The variation with space and time of the neutron energy distribution in pulsed BeO assemblie

A.I.M. Ritchie
Wollongong University College

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THE VARIATION WITH SPACE AND TIME
OF THE NEUTRON ENERGY DISTRIBUTION
IN PULSED BeO ASSEMBLIES

A.I.M. Ritchie B.Sc.(Hons.)

April 1970
Time dependent spatial distributions have been measured in BeC assemblies covering the buckling range 0.0083 cm$^{-2}$ to 0.0399 cm$^{-2}$, referred to a density of 2.96 gm cm$^{-3}$. This buckling range covers assemblies with decay curves that can be described by a discrete decay constant ($\lambda < \lambda^*$), a pseudodiscrete decay constant ($\lambda^* < \lambda < \lambda_K$) and a decay constant that changes with time ($\lambda > \lambda_K$). It was found that the extrapolated length derived from these spatial distributions did not change with time if $\lambda < \lambda^*$. At a buckling of 0.0292 cm$^{-2}$, ($\lambda \sim \lambda_K$), the extrapolated length changed by $(0.5 \pm 0.25)$ per cent. /1,000 $\mu$s in the time range 1.4 ms to 2.9 ms and by $(2.0 \pm 0.5)$ per cent. /1,000 $\mu$s over a similar time range at the largest buckling, 0.0399 cm$^{-2}$, ($\lambda > \lambda_K$).

The decay constant of the next higher spatial mode (the 211 mode) did not lie on the $\lambda(B^2)$ curve of the fundamental. In the region $\lambda^* < \lambda < \lambda_K$ the decay constants were in general higher than those for the fundamental. For $\lambda > \lambda_K$ the decay constants tended to lie below the $\lambda(B^2)$ curve, in particular at long ($\sim 1.5$ ms) times after the pulse.

The effective decay constant of the lowest mode of decay in the assembly with $\lambda > \lambda_K$ was found to be only marginally space dependent. At times $\sim 2.2$ ms after the pulse the decay constant at the centre of the assembly was 2 per cent. lower than the decay constant 8.5 cm from the centre, the errors being 0.8 per cent. The decay constants at the positions $\pm 8.5$ cm were the same within the errors at these times.
SUMMARY (continued)

Calculations were done, based on a multigroup diffusion code which approximated leakage by a DB$^2$ term. The time dependent energy spectra corresponding to a number of Fourier modes were calculated and these added together, assuming a delta function source to define the initial amplitudes, to give the energy spectrum as a function of space and time. The time dependent spatial distributions corresponding to a 1/v reaction rate were calculated and then analysed in exactly the same way as the experimental results. The multigroup cross section set was a 41 group set covering the range 1.125 eV to 9.36 x 10$^{-6}$ eV with 18 groups below the Bragg cut-off and 23 groups above.

The neutron density averaged mean free path did show an increase with time in the assemblies of largest buckling but the corresponding increase in the extrapolation distance and hence of the extrapolated length was a factor of four too small to explain the experimental results. The theory did show good agreement with the decrease of extrapolation distance with buckling seen in the assemblies with $\lambda < \lambda^*$. Theory did not predict the correct dependence of the decay constant with both time and position although the agreement with experiment at the central position (essentially the fundamental alone) was very good.

The amplitude of the asymmetric modes, in particular the amplitude of the 211 mode, was overestimated by the theory. The discrepancy increased with buckling and was a factor of ten different
from experiment at a buckling of 0.0399 cm\(^{-2}\). This should be compared with the excellent agreement obtained between theory and experiment in a similar measurement in BeO when the pulse length was 20 \(\mu\)s rather than the 500 \(\mu\)s used in the present series of experiments.

Theory also predicts an increase in the asymmetry at long times, 3 to 5 ms, after the pulse. This is consistent with the idea that the long term distribution is dominated by far sub-Bragg neutrons. Proper experimental verification of this, in particular observation of the return to a symmetric distribution at very long times (~140 ms after the pulse) would require impossibly intense neutron sources.
ACKNOWLEDGEMENTS

I would like to thank the Australian Atomic Energy Commission for the award of a scholarship which enabled me to complete the work contained in this thesis. I would further like to thank the members of the A.A.E.C.'s Physics Division for all their help, in particular Mr. L. Russell and Mr. D. Stevenson, for their assistance in setting up the experiments, and the operating team of the 3 MeV Van de Graaff, Mr. A. van Heughten, Mr. H. Broe, Mr. J. Copland and Mr. A. Croal, for their aid at various times of the experimental programme.

My thanks are due to my two supervisors, Professor A. Keane and Dr. J. Symonds, for their continued help and advice during the project. The assistance of Mr. M. Rainbow and Mr. S. Whittlestone in the running of the experiments through week-ends, public holidays and the "dead watches of the night" is also greatly appreciated. They, together with Mr. K. Maher, helped to clarify, in the course of many illuminating, if at times acrimonious, discussions many of the problems that appeared in both the experimental and theoretical work. In the same vein I must thank Mr. G. Trimble for all his help in running and preparing the various computer codes used in this work.

Finally I greatly appreciate all the hard work done by Mr. D. Hyde and Miss Sherri Bond on the diagrams and typing required in the preparation of this thesis.
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GLOSSARY OF SYMBOLS

$\mathbf{v}$ Neutron speed; usually measured in units of cm sec$^{-1}$.

$E$ Neutron energy; usually measured in units of eV.

$r \equiv (x,y,z)$ The spatial position of the neutron referred to some origin.

$\hat{\Omega}$ The unit vector describing the direction in which the neutrons travel. Components are, using spherical polar coordinates, $(\cos \psi \cdot \sin \theta, \sin \psi \cdot \sin \theta, \cos \theta)$.

$\mu \equiv \cos \theta$ Commonly used to replace $\theta$ in cylindrically symmetric systems.

$n$ The neutron density. In the most general form it is a function of the variables $(r, E, t, \Omega)$ or $(r, v, t, \Omega)$ in which case it is the number of neutrons/unit volume of phase space at the point $r$ and time $t$ which have energy $E$ and travel in the direction $\Omega$.

In some cases the phase space variables are reduced e.g. in one velocity diffusion theory $n = n(r, t)$.

$f$ Vector flux; in general a function of the variables $(r, E, t, \Omega)$. $f(r, E, t, \Omega) dE d\Omega$ represents the number of neutrons crossing unit area in unit time at the point $r$ and at time $t$ and which have energy $E$ and travel in the direction $\Omega$. Like $n$, $f$ can be a function of a reduced set of variables. $f$ and $n$ are related to each other by $f = nv$. 
Scalar flux $\phi = \int d\Omega f(\mathbf{r}, E, t, \Omega)$.

Total neutron density $N(\mathbf{r}, t) = \int d\Omega dE n(\mathbf{r}, E, t, \Omega)$.

Total macroscopic cross section. The cross section is assumed to be a function of energy (velocity) alone i.e. homogeneous systems only are considered. Unless otherwise stated $\Sigma(E) = \Sigma_s(E) + \Sigma_a(E)$. Units are cm$^{-1}$.

$\Sigma_s$ The macroscopic scattering cross section.

$\Sigma_{el}$ The elastic part of the macroscopic scattering cross section.

$\Sigma_{inel}$ The inelastic part of the macroscopic scattering cross section.

$\Sigma_a$ The macroscopic absorption cross section. In general this is assumed to be proportional to $1/v$ i.e.

$$\Sigma_a = \lambda_a/v.$$

$\Sigma(E' \rightarrow E, \Omega' \rightarrow \Omega)$ Differential macroscopic scattering cross section;

$\Sigma(E' \rightarrow E, \Omega' \rightarrow \Omega) dE d\Omega$ represents the macroscopic cross section for scattering events that change the neutron energy from $E'$ to $E$ and the direction of travel from $\Omega'$ to $\Omega$.

$\bar{\mu}$ The average value of the cosine of the angle through which the neutrons are scattered after a collision.

$$\bar{\mu} = \bar{\mu} \int \int \int \Sigma(E' \rightarrow E, \Omega' \rightarrow \Omega) d\Omega dE \int \Sigma(E' \rightarrow E, \Omega' \rightarrow \Omega) d\Omega dE.$$
Transport cross section $\Sigma_{tr} = \Sigma_a + \Sigma_s (1-\mu)$.

Diffusion coefficient $D(E) = 1/3 \Sigma_{tr} (E)$.

Transport mean free path $\ell_{tr} (E) = 1/\Sigma_{tr} (E)$.

Transport mean free path averaged over the neutron density.

Some other average of the transport mean free path.

In particular the value derived from the diffusion coefficient measured in pulsed neutron experiments.

The extrapolation distance. Usually related to the neutron mean free path in the one velocity case by $E = 0.71 \ell_{tr}$.

The extrapolated length of the x dimension. In a parallelepiped it is related to the physical dimension $l_x$ by $L_x = l_x + 2E$.

A propagation vector which can be used to define the spatial distribution. Sometimes used as the integration variable in a three dimensional Fourier transform. Units are cm$^{-1}$.

The boundary conditions frequently impose limits on the allowable values of $B$ which gives rise to the concept of spatial modes.
The (geometric) buckling. \( B^2 = B_x B_y B_z = (B_x^2 + B_y^2 + B_z^2) \).

Also reserved for the buckling of the lowest spatial mode.

The buckling of the \( m \)th spatial mode. In a parallelepiped three integers are required to specify the \( m \)th mode i.e. \( m = (i,j,k) \)

\[
B_{ijk}^2 = \pi^2 \left[ i^2/L_x^2 + j^2/L_y^2 + k^2/L_z^2 \right].
\]

In general a decay constant. Sometimes used as the integration variable in a Laplace transform. Units are sec \(^{-1} \).

The velocity corresponding to the first major Bragg peak in the elastic scattering cross section.

Corngold's limit; \( \lambda^* = (vE)_{\text{min}} \). In particular for a polycrystalline scatterer \( \lambda^* = (vE_{\text{inel}})_{\text{min}} \).

The buckling corresponding to \( \lambda^* \) in the \( \lambda(B^2) \) dispersion relation.

Kothari's limit; \( \lambda_K = \left[ vE_{\text{inel}} + vDB^2 \right]_{v=v_B} \).

Limit point for elastic continuum of Conn and Corngold;

\[
\lambda_{el} = \left[ vE_{\text{inel}} + vD_{el}B^2 \right]_{v=v_B}.
\]

Diffusion coefficient for purely elastically scattered neutrons.
R(t,x,y,z) The reaction rate for a particular reaction as a function of space and time.

r(t,x) The reaction rate measured in one particular direction.

A_l The amplitude of the lowest Fourier mode in r(t,x).

A_s The amplitude of the first asymmetric (sine) mode in r(t,x).

A_c The amplitude of the second lowest symmetric mode in r(t,x).

A_{ijk} The amplitude of the i,j,k spatial mode in R(t,x,y,z).

\phi_i The flux in the i^{th} energy group.

D_i The diffusion coefficient averaged over the flux in the i^{th} energy group.

\Sigma_i The total cross section averaged over the flux in the i^{th} energy group.

\Sigma_{j \rightarrow i} The differential cross section averaged such that
\phi_j \Sigma_{j \rightarrow i} represents the number of neutrons/cm^3/sec scattered from the j^{th} to the i^{th} energy group.

\frac{1}{v_i} The reciprocal of the velocity averaged over the i^{th} group.

\sigma Describes a microscopic cross section e.g. \sigma_{tr} is the microscopic transport cross section. Units of cm^2.
1. INTRODUCTION

The pulsed neutron technique provides a method for both measuring parameters of interest in particular reactor systems and assessing the accuracy of calculational methods used to describe reactor systems. The method consists of measuring the time dependent response of the neutron population in the system following the injection of a pulse of neutrons into the system. The technique is attractive in that the characteristic times associated with most non-critical reactor systems are short (in the range 100 ns to 100 ms) compared to everyday time scales. This allows repetitive pulsing at comparatively high rates and the consequent collection of good counting statistics in a relatively short period of time. The technique is also inherently more accurate than analoguous stationary techniques since the time dependence of neutron events can generally be measured with greater precision than the spatial dependence.

