyset} 1_{U \cap G_n}$ where each $U \cap G_n$ is a graded compact open bisection. So fix $n \in \mathbb{N}^k$ and a compact open n-graded bisection V. It suffices to show that $1_V \in \operatorname{span}\{1_{Z(\mu,\nu)} : s(\mu) = s(\nu)\}$. Because V is compact and the sets $Z(\mu, \nu)$ form a basis for the topology on G_Λ [11, Proposition 2.8], we can write $V = \bigcup_{(\mu,\nu) \in F} Z(\mu, \nu)$ for some finite set $F \subseteq \{(\mu, \nu) \in \Lambda \times \Lambda : s(\mu) = s(\nu)\}$. Since V is n-graded, we have $d(\mu) - d(\nu) = n$ for all $(\mu, \nu) \in F$. Let $p := \bigvee_{(\mu,\nu) \in F} d(\mu)$. Then for each $(\mu, \nu) \in F$ we have $Z(\mu, \nu) = \bigcup\{Z(\mu\alpha, \nu\alpha) : \alpha \in s(\mu)\Lambda^{p - d(\mu)}\}$. Let $H := \{(\mu\alpha, \nu\alpha) : (\mu, \nu) \in F, \alpha \in s(\mu)\Lambda^{p - d(\mu)}\}$. Then $Z(\eta, \zeta) \cap Z(\eta', \zeta') = \emptyset$ for distinct $(\eta, \zeta), (\eta', \zeta') \in H$, so $V = \bigsqcup_{(\eta,\zeta) \in H} Z(\eta, \zeta)$. Hence $1_U = \sum_{(\eta,\zeta) \in H} 1_{Z(\eta,\zeta)}$, and it follows that ϕ is surjective. □

Remark 4.4. When $k = 1$ in the preceding proposition, Λ is the path category of the directed graph $E = (\Lambda^0, \Lambda^1, r, s)$ and, in this case, the proposition specialises to the statement that $A(G)$ is isomorphic to the Leavitt path algebra of [1].

5. The uniqueness theorems

Interestingly, in the situation of groupoids, the graded uniqueness theorem is a corollary of the natural generalisation of the Cuntz-Krieger uniqueness theorem. This in turn is essentially Renault's structure theorem for the reduced C^*-algebra of a groupoid in which the units with trivial isotropy are dense in the unit space. This condition has been referred to, variously, as "topologically free", "topologically principal", "essentially free."

Given a unit u, it is standard to denote the isotropy subgroup $\{\alpha \in G : r(\alpha) = s(\alpha) = u\}$ by either $G(u)$ or G_u^u. Here we have chosen the more suggestive notation uGu, which is in keeping with the notation established in (2.1). Likewise, we write Gu for $s^{-1}(u)$.

Theorem 5.1. *Let G be a locally compact, Hausdorff, étale groupoid with totally disconnected unit space. Suppose that $\{u \in G^{(0)} : uGu = \{u\}\}$ is dense in $G^{(0)}$. Let $\pi : A(G) \to B$ be a *-homomorphism into a complex *-algebra B. Suppose that* $\ker(\pi) \neq \{0\}$. *Then there is a compact open subset $K \subseteq G^{(0)}$ such that $\pi(1_K) = 0$.*

Remark 5.2. To see why the hypothesis that the units with trivial isotropy are dense is needed in Theorem 5.1, consider the situation where $G = \mathbb{Z}/2\mathbb{Z}$ regarded as a groupoid with one unit 0. Then $A(G)$ is the group algebra $\mathbb{C}\delta_0 + \mathbb{C}\delta_1$, and the map $\pi : A(G) \to \mathbb{C}$ such that $\pi(\delta_0) = \pi(\delta_1) = 1$ is a *-homomorphism of $A(G)$ which is not injective, but which restricts to an injective representation of $C_c(G^{(0)}) = \mathbb{C}\delta_0$. A related construction applies for arbitrary G — see [5, Proposition 4.4].

To prove Theorem 5.1, we need a technical lemma.

