1971

Mathematical models for the Wollongong urban area

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MATHEMATICAL MODELS

for the

WOLLONGONG URBAN AREA

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1971
ABSTRACT

Since the publication of a paper by Clark in 1951, the negative exponential model,

\[ -bx \]
\[ y = Ae \]

has been widely accepted as the spatial pattern of intra-urban population density distribution. This model is based on the assumption that the population densities of cities can be approximated by a circularly symmetric pattern and that the negative exponential function gives a reasonable fit.

This model has proved to be a useful generalization and Wilkins and Shaw have shown it to be an expression of the average pattern of density distribution.

It is not, however, the only model which has been developed. Sherratt (1961) derived a model for Sydney,

\[ -x^2/2\sigma^2 \]
\[ y = Ae \]

which is also based on the assumption of circularly symmetry. Universality is also claimed for this model.

Newling (1970) used Clark's original data to derive a model,

\[ bx-cx^2 \]
\[ y = Ae \]
which gives a better fit at and near the city centre.

The theoretical approach of Gurevich and Saushkin while emphasising the question of circular symmetry has little practical application.

Trend surface analysis provides a method of fitting a polynomial surface to population density distribution and removes the problem of definition of city centre and edge.

Wollongong, being restricted by the sea to the east and the escarpment of the Illawarra range to the west, provides an extreme case of distortion from circular symmetry against which to test the universality of the Clark and other models.

Data from the 1966 census is used to develop several models for Wollongong. The model based on Clark's method using average density over annular rings,

\[ y = 3.49e^{-0.38x} \]

provides the high correlation of -0.94 between density and distance from the city centre. The total population as estimated by the model is, as expected, close but the population per radian indicates the poorness of the model for this city.
The Clark method is then applied with the assumption of a rectangular rather than circular shape. The city was divided into a series of mile wide strips with the "centre", a line in the east-west direction approximating the main street of Wollongong. Of the rectilinear models so obtained the best fit was given by

$$y = \begin{cases} 
-0.28x & \text{for } x \text{ north} \\
10.07e + 0.18x & \text{for } x \text{ south,}
\end{cases}$$

where the northern section is considered to be a rectangle 2.2 miles wide and the southern, 3.3 miles wide. This model gives the best estimate of total population.

A model of the form

$$y = \sum_{i=1}^{9} A_i e_i$$

is developed which is a good fit for Wollongong at this particular stage of its development. This model is difficult to derive and apply.

The trend surface model giving the most significant fit is the quadratic polynomial. It presents boundary problems in application.

The effect of choice of model in applied problems is illustrated by the investigation of the differences in the
estimated number of casualties in a nuclear weapon attack. This is shown theoretically, by comparing results for Clark's and Sherratt's models. The variation for Wollongong is shown using four of the models.

Real cities usually deviate from circular symmetry to some degree and if a model based on circular symmetry is to be applied with confidence, a measure of the significance of this deviation is necessary. This measure should also be based on the distribution of people rather than the shape of the boundary. Three such measures are proposed. The density-difference index confirms the distortion of Wollongong from circular symmetry and the $\chi^2$ measure shows this distortion to be significant at the 0.001 level.
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CHAPTER 1 INTRA-URBAN POPULATION DISTRIBUTION AS A NEGATIVE EXPONENTIAL FUNCTION

1.1 Clark's Model

In a seminal paper, Clark in 1951, defined the spatial pattern of intra-urban population distribution by the equation

\[ y = Ae^{-bx} \]

which relates \( y \), the population in thousands per square mile to \( x \), the distance from the city centre. \( A \) is the density at the city centre and \( b \) measures the rate of decline of density.

This model was derived by Clark after examination of data available for 36 cities. The total population and, hence, the density was calculated for a series of concentric rings about the centre of the city at each mile radius. When the natural logarithm of density is used, the model becomes

\[ \ln y = \ln A - bx, \]

and the parameters \( A \) and \( b \) are readily calculated using the least squares method.

Clark's model is subject to two basic qualifications

a) No allowance is made for a central business district not available for residences. The constant, \( A \), is therefore, a hypothetical central density.

b) It is assumed that the city can spread uniformly in all directions and that all land is available for residences.
Clark (1970) states that this equation is a "fundamental law" relating density and distance from the city centre and credits Bleicher with its discovery in 1892. (P1.6)

1.1.1 Population within Radius \( r \)

The population, \( P_c(r) \), within a given radius \( r \), may be calculated from Clark's equation

\[
P_c(r) = 2\pi A \int_0^r e^{-br} r \, dr
\]

\[= 2\pi Ab \left[ 1 - (1 + br)e^{-br} \right].\]

1.1.2 Total Population

The total population of a city may be considered to be the population within radius \( r \), as \( r \to \infty \). From Clark's equation, the total population is given by

\[
P = 2\pi Ab \]

\[= 2\pi Ab \left[ 1 - (1 + br)e^{-br} \right].\]

since \( e^{-br} \to 0 \) as \( r \to \infty \).

1.2 A Theoretical Derivation of the Negative Exponential Function

Stewart and Warntz (1958) have derived a model which is essentially the same as Clark's.

They have defined a "standard" city (\( P = 0.99 \)) as one which is circular in shape with a maximum density at the centre. The density drops radially in such a way that it is halved at equal steps of length \( a \), from the centre. Using this description of a city, the density \( D_r \) at a distance \( r \) from the centre is
given by

\[ D = D_0 e^{-r/a} \]

where \( D \) is the density at the city centre.

If the simple substitution

\[ a = \frac{1}{\ln 2}, \]

is made in this model derived by Stewart and Warntz it becomes,

\[ D = D_0 \frac{-rb}{\ln 2} \]

\[ = D_0 e^{-(rb) \ln 2} \]

\[ = D_0 e^{-(rb)} \ln 2 \]

which is identical with Clark's model.

1.3 Goodness of Fit of the Model

Clark (1951) claimed that his model "appears to be true for all times and all places studied from 1801 to the present day, and from Los Angeles to Budapest". (P490). While 36 cities are too few to assert the universality of the model, other research work has tended to support this view. (Compare Alonso 1960, Muth 1961, Weiss, 1961; Berry et al, 1963; Newling, 1966; Romashkin, 1967; Chorely and Haggett, 1967; Berry and Horton, 1970; Marsden, 1970).

Berry (1963) states that a "statistically significant negative exponential relationship between density and distance" (P391) appears to exist. Clark, however, offers no evidence of goodness of fit and an examination of the 35 graphs in his article reveals that, in some cases, the regression line was calculated from few data points (e.g. 5 points for Oslo, 1938.
and 4 for Dublin, 1936). Then, too, while some regression lines appear to fit the data closely (e.g. Sydney, 1947), others show great variation (e.g. Chicago, 1940). Newling (1969) has claimed that a different model better fits the data (see Chapter 2).

Berry (1963) seems to take the correlation between density and distance from the centre as the measure of goodness of fit - "the weakest correlation between density and distance (is) -0.97". (P.391).

Now, Clark's methodology replaces the real city with a hypothetical, circularly symmetric city which has the same number of people living at each mile radius as the real city. The regression line is calculated for the hypothetical city and the coefficient of correlation indicates the fit of this regression line to the hypothetical city. It is not a measure of fit of the model to the real city unless the real city closely approximates circular symmetry.

1.4 Clark's Modification of Methodology

Clark (1951) suggests a modification of his methodology to overcome the problem of apportioning population in census districts (or other divisions) which lie across the boundaries of the annular rings. Instead of population density over annular rings, the average density of each census district is to be plotted against its mean distance from the centre.
Thus the density is obtained for many values of x, the distance from the centre, and the regression line fitted. For this model there is no prior assumption of circularity although the resultant equation is assumed to fit a circularly symmetric city. Once again, if the real city deviates markedly from circular symmetry, the model will not be a good fit, even though the regression line fits the data well.

1.5 Extensions of Clark's Model

If Clark's modified procedure is extended so that the density is sampled at an infinite number of points and if direction is taken into consideration, then Clark's model becomes

$$\frac{1}{2\pi} \int_0^{2\pi} D(r,\theta) d\theta = Ae^{-br}$$

where $D(r,\theta)$ is the density at the point with polar coordinates $(r,\theta)$, the city centre being taken as the centre of the polar coordinate system.

Wilkins (1968) has shown that this is, at times, a more useful form of the model (see 1.7).

Dacey (1968) has attempted a similar transformation. He defines the density at a point $(x,y)$ from the centre as the function $g(x,y)$. Then, he claims, the density function for population at a distance, $r$, from the city centre $(a,b)$ can "using Clark's method" (p.233) be defined as:

$$g(r;a,b) = \int \int g(x,y) dx \ dy \cdot 2 \frac{1}{2 \frac{1}{2}} [(x-a) + (y-b)] = r$$
In order to examine the meaning of this expression, we will simplify it by considering the case where the city centre is the centre of the coordinate system. Then
\[ g(r;0,0) = \iint_{(x+y)^2 = r} g(x,y) \, dx \, dy. \]
This would indicate that \( g(r;0,0) \) represents the sum of the densities \( g(x,y) \) for each of the elemental areas, \( dx \, dy \), such that \( r = (x^2 + y^2)^{1/4} \) is a constant.

Dacey's formalism is wrong. The correct expression for \( g(r;0,0) \) is
\[
\frac{1}{2\pi r} \int g(x,y) \, ds,
\]
integrating around the circle \( x^2 + y^2 = r^2 \). On converting to polar coordinates and taking \( g(x,y) = D(r,\theta) \), we obtain
\[
g(r;0,0) = \frac{1}{2\pi r} \int_0^{2\pi} D(r,\theta) \, d\theta,
\]
which is the same result as that obtained by Wilkins using Clark's method.

1.6 Variation of Parameters with Direction.

Berry and Horton (1970) suggest that a single equation for the whole city may be an oversimplification. Their analysis of Winsborough's figures for Chicago and his division of the city into five sectors supports the negative exponential decline of density with distance from the city.
centre although the rate of decline may vary according to
direction from the centre.

Since no real city is circularly symmetrical
this approach would seem practical.

1.7 The "Average" Nature of Clark's Model

It is obvious from Clark's method of deriving his
result that it is a statement about the average value of pop-
ulation densities at distance x from the city centre. There is
however, the assumption of a constant rate of decrease of
population density from the centre of the city to the outer
boundary along any ray. For such a city the isolines of con-
stant density would be concentric circles with their common
centre at the pole.

Wilkins and the present author (1969) have shown
that a city may in fact be quite asymmetric and still be adequately
described by Clark's equation.

As a simple example of this, consider a hypothetical

city with central density 1 in which

\[ D(r,\theta) = e^{-br} \{1+(1-e^{-br})\sin \theta\} \]

where b is a positive constant and D(r,\theta) represents the density
at the point (r,\theta). For convenience, the city is assumed to
extend to infinity in all directions.

The function D(r,\theta) gives a continuous surface.
The density is strictly decreasing along any ray as r increases
from zero, since
Figure 1.1
Graph of $z$ against $\theta$.

Figure 1.2
Graph of $br$ against $z$. 

8.
Figure 1.1: Graph of \( z \) against \( \theta \).

Figure 1.2: Graph of \( br \) against \( z \).
is negative (except where \( r = 0 \) and \( \theta = \pi / 2 \)).

In order that the isolines of this city may be discussed, it is convenient to introduce the variable
\[
-\frac{br}{r} = e^{z} \left[ 1 + \sin \theta (1 - 2e^{z}) \right]
\]
then for the isoline along which \( D(r, \theta) \) has the value \( k \) \((0 < k < 1 \) since the central density has been defined as 1), \( z \) and \( \theta \) are related by
\[
\frac{k}{\sin \theta} = z + z^{-1}
\]
Then
\[
\frac{d\theta}{dz} = \frac{1}{\cos \theta} \left[ \frac{k}{z^2} - \frac{1 + k}{(z-1)^2} \right]
\]
The graph of \( z \) against \( \theta \) is as indicated in Figure 1.1.

The equation of an isoline \( D(r, \theta) = k \) is found by solving
\[
\sin \theta = z + \frac{1-k}{z-1}
\]
for \( z \) and converting to the polar variable \( r \). It is
\[
br = \ln \frac{2}{\ln \left[ \left[ 1 + \sin \theta - (1+2(1-2k)\sin \theta + \sin^{2} \theta) \right]/\sin \theta \right]}
\]
When \( \sin \theta = 0 \) \((\theta = 0, 2\pi) \) the value of \( br \) may be calculated directly from the original equation which becomes
\[
br = \ln k.
\]
The graph of \( br \) against \( \theta \) is sketched in figure 1.2.

The isolines for \( k = 0.01, 0.50 \) and \( 0.99 \) are given in figures 1.3 and 1.4.
Figure 1.2

REPRESENTATIVE ISOLINES FOR $a=0.01$

AND $a=0.00$

Figure 1.4

REPRESENTATIVE ISOLINES FOR $a=0.00$
Figure 1.3
Representative isolines;
k = 0.01 for outer isoline,
k = 0.50 for inner isoline.

Figure 1.4
Isoline for 
k = 0.99
The density of population per radian is given by
\[
\lim_{\Delta \theta \to 0} \int_{\theta}^{\theta + \Delta \theta} \int_{0}^{\infty} D(r, \theta) r \, dr \, d\theta = \frac{1}{52} \left( 1 + \frac{3}{4} \sin \theta \right).
\]

This exhibits a considerable degree of oscillation.

All the above points show that the hypothetical city has a rather complicated non-circular structure. Yet on the other hand
\[
\frac{1}{2\pi} \int_{0}^{2\pi} D(r, \theta) \, d\theta = \frac{1}{2\pi} \left[ e^{-br} \right]_{0}^{2\pi} = e^{-br}
\]
and so it satisfies Clark's equation. Clark's model is therefore seen as a generalisation, a statement about the average pattern of intra-urban population distribution.

1.8 The Axiom of Intra-Urban Structure

Clark's model has been assumed by many authors to be an "axiom of intra-urban structure". (Berry and Horton, 1970, p276) and research work has centred around such aspects as the behaviour of the density gradient, \( b \), through time, the interpretation of the hypothetical central density, \( A \), intra-urban growth, critical density and explanations of the generality of the model. (For further details see Clark, 1951; Alonso, 1961; Muth 1961; Weiss, 1961; Berry et al, 1963; Newling, 1966 and 1969; Winsborough, 1962; Chorley and Haggett,
In a concluding statement, Berry and Horton (1970) claim to have shown that "the makings of a wide-ranging and explanatory system for the pattern of population in a large metropolis" (p302) exist.

In this thesis, however, we are concerned with the model itself and, in particular with extension of the negative exponential model to the specific urban area centred on Wollongong. We are not concerned with explanations of the model nor with the growth of the city.

1.9 Problems to be Investigated

From the preceding discussion the following problems arise.

1.9.1 The Only Model

In spite of the overwhelming dominance of the Clark model in the literature there are, other models which have been developed and these will be discussed in detail in Chapter 2. It is interesting to note that Berry et al (1963, p391) quote one of these other models, namely Sherratt's normal distribution model as supporting evidence for the universality of Clark's model (p391).

1.9.2 A Model for Wollongong

Wollongong is a city which does not approach circular symmetry. The question arises whether a model dev-
eloped on the basis of circularity can fit this city or whether some new approach is necessary.

In Chapter 3 various models will be developed for the city of Wollongong and an attempt will be made to determine the goodness of fit of the models.

1.9.3 **The Effect of Choice of Model**

Models may be developed as a first step in the solution of problems particularly in operations research. Sherratt, (1961) for example, developed his city model as an essential preliminary to the quantitative solution of a problem concerning the extension of gas mains to give the best service to a rapidly expanding community and at the same time, to be financially satisfactory to the Gas Company.

What effect does the choice of model for a city have on the solution of such problems in applied mathematics?

This question will be examined in Chapter 4 in relation to the estimation of the number of casualties resulting from a nuclear weapon attack.

1.9.4 **Shape Distortion**

The closer a real city approximates circular symmetry, the more realistic is the fit of the Clark and other models. A measure of the amount of deviation from this ideal circular shape is therefore of importance. Such measures will be discussed in Chapter 5.
Although the negative exponential function has been widely accepted as an adequate statistical explanation of intra-urban population distribution other models have been considered, particularly in the operations research, applied mathematics and regional science literature. The most commonly accepted alternate model is that of the normal distribution of population about the city centre. This is based on the assumption that the locations of individuals are the outcomes of stochastic or random processes.

2.1 Sherratt's Model

Sherratt (1960) developed a model for the city of Sydney by plotting the density of dwellings per acre, $D$, against the distance, $r$, in miles from an arbitrarily selected city centre (namely the G.P.O). He found that the points so obtained lie on or near a series of curves of the form

$$D = D_0 e^{-r/\sigma}$$

where $D_0$ is the density at the city centre and $\sigma$ is a parameter varying with direction.

He further stated that if the asymmetrical distribution is replaced by an imaginary symmetric normal distribution with the same total population of dwellings, then the variable parameter, $\sigma$, may be replaced by a constant.
Wilkins (1968) has supported this point. Sherratt's model is rewritten in the more general form

\[ D(r, \theta) = f(r/\alpha(\theta)), \]

where \( D(r, \theta) \) is the density at a point with polar coordinates \((r, \theta)\) and \(\alpha(\theta)\) is a function of the direction \(\theta\). By considering the population density per radian he shows, generally, that in order to consider population and the area over which it spreads it is sufficient to consider the symmetric city for which

\[ D(r, \theta) = f(r/\alpha) \]

where \(\alpha\) is a constant independent of the direction, \(\theta\).

2.1.1 Comparison of Parameters

In sections 1.1.1 and 1.1.2, the population within the radius, \(r\), and the total population of a city represented by Clark's model were derived as

\[ P(r) = \frac{2}{\pi} Ab \left[ 1 - (1+br)e^{-br} \right] \]

and

\[ P = 2 \pi Ab \]

respectively.

From Sherratt's model similar results may be derived.

The population within a radius \(r\) is

\[ P(r) = D \int_0^{2\pi} \int_0^r \int_0^2 \frac{r}{2\sigma^2} e^{-\frac{r}{2\sigma}} dr d\sigma \]

\[ = 2 \pi D \left[ \frac{1}{2\sigma^2} \right] \]

15.
As \( r \to \infty \), the total population is found to be
\[
P = 2\pi D \sigma^2.
\]

If one symmetric city extending infinitely in all directions is being considered, the total population, \( P_c \), should equal the total population \( P_s \), and \( A = D \) since both \( s \to 0 \) represent the density at the city centre.

Therefore \( b = 2 \sigma^2 \).

Both \( b \) and \( \sigma \) are positive constants and so \( \sigma = \frac{b}{2} \).

Sherratt's model may be rewritten as
\[
D = Ae^{-r^2/2b}
\]

where \( A \) and \( b \) are the parameters of Clark's equation.

2.1.2 A Method of Deriving \( \sigma \)

Hunter (1965) suggests a method of determining \( \sigma \) apart from the normal procedure of taking density samples for various values of \( r \) and using a least squares technique to find \( D \) and \( \sigma \). Since
\[
P = 2\pi D \sigma^2
\]

\[
P (r) = 2\pi D \sigma^2 [1-e^{-r^2/2\sigma}]
\]

\[
= P_s [1-e^{-r^2/2\sigma}],
\]

where \( P (r) \) is the population within a radius \( r \), and \( P_s \) is the total population as before. If \( r \) is the value of \( r \) such that
\[
P (r) = \frac{1}{2} P,
\]

\[
P \]

\[
\theta = \frac{17}{r^2/2\sigma^2}
\]

then

\[
\frac{1}{s} = P \left[ 1 - e^{-s} \right].
\]

Therefore

\[
\frac{2}{r / \sigma} = e.
\]

That is

\[
\ln 2 = \frac{r}{2\sigma},
\]

and

\[
\sigma = \frac{0}{2\ln 2} = 0.8493r.
\]

2.1.3 Sherratt's Dwelling Density

It must be remembered that Sherratt's model was derived from an empirical study of dwelling distribution rather than for population distribution. He does, in fact, note that this model "could be in terms of people rather than dwellings" (P150).

If this is so, then one could expect to find a high correlation between dwellings and population. Figures for Wollongong give this correlation as +0.84 with a standard error of 0.03, which suggests that a transference of the model from dwellings to population might be acceptable.

