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Some new constructions of orthogonal designs

Tianbing Xia

University of Wollongong, txia@uow.edu.au

Jennifer Seberry

University of Wollongong, jennie@uow.edu.au

Mingyuan Xia

Central China Normal University, xiamy@mail.ccnu.edu.cn

Shangli Zhang

Beijing Jiaotong University, shlzhang@center.njtu.edu.cn

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Keywords

era2015, constructions, designs, orthogonal

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Some new constructions of orthogonal designs

TIANBING XIA JENNIFER SEBERRY

*School of CS and SE, Informatics
University of Wollongong
NSW 2522
Australia*

txia@uow.edu.au j.seberry@uow.edu.au

MINGYUAN XIA

*School of Mathematics and Statistics
Central China Normal University
Wuhan, Hubei 430079
China*

xiamy@mail.ccnu.edu.cn

SHANGLI ZHANG

*School of Science
Beijing Jiaotong University
Beijing
China*

shlzhang@center.njtu.edu.cn

Abstract

In this paper we construct $\text{OD}(4pq^r(q+1); pq^r, pq^r, pq^r, pq^r, pq^{r+1}, pq^{r+1}, pq^{r+1}, pq^{r+1})$ for each core order $q \equiv 3 \pmod{4}$, $r \geq 0$ or $q = 1$, p odd, $p \leq 21$ and $p \in \{25, 49\}$, and $\text{COD}(2q^r(q+1); q^r, q^r, q^{r+1}, q^{r+1})$ for any prime power $q \equiv 1 \pmod{4}$ (including $q = 1$), $r \geq 0$.

1 Introduction

An orthogonal design (OD) X of order n and type (s_1, \dots, s_m) , s_i positive integers, is an $n \times n$ matrix with entries $\{0, \pm x_1, \dots, \pm x_m\}$ (the x_i are commuting indeterminates) satisfying

$$XX^T = \left(\sum_{i=1}^m s_i x_i^2 \right) I_n,$$

where I_n is the identity matrix of order n . This is denoted by $\text{OD}(n; s_1, \dots, s_m)$.

Such generically orthogonal matrices have played a significant role in the construction of Hadamard matrices (see, e.g., [3], [6]) and they have been extensively used in the study of weighing matrices (e.g. [3] and [8]).

Since Baumert and Hall [9] gave the first example of Baumert-Hall arrays, or $OD(4t; t, t, t, t)$, and Plotkin [7] defined Plotkin arrays, or $OD(8t; t, t, t, t, t, t, t, t)$, to construct Hadamard matrices, many research results have been published for T -matrices that are used in the construction of Plotkin arrays (see [3], [5], [9], [10]).

Turyan [11] introduced the notion of a complex Hadamard matrix, i.e., an $n \times n$ matrix C whose entries are chosen from $\{\pm 1, \pm i\}$ and satisfy $CC^* = nI_n$ ($*$ is conjugate transpose). He further showed how such matrices could be used to construct Hadamard matrices, and gave several examples. Further examples of such matrices are given in [3] and [4].

For a complex analogue of orthogonal designs there are several possible generalizations; we choose the one which gives real orthogonal designs as a special case.

A complex orthogonal design (COD) [4] of order n and type (s_1, \dots, s_m) (s_i positive integers) on the real commuting variables x_1, \dots, x_m is an $n \times n$ matrix X , with entries chosen from $\{\varepsilon_1 x_1, \dots, \varepsilon_m x_m : \varepsilon_i \text{ a fourth root of } 1\}$ satisfying

$$XX^* = \left(\sum_{i=1}^m s_i x_i^2 \right) I_n.$$

For further discussion we need the following definitions from [6].

Definition 1 [Amicable Matrices; Amicable Set] Two square real matrices of order n , A and B , are said to be *amicable* if $AB^T - BA^T = 0$.

A set $\{A_1, \dots, A_{2n}\}$ of square real matrices is said to be an *amicable set* if

$$\sum_{i=1}^n (A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T) = 0.$$

It is easy to generalize an amicable set to the case of square complex matrices. For this, we just need to replace A^T by A^* , the conjugate transpose of A .

Definition 2 [T -matrices] $(0, \pm 1)$ type 1 matrices T_1, T_2, T_3 and T_4 of order n are called *T -matrices* if the following conditions are satisfied:

- (a) $T_i * T_j = 0$, $i \neq j$, $1 \leq i, j \leq 4$, where $*$ denotes Hadamard product;
- (b) $\sum_{i=1}^4 T_i T_i^T = nI_n$.