Lemma 5.3. *Let G be a locally compact, Hausdorff, étale groupoid. Fix $\alpha \in G$ and a precompact neighbourhood V of α. Suppose that $r(\alpha)Gs(\alpha) = \{\alpha\}$. Then there exist neighbourhoods X of $r(\alpha)$ and Y of $s(\alpha)$ such that XVY is a precompact open bisection.*

Proof. Suppose, to the contrary, that for every neighbourhood X of $r(\alpha)$ and every neighbourhood Y of $s(\alpha)$, XVY fails to be a bisection. Let U be an open bisection containing α. Fix a fundamental sequence of neighbourhoods $(Y_i)_{i=1}^\infty$ of $s(\alpha)$, and for each i, let $X_i := r(UY_i)$, so that $(X_i)_{i=1}^\infty$ forms a fundamental sequence of neighbourhoods of $r(\alpha)$. Since each X_iVY_i fails to be a bisection, for each i there exist $\beta_i, \gamma_i \in X_iVY_i$ with $\beta_i \neq \gamma_i$ such that either $s(\beta_i) = s(\gamma_i)$ or $r(\beta_i) = r(\gamma_i)$ for all i. The sequence $\big((\beta_i, \gamma_i)\big)_{i=1}^\infty$ belongs to the precompact set $V \times V$, so by passing to a subsequence and relabelling we may assume that $\beta_i \to \beta$ and $\gamma_i \to \gamma$. Since the X_i and Y_i are fundamental sequences of neighbourhoods, it follows that $r(\beta_i), r(\gamma_i) \to r(\alpha)$ and $s(\beta_i), s(\gamma_i) \to s(\alpha)$. Since $r, s : G \to G^{(0)}$ are continuous and $G^{(0)}$ is Hausdorff, $r(\beta) = r(\alpha) = r(\gamma)$ and $s(\beta) = s(\alpha) = s(\gamma)$. By hypothesis, $s(\alpha)Gr(\alpha) = \{\alpha\}$, so we have $\beta = \gamma = \alpha$. Since U is a neighbourhood of α, we then have $\beta_i, \gamma_i \in U$ for large i. Fix i such that $\beta_i, \gamma_i \in U$. Then $\beta_i \neq \gamma_i$ but either $r(\beta_i) = r(\gamma_i)$ or $s(\beta_i) = s(\gamma_i)$, contradicting that U is a bisection. □

Proof of Theorem 5.1. Fix $f \in \ker(\pi) \setminus \{0\}$. Since s is a local homeomorphism, it is an open map, and since f is locally constant, we deduce that $s(\mathrm{supp}(f)) \subseteq G^{(0)}$ open. Because $\{u \in G^{(0)} : uGu = \{u\}\}$ is dense in $G^{(0)}$, there exists $u \in s(\mathrm{supp}(f))$ such that $uGu = \{u\}$. Fix $\alpha \in \mathrm{supp}(f)$ with $s(\alpha) = u$. Then $r(\alpha)Gs(\alpha) = \alpha(\alpha^{-1}Gu) \subseteq \alpha(uGu) = \{\alpha\}$.

By Lemma 5.3, there exist compact open neighbourhoods X of $r(\alpha)$ and Y of $s(\alpha)$ such that $X\,\mathrm{supp}(f)Y$ is a bisection containing α. Because r and s are continuous,

$X\operatorname{supp}(f)Y = r^{-1}(X) \cap \operatorname{supp}(f) \cap s^{-1}(Y)$ is compact. Since f is locally constant, $X\operatorname{supp}(f)Y$ is also open and there exist subneighbourhoods $X_0 \subseteq r(X\operatorname{supp}(f)Y)$ of $r(\alpha)$ and $Y_0 \subseteq s(X\operatorname{supp}(f)Y)$ of $s(\alpha)$ such that $X_0\operatorname{supp}(f)Y_0$ is a compact open bisection and $f(\beta) = f(\alpha)$ for all $\beta \in X_0\operatorname{supp}(f)Y_0$.