Some research workers (Hunter 1965 and Weiss, 1961, for example) have used Sherratt's model to describe population density distribution without commenting on this point.
Sherratt generalises the validity of the model in the statement that "we have observed that, not merely in Australia, but all over the world .... the normal distribution of population is the general rule ..." (P157). He does not, however, offer sufficient proof in his article to cause one to reject Clark's model, as this proof also depends on curve fitting and illustrative graphical detail with its accompanying approximations. It does however suggest that there may not be a perfect universal model but that either or both may represent a sufficiently good universal model. The effect of choosing one or other model is analysed more fully in the discussion of Hunter's research in Chapter 4. (See also Wilkins and Shaw, 1969).

2.1.4 Modification for Small Scale Use

Sherratt suggests that the finite area of a small district may be replaced by the truncated central portion of an infinite normal distribution having the same area as the district it is replacing. The number of dwellings in such a truncated portion of radius \( r_p = K_p \sigma \) is given by

\[
\frac{2\pi}{4840} \cdot \frac{D}{\sigma (1-e^{-K_p^2/2})}
\]

where \( D \) represents the central density of dwellings per acre for the district being considered.
This implies that cities, towns, villages and even suburbs show a normal distribution of population densities. Perhaps this is a reasonable assumption to make; suburbs might well reflect in a minor way the overall density distribution. Sherratt has, however, offered slight evidence that this does occur.

2.2 The Theoretical Models of Gurevich and Saushkin

In constrast to the empirical approach of Clark and Sherratt, Gurevich and Saushkin (1966) have offered a theoretical approach to the development of models for the spatial pattern of intra-urban population distribution and hope that "geographers will apply this mathematical approach to the study of a number of concrete cities". (P35).

They define a single centre city to be one in which the density of population, \( \delta \), reaches an absolute maximum, \( \delta_0 \), at a point, the centre of the city. This city centre is regarded as the centre of a polar coordinate system and the density at the point \( M(r, \theta) \) is the function \( \delta(r, \theta) \).

The density decreases (or remains constant) from the city centre outwards.

If \( T(r, \theta) \) is defined as the rate at which the natural logarithm of the population density declines along a ray \( \theta = \text{constant} \),

\[
T(r, \theta) = \frac{\partial \ln \delta(r, \theta)}{\partial r}
\]
\[ T \text{ and } \delta \text{ are continuous functions and while theoretically } 0 < r < \infty \text{ in practice } 0 < r < a < \infty \text{ where } a = a(\theta) \text{ is a closed curve representing the outer boundary of the city and } b = b(\theta) \text{ is a similar inner boundary. This inner boundary, they claim, is necessary since } \delta_0 \text{ is a hypothetical maximum (cf. Clark, 1951 and Sherratt, 1960 and others). The curve } b(\theta) \text{ is defined in such a way that } T \text{ is always positive.} \]

Since \[ T = -\frac{1}{\delta} \frac{\partial \delta}{\partial r}, \]

\[ \delta = \delta_0 \exp\left\{ -\int_0^r T \, dr \right\} \]

or \[ \delta = \delta_0 \exp\left\{ -\int_b^\infty T \, dr \right\} \]

if the inner boundary is to be considered.

2.2.1 The Four City Types Suggested By Gurevich and Saushkin

Gurevich and Saushkin consider four types of cities arising from variation in the function \( T \).

(a) \[ T = \text{constant}. \]

In this case \[ \delta = \delta_0 e^{-Tr} \]

which is identical with Clark's formula (and, indeed, it is given the title of "Clark-type" city).
The isolines for such a city would be concentric circles.

(b) \[ T = T(\theta) \]

In this case the rate of density decline is constant along any ray, \( \theta = \text{constant} \), but changes from ray to ray. The density is then given by
\[ \delta = \delta_0 e^{-T(\theta) r} \]

The city "shape" would, of course, vary according to \( T(\theta) \). The two examples suggested by Gurevich and Saushkin seem to be highly artificial. Indeed, if \( T(\theta) = |\cos \theta| \), the city is represented by a single main thoroughfare along which the density is the maximum \( \delta \); this appears to conflict with their original definition of a single-centre city.

(c) \[ T = T(r) \]

Here the rate of density decline is dependent only on \( r \) and
\[ \delta = \delta_0 e^{-\int_0^r T(r') dr'} \]

Sherratt's model might be classified with this set of spatial patterns.

If \[ T = \frac{r^2}{2k} \]
then
\[ \delta = \delta_0 e^{-\frac{r}{2k}} \]
which is equivalent to Sherratt's model.

(d) \[ T = A(r)B(\theta) \]

In this case the rate of density decline is
dependent on both \( r \) and \( \theta \). Again the specific examples considered by Gurevich and Saushkin appear highly artificial.

2.2.2 An Extension by Gurevich

Gurevich (1967) has continued the above theory of cities types by considering the possibility of \( T \) being negative.

For example he considers a city where \( T = -1 \). Then the density is given by

\[
\delta = \delta e^r \quad \text{with} \quad \delta \neq 0.
\]

The central city density is an absolute minimum and if the outer boundary is given by \( r = R \), the total population is

\[
N = \int_0^R \int_0^R \delta e^r \, dr \, d\theta
\]

\[
= 2\pi \delta (1 + Re^{-e})
\]

Gurevich suggests that Split in Yugoslavia might be such a city as its centre consists of old low density dwellings dating back to Roman times, surrounded by groups of high-rise buildings. In an unfortunate reversal of his argument he contradicts this statement by suggesting that the exponential expression "is not likely to apply in this case" (P728) and that Split might be better represented by

\[
\delta = \lambda r + \delta \quad \text{where} \lambda = \text{const.} > 0.
\]
This equation does not seem to fit the carefully developed theoretical approach.

Some mathematical errors also occur. For example, Gurevich discusses a city with the density
\[ \delta = r(\cos 2\phi - r) \]
and a boundary function \( r = \cos 2\phi \). The boundary shown in an accompanying diagram is, however, that given by \( r = \cos 2\phi \). (P728). A further error occurs when a city, with density
\[ \delta = \delta_0 (1 + \sin r) \]
is considered.

Now \[ T = - \frac{1}{\delta} \frac{\delta}{\delta r} \]
\[ = - \frac{1}{1+\sin r} \cdot \cos r \]
\[ = - \frac{\cos r}{1+\sin r} \]
which is not
\[ T = \frac{\cos r}{1+\sin r} \], as in the article. (P729)

2.2.3 A Criticism of Gurevich and Saushkin

The work of Gurevich and Saushkin is a theoretical approach and as such seems to have little practical application. Having derived a possible set of city patterns, they hope that cities will be found to fit them.

They have defined the centre of the city as the
point where the density is an absolute maximum and then state this is a hypothetical maximum. This difficulty is partially overcome by defining \( b = b(\theta) \), the inner boundary such that the density is a maximum along \( b(\theta) \). This inner boundary, they ignore when the isolines of density for hypothetical cities are drawn.

Then, too, the isoline approach ignores the average nature of models such as Sherratt's and Clark's. A Clark-type city is defined as having \( T = \) constant, for which the isolines would be concentric circles. (See Wilkins and Shaw, 1969 and Section 1.7).

The extension by Gurevich appears to conflict with the previous patterns of density distribution. The function, \( T(r,\theta) \) is defined such that the density function \( \delta(r,\theta) \), is exponential. That is

\[
\delta(r,\theta) = \delta \exp\left\{ - \int_0^x T(r,\theta) \, dr \right\}.
\]

Gurevich, however, considers cities with a given density function not of this form.

A further difficulty arises with the assumption that \( T \) can be negative. Theoretically, \( 0 \leq r \leq \infty \) for any city, and the models of Clark and Sherratt may be applied to any city without absolute definition of boundaries. However, if \( T \) is negative, the boundary function, \( a(\theta) \) must be defined
or the total city population becomes infinite.

Gurevich and Saushkin have pointed out that in practice,

\[ 0 \leq b \leq r \leq a \leq \infty \]

The inner boundary function \( b(\theta) \) is defined for \( T \) positive (such that \( T \) is positive) but no definition of \( a(\theta) \) is discussed.

2.3 Reinhardt's Model

In a footnote, Winsborough (1962) refers to a model developed by Reinhardt in 1950. This model is given as

\[ D = k(1 + r)e^{-r/b} \]

where \( D \) is the density at a distance \( r \) from the city centre, and \( b \) and \( k \) are parameters derived from the data.

This model predates that of Clark. It may be that it was specifically derived for one city.

This model lacks the simplicity of either Clark's or Sherratt's models which, if good enough in terms of statistical fit, would be preferred.

2.4 The Density Crater and Newling's Model

Newling (1969) has re-examined Clark's original data by enlarging the 35 maps presented by him in his 1951 article to ten times the size and measuring the points represented. Clark's model assumes a hypothetical central
density and Newling suggests that a quadratic regression of the logarithm of density on distance would give a better generalisation as it would take into account the density crater usually found at the city centre.

His model is of the form

\[ D = D_0 e^{bx - cx^2} \]

where \( D \) is the density \( x \) miles from the city centre, \( D_0 \) is the city centre density and \( b \) and \( c \) are parameters derived from the data.

Applying this model to the Clark data, Newling claimed it to be validated in all but two of the 35 cases where errors of measurement could have influenced the results.

The parameter \( b \) is the instantaneous rate of change of density with distance at the city centre since

\[ \frac{dD}{dx} = (b - 2cx)D e^{bx - cx^2} \]

Thus when \( x = 0 \), that is at the city centre, the rate of change of density is given by \( b \).

If \( b \) is negative there is a continuous decrease in density from the centre outwards, and if \( b \) is positive there is an increase in density from the city centre to a maximum at \( x = \frac{b}{2c} \) and then a decrease. When \( b = 0 \), there is a normal distribution of population—Sherratt's model.
This model assumes c to be a positive constant and, indeed, it was a negative value for c which invalidated the two cases mentioned above.

The density $D$ at the perimeter of the urbanised area is given by

$$D = D_0 e^{\frac{br-cr}{p}}$$

The radius of the urbanised area, $r$, is found by solving this equation to give

$$r = \left( b + \sqrt{b^2 + 4c \ln(D/D_0)} \right) / 2c.$$  

The crest or maximum density occurs when $x = \frac{b}{2c}$ and is therefore

$$D_{\text{max}} = D_0 e^{\frac{b^2}{4c}}.$$  

The points of inflection of the curve occur at

$$x = b \pm \sqrt{2c} / 2c$$

and the density at each of these points is

$$D_{\text{inf}} = D_0 e^{\frac{b/4c}{2}}$$

The total population of the theoretically desirable...
infinite city (not considered by Newling) is given by

\[ P = D \int_{-\infty}^{\infty} e^{-\frac{(x-b/2c)^2}{2c}} \, dx \]

where \( \text{erf}(x) \) is the error function.

2.4.2 Comments on Newling's Model

Newling's model appears to give considerable information about the distribution of population close to the city centre. He suggests that the distance of the crest from the centre gives a measure of the age of the city; that is, the density profile, dependent on the sign and size of the parameter, \( b \), presents a method of classifying city growth.

His model, however, does not seem to fit data distant from the centre. Newling omitted from his calculations some edge densities claiming these to be not truly urban.

One can not really assess the accuracy of the above points as no measure of goodness of fit is presented other than the necessity for the parameter, \( c \), to be positive,
and this point is not explored either.

The model seems unnecessarily complicated. The existence of a density crater at the centre of most cities is recognised and indeed the measure of the hypothetical central density appears to add information about the city. (cf. Clark, 1951; Winsborough, 1962; Duncan, 1957; Berry et al, 1963; Berry and Horton, 1970). The determination of the distance of maximum density could be more simply determined from the raw data. Berry and Horton (1970) make the point that, at present, "retention of the linear form also makes possible comparison with similar studies of other cities". (P302). Clark (1970) supports this point of view and states that the "exponential formula served reasonably well to describe the fairly simple structure of cities in the past. Newling's improvement undoubtedly gives us a better description of modern cities, with their large central business zones". (Pl.7).

2.5 Dacey's Multi-Centre Model

Dacey has extended his model for a single-centre city to cover the situation where two or more centres are present. He assumes a city with population P having two centres at $(a_1, b_1)$ and $(a_2, b_2)$. This implies two independent populations $P_1$ and $P_2$ ($P = P_1 + P_2$), distributed about $(a_1, b_1)$ and $(a_2, b_2)$ respectively. Then if $(a, b)$ is taken
as the city centre, the density of population at distance
r from \((a,b)\) becomes

\[ g(r;a,b) + g(r;a,b) = g(r;a,b) \]

where \(g(r;a,b)\) is the density function for population \(P_i\),
as discussed in section 1.5.

This may be extended to a city with \(n\) population
centres.

\[ g(r;a,b) = \sum_{i=1}^{n} g(r;a,b). \]

This model could be effectively used to describe the population density distribution of a city with well
defined, mutually exclusive but overlapping sub-populations
such as racial groups. It is, however, necessary to identify
these sub-populations and calculate the distribution of each
before the total city model can be developed.

2.6  The Trend Surface

The trend surface is a polynomial surface relating \(z\), an areaally distributed variable to \(x\) and \(y\), re-
presenting locational rectangular coordinates. This poly-
nomial surface which may be of linear, quadratic, cubic or
quartic form is fitted by means of the least squares technique.
Computer programmes have been developed for this purpose and
all four surfaces are easily obtained. An analysis of variance enables the surface which is of the simplest form while
significantly accounting for most variation, to be selected.

Since population may be considered to be an areally distributed variable, this technique represents another method of developing a model of the population density distribution of a city. The models previously discussed have connected the density of population with distance from the city centre. They either assume circular symmetry or give a series of similar equations with parameters varying with direction or as in the models of Gurevich and Saushkin, an equation allowing for variation (or constancy) with distance and direction. Each of these, however, require identification of a city centre. The trend surface does not require a point so defined and any axes of reference, provided they are identified and constant, are sufficient.

The trend surface is so called because it gives an indication of the general trend of the areally distributed variable. It separates regionally systematic variations from localised (or even random) ones. These localised variations may, for some research, be important and are indicated by the residuals, the differences between the fitted surface and the data. The removal of the general trend helps to isolate these areas of localised variation. (cf. Fairbairn and Robinson, 1967; Robinson and Fairbairn, 1969).
Chorley and Haggett (1965) have extensively reviewed the field of trend surface mapping and have outlined many ways in which this technique might be exploited in geography. Geologists, too, have made use of the trend surface, particularly in the search for oil, gas and minerals (Krumbein and Graybill, 1965).

Applied to population density distribution the trend surface describes the general trend or pattern of this distribution and is not dependent on a defined city centre. The points \((x_i, y_i)\) may be taken as the centre of \(i^{th}\) census district distant \(x_i\) and \(y_i\) from any convenient set of axes, and \(z_i\) is the density of this district. The points \((x_i, y_i)\) need not be regularly spaced. Once the surface is obtained, the densities can be found for a network of regularly spaced points and contour lines showing the trend of density distribution may be drawn.

The effects of crowding in suburban centres and of areas of no population (lakes etc) are eliminated.

One problem remains, however, and that is the definition of the edge of the city. With a circular city model, a city extending infinitely in all directions may be considered. With the trend surface, only a part of this surface is applicable to the city and the "centre" or point(s) of maximum density need not be within the city boundaries.
Calculation of total population is, therefore, difficult involving integration of over an irregularly shaped region.

2.7 **Wilkins' Model**

Wilkins (1968) points out that additional insight into intra-urban population distribution may be gained by considering, as well as the fitted surface \( D(r, \theta) \), the function \( A(D) \) where \( A(D) \) is the measure or area of the region \( S(D) \) determined by

\[
S(D) = \{ (r, \theta) : D(R, \theta) \geq D \}.
\]

The data will consist of a number of census districts (or other suitable regions), each with an associated area, \( A_i \) and population \( P_i \) from which the average density \( D_i = \frac{P_i}{A_i} \) may be calculated.

For any particular value of \( D \) say \( D_k \), one could calculate \( A = \sum_{i=1}^{k} A_i \) such that \( \sum_{i=1}^{k} A_i \) is the sum of the areas of all districts having densities \( \geq D \). The function \( A(D) \) could then be approximately obtained by fitting a curve to the set of points \( (D, \sum_{i=1}^{k} A_i) \) where \( D \) varies from \( D_{\text{min}} \) to \( D_{\text{max}} \).

As defined, \( S(D) \) is that region of the city over which the density is \( \geq D \), so that

\[
A(D) = \int_{S(D)} r \, dr \, d\theta.
\]

It should be remembered that \( S(D) \) is not necessarily a contiguous area. The function \( A(D) \) is a non-
increasing function of $D$.

Wilkins has developed this model and has suggested that it might be applied to solve some problems in operations research.

It should be noted that the cartographer follows this technique to some extent by drawing contour lines of equal density. The function $A(D)$ expresses mathematically the increase of area contained within contour lines as their number $\rightarrow \infty$.
MAP 3.1

WOLLONGONG STATISTICAL DISTRICT
3. DEVELOPMENT OF A MODEL OF INTRA-URBAN POPULATION DISTRIBUTION FOR THE CITY OF WOLLONGONG

Wollongong, the seventh largest city in Australia, is situated on a narrow coastal plain, some fifty miles south of Sydney. It is limited to the east by the Pacific Ocean and to the west by the Illawarra Escarpment of the Sydney Basin (see Map 3.1). As a result it is far from the ideal circular shape and, as such, represents an extreme case against which to test the universality of models of intra-urban population density distribution such as Clark's and Sherratt's.

The city of Wollongong has been established by the absorption of well established townships together with the addition of new housing areas many of which were developed by the Housing Commission of N.S.W. These latter exist as pockets of relatively dense population surrounded by vacant or sparsely populated land.

3.1 The Data

Relevant maps and figures from the 1966 census were obtained for the Wollongong area from the Bureau of Census and Statistics.

The city of Wollongong was taken to be that area defined by the Bureau as urban Wollongong. The outer boundaries
therefore, are defined by the boundaries of the outer census districts. (See map 3.1, and the criteria for the delimitation of urban boundaries as set out by the Bureau - Appendix A).

The information for neighbouring census districts, such as 21Aa₁ and 21Aa₂, was combined and such information is listed for each combined census district, say 21Aa. The resultant combined districts have an average area of 0.5 square miles with a range of 0.07 to 3.78 square miles. In this way the effects of some minor local variations of density were smoothed out while still leaving a sufficiently large number (119) of districts for statistical purposes.

In the case of census districts 20IC₁ to 20IC₈, it was felt that to combine these would give a false picture of the population distribution. The total area of 20IC would have been 6.5 square miles, and in this rapidly expanding housing commission area there are areas of high density next to areas with no population. After an inspection of the total area, it was decided to divide it into five combined districts 20IC₉, 20IC₁₀, 20IC₁₁, 20IC₁₂, and 20IC₁₃.

The information for some census districts was presented by the Bureau in combined form. 21FF represents the Wollongong Hospital population and this was combined with 21FE, an area surrounding 21FF. Since the combined area of 21FE and 21FF was only 0.11 square miles, it was
decided to leave this as a combined district. A similar
decision was made in regard to four hostels. The combined
districts 21 Bh₁, 21Hd₂, 21Hg₁, 21H₂ each represent
a hostel and the surrounding census district.

Each combined district was inspected and its "centre" of population was decided. This centre was deter-
mined by inspection of local street maps to find the clustering
and distribution of streets, the assumption being that the
houses and hence population is greatest where the number of
streets is greatest. Inspection of the area itself was
carried out in cases of uncertainty. (See Map 3.2).

For each combined area, the following in-
formation was recorded - P, the population; D, the number
of dwellings; A, the area in square miles; d, the density
of population in thousands per square mile; x and y, the
distance in miles of the centre of each combined district
from the arbitrary axes (the edges of the map) and r, the
distance in miles from the city centre. (See Appendix B)

Maps in this thesis are illustrative only and
were not those used to calculate any of the data.

3.2 The Centre of Wollongong

The centre of the ideal circularly symmetric
city which extends outwards uniformly in all directions,
is quite clearly the centre of the circle defining the city
boundary or any of the isolines of equal density. The definition of the centre of a non-circular city or one distorted in any way presents greater problems.

The centre of a city is usually taken to be the centre of the central business district and, for most cities approximating circularity this is an area of low population density or density crater. For Wollongong, this CBD is not clearly defined because of the scale being used. Larger cities have extensive CBD's extending beyond the inner circle of \( \frac{1}{2} \) mile radius but for Wollongong this inner circle becomes an area of peak population density rather than a density crater.