The technique first came into prominence in the mid-1950's as a method for measuring the thermal neutron diffusion parameters of the common moderators. When a pulse of energetic neutrons is injected into a large block of such a material the neutrons slow down and migrate through the mass of the material. Quite quickly (within 100 μs or so for a polycrystalline moderator) the average neutron energy approaches that of the scattering centres in the material. If the dimensions of the block are very much greater than the mean free path of the neutrons between collisions then it is reasonable to compare the neutrons to a gas enclosed within the confines of the block. Such a
gas will come into thermal equilibrium with the scattering centres in the block and have a Maxwellian distribution of energies corresponding to the temperature of the block. The time dependent spatial distribution is then adequately described by the diffusion equation

$$\frac{\partial n}{\partial t} - \frac{\nabla^2 n}{vD} + \frac{\overline{\Sigma}}{\overline{\alpha}} n = 0$$  \hspace{1cm} (1.1)

where $n = n(\xi, t)$ is the neutron density and the bars denote averages over a Maxwellian energy distribution.

Now the equation

$$\nabla^2 \chi_m(\xi) + B_m^2 \chi_m(\xi) = 0$$  \hspace{1cm} (1.2)

is well known in physics and its solutions are known to form a complete orthonormal set. Hence we can expand the neutron density in the form

$$n(\xi, t) = \sum_m a_m(t) \chi_m(\xi)$$  \hspace{1cm} (1.3)

It is then easy to show using the orthonormal properties of the $\chi_m$ that

$$a_m(t) = a_m^0 \exp(-\lambda_m t)$$  \hspace{1cm} (1.4)

where

$$\lambda_m = \overline{\Sigma} + \overline{vD} B_m^2$$  \hspace{1cm} (1.5)

Now the actual form of the $\chi_m$ and the $B_m^2$ depend on the shape of the block and the boundary conditions assumed in the solution of
equation 1.2. In general the quantity $B_m^2$, which is normally called the buckling of the $m^{th}$ spatial mode, is given by an expression of the form

$$B_m^2 = m^2 \cdot \text{constant/(characteristic length)}^2$$  \hspace{1cm} (1.6)

The characteristic length is related to the physical dimensions of the assembly and for an assembly with dimensions much larger than the neutron mean free path is little different from the actual physical dimensions of the assembly.

The physical interpretation of equations 1.3, 1.4 and 1.5 is as follows. Soon after the end of the fast neutron pulse the neutron density in the assembly can be described by a set of spatial modes, the amplitudes of which decay exponentially with time. The higher the order of the mode the larger the buckling and the greater the rate of decay associated with that spatial mode. This follows from the form of equation 1.5 and reflects the fact that the leakage rate of neutrons in the $m^{th}$ spatial mode is $\nu DB_m^2$ which is large for large bucklings. At sufficiently long times after the pulse the higher modes vanish and leave only the fundamental which then describes the space and time dependence of the neutron density. The count rate from a detector sampling the neutron density at these late times decays exponentially with time so providing a method for measuring the decay constant of the fundamental mode. This means that for a particular sized assembly the $\lambda$ and $B^2$ in the dispersion relationship 1.5 can be found and this dispersion relation used to find the constants $\underline{\nu_S}$ and $\underline{\nu_D}$ which are the thermal neutron diffusion parameters of the material in the assembly.
This then is the basic idea behind the pulsed neutron technique. The decay constant $\lambda$ of a particular assembly can be measured and the buckling $B^2$ evaluated from a relationship of the form 1.6. A dispersion relation of the form 1.5 can then be used to determine properties of the system with which the neutrons interact.

If in fact a decay constant can be measured for a system and if the parameter $B^2$ can be confidently evaluated from the dimensions of the assembly the parameters $\overline{vE_a}$ and $\overline{vD}$ can be found to a high degree of accuracy provided the dispersion relation 1.5 is correct. Now it has been assumed that the neutrons behave like a gas enclosed within the bounds of the moderating material. However unlike a gas, neutrons can escape very easily through the surface of the moderator and, as can be inferred from equation 1.1, the rate of leakage is proportional to $vD\nabla n$. Since the product $vD$ is in general energy dependent, neutrons of different energies will leak from the system at different rates. Now the system must have measurable leakage (high buckling) to make use of the dispersion relation 1.5 at all, so that it is reasonable to expect the neutron energy spectrum to differ from that of a Maxwellian. The dispersion relationship (1.5) is now no longer adequate and needs to be modified to take account of the shift of the energy spectrum away from a Maxwellian.

This was the first stage in the development of the pulsed neutron technique as applied to thermal homogeneous systems. More accurate forms of the dispersion relation 1.5 were found and efforts were made to measure decay constants with greater and greater accuracy.
The second stage in the development came when it was realised that a unique decay constant need not exist for a particular geometric assembly and that there was no simple one to one relationship between the rate of decay in the assembly and the size of the assembly. Pulsed neutron experiments then began to change their character and became in many instances tools for investigating various approximate solutions to the transport equation.

This thesis describes pulsed neutron experiments and calculations which belong to the second phase of the pulsed neutron experiment outlined above. The time dependent spatial distribution of a 1/v detector has been measured in small pulsed BeO assemblies, some of which are known to have a decay curve which does not decay exponentially with time, in order to investigate experimentally the existence of a buckling in these assemblies. The measured time dependent spatial distributions have been calculated using an extension of a method which has had great success in calculating the time dependence of the 1/v reaction rate in these small systems, in an effort to see if the method can describe the space-time-energy behaviour of the neutron flux.

The two phases of the pulsed neutron experiment outlined above are described in much greater detail in the next chapter which also contains a discussion of the present state of understanding of the physical processes taking place in a small pulsed polycrystalline system. A more detailed description of the subject matter of this thesis and its relevance to the present understanding of these physical processes will also be deferred until the next chapter.
2. HISTORICAL REVIEW

Von Dardel in 1954 (1,2) was the first to note that the simple dispersion relation derived in the last chapter was inadequate. A number of theoretical papers then appeared (3,4,5) which showed that the decay constant of the persisting mode in a pulsed non-multiplying homogeneous assembly could be written more generally as

\[ \lambda = \overline{v\Sigma_a} + \overline{vDB^2} - CB^4 + FB^6 \]

The bars denote flux averages over a Maxwellian distribution and the quantities \( v, \Sigma_a \) and \( D \) have their usual meaning. The quantities \( C, F \) were products of Maxwellian averages of various moments of the scattering kernel. In particular, \( C \) the diffusion cooling constant contained the Maxwellian average of the second moment of the scattering kernel. Hence it appeared that pulsed experiments in the common moderators could provide integral checks on the scattering law data that was appearing from such centres as Chalk River and Harwell (see for example 6).

The apparatus for these pulsed experiments was neither very complex nor expensive; 500 keV Van de Graaff or Cockcroft Walton accelerators, which were then being superseded by higher energy machines in low energy nuclear physics research, produced sufficiently intense neutron pulses (typically \( \sim 5 \times 10^7 \) neutrons/pulse using the \( \text{D(T,n)He}^4 \) reaction and pulses \( \sim 1 \) ms long) and multichannel pulse height analysers readily converted to multichannel time analysers were available. Many experimental groups throughout the world set up
apparatus to study the common moderators using the pulsed neutron technique. By the early 1960's there was good agreement between the various pulsed experiments carried out on water and heavy water but there were rather puzzling inconsistencies in the results for the polycrystalline moderators beryllium, beryllium oxide and graphite at the higher bucklings (smaller geometrical assemblies) required to obtain good estimates of the diffusion cooling constant.

Silver (7) had noted persisting changes in the decay constant of small beryllium blocks, particularly at low temperatures, which were absent from similar small blocks of ice even when the temperature was reduced to -90°C. De Saussure (8) in a companion paper attributed this to the "trapping" of neutrons in the Bragg peaks. The argument was that as the elastic cross section is very high (~10 barns) and the inelastic cross section is very low (~0.5 barns) in these peaks the neutrons tend to be trapped at the maxima of the cross section. Furthermore at the lowest energy Bragg peak the scattering is predominantly backward leading to a greatly enhanced transport cross section and consequent reduction in leakage rate. The consequent loss rate of neutrons with this particular energy was lower than many "decay constants" measured in polycrystalline assemblies of large buckling and de Saussure argued that the measured decay constants were not those of the asymptotic decay. In fact he went further and raised the possibility that there was no asymptotic mode in these systems.

The loss rate of neutrons from a particular energy interval, due to leakage, inelastic scatter and absorption, has in the case of
Figure 1. The quantity \( (\nu \Sigma_{\text{inel}} + \nu DB^2) \) as a function of energy and buckling in BeO.
polycrystalline materials, a local minimum at energies in the region of the Bragg peaks (usually the first Bragg peak). This can be seen in Figure 1 which shows the loss rate curve for BeO. Now it is obvious that the decay constant of the asymptotic mode describing the decay of the neutron population as a whole must be less than the loss rate at any particular energy. If it is not, the energy spectrum will change continuously with time until the decay of the ensemble is the decay of the slowest, and we will be left with a rather singular looking energy distribution. We can investigate this more precisely by looking at the Boltzmann equation which describes the neutron population in these assemblies.

The equation takes the following form at times after the end of the neutron pulse

$$\frac{\partial n}{\partial t} + v \Omega \cdot \text{grad} n + v \Sigma n = \int d\Omega' d\Omega \, \Sigma(\Omega' \rightarrow \Omega, \Omega \rightarrow \Omega') \, v' \, n(r, v', \Omega', t) \quad (2.1)$$

The experimenter would like to measure a reaction rate which is asymptotically described by an exponential, i.e., we want solutions of the form

$$n(r, v, \Omega, t) = \psi(r, v, \Omega) \exp(-\lambda t) \quad (2.2)$$

where

$$(v \Omega \cdot \text{grad} \psi + (v \Sigma - \lambda) \psi) = \int d\Omega' \, d\Omega \, \Sigma(\Omega' \rightarrow \Omega, \Omega \rightarrow \Omega') \, v' \psi(r, v', \Omega') \quad (2.3)$$
Now we expect $\psi$ to be positive everywhere since physically we do not obtain negative reaction rates. Hence the right hand side of equation 2.3 is positive. This means that in order to keep the left hand side positive there must be an upper bound on $\lambda$ to allow solutions of the form 2.2.

In many cases the experimenter would like to extend equation 2.2 and look for solutions of the form

$$n(r, v, \Omega, t) = \sum_{n}^{N} \psi_{n}(r, v, \Omega) \exp(-\lambda_{n}t)$$

(2.4)
as he can associate the higher decay schemes with spatial or energy modes. We can now no longer expect the $\psi_{n}(r, v, \Omega)$ to be positive everywhere since the total reaction rate will be made up of sums of the $\psi_{n}(r, v, \Omega)$ but we certainly would not expect them to be singular so that again $\lambda$ must be bounded above. The questions that were being asked in the early 1960's were:

1. What is the upper bound on $\lambda$?
2. To what energy does this upper bound correspond?
3. Do solutions of the form 2.4 form a complete set?
   If not what does a complete set look like?
4. Is the sum in 2.4 finite?
5. If it is finite are the $\lambda_{n}$ very limited in number?
6. Are there some cases where there are no $\lambda_{n}$?

The last two questions were of particular interest to experimenters. Many experiments (reviewed most succinctly by Beckurts (9)) had been carried out to measure the thermalisation
time constant which is one of the $\lambda_n$ in 2.4. If in fact there are no higher discrete modes then a thermalisation time constant as such does not exist and an experimenter must be careful to specify how he has analysed his data. Similarly if there are no discrete modes for an assembly the experimenter cannot sensibly measure a lowest decay constant and the corresponding points on the $\lambda(B^2)$ curve for that assembly.

Lehner and Wing (10) had studied the one velocity form of equation 2.1 in slab geometry and had shown that the complete solution had the form

$$n(z,\mu,t) = \sum_{n=1}^{N} \psi(z,\mu) \exp(-\lambda_n t) + \int_{\lambda^*}^{\lambda^*+i\infty} d\lambda \exp(-\lambda t) \psi(\lambda,z,\mu)$$

where

$$\lambda^* = v\Sigma \quad \text{and} \quad \lambda_1 < \lambda_2 < \lambda_3 \cdots < \lambda_N < \lambda^*.$$ 

$N$ was finite and in particular $N \gg 1$ so that there was always a solution of the form 2.2 which persisted at long times. Van Norton (11) studied the one velocity case in a sphere and showed that there was always a discrete decay constant for any size of sphere and that there was no continuum. This seemed to indicate that the continuum was a peculiarity of a slab where neutrons could travel parallel to the face of the slab and never escape. The existence of a discrete mode of decay for all sphere sizes opened the possibility that in a
finite system there would always be a discrete mode which would describe the decay of the population at long times. The likelihood of this being the case was increased by the work of Jörgens (12) which showed that in the multivelocity case there is always a discrete eigenvalue for any size of sphere provided the velocity is bounded away from zero.