We have $1_{X_0}, 1_{Y_0} \in A(G)$. By Lemma 3.3, f may be written as a linear combination of characteristic functions of compact open bisections. [24, Proposition 4.5 (3)] together with bilinearity of multiplication implies that for $\beta \in G$,

$$(1_{X_0} * f * 1_{Y_0})(\beta) = 1_{X_0}(r(\beta))f(\beta)1_{X_0}(s(\beta)) = 1_{X_0\operatorname{supp}(f)Y_0}(\beta)f(\beta) = 1_{X_0\operatorname{supp}(f)Y_0}(\beta)f(\alpha).$$

Thus $f_0 := 1_{X_0} * f * 1_{Y_0} = f(\alpha)1_{X_0\operatorname{supp}(f)Y_0}$. Since $\pi(f) = 0$, we have $\pi(f_0) = 0$. We have $(X_0\operatorname{supp}(f)Y_0)^{-1}(X_0\operatorname{supp}(f)Y_0) = Y_0$ because $X_0\operatorname{supp}(f)Y_0$ is a bisection. [24, Proposition 4.5 (3)] implies that

$$f_0^* * f_0 = |f(\alpha)|^2 1_{(X_0\operatorname{supp}(f)Y_0)^{-1}(X_0\operatorname{supp}(f)Y_0)} = |f(\alpha)|^2 1_{Y_0}.$$

Hence $K := Y_0$ satisfies $\pi(1_K) = \frac{1}{|f(\alpha)|^2}\pi(f_0^* * f_0) = 0$ as required. □

Our graded uniqueness theorem now follows from a bootstrapping argument.

Theorem 5.4. *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. Suppose that $\{u \in G^{(0)} : uG_eu = \{u\}\}$ is dense in $G^{(0)}$. Let B be a complex *-algebra and let $\pi : A(G) \to B$ be a graded *-homomorphism. Suppose that* $\ker(\pi) \neq \{0\}$. *Then there is a compact open subset $K \subseteq G^{(0)}$ such that $\pi(1_K) = 0$.*

Proof. We first claim that there exists nonzero $f \in A_e$ such that $\pi(f) = 0$. To see this, observe that since $\ker(\pi) \neq 0$, there exists $g \in \ker(\pi) \setminus \{0\}$ such that $\pi(g) = 0$. Since g is an element of the graded algebra $A(G)$, g can be expressed as a finite sum of graded components $g = \sum_{h\in F} g_h$ where $F \subseteq \Gamma$ and each $g_h \in A_h$. Now $\pi(g) = \sum_{h\in F} \pi(g_h) = 0$, and each $\pi(g_h) \in B_h$ because π is a graded homomorphism. Because the graded subspaces of B are linearly independent, it follows that each $\pi(g_h) = 0$. Since $g \neq 0$, there exists $k \in F$ such that $g_k \neq 0$. By Lemma 3.5, we can write g_k as $\sum_{V\in K} a_V 1_V$ where K is a finite set of mutually disjoint elements of $B_k^{\mathrm{co}}(G)$. Note that $g_k^* = \sum_{V\in K} \overline{a_V} 1_{V^{-1}}$; define $f := g_k^* * g_k$. We claim that $f \in A_e \setminus \{0\}$ and $\pi(f) = 0$. To see this, first notice that

$$f = \Big(\sum_{V\in K} \overline{a_V} 1_{V^{-1}}\Big) * \Big(\sum_{W\in K} a_W 1_W\Big) = \sum_{V,W\in K} \overline{a_V} a_W 1_{V^{-1}} * 1_W = \sum_{V,W\in K} \overline{a_V} a_W 1_{V^{-1}W}$$

by [24, Proposition 4.5 (3)]. Now, because each $V \in K$ is a subset of G_k, each $V^{-1}W \subseteq G_{k^{-1}k} = G_e$, and thus $f \in A_e$ as claimed. We have $\pi(f) = 0$ because $\pi(g_k) = 0$.