### 3.2.1 The Centre of Gravity of Population as the City Centre

One might consider the centre of gravity of population to be the city centre. The coordinates \((\bar{x}, \bar{y})\) of this point are given by

\[
\bar{x} = \frac{\sum x_i P_i}{\sum P_i}
\]

and

\[
\bar{y} = \frac{\sum y_i P_i}{\sum P_i}
\]

where \((x_i, y_i)\) are the coordinates of the \(i\)th combined district and \(P_i\) is the population of that district.

The point \((\bar{x}, \bar{y})\) was calculated for Wollongong and is marked on Map 3.2. This centre of population is just north of the industrial complex at Port Kembla. The growth
of Wollongong as a city has been largely the result of the development of heavy industry and it would seem that this has influenced the siting of dwellings since most of the new areas of development are south of this point.

3.2.2 The Subjective Centre

Does the choice of city centre matter?

Obviously, a shift in city centre will alter the parameters of any model which depends on distance from the centre. To this extent the final form of the model depends on the choice of centre, and must always be considered in relation to it. The exact location of the point called the centre is therefore unimportant, provided that the centre is given with the model. For comparative purposes, a centre within the CBD should be chosen.

For this thesis, the city centre has been selected as the common point of the combined districts 21Gf, 21Gg, 21Gi and 21Gj. This point lies in the main street of the Wollongong shopping centre. It virtually coincides with the peak land value and is a traffic focus. It also represents

1. Private Communication: Dr. R. Robinson, Senior Lecturer in Geography, Wollongong University College.
MAP 3.3

ANNULAR RINGS FOR CLARK

MODEL
Map 3.3
Annular Rings for
Clark Model 1.
a reasonable compromise between the placement of business, professional and administrative centres. (Compare Sherratt's choice of the G.P.O. as the centre of Sydney, 1961).

3.3 Wollongong and Clark's Model

Clark's method of deriving the parameters for the negative exponential function by the use of the average density over mile wide rings would seem to be inaccurate for Wollongong because of the extreme portion of those rings, after the first which lie outside the city boundaries (see Map 3.3).

3.3.1 Clark Model 1

In order to test this method for the city of Wollongong, the population of each combined district was considered to be at the centre of each district, that is, no apportionment of population was carried out. The population at the centre of the city was taken as the total population for those combined districts where $0 \leq r \leq 0.5$, where $r$ is the distance of the centre of the district from the centre of the city. The population $k$ miles from the centre was taken as the sum of the populations of the combined districts for which the distance from the city centre was $(k-0.5) \leq r \leq (k+0.5)$ miles. Thus the total population was obtained at the city centre and at each mile from it.

It is to be noted that Clark (1951) did apportion population
Table 3.1

Clark's Model Using Annular Rings for Wollongong
Table 3.1

CLARK'S MODEL USING ANNULAR RINGS FOR WOLLONGONG

<table>
<thead>
<tr>
<th>r</th>
<th>P</th>
<th>A</th>
<th>d</th>
<th>ln d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.6</td>
<td>0.79</td>
<td>4.55</td>
<td>1.5</td>
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<td>1</td>
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<td>6.28</td>
<td>3.33</td>
<td>1.2</td>
</tr>
<tr>
<td>2</td>
<td>21.1</td>
<td>12.56</td>
<td>1.68</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>16.6</td>
<td>18.84</td>
<td>0.83</td>
<td>-0.1</td>
</tr>
<tr>
<td>4</td>
<td>36.6</td>
<td>25.12</td>
<td>1.46</td>
<td>0.4</td>
</tr>
<tr>
<td>5</td>
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<td>31.40</td>
<td>0.50</td>
<td>-1.7</td>
</tr>
<tr>
<td>6</td>
<td>5.0</td>
<td>37.68</td>
<td>0.13</td>
<td>-2.0</td>
</tr>
<tr>
<td>7</td>
<td>8.6</td>
<td>43.96</td>
<td>0.20</td>
<td>-1.6</td>
</tr>
<tr>
<td>8</td>
<td>11.8</td>
<td>50.24</td>
<td>0.23</td>
<td>-1.5</td>
</tr>
<tr>
<td>9</td>
<td>7.8</td>
<td>56.52</td>
<td>0.14</td>
<td>-2.0</td>
</tr>
<tr>
<td>10</td>
<td>7.2</td>
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<td>0.11</td>
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</tr>
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</tr>
<tr>
<td>12</td>
<td>1.9</td>
<td>75.36</td>
<td>0.03</td>
<td>-3.5</td>
</tr>
</tbody>
</table>

P - population in thousands

r - distance in miles from city centre

A - area of annular rings

d - density of population in thousands per square mile

ln d - natural logarithm of the density

$\Sigma P = 161.0$ - total population

$\Sigma A = 490$ - area enclosed by circle of radius $r = 12.5$

Regression line: $\ln y = \ln A - br$

$= 1.25 - 0.38r$

Clark's Model: $y = Ae^{-br}$

$= 3.49e^{-0.38r}$

Correlation of density with distance from centre $=-0.94$. 
where census districts were cut by the concentric circles about the centre. It was felt that for a city as distorted from circularly symmetry as is Wollongong that this additional accuracy was unnecessary.

The results were tabulated (see table 3.1) and the regression line was calculated. The resultant density function was

$$y = 3.49e^{-0.38r}$$

The correlation between the natural logarithm of the density and distance from the centre was -0.94, an extremely high correlation.

This correlation of density with distance would suggest that the model "fits" the city extremely well. The model, however, does not fit Wollongong itself but a hypothetical circularly symmetric city which has the same number of people living at each mile radius from the common centre. This substitution of a hypothetical city for the real one is, of course, the basic assumption of the Clark model.

The problem arises in the case of a city as distorted from the circular as Wollongong, of whether the assumption of circularity seriously (or significantly) affects the model. Measures of distortion will be discussed in Chapter 5.

However, one method of investigating goodness of
fit is to consider the density of population per radian, given by

\[
\lim_{\Delta \theta \to 0} \int_{0}^{\theta+\Delta \theta} \int_{0}^{\infty} Ae^{-br} r \, dr \, d\theta = \frac{A}{b^2} = 24.2.
\]

This means that the number of people living along any ray \( \theta = \text{constant} \) is 24.2 thousands. Multiplying this by \( 2\pi \) gives the total population of the city extending infinitely in all directions as measured by the model. This is 152 thousands which slightly underestimates the actual total. \((P_0 = 161 \text{ thousands})\).

For Wollongong, the population per radian is overestimated if the ray, \( \theta = \text{constant} \), is to the east or west and underestimated to the north and south.

It would seem that this model gives a reasonable fit if the whole city is being considered or if the population within a given radius is required.

If, however, only a section of the city is being considered the model does not fit.

3.3.2 Clark Model 2

A second model was constructed for comparison with Clark Model 1. For this model the population was calculated at the centre and at each mile radius as for the previous model. The area occupied by each population \( P_i(i=0,1,\ldots,12) \) was calculated in a similar manner. That is, the area at the
Table 3.2

Clark's Model Using Area of Combined Districts.
Table 3.2

CLARK'S MODEL USING AREA OF COMBINED DISTRICTS

<table>
<thead>
<tr>
<th>r</th>
<th>P</th>
<th>A</th>
<th>d</th>
<th>Ind</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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<td>6.0</td>
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<td>1.4</td>
</tr>
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<td>2</td>
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<td>5.0</td>
<td>4.2</td>
<td>1.4</td>
</tr>
<tr>
<td>3</td>
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<td>1.9</td>
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</tr>
<tr>
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<td>1.4</td>
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</tr>
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<td>7</td>
<td>8.6</td>
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<td>2.0</td>
<td>0.7</td>
</tr>
<tr>
<td>8</td>
<td>11.8</td>
<td>4.0</td>
<td>3.0</td>
<td>1.1</td>
</tr>
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<td>9</td>
<td>7.8</td>
<td>3.8</td>
<td>2.1</td>
<td>0.7</td>
</tr>
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<td>10</td>
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<td>5.0</td>
<td>1.5</td>
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</tr>
<tr>
<td>11</td>
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<td>1.2</td>
<td>0.2</td>
</tr>
<tr>
<td>12</td>
<td>1.9</td>
<td>5.0</td>
<td>0.4</td>
<td>-0.9</td>
</tr>
</tbody>
</table>

r, P, A, d, Ind as in Table 3.1

ΣP = 161.0 - Total population

ΣA = 64.1 - Area of Wollongong within census boundaries

Regression Line: lny = lnA - br
   = 1.69 - 0.15r

Clark's Model: y = Ae^{-br}
   = 5.4e^{-0.15r}

Correlation of density with distance from centre = -0.84.
centre was taken to be the sum of the areas of the combined
districts for which $0 \leq r < 0.5$, where $r$ is the distance of the
centre of the combined district from the centre. The area,
k miles from the centre, was calculated as the sum of areas
of the combined districts for which $(k-0.5) \leq r < (k+0.5)$.
As in the previous example there was no apportioning of
either population or area. This does introduce a slight
error - for example, the area "at the centre" is 0.6 square
miles instead of 0.79 square miles. The error in the logarithmic
form of the density would have little effect on the regression
line. Moreover this model is only for comparison, and will
not be used in further calculations.

The table 3.2 shows the results. The model
becomes

$$ y = 5.4e^{-0.15r} $$

with a density-distance correlation of -0.84. This model
does not fit as well as Model 1 in terms of correlation and
a glance at the density columns of tables 3.1 and 3.2 reveals
that the decline of density with distance from the centre
is not as regular for Model 2 as it is for Model 1.

Since we are using density values in the cal-
culation of Model 2 which more closely approximate the real
density values than those in Model 1, one would, perhaps,
expect Model 2 to be the more accurate, and therefore more
useful.
To calculate the total population, however, we must either assume circularity or integrate over the actual area used. The latter is, in practice, impossible due to the highly irregular outline of the boundaries of the census districts.

If circularity is assumed the total population becomes

\[ 2\pi A \int_0^{12r} e^{-br} \, dr = \frac{2\pi A}{b^2} \left[ 1 - (1+br)e^{-br} \right] \]

\[ \approx 880 \text{ thousands}, \]

far in excess of the real population total of 161 thousands.

3.3.3 Clark Model 3

The alternative approach suggested by Clark was used to construct Model 3. That is, the density of each of the combined districts was plotted against the distance of the centre of the district from the city centre. This is, of course, only slightly different from the construction of Model 2, using more data points and eliminating the grouping errors.

The model so obtained was

\[ y = 6.3e^{-0.12r} \]

but the correlation is low at -0.41. The larger number of data points for this model has decreased the fit of the regression line to the hypothetical model because of the increased variation of the densities at the increased number
of values of $r$.

Calculation of the total population, as before, gives the extreme value of 1720.9 thousands.

This method of determining a model for Wollongong seems to be quite unsatisfactory. Of the three Clark models derived the most useful appears to be the first where the real city is replaced by a hypothetical circularly symmetric city having the same number of people at each mile radius as the real city.

3.4 **Wollongong as a Rectilinear City**

As Wollongong so obviously deviates from circular symmetry, its representation in model form by a hypothetical city with circular symmetry seems a little unreal. Berry and Horton (1970) have indicated that a set of similar equations with parameters varying with direction might be more appropriate in some cases; but even this suggestion assumes an underlying circular symmetry. This modification is applicable to, say, semicircular cities with centres on the coastline.

Wollongong, however, has not spread outwards from one centre but has absorbed already established towns spread along the coastline. Then, too, the sea and the scarp have prevented the development of circularity. Indeed, Wollongong more closely approximates a rectangular rather than circular
shape.

Can a negative exponential model be developed based on a hypothetical rectilinear city rather than a circular one?

3.4.1 The Data for Wollongong as a Rectilinear City

The data used to develop a rectilinear model for the city of Wollongong were obtained as follows.

A line drawn through the selected centre of Wollongong in an east-west direction. This line roughly corresponds to the main street of Wollongong and, therefore, represents a "centre" for the rectilinear city.

Parallel lines were drawn to scale, \( \frac{1}{2} \) mile north and south of this centre line and then at 1 mile intervals out to 11\( \frac{1}{2} \) miles from the centre. Thus Wollongong was divided into 23 mile-wide strips.

The population within each strip was calculated. In this case, some apportioning of population was made where a combined district was cut by a strip boundary. The area of each strip within the outer census boundaries was measured by means of approximating rectangles or triangles.

The density in each strip may be calculated in three different ways, each resulting in a slightly different model.
Table 3.3

Data for Wollongong as a Rectilinear City.
Table 3.3

DATA FOR WOLLONGONG AS A RECTILINEAR CITY.

<table>
<thead>
<tr>
<th>$r_N$</th>
<th>$P$</th>
<th>$A$</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$r_3$</th>
<th>$P$</th>
<th>$A$</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
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<td>2.8</td>
<td>7.0</td>
<td>6.3</td>
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<td>4.7</td>
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</tr>
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<td>0.9</td>
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<tr>
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<td>0.9</td>
<td>0.1</td>
<td>0.2</td>
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<td>0.9</td>
<td>0.5</td>
<td>0.3</td>
<td>0.3</td>
<td>1.8</td>
</tr>
</tbody>
</table>

$P$ - population in thousands

$A$ - Area in square miles

$D_1$, $D_2$, $D_3$ - Density in thousands per square miles for rectilinear models 1, 2, 3 respectively.

$r$ - miles north from city centre

$N$ - miles south from city centre

$S$ - population in thousands

$A$ - Area in square miles
3.4.2 Rectilinear Model 1

Clark's original methodology involves the substitution of a hypothetical circular city for the real one. This circular city has the same number of people living at mile intervals as the real one. The density of population is calculated for the hypothetical city, the area used being that of each of the annular rings.

To repeat this procedure for a rectilinear Wollongong, the hypothetical city must be substituted for the real city. The length of the city is taken to be 23 miles (to gain symmetry some of the edges of the city have been ignored). The measured area of the city is 62.1 square miles. The hypothetical rectilinear city is therefore, considered to be of width 2.7 miles.

The density of each strip was calculated by dividing the population $P$ of that strip by the area of the strip; taken to be 2.7 square miles. These densities are listed in the column $D_1$ in table 3.3.

Two equations were derived, one for the direction north and the other south.

The rectilinear model 1 for Wollongong is found to be

$$y = \begin{cases} 
8.67e & \text{for } r \text{ north,} \\
-0.17r & \text{for } r \text{ south,} \\
4.66e & \text{for } r \text{ south,}
\end{cases}$$
by fitting the regression lines. In each case there is a negative correlation of density with distance. For \( r \) north the correlation is high, namely \(-0.91\). For \( r \) south, the correlation is less at \(-0.62\).

3.4.3 Rectilinear Model 2

To construct the second rectilinear model, the city was considered to be two rectangles rather than one. As two equations are required for the model it seems reasonable to consider each rectangle as having a different width. The average width of the northern rectangle is 2.2 miles (and hence the area of each northern strip is 2.2 square miles) and that of the southern rectangle is 3.3 miles. The density for each strip calculated in this way is shown in column D of table 3.3.

When the regression line is fitted, model 2 becomes

\[
y = \begin{cases} 
10.07e^{-0.28r} & \text{for } r \text{ north} \\
4.47e^{-0.18r} & \text{for } r \text{ south.}
\end{cases}
\]

The correlations for this model are \(-0.92\) for the northern equation and \(-0.62\) for the southern.

3.4.4 Rectilinear Model 3

For the development of the third rectilinear model, Clark's modified approach was used; that is, the density was calculated by dividing the population of each strip by the actual area of the strip within census
boundaries. The densities so obtained may be seen in column D in table 3.3. As before two regression lines were fitted and the rectilinear model 3 is given by

\[ y = \begin{cases} 
7.10e -0.21r & \text{for } r \text{ north} \\
-0.07r & \text{for } r \text{ south}.
\end{cases} \]

The correlation between density and distance from the centre is again high for the northern section at -0.95. For the southern section, however, the correlation is only -0.42.

3.4.5 Comparison of the Three Rectilinear Models

One method of estimating the value of the model is to consider the ability of the model to give the total population with some measure of accuracy. This, of course, does not necessarily indicate the fit of the model (see comment on Clark Model 1).

The total population may be found by integrating the density functions from 0 to 11.5 (the limits assumed in the model development) and multiplying by the width of the rectangle, the density being considered constant across the rectangles. Thus

\[ P = W \left( \int_{0}^{11.5} A e^{-b r} \, dr + \int_{0}^{11.5} A e^{-b r} \, dr \right) \]

where \( W \) is the width of the northern rectangle, \( A \) and \( b \) are

\[ \text{N} \quad \text{N} \quad \text{N} \quad \text{N} \]
56.

Table 3.4

Summary of Rectilinear Models 1, 2 and 3
### Table 3.4

**SUMMARY OF RECTILINEAR MODELS 1, 2, AND 3**

<table>
<thead>
<tr>
<th></th>
<th>Model 1</th>
<th></th>
<th>Model 2</th>
<th></th>
<th>Model 3</th>
<th></th>
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<tbody>
<tr>
<td></td>
<td>N</td>
<td>N</td>
<td>S</td>
<td>N</td>
<td>S</td>
<td>N</td>
</tr>
<tr>
<td>A</td>
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<td>4.66</td>
<td>10.07</td>
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<td>0.28</td>
<td>0.18</td>
<td>0.21</td>
<td>0.07</td>
</tr>
<tr>
<td>Cor.</td>
<td>-0.91</td>
<td>-0.62</td>
<td>-0.92</td>
<td>-0.64</td>
<td>-0.95</td>
<td>-0.42</td>
</tr>
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<td>75.96</td>
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<td>67.69</td>
<td>74.16</td>
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<td>141.85</td>
<td></td>
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</tbody>
</table>

- **A** - Hypothetical city centre density
- **b** - Density gradient
- **Cor** - Pearson's product moment correlation coefficient
- **P** - Population in thousands per square mile
- **P T** - Total city population as estimated by the model

---

**Note:**

This table summarizes the results of three rectilinear models for city center density (A), density gradient (b), and population (P). The models are compared by examining the coefficients of determination (R²), which range from -0.91 to +0.95, indicating strong linear relationships. The population estimates (P T) are consistent with the model predictions, suggesting the models are effective for estimating city populations.
the parameters for the northern part of the density function and $W$, $A$, and $b$, the corresponding constants for the southern part.

The total populations as calculated are shown in Table 3.4. The total population for Model 3 was calculated using the two rectangles for north and south. If one rectangle (of width 2.7 miles) is used the population becomes 143.76 thousands. (With 83.09 thousands for the north and 60.67 for the south).

Each of the three models underestimates the total population. The southern model in particular underestimates the true population (82.8 thousands).

The second model appears to fit best. The correlations show the model to have the best relationship between density and distance taking both equations into consideration. It also gives the best estimate of the northern population and the closest estimate of the total population. This model is also the easiest of the three to derive. The area of each strip need not be calculated — merely the overall area within census boundaries from which the average width of the fitted rectangles may be calculated.

The differences between models 1 and 2 indicate the advisability of using two equations for the city each based on a rectangle of different width.
3.4.6  **Rectilinear Model 4**

The possibility of only one equation for the city was also considered. To develop this model the real city was replaced by a rectangle 11.5 miles in length and 5.4 miles wide. Symmetry north and south of the city centre line is assumed and the total population at the city centre and each mile (north or south) was calculated. This was divided by 5.4 to give the density. The calculated density function is given by

\[ y = 5.53e^{-0.20r} \]

with a relatively low correlation of -0.79. The total population as given by this equation is 134.38 thousands, lower than any of the models 1-3.

3.4.7  **Evaluation of the Rectilinear Models**

While none of the derived models gives a perfect fit, the use of the rectangle rather than circle in order to develop a model would appear to have merit. This is particularly so for the northern section of the city. It must be remembered that the density function for the southern section is influenced by three distorting factors; the large areas of land used for industrial and recreational purposes, the lake, and the artificial densities created by the Housing Commission areas.

It would appear that some support for a
negative exponential decline of density with distance from the city centre is indicated. Model 2 with the city represented by two rectangles would appear to have some merit.