A number of very important papers followed which considered the full energy range but used a number of different approximations and simplifications. Koppel (13), Shapiro and Corngold (14) and Küscer and Corngold (15) were able to show for the infinite medium case a solution of the form 2.4 was not complete, that in general \( N \) was finite (but infinite for a proton gas and possibly for a monatomic gas), and that particularly in a polycrystal the value of \( N \) was very limited (\( N \sim \text{unity} \)). In all cases the eigenvalues were bounded above by \( \lambda^* = (v \Sigma_{\text{inel}})_{\text{min}} \). A further set of papers, Nelkin (16), Corngold (17), Corngold and Michael (18), Albertoni and Montagnini (19) and Bednarz (20) where a number of simplifying assumptions were made about either the streaming operator (\( \nabla \cdot \text{grad} \)) or the scattering operator, showed that in a finite system the set of eigenvalues below \( \lambda^* \) was finite and that for a sufficiently small system there were no eigenvalues below \( \lambda^* \).

The results can best be summarised in the language of functional analysis. We can write the Boltzmann equation 2.1 as

\[
\frac{\partial n}{\partial t} = A_n \tag{2.5}
\]
Figure 2. The spectrum of the operator appearing in various approximate forms of the Boltzmann equation.
where the operator

\[ A = -v \nabla \cdot \nabla + \int d\Omega' \int_\Omega \Sigma(v' \to v, \nabla' \to \nabla) \]  \hspace{1cm} (2.6)

Now if we use Laplace transform methods to solve equation 2.5 we can write down a formal solution if we know the spectrum of the operator A. The work outlined above, in particular that of Albertoni and Montagnini, and Bednarz showed that the spectrum of A consisted of a discrete set on the real line and a region to the left of the line \( \text{Re}(\lambda) = -\lambda^* \) which certainly did not belong to the resolvent set but which could not be classified as discrete, continuous or residual spectrum. This is illustrated in Figure 2a. Figure 2b shows the spectrum obtained by Corngold (17) who assumed that the spatial distribution could be approximated by a term \( \exp(iB \cdot \rho) \). Now we can use the heuristic argument here that this approximation considers only one spatial Fourier mode and that the general solution must contain an infinity of B values. If B is allowed to tend to infinity, Corngold's spectrum tends to the spectrum of Albertoni and Montagnini where the streaming term was handled exactly.

As far as the experimenter was concerned these theoretical results, in particular those of Corngold (17) showed that there was an upper bound \( \lambda^* \) to the discrete set of eigenvalues and an associated upper limit to the buckling, \( B^2_\star \). In assemblies with bucklings greater than \( B^2_\star \) the time decay would not be described asymptotically by an exponential and a well defined decay constant would not be found in these assemblies. The upper bound \( \lambda^* \) for all the polycrystalline
materials was found to be even lower than the bound derived by de Saussure (8).

There was however still one weak link in the above line of argument which prevented the results being applied to polycrystalline assemblies in general. The spectrum of the operator which contained the exact form for the streaming term and a term which described elastic scatter, was still unknown.

Concurrent with this analytical work, Ghatak and Honeck (21) carried out some numerical calculations on a graphite system using multigroup diffusion theory and approximating the streaming by a $DB^2$ term. They used a realistic scattering kernel (Parks model) to describe the inelastic scattering, derived the elastic scattering from the experimental values for the total cross section and evaluated a great number of eigenvalues (all discrete since it was a multigroup calculation), and their associated eigenfunctions. The eigenvalues plotted as a function of buckling had some startling changes of slope. In particular the lowest eigenvalue had a marked change of slope at $\lambda = \lambda^*$ where it looked as though it was going to cross the next lowest eigenvalue. It was not really clear how these results fitted into the picture although below $\lambda^*$ there was reasonable agreement between calculation and experiment while above $\lambda^*$ the eigenvalues were quite sensitive to the group structure indicating that they were not true eigenvalues but possibly some sample from a continuum.

The experimental picture was even more confused. In all of the common polycrystalline moderators experimenters had measured
"decay constants" well above $\lambda^*$. In BeO for example, the material studied in the present thesis, Joshi et al. (22), Zhezherun (23) and Ritchie (24) had all measured decay constants for assemblies with buckling up to 0.03 cm$^{-2}$ and greater where $B^2 \sim 0.02$ cm$^{-2}$.

However, Fullwood et al. (25) while measuring the time dependent energy spectra in a small ($B^2 = 0.073$ cm$^{-2}$) beryllium sample noted that the decay at $\sim 1.0$ ms after the pulse was not exponential.

In an attempt to explain these results Kothari (26) suggested that it was physically unreasonable to consider the behaviour of zero velocity neutrons since it was unlikely that the experimenters would encounter these. Rather one should look for the minimum of the loss rate in the region of maximum neutron density. In diffusion theory with DB$^2$ leakage the loss rate at any energy is given by

$$\lambda = (v\Sigma_{\text{inel}} + vDB^2)$$

In most polycrystalline moderators the minimum occurs at the first Bragg peak and we get

$$\lambda_K = (v\Sigma_{\text{inel}} + vDB^2)_{v=v_B}$$

where $v_B$ = velocity corresponding to the first Bragg peak. This minimum can be seen in Figure 1. Kothari claimed that this limit should be used rather than Corngold's limit of $\lambda^*$. It explained why experiments in polycrystalline moderators produced decay constants greater than $\lambda^*$ and was consistent with the observations of Fullwood.
et al. that at sufficiently large bucklings the decay was not exponential. However it did not explain why in systems with $B^2 > B_\star^2$ where $\lambda^\star$ did represent the absolute minimum loss rate, the neutron population did not eventually decay with this decay constant.

Ritchie and Rainbow (27) in some further measurements in BeO noted that although they could measure well defined decay constants for $B^2 > B_\star^2 \sim 0.02 \text{ cm}^{-2}$, in the region of $\lambda_K (B^2 \sim 0.03 \text{ cm}^{-2})$ the decay constants possibly changed with time. When $B^2 = 0.039 \text{ cm}^{-2}$ the decay constant changed markedly with time at times $\sim 1.5 \text{ ms}$ after the pulse by which time asymptotic decay should have been established. This was in apparent agreement with Kothari's hypothesis. Corngold and Durgun (28), however, were able to show using diffusion theory with a $DB^2$ leakage that the reaction rate in a $1/v$ detector had the form

$$R(t) = \sum_n a_n \exp(-\lambda_n t) + \int_{\lambda^\star}^{\infty} a(\lambda) \exp(-\lambda t) d\lambda$$

When $\lambda > \lambda^\star$ there was no discrete mode as expected but the function $a(\lambda)$ in polycrystalline materials had a pronounced peak which behaved rather like a $\delta$ function. Hence an experimenter would observe a decay not much different from an exponential. As $\lambda$ increased the peak spread out and the decay became less and less like an exponential. Numerical calculations by Wood (29) using diffusion theory and realistic scattering kernels for graphite, beryllium and beryllium oxide had led him to the same conclusions. However neither theoretician really explained the importance of $\lambda_K$ although each was
\[ \lambda_{el}, \lambda_{el} \text{ derived from Lucas Heights Data (ref 5b)} \]

\[ \Sigma_{el} \text{ corrected for extinction} \]

Figure 3. The experimental values of the \( \lambda(b^2) \) relationship for BeO.
able to show that Kothari's ad hoc method of bounding the velocity away from zero was not necessary to explain some of the experimental results.

Borysiewicz and Mika (30) now extended the work of Albertoni and Montagnini to the case where there were elastic terms in the scattering kernel. The spectrum of this new operator turned out to be identical in form to the earlier one shown in Figure 2a. Their proof was limited to the case where the elastic scattering was isotropic but it seemed likely that anisotropic elastic scattering would not alter the result in any major way.

Ritchie and Rainbow (31) also extended their experimental results on BeO in the region of $\lambda^*$ and $\lambda_K$. Their major results are summarised in Figure 3. In the region between $\lambda^*$ and $\lambda_K$ decay curves could be described by an exponential within the experimental errors. Beyond $\lambda_K$ the decay constants changed with time at an increasingly faster rate as the buckling increased. The envelope drawn round their experimental points for $\lambda > \lambda_K$ indicated that the region $\lambda = \lambda_K$ was a transition region between different modes of decay. Hence any theory describing the physical processes should explain the transition region and also the fairly rapid change in decay constant above $\lambda_K$. Zhezherun (32) also reported non exponential behaviour in small BeO samples. However he claimed exponential decay in stacks which had decay constants much higher than the value of $\lambda_K$ derived from the first Bragg peak. He pointed out that primary extinction could reduce the elastic scattering cross section so much at low
energies that the minimum of \((v\Sigma_{\text{inel}} + vDB^2)\) occurred at the second Bragg peak. The corresponding value of \(\lambda_K\) is shown in Figure 3 where it can be seen that it is much larger than the value evaluated at the first peak. This underlines the importance that the physical form of the moderating material may have on the experimental results. The results of Zhezherun and other experimenters are discussed more fully by Rainbow and Ritchie (31).

The major features of Rainbow and Ritchie's results were qualitatively explained by the theoretical work of Conn and Corngold (33,34). These authors extended the technique used by Corngold and Durgun to the transport equation in which they assumed that the solution had to be of the form

\[
n(r,v,\Omega,t) = \varphi(B,v,\Omega,t) \exp(iB.r) \tag{2.7}
\]

and that the elastic scattering was isotropic. The transport operator \(A\) in equation 2.5 now took the form

\[
A = ivB\mu - v\Sigma + \int \int dv' d\mu' v' \Sigma(v' \rightarrow v, \mu' \rightarrow \mu)
\]

Conn and Corngold were able to show by considering those regions of the \(\lambda\)-plane on which the Laplace transform of \(\varphi(B,v,\mu,t)\) was analytic, that the spectrum of the above operator had the form shown in Figure 2c which has the scale based on scattering cross sections in BeO. The importance of this spectrum is that there are discrete poles beyond \(\lambda\) and that a point \(\lambda_{e1} \sim \lambda_K\) plays an important role in describing the evolution of the neutron population.
In general the decay of the population will be described by contributions from the cut $\Gamma_{SB}$, the poles, the line cut $\Gamma_{e1}$ from $\lambda_{e1}$ to $\lambda^*$, and the area to the left of $\lambda_A$. When $\lambda < \lambda^*$ the contribution from the pole will dominate at long times. When $\lambda^* < \lambda < \lambda_{e1}$ the contributions from $\Gamma_{SB}$ which represents the evolution of sub-Bragg neutrons will eventually dominate. However Conn and Corngold were able to show, by using a one term degenerate kernel, that the contribution from $\Gamma_{SB}$ was small at early times and combined with the contribution from the pole in such a way that the decay curve appeared exponential with an exponent slightly less than the value of the discrete pole. They were further able to show that as the buckling increased and the pole moved to the left there was quite a sudden transition from a situation in which the pole dominated to a situation in which the contribution from the line cut, $\lambda_{e1}$ to $\lambda_A$, dominated.

When the buckling was so large that the discrete pole disappeared into the line cut $\lambda_{e1}$ to $\lambda_A$ the decay was dominated at early times by the continuum from this line cut and the decay was markedly non-exponential.

Perhaps most important of all they were able to show that the effective decay constant for curves with $\lambda \gtrsim \lambda^*$ was quite well described by the continuation of the $\lambda(B^2)$ dispersion relation that described the poles to the right of $\Gamma_{SB}$. Hence an experimenter would measure a smooth $\lambda(B^2)$ curve as $\lambda$ increased through $\lambda^*$.