To show that f is nonzero, fix $\alpha \in G_k$ such that $g(\alpha) \neq 0$. Since the elements of K are mutually disjoint, there is a unique $V_\alpha \in K$ such that $\alpha \in V_\alpha$, and then $a_{V_\alpha} = g(\alpha) \neq 0$. Since s is a local homeomorphism, $Gs(\alpha)$ is a discrete space. Write $C_c(Gs(\alpha))$ for the space of finitely supported functions from $Gs(\alpha)$ to $\mathbb{C}$ and for each $\beta \in Gs(\alpha)$ let δ_β denote the point-mass at β so that $C_c(Gs(\alpha)) = \operatorname{span}\{\delta_\beta : \beta \in Gs(\alpha)\}$. For $f \in C_c(G)$, let $\rho(f)$ be the linear map on $C_c(Gs(\alpha))$ determined by

$$\rho(f)\delta_\beta = \sum_{s(\alpha)=r(\beta)} f(\alpha)\delta_{\alpha\beta}.$$

Let $(\cdot \mid \cdot)$ be the standard inner product on $C_c(Gs(\alpha))$, that is $(f|g) = \sum_\beta \overline{f(\beta)}g(\beta)$. Since the elements of K are mutually disjoint, $\big(\rho(1_V)\delta_{s(\alpha)} \mid \rho(1_W)\delta_{s(\alpha)}\big) = 0$ for distinct $V, W \in$

K. A calculation shows that for $V \in K$ and $\beta, \gamma \in Gs(\alpha)$, we have $(\delta_\beta | \rho(1_{V^{-1}})\delta_\gamma) = (\rho(1_V)\delta_\beta | \delta_\gamma)$. Hence

$$\begin{aligned}\big(\rho(f)\delta_{s(\alpha)} \mid \delta_{s(\alpha)}\big) &= \big(\rho(g_k)\delta_{s(\alpha)} \mid \rho(g_k)\delta_{s(\alpha)}\big) \\ &= \sum_{V,W \in K} \overline{a_V} a_W \big(\rho(1_W)\delta_{s(\alpha)} \mid \rho(1_V)\delta_{s(\alpha)}\big) = \sum_{\substack{V \in K,\\ s(\alpha) \in s(V)}} |a_V|^2 \geq |a_{V_\alpha}|^2.\end{aligned}$$

Hence $\rho(f) \neq 0$ which forces $f \neq 0$.

By hypothesis $\{u \in G^{(0)} : uG_e u = \{u\}\}$ is dense in $G^{(0)}$. By definition, A_e is equal to the space of locally constant, continuous, compactly supported functions on G_e, so we may apply Theorem 5.1 to see that $\pi|_{A_e} : A_e \to B$ annihilates 1_K for some compact open $K \subseteq G_e^{(0)} = G^{(0)}$. □

Corollary 5.5. *Let Γ be a discrete group, G a locally compact, Hausdorff, étale groupoid with totally disconnected unit space, and $c : G \to \Gamma$ a continuous cocycle. Suppose that $\{u \in G^{(0)} : uG_e u = \{u\}\}$ is dense in $G^{(0)}$. Let B be a Γ-graded complex algebra and let $\{t_U : U \in B_*^{\mathrm{co}}(G)\}$ be a representation of $B_*^{\mathrm{co}}(G)$ in B. Suppose that $t_U \in B_n$ whenever $U \in B_n^{\mathrm{co}}(G)$ and that $t_K \neq 0$ for each compact open $K \subseteq G^{(0)}$. Then the homomorphism $\pi : A(G) \to B$ obtained from Theorem 3.10 is injective.*

Proof. Since each $A(G)_n$ is spanned by $\{1_U : U \in B_n^{\mathrm{co}}(G)\}$, the homomorphism π is graded. Since $\pi(1_K) = t_K \neq 0$ for all compact open $K \subseteq G^{(0)}$, it follows from Theorem 5.4 that $\ker(\pi) = \{0\}$. □

Remark 5.6. Suppose that G is a locally compact, Hausdorff, étale groupoid with totally disconnected locally compact unit space such that $\{u \in G^{(0)} : uGu = \{u\}\}$ is dense in $G^{(0)}$. We may apply Corollary 5.5 with c the trivial cocycle to prove that $A(G)$ is the unique algebra generated by nonzero elements $\{t_U : U$ is a compact open bisection of $G\}$ satisfying

(1) $t_\emptyset = 0$;
(2) $t_U t_V = t_{UV}$ for all compact open bisections U, V; and
(3) $t_U + t_V = t_{U \cup V}$ whenever U and V are disjoint compact open bisections whose union is a bisection.