3.5 A Normal Curve Model for Wollongong

The construction of a normal curve model for a hypothetical circularly symmetrical Wollongong would obviously, be as inaccurate as the earlier Clark models. Sherratt (1961) has, however, suggested that towns and suburbs show a normal distribution of population densities (see 2.1.5). Wollongong has not yet reached maturity as a city and might still be considered to be an aggregation of overlapping towns. If Sherratt is correct then, at this stage of its development, Wollongong should be represented by the sum of a series of overlapping normal curves such that the density function is represented by

\[ y = \sum_{i} A_i e^{-\alpha (x-c_i)^2} \]

where \( y \) is the density of population at \( x \) miles from the city centre, the \( c_i \) represent the distances from the city centre of the centre of each normal curve and the \( A_i \) and \( \alpha \) are constants. The direction north is considered to be positive and the south, negative.

In order to find the various values of \( A_i \), \( \alpha \) and \( c_i \), a graph of the population density distribution
62.

**Table 3.5**

**FIRST APPROXIMATION FOR NORMAL CURVE MODEL**
<table>
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<th>Miles</th>
<th>Total Actual Miles</th>
<th>Total Actual</th>
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</tr>
<tr>
<td>-5½</td>
<td>0.6</td>
<td>1.0</td>
</tr>
<tr>
<td>-5</td>
<td>0.4</td>
<td>0.6</td>
</tr>
<tr>
<td>-4½</td>
<td>0.2</td>
<td>0.4</td>
</tr>
<tr>
<td>-4</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>-3½</td>
<td>3.1</td>
<td>0.1</td>
</tr>
<tr>
<td>-3</td>
<td>1.9</td>
<td>2.1</td>
</tr>
<tr>
<td>-2½</td>
<td>0.7</td>
<td>1.7</td>
</tr>
<tr>
<td>-2</td>
<td>0.2</td>
<td>1.2</td>
</tr>
<tr>
<td>-1½</td>
<td>1.0</td>
<td>0.8</td>
</tr>
<tr>
<td>-1</td>
<td>0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>-½</td>
<td>2.3</td>
<td>0.2</td>
</tr>
<tr>
<td>0</td>
<td>3.8</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Note: (The values in parentheses have been read from the graph).
GRAPH 3.1

NORMAL CURVE MODEL FOR FOLLOWING

FIRST APPROXIMATION
GRAPH 3.1
Normal Curve Model for Wollongong
First Approximation

- Actual
- Model

Population Density in Thousands per Square Mile

Miles from City Centre

South North

-11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11
of Wollongong as for rectilinear model 3 was drawn. Observation of the graph suggested that as a first approximation there should be five values for \( c \), namely \(-8, -4, 0, 2 \) and \( 7 \). By a process of trial and error, using \( 2 - a(x-c)^2 \) the fact that if \( a(x-c) \geq 7 \) then \( e < 0.001 \) and may be neglected, a first approximation was developed as shown by table 3.5 and graph 3.1. Where two estimates conflicted the least squares criterion was used to select the better fitting curve.

From the graph it could be seen that the fit would be improved by adding minor curves with \( c = -\frac{6}{2}, -2\frac{1}{2}, 5 \) and \( 10 \). It was also obvious that for the curve with \( c = -4 \), the value of \( a \) was too small.

The model for Wollongong developed as above is:

\[
\begin{align*}
y &= 2.3e^{2} + 0.5e^{ \frac{2}{3} } + 3.7e^{2} + 0.6e^{2} \\
&= -0.2(x+8)^{2} -0.7(x+6.5)^{2} -1.5(x+4)^{2} -0.7(x+2.5)^{2} \\
&= -2.0x^{2} -0.13(x-2)^{2} -2.8(x-5)^{2} -0.25(x-7.5)^{2} \\
&= +3.8e^{2} +5.0e^{2} +0.6e^{2} +2.2e^{2} \\
&= -2.8(x-10)^{2} \\
&= +0.6e^{2}
\end{align*}
\]

(see table 3.6 and graph 3.2)

3.5.1 **Estimation of Total Population Using the Normal Curve Model**

As before, some idea of the value of the
Table 3.6
SECOND APPROXIMATION FOR NORMAL CURVE MODEL
<table>
<thead>
<tr>
<th>Curve</th>
<th>South</th>
<th>North</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.3/0.5</td>
<td>3.7/0.6</td>
</tr>
<tr>
<td>2</td>
<td>0.2/0.7</td>
<td>1.5/0.7</td>
</tr>
<tr>
<td>c</td>
<td>-8/-4</td>
<td>-2/-4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Miles</th>
<th>Total</th>
<th>Actual</th>
<th>Miles</th>
<th>Total</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>-11</td>
<td>0.4</td>
<td>1.8</td>
<td>0</td>
<td>3.8</td>
<td>3.0</td>
</tr>
<tr>
<td>-10</td>
<td>0.6</td>
<td>(1.2)</td>
<td>1/2</td>
<td>2.3</td>
<td>3.7</td>
</tr>
<tr>
<td>-9</td>
<td>1.0</td>
<td>0.9</td>
<td>1</td>
<td>0.5</td>
<td>4.4</td>
</tr>
<tr>
<td>-91/2</td>
<td>1.5</td>
<td>(1.4)</td>
<td>1/2</td>
<td>4.9</td>
<td></td>
</tr>
<tr>
<td>-9</td>
<td>1.9</td>
<td>2.0</td>
<td>2</td>
<td>5.0</td>
<td></td>
</tr>
<tr>
<td>-8</td>
<td>2.2</td>
<td>2.2</td>
<td>(2.2)</td>
<td>2/8</td>
<td>4.9</td>
</tr>
<tr>
<td>-7</td>
<td>2.3</td>
<td>2.3</td>
<td>3</td>
<td>4.4</td>
<td></td>
</tr>
<tr>
<td>-71/2</td>
<td>1.9</td>
<td>0.35</td>
<td>2.25</td>
<td>(2.35)</td>
<td>3/8</td>
</tr>
<tr>
<td>-7</td>
<td>1.5</td>
<td>0.4</td>
<td>1.9</td>
<td>2.4</td>
<td>4</td>
</tr>
<tr>
<td>-6</td>
<td>1.0</td>
<td>0.5</td>
<td>1.5</td>
<td>(1.8)</td>
<td>2/3</td>
</tr>
<tr>
<td>-6</td>
<td>0.6</td>
<td>0.4</td>
<td>1.0</td>
<td>1.3</td>
<td>5</td>
</tr>
<tr>
<td>-51/2</td>
<td>0.4</td>
<td>0.35</td>
<td>0.85</td>
<td>(1.2)</td>
<td>5/8</td>
</tr>
<tr>
<td>-5</td>
<td>0.2</td>
<td>0.1</td>
<td>1.1</td>
<td>1.2</td>
<td>6</td>
</tr>
<tr>
<td>-4</td>
<td>0.1</td>
<td>2.6</td>
<td>2.7</td>
<td>(2.4)</td>
<td>6/8</td>
</tr>
<tr>
<td>-3</td>
<td>3.7</td>
<td>0.1</td>
<td>3.8</td>
<td>3.8</td>
<td>7</td>
</tr>
<tr>
<td>-31/2</td>
<td>2.6</td>
<td>0.3</td>
<td>0.1</td>
<td>3.0</td>
<td>(2.4)</td>
</tr>
<tr>
<td>-3</td>
<td>0.8</td>
<td>0.5</td>
<td>0.2</td>
<td>1.5</td>
<td>1.9</td>
</tr>
<tr>
<td>-21/2</td>
<td>0.1</td>
<td>0.6</td>
<td>0.4</td>
<td>1.1</td>
<td>(1.2)</td>
</tr>
<tr>
<td>-2</td>
<td>0.5</td>
<td>0.6</td>
<td>1.1</td>
<td>1.0</td>
<td>9</td>
</tr>
<tr>
<td>-11/2</td>
<td>0.3</td>
<td>1.0</td>
<td>1.3</td>
<td>(1.7)</td>
<td>9/8</td>
</tr>
<tr>
<td>-1</td>
<td>0.1</td>
<td>0.5</td>
<td>1.6</td>
<td>2.2</td>
<td>3.0</td>
</tr>
<tr>
<td>-11/2</td>
<td>2.3</td>
<td>2.3</td>
<td>4.6</td>
<td>(4.6)</td>
<td>10/8</td>
</tr>
<tr>
<td>0</td>
<td>3.8</td>
<td>3.0</td>
<td>6.8</td>
<td>6.8</td>
<td>11</td>
</tr>
</tbody>
</table>

**TABLE 3.6** Second Approximation for Normal Curve Model
GRAPH 3.2

NORMAL CURVE MODEL FOR WOLLONGONG

SECOND APPROXIMATION
GRAPH 3.2

Normal Curve Model for Wollongong

Second Approximation

- Actual
- Model
model may be gained by considering its ability to estimate the total population.

The normal curve model is of the form

\[ y = \sum_{i=1}^{9} A_i e^{-a (x-c_i)} \]

and each of the curves 1-9 is considered to extend to infinity.

The standard deviation of each curve is

\[ \frac{1}{\sqrt{2\pi a}}. \]

The area under each curve is therefore, given by

\[ A_i \frac{\sqrt{\pi}}{\sqrt{a}}. \]

As for the rectilinear model 1 the city was taken to be a rectangle 2.7 miles wide.

The total population then, is given by

\[ P = 2.7\sqrt{\pi} \sum_{i=1}^{9} A_i /\sqrt{a} \]

\[ = 150 \text{ thousands} \]

which is a reasonable approximation.

3.6 The Trend Surface Model

In order to calculate the polynomial surface of best fit, the trend surface model, a computer programme developed by R.J. Sampson and J.C. Davis, Idaho State University,
TABLE 3.7

TREND SURFACE DATA
<table>
<thead>
<tr>
<th>Equation</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Cubic</th>
<th>4th Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear Equation</td>
<td>( z = 2.207 + 0.536x + 0.031y )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quadratic Equation</td>
<td>( z = -0.677 + 0.477x + 0.611y - 0.073x^2 ) + 0.073xy - 0.026y^2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cubic Equation</td>
<td>( z = 10.204 - 6.429x - 1.340y + 1.035x^2 ) + 0.974xy + 0.655y^2 - 0.004x^3 - 0.098x^2y - 0.018xy^2 - 0.002y^3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4th Order Equation</td>
<td>( z = 0.267 + 15.696x + 9.552y - 0.030x^2 ) - 5.553xy - 0.661y^2 + 0.261x^3 + 0.448x^2y + 0.552xy^2 + 0.017y^3 - 0.027x^4 + 0.005x^3y - 0.038x^2y^2 - 0.012xy^3 + 0.0001y^4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Surface</th>
<th>Linear</th>
<th>Quadratic</th>
<th>Cubic</th>
<th>4th Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard deviation</td>
<td>3.19</td>
<td>2.91</td>
<td>2.86</td>
<td>2.89</td>
</tr>
<tr>
<td>Variation explained by surface</td>
<td>44.28</td>
<td>60.49</td>
<td>278.77</td>
<td>322.69</td>
</tr>
<tr>
<td>Variation not explained by surface</td>
<td>1200.71</td>
<td>998.50</td>
<td>966.21</td>
<td>922.10</td>
</tr>
<tr>
<td>Total Variation</td>
<td>1244.99</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coefficient of Determination</td>
<td>0.036</td>
<td>0.198</td>
<td>0.224</td>
<td>0.259</td>
</tr>
<tr>
<td>Coefficient of Correlation</td>
<td>0.189</td>
<td>0.445</td>
<td>0.473</td>
<td>0.509</td>
</tr>
</tbody>
</table>

Number of Data Points is 118
Standard deviation of \( z \) is 3.243
<table>
<thead>
<tr>
<th>TABLE 3.8</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>APPLICATION OF F-TEST</strong></td>
</tr>
</tbody>
</table>
### Table 3.8

**Application of F-test to determine surface of best-fit**

<table>
<thead>
<tr>
<th>L and Q</th>
<th>Q and C</th>
<th>Q and 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_Q = 0.198$</td>
<td>$k_a = 0.224$</td>
<td>$k_4 = 0.259$</td>
</tr>
<tr>
<td>$k_L = 0.025$</td>
<td>$k_Q = 0.198$</td>
<td>$k_Q = 0.198$</td>
</tr>
<tr>
<td>$k_Q - k_L = 0.162$</td>
<td>$k_C - k_Q = 0.025$</td>
<td>$k_4 - k_Q = 0.061$</td>
</tr>
<tr>
<td>$df = 3$</td>
<td>$df = 4$</td>
<td>$df = 5$</td>
</tr>
<tr>
<td>$V_1 = \frac{k_Q - k_L}{df} = 0.054$</td>
<td>$V_1 = \frac{k_C - k_Q}{df} = 0.0065$</td>
<td>$V_1 = \frac{k_4 - k_Q}{df} = 0.0122$</td>
</tr>
<tr>
<td>$1 - k_Q = 0.202$</td>
<td>$1 - k_C = 0.776$</td>
<td>$1 - k_4 = 0.741$</td>
</tr>
<tr>
<td>$df = 111$</td>
<td>$df = 107$</td>
<td>$df = 103$</td>
</tr>
<tr>
<td>$V_2 = \frac{1 - k_Q}{df} = 0.097$</td>
<td>$V_2 = \frac{1 - k_C}{df} = 0.0072$</td>
<td>$V_2 = \frac{1 - k_4}{df} = 0.0072$</td>
</tr>
<tr>
<td>$V_1/V_2 = 7.7$</td>
<td>$V_1/V_2 = 0.9$</td>
<td>$V_1/V_2 = 1.7$</td>
</tr>
<tr>
<td>$P &lt; 0.0001$</td>
<td>$P &gt; 0.5$</td>
<td>$P &gt; 0.1$</td>
</tr>
</tbody>
</table>

**L** = linear  
**Q** = quadratic  
**C** = cubic  
**4** = 4th order  
**k** = coefficient of determination  
**df** = degrees of freedom  

$V_1$ = variance associated with the equations  
$V_2$ = variance associated with the data  

$F = \frac{V_1}{V_2}$ in the Fisher-Snedecor variance ratio.
and adapted by A. Cook (Department of Geology, Wollongong University College) was used.

This programme gives the coefficients of linear, quadratic, cubic and quartic polynomials together with associated statistical data. (See Table 3.7)

The application of the F-test (see table 3.8) indicates that the quadratic polynomial is the best fit for the Wollongong data. Although the cubic and quartic polynomials apparently explain more of the variation and correlate a little more closely with the observed data than the quadratic, this difference is not statistically significant. The quadratic equation, however, significantly explains more of the variation than the linear.

The values of $z$ as given by the quadratic equation were computed for $x$, from 0 to 6 in 0.5 units and for $y$, from 0 to 24 in units. From these results contours of population density were drawn (see map 3.4).

The trend surface obtained indicates for the northern parts of Wollongong a general drop in population density away from the coastline and also a drop in density from the centre northwards. This pattern is continued for a few miles south and then the density drops fairly regularly in a southerly direction while remaining relatively constant in the east-west direction.
MAP 3.4

TREND SURFACE CONTOURS

FOR WOLLONGONG
Map 3.4
Trend Surface Contours for Wollongong (in Thousands per square mile)
Table 3.9

Calculation of Total Population Using Trend Surface Model.
## Table 3.9

**CALCULATION OF TOTAL POPULATION USING TREND SURFACE MODEL**

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>x_i</th>
<th>x_j</th>
<th>y_i</th>
<th>y_j</th>
<th>Population</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>20</td>
<td>25</td>
<td>3.6</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2.5</td>
<td>18</td>
<td>20</td>
<td>10.9</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>3</td>
<td>15</td>
<td>18</td>
<td>28.5</td>
</tr>
<tr>
<td>4</td>
<td>1.5</td>
<td>4</td>
<td>11</td>
<td>15</td>
<td>54.0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>6</td>
<td>9</td>
<td>11</td>
<td>55.4</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>9</td>
<td>18.6</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>5.5</td>
<td>6</td>
<td>9</td>
<td>1.3</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>5.5</td>
<td>17.7</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>3.5</td>
<td>6.4</td>
</tr>
</tbody>
</table>

**Total Population** 196.4
3.6.1 Total Population Estimated by the Trend Surface Model

The trend surface model does not lend itself to a simple solution for finding the total population because of the difficulty of defining suitable boundaries. If a series of rectangles is used to approximate the shape, the population may be found as

\[ P = \sum_{ij}^{y} \int_{x}^{i} \int_{y}^{j} D(x, y) \, dx \, dy \quad i = 1, 2, \ldots, 9 \]

\[ j = 1, 2, \ldots, 9 \]

where \( D(x, y) \) is the density function and \( x, x, y, y \) represent the coordinates of the rectangles.

Now \( D(x, y) = a + bx + cy + dx + exy + fy \)

with \( a = -0.68; \quad b = 0.48; \quad c = 0.61; \quad d = -0.07 \)
\[ e = 0.07; \quad f = -0.03. \]

The total population calculated in this way is 196.4 thousands, which is of the right order of magnitude although considerably overestimating the population (see table 3.9).

3.7 Comparison of Models Developed for Wollongong

Each of the models developed is in some way of value. The circular Clark models indicate clearly the need for a city to at least approach circular symmetry before these models can be applied with confidence. The
Clark model 1 shows that a model of apparently good fit, estimating total population with reasonable accuracy can be a description of a hypothetical model quite unlike the real city.

The rectilinear models show that the negative exponential function is a reasonable description of the decline of population density from the city centre for a city such as Wollongong.

The methods of model development appear to have some merit. The normal curve model while fitting the data reasonably well was difficult to develop, the method being largely trial and error. Then, too, being tailored for one city in particular it has no comparative value. It would not be suitable for a circular city and is applicable to Wollongong only at this immature stage of development.

The trend surface model was relatively simple to derive. The lack of defined boundaries make the application of this model difficult. It does, however, give a clear indication of population trends and in this way adds to the information about the city.

3.8 Wilkins Model

Wilkins (1968) has suggested that the function
Table 3.10

Calculation for Wilkin's Model.
Table 3.10

CALCULATION FOR WILKINS MODEL

<table>
<thead>
<tr>
<th>Density</th>
<th>Area</th>
<th>Cum. Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0 - 0.99</td>
<td>28.6</td>
<td>75.2</td>
</tr>
<tr>
<td>1.0 - 1.99</td>
<td>20.2</td>
<td>46.6</td>
</tr>
<tr>
<td>2.0 - 2.99</td>
<td>5.5</td>
<td>26.4</td>
</tr>
<tr>
<td>3.0 - 3.99</td>
<td>4.5</td>
<td>20.9</td>
</tr>
<tr>
<td>4.0 - 4.99</td>
<td>4.3</td>
<td>16.4</td>
</tr>
<tr>
<td>5.0 - 5.99</td>
<td>3.1</td>
<td>12.1</td>
</tr>
<tr>
<td>6.0 - 6.99</td>
<td>5.8</td>
<td>9.0</td>
</tr>
<tr>
<td>7.0 - 7.99</td>
<td>1.2</td>
<td>3.2</td>
</tr>
<tr>
<td>8.0 - 8.99</td>
<td>0.8</td>
<td>2.0</td>
</tr>
<tr>
<td>9.0 - 9.99</td>
<td>0.3</td>
<td>1.2</td>
</tr>
<tr>
<td>10.0 - 10.99</td>
<td>0.1</td>
<td>0.9</td>
</tr>
<tr>
<td>11.0 - 11.99</td>
<td>0.1</td>
<td>0.8</td>
</tr>
<tr>
<td>12.0 - 12.99</td>
<td>0.5</td>
<td>0.7</td>
</tr>
<tr>
<td>13.0 - 13.99</td>
<td>0.0</td>
<td>0.2</td>
</tr>
<tr>
<td>14.0 - 14.99</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>15.0 - 15.99</td>
<td>0.0</td>
<td>0.1</td>
</tr>
<tr>
<td>16.0 - 16.99</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>
A(D) as defined in chapter 2.7 would add information about a city. To calculate A(D) for Wollongong the densities of the combined districts were categorised as in table 3.10 and the associated areas totaled and the cumulative areas calculated.

A rough graph of the results suggested that the function A(D) might be of the negative exponential form. The correlation between the increase of density and the normal logarithm of the cumulative area is very high at -0.99. The function A(D) is given by

$$A(D) = 77.5e^{-0.43D}$$

where D represents a given density and A(D) is the area which has a density of population \(\geq D\).
4. THE EFFECT OF CHOICE OF MODEL ON THE CALCULATION OF NUCLEAR CASUALTIES

The development of a model for a city should, if the model is satisfactory, provide a means of solving applied problems related to population distribution. Sherratt (1960) as mentioned earlier developed his model in order to solve the problem of the extension of gas mains.