Yet further substantiation of these major features of the $\lambda(B^2)$ curve for $\lambda < \lambda^*$, $\lambda^* < \lambda < \lambda_K$ and $\lambda > \lambda_K$ has been provided by
Ritchie et al. (35) who have reported some multigroup diffusion theory calculations in BeO in which the leakage was approximated by a DB² term. The code used produced the energy spectrum as a function of time after the initial neutron pulse instead of the more usual lowest eigenvalue and its associated eigenfunction. The reaction rate corresponding to a 1/v detector was calculated and the reaction rates processed using the same methods as Rainbow and Ritchie. The "pseudo-discrete" region beyond λ* was predicted and the increased deviation from exponential behaviour beyond the Kothari -λel limit reproduced. Perhaps most interesting of all the calculations showed that between λ* and λK the rate of change of the reaction rate was not quite flat but decreased slowly with time getting closer and closer to λ*. However the time taken to approach λ* was very long (~78 ms) and well beyond the times accessible to experiment.

Once again there appears to be a measure of accord between theory and experiment. The physical picture is reasonably clear and can be described by the help of Figure 1. When λ ~ λ₁ < λ* there is at least one discrete mode of decay. When λ ~ λ₂ > λ* the system must eventually decay with λ* and the energy distribution is singular, peaking at ν = 0. However since in the region below the Bragg cut off the time between collisions is about 450 μs, the length of time for this distribution to be set up is very long and the experimenter is largely ignorant of the existence of these neutrons. However when λ > λK the neutron population will in the first instance continuously change its decay constant towards the value λK. Now since the mean
time between collisions for these neutrons is typically \( \sim 6 \ \mu s \) the process is quite rapid and easily seen in the times explored by the experimenter.

Immediately the question springs to mind. Why is \( \lambda_{el} \) important in the analysis of Conn and Corngold when on physical grounds one would expect \( \lambda_k \), the total loss rate in the region of the Bragg peaks? \( \lambda_{el} \) describes the loss rate of neutrons whose diffusion coefficient is determined purely by the elastic scattering cross section and does not describe a proper physical process in the system. A number of other questions soon follow.

The spectrum of the operator used by Conn and Corngold is significantly different from that of the very general operator studied by Borysiewicz and Mika. These last authors prove that the whole half plane \( \Re(\lambda) < \lambda^* \) belongs to the spectrum whereas Conn and Corngold prove that a large part of this area belongs to the resolvent set. The \( \exp(iB \cdot r) \) spatial distribution assumes one Fourier mode. If we use the same heuristic argument as before and consider the spectrum due to an infinity of \( B \) values, the spectrum of Conn and Corngold's operator has the form shown in Figure 2d. Unlike the case where the spectrum of Figure 2b tended in some way to that of Figure 2a Conn and Corngold's spectrum remains quite different from that of Borysiewicz and Mika. If we assume that apart from the restriction of isotropic elastic scattering, the operator considered by Borysiewicz and Mika adequately describes the physical situation in pulsed polycrystalline systems (but see the remark by Conn and Corngold (33)),
then we must assume that the operator studied by Conn and Corngold is different from that in the general transport equation. A possible clue is the restriction that Conn and Corngold put on the spatial variation of the neutron density. This could mean that their operator is a restriction (36) in the functional analysis sense of that used by Borysiewicz and Mika and may have quite a different spectrum.

There is the further point that the \( \exp(iB \cdot r) \) assumption is formally the same as carrying out a Fourier transform with respect to the space variable. In view of the continuous regions of singularities found by Conn and Corngold for the transform \( \phi(B,v,Q,\lambda) \) it seems reasonable to expect that the inverse transform of this with respect to space will contain integrals over the variable \( B \). In other words the experimenter will expect to measure a spatial distribution in which the buckling varies with time. Again it seems reasonable that when there is a discrete eigenvalue for \( \lambda > \lambda^* \) there will be either a discrete value of \( B \) or such a peak in the distribution function, after the form of Corngold and Durgun, that the experimenter will measure only one buckling. When there is no discrete mode with respect to \( \lambda \) then one would expect to see a continuous change in buckling. In other words the energy spectrum changes continuously with both space and time. This point has been dealt with in some detail by Williams (37).

This then is the motivation behind this thesis. How far is the use of one spatial mode justified in both the work of Conn and Corngold and Ritchie et al.? Does the buckling change with time for
Does the decay constant change with position at long times after the pulse in small assemblies? Do the higher spatial modes behave in the same way as the fundamental distributions with the same buckling? This last point is of particular interest since for intensity reasons the time dependent spectra in small systems have to be measured at short times after the pulse where there are still a number of higher spatial modes present.

The next chapter, Chapter 3, describes some experiments in BeO in which the time dependent spatial distribution was measured using a $1/v$ detector in a range of assemblies which had decay constants in the regions $\lambda < \lambda^*$, $\lambda^* < \lambda < \lambda_K$ and $\lambda > \lambda_K$. These distributions have been analysed to find if the shape of the spatial distribution changed with time and, if so, was the change correlated with changes in the behaviour of the $\lambda(B^2)$ curve. As part of this investigation the time dependent amplitudes of some of the higher spatial modes were determined. The variation of the effective decay constant as a function of both space and time was measured to see if there was very marked change of the decay constant with position at long times.

Chapter 4 describes some time dependent multigroup diffusion theory calculations of the space dependence of the $1/v$ reaction rate in BeO assemblies covering the range of sizes investigated experimentally. The time dependent spatial distributions were analysed in exactly the same way as the experimental ones to produce parameters that were directly comparable with experiment. The time dependent average mean free path was also evaluated to see if this could explain changes in shape of the spatial distribution.
Chapter 5 contains a discussion of the differences between the theoretical and experimental results and discusses some of the implications of the theoretical results.

Chapter 6 summarises the general conclusions.
3. MEASUREMENT OF TIME DEPENDENT SPATIAL DISTRIBUTIONS

The pulsed measurements described in this chapter were carried out on BeO assemblies varying in size from $55.88 \times 60.96 \times 60.96$ cm to $20.32 \times 30.48 \times 30.48$ cm representing a buckling range of $0.0083 \text{ cm}^{-2}$ to $0.0399 \text{ cm}^{-2}$ when normalised to a reference density of $2.96 \text{ gm cm}^{-3}$. The corresponding decay constants cover the range $1.0 \times 10^3 \text{ sec}^{-1}$ to $4.5 \times 10^3 \text{ sec}^{-1}$, which includes the two limits $\lambda_0$ and $\lambda_k$. Such measurements require a pulsed source with repetition rates variable in the range $250 \text{ c/s}$ to $80 \text{ c/s}$ and pulse lengths of about $500 \mu\text{s}$ to $2 \text{ ms}$ depending on the source strength available. Similarly the time analyser must be capable of sweep times from about $4 \text{ ms}$ to $12 \text{ ms}$ and channel widths from about $80 \mu\text{s}$ to $500 \mu\text{s}$. The pulsed neutron source and time analyser used in the experiments are described in Section 3.1 below where it will be seen that these units more than cover the range indicated above.

As indicated in the last chapter the experiments were designed primarily to measure time dependent spatial distributions but were also designed to measure the spatial variation (if any) of the decay constant in particular assemblies. The latter experiment is relatively simple and requires only a suitable detector free to move through the assembly as well as the pulsed source and time analyser. The spatial distribution measurements, however, require a movable detector and a means of normalising the count rate from it to some constant source strength. The details of the method used are given in Section 3.1 together with a description of the BeO.
assembly and the technique used to control the position of the movable detector.

Section 3.2 gives details of the experimental technique and the data analysis methods used while Section 3.4 describes the experimental results. An assessment of the errors in the various measurements is contained in Section 3.5.

3.1 Apparatus

3.1.1 Pulsed source

A pulsed neutron source for this kind of experiment should have the following properties.

(i) High intensity; $\sim 10^9$ neutrons/pulse into $4\pi$ seems necessary.

(ii) Low background level; the off pulse level to on pulse level needs to be better than $10^{-5}:1$.

(iii) Variable width and repetition rate so that these can be matched to the decay constant of the assembly under study.

(iv) Synchronous timing; this allows time analyser gates etc. to be operated at well defined times with respect to the neutron pulse.

(v) Reliability; the system should operate continuously for periods of 60 hours or more.

The first two properties are very important since the decay in small pulsed polycrystalline assemblies needs to be followed over at least 4.0 ms after the pulse. This represents in some cases $\sim 15$
Figure 4. Schematic diagram of 3 MeV Van de Graaff square wave pulsing system.
mean lives and greater than $10^6$ change in count rate. Hence the intensity must be high to have a reasonable count rate at the times of interest (~2.0 ms after the pulse) and the background low to see the decay at ~4.0 ms. The third property is useful to obtain optimum conditions when the intensity is marginal while the last two are not essential but greatly improve the ease with which the experiments can be done.

The pulsed neutron source for these experiments was obtained by bombarding a suitable target with the beam from a 3 MeV Van de Graaff accelerator. The major part of the modulation took place in the deflection chamber which was a standard piece of equipment (H.V.E.C. part no. D.K. TU-35) situated in the top terminal and which consisted of a focus electrode, two sets of deflection plates, an einzel lens and a chopping aperture. The beam entering the deflection chamber from the ion source passed between the two sets of deflection plates and was focussed on the chopping aperture which was a hole 0.125 ins in diameter (Figure 4). Any voltage applied to the deflection plates swung the beam away from the hole and modulated the intensity of the beam passing down into the accelerator tube.

A square wave pulse of about 1 kV was applied across the bottom set of deflection plates. The deflection voltage could be triggered to zero by an "on" command pulse fed to the top terminal through a pulsed light source and light pipe system. A similar "off" channel switched the deflection voltage back on. With such a system there was full control of the pulse width and repetition rate of the
pulsed beam. It was also a simple matter to drive the pulsed beam and any other timing equipment from a master control unit.

There was also a back up deflection system which operated on the high energy beam from the accelerator. This post acceleration deflection system applied a 10 kV deflection voltage across the beam during the off pulse period to remove any residual beam that had escaped from the deflection chamber into the accelerator tube. The deflection voltage was triggered to zero by the "on" pulse while the duration of the zero deflection was controlled by a monostable that was variable over the range 100 μsec - 2.0 ms. The post acceleration deflection system reduced the off pulse level to less than $10^{-5}$ of the on pulse level.

The details of both these systems are given by Fraser et al. (38).

The $^9$Be$(d,n)^{10}$Be reaction was used throughout the experiment to provide the source of neutrons. The deuteron energy ranged from about 2.5 mV to 2.9 mV depending very much on the performance of the Van de Graaff on the day of operation. At these energies the neutron yield is about $2 \times 10^9$ neutrons/sec/μA into $4\pi$ solid angle and in the $0^o$ direction the neutron energy distribution peaks at $\sim 2$ MeV (39) although there are neutrons with energies up to $\sim 7$ MeV. The target current in the peak of the pulse varied from about 300 μA to 1,000 μA depending on the size of the stack being investigated and to some extent on the performance of the ion source.

3.1.2 BeO assembly

The BeO assembly and its shielding is shown schematically in
Figure 5. Schematic diagram of Beryllium oxide assembly and associated shielding.
Possibly the most important part of the system was the shielding arrangement which was devised to minimise the problem of room return neutrons in the course of experiments (40) to measure the $\lambda(8^2)$ for BeO. The inner part of the shield was 1/4" Boral in intimate contact with the BeO to ensure that a vacuum boundary condition was achieved as closely as possible. Outside the Boral was a layer of about six centimetres of borated paraffin (equal parts by weight of paraffin and boric acid) and a further 1/4" of Boral. Boral, rather than the more usual cadmium, was used as a thermal neutron absorber since more than 25 per cent. of the $\gamma$ rays from neutron capture in cadmium have energies greater than the $(\gamma,n)$ threshold in beryllium and might give rise to a time dependent background. The success of the shielding arrangement is discussed in more detail by Ritchie (41) and Rainbow and Ritchie (31).

The whole assembly could move vertically and horizontally to allow the neutron source to be set at the centre of one face of the BeO block.

The BeO was in the form of sintered tiles 15.24 x 15.24 x 2.54 cms of average density 2.87 gm cm$^{-3}$. Some of the tiles had slots cut in the edges to allow the construction of probe holes of cross section 1.016 x 1.016 cm. These probe holes were invariably set up with their axes running parallel to the direction of the incoming neutron beam since this was the direction richest in higher spatial modes. The position of the probe holes with respect to the stack centre varied from assembly to assembly and will be given in
Figure 6. Schematic diagram of counter drive system and counter holders.
detail later. For future reference it is convenient to set up the system of cartesian coordinates that are shown in Figure 5.

