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Shi's local estimates

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Abstract

The talk is concerned with some of the analytic results and techniques that are fundamental to the study of the qualitative behavior of solutions of the Ricci flow, later used in singularity analysis. In particular we focus on derivatives estimates, useful for proving long time existence of solutions and obtaining local control of solutions.

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In particular we focus on derivatives estimates, useful for proving long time existence of solutions and obtaining local control of solutions.

The following result is due to W.-X. Shi:

Proposition 1 ([1]). *(Local derivative of curvature estimates). For any α, K, r, n and $m \in \mathbf{N}$, there exists C depending only on α, K, r, n and m such that if M^n is a manifold, $p \in M$, and $g(t)$, $t \in [0, T_0]$, $0 < T_0 < \frac{\alpha}{K}$, is a solution to the Ricci flow on an open neighborhood U of p containing $\overline{B}_{g(0)}(p, r)$ as a compact subset, and if*

$$|Rm(x, t)| \le K \quad for \quad all \quad x \in U \quad and \quad t \in [0, T_0],$$

then

$$|\nabla^m Rm(y, t)| \le \frac{C(\alpha, K, r, n, m)}{t^{m/2}}$$

for all $y \in B_{g(0)}(p, r/2)$ and $t \in (0, T_0]$.

The talk will detail Hamilton's proof of Shi's first derivative estimate, in a slightly better version than the main theorem stated above:

Proposition 2 ([1]). *(Interior first derivative estimates) curvatures). There exists a constant $C(n)$ depending only on n such that if M^n is an n-manifold, $p \in M$, and $g(t)$, $t \in [0, \tau]$, is a solution to the Ricci flow on an open neighborhood U of p containing $\overline{B}_{g(0)}(p, r)$ as a compact subset, and if*

$$|Rm(x, t)| \le K \quad for \quad all \quad x \in U \quad and \quad t \in [0, \tau],$$

then

$$|\nabla^m Rm(y, t)| \le C(n) K \left(\frac{1}{r^2} + \frac{1}{\tau} + K \right)^{1/2}.$$

For the proof we follow details of [1] and rely on applying a barrier argument to a quantity containing the first derivative, which has good evolution equation. The main ideas of the proof can be enclosed into 4 steps.

The first of these is based on a simple note about the very estimate we want to obtain. The fact that constant K appears in the right hand side of the estimate allows us to assume without loss of generality in the proof that $\tau \in (0, 1/K]$ and $r \in (0, 1/\sqrt{K}]$.

The next step is finding the good first derivative quantity and derive its evolution equation:

$$G := \frac{c(n)}{K^4}(16K^2 + |Rm|^2)|\nabla Rm|^2$$

$$\frac{\partial G}{\partial t} \le \Delta G - G^2 + K^2$$

The following step is more technical and consists of construction of the barrier function. This can be done by using the existence of a good cutoff function with bounded first and second derivative on a manifold with bounded curvature, which is our case.

The last step links the estimates of the quantity defined in the second step with those of the first and second derivative of the cutoff function. We prove that as long as our good quantity G is dominated by a certain comparison function, the derivatives of the cutoff function satisfy also good bounds. But on the other hand these are easily obtained from the bounded curvature hypothesis.

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