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Monotonicity of reduced volume; local non-collapsing

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Abstract

The talk is divided in two parts. In the first one we give the definition of the reduced volume and by means of reduced length we prove it to be non-increasing in backwards time.

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Monotonicity of reduced volume; local non-collapsing

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The talk is divided in two parts. In the first one we give the definition of the reduced volume and by means of reduced length we prove it to be non-increasing in backwards time.

Definition 1 ([2]). *The reduced volume of a backwards Ricci flow solution $(M, g(\tau))$ is defined as*

$$\tilde{V}(\tau) = \int_M \tau^{-\frac{n}{2}} e^{-l(q,\tau)} dq.$$

Proposition 1 ([2]). *$\tilde{V}(\tau)$ is non-increasing in τ (non-decreasing in t).*

The proof comes from writing the reduced volume in terms of the exponential map as (details in [3]):

$$\tilde{V}(\tau) = \int_{T_pM} \tau^{-\frac{n}{2}} e^{-l(\mathcal{L}exp_\tau(v),\tau)} \mathcal{J}(v,\tau) \aleph_\tau(v) dv$$

where $\mathcal{J}(v,\tau) = det\ d(\mathcal{L}exp_\tau(v))$ and $\aleph_\tau(v)$ is a cut off function related to the $\mathcal{L}$-cut locus of point $p \in M$.
The tangential injectivity domain and the Jacobian will have the following properties:

$$\Omega^{T_p}(\tau_2) \subset \Omega^{T_p}(\tau_1), \quad \forall \quad \tau_1 \leq \tau_2$$
$$\frac{d}{d\tau}|_{\tau=\overline{\tau}} ln\mathcal{J}(v,\tau) \leq \frac{n}{2\overline{\tau}} - \frac{1}{2}\overline{\tau}^{-\frac{3}{2}} K(v,\overline{\tau})$$

Here, $K(v,\tau)$ is the analog in the tangential space of the one defined in the previous talk.

The next part concentrates on the local non-collapsing theorem. We start by defining what it means for a solution to be κ-collapsed:

Definition 2 ([2]). *A solution to the Ricci flow $(g_{ij})_t = -2R_{ij}$ is said to be κ-collapsed at (x_0, t_0) on scale $r > 0$ if $|Rm|(x,t) \leq \frac{1}{r^2}$ for all (x,t) satisfying $dist_{t_0}(x, x_0) < r$ and $t_0 - r^2 \leq t \leq t_0$, and the volume of the metric ball $B(x_0, r^2)$ at time t_0 is less than κr^n.*

Our goal is to prove the local non-collapsing theorem.

Theorem 1 ([2]). *For any $A > 0$ there exists $\kappa = \kappa(A)$, $\kappa > 0$ with the following property : if $g_{ij}(t)$ is a smooth solution to the Ricci flow on the time interval $0 \leq t \leq {r_0}^2$, which has $|Rm|(x,t) \leq \frac{1}{{r_0}^2}$ for all (x,t) satisfying $dist_{t_0}(x,x_0) < r_0$, and the volume of the metric ball $B(x_0, r^2)$ at time 0 is at least $A^{-1}{r_0}^n$, then $g_{ij}(t)$ can not be κ-collapsed an the scales less than r_0 at points $(x, {r_0}^2)$ with $dist_{{r_0}^2}(x, x_0) < Ar_0$.*

As an outline, the proof is based on the separation of the region around the problem point in two parts: one with bounded geometry obtained from the assumptions of the theorem and one in which we use the properties of the reduced distance l in the backwards time, previously proved. Following details of [1] we use the Bishop-Gromov inequality to prove that there exists a lower bound for the volume of the geodesic ball at time t from the one at initial time 0. This along with a uniform sectional curvature bound will give us the bounded geometry on the first time region, thus a lower bound on the reduced volume of that region.

For the second part of the argument an effective upper bound on the minimum of the reduced distance will give a lower bound on the reduced volume of the region. This can be computed by using the properties of the $\mathcal{L}$-length and reduced distance l, obtained in the previous talk, localized around the problem point using a ingeniously chosen radial function,depending on the reduced distance l for which we apply a maximum principle. This gives us again a lower bound of the reduced volume also on the second region.

Putting the two bounds together we will get a lower bound of the reduced volume in a region where we have assumed, by contradicting the theorem, that the solution is κ-collapsed at scales less than r_0. This contradict the definition of κ-collapsed and finishes the proof.

References

[1] Bruce Kleiner, John Lott. Notes on Pereman's papers. http://arxiv.org/abs/math.DG/0605667

[2] Grisha Perelman. The entropy formula for the Ricci flow and its geometric application. arxiv.org/abs/math.DG/0211159

[3] John Morgan, Gang Tian. Ricci flow and the Poincaré conjecture. arxiv.org/abs/math.DG/0607607

[4] Richard Hamilton. The Harnack estimate for the Ricci flow. J. Differential Geom. 41 (1995),215-226.