T -matrices can be used to construct orthogonal designs (see [1]).

The following definition was first used by Holzmann and Kharaghani in [5].

Definition 3 [Weak amicable] The T -matrices T_1, T_2, T_3 and T_4 are said to be *weak amicable* if

$$T_1(T_3 + T_4)^T + T_2(T_3 - T_4)^T = (T_3 + T_4)T_1^T + (T_3 - T_4)T_2^T.$$

Definition 4 [Core] Let Q be a matrix of order n , with zero diagonal and all other elements ± 1 satisfying

$$QQ^T = nI_n - J_n, \quad QJ_n = J_nQ = 0,$$

where J_n is the matrix of order n , consisting entirely of 1's. Further if $n \equiv 1 \pmod{4}$, $Q^T = Q$, and if $n \equiv 3 \pmod{4}$, then $Q^T = -Q$. Here Q is called the *core* and n is the *core order*.

If $H = I_n + K$ is an Hadamard matrix of order n with $K^T = -K$, we call it skew type Hadamard matrix.

Here we rewrite the following theorem as

Theorem 1 ([12]) *If there exists a skew type Hadamard matrix of order $q + 1$, then there exists a core of order q .*

It is well-known that if $q + 1 = 2^t n_1 \dots n_s$, each n_i of the form $p^r + 1 \equiv 0 \pmod{4}$, and p is prime, then q is a core order. Moreover, if $q \equiv 3 \pmod{4}$ is a core order, then q^r is a core order for any odd $r \geq 1$ (see [9], p. 497).

In Section 2 we give an infinite class of OD with 8 variables. In Section 3 we construct several families of COD with 4 variables. In Section 4 we construct weak amicable T -matrices.

2 The construction of OD

The Goethals-Seidel (or Wallis-Whiteman) array has been proven to be a very useful tool for construction of orthogonal designs. Such arrays are essential for construction of orthogonal designs with more than four variables.

For convenience we need following definition:

Definition 5 [Additive property] A set of matrices $\{B_1, \dots, B_m\}$ of order n with entries in $\{0, \pm x_1, \dots, \pm x_k\}$ is said to satisfy the *additive property*, with weight $\sum_{i=1}^k s_i x_i^2$, if

$$\sum_{i=1}^m B_i B_i^T = \left(\sum_{i=1}^k s_i x_i^2 \right) I_n. \tag{1}$$

Kharaghani [6] gave an infinite number of arrays which are suitable for any amicable set of 8 type 1 matrices. Here **suitable** means a set of matrices satisfying the **additive property**. If one substitutes the matrices in an orthogonal design, or the Goethals-Seidel array, one can get an orthogonal design. We rewrite the following theorems without proof.

Theorem 2 ([6]) *There is an 8×8 array which is suitable to make an $8n \times 8n$ orthogonal matrix for any amicable set of 8 type 1 matrices of order n satisfying an additive property.*

Theorem 3 ([6]) *For each prime power $q \equiv 3 \pmod{4}$ there is an array suitable for any amicable set of eight matrices A_i satisfying*

$$\sum_{i=1}^4 (A_{2i-1}A_{2i}^T + A_{2i}A_{2i-1}^T) = cI_{q+1},$$

where c is a constant expression.

More general results are given in [2]. As an application we give an example of such an OD.

If A is a circulant matrix of order n with the first row (a_1, \dots, a_n) , we denote it by

$$A = \text{circ}(a_1, \dots, a_n).$$

Example 1 Let x_1, x_2, x_3, x_4 and x_5 be real commuting variables and

$$\begin{aligned} A_1 &= \text{circ}(x_1, x_2, x_3, x_4, -x_4, -x_3, x_2), & A_2 &= \text{circ}(-x_1, x_2, x_3, -x_4, x_4, -x_3, x_2), \\ A_3 &= \text{circ}(x_1, -x_2, x_3, -x_4, x_4, -x_3, -x_2), & A_4 &= \text{circ}(x_1, x_2, -x_3, -x_4, x_4, x_3, x_2), \\ A_5 &= \text{circ}(x_5, x_2, x_3, x_4, x_4, x_3, -x_2), & A_6 &= \text{circ}(-x_5, x_2, x_3, x_4, x_4, x_3, -x_2), \\ A_7 &= \text{circ}(x_5, -x_2, x_3, -x_4, -x_4, x_3, x_2), & A_8 &= \text{circ}(-x_5, -x_2, x_3, -x_4, -x_4, x_3, x_2). \end{aligned}$$

It is easy to verify that

$$\sum_{i=1}^4 (A_{2i-1}A_{2i}^T - A_{2i}A_{2i-1}^T) = 0 \text{ and } \sum_{i=1}^8 A_iA_i^T = (4(x_1^2 + x_5^2) + 16(x_2^2 + x_3^2 + x_4^2))I_7.$$

From the proof of Theorem 2, using the method in [6], one can construct an $OD(56; 4, 4, 16, 16, 16)$.