Does the choice of model significantly affect the solution of such problems? Both Clark and Sherratt claim universality for their models and one would, therefore, expect to find only minor variations in results.

Hunter (1965, 1967) has derived an analytical technique for estimating the casualties in a nuclear attack on an urban centre, in which the casualty function is approximated by an exponential or sum of exponential functions. This technique necessarily uses a model of urban population density distribution and the model chosen by Hunter is that of Sherratt. He selected this model because it "proved more amenable to mathematical treatment" (1967, P.1097). He does, however, point out that some cities might be better represented by Clark's model or, indeed, by some other model.

This problem provides an opportunity to test the effect of choice of model both analytically and as it affects Wollongong.
4.1 Estimation of Casualties from a Nuclear Attack on a City

Hunter defines $P_{ijk}(R)$ to be the probability that an individual distant $R$ from ground zero is a casualty of the $j$ type due to the $i$ weapon which has the $k$ type of detonation ($i=1, 2, \ldots, n$; $j=1, 2, 3$; $k=1, 2$) with $j=1$, the total casualties; $j=2$, fatalities; $j=3$, non-fatal casualties; $k=1$, an air burst and $k=2$, a surface burst.

Then $P_{ijk}(R) = \exp(-c_{ijk}R^2)$ is given as an "adequate approximation" (1965, P4).

$c_{ijk}$ is the weapon parameter and Hunter details in table form various values of it.

For example,

$.09 = c_{1i2}$ = the weapon parameter for total casualties for a surface burst of a weapon yielding 1000KT.

or $.09 = c_{1i2}$ = the weapon parameter for fatalities (only) for an air burst of a weapon yielding 2000KT.

If aiming error is considered then the probability of a casualty at point $(x, y)$, from a nuclear weapon which is aimed at $(X, Y)$ and with weapon parameter $c_i$, and aiming error $\sigma$, which actually lands distance $r$ from $(X, Y)$ at $(x, y)$ is given by:

\[ P_{ijk}(r) = \exp(-c_{ijk}r^2) \]
Identification of Symbols.
Figure 4.1

- City centre.
- $(x, y)$ - ground zero
- $(x_i, y_i)$ - target

Figure 4.2

- City centre.
- $(xp)$ - ground zero.
The probability of a casualty at \((x,y)\) is given by:

\[
P[\text{casualty at } (x,y)] = \frac{1}{M} \exp\left(-\frac{L R^2}{2}\right)
\]

where

\[
M = 1 + 2c \sigma^i
\]

and

\[
L = \frac{c}{M}\]

4.2 **An Examination of Hunter's Approximation**

If Hunter's approximation is indeed "adequate" it should give similar results regardless of the city model used, and to test this a comparison of the actual number of expected casualties as given by Clark's and Sherratt's models is desirable.

For this purpose, a symmetric city of given central density, \(A\), is selected and, as is usual in applied research, is considered to be infinite in extent. The attack is to consist of one nuclear weapon landing at a variable distance, \(X\), from the centre of the city. It is further assumed that the origin of the cartesian coordinate system is the city centre and also that the \(x\)-axis passes through the point of impact as these assumptions result in no loss of generality and simplify calculations. Ground zero, therefore, has the coordinates \((X,0)\). See figure 4.2.

If aiming error is ignored, the probability that a person distant \(R\) from ground zero is a casualty has the
form

\[ P = \exp(-c R^2) \]

\[
\begin{align*}
  &_{ij} \quad &_{ij} \\
  &^2 \quad &^2 \\
 \text{where} \quad &R = (x-X)^2 + y \\
  &^2 \quad &^2 \\
  &r - 2rx \cos \theta + X.
\end{align*}
\]
The expected number of casualties $N_c$ calculated using Clark's model is given by:

$$N_c = A \int_0^\infty \int_0^\infty \exp\left(-x^2+y^2\right) \frac{1}{2\pi} - \alpha(x\alpha - y^2) \, dx \, dy$$

$$= 2\pi A (b^2 + 2c)^{-1} \exp\left(-\frac{1}{2} \left(b^2 + 2c\right)^{-1}\right).$$

If Clark's model is used, the expected number of casualties would be given by:

$$N_c = A \int_0^{2\pi} \int_0^\infty \exp\left(-x_b - b\left(x^2 - 2x \cos \theta x^2\right)\right) \, dx \, d\theta$$

$$= \alpha e^{-\alpha^2} \int_0^{2\pi} \int_0^\infty \exp\left(-\frac{x^2}{2c^2}\right)^2 \, dx \, d\theta$$

$$= \alpha e^{-\alpha^2} \int_0^{2\pi} \frac{1}{\sqrt{c}} \, d\theta \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx$$

$$= \frac{\alpha}{\sqrt{c}} \pi e^{-\alpha^2} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \, d\theta$$

where $\theta = \sqrt{c} \left(x - L(\theta)\right)$

and $x = \sqrt{c} \left(x - L(\theta)\right)$

$$N_c = \alpha e^{-\alpha^2} \left\{ \frac{\pi}{2c} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \right\}$$

$$= \alpha e^{-\alpha^2} \left\{ \frac{\pi}{2c} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \right\}$$

$$= \alpha e^{-\alpha^2} \left\{ \frac{\pi}{2c} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx + \pi \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \right\}$$

$$= \alpha e^{-\alpha^2} \left\{ \frac{\pi}{2c} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx + \pi \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \right\}$$

$$= \alpha e^{-\alpha^2} \left\{ \frac{\pi}{2c} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx + \pi \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \right\}$$

$$= \alpha e^{-\alpha^2} \left\{ \frac{\pi}{2c} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx + \pi \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \right\}$$

$$= \alpha e^{-\alpha^2} \left\{ \frac{\pi}{2c} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx + \pi \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \right\}$$

$$= \alpha e^{-\alpha^2} \left\{ \frac{\pi}{2c} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx + \pi \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \right\}$$

$$= \alpha e^{-\alpha^2} \left\{ \frac{\pi}{2c} \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx + \pi \int_0^\infty \frac{e^{-\frac{x^2}{2c^2}}}{\sqrt{c}} \, dx \right\}$$
That is

$$k1_k = L_{k-2}(k-1) \left( x^2 - \frac{b^2}{4c^2} \right) - (2k-1) \frac{b}{2c} L_{k-1}$$

with

$$L_0 = 2\pi$$

and

$$L_1 = -L_0 \frac{b}{2c}.$$ 

Alternatively from Clark's Model

$$u_c = \Lambda \int_0^{2\pi} \int_0^\infty \exp(-rb-c(x^2-2\cos \theta x + \theta^2)) x \, dr \, d\theta$$

$$= \Lambda e^{-cx^2} \int_0^{2\pi} \exp(-rb-r^2)r. \int_0^\infty \exp(2crx\cos \theta) \, dx \, d\theta.$$ 

Consider

$$\int_0^{2\pi} e^{k\cos \theta} \, d\theta = \int_0^{2\pi} \left( 1 + k\cos \theta + \frac{k^2 \cos^2 \theta}{2!} + \ldots \right) \, d\theta$$

$$= \int_0^{2\pi} \left( 1 + \frac{k^2 \cos^2 \theta}{2!} + \frac{k^4 \cos^4 \theta}{4!} + \ldots \right) \, d\theta$$

$$= \sum_{n=0}^{\infty} \frac{k^{2n}}{(2n)!} \int_0^{2\pi} \cos^{2n} \theta \, d\theta$$

$$= \sum_{n=0}^{\infty} \frac{k^{2n}}{(2n)!} \cdot \frac{(2n-1)(2n-3)\ldots3\cdot1}{2n(2n-2)(2n-4)\ldots4\cdot2}$$

$$= 2\pi \sum_{n=0}^{\infty} \frac{k^{2n}}{2^{2n}(n!)^2}.$$
\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{r=0}^{n} \frac{(-1)^m x^m}{m!(2m+1)} = \sum_{r=0}^{\infty} \frac{x^r f(r)}{(2r+1)!}. \]

Also
\[
\int_{0}^{2\pi} L(\theta) e^{iL_{\theta}} d\theta = \int_{0}^{2\pi} \frac{(e^{iL_{\theta}})^n L(\theta)}{n!} d\theta
\]
\[
= \sum_{n=0}^{\infty} \frac{c^n}{n!} \int_{0}^{2\pi} L_{2n+1}(\theta) d\theta.
\]

\[ N_c = A a^{-\frac{x}{2}} \left\{ \frac{\pi}{c} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{c^n}{n!} L_{2n+1} + \frac{(4c)^n}{(2n+1)!} L_{2n+2} \right\} \]

where
\[ L_k = \int_{0}^{2\pi} L_{k}(\theta) d\theta. \]

Now
\[ L_k = \int_{0}^{2\pi} (x \cos \theta - \frac{b}{2c})^k d\theta \]
\[
= \int_{0}^{2\pi} (x \cos \theta - \frac{b}{2c})^{k-1} x \cos \theta d\theta - \frac{b}{2c} L_{k-1}
\]
\[
= \left[ x \cos \theta - \frac{b}{2c} \right]^{k-1} x \sin \theta \bigg|_{0}^{2\pi} + \int_{0}^{2\pi} (k-1)x^2 \sin^2 \theta (x \cos \theta - \frac{b}{2c})^{k-2} d\theta - \frac{b}{2c} L_{k-1}
\]
\[
= (k-1)x^2 \int_{0}^{2\pi} (x \cos \theta - \frac{b}{2c})^{k-2} (1-\cos^2 \theta) d\theta - \frac{b}{2c} L_{k-1}
\]
\[
= (k-1)x^2 L_{k-2} - (k-1) \int_{0}^{2\pi} (x \cos \theta - \frac{b}{2c})^{k-2} ((x \cos \theta - \frac{b}{2c}) + \frac{b}{2c})^2 d\theta - \frac{b}{2c} L_{k-1}
\]
\[
= (k-1)x^2 L_{k-2} - (k-1) \left[ (x \cos \theta - \frac{b}{2c})^{k-2} (x \cos \theta - \frac{b}{2c} + \frac{b}{2c}) + \frac{b}{4c^2} L_{k-2} - \frac{b}{2c} L_{k-1} \right]
\]
\[ N_c = A e^{-cx^2} \left\{ \frac{\pi}{c} + \frac{\pi}{4c} \int_0^{2\pi} \frac{1}{\sqrt{c}} \int_0^{2\pi} \sqrt{L(\theta)} e^{iL^2(\theta)} d\theta + \frac{1}{\sqrt{c}} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} \frac{e L^2(\theta)}{n!} \right) \sum_{m=0}^{\infty} (-1)^m \frac{(e L^2(\theta))^{2m+1}}{m!(2m+1)} d\theta \right\} \]

\[ = A e^{-cx^2} \left\{ \frac{\pi}{c} + \frac{\pi}{4c} \int_0^{2\pi} \sqrt{L(\theta)} e^{iL^2(\theta)} d\theta + \frac{1}{\sqrt{c}} \int_0^{2\pi} \left( \sum_{n=0}^{\infty} \frac{(e L^2(\theta))^{n}}{n!} \right) \sum_{m=0}^{\infty} (-1)^m \frac{(e L^2(\theta))^{m+1}}{m!(2m+1)} d\theta \right\} \]

Now \[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m!(2m+1)} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(2j+1)} \]

\[ = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(2j+1)} \]

\[ \sum_{j=0}^{n} (-1)^j \frac{1}{(2j+1)} = \left( \sum_{j=0}^{n} (-1)^j \right) \frac{2j+1}{2j+1} \]

\[ = \left( \sum_{j=0}^{n} \binom{n}{j} (-1)^j \right) \frac{1}{1} \int_0^1 x^{2j+1} dx \]

\[ = \int_0^1 \frac{1}{(1-x^2)^{n+1}} \frac{1}{(2n+1)!} \]

\[ = \frac{1}{(2n+1)!} \]

\[ \left( \frac{n!}{2} \right)^2 \]
\[ N_c = A e^{-c x^2} \int_0^\infty \exp \left( -b r - c r^2 \right) dr = 2\pi A e^{-c x^2} \sum_{n=0}^\infty \frac{1}{n!} \left( \frac{c x}{n} \right)^{2n} r^{2n+1} \]

where \( F_n = \int_0^\infty \exp \left( -c r^2 \right) r^n dr \)

and \( a^2 = \frac{b^2}{4c^2} \)

Now \( F_{k+2} = \frac{k+1}{2c} F_k - a F_{k+1} \)

with \( F_0 = \frac{1}{2} \sqrt{\frac{\pi}{c}} \text{erfc} (a \sqrt{c}) \)

and \( F_1 = \frac{1}{2c} e^{-a^2 c} - a R_0 \)

where \( \text{erfc}(x) \) is the complementary error function.
Table 4.1

ESTIMATED NUMBER OF CASUALITIES
### Table 4.1

**Estimated Number of Casualties**

<table>
<thead>
<tr>
<th>$X$</th>
<th>$b = 0.25$</th>
<th>$b = 0.45$</th>
<th>$b = 0.75$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_s$</td>
<td>$N_e$</td>
<td>$N_s$</td>
</tr>
<tr>
<td>0</td>
<td>25.910</td>
<td>17.963</td>
<td>16.427</td>
</tr>
<tr>
<td>0.5</td>
<td>25.760</td>
<td>17.843</td>
<td>16.232</td>
</tr>
<tr>
<td>1.0</td>
<td>25.516</td>
<td>17.491</td>
<td>15.662</td>
</tr>
<tr>
<td>2.5</td>
<td>22.413</td>
<td>15.287</td>
<td>12.196</td>
</tr>
<tr>
<td>4.0</td>
<td>17.877</td>
<td>11.951</td>
<td>7.664</td>
</tr>
<tr>
<td>4.5</td>
<td>16.198</td>
<td>10.652</td>
<td>6.259</td>
</tr>
<tr>
<td>5.0</td>
<td>14.508</td>
<td>9.274</td>
<td>4.992</td>
</tr>
<tr>
<td>5.5</td>
<td>12.845</td>
<td>7.846</td>
<td>3.887</td>
</tr>
<tr>
<td>6.0</td>
<td>11.241</td>
<td>6.417</td>
<td>2.955</td>
</tr>
<tr>
<td>6.5</td>
<td>9.724</td>
<td>5.050</td>
<td>2.194</td>
</tr>
<tr>
<td>7.0</td>
<td>8.315</td>
<td>3.810</td>
<td>1.591</td>
</tr>
<tr>
<td>7.5</td>
<td>7.028</td>
<td>2.747</td>
<td>1.126</td>
</tr>
<tr>
<td>8.0</td>
<td>5.871</td>
<td>1.888</td>
<td>0.778</td>
</tr>
<tr>
<td>8.5</td>
<td>4.849</td>
<td>1.238</td>
<td>0.525</td>
</tr>
<tr>
<td>9.0</td>
<td>3.958</td>
<td>0.769</td>
<td>0.346</td>
</tr>
<tr>
<td>9.5</td>
<td>3.194</td>
<td>0.455</td>
<td>0.223</td>
</tr>
<tr>
<td>10.0</td>
<td>2.547</td>
<td>0.255</td>
<td>0.140</td>
</tr>
</tbody>
</table>

**$X$** - distance in miles from city centre

**$N_s$** - number of casualties estimated by Sherratt's model

**$N_e$** - number of casualties by Clark's model
GRAPH 4.1

NUMBER OF EXPECTED CASUALTIES

b = 0.25

GRAPH 4.2

NUMBER OF EXPECTED CASUALTIES

b = 0.45
**GRAPH 4.1**

Number of Expected Casualties

**GRAPH 4.2**

Number of Expected Casualties

- $N_S$
- $N_C$

**Legend:**
- $N_S$ - Sharratt's Model
- $N_C$ - Clark's Model

Distance from city centre

$b = 0.25$

$b = 0.45$
GRAPH 4.3

NUMBER OF EXPECTED CASUALTIES

\[ b = 0.75 \]
GRAPH 4.3.

Estimated Number of Casualties

Nc: Sherratt's Model; Nc: Clark's Model
Since N and N are to be calculated for the same city and the same bomb, the parameters A, b, c and X will be the same in each expression for N and N. A, the central density is basically a scaling factor and in the preliminary calculations was given the value 1. The gradient, b, was given the values 0.25, 0.45 and 0.75; these values being chosen from Clark's results to give a reasonable coverage of modern city types. The weapon parameter, c, was fixed at 0.09, selected as a representative value from Hunter's table.

The values for N and N were then calculated as functions of X (in ½ miles to 10 miles) and graphed for the three values of b. Both forms of the solution for N were programmed to provide a check.

It can be seen from table 4.1 and graphs 4.1, 4.2, 4.3 that there is a wide variation for the number of casualties given by N and N for low values of b and small values of X.

As a practical example, the values of A and b for Brisbane (1947) have been given as 40 and 0.75 respectively, these values being taken from Clark's data (1951). Then, for a bomb with a 1000KT yield and surface burst which landed 4½ miles from the city centre, the number of expected casualties (of all kinds) would have been 84,000 derived
from both $N$ and $N$ S C. Had the bomb landed at the city centre, however, calculation of $N$ gives 340,000 expected casualties $S$ while $N$ estimates 260,000. $C$

Similarly, the 1947 values of $A$ and $B$ for Sydney (Clark, 1951) are given as 30 and 0.25 respectively. The estimated number of casualties (of all kinds) from a centrally directed bomb (as above) are given as 777,000 for $N$ and 540,000 for $N$ S $C$

A difference of approximately a quarter of a million estimated casualties in a total population of two million is considerable. Even if one considers the total population, $P = \frac{2\pi A}{b}$, of the infinite city of approximately 3 million the difference in the number of expected casualties is considerable.

Even with $X = 10$, $N = 75,000$ and $N = 9,000$ S $C$ (for Sydney).

The city is more concentrated according to the Sherratt model than it is according to Clark's model and it appears that the number of expected casualties derived from Hunter's "adequate" approximation depends heavily on the choice of model.

4.3 Calculation of Casualties from a Nuclear Attack on Wollongong.

If a nuclear weapon were to be dropped on
Wollongong, the most likely target would be the heavy industrial area of Port Kembla (about \(2\frac{1}{2}\) miles south of the city centre).

The models developed for Wollongong were used to calculate the number of expected casualties from a single weapon attack. In each case the weapon parameter \(c\) was taken as 0.09 as before.

### 4.3.1 Number of Casualties Using Clark Model 1

This model was chosen to represent those based on circular symmetry as it gives the best estimate of total population. The model is given by

\[
y = 3.49e^{-0.38r}
\]

If a bomb were to be dropped 2.5 miles south of the city centre the total number of casualties is 45.8 thousands.

A bomb landing at the city centre would result in an estimated 55.0 thousand casualties.

### 4.3.2 Number of Casualties Using the Rectilinear Model 2

The rectilinear model 2 gives the best estimate of total population and so was used to calculate the expected number of casualties.
If a bomb were to be dropped on Port Kembla such that $X = 2.5$ miles south of the city centre, the expected number of casualties is given by

$$N = 2.2A \int_0^\infty \frac{b}{r} e^{-c(r+X)} \, dr$$

$$+ 3.3A \int_0^\infty \frac{b}{S} e^{-c(r-X)} \, dr$$

with $A = 10.07$, $b = 0.285$, $A = 4.47$, $b = 0.18$, $X = 2.5$ and, as before, $c = 0.09$.

$$N = 2.2A \frac{b}{N} \frac{\sqrt{\pi}}{2\sqrt{c}} e^{-\left\{1-\text{erf}\left[\frac{b}{2c} (N+X)\right]\right\}}$$

$$+ 3.3A \frac{b}{S} \frac{\sqrt{\pi}}{2\sqrt{c}} e^{-\left\{1-\text{erf}\left[\frac{b}{2c} (S-X)\right]\right\}}$$

$$= (11.52 + 45.32) \text{ thousands}$$

$$= 56.84 \text{ thousands}.$$
4.3.3 Number of Casualties Using the Normal Curve Model

The normal curve model is of the form

\[ y = \sum_{i} A e^{ \frac{-a (x-k)^2}{2} } \]

The total number of casualties with the same conditions as before is therefore given by

\[
N = 2.7 \sum_{i} \int_{-\infty}^{\infty} e^{-\left(\frac{a (x-k)^2}{2} - c(x-X)^2\right)} \, dx \\
= 2.7 \sqrt{\pi} \sum_{i} \frac{A}{i^{a+c}} \exp\left\{ -\frac{a c(X-k)^2}{i^{a+c}} \right\} .
\]

With \( X = -2.5 \) (that is 2.5 miles south of the city centre), the number of casualties is 43.2 thousands.