The whole assembly was enclosed in a double walled polythene tent to avoid contamination of the surrounding experimental area from the BeO. The enclosed volume was temperature controlled and, partly helped by the good insulating properties of the polythene, could be kept to 22.0 ± 0.3°C.

3.1.3 Counter drive system

Basically the counter drive system consisted of a counter holder which was attached to a motorised screw. Two kinds of counter holders were used both of which were made of aluminium and both of which are shown schematically in Figure 6. Holder I slid in the bare probe hole whereas Holder II slid in an aluminium guide tube 0.853 cms in diameter of wall thickness 0.159 cms. Both counter holders were designed to carry 20th Century 1/4 inch 5 EB 70 BF₃ detectors.

Again two different drive systems were used (Figure 6) one of which was a much more sophisticated version of the other. The original system consisted of a lead screw which was moved by rotating a large nut. The nut could be rotated by a motor via a pulley or by hand. The position could be read off on a vernier scale attached to the lead screw.

The second system was essentially a motorised lathe bed, the counter holder being attached to the cross and vertical adjustment movements (see Figure 6) of the lathe. The motor was connected to the lead screw through a two speed gear box fitted with magnetic
clutches. These clutches and the power supply to the motor were controlled by microswitches operated by cams on the lead screw. The basic idea was that when the motor was energised by a push button control it would run until the lead screw had made a complete revolution then would be stopped by one of the microswitches. Just before the lead screw made a complete revolution another microswitch was operated which energised one of the magnetic clutches and reduced the speed of rotation by a factor of eight. This meant that the lathe bed approached its final position slowly, which prevented overrun of the system and led to greater precision in positioning.

The first counter drive system was essentially an analogue one as the counter could be positioned anywhere whereas the second system was essentially digital as the counter moved by an amount determined by an integral number of lead screw revolutions. The lead screw was designed so that one revolution moved the detector through 0.5 cms, this being the least displacement likely to be required in the experiment.

The digital nature of the more sophisticated counter drive allowed it easily to be interfaced to a PDP-7 computer. A code was written for the PDP-7 which allowed time analysis of detector events to occur until some preset level was reached (usually a fixed count from a monitor detector). The time analysis was stopped, the contents of the time analyser store read out, other relevant information such as the time taken by the run read out and the contents of the time analyser and all other scalers zeroed. The
Figure 7. Block diagram of counting, timing and automatic counter positioning system.
PDP-7 then signalled the counter drive system to move to a new position, the positioning information being supplied by a rev. counter attached to the lead screw of the lathe bed. The programme allowed any number of detector spatial positions to be specified and fed into the computer at the start of an experiment. At each position the PDP-7 carried out a string of commands that allowed time analysis and the completed time distributions to be typed out or punched on paper tape for further analysis before moving to the next position.

The complete system (shown diagrammatically in Figure 7) considerably eased the labour of this type of experiment. Some individual time distributions took about 1.5 hours to accumulate and a complete time-space distribution, of a total of 30 space points, some 45 hours. During this time the experiment and Van de Graaff essentially ran themselves requiring only periodic checking to make sure the results were sensible.

3.1.4 Electronics

The underlying scheme of the electronics was to:

(i) Allow time analysis of the output from the movable detector for a time determined by a preset number of monitor counts.

(ii) Allow repositioning of the movable detector and further time analysis for the same preset monitor count.

(iii) Allow the time analyser starting time to be varied at will with respect to the initial neutron pulse.
A block diagram of the electronics is shown in Figure 7.

The detectors used throughout the experiment were 20th Century 1/4 inch 5EB 70 BF₃ proportional detectors. The pulse amplifiers and discriminators were fairly standard and the types of unit actually used shown in Figure 7. The important feature of this part of the system was the "EHT box" which fed both the monitor and movable detectors from a common EHT unit so minimising the gain drift between the two systems due to changes in EHT. This could easily have been done by feeding both preamplifiers from the same EHT unit but it was found that the ceramic blocking capacitors used in these units were noisy at the EHT levels used in the experiment and that paper capacitors were more reliable although unfortunately more bulky. Moreover it was found necessary to differentiate the signal very hard at the input to the preamplifier to prevent pulse pile up and excessive shift in D.C. levels during the period when the neutron pulse was present. This necessitated changes in the values of the R.C. filters used to remove 50 and 100 c/s ripple from the EHT supplied to the detectors. The values of the components actually used in the "EHT box" are shown in Figure 7.

The time analyser was a multiscaling facility fitted to the PDP-7 computer. In this two fast scalers were operated alternately and emptied their contents into a temporary storage region in the PDP-7 core. The length of time the scalers scaled was determined by a digital timing system operated from a 10 M c/s crystal clock. At the end of the selected time period the scaler in operation was stopped
and its contents deposited in a given location using the data break facility of the PDP-7. The signal to stop the first scaler started the second scaler which counted while the transfer process was taking place. The second scaler in its turn was stopped, the first started and the contents of the second transferred to the temporary store. This sequence of events continued until a predetermined number of time channels was reached. At this point a string of commands was executed to transfer the data from the temporary store to a definite data region in the PDP-7 core.

It is obvious that there are three important times involved; the time taken in switching between scalers which will define the uncertainty in the channel widths, the time taken to transfer information to the buffer store which will determine the minimum channel width and the time taken at the end of each cycle to transfer information from the buffer store which will determine the dead time or more precisely the useless time between each time analyser cycle. The switching time on the scalers, gate fully open to gate fully shut, was $\sim 30 \text{ ns}$ which was negligible compared to the channel widths of $\sim 100 \text{ µs}$, while the minimum channel width possible was $3 \text{ µs}$ which was much less than that required in the experiment. The useless time between cycles worked out at $\sim 19 \text{ µs/channel}$ which meant care had to be taken in not using more channels than were really necessary.

The number of channels could be set at 16, 32, 128, 256 or 512, and their width changed from a nominal 1 µs (but see above) to 511 µs by 1 µs steps.
The delay boxes shown in Figure 7 were essentially fast scalers that scaled pulses from a digital clock. This digital clock had as its basic timing unit a 10 M c/s crystal oscillator which it counted down to produce timing pulses covering the range 10 M c/s to 100 c/s by decade steps. The delay box started to scale on the receipt of a signal and gave out a logic pulse when it reached the preset delay (1 to 99 clock pulses). The delay boxes could be used in cascade in such a way as to effectively multiply the delay times. This was a very useful attribute since there is a time jitter in each delay of half a clock pulse representing the average time difference between an asynchronous trigger pulse and the arrival of a clock pulse. Throughout the experiment the delay boxes were used in such a way as to give delays of up to 5 ms with a 1 µs jitter.

The monitor gate gated the logic pulses coming from the monitor discriminator. The gate was designed to work in conjunction with pulses from the digital delay boxes and was gated on the receipt of one pulse and ungated on the receipt of another. This system allowed the gate length and the timing of the gate period to be both accurately controlled and have long term stability.

3.2 Experimental technique

One of the parameters of interest in this experiment was the decay constant of the next higher spatial mode (the 211 mode). As this required a measurement of the asymmetry in the system it was important to know precisely the location of the centre of the stack and the centre of the sensitive counting volume of the BF$_3$ detectors.
Once these were known the detector could be set up at the centre of the stack and any asymmetry easily detected. It will be shown later that a total accuracy of better than 1 mm is required in locating the centre of the detector at the centre of the assembly.

The effective length of the BF$_3$ detectors was taken to be the length along the central anode between the guard rings, and was measured from X-ray photographs of the detectors. However it was found that such measurements could be in error by up to 1.5 mm both because the X-ray source was a point source with a consequent magnification of the detector as seen by the X-ray plate and because the X-ray plates distorted on development. To avoid both these problems a test piece was made with dimensions similar to the detectors and X-rayed with the detectors. Any distortion of the real physical dimensions was then easily checked. In practice the distortion varied between 1 and 2.5 per cent.

The actual length of the stack from front Boral plate to back Boral plate was measured by a specially constructed depth gauge which could be inserted into the same probe hole used to carry out the spatial distribution measurements. In practice it was found that the actual length of the stack varied from the nominal length by about 1 mm.

Once the stack dimensions and the distance from the counter end to the centre of the counter sensitive volume were known the detector was run into the probe hole until its end just touched the inside of the front Boral plate. It was then known how far the
counter had to be withdrawn to set its centre at the centre of the stack. This movement was straightforward on the earlier version of the counter drive system. On the later version the counter was withdrawn by the multiple of 0.5 cms which was closest to the position required. The final movement was done using the calibrated screw on the cross adjustment of the lathe.

3.2.2 Measurement of time dependent spatial distribution

When the detector had been set up at the centre of the stack, the machine output was adjusted to give ~2 per cent. dead time corrections in the first time channel of interest. At the same time the waveform of the pulses from the amplification system were monitored to make sure that there were no signs of gain shift or excessive overshoot following the initial neutron pulse. In all cases a 500 µs long neutron pulse was used. This gave a sufficiently high count rate in the experiments on large stacks and, since it was about twice the mean life of neutrons in the smaller assemblies, produced almost the maximum neutron density possible in these smaller stacks. The channel width of the time analyser was chosen as compromise between good counting statistics and the assumption that the change of the time distribution across the channel was linear. The choice of channel number depended on the total time of interest. Usually 16 channels were sufficient but in cases when a decay constant was also measured 32 or 64 channels were chosen to get a good estimate of the background.

Once the time analyser start time, the channel width and channel number were chosen, the time between the start of the neutron
pulse and the time at which the time analyser was ready to function again was defined. This time was typically 3.5 ms using 16 channels of 125 µs and a time analyser start time of 1.0 ms. This then represented the smallest time between neutron pulses. In practice a larger time between pulses was used to ensure that a vanishingly small number of neutrons were left in the system from the previous pulse. For the smaller assemblies the repetition time varied from 4.5 to 6.0 ms.

When the time analyser parameters and the repetition time had been fixed the gate opening time of the monitor was chosen to take place at a time when the count rate from the monitor detector changed almost exponentially with time. The gate length was chosen to give a monitor count similar to that in the first channel of interest in the time analyser. This meant that for this particular channel the statistical error in the monitor counts was the same order as that for the channel counts but at later times the statistical error in the monitor counts decreased relative to the channel counts.

The time distributions at each spatial point were measured for a total time that gave \( \sim 250,000 \) counts in the first channel of interest when the detector was at the position corresponding to the centre of the stack. This time varied from about 10 to 90 minutes depending on the size of the stack and the delay time after the initial pulse before observations started. The actual space points were chosen to give as wide a scan as possible but were limited by the physical size of the detector (\( \sim 6.6 \text{ cm} \) from end to centre of...
sensitive volume) and by the desire to keep at least two mean free paths from the edge of the system since boundary transients might then become important. (Wood and Williams (40), Ritchie (41)). In general a complete time-space distribution measurement consisted of two separate scans each of ~13 space points.

The temperature of the stack was monitored throughout the experiment and periodic counts were carried out with no source to check for noise and "pick up" in the system.

3.3 Data analysis

Two IBM 360-50 data processing codes, EXPFIT and FORFIT, were used to analyse the experimental data. The code EXPFIT could fit the time distributions measured at any point with the sum of an exponential and a background. A least squares criterion was used to obtain the best fit and the errors were derived in the usual way. This code and its variants have been described elsewhere (Ritchie (24), Rainbow and Ritchie (31)).