Theorem 4 *Let $q \equiv 3 \pmod{4}$ be a core order. Then there is an $OD(4q^r(q + 1); q^r, q^r, q^r, q^r, q^{r+1}, q^{r+1}, q^{r+1}, q^{r+1})$ for any integer $r \geq 0$.*

Proof. Let Q be a core of order q , and let a_1, \dots, a_8 be real commuting variables. Set

$$A_{2i-1}(0) = a_{2i}, \quad A_{2i}(0) = a_{2i-1}, \quad i = 1, 2, 3, 4.$$

It is clear that, as $A_i(0)$ are commuting variables,

$$\begin{aligned} A_1(0), \dots, A_8(0) &\text{ are type 1,} \\ A_{2i-1}(0)A_{2i}^T(0) &= A_{2i}(0)A_{2i-1}^T(0), \quad i = 1, 2, 3, 4, \end{aligned}$$

and (with $q^0 = 1$),

$$A_{2i-1}(0)A_{2i-1}^T(0) + qA_{2i}(0)A_{2i}^T(0) = q^0(qa_{2i-1}^2 + a_{2i}^2)I_{q^0}, \quad i = 1, 2, 3, 4.$$

Suppose that for $r \geq 1$ we have

$$\begin{aligned} &A_1(r-1), \dots, A_8(r-1) \text{ are all type 1} \\ &A_{2i-1}(r-1)A_{2i}^T(r-1) = A_{2i}(r-1)A_{2i-1}^T(r-1), \text{ and} \\ &A_{2i-1}(r-1)A_{2i-1}^T(r-1) + qA_{2i}(r-1)A_{2i}^T(r-1) = q^{r-1}(qa_{2i-1}^2 + a_{2i}^2)I_{q^{r-1}}, \\ &i = 1, 2, 3, 4. \end{aligned}$$

Write

$$A_{2i-1}(r) = J_q \times A_{2i}(r-1), \quad A_{2i}(r) = I_q \times A_{2i-1}(r-1) + Q \times A_{2i}(r-1),$$

where \times is the Kronecker product. Then $A_1(r), \dots, A_8(r)$ are type 1 of size q^r .

It is easy to verify that

$$\begin{aligned} &A_{2i-1}(r)A_{2i}^T(r) = A_{2i}(r)A_{2i-1}^T(r), \\ &A_{2i-1}(r)A_{2i-1}^T(r) + qA_{2i}(r)A_{2i}^T(r) = q^r(qa_{2i-1}^2 + a_{2i}^2)I_{q^r}, \quad i = 1, 2, 3, 4. \end{aligned}$$

Now let B_i of size $(q+1)q^r$ be given by

$$B_i = I_{q+1} \times A_{2i-1}(r) + K \times A_{2i}(r), \quad i = 1, 2, 3, 4, \quad K = \begin{bmatrix} 0 & e^T \\ -e & Q \end{bmatrix},$$

where $e^T = (1, \dots, 1)$ is a row vector with q components.

Then B_1, B_2, B_3 and B_4 are of type 1 and

$$\sum_{i=1}^4 B_i B_i^T = \sum_{i=1}^4 q^r (qa_{2i-1}^2 + a_{2i}^2) I_{q^r(q+1)}.$$

From Theorem 3 it follows that there is an $OD(4q^r(q+1); q^r, q^r, q^r, q^r, q^{r+1}, q^{r+1}, q^{r+1}, q^{r+1})$. □

Note that Corollary 5 of [6] is a special case of Theorem 4 with $r = 0$.

If there are type 1 T -matrices of order n , then there exist an $OD(4n; n, n, n, n)$ (see [9]). Further, from [5], weak amicable sets can be used to get the following.