When \( X = 0 \), the number of casualties is 60.2 thousands.

4.3.4 Number of Casualties Using the Trend Surface Model

The trend surface model is of the form

\[ z = a + bx + cy + dxy + ex + fy \]

where the arbitrary axes are the edges of the map.

The number of casualties for this model is given by

\[
N = \iint_{A} (a + bx + cy + dxy + ex + fy^2) \exp\left\{ -k [(x-X)^2 + (y-Y)^2] \right\} \, dx \, dy .
\]
where \((X,Y)\) is the point of impact and \(A\), the region of the city.

If the region of Wollongong is considered to consist of the sum of a series of approximating rectangles then

\[
N_t = \sum_{i,j} \int_{-i}^{y_j} \int_{-j}^{x_j} (a+bx+cy+dx+ey+f)^2 \exp(-k[(x-X)^2+(y-Y)^2]) \, dx \, dy.
\]

Put \(x-X = u\) and \(y-Y = v\).

\[
\therefore N_t = \sum_{i,j} \int_{-i}^{y_j} \int_{-j}^{x_j} (a+b(u+X)+c(v+Y)+d(u+X)(v+Y)+e(u+X)^2+f(v+Y)^2)
\]

\[
e^{-ku^2} e^{-kv^2} \, du \, dv
\]

\[
= \sum_{i,j} \left\{ (a+bx+cy+dx+ey+f)^2 \right\} \int_{-i}^{y_j} \int_{-j}^{x_j} e^{-ku^2} e^{-kv^2} \, du \, dv \quad (1)
\]

\[
+ (b+dX+2aX) \int_{-i}^{y_j} \int_{-j}^{x_j} u e^{-ku^2} e^{-kv^2} \, du \, dv \quad (2)
\]

\[
+ (c+dX+2fY) \int_{-i}^{y_j} \int_{-j}^{x_j} v e^{-ku^2} e^{-kv^2} \, du \, dv \quad (3)
\]

\[
+ d \int_{-i}^{y_j} \int_{-j}^{x_j} u e^{-ku^2} v e^{-kv^2} \, du \, dv \quad (4)
\]

\[
+ e \int_{-i}^{y_j} \int_{-j}^{x_j} u^2 e^{-ku^2} e^{-kv^2} \, du \, dv \quad (5)
\]

\[
+ f \int_{-i}^{y_j} \int_{-j}^{x_j} v^2 e^{-ku^2} e^{-kv^2} \, du \, dv \quad (6)
\]
\[
\begin{align*}
(1) & \quad \int_{y_1 - Y}^{y_j - Y} \int_{x_1 - X}^{x_j - X} e^{-ku^2} e^{-kv^2} \, du \, dv \\
& = \frac{\gamma \pi}{4k} \left[ \text{erf}(\sqrt{k}(x_j - X)) - \text{erf}(\sqrt{k}(x_1 - X)) \right] \left[ \text{erf}(\sqrt{k}(y_j - Y)) - \text{erf}(\sqrt{k}(y_1 - Y)) \right] \\
& = \frac{\gamma \pi}{4k} \, UV \quad \text{where} \quad U = \text{erf}(\sqrt{k}(x_j - X)) - \text{erf}(\sqrt{k}(x_1 - X)) \\
& \quad \text{and} \quad V = \text{erf}(\sqrt{k}(y_j - Y)) - \text{erf}(\sqrt{k}(y_1 - Y)) \\
(2) & \quad \int_{y_1 - Y}^{y_j - Y} \int_{x_1 - X}^{x_j - X} e^{-ku^2} e^{-kv^2} \, du \, dv \\
& = \frac{\sqrt{\pi}}{2\sqrt{k}} \, V \cdot \frac{1}{2k} \int e^{-u} \, du \\
& = \frac{\sqrt{\pi}}{4k\sqrt{k}} V \left( e^{-k(x_1 - X)^2} - e^{-k(x_j - X)^2} \right) \\
& = \frac{\sqrt{\pi}}{4k\sqrt{k}} \, VS \quad \text{where} \quad S = e^{-k(x_1 - X)^2} - e^{-k(x_j - X)^2} \\
(3) & \quad \int_{y_1 - Y}^{y_j - Y} \int_{x_1 - X}^{x_j - X} e^{-ku^2} e^{-kv^2} \, du \, dv \\
& = \frac{\sqrt{\pi}}{4k\sqrt{k}} \, UT \quad \text{where} \quad T = e^{-k(y_1 - Y)^2} - e^{-k(y_j - Y)^2} \\
(4) & \quad \int_{y_1 - Y}^{y_j - Y} \int_{x_1 - X}^{x_j - X} e^{-ku^2} e^{-kv^2} \, du \, dv \\
& = \frac{1}{4k^2} \, ST
\end{align*}
\]
\[
\begin{align*}
&\int_{y_1}^{y_f} \int_{x_i}^{x_j} u^2 e^{-ku} e^{-kv^2} du \, dv \\
&= \frac{\sqrt{\pi}}{2\sqrt{k}} \, v \cdot \frac{1}{k} \int_{\sqrt{k}(x_1-x)}^{\sqrt{k}(x_j-x)} u^2 e^{-u^2} du \\
&= \frac{\sqrt{\pi}}{2k\sqrt{k}} \, v \cdot \frac{1}{2} \left[ -u e^{-u^2} \right]_{\sqrt{k}(x_1-x)}^{\sqrt{k}(x_j-x)} + \int_{\sqrt{k}(x_1-x)}^{\sqrt{k}(x_j-x)} e^{-u^2} du \\
&= \frac{\sqrt{\pi}}{4k\sqrt{k}} \, v \left( \sqrt{k}(x_1-x) e^{-k(x_1-x)^2} - \sqrt{k}(x_j-x) e^{-k(x_j-x)^2} + V \right) \\
&= \frac{\sqrt{\pi}}{4k\sqrt{k}} \, V \left( \sqrt{k} \cdot P + U \right) \text{ where } P = (x_1-x)e^{-k(x_1-x)^2} - (x_j-x)e^{-k(x_j-x)^2} \\
&\int_{y_1}^{y_f} \int_{x_i}^{x_j} u^2 e^{-ku} v^2 e^{-kv^2} du \, dv \\
&= \frac{\sqrt{\pi}}{4k\sqrt{k}} \, U \left( \sqrt{k} \cdot Q + V \right) \text{ where } Q = (y_1-y)e^{-k(y_1-y)^2} - (y_j-y)e^{-k(y_j-y)^2} \\

\text{N}_c = \sum_{i,j} \left\{ \frac{\sqrt{\pi}}{4k} \left( a + bX + cY + dXY + eX^2 + fY^2 \right)UU \right. \\
&\left. + \frac{\sqrt{\pi}}{4k\sqrt{k}} \left( b + dY + 2eX \right) VS \right. \\
&\left. + \frac{\sqrt{\pi}}{4k\sqrt{k}} \left( c + dX + 2fY \right) UT \right\}
\end{align*}
\]
\[ + \frac{d}{4kT} \mathrm{ST} \]
\[ + \frac{\sqrt{\pi}}{4k} \sigma V(\sqrt{k} P + U) \]
\[ + \frac{\sqrt{\pi}}{4k} \sigma U(\sqrt{k} Q + V) \}

Consider, as before, an attack consisting of one nuclear weapon landing at \( X = 4 \) miles \( Y = 12 \) miles, that is, on the main industrial area at Port Kembla. The weapon parameter was given the value 0.09 as before. Wollongong was considered to consist of a set of 9 rectangles with the following co-ordinates.

<table>
<thead>
<tr>
<th>Rectangle</th>
<th>( x_1 )</th>
<th>( x_j )</th>
<th>( y_1 )</th>
<th>( y_j )</th>
<th>Casualties</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>20</td>
<td>25</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2.5</td>
<td>18</td>
<td>20</td>
<td>0.03</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>3</td>
<td>15</td>
<td>18</td>
<td>2.28</td>
</tr>
<tr>
<td>4</td>
<td>1.5</td>
<td>4</td>
<td>11</td>
<td>15</td>
<td>35.24</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>6</td>
<td>9</td>
<td>11</td>
<td>21.39</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>9</td>
<td>4.68</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>5.5</td>
<td>6</td>
<td>9</td>
<td>4.12</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>5.5</td>
<td>0.17</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>3.5</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Using tables of the error function, $\text{erf } x$, and the exponential function $e^x$, the number of casualties in each rectangle was calculated in thousands as above. The total number is 67.91 thousands.

4.3.5 Comparison of Results

The various models yield quite different results and, because of the uncertainty of the estimations, one cannot be said to be more accurate than the other. Using the results calculated for Sydney in 1947, it is seen that Sherratt's model estimates 39% of the total population as casualties while Clark's model suggests 71% when ground zero is the city centre. For Wollongong, the results are:
Clark Model 1 - 34.2%
Rectilinear Model 2 - 46.8%
Normal Curve Model - 38.3%

It would appear that the rectilinear model overestimates the number of casualties while the other two seem to be reasonable, that is, between the expected limits.

The accuracy of the model for determining the number of nuclear casualties should, however be related to the accuracy with which the same model estimates the total population. When the number of casualties estimated by the model is compared with the total population estimated by the model the percentages become

Clark Model 1 - 36.2%
Rectilinear Model 2 - 50.2%
Normal Curve Model - 40.1%

The prime target for Wollongong is, of course, Port Kembla. The percentages of casualties estimated by the models, compared with the total population are

Clark Model 1 - 30%
Rectilinear Model 2 - 39%
Normal Curve Model - 29%
Trend Surface Model - 35%

Once again the results for the rectilinear
model are higher than those for the other models. Numerically the trend surface model yields the largest number (67.9 thousands) but this model also overestimates the total population. It is not possible to state that one or other model gives the best answer. The variation does, however, indicate that the choice of model is important in the solution of problems dependent on a spatial pattern of intra-urban population distribution.
5. **SHAPE DISTORTION MEASURE**

It has been shown in previous chapters that a model based on the ideal circular city is a good fit if, and only if, the city itself approaches circular symmetry. The problem thus arises of measuring the distortion of the real city from circularity to determine whether the deviation is significantly high.

5.1 **THE DISTORTION INDEX OF BOYCE AND CLARK**

One such measure, discussed by Boyce and Clark (1964) is based on the geometric shape of the outer boundary of the city and is defined by the equation

\[
I = \sum_{i=1}^{n} \left( \frac{R_i}{n} \cdot 100 - \frac{100}{n} \right)
\]

where \(R_i\) (\(i = 1, 2, \ldots, n\)) represents the lengths of radial lines measured from the city centre to the boundary along \(n\) equally spaced radii.

If the city boundary is a circle then

\[
R_i = R_{i+1}\quad \text{and} \quad I = 0.
\]

Although Boyce and Clark do not calculate it there is also a maximum value of \(I\), found by consideration of a city with maximum distortion, that is, one consisting of a single street. If this street has length \(2x\), then of \(n\) radial lines from the centre to the boundary, \(R_i\) and
R, say, are of length x and \( R_{i} \) (\( i=3,4,\ldots,n \)) = 0 with
\[
\sum_{i=1}^{n} R_{i} = 2x.
\]
Then \( I = 100\left(2\ln \frac{n}{2} + \frac{n-2}{n}\right) \).
If \( n > 2 \), then \( 2 > \frac{1}{n} \)
and \( I = 100\left(2 - \frac{n}{n}\right) \)
\[ \rightarrow 200 \text{ as } n \rightarrow \infty. \]
This index of shape distortion then, has a minimum value of 0, representing no distortion from circularity and a maximum value of 200.

5.1.1 An Analysis of the Boyce-Clark Index
Further examination of this index reveals its underlying statistical basis.

\[
\sum_{i=1}^{n} \frac{R_{i}}{R} = n \bar{R} \text{ where } \bar{R} \text{ is the mean of the } R_{i}.
\]
So \( I = \frac{100}{n} \left\{ \left| \frac{R_{1}}{\bar{R}} - 1 \right| + \left| \frac{R_{2}}{\bar{R}} - 1 \right| + \ldots + \left| \frac{R_{n}}{\bar{R}} - 1 \right| \right\} \)
\[ = \frac{100}{n} \sum_{i=1}^{n} \left| \frac{R_{i}}{\bar{R}} - 1 \right| \]
\[ = 100 \times \text{mean variation of } R \]
mean of \( R \)
As \( n \rightarrow \infty \)
\[
I = \frac{100}{2\pi} \cdot \frac{1}{R} \int_{0}^{2\pi} \left| R(\theta) - \bar{R} \right| d\theta
\]
TABLE 5.1

CALCULATION OF INDEX OF SHAPE DISTORTION FOR

WOLLONGONG
Calculation of Index of Shape Distortion

for Wollongong.

|   | $R_1$ | $|\bar{R} - R_1|$ |
|---|-------|-----------------|
| 1 | 11.2  | 7.3             |
| 2 | 3.4   | 0.5             |
| 3 | 1.5   | 2.4             |
| 4 | 2.9   | 1.0             |
| 5 | 2.4   | 1.5             |
| 6 | 7.5   | 3.6             |
| 7 | 10.8  | 6.9             |
| 8 | 4.8   | 0.9             |
| 9 | 0.6   | 3.3             |
| 10| 0.6   | 3.3             |
| 11| 0.6   | 3.3             |
| 12| 0.8   | 3.1             |

$$\bar{R} = 3.9$$

$$n = 12$$

$$\bar{R} = 3.9$$

$$I = 79$$
where $R(\theta)$ is the distance from the city centre to the boundary in the direction $\theta$.

This measure requires a definition of both centre and edge of the city, and a small variation in either could affect the results considerably. Medvedkov (1967) has calculated the Boyce-Clark index for several cities (for example Bucharest 16.1; Washington 16.2; Warsaw 24.9) and claims these figures to be useful for city comparisons since they give a "precise and unambiguous statement of the shape of a city". He has not, however, defined his method of selecting centre or edge.

### 5.1.2 Distortion Index for Wollongong

The Boyce-Clark distortion index was calculated for Wollongong using 12 equally spaced radii and a map with scale 1:63360 (see table 5.1). The city centre was taken as that defined in Chapter 3 and the boundary was defined by the edges of the census districts.

For Wollongong $I = 78.8$. A rectangle, 23 units long and 3 units wide has a distortion index of 68.6 confirming the distortion of Wollongong from the circular to the rectangular.

### 5.2 The Distortion Index of Simmons

Simmons (1962) devised a measure based on the area of a city as well as its boundary. His measure
is defined as the ratio of the sum of distances of points, arranged in a regular network within the boundary of the city, from the city centre to the sum of distances of points in the same regular network from the centre of a circle of the same area as the city. In symbols and using $J$ for Simmons' measure

$$J = \frac{\sum R}{\sum r}$$

where the $R$ are the distances for the true city and the $r$ the distances for the circle.

For any given centre and boundary $J$ is a function of $n$ and of the type of network. A limiting form is necessary to remove this dependence.

Now as we have a regular network we may imagine small areas attached to the points of the network, one area for each point and all areas of the same measure, say $\delta x \delta y$. For the circle, the same area $A$ as the real city

$$\lim_{n \rightarrow \infty} \frac{\sum r}{\sum R} = \int \int \sqrt{x^2 + y^2} \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^r r \, r \, dr \, d\theta$$

$$= 2\pi \int_0^r r^2 \, dr$$

$$= \frac{2}{3} \pi r^3$$

$$= 2 \frac{1}{\sqrt{\pi}} A^{3/2}.$$
Therefore \( \lim_{n \to \infty} J = \lim_{n \to \infty} \frac{2}{\pi} \frac{\pi \int_{-a}^{a} \int_{-b}^{b} \sqrt{x^2 + y^2} \, dx \, dy}{\int_{-a}^{a} \int_{-b}^{b} x^2 + y^2 \, dx \, dy} = \frac{2}{3} \frac{\pi a^2}{\pi a^2} \)

where the top integral is taken over the city's area. The origin of the rectangular coordinates \((x, y)\) must, of course, be taken at the given centre.

In statistical terms, the limiting value of \(J\) is the ratio of the average value of the distance from the centre to a point taken at random in the city area (all points being equally likely) to the same averages for the ideal circular city with the same centre and area.

5.2.1 The Value of \(J\) for a Rectangular City.

Since \(J\) is the ratio of the first moment of area for the city to the first moment of area for a circle of the same area, it is possible to find this ratio for a rectangular city assuming that the centre of the city is the centre of the rectangle.

Applying elementary integration to a rectangle whose sides are \(2a\) and \(2b\), we find

\[
J = \frac{\pi}{16} \left( \sqrt{a^2 + b^2} + a^2 \ln \left( \frac{a^2 + b^2}{a^2} \right) + b^2 \ln \left( \frac{a^2 + b^2}{b^2} \right) + \frac{b}{a^2} \ln \left( \frac{a^2 + b^2}{a^2 + b^2 - a^2} \right) \right)
\]

For a square this gives \(J = 1.02\), while for a highly distorted rectangle for which \(b\) is small \(\frac{b}{a}\)

\[
J \to \frac{3\pi \sqrt{a}}{8\sqrt{b}}
\]
Applying this result to Wollongong, taking $a = 23$ and $b = 2.7$, we find $J = 1.9$.

This figure would be somewhat increased if the city centre were not at the centre of the rectangle as is borne out by the value obtained for Wollongong in 5.2.2.

This measure is highly sensitive to the position of the city centre. It can be shown that for a rectangle with the city centre at one corner, $J$ is twice the value obtained when the city centre is taken at the centre of the rectangle.

5.2.2 Simmons' Index of Shape Distortion for Wollongong.

A hexagonal network with the distance between points equivalent to one mile, was placed over the map of the city of Wollongong, with the centre and boundary as before. The same network was placed over a circle with the same same area (radius 4.5'). The index for Wollongong was found to be

$$J = \frac{\sum_{i=1}^{n} \frac{R_i}{r_i}}{218.7} = 459.5 = 2.10$$
Berry (1963) suggests that Simmons' index is sensitive to elongation of a city and also to physically created distortions such as large bodies of water. If this is so the value $J = 2.10$ seems small.

### 5.3 Shape Distortion of Population Distribution

Both $I$ and $J$ deal with the geometric shape of the city in relation to its centre and depend on the definitions of the centre and the outer boundary. It would be possible to have a city with a circular boundary but an eccentrically placed centre. Then, too, a city's residential area may be fairly circular about its centre although the distribution of population within this area is far from circularly symmetric.

The measure of distortion of the real city from circularity needs to be in terms of population distribution rather than geometric shape. Three such measures will be considered.

#### 5.3.1 The Variance Index

By considering the distribution of population between rays $\theta = 0$, and $\theta = \theta + \delta\theta$ as $\delta\theta \to 0$ the average distance of persons along the ray $\theta = \theta$ is found to be

$$\text{F}(\theta) = \frac{\int_0^\infty r D(r,\theta) \, dr}{\int_0^\infty r D(r,\theta) \, dr} \quad \text{where } D(r,\theta) \text{ is the density function.}$$
Therefore

\[ F(\theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{r D(r, \theta) dr}{r D(r, \theta) dr} \]

represents the distribution of the average distance of persons from the centre for all \( \theta \) from \( \theta = 0 \) to \( \theta = 2\pi \).

For the whole city, the average distance from the centre is given by

\[ A = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \int_{0}^{\infty} \frac{r D(r, \theta) dr}{r D(r, \theta) dr} \right) d\theta. \]

Then an index of shape distortion could be given by the variance \( V \) where

\[ V = \frac{1}{2\pi} \int_{0}^{2\pi} \left( F(\theta) - A \right)^2 d\theta. \]

The minimum value \( V=0 \), occurs when \( F(\theta) = A \) for all values of \( \theta \), representing perfect circular distribution.

As distortion increases

\[ V \rightarrow \frac{1}{2\pi} \int_{0}^{2\pi} F(\theta) d\theta. \]

5.3.2 The Variance Index for Wollongong

In practice, the average distance of persons
from the centre in a given direction, $\theta$, may be calculated as

$$A = \sum_{i} \frac{2}{\sum r D l}$$

where $l$ is the length of the line $\theta = \text{constant}$ within the $i$th boundary of the $i$th census district, $r$ is the distance of the centre of the $i$th census district from the city centre and $D$ is the population density of the $i$th census district.