The other code FORFIT was essentially a code for Fourier analysing the spatial distributions. The normalised time distributions at each space point were first corrected for dead time in both the movable and monitor detectors. The spatial distribution at some given time (some specified channel number) could then be fitted with a number of Fourier modes, a weighted least squares criterion again being used to define the best fit to the data. Any combination of Fourier modes could be used. The extrapolated length of the stack could be treated either as a parameter to be fitted or else fed in as a fixed quantity.
TABLE 1 - Details of geometry, time range and counting statistics for the various measurements of time dependent spatial distributions

<table>
<thead>
<tr>
<th>Stack Dimensions (x, y, z) (cm)</th>
<th>Probe Hole (y, z) (cm)</th>
<th>Holder Type</th>
<th>Time Range Covered (ms)</th>
<th>Spatial Scan (cm)</th>
<th>Counting Statistics at Position Zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>55.88, 60.96, 60.96</td>
<td>10.16, 0.508</td>
<td>I</td>
<td>2.6 to 5.9</td>
<td>+20.0 to -20.0</td>
<td>1.5 (\times) 10^5 at 2.6 ms</td>
</tr>
<tr>
<td>55.88, 45.72, 45.72</td>
<td>2.54, 3.508</td>
<td>I</td>
<td>1.9 to 6.9</td>
<td>+20.0 to -20.0</td>
<td>2.0 (\times) 10^5 at 1.9 ms</td>
</tr>
<tr>
<td>50.8, 45.72, 45.72</td>
<td>2.54, 3.508</td>
<td>I</td>
<td>2.0 to 6.9</td>
<td>+18.0 to -18.0</td>
<td>4.5 (\times) 10^5 at 2.0 ms</td>
</tr>
<tr>
<td>35.56, 45.72, 45.72</td>
<td>2.54, 3.508</td>
<td>I</td>
<td>1.38 to 4.58</td>
<td>+10.2 to -10.2</td>
<td>5.0 (\times) 10^5 at 1.38 ms</td>
</tr>
<tr>
<td>30.48, 30.48, 30.48</td>
<td>5.08, -0.508 and 0.508, -5.08</td>
<td>II</td>
<td>0.26 to 3.05</td>
<td>+11.5 to -8.5</td>
<td>8.0 (\times) 10^5 at 0.26 ms</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2 regions</td>
<td></td>
<td>region 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4.0 (\times) 10^5 at 1.2 ms region 2</td>
</tr>
<tr>
<td>30.48, 20.32, 30.48</td>
<td>0.0, -0.508</td>
<td>II</td>
<td>0.26 to 3.05</td>
<td>+11.5 to -8.5</td>
<td>4.0 (\times) 10^5 at 0.26 ms region 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2 regions</td>
<td></td>
<td>4.0 (\times) 10^5 at 1.2 ms region 2</td>
</tr>
</tbody>
</table>
Details of the weights used in these two codes are given in Section 3.5 below.

3.4 Results

It is obvious from the experimental results of Rainbow and Ritchie (31) and the theoretical results of Conn and Corngold (33,34) that there are three regions of interest in the $\lambda(B^2)$ relationship of polycrystalline moderators.

I  The region $B^2 < B^2_0$, $B^2_0 \approx 0.02 \text{ cm}^{-2}$ in BeO

II  The region $B^2_0 < B^2 < B^2_{\exp}$, $B^2_{\exp} \approx 0.03 \text{ cm}^{-2}$ in BeO

III The region $B^2 > B^2_{\exp}$

where $B^2_{\exp}$ has been defined by Rainbow and Ritchie and approximates the buckling at which the $\lambda(B^2)$ curve meets the $\lambda_K$ curve of Kothari.

For this reason the time dependent spatial distributions were done in assemblies with buckling representative of these regions. The assemblies actually studied are given in Table 1, which also gives details of the probe hole positions, spatial points chosen and approximate values of the counting statistics at various times after the pulse.

The assembly chosen to represent the buckling region II was the smallest assembly to which Rainbow and Ritchie were able to assign an exponential mode of decay since it was felt that any departure of the spatial distribution in the "pseudodiscrete" region from that described by a buckling would be most noticeable in this assembly.
The assembly representative of region III was chosen as one having a buckling sufficiently high to provide a marked change of the decay constant with time but sufficiently low to obtain good counting statistics in a reasonable length of time.

### 3.4.1 The time dependence of the extrapolated length

If a buckling can in fact be used to describe the spatial variation of the flux in a pulsed polycrystalline assembly then at long times after the pulse we would expect the spatial dependence of the reaction rate measured in a parallelepiped to have the form (see for example 42)

\[
R(t,x,y,z) = A(t) \cos B_x x \cos B_y y \cos B_z z
\]  \hspace{1cm} (3.1)

where the origin is assumed to be at the centre of the assembly and the buckling \( B^2 \) is given by

\[
B^2 = B_x^2 + B_y^2 + B_z^2
\]

and

\[
B_i = \frac{\pi}{L_i}, \ i = x, y, z
\]

The parameters \( L_x, L_y, L_z \) are the extrapolated dimensions of the assembly and are related to the physical dimensions \( l_x, l_y, l_z \) by the relationship

\[
L_i = l_i + 2 \xi \hspace{1cm} i = x, y, z
\]
The parameter $\mathcal{E}$ is the extrapolation distance and is normally related (but see 43, 44, 45 and 46 for further discussion) to the average transport mean free path $\overline{\ell}_{tr}$ by the relationship

$$\mathcal{E} = 0.71 \overline{\ell}_{tr} \quad (3.2)$$

A spatial distribution measurement in say the $x$ direction at fixed $y$ and $z$ should give

$$r(t,x) = C(t) \cos B_x X$$

and a fit with a cosine should yield the parameter $L_x$. If the value of $L_x$ does not change with time then it is reasonable to assume that a buckling can describe the evolution of the time dependent energy spectrum in these assemblies. If the extrapolated length does change with time then either the form 3.1 does not describe the spatial distribution and the Fourier mode is only a good approximation to it or the distribution is a cosine but the spectrum changes sufficiently to change the value of the extrapolation distance derived from expression 3.2. Previous measurement (41) by the present author in BeO assemblies with $B_2^2 < B_m^2$ have shown that the spatial distribution can be described by a cosine for a long time after the pulse but that the value of the extrapolation distance changes from assembly to assembly. This can be attributed to the change in the asymptotic energy spectrum as the size of the assembly decreases and will be discussed in more detail later (Section 4.3.1).
Figure 8. Variation of the extrapolated length with time in assemblies of buckling $B^2=0.0122 \text{ cm}^{-2}$, $B^2=0.0292 \text{ cm}^{-2}$ and $B^2=0.0399 \text{ cm}^{-2}$. 
Figure 9. The variation of the extrapolated length as a function of the number of spatial modes and the number of space points in an assembly with $B^2=0.0399\text{ cm}^{-2}$. 
Figure 8a shows the variation of the extrapolated length $L_x$ in an assembly of buckling $0.01217 \text{ cm}^{-2}$ i.e. less than $B^2_x$ where the physical length $l_x$ was $55.98 \text{ cms}$. The results shown are those obtained by fitting the curve with both a cosine $(A_1)$ and sine $(A_s)$ term using the fitting routine described in Section 3.3 above since, as will be seen later the asymmetric component was still $\sim 2$ per cent. of the fundamental at $\sim 3 \text{ ms}$ after the pulse. However it is interesting to note that the results for the extrapolated length obtained by fitting with the fundamental alone agree within the errors of Figure 8a and in most cases are indistinguishable from the points plotted using the scale of Figure 8a. It can be seen that the results show no tendency within the errors to change with time over the time range $1.8 - 3.5 \text{ ms}$ and that hence a buckling can be used to describe the spatial distribution. This result was typical for all the assemblies with $B^2 < B^2_x$ in agreement with the previous observations (41).

Figures 8b and 8c show the variation of the extrapolated length $L_x$ in the assembly of buckling $0.0292 \text{ cm}^{-2}$ and $0.0399 \text{ cm}^{-2}$. Again the spatial distributions were fitted with a three parameter fit $(A_1, A_s$ and $L_x)$ since at $\sim 1.5 \text{ ms}$ the asymmetric component was still $\sim 1$ per cent. of the fundamental. Again the two parameter fit $(A_1, L_x)$ produced results which agreed within the errors for any particular spatial distribution measurement although there was a tendency at times between $1.0 \text{ ms}$ and $2.0 \text{ ms}$ for the two parameter fit to produce a lower value of $L_x$. This can be seen in Figure 9.
Because of the wide time range investigated in these two assemblies and the consequent large range of count rate (\( \sim \) four decades) the time range was broken up into the overlapping regions. This allowed measurements to be made at long times with good counting statistics while avoiding excessive dead time corrections at the earlier times. The first time region spanned the times 264 to 2312 \( \mu \text{s} \) after the pulse and the second the times 1064 to 3112 \( \mu \text{s} \) after the pulse, the second being the major period of interest in this particular measurement. In this time period three independent spatial distribution measurements were made in the assembly of buckling 0.0292 cm\(^{-2}\) and four independent measurements for the buckling of 0.0399 cm\(^{-2}\). Each measurement was made up of the two scans across the stack and the stack was disassembled and reassembled between each measurement. The results shown in Figures 8b and 8c are means of the separate measurements. The method of estimating the errors are dealt with in Section 3.5.

It can be seen that in these assemblies the value of \( L_x \) decreases with time until about 900 \( \mu \text{s} \) after the end of the pulse. The variation is more marked in the stack of buckling 0.0292 cm\(^{-2}\) than in the stack of buckling 0.0399 cm\(^{-2}\). This variation is almost certainly due to the fact that the spatial distribution was fitted with only a three parameter fit \( (A_1, A_3, L_x) \) whereas there is, as will be shown later, a measurable second cosine \( (A_c) \) component at these times. This component, which has a negative amplitude will tend to flatten the distribution and give an apparently large value of \( L_x \). It can be
TABLE 2 - Results of fitting time variation of $L_x$ in the assemblies of $B^2 = 0.0292 \text{ cm}^{-2}$ and $0.0399 \text{ cm}^{-2}$ with a constant and with a straight line

<table>
<thead>
<tr>
<th>Functional Form of Fit</th>
<th>Constant $a$ (cm)</th>
<th>$\delta a$ (cm)</th>
<th>Slope $b$ (cm/channel)</th>
<th>$\delta b$ (cm/channel)</th>
<th>Variance</th>
<th>Slope per cent/1,000 (\mu \text{s})</th>
<th>Time Range (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>30.48 x 30.48 x 30.48 cm Stack</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>33.229</td>
<td>0.024</td>
<td>—</td>
<td>—</td>
<td>1.03</td>
<td>—</td>
<td>1.4 to 2.9</td>
</tr>
<tr>
<td>Straight line</td>
<td>33.176</td>
<td>0.036</td>
<td>0.022</td>
<td>0.011</td>
<td>0.86</td>
<td>0.5</td>
<td>—</td>
</tr>
<tr>
<td><strong>30.48 x 20.32 x 30.48 cm Stack</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>33.333</td>
<td>0.026</td>
<td>—</td>
<td>—</td>
<td>2.21</td>
<td>—</td>
<td>1.4 to 2.3</td>
</tr>
<tr>
<td>Straight line</td>
<td>33.182</td>
<td>0.044</td>
<td>0.087</td>
<td>0.021</td>
<td>1.64</td>
<td>2.0</td>
<td>—</td>
</tr>
</tbody>
</table>
shown quite easily in these assemblies that a 0.5 per cent. \( A_c \)
component will produce a systematic increase in \( L_x \) of \( \sim 0.5 \) per cent.
and that the error is linear with changes in the \( A_c \) component. At
648 \( \mu \)s the \( A_c \) component in the larger stack \( (B^2 = 0.292 \text{ cm}^{-2}) \) is
\( \sim 0.5 \) per cent. whereas that in the smaller stack \( (B^2 = 0.0399 \text{ cm}^{-2}) \)
is \( \sim 0.2 \) per cent., both of which are consistent with the increased
value of \( L_x \) at these times. Attempts to fit the distributions to a
four parameter fit did not prove to be too useful for although the
results were in accord with those of the three parameter fit the
errors on the result were much larger. However the systematic
error in the extrapolated length \( L_x \) at early times was not considered
important since these measurements were done to find when a three
parameter fit was realistic, the region of real interest being for
times greater than \( \sim 1.0 \) ms.

It can be seen that for times from \( \sim 1.0 \) ms to about 1.5 ms
the results for \( L_x \) are sensibly constant with time or at least a
measurement limited to these times would lead one to this conclusion.
At times greater than \( \sim 1.5 \) ms there is a tendency for the values of
\( L_x \) to increase the tendency being more marked in the assembly of
buckling 0.0399 \( \text{ cm}^{-2} \) than in the assembly of smaller buckling.