Lemma 1 *For p odd, $1 \leq p \leq 21$, $p \in \{25, 49\}$, there exists an $OD(8p; p, p, p, p, p, p, p, p)$.*

Proof. For each $p \in \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 25, 49\}$, there exist T -matrices T_1, T_2, T_3 and T_4 of order p satisfying weak amicability.

The explicit construction of such T -matrices of these orders can be found in Table 1 of [5] and the Appendix of this paper. From Theorem 5 and Corollary 6 of [5], there exist $OD(8p; p, p, p, p, p, p, p, p)$. □

Theorem 5 *Let T_1, T_2, T_3 and T_4 be T -matrices of order p with weak amicability. Then there is an $OD(4pq^r(q+1); pq^r, pq^r, pq^r, pq^r, pq^{r+1}, pq^{r+1}, pq^{r+1}, pq^{r+1})$ for each core order $q \equiv 3 \pmod{4}$ and $r \geq 0$.*

Proof. Write

$$f(a, b, c, d) = T_1a + T_2b + T_3c + T_4d.$$

Here a, b, c and d are real commuting variables. Let A_1, \dots, A_8 be defined as follows:

$$\begin{aligned} A_1 &= f(x_1, x_2, x_3, x_4), & A_2 &= f(-x_8, -x_7, x_6, x_5), \\ A_3 &= f(x_2, -x_1, x_4, -x_3), & A_4 &= f(x_7, -x_8, -x_5, x_6), \\ A_5 &= f(x_3, -x_4, -x_1, x_2), & A_6 &= f(x_5, x_6, x_7, x_8), \\ A_7 &= f(x_4, x_3, -x_2, -x_1), & A_8 &= f(x_6, -x_5, x_8, -x_7), \end{aligned}$$

where x_1, \dots, x_8 are real commuting variables. Set

$$A_{2i}(0) = A_{2i-1}, \quad A_{2i-1}(0) = A_{2i}, \quad i = 1, 2, 3, 4.$$

For $r \geq 1$ let

$$\begin{aligned} A_{2i-1}(r) &= J_q \times A_{2i}(r-1), \\ A_{2i}(r) &= I_q \times A_{2i-1}(r-1) + Q \times A_{2i}(r-1), \quad i = 1, 2, 3, 4, \end{aligned}$$

where Q is a square matrix of order q defined as in Theorem 4. Replacing

$$\begin{aligned} A_{2i-1}(r)A_{2i}^T(r) &= A_{2i}(r)A_{2i-1}^T(r), \\ A_{2i-1}(r)A_{2i-1}^T(r) + qA_{2i}(r)A_{2i}^T(r) &= q^r(qa_{2i-1}^2 + a_{2i}^2)I_{q^r}, \quad i = 1, 2, 3, 4, \quad r \geq 0, \end{aligned}$$

by

$$\begin{aligned} \sum_{i=1}^4 (A_{2i-1}(r)A_{2i}^T(r) - A_{2i}(r)A_{2i-1}^T(r)) &= 0, \\ \sum_{i=1}^4 (A_{2i-1}(r)A_{2i-1}^T(r) + qA_{2i}(r)A_{2i}^T(r)) &= pq^r \sum_{i=1}^4 (qx_i^2 + x_{i+4}^2)I_{pq^r}, \end{aligned}$$

respectively, and repeating the procedure of the proof of Theorem 4, one can obtain the theorem. □

Corollary 1 *For p odd, $1 \leq p \leq 21$ and $p \in \{25, 49\}$, there exists an $OD(8pq^r(q+1); pq^r, pq^r, pq^r, pq^r, pq^{r+1}, pq^{r+1}, pq^{r+1}, pq^{r+1})$ with each core order $q \equiv 3 \pmod{4}$ and integer $r \geq 0$.*

3 The construction of COD

In this section we give several infinite classes of COD.

Theorem 6 *There exists a $COD(2q^r(q+1); q^r, q^r, q^{r+1}, q^{r+1})$ for each prime power $q \equiv 1 \pmod{4}$ and $r \geq 0$.*

Proof. Let Q be the symmetric core of order $q \equiv 1 \pmod{4}$.