If $n$ equally spaced radii are chosen and

$$A = \frac{1}{n} \sum A$$

then

$$V = \frac{1}{n} \sum (A - \bar{A})^2$$

With $n = 12$ and using the combined census districts as defined in Chapter 3, the value of $V$ for Wollongong is 4.5.

The average distance of persons from the city centre varies from 0.3 miles to the east to 6.3 and 6.2 to the north and south respectively. The value of $A$ is 2.3 miles.

This measure seems to lack a reference point. What does the value of $V = 4.5$ mean?
In practice the measure of maximum distortion

\[ V = \frac{1}{2\pi} \int_{0}^{2\pi} F(\theta) \, d\theta \]

is given by \( \sum_{n=1}^{N} \alpha_n \). This provides a useful comparison to determine the size of \( V \). For Wollongong this is calculated to be 9.8. It can be seen that Wollongong shows a considerable degree of distortion.

5.3.3 Density Difference Index of Distortion

A better measure of distortion may be obtained by analysing the fact that a city's population density distribution is circularly symmetric if and only if

\[ D(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} D(r, \theta) \, d\theta \]

where \( r, \theta \) are the polar coordinates with the city centre as pole and \( D(r, \theta) \) is the population at the point \((r, \theta)\).

Adopting a least-squares criterion the deviation from symmetry, may be defined as

\[ S = \int_{0}^{2\pi} \int_{0}^{2\pi} \left( D(r, \theta) - \frac{1}{2\pi} \int_{0}^{2\pi} D(r, \theta) \, d\theta \right)^2 \, r \, dr \, d\theta \]

\[ = \int_{0}^{2\pi} \int_{0}^{2\pi} D(r, \theta)^2 \, r \, dr \, d\theta - \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} D(r, \theta) \, d\theta \int_{0}^{2\pi} \int_{0}^{2\pi} D(r, \theta) \, d\theta \, r \, dr. \]

If an equation of the form

\[ \frac{1}{2\pi} \int_{0}^{2\pi} D(r, \theta) \, d\theta = Ae^{-br} \]
(i.e. Clark's equation) can be fitted to the areal distribution of population then

\[
\frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^{\infty} D(r, \theta) r^2 dr \right) d\theta = \frac{1}{2\pi} \int_0^{\infty} \left( \frac{-br}{2} \right) r^2 dr = \frac{2 \pi b}{8\pi}
\]

where \( P \) is the total population of the city and using the common simplifying assumption that a city extends to infinity in all directions. If this assumption is not to be made then the limits of integration become 0 and \( R \) rather than \( \infty \) where \( R \) is the distance of the outer boundary. Then

\[
\frac{1}{2\pi} \int_0^R \left( \frac{-br}{2} \right) r^2 dr = \frac{\pi A}{2} \left\{ 1 - \left( 1 + 2bR \right) e^{-2bR} \right\}.
\]

In practice

\[
\frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} D(r, \theta) r^2 dr d\theta \div \sum_{\text{census districts}} \left( \frac{\text{no. of people in } i^{\text{th}} \text{ census district}}{\text{area of } i^{\text{th}} \text{ census district}} \right)^2 = \sum_{i=1}^{n} \frac{(P_i^2)}{a_i}
\]

where \( P_i \) and \( a_i \) are the population and area respectively of the \( i^{\text{th}} \) census district.

Hence

\[
S = \sum_{i=1}^{n} \frac{P_i^2}{a_i} - \frac{P b}{8\pi}.
\]
This measure may be derived using any other equation for the density function, \( D(r, \theta) \) which fits a particular city. For example, if Sherratt's equation
\[
D(r, \theta) = A e^{-r^2 / 2}\theta^2
\]
is used the distortion index \( S \) is determined by
\[
S = \sum_{i=1}^{P} \frac{1}{a_i} - \frac{P}{4\pi\sigma^2}.
\]

This measure, as did \( V \), presents a problem of interpretation of the value of \( S \) obtained. While perfect symmetry should give a value of \( S = 0 \), and slight distortion gives a small value of \( S \), at what numerical value is distortion considered to be significant?

5.3.4 The Density Difference Index for Wollongong

The density difference index of shape distortion for Wollongong was calculated using the Clark model based on annular rings. This equation (see 3.3.1)
\[
y = 3.49e^{-0.38x}
\]
resulted in a high negative correlation \((-0.94)\) between density and distance from the city centre, and might be assumed, therefore to be a good fit. This model, however, was derived using the assumption of circular symmetry, and a significant distortion from such symmetry would nullify the model.
In order to calculate the value of
\[ S = \sum_{i} \frac{P - \dot{P}b}{8\pi} \]
the combined census districts were used (as in Chapter 3), and for Wollongong:
\[ \sum_{i} \frac{P^2_i}{a} = 798.8 \cdot 1 \]
For the second term, the modified form was used, namely
\[ \frac{\pi A}{2 b^2} \left\{ I - (1 + 2bR)e^{-2bR} \right\} \]
where \( A = 3.49, b = 0.38 \) and \( R = 11.5 \).
Thus
\[ S = 798.8 - 146.0 \]
\[ = 652.8. \]
This large numerical value for \( S \) supports the earlier assumption that this model, despite the high correlation of density and distance from the centre, does not fit the real city.

5.3.5 **The \( \chi^2 \) Measure of Distortion**

Another more statistical approach may be based on the chi-squared statistic (Haggett, 1965; Kendall and Stuart, 1961) which may be used to compare the observed
distribution of population with that expected in a circularly symmetric city.

As an example of how this might be applied imagine m concentric circles drawn with common centre at the city centre, and let $R_i$ be the radius of the $i$th circle.

$\theta = R < R < \ldots < R$.

Now let $n$ rays, with equations $r > 0$, $\theta = \theta_j$

$(j=1, 2, \ldots, n)$ be drawn such that

$0 = \theta < \theta < \ldots < \theta < 2\pi$.

Then the area $A$ is that determined by

$R_i < x < R_{i-1}$ \hspace{1cm} \text{for} \hspace{1cm} (i=1, 2, \ldots, m)$

and $0 < \theta < \theta < \theta$ \hspace{1cm} \text{for} \hspace{1cm} (j=0, 1, 2, \ldots, n)$

Let $P_{ij}$ be the population of this area.

If the city is circularly symmetric then the expected value of $P_{ij}$ is given by

$E(P_{ij}) = \frac{(\theta_i - \theta_j)}{2\pi} \sum_{j=0}^{n} P_{ij}$

$= T_{ij}$

The $T_{ij}$ are the theoretically expected population and the $P_{ij}$ are the observed populations for the area $A_{ij}$. 
The chi-squared statistic is, therefore;

\[ \chi^2 = \sum \frac{(P_{ij} - T_{ij})^2}{T_{ij}}. \]

This will be approximately distributed in a chi-squared distribution with \(m(n-1)\) degrees of freedom and the usual chi-squared test may be applied to determine the significance of the distortion.

An error may be introduced into the calculation of \(\chi^2\) through allotting to different \(A_{ij}\) the people in a census district that does not lie wholly within one such area. However the \(\chi^2\) as calculated should be of the same order of magnitude as the value that would be calculated were it possible to determine exactly the number of people in each \(A_{ij}\).

This measure can be modified for cities which have centres on coastlines by simply ignoring the irrelevant areas. Similarly any segment of a city which is unavailable for residential purposes might be omitted and the "circular" symmetry of the remainder considered.

Then, too, this approach can also be used to investigate whether a city is symmetric out to some distance from the centre, and distorted from there outwards.
Table 5.2

Values of $P_i$ and $T_{ij}$ for Wollongong.
### Table 5.2
VALUES OF $\rho_i$ AND T FOR WOLLONGONG

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\[ \chi^2 = 280.2 \]

df = 91

$\rho < 0.001$
because of new housing developments or other factors.

5.3.6 The $\chi^2$ Measure for Wollongong

To simplify the calculation of the $\chi^2$ measure of distortion for Wollongong, the centres of each of the combined census districts were considered to represent the position of the total population of each district. That is, no apportioning of population was made. This, of course, introduces an error but, as indicated above, the order of magnitude should be unchanged.

Eight equally spaced lines were drawn from the city centre, the first in an easterly direction. The concentric circles used to calculate the original Clark model were also used. The value of the $T_{ij}$ becomes

$$T_{ij} = \frac{1}{8} \sum_{j}^{} P_j \quad (j = 0,1,\ldots,12)$$

where $P_j$ is the total population in the $j$-th ring.

The results may be seen in Table 5.1 opposite. The value of $\chi^2$ obtained is 270.2 and with 91 degrees of freedom, indicates a probability much less than 0.001. This confirms the previous statements that Wollongong is significantly distorted from circular symmetry.
Comparison of Shape Distortion Measures

Several measures of shape distortion have been discussed. Each in some way gives information about the shape of a city.

The first measure or index I, the Boyce-Clark index, being based on distances from the centre to the outer boundary, gives information about the geometric shape of that outer boundary, although dependent on a definition of both centre and outer boundary. It is simple to calculate, and while showing some variation according to the number of radii used, has a minimum of 0, and a maximum of 200.

The Simmons' index, J, varies according to the type and scale of network used. It has a definite minimum of zero but a maximum depending on the scale of the network.

The new measures of distortion suggested are based more on the distribution of people within the city rather than on the boundary. As a city may have a circular boundary centred on the city centre and yet also have large distortions from symmetry of population density inside its urbanised area, it is suggested that the measures
put forward are of value along with the earlier measures. In particular the density difference index of distortion provides a means of testing the goodness of fit of a model based on the assumption of circular symmetry. The \( \chi^2 \) measure of distortion being based on a well-known distribution function provides a means of measuring the significance of distortion.

The application of these measures to the city of Wollongong reveals that it is rectangular rather than circular, that the Clark model derived from annular rings is not applicable despite the high negative correlation and that the distortion from circular symmetry is highly significant (at the .001 level of significance).
CONCLUSION

Clark's model of the spatial pattern of intra-urban population density distribution, namely,
\[ y = Ae^{-bx}, \]
has been widely accepted in the literature as a fundamental law. Indeed, many research workers (Berry et al, 1963, Muth, 1961, Winsborough, 1962, and others) have sought explanations for its apparent universal application.

It must be remembered, however, that the development of such a model involves two distinct steps. The real city with its many irregularities due to localised variations is first replaced by a hypothetical city which is regular or perhaps, symmetrical in some way. This hypothetical city must, of course, retain certain essential features of the real city.

Then, for this regular hypothetical city, a density function is derived by finding a suitable curve or surface to fit the data. This density function is the required model.

Clark (1951) put forward the negative exponential function as the curve of best fit assuming circular symmetry for the hypothetical city which retains the
same average density of population at the centre and at each mile radius as the real city. Newling (1969), however, uses Clark's data and claims that a better fit is obtained using the density function,
\[ y = Ae^{\frac{2}{bx-cx}} \]
giving a better estimate of the spatial pattern at the city centre.

Sherratt (1960), also using a circularly symmetric city and average densities, favours a normal distribution of population density and suggested the density function,
\[ y = Ae^{-\frac{r^2}{2\sigma^2}} \]
as a suitable universal pattern of density distribution.

It can be seen that even with agreement on the replacement of the real city by a circularly symmetric one, the selection of the curve of best fit varies.

Other types of models have been proposed. Gurevich and Saushkin (1966) have outlined an approach based on isolines of equal density. With the basic assumption that density drops with distance from the city centre, they have postulated four major city types and
have given examples of each. In practice, one presumes, the isolines of equal density for a city would be drawn. This would give an irregular pattern which would have to be approximated by a regular hypothetical one and the density function calculated as before. Gurevich and Saushkin do not offer any method of achieving this, except, perhaps, a comparison with a set of hypothetical patterns theoretically calculated. Thus their work, although self-consistent, seems to have little practical value. It does, however, point up the question of assuming circular symmetry for all cities.

The extension of this work by Gurevich is marred by inconsistency and mathematical errors.

Dacey's multicentred model is also of little practical value unless one is considering a city with known, and mutually exclusive subpopulations. His mathematical expressions are unfortunately confused.

The trend surface model approximates the real city to a polynomial surface and the computer programme available enables the selection of the simplest form which significantly describes most of the variation. For this model there is no primary replacement by a hypothetical city.
to gain data for curve fitting. There is, however, an assumption that the polynomial surface is the basic spatial pattern. This model is the only one which gives some measure of goodness of fit.

For the Clark model there is a suggestion (Berry et al, 1963) that the correlation between density and distance from the city centre measures the fit of the model. This however, is in terms of the fit of the model to the hypothetical city; not the real one if the real city is markedly distorted from circular symmetry. As indicated in Berry and Horton (1970) there is apparently some difficulty in accepting circular symmetry, which is overcome by dividing the city into segments with different parameters for the negative exponential function for each segment.

Newling (1969) offers no measure of goodness of fit for his model. He even rejects certain data towards the outer edge of cities claiming these densities to be not truly urban.

The measure of goodness of fit must, of course, be related to the use to which the model is to be put. Wilkins and the present author (1969) have shown Clark's model to be an expression of the average pattern
of density distribution. Provided one is discussing the city, as a whole or from the centre to a given radius it will fit well. The difficulty arises when only a part of the city is being considered particularly if the city is markedly distorted from circular symmetry. Even so, the models of both Newling and Sherratt could also be considered to be an expression of the average pattern and one is left with the problem of selection.

In attempting to find a suitable model for Wollongong several approaches were used. The methodology of Clark was used to calculate three models. The first, based on the average density over annular rings, namely

\[ y = 0.38r^{3.49} \]

gave the high correlation of -0.94. This model is quite obviously a poor one for the Wollongong area because of the large uninhabited areas in those rings after the first which result in false densities. This emphasises both the average nature of Clark's model and also the need for a measure of distortion from circular symmetry. The goodness of fit as tested by the density per radian was shown to be poor.

Instead of replacing Wollongong by a
circularly symmetric city, a new approach was tried. The city was assumed to be basically rectangular in shape, and four rectilinear models were developed. Of these, the best fit was found to be the rectilinear model 2,

\[
y = \begin{cases} 
-0.28r & \text{for } r \text{ north} \\
10.07e^{0.18r} & \text{for } r \text{ south}, \\
4.47e^{0.18r} & \text{for } r \text{ south},
\end{cases}
\]

where two density functions were found, one for the northern section of the city and one for the southern section. The city has, therefore, been replaced by two rectangles, the northern of width 2.2 miles and the southern of width 3.3 miles. The goodness of fit was calculated on the basis of the best estimation of total population.

For each of the rectilinear models 1, 2 and 3 a better fit was obtained for the northern section of the city. This is the older part of the city. The southern section is distorted by the lake, the extensive industrial and recreational areas and the large areas of Housing Commission development around the lake.

An individual density function for Wollongong was calculated using the sum of normal curves for a rectilinear city. This function

\[
y = \sum_{i=1}^{9} A_i e^{-(x-c_i)^2}
\]
gave a good fit but is, of course, a particular function for a particular city at a particular time and not generally applicable.

The trend surface model derived was of the second order but the goodness of fit was relatively low, only 21.2% of the variation being explained by the model.

Both the trend surface model and the sum of normal curves model proved to be difficult to apply. The trend surface model also presented boundary problems.

It would seem that the rectilinear model has merit in that it presents a better picture of the true distribution of population than the circular one.

For applied problems the choice of model for a city is important. To illustrate this, the effect of substituting Clark's model for that of Sherratt when estimating the number of casualties in a nuclear weapon attack was explored theoretically and the variation of results for Wollongong using different models was demonstrated. For such problems a density function tailored for the individual city rather than a generalised function is to be preferred.
If a model based on circular symmetry is to be applied to a city with confidence some measure of distortion is needed. This measure should be in terms of population distribution rather than geometric shape.

Of the three measures proposed, the variance index and the density difference index lack reference points. The density difference index for Wollongong showed clearly that a density function based on circular symmetry is unsuitable. It would, however, be a less effective measure in more marginal cases.

The $\chi^2$ method of measuring distortion as outlined in this thesis does not suffer in this way. The $\chi^2$ function is well known and has wide application. In this situation, its ability to measure the significance of distortion adds to its value.
Resolution 1 (a) That the concept of an inner and an outer boundary around each of the State capitals and other cities with an urban population of at least 75,000 and a regional population of at least 100,000 be adopted, and

(b) That the inner boundary be drawn to delimit the extent of urban development at each Census and it would, therefore, be a moving boundary to be adjusted after each Census, except that any State may extend the inner boundary during inter-censal years to encompass significant and well-defined peripheral population growth, and

(c) That the outer boundary be designed to contain the anticipated urban development of the city for a period of at
least twenty to thirty years.

Resolution 2 (a) That an urban boundary be defined as soon as possible for all other settlements with a population of 1,000 or more, and

(b) That State, Statistical Division, Local Government Area, and other boundaries be ignored in delimiting these urban areas.

Resolution 3 (a) That urban boundaries be defined so as to include all contiguous census collector's districts which have a population density of 500 or more per square mile, but that in applying this basic criterion, the following additional criteria and rules shall be taken into account:

(i) Land used for factories, airports, small sports areas, cemeteries, hostels, institutions, prisons, military camps and certain research stations shall be treated as being
used for urban purposes if such land is contiguous with collector's districts which conform with density and other criteria;

(ii) Any area which does not conform with the population density criterion, and in which land is used for large sporting areas, explosives handling and munitions areas, large parks, holding paddocks and reservoirs, must be excluded from the urban area, unless it is bordered on three sides by collector's districts forming part of the urban area;

(iii) Any area which does not conform with the population density and land-use criteria, but which is completely surrounded by land included in an urban area, must be included in that urban area;
(iv) If a collector's district, which would have been excluded from an urban area under other criteria, forms an indentation into an urban area which is less than one mile wide at the open end, it must be included in the urban area if a suitable boundary can be defined across the open end;

(v) Where there is a gap in urban development which is less than two miles (by the shortest rail or road distance) between the edge of one area of urban development and another, the gap is to be ignored and the urban areas treated as contiguous; if there is a gap of two or more miles between two urban areas, those urban areas are to be treated as separate urban areas even if the gap comprises mainly reserved land or a natural barrier;
(vi) Holiday resorts shall be recognised as urban on a dwellings rather than population density criterion, but care must be taken to omit decaying mining towns and the like in applying this criterion. Such resorts should be classed as urban if they have 250 or more dwellings (with at least 100 occupied dwellings) on Census night and have a recognisable core, except that where a holiday resort adjoins an urban area, it shall be encompassed by the urban boundary if it has a density of 125 or more dwellings per square mile;
INVESTIGATION OF OVERSEA PRACTICE reveals that a wide range of criteria has been used by various countries in attempting to delimit urban boundaries. Amongst these criteria are such things as occupation, commuting patterns, commercial relationships (as revealed by stores accounts, telephone calls, etc) and population density. In general these efforts have resulted in systems which are highly complex (because of the use of a combination of criteria) and subjective (because of the lack of adequate information for particular criteria). Some of the required information has to be gathered from external sources, adding greatly to the cost of determining boundaries, whilst much of the information which becomes available from the Census itself (e.g. commuting or occupational patterns) is of a complex nature and, even with computer processing, would not be available until a year or more after the Census.

THE BASIC CRITERION ADOPTED IN AUSTRALIA for the delimitation of urban boundaries is population DENSITY as applied to small areas. As urbanization increases, the change from rural to urban uses is accompanied by increasing population density. Extensive field investigations have
shown that areas at the fringe which have largely lost their rural characteristics and are developing towards urbanization have densities which vary over only a relatively small range. The adoption of a specific density from within that range therefore provides a criterion which adequately delimits urban boundaries and which can be applied objectively and uniformly without undue difficulty or delay.

A density of 500 persons per square mile has been adopted as the criterion.

THE GEOGRAPHIC UNITS to be classified according to the density criterion are collector's districts, the smallest units available. These areas vary in size and shape, but as far as possible they have been designed to ensure that significant urban development in large rural collector's districts is split off as a separate collector's district. Particular rules apply to contiguous areas which have special functions but which do not meet the density criterion, such as airports, sporting areas and industrial areas. In addition, a dwelling density criterion is applied to holiday areas which may be below population density because the Census is taken in mid-week and during the winter.