Straight line fits were made to the experimental points and
the results summarised in Table 2. It can be seen that the best fit
to the 0.0292 \( \text{ cm}^{-2} \) data does indicate a slight increase with time
which amounts to some \( (0.5 \pm 0.25) \) per cent./1,000 \( \mu \)s. It is recalled
that Rainbow and Ritchie assigned an exponential decay to an assembly
if the decay constant changed by less than 0.5 per cent. at times greater than 1.5 ms, the effective time range for this assembly being 1.4 to 2.9 ms. Using that criterion this assembly could be described as having an indefinite buckling again using the language of their paper. The larger buckling shows a change of $(2.0 \pm 0.5)$ per cent./1,000 μs which would imply that a well defined buckling cannot be assigned to this assembly. If this result could be applied to all the dimensions there would be an increase of 4 per cent./1,000 μs in the buckling which, as we will see later, corresponds quite closely to the decrease in the decay constant of 5.2 per cent./1,000 μs at the centre and 3.7 per cent./1,000 μs at the edge of this assembly over the same time range.

It is obviously important to estimate the effect of boundary transients (Wood and Williams (40)) on the spatial distributions. Figure 9 shows the extrapolated length as a function of time in the stack of buckling $0.0399 \text{ cm}^{-2}$ when the fits were done using only 13 points, the points omitted being those at -10.0 cm and -11.5 cm which are the ones nearest the boundary. It can be seen that the omission of these points has little effect on the value of $L_x$.

### 3.4.2 Variation of decay constant with position

The analysis of Conn and Corngold and other theoreticians assumes that the decay rate is a property of the assembly as a whole and does not vary from position to position in the assembly once higher spatial modes have died away. However as was pointed out in Chapter 2, it is reasonable to expect that when there is no discrete
<table>
<thead>
<tr>
<th>Stack Size x, y, z (cm)</th>
<th>Probe Hole Position (y, z) (cm)</th>
<th>Time Range (ms)</th>
<th>Statistics at First Channel of Interest</th>
<th>Buckling (cm⁻²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>35.56, 45.72, 45.72</td>
<td>2.54, 3.508</td>
<td>1.4 - 4.6</td>
<td>3 x 10⁵</td>
<td>0.0164</td>
</tr>
<tr>
<td>30.48, 30.48, 30.48</td>
<td>5.08, -0.508</td>
<td>1.2 - 6.0</td>
<td>3 x 10⁵ region one</td>
<td>0.0292</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3 x 10⁵ region two</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>10⁵ region three</td>
<td></td>
</tr>
<tr>
<td>30.48, 20.32, 30.48</td>
<td>0.0, -0.508</td>
<td>1.0 - 5.5</td>
<td>7 x 10⁵ region one</td>
<td>0.0399</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1 x 10⁵ region two</td>
<td></td>
</tr>
</tbody>
</table>
mode of decay with respect to time then there will be no discrete mode with respect to space and the time dependent energy spectrum will vary with position in the stack. If the variation is very marked then the time dependent decay constants produced by Rainbow and Ritchie are not very useful parameters to measure since they will depend on counter position, orientation and size.

In fact Rainbow and Ritchie did look for this possibility by moving their detector positions, which were normally placed at the top centre of the assembly, to one edge of the stack but still with the sensitive volumes uniformly distributed about the centre line of the stack in the beam direction. They noted that the "decay constant" was $\sim 1$ per cent. lower in an assembly of buckling 0.0454 cm$^{-2}$ at the edge compared to the centre over the time range 1.0 - 2.0 ms. This did not appear to be a very marked variation. However it seems reasonable to expect any differences to be most marked along the beam direction. Hence the present set of measurements consisted of measurements of the time dependence of the "decay constant" at various positions along the beam direction.

The assemblies investigated, again representative of buckling regions I, II and III, are shown in Table 3 together with details of probe hole positions, spatial positions and the time ranges investigated. The decay constants were derived using the code EXPFIT (see Section 4.3) and the time dependence was estimated by letting the time at which the fit started vary but keeping the last time point used in the fit fixed. This was the technique used by Rainbow and Ritchie.
35.56 x 45.72 x 45.72 cm Stack

$B^2 = 0.0164 \text{ cm}^{-2}$

- Position $x = -10.2 \text{ cm}$
- Position $x = 0.0 \text{ cm}$
- Position $x = 10.2 \text{ cm}$

Channel width = 80 $\mu$s

Figure 10. Variation of the decay constant with position and time at $B^2 = 0.0164 \text{ cm}^{-2}$. 
Figure 11. Variation of the decay constant with position and time at $B^2 = 0.0292$ cm$^{-2}$

lla. Experimental results.
lib. Variation with time of the mean value of results from the three positions.
llc. Theoretical results.
Figure 10 shows the variations of the decay constant with time and distance in the stack of buckling 0.0164 cm⁻². From about 2.0 ms on the results at the three positions agree within the errors and do not show any variation with time whereas before 2.0 ms there is a significant difference between decay constants at the three positions. This is consistent with the concept of a higher spatial mode (the 211 mode) feeding neutrons in towards the centre from an initially asymmetric distribution. Within the errors on the measurement the results show, as is expected in an assembly with \( B^2 < B^2 \), that the decay constant does not change with position once higher spatial modes have decayed away.

The variation of the decay constant with time and position in the assembly of buckling 0.0292 cm⁻² is shown in Figure 11a. The general form is similar to that in the smaller buckling except that at early times there is no difference within the errors and there is a steady decrease of the value of the decay constant from 1.0 ms to 2.0 ms. From 2.0 ms to 2.9 ms the decay constant is essentially constant within the errors. There is a tendency for \( \lambda(8.5) < \lambda(0.0) < \lambda(-8.5) \) in particular at times after 2.0 ms. It is of interest to take the mean value of the results at times greater than 2.0 ms since such a result should be equivalent to a measurement with a long detector and comparable to the earlier result of Rainbow and Ritchie. This is shown in Figure 11b where it is obvious that the mean value does not change with time for times greater than 2.0 ms.
Figure 12. Variation of the decay constant $\lambda$ with position and time at $B^2 = 0.0399$ cm$^{-2}$.

12a. Experimental Results. 12b. Theoretical Results.
Figure 13. Decay curve at positions 0.0 cm and 8.5 cm at times greater than 1.5 ms after the pulse in the assembly with $B^2 = 0.0399 \, \text{cm}^{-2}$. 
The results for the assembly were derived from measurements using three overlapping time regions which spanned the times 1064 - 3112 µs, 1768 - 3112 µs, and 2064 - 6048 µs after the pulse.

Figure 12a shows the variation with space and time of the decay constant in the assembly of buckling 0.0399 cm$^{-2}$. The behaviour up to ~1.6 ms is much like that of the other two assemblies at relatively early times in that $\lambda(8.5) < \lambda(0.0) < \lambda(-8.5)$, although the change of decay constant with time is very marked. In the region of 1.6 ms the values of the "decay constant" at the three spatial positions are indistinguishable within the errors. This remains true for the values at 8.5 cm and -8.5 cm throughout the rest of the time range considered. The value of the decay constant at the centre position shows a tendency to fall below those at the edges, a tendency that becomes more marked as time increases. At 2235 µs after the pulse the difference is some 2.3 per cent. when the errors on the points are ~0.8 per cent.

Again the results were obtained from measurements in overlapping time regions, in this case the time regions covering the range 1049 - 3148 µs and 1549 - 3648 µs. In each region three separate time distributions were measured at the three different positions. Figure 13 shows the decay curve at the zero and -8.5 positions to give some idea of the counting statistics obtained and the backgrounds actually encountered in the experiment. The total counting time to obtain these distributions was some four and a half hours so that the backgrounds can be taken as an upper limit on all the measurements done since in all the other cases the running times were much less.
Figure 14. Decay of the fundamental and first two higher spatial modes derived from experimental data $B^2 = 0.0399 \text{cm}^{-2}$ and $B^2 = 0.0292 \text{cm}^{-2}$. 

Figure 14a
- $A_s$ Stack $20.32 \times 30.48 \times 30.48 \text{ cms}$
- $A_c$ Stack $B = 0.0399 \text{ cm}^{-2}$

Figure 14b
- $A_s$ Stack $30.48 \times 30.48 \times 30.48 \text{ cms}$
- $A_c$ Stack $B = 0.0292 \text{ cm}^{-2}$

Amplitude (Counts/Channel)

Time after End of Pulse (µs)
3.4.3 Amplitudes of the higher modes as a function of time

The behaviour of the decay constant as a function of time in the assemblies of buckling 0.0292 cm\(^{-2}\) and 0.0399 cm\(^{-2}\) would indicate that there is a non-negligible asymmetric component in the assemblies up to \(\sim 1.5\) ms after the pulse. The behaviour of the extrapolated length at early times in these assemblies would indicate that there is a second cosine component. Hence it is obviously of interest to analyse the time dependent spatial distributions and obtain the decay curve for the higher spatial modes. This can also be done for larger assemblies and the decay constants derived from the decay curves compared with the decay constants of the fundamental with nominally the same buckling.

Figure 14 shows the variations of the amplitudes \(A_s\), \(A_s\) and \(A_c\) in the stacks of buckling 0.0292 cm\(^{-2}\) and 0.0399 cm\(^{-2}\). In the case of the smaller buckling the results are derived from two separate measurements over the time ranges 264 - 2312 \(\mu\)s and 1064 - 3112 \(\mu\)s after the pulse, the results for \(A_s\) being normalised to each other by equating the \(A_1\) components. The results for the larger buckling were derived from spatial distributions in only the first time region. In both cases the value of the extrapolated length was fixed. It can be seen that for the smaller buckling \(A_s/A_1\) is \(\sim 1\) per cent. at 1.4 ms and is still \(\sim 0.7\) per cent. at 2.0 ms. This persistence of the higher spatial mode could explain the tendency at \(\sim 2.0\) ms and greater for the decay constant to differ with position. In the stack of larger buckling \(A_s/A_1\) is \(\sim 0.4\) per cent. at \(\sim 1.5\) ms which is consistent
Figure 15. Decay of the first higher spatial mode (2.1.1 mode) in assemblies with the fundamental value of $B^2$ less than $B_0^2$. 

- Figure 15a: Amplitude of 2.1.1 Mode, $A_{2.1.1}$ (counts/channel) vs. Time after Pulse (ms) for different stack sizes.

- Figure 15b: Decay curve with decay constant $\lambda = 3.4 \times 10^3$ sec$^{-1}$ and zero position shifted 0.2 cm and -0.08 cm.

Legend:
- 0: $55 \times 88 \times 60 \times 96$ cm Stack (x10)
- 1: $50 \times 80 \times 45 \times 72$ cm Stack
- 2: $55 \times 88 \times 45 \times 72$ cm Stack
- 3: $50 \times 80 \times 45 \times 72$ cm Stack
- 4: $50 \times 80 \times 30 \times 48$ cm Stack
with the fact that in this assembly the decay constant at opposite
sides of the stack is virtually the same after $\sim 1.5$ ms.

As noted earlier in both these assemblies there is a
measurable $A_c$ component at $\sim 500$ $\mu$s after the pulse. At a buckling
of $0.0292 \text{ cm}^{-2}$ $A_c/A_1$ is $\sim 0.1$ per cent. at $830$ $\mu$s after the pulse while
it is only $0.05$ per cent. in the assembly of buckling $0.0399 \text{ cm}^{-2}$.
These results would explain the high values of the extrapolated
length at early times.