Remark 5.7. In the proof of Theorem 5.4, to see that the function $g_k^* * g_k$ was nonzero, we really just checked that its image under Renault's left-regular representation of G associated to the unit $s(\alpha)$ is nonzero. However, since we are not working in a C^*-completion, we can do everything at the level of linear algebra rather than on Hilbert space. We could instead have appealed to the C^*-identity by regarding $A(G)$ as a subalgebra of $C_r(G)$, but chose a more elementary argument: our argument is essentially that used by Renault to show that the reduced norm is positive definite on $C_c(G)$.

Remark 5.8. Recall from [9] that if Λ is a finitely aligned k-graph, then the k-graph groupoid G_Λ is totally disconnected and locally compact, and carries a $\mathbb{Z}^k$-grading such that $\{u \in G^{(0)} : uG_e u = \{u\}\}$ is dense in $G^{(0)}$. So our graded uniqueness theorem applies to $A(G_\Lambda)$ for any finitely aligned k-graph. Likewise, Remark 5.6 suggests a Cuntz-Krieger uniqueness theorem for $A(G_\Lambda)$. But in practise the relations described in Definition 3.9 and Remark 5.6 are much harder to verify than those of [3, Definition 3.1].

We do not, at this stage, have any invariants at our disposal to decide whether, given groupoids G and G' satisfying our hypotheses, the algebras $A(G)$ and $A(G')$ are or are not isomorphic. It would be very interesting to develop computable algebraic invariants of $A(G)$ for this purpose, but it is beyond the scope of this paper.

However, as an indication that our construction is more flexible the construction of Kumjian-Pask algebras in [3], we describe a class of groupoids that satisfy our hypotheses but do not obviously arise from k-graphs.

Example 5.9. Let $T : X \to X$ be a surjective local homeomorphism of a totally disconnected, compact, Hausdorff space X. Define $T^0 := \mathrm{id}$ and for $k \geq 2$ let $T^k := T \circ \cdots \circ T$ be the k-fold self-composite of T. Let G be the Deaconu-Renault groupoid defined in [10, Section 3]. So

$$G = \{(x, n, y) \in X \times \mathbb{Z} \times X \ : \ T^k(x) = T^l(y), n = k - l\}.$$

Let $G^{(0)}$ be the subset $\{(x, 0, x) : x \in X\}$, which we identify with X in the obvious way. The range and source maps are given by $r(x, n, y) = x$ and $s(x, n, y) = y$. Hence triples (x_1, n_1, y_1) and (x_2, n_2, y_2) are composable if and only if $x_2 = y_1$, in which case $(x_1, n_1, y_1)(x_2, n_2, y_2) := (x_1, n_1 + n_2, y_2)$. The inverse of (x, n, y) is $(y, -n, x)$. For open subsets $U, V \subseteq X$ and $k, l \geq 0$ such that $T^k|_U$ and $T^l|_V$ are homeomorphisms and $T^k(U) = T^l(V)$, define

$$Z(U, V, k, l) := \{(x, k - l, y) \in G : x \in U, y \in V\}.$$

Then

$$\{Z(U, V, k, l) : U, V \subseteq X \text{ are compact open}, k, l \geq 0,$$
$$T^k|_U \text{ and } T^l|_V \text{ are homeomorphisms and } T^k(U) = T^l(V)\}$$

is a basis of compact open sets for a topology on G under which it becomes a locally compact, Hausdorff groupoid with totally disconnected unit space X. Fix $(x, n, y) \in G$ and k, l such that $k - l = n$ and $T^k(x) = T^l(y)$. The source map on G restricts to a homeomorphism on each basic open set $Z(U, V, k, l)$ so is a local homeomorphism. Moreover, the map $c : G \to \mathbb{Z}$ defined by $c((x, n, y)) = n$ is a cocycle and is continuous because each basic open set belongs to some $c^{-1}(n)$. Hence (G, c) satisfies our hypotheses, and $A(G)$ is a sensible candidate for the Leavitt algebra of (X, T).

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