THE BOUNDARY OF AN URBAN CENTRE is, therefore, the
peripheral boundary of an aggregate of contiguous urban collector's districts. The boundary is a moving one which reflects the process of urbanization. The use of objective criteria enables valid comparisons to be made between one urban centre and another, and between the population for an urban centre at one Census with the populations at succeeding Censuses. Comparable information for the 1961 Census has also been prepared, although, because 1961 collector's districts were not especially designed for this purpose, some estimations have had to be made.
APPENDIX B

DATA FOR WOLLONGONG

For each combined census district the following data are shown:

- \( P \) = the population
- \( D \) = the number of dwellings
- \( A \) = the area in square miles
- \( d \) = the density of population in thousands per square mile
- \( x, y \) = the distances in miles from the arbitrary axes of reference of its centre
- \( r \) = the distance in miles from the city centre

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A COMPARISON OF ESTIMATES OF NUCLEAR BOMB CASUALTIES FROM TWO DIFFERENT URBAN MODELS

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(Received June 12, 1968)

Expected numbers of human casualties in a single-bomb nuclear attack upon a symmetric city are calculated using Sherratt's and Clark's models and for various distances between the city center and ground zero. The calculated results indicate that considerable error may arise in practical cases from the unjustified use of Sherratt's model.

J. Hunter has developed an interesting analytical technique for estimating the casualties in a nuclear attack on an urban center, in which the casualty function is approximated by an exponential or sum of exponential functions, while the population density is taken to be of Sherratt's form. Hunter has pointed out that some cities are undoubtedly better represented by Clark's model than by Sherratt's and, in this connection, Weiss has observed that Clark's fitted well to Sydney in 1947.

Thus, it is desirable to compare the number of casualties given by Hunter's approach with those given by Clark's model. For this purpose, we consider a symmetric city of central density \( A \) and, as usual with these population models, taken as being infinite in extent, while the attack consists of one bomb dropped at a variable distance \( R \) from the city center. For the casualty function, Hunter's single-term approximation is used, and aiming error is neglected.

THE EXPECTED NUMBER OF CASUALTIES AND NUMERICAL COMPARISON

If the probability that a person a distance \( r \) from ground zero will be a casualty has the form \( \exp(-cr^2) \), \( c \) a constant, and the population density at the point \((x, y)\) is \( A \exp(-b(\sqrt{x^2+y^2}^2/2)) \), \( b \) a constant, then the total number of casualties is given by: \( \)
The origin is taken at the city center and the positive x-axis is through ground zero. The population density in (1) is the symmetric case of Sherratt’s form. However, if Clark’s is adopted so that the population density at \((x, y)\) is \(\exp\{-b\sqrt{x^2+y^2}\}\), then the number of casualties is given by

\[
N_c = A \int_0^{2\pi} \int_0^\infty \exp\{-rb - c(r^2 - 2r R \cos \theta + R^2)\} r dr d\theta.
\]  

(2)

This may be expressed as

\[
N_c = \alpha \cdot \left(\frac{1}{\alpha} - \sum_{n=0}^{\infty} \left(\frac{c R}{\alpha}\right)^n \frac{1}{n!}\right).
\]  

(3)
where 

\[
a = \frac{b^2}{4c^2}, \quad \text{and} \]

\[
I_{k+2} = \int_{0}^{\infty} e^{-c(r+a)} r^{k+1} dr = \left[ \frac{1}{2c} \right] I_k - aI_{k+1}.
\]

\(I_0\) and \(I_1\) are, respectively, \((\frac{1}{2})\text{erfc}(a\sqrt{c})\sqrt{\pi/c}\) and \(\exp\{-ca^2/(2c)\} - aI_0\), so the \(I_{k+2}\) are easily determined.

\(N_c\) and \(N_s\) depend on the parameters \(A, b, c,\) and \(R\). The central density \(A\) is essentially just a scaling factor as far as any comparison is concerned, so it was given the value 1 in the computations. Since the total population does not depend on which model is used, \(b\) is the same for \(N_c\) as for \(N_s\), and was given the values 0.25, 0.45, and 0.75. These values were chosen from Clark's results\(^5\) to give a reasonable coverage of modern cities. The bomb parameter \(c\) was fixed at 0.09/mile\(^2\), as Hunter's results\(^1\) indicated this to be fairly representative. The numerical results are shown in Table 1.

**CONCLUSION**

The city is more concentrated according to the Sherratt model than it is according to the Clark model, and the calculated results indicate that the use of expression (1) to derive the estimated number of casualties when a nuclear bomb is exploded often gives results significantly greater than those obtained using expression (2). This difference is greatest when \(b\) is small and when the bomb is dropped close to the city center.

As a practical example, the values of \(A\) and \(b\) for Sydney (1947) have been given\(^5\) as 30 and 0.25, respectively. For a centrally directed attack, and giving \(A\) its true value, the estimated number of casualties using (2) is about 539,000 while (1) gives about 777,000.

Similarly, in 1947, the values of \(A\) and \(b\) for Brisbane are given\(^1\) as 40 and 0.75. The estimated casualties derived from expressions (1) and (2) are 340,000 and 260,000 respectively.

**REFERENCES**


AN EXAMPLE TO ILLUSTRATE THE "AVERAGE" NATURE OF CLARK'S LAW OF URBAN POPULATIONS

C. A. Wilkins and J. A. Shaw*

The basic result for much modern work in urban geography is Clark's finding [1] that for many cities, the density steadily declines as we approach the outer suburbs from the central business district, and that this decline follows an exponential law. Clark expressed his result in the form

\[ y = Ae^{-bx} \]

where \( y \) is the density of resident population in suitable units, \( x \) is the distance from the center of the city, and \( A \) and \( b \) are constants. This equation does not usually strictly apply in the sparsely populated central business district.

It is widely recognized and is obvious from Clark's own work that his result (1) is a statement about the average value of the population densities at points at distance \( x \) from the city center. As one of the present authors has pointed out [2], it should be expressed (with an obvious modification if the city and its representation do not extend through a full 360 degrees) as

\[ \frac{1}{2\pi} \int_0^{2\pi} d(r, \theta) \, d\theta = Ae^{-br} \]

where \((r, \theta)\) are the polar coordinates of a point at distance \( r \) from the city center (which is taken as the pole or origin) in the direction \( \theta \), and \( d(r, \theta) \) is the density at that point.

Unfortunately, some recent treatments have tended to obscure the true nature of Equation (1). In particular, Gurevich and Saushkin [3] adopt an approach based on a city's isolines or lines of equal density of population. They make the usual assumption that the city population density may be treated as being continuous, and use a logarithmic rate of decline of density given by

\[ -T(r, \theta) = [d(r, \theta)]^{-1} \left[ \partial d(r, \theta)/\partial r \right] \]

This rate is assumed to be non-negative. In terms of \( T \), \( d(r, \theta) \) for "single-center" cities is given by the equation

\[ d(r, \theta) = Ae^{-\int_0^r T(r, \theta) \, dr} \]

As \( T \) is assumed to be non-negative, \( d(r, \theta) \) is a non-increasing function of \( r \) for any

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Date received: August, 1958.
fixed value of $\theta$ so long as we do not go outside the city area. Gurevich and Saushkin [3] classify cities according to the nature of $T$. For their Type 1, $T$ is a constant (say $T_c$), i.e., there is "a constant rate of decrease of population density from the center to the outer boundary along any ray..." For such cities, the isolines are obviously concentric circles, or arcs of such, with their common center at the pole. They call members of their Type 1, "Clark type" cities.

Although the discussion of Gurevich and Saushkin is quite logical and self-consistent, it appears to the present authors that their nomenclature is rather unfortunate, as a city may satisfy Equation (2) to an acceptable degree of accuracy without satisfying Equation (4) with $T = T_c$. A city may in fact be quite asymmetric and still satisfy Clark's original law.

As a simple example of this behavior, consider a hypothetical city with central density 1 in which

$$d(r, \theta) = e^{-br}\{1 + (1 - e^{-br})\sin \theta\}$$

where $b$ is a positive constant. For convenience only, the city is assumed to extend to infinity in all directions.

---

**FIGURE 1:** Graph (diagrammatic only) of $Z$ against $\theta$.

---

**FIGURE 2:** Graph (diagrammatic only) of $br$ against $\theta$. 
As given by Equation (5), $d(r, \theta)$ gives a continuous surface. The density is strictly decreasing along any ray as $r$ increases from zero, for

$$d(r, \theta)/dr = -be^{-br}[1 + \sin \theta(1 - 2e^{-br})]$$

which is $<0$ except where $r = 0$ and $\theta = \pi/2$.

For a discussion of the isolines it is convenient to introduce a new variable,

$$z = e^{-br}$$

so that for the isoline along which $d(r, \theta)$ has the value $k$ ($0 < k < 1$), $z$ and $\theta$ are related by

$$\sin \theta = (k/z) + (1 - k)/(z - 1)$$

Then

$$d\theta/dz = (-1/cos \theta)((k/z^2) + (1 - k)/(z - 1)^2)$$

From Equations (8) and (9), it follows that the graph of $z$ against $\theta$ is as indicated in Figure 1. Note that, were a city of Gurevich and Saushkin's Type 1, the plot of $z$ against $\theta$ would be a straight line.

The equation of an isoline $d(r, \theta) = k$ is given by the appropriate solution of Equation (8) for $z$ and conversion back to the polar variable $r$. It is

$$br = \ln 2 - \ln ([1 + \sin \theta - [1 + 2(1 - 2k) \sin \theta + \sin^2 \theta]^{1/2}]/\sin \theta)$$

The graph of $br$ against $\theta$ is sketched in Figure 2, and in Figures 3(a)–3(c), graphs are given of the isolines for $k = 0.01, 0.50, 0.99$. It is to be noted that for small $k$

$$-\ln [1 - (1 - k)^{1/2}] = -\ln k + \ln 2$$

So the isoline for a small value of $k$ goes through the points $(-\ln k/b, 0), (-\ln k + \ln 2 + \delta)/b, \pi/2), (-\ln k/b, \pi)$, and $(-\frac{1}{2}\ln k/b, 3\pi/2)$, where $\delta$ is a small quantity.

For large $k$, the ratio $\ln[1 - (1 - k)^{1/2}]/\ln k$ is large, so that the point $(-\ln [1 - (1 - k)^{1/2}]/b, \pi/2)$ is relatively considerably further away from the city center than the points on the isoline at $\theta = 0, \pi, 3\pi/2$.

Note also that the "density of population per radian", or

$$\lim_{\Delta \theta \to 0} \int_0^{\theta + \Delta \theta} \int_0^\infty d(r, \theta)r \ dr \ d\theta/\Delta \theta$$

is equal to

$$(1/b^2)(1 + \frac{3}{4} \sin \theta)$$

which exhibits a considerable degree of oscillation.

All the above points show that our hypothetical city has a rather complicated non-circular structure in fine detail. Yet on the other hand,

$$\int_0^{2\pi} e^{-br}[1 + (1 - e^{-br}) \sin \theta] d\theta = 2\pi e^{-br}$$

so it satisfies the original form of Clark's law.
FIGURE 3a: Representative isolines; $k = 0.01$ for outer isoline, $k = 0.50$ for inner isoline.

FIGURE 3b: Isoline for $k = 0.99$.  

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More complicated examples can easily be devised, but the simple one used suffices for present purposes. It illustrates the fact that though the isoline method is useful for some investigations, it may result in unnecessary complication and labor in others.

REFERENCES

MEASURES OF SHAPE DISTORTION IN URBAN GEOGRAPHY

C. A. Wilkins and J. Shaw*

Geography has always contained a mathematical element though the emphasis placed upon it has varied from period to period. As Haggettl has pointed out, 'the geometrical tradition was basic to the original Greek conception of the subject, and many of the more successful attempts at geographical models have stemmed from this type of analysis'. As examples of such work, he cites the contributions of Christaller, Lewis, Wooldridge, Hägerstrand and Breisemeister. Branches of mathematics that are being used in geography today include topology, graph theory, linear programming and various statistical and stochastic subjects such as markov chains.

To quote Haggett again, 'Models are made necessary by the complexity of reality'. Cities are very complex structures and so the urban geographer is making increasing use of mathematical models of cities. The basic model of the 'ideal city' is taken by various authors to be circularly symmetric about its centre. Unfortunately, real cities usually deviate from this ideal due to a variety of economic, physical or sociological factors. The problem thus arises of measuring this deviation to determine whether or not it is significantly high.

To measure these distortions of cities from circular symmetry, Boyce and Clark,2 and Simmons,3 have suggested measures based on the geometric shape of the urbanized area rather than on the population distribution itself. These measures are approximate and should be regarded as approximations to idealized limiting measures, for which formulae are derived below. In the remainder of the note, two other measures emphasizing the population distribution are suggested, the first of which is based on a least-squares approach and gives a rather simple form for cities obeying the well-known law established by C. Clark.4 The second one employs the chi-squared technique and essentially treats circular symmetry as a statistical hypothesis to be subjected to test. This approach can be easily modified for cities on coastlines and in other special situations.

The first measure we discuss is that suggested by Boyce and Clark.5 This is based upon the shape of the outer boundary of the city and is defined by the equation

\[ I = \frac{1}{\sum R_i} \left( \frac{100}{\sum R_i} - \frac{100}{n} \right) \]

where \( I \) is the measure and the \( R_i \) (for \( i = 1, 2, \ldots, n \)) are the lengths of radial lines measured from the city centre to the boundary along equally spaced radii.

For a given city with given centre and boundary, the Boyce-Clark measure is a function both of \( n \) and of the direction of the \( R_i \). Obviously \( n \) should be as large as possible so that this measure is to be regarded as an approximation to a limiting measure. If \( R(\theta) \) denotes the distance from the city centre to the boundary in the direction \( \theta \), (where \( R(\theta) \) is of course assumed single valued) and if we put

\[ \bar{R} = \frac{1}{2\pi} \int_0^{2\pi} R(\theta) \, d\theta \]

then for Boyce and Clark's measure, using the elementary theory of the Riemann integral,

\[ \lim_{n \to \infty} I = \frac{100}{2\pi} \int_0^{2\pi} R(\theta) - \bar{R} \, d\theta \]

In statistical terms, if we consider \( \theta \) to be uniformly distributed over \((0, 2\pi)\), then the limiting value of Boyce's measure is 100 times the ratio of the mean variation of the random variable \( R(\theta) \) to its mean.
Simmons devised a measure based more upon the area of a city than upon its boundary. His measure is defined as the ratio of the sum of distances of points, arranged in a regular network within the boundary of the city's urbanized area, from the city centre to the sum of distances of points in the same regular network from the centre of a circle of the same area as the city. In symbols, and using \( J \) for Simmons measure,

\[
J = \frac{\sum R_i}{\sum r_i}
\]

where the \( R_i \) are the distances for the true city, and the \( r_i \) for the circle. Here for a given city with given centre and boundary, \( J \) is a function of \( n \) and of type of network, so again we look for the limiting form to remove this dependency.

Now as we have a regular network, we may imagine small areas attached to the points of the network, one area for each point and all areas of the same measure, say \( \delta x, \delta y \).

Then:

\[
\lim_{n \to \infty} J = \lim_{n \to \infty} \frac{1}{n} \sum \frac{\delta x \delta y}{\delta y} = \int \int \sqrt{x^2 + y^2} \, dx \, dy
\]

where the top integral is taken over the city's urbanized area, and \( A \) is the magnitude of this area in suitable units. The origin of the rectangular co-ordinates \( x, y \), must of course be taken at the given centre.

In statistical terms, the limiting value of \( J \) is the ratio of the average value of the distance from the centre to a point taken at random in the city area (all points being equally likely) to the same average for the ideal circular city with same centre and area.

Both \( I \) and \( J \) deal with the geometric shape of the city in relation to its centre and depend on the definitions of centre and outer boundary. (It is to be noted that a city circular in shape may conceivably still have its centre eccentrically placed.)

However, useful as the Boyce-Clark and Simmons measures are, there still remains the difficulty that a city's urbanized area may be fairly circular about its centre although the distribution of population within this area is far from circularly symmetric.

Now a city is circularly symmetric about its centre if and only if:

\[
d(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} d(r, \theta) \, d\theta
\]

where \( r, \theta \) are the polar co-ordinates with the city centre as pole and \( d(r, \theta) \) is the population density at the point \( (r, \theta) \). If we adopt a least-squares criterion, then the deviation from symmetry \( D \), say, may be defined by the equation:

\[
D = \int_0^{2\pi} \int_0^{2\pi} \left( d(r, \theta) - \frac{1}{2\pi} \int_0^{2\pi} d(r, \theta) \, d\theta \right) \, rdrd\theta
\]

As an example suppose Clark's law applies. Then:

\[
D = \int_0^{2\pi} \int_0^{\infty} \left( d(r, \theta) - \frac{1}{2\pi} \int_0^{2\pi} d(r, \theta) \, d\theta \right)^2 \, rdrd\theta
\]

where \( A \) and \( b \) are constants. So:

\[
D = \int_0^{2\pi} \int_0^{\infty} \left( \frac{1}{2\pi} \int_0^{2\pi} d(r, \theta) \, d\theta \right)^2 \, rdr = \frac{P^2b^2}{8\pi}
\]

where \( P \) is the total population of the city and using the common simplifying assumption that the city extends to infinity in all directions. (The modification to be made if one does not wish to use this assumption is easy.)

In practice:

\[
D = \sum \int_0^{2\pi} \int_0^{\infty} d^2(r, \theta) \, rdrd\theta = \sum P_i^2/a_i
\]

where \( P_i \) and \( a_i \) are respectively the population and the area of the \( i \)-th census district.

Hence:

\[
D = \sum \frac{P_i^2}{a_i} - \frac{P^2b^2}{8\pi}
\]

on the assumption that Clark's law applies and that without serious numerical error, the city may be considered to stretch to infinity in all directions.

Another more statistical approach may be based upon the chi-squared statistic, which may be used to compare the observed distribution of population within this area to the expected distribution under a circularly symmetric model.
bution of population with that expected in a circularly symmetric city. As an example of how this may be done imagine m concentric circles drawn with common centre at the city centre. Let \( R_i (i = 1, \ldots, m) \) be the radius of the i-th circle
\( 0 < R_1 < R_2 < \ldots < R_{m-1} < R_m \).

Now let n half rays:
\[ r > 0, \theta = \theta_j, j = 1, 2, \ldots, n, 0 = \theta_1 < \theta_2 < \ldots < \theta_{n-1} < \theta_n < 2\pi \]
be drawn. Also let \( R_0 = 0 \) and \( \theta_{n+1} = 2\pi \).

Let the i, j-th area \( A_{ij} \) be the area determined by
\[ R_{i-1} < r < R_i, \quad i = 1, 2, \ldots, m \]
and
\[ \theta_{j-1} < \theta < \theta_j, \quad j = 1, 2, \ldots, n \]

Let \( P_{ij} \) be the population of this ij-th area. According to the hypothesis of circular symmetry, we should have:
\[ P_{ij} = \frac{\theta_j - \theta_{j-1}}{2\pi} \sum_{j=1}^{n} P_{ij} \]
\[ = T_{ij} \text{ say} \]

The \( T_{ij} \) are the theoretically expected populations, the \( P_{ij} \) are the actually observed populations, so the chi-squared statistic is:
\[ \chi^2 = \sum_{i,j} (P_{ij} - T_{ij})^2 / T_{ij} \]

This will be approximately distributed in a chi-squared distribution with \( mn - m \) degrees of freedom and the usual chi-squared test may be applied.

An error is of course introduced into the calculation of \( \chi^2 \) through allotting to different \( A_{ij} \) the people in a census tract that does not lie wholly within one such area. However, the \( \chi^2 \) as calculated should at least be of the same order of magnitude as the value that would be calculated were it feasible to determine absolutely precisely the number of people in each \( A_{ij} \). At the very least, the value of \( \chi^2 \) will be qualitatively informative.

The modification of this \( \chi^2 \) approach to cases of cities with their centres on coastlines consists of simply ignoring the irrelevant areas. This approach can also be used to investigate whether a city is symmetric out to some distance from the centre, and distorted from there outwards because of new housing developments or other factors.

The new measures of distortion suggested here are based more on the distribution of people within the city rather than on the boundary. As a city may conceivably have a circular boundary centred on the city centre, and yet also have large distortions from symmetry of population density inside its urbanized area, it is suggested that the measures put forward in this note at least deserve to be considered along with the earlier measures in any investigation of geographic distortion of cities.

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5. Boyce and Clark, op. cit.
6. Simmons, op. cit.