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Perelman's l-distance

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Abstract
This talk is a preparation of the necessary tools for proving the non-collapsing results. The L-length defined by Perelman is the analog of an energy path, but defined in a Riemannian manifold context. The length is used to define the l reduced distance and later on, the reduced volume. So far the properties of the l-length have two applications in the proof of the Poincare conjecture. Associated with the notion of reduced volume, they are used to prove non-collapsing results and also to study the K-solutions.

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Perelman's $l$-distance

Valentina Vulcanov

This talk is a preparation of the necessary tools for proving the non-collapsing results. The $\mathcal{L}$-length defined by Perelman is the analog of an energy path, but defined in a Riemannian manifold context. The length is used to define the $l$ reduced distance and later on, the reduced volume. So far the properties of the $l$-length have two applications in the proof of the Poincaré conjecture. Associated with the notion of reduced volume, they are used to prove non-collapsing results and also to study the $\kappa$-solutions.

The first step is introducing the reduced distance by means of the $\mathcal{L}$-length defined by Perelman, [2].

Consider a backward solution of the Ricci flow $(M, g(\tau))$: $$\frac{\partial}{\partial \tau} g_{ij}(\tau) = 2 \text{Ric}_{ij}(g(\tau)),$$
where $\tau = T - t$ ($T$ is the final time). Let $\gamma : [\tau_1, \tau_2] \to M$ be a curve on the manifold parametrized by backward time.

**Definition 1 ([2]).** The $\mathcal{L}$-length of a curve $\gamma$ is

$$\mathcal{L}(\gamma) := \int_{\tau_1}^{\tau_2} \sqrt{\tau} (R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2) d\tau,$$
where $R(\gamma(\tau))$ is the scalar curvature at the point $\gamma(\tau)$.

Considering a variation of the curve $\gamma$, $\tilde{\gamma}(s, \tau)$, $s \in (-\epsilon, +\epsilon)$, $t \in [\tau_1, \tau_2]$ we can define the tangential and variational vector fields by $X = \frac{\partial \tilde{\gamma}}{\partial \tau}$ and $Y = \frac{\partial \tilde{\gamma}}{\partial s}$.

The first properties obtained are the (Euler-Lagrange) equations of $\mathcal{L}$-geodesics:

**Proposition 1 ([2]).** There holds

$$\nabla_X X - \frac{1}{2} \nabla R + 2 \text{Ric}(X, \cdot) + \frac{1}{2\tau} X = 0.$$

The proof comes easily from the first variation formula for the $\mathcal{L}$-length.

Let $\tau_1 = 0$ and $\tau_2 = \overline{\tau}$ we consider furthermore variations of curves on $M$, connecting points $p, q \in M$ with fixed starting point $p$ and moving end point $q = q(\overline{\tau})$.

**Definition 2 ([2]).** Denote by $L(q, \overline{\tau})$ the $\mathcal{L}$-length of the $\mathcal{L}$- shortest curve $\gamma(\tau)$, $0 \leq \tau \leq \overline{\tau}$ connecting $p$ and $q$.

The reduced length is defined as $l(q, \tau) = \frac{L(q, \tau)}{2\sqrt{\tau}}$. 
From the definition one can see that properties of the reduced distance can be easily obtained if we have the corresponding ones for the $L$-length. We are concentrating the last mentioned ones in two main propositions:

**Proposition 2 ([2]).** There holds

$$L_{\tau}(q, \tau) = 2\sqrt{\tau} R(q) - \frac{1}{\tau} K - \frac{1}{2\tau} L(q, \tau),$$

$$|\nabla L|^2(q, \tau) = -4\tau R(q) + \frac{2}{\sqrt{\tau}} L(q, \tau) - \frac{4}{\tau} K.$$

where $K = K(\gamma, \tau) = \int_0^\tau \tau \frac{1}{2} H(X(\tau)) d\tau$ and $H(X)$ is the trace of the expression appearing in Hamilton’s Harnack inequality, [4].

**Proposition 3 ([2]).** There holds

$$\Delta L \leq \frac{n}{\sqrt{\tau}} - 2\sqrt{\tau} R - \frac{1}{\tau} \int_0^\tau H(X) d\tau.$$

For the proof of the last one we have followed the detailed steps of [1, 3]. One starts by computing the second variation and then the Hessian of the $L$-length. We define the $L$-Jacobi fields along $L$-geodesics and prove that they are minimizers of Hessian of the $L$-length. Then making a special choice of orthonormal basis for $T_{\gamma(\tau)} M$ we obtain the result.

Using the above properties of the $L$-length we can also obtain the reduced length $l(q, \tau)$ properties, which will be used in the following to prove monotonicity of reduced volume and non-collapsing results:

**Proposition 4 ([2]).** One has

$$l_{\tau} - \Delta l + |\nabla l|^2 - R + \frac{n}{2\tau} \geq 0$$

$$2\Delta l - |\nabla l|^2 + R + \frac{l - n}{\tau} \leq 0$$

$$\min_{\tau} l(\cdot, \tau) \leq \frac{n}{2}$$

$$\frac{d}{d\tau}|_{\tau=\tau} \tilde{Y}|^2 \leq \frac{1}{\tau} - \frac{1}{\sqrt{\tau}} \int_0^\tau \sqrt{\tau} H(X, \tilde{Y}) d\tau,$$

where $\tilde{Y}$ is any $L$- Jacobi field along $\gamma(\tau)$.

**References**