Now let

$$A_{2i-1}(0) = a_{2i-1}, \quad A_{2i}(0) = a_{2i}, \quad i = 1, 2,$$

where a_1, a_2, a_3 and a_4 are real commuting variables. Note that $q^0 = 1$. It is clear that

$$\begin{aligned} A_{2i-1}(0)A_{2i}^*(0) &= A_{2i}(0)A_{2i-1}^*(0), \\ A_{2i-1}(0)A_{2i-1}^*(0) + qA_{2i}(0)A_{2i}^*(0) &= q^0(a_{2i-1}^2 + qa_{2i}^2)I_{q^0}, \quad i = 1, 2, \\ A_i(0)A_j(0) &= A_j(0)A_i(0), \quad 1 \leq i, j \leq 4. \end{aligned}$$

Suppose that for $r \geq 1$ we have

$$\begin{aligned} A_{2i-1}(r-1)A_{2i}^*(r-1) &= A_{2i}(r-1)A_{2i-1}^*(r-1), \\ A_{2i-1}(r-1)A_{2i-1}^*(r-1) + qA_{2i}(r-1)A_{2i}^*(r-1) &= q^{r-1}(a_{2i-1}^2 + qa_{2i}^2)I_{q^{r-1}}, \\ &\quad i = 1, 2, \\ A_i(r-1)A_j(r-1) &= A_j(r-1)A_i(r-1), \\ &\quad 1 \leq i, j \leq 4. \end{aligned}$$

Write

$$A_{2j-1}(r) = J_q \times A_{2j}(r-1), \quad A_{2j}(r) = I_q \times A_{2j-1}(r-1) + iQ \times A_{2j}(r-1),$$

$i = \sqrt{-1}, j = 1, 2$. It follows that

$$\begin{aligned} A_{2i-1}(r)A_{2i}^*(r) &= A_{2i}(r)A_{2i-1}^*(r), \\ A_{2i-1}(r)A_{2i-1}^*(r) + qA_{2i}(r)A_{2i}^*(r) &= q^r(a_{2i-1}^2 + qa_{2i}^2)I_{q^r}, \quad i = 1, 2, \\ A_i(r)A_j(r) &= A_j(r)A_i(r), \quad 1 \leq i, j \leq 4. \end{aligned}$$

Let

$$K = \begin{bmatrix} 0 & e^T \\ e & Q \end{bmatrix}.$$

Put

$$F_j = I_{q+1} \times A_{2j-1}(r) + iK \times A_{2j}(r), \quad i = \sqrt{-1}, j = 1, 2.$$

We have

$$\begin{aligned} F_j F_j^* &= q^r(a_{2j-1}^2 + qa_{2j}^2)I_{q^r(q+1)}, \quad j = 1, 2, \\ F_1 F_2 &= F_2 F_1. \end{aligned}$$

Finally, let

$$X = \begin{pmatrix} F_1 & F_2 \\ -F_2^* & F_1^* \end{pmatrix}.$$

Then X is a COD($2q^r(q+1); q^r, q^r, q^{r+1}, q^{r+1}$), as required. \square

From the proof of Theorem 6 we can obtain the following theorem.

Theorem 7 *There is a COD($q^r(q+1); q^r, q^{r+1}$) for each prime power $q \equiv 1 \pmod{4}$ and $r \geq 0$.*

4 The construction of weak amicable T -matrices

It is convenient to use the group ring $Z[G]$ of the group G of order p over the ring Z of rational integers with the addition and multiplication. Elements of $Z[G]$ are of the form

$$a_1g_1 + a_2g_2 + \cdots + a_pg_p, \quad a_i \in Z, \quad g_i \in G, \quad 1 \leq i \leq p.$$

In $Z[G]$ the addition, $+$, is given by the rule

$$\left(\sum_g a(g)g \right) + \left(\sum_g b(g)g \right) = \sum_g (a(g) + b(g))g.$$

The multiplication in $Z[G]$ is given by the rule

$$\left(\sum_g a(g)g \right) \left(\sum_h b(h)h \right) = \sum_k \left(\sum_{gh=k} a(g)b(h) \right) k.$$

For any subset A of G , we define

$$\sum_{g \in A} g \in Z[G],$$

and by abusing the notation we will denote it by A .

Let a set $\{X_1, \dots, X_8\}$ be a C -partition of an abelian additive group G of order p , i.e.,

$$X_i \subset G, \quad X_i \cap X_j = \emptyset, \quad i \neq j,$$

and

$$\sum_{i=1}^8 X_i = G, \quad \sum_{i=1}^8 X_i X_i^{(-1)} = p + \sum_{i=1}^4 \left(X_i X_{i+4}^{(-1)} + X_{i+1} X_i^{(-1)} \right),$$

where the equations above hold in the group ring $Z[G]$; (see [13]).

For any $A \subset G$, set

$$I(A) = (a_{ij})_{1 \leq i, j \leq n}, \quad a_{ij} = \begin{cases} 1, & \text{if } g_j - g_i \in A, \\ 0, & \text{otherwise,} \end{cases}$$

where g_1, \dots, g_p are elements of G in any order. That is, $I(A)$ is the $(0, 1)$ incidence matrix of A of type 1. Now let

$$T_i = I(X_i) - I(X_{i+4}), \quad i = 1, 2, 3, 4;$$

then T_1, T_2, T_3 and T_4 are T -matrices of order p .

Let $\sum_g a(g)g \in Z[G]$ where $a(g) \in Z$ and $g \in G$. If, for any $g \in G$, we have $a(g) = a(-g)$, then we call $\sum_g a(g)g$ symmetric in the group ring $Z[G]$.

It is clear that T -matrices T_1, T_2, T_3 and T_4 of order p satisfy weak amicability, if and only if $T_1(T_3 + T_4)^T + T_2(T_3 - T_4)^T$ is symmetric, and if and only if $(X_1 - X_5)(X_3^{(-1)} - X_7^{(-1)} + X_4^{(-1)} - X_8^{(-1)}) + (X_2 - X_6)(X_3^{(-1)} - X_7^{(-1)} - X_4^{(-1)} + X_8^{(-1)})$ is symmetric in the group ring $Z[G]$.

The following theorem and corollary will simplify the verification of weak amicability in some cases.

Theorem 8 *Let G be an abelian group of order n and let $\{X_1, \dots, X_8\}$ be a C -partition of G . If both $X_1 - X_5 + X_2 - X_6$ and $X_3 - X_7 + X_4 - X_8$ are symmetric in the group ring $Z[G]$, then there exist T -matrices of order n satisfying weak amicability if and only if $(X_2 - X_6)(X_4^{(-1)} - X_8^{(-1)})$ is also symmetric in the group ring $Z[G]$.*

Using the same assumptions as in Theorem 8, we have the following corollary.

Corollary 2 *If $X_4 = X_8 = \emptyset$, then there exist T -matrices of order n satisfying weak amicability.*

Appendix

Now we give decomposition of the sum of four squares and the new sets of T -matrices which have weak amicability for $p = 9, 25, 49$. The values $1 \leq p \leq 21$ are given in Holtzmann and Kharaghani [5].

$$\begin{aligned}
 p = 9 &= 3^2 + 0^2 + 0^2 + 0^2, & Q_1 &= \{0, 1, x + 1\}, Q_2 = \{2\} - \{x + 2\}, \\
 & & Q_3 &= \{2x\} - \{2x + 2\}, Q_4 = \{2x + 1\} - \{x\}. \\
 p = 25 &= 5^2 + 0^2 + 0^2 + 0^2, & Q_1 &= \{0\} - E_0 \cup E_1, Q_2 = E_2 - E_6, Q_3 = E_3 - E_7, \\
 & & Q_4 &= E_4 - E_5, \\
 & & & \text{where } E_i = \{g^{8j+i} : j = 0, 1, 2\}, i = 0, \dots, 7, \\
 & & & \text{and } g = x + 1 \pmod{x^2 - 3, 5} \text{ is a generator} \\
 & & & \text{of GF}(25). \\
 p = 49 &= 7^2 + 0^2 + 0^2 + 0^2, & Q_1 &= \{0\} \cup E_0 \cup E_1 \cup E_6 \cup E_{12} - E_3 \cup E_7, \\
 & & Q_2 &= E_4 \cup E_{10} \cup E_{15} - E_8 \cup E_{11} \cup E_{13}, \\
 & & Q_3 &= E_9 - E_2, Q_4 = E_5 - E_{14}, \text{ where} \\
 & & & E_i = \{g^{16j+i} : j = 0, 1, 2\}, i = 0, \dots, 15, \text{ and} \\
 & & & g = x + 2 \pmod{x^2 + 1, 7} \text{ is a generator} \\
 & & & \text{of GF}(49).
 \end{aligned}$$

Remark. Holtzmann and Kharaghani [5] have given constructions of weak amicable T -matrices of order 9 in Z_9 and for $9 = 2^2 + 2^2 + 1^2 + 0^2$. However, our construction is given in $\text{GF}(9)$ and for $9 = 3^2$. These constructions are different in essence.

Conjecture ([5]) There exist infinite orders of T -matrices satisfying weak amicability for all odd integers.

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