Figure 15a shows the variation of $A_s$ as a function of time in
assemblies with fundamental buckling less than $B_2$. It can be seen
that in all cases the decay may be followed over at least one decade
and a reasonable estimate of the decay constant is possible. It
should be noted that in the case of the $55.88 \times 45.72 \times 45.72 \text{ cm}$
stack and more marked in the case of the $50.80 \times 45.72 \times 45.72 \text{ cm}$
stack the decay curve has a tendency to increase its slope with time.
Now the amplitude of the asymmetric mode is quite sensitive to the
error in locating the centre of the detectors at the centre of the
assembly. This appears as a change in amplitude of some $50$ per cent./$\text{mm}$
zero shift in the zero position $\sim 3.0$ ms after the pulse in the $50.80 \times
45.72 \times 45.72 \text{ cm}$ stack and a consequent $3.5$ per cent./$\text{mm}$ zero shift
change in the estimate of the decay constant. However the estimate
of the error in the central position is $0.3$ mm which would imply only
$\sim 1$ per cent. uncertainty in the decay constant. Figure 15b shows a
decay curve for this assembly in which the zero position has been
moved $0.12$ cm (four times the estimated error) in a direction to
TABLE 4 - Details of assembly sizes, probe hole positions and results for decay constant measurements of the 211 mode

<table>
<thead>
<tr>
<th>Stack Size (cm)</th>
<th>Probe Hole Position (cm)</th>
<th>Density (g cm(^{-3}))</th>
<th>Buckling of 211 mode (cm(^{-2}))</th>
<th>((\lambda - \lambda_3) \times 10^{-3}) (sec(^{-1}))</th>
<th>((\delta\lambda) \times 10^{-3}) (sec(^{-1}))</th>
<th>Time at which Decay Constant was Evaluated (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>55.88, 60.96, 60.96</td>
<td>10.16, 0.508</td>
<td>2.87</td>
<td>1.759</td>
<td>2.31</td>
<td>0.08</td>
<td>2.98</td>
</tr>
<tr>
<td>55.88, 45.72, 45.72</td>
<td>2.54, 3.508</td>
<td>2.87</td>
<td>2.144</td>
<td>2.94</td>
<td>0.01</td>
<td>2.46</td>
</tr>
<tr>
<td>50.8, 45.72, 45.72</td>
<td>2.54, 3.504</td>
<td>2.87</td>
<td>2.388</td>
<td>3.33</td>
<td>0.06</td>
<td>2.46</td>
</tr>
<tr>
<td>50.8, 45.72, 30.48</td>
<td>-0.508, 2.54</td>
<td>2.87</td>
<td>2.908</td>
<td>3.59</td>
<td>0.16</td>
<td>2.49</td>
</tr>
<tr>
<td>35.56, 30.48, 45.72</td>
<td>-0.508, 2.54</td>
<td>2.865</td>
<td>4.348</td>
<td>3.87</td>
<td>0.34</td>
<td>2.0</td>
</tr>
<tr>
<td>30.48, 30.48, 30.48</td>
<td>5.08, -0.508</td>
<td>2.865</td>
<td>5.844</td>
<td>5.48</td>
<td>0.08</td>
<td>0.75</td>
</tr>
<tr>
<td>30.48, 20.32, 30.48</td>
<td>0.0, -0.508</td>
<td>2.865</td>
<td>6.909</td>
<td>6.62</td>
<td>0.16</td>
<td>0.75</td>
</tr>
</tbody>
</table>

Note: 1) Decay constants normalised to a density of 2.96 g cm\(^{-3}\)
2) \(\lambda_a\) taken as \(1.96 \times 10^2\) sec\(^{-1}\)
Figure 16. Comparison of the $\lambda(B^2)$ dispersion relation for 2,1,1 mode with that for the fundamental.
minimise the tendency of the decay curve to increase its slope. It can be seen that the tendency for the slope to increase with time is still there. It is considered that the tendency is significant and should be contrasted with the tendency of the decay curve in the assembly of buckling 0.292 cm\(^{-2}\) to flatten out with time.

Table 4 lists all the assemblies in which the decay curve of the higher spatial mode was measured. It also shows the values of the decay constant of the higher mode and other parameter of interest such as densities, stack orientation and probe hole positions. In all cases the decay constants were estimated from graphs of the decay curves since this method is usually accurate to about 1 per cent. or so which was as accurate a technique as the errors in the amplitudes warranted.

Figure 16 shows the decay constants of the higher modes corrected for absorption and density and plotted on the \(\lambda(B^2)\) curve derived from earlier experiments (24,31). It is at once obvious that in the region of \(B^2 < B_{exp}^2\) the higher spatial modes have markedly higher decay constants than the fundamental modes of nominally the same buckling, the difference being greatest in the middle of the region. The behaviour for \(B^2 > B_{exp}^2\) is generally consistent with the fundamental except that at long times the decay constants of the higher mode in the two assemblies that could be investigated at long times, lie well below the envelope drawn to indicate the variation with time of the "fundamental".
Also shown in Figure 16 are three evaluations of Kothari's limit which, we recall, represents the lowest rate at which predominantly elastically scattered neutrons can be lost from the system. The lowest of the three is that reported by Rainbow and Ritchie (31) which was derived assuming that $\Sigma_{\text{inel}}$ varies as $1/v$ in the region of the Bragg cut-off and that the diffusion coefficient corresponds to ideal polycrystalline BeO. The other two evaluations are derived from the calculations of Ritchie et al. (35), one assuming ideal polycrystalline BeO and the other applying a correction for extinction. It can be seen that the decay constants for these assemblies with $B^2 > B^2_{\text{exp}}$ in which the decay constant could be evaluated at long times after the pulse, lie very close to the lowest value of Kothari's limit. This point will be developed later.

3.5 Estimates of errors

3.5.1 Errors in decay constants

The decay constant of a decay curve was determined either by using a least squares fitting routine, EXPFIT, or by fitting a straight line by eye to a log-linear plot of the decay curve. The first method was obviously the more accurate of the two and was used when investigating the variation of the decay constant with space and time where an accuracy of $\sim 0.5$ per cent. was required. The graphical method which is accurate to $\sim 1$ per cent. was used only in estimating the decay constant from the decay curve of the 211 spatial mode where the errors in the amplitude were such ($\sim 5$ per cent.) that a more accurate method was not justified.
The code, EXPFIT, used a weighted least squares criterion to find the best fit to the data. The weight of the $i^{th}$ time point was taken to be

$$w_i = 1/y_i$$

$y_i$ = the count in the $i^{th}$ time channel

since it was assumed that the random error due to the statistical error in the counts of the $i^{th}$ channel was always greater than the error due to changes in the time analyser channel widths. This seemed reasonable since the statistical error was generally about 0.2 per cent. in comparison with which variations in the time analyser channel width due to variations in the crystal frequency (0.01 per cent.) and variations in the frequency dividing system (estimated to be less than 0.01 per cent.) were negligible. The actual error in the decay constant was determined from the inverse matrix of the Normal Equations in the usual way.

The variance of fit was calculated for each set of parameters derived by EXPFIT. This function has the form

$$\text{VAR} = \sqrt{\sum_i w_i(y_{ie} - y_{ic})^2 / (N - M)}$$

$y_{ie}$ = experimental value at the $i^{th}$ point

$y_{ic}$ = calculated value at the $i^{th}$ point

$N$ = number of time points in the fit

$M$ = number of parameters in the fit
and should have a value close to unity if the differences between the calculated and experimental values of the decay curve at each point are close to their estimated errors. This was the case in all the fits done and justified the assumptions made about the random errors.

There was also the possibility of systematic errors appearing in the values of the decay constants due to

(i) dead time in the counters,

(ii) room return.

The maximum possible systematic error due to incorrect dead time corrections was estimated to be 0.05 per cent. The effect of room return was much harder to assess. Table 5 shows the results from an earlier experiment (Ritchie (24)) where the decay constant of an assembly of relatively large buckling was measured at three widely spaced points in time with different amounts of massive shielding in the vicinity of the assembly. In the first measurement the main source of reflected neutrons would have been the floor. In the last, the reflections from the floor, side walls and ceiling would have been of the same order. As there did not seem to be any systematic trends within the errors it was assumed that there was no systematic error due to room return.
TABLE 5 - Comparison of decay constants for a 22.86 x 60.96 x 60.96 cm assembly ($B^2 = 0.0203 \text{ cm}^{-2}$) with various configurations of concrete shielding

<table>
<thead>
<tr>
<th>Date</th>
<th>$\lambda(\text{sec}^{-1}) \times 10^{-3}$</th>
<th>$\delta\lambda(\text{sec}^{-1}) \times 10^{-3}$</th>
<th>Concrete shielding configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.10.65</td>
<td>2.8008</td>
<td>0.0027</td>
<td>Floor 2 ft. below stack</td>
</tr>
<tr>
<td>12.12.65</td>
<td>2.7973</td>
<td>0.0076</td>
<td>Floor + 2 ft. of concrete 2 ft. from stack near target</td>
</tr>
<tr>
<td>14. 4.67</td>
<td>2.8046</td>
<td>0.0030</td>
<td>Floor + concrete at target + 2 ft. of concrete 3 ft. from roof of assembly and 2 ft. of concrete 3 ft. from sides nearest the target</td>
</tr>
</tbody>
</table>

In view of the above comments on the relative importance of various random errors it is obvious from Figures 14 and 15 that the error in $A_s$, the amplitude of the 211 spatial mode, will be the dominating factor in determining the error in the decay constant of this mode. The error in the decay constants were estimated in the following way.

Let us assume that we have to fit a curve to the form

$$y_i = A \exp(-\lambda t_i)$$

i.e. $y_i = C - \lambda t_i$ ; $Y_i = \log y_i$
If we assume that the points all have the same weight and we use a least squares method to obtain the parameters C and \( \lambda \) then we get

\[
\begin{bmatrix}
N & \sum_i t_i \\
\sum_i t_i & \sum_i t_i^2
\end{bmatrix}
\begin{bmatrix}
C \\
-\lambda
\end{bmatrix}
= 
\begin{bmatrix}
\sum_i Y_i \\
\sum_i Y_i t_i
\end{bmatrix}
\]

and \( (\Delta \lambda)^2 = N(Y_{ie} - Y_{ic})^2 / (N-2) \left[ N \sum_i t_i^2 - \left( \sum_i t_i \right)^2 \right] \)

where \( \Delta \lambda \) = the estimated error in the decay constant

\( N \) = number of time points

\( Y_{ic} \) = calculated value of \( Y_i \) at the \( i^{th} \) time point

\( Y_{ie} \) = experimental value of \( Y_i \) at the \( i^{th} \) time point

Now in general the time points are equally spaced

\[
\sum_i t_i = N(N-1)/2 ; \quad \sum_i t_i^2 = N(N-1)(2N-1)/6
\]

Let us now assume that the errors in amplitude at each time point of the decay curve are the same and provide good estimates of the differences between the calculated and the experimental values of the decay curve

i.e. \( (Y_{ie} - Y_{ic})^2 = (\Delta Y_i)^2 = (\Delta y_i)^2/y_i^2 = (\Delta A/A)^2 \)
This was used to estimate the error in the decay constant of the 211 spatial mode from the errors in the amplitude calculated by the fitting routine FORFIT.

3.5.2 Errors in parameters derived from time dependent spatial distributions

The extrapolated length and the amplitudes of the various spatial modes were derived from the time dependent spatial distributions by the least squares fitting routine FORFIT. This routine also evaluated the errors in these parameters in the usual way from the inverse matrix of the Normal Equations derived from the least squares criterion. These errors only reflected the errors in the actual space-time distribution used and did not take account of the variations between different measurements of nominally the same time dependent spatial distribution. Hence we need to consider:

(i) errors appearing at each point of the spatial distribution,

(ii) variations between different measurements of the same spatial distribution.

The following make up the total error at each point of the spatial distribution.

(i) The statistical error in the count of the movable detector.
(ii) Changes in count rate in the movable detector caused by amplifier gain, EHT and discriminator bias drifts.

(iii) Errors in normalisation due to statistical error in monitor.

(iv) Errors in normalisation caused by gain, EHT, and discriminator drifts.

(v) Errors in normalisation caused by changes in timing and width of the monitor gate.

(vi) Error in positioning the movable detector.

The statistical error in the counts registered by the movable detector ranged from about 0.2 per cent. to 3 per cent. depending on the time after the initial neutron pulse at which the distribution was measured. In practically all cases the monitor counts were $2^{18}$ which implied a statistical error of about 0.2 per cent. In contrast with these the total errors due to gain, EHT and discriminator drifts were less than 0.05 per cent. and were neglected. The normalisation error contributed by the monitor gate was also ignored since the gate was operated synchronously with the beam pulsing system and had its width controlled by a crystal clock (accuracy 0.01 per cent.). The estimated error in the counter position was 0.03 cm and gave rise to errors of between 0.3 per cent. and zero depending on the gradient of the flux distribution at the point of interest.

These errors were compounded to estimate the weight of a particular point of the spatial distribution in the following way.
The weight \( w_i \) of the \( i^{th} \) spatial point was taken to be

\[ w_i = \frac{1}{e_i^2} \]

\[ e_i^2 = y_i^2 + \left(\frac{\partial y_i}{\partial x}\right)^2 \Delta x^2 \]

where \( e_i \) = estimate of the error at the \( i^{th} \) point of the spatial distribution

\( y_i \) = dead time corrected, normalised count at the \( i^{th} \) point

\( \Delta x \) = estimate of error in positioning the detector

The gradient of the spatial distribution was evaluated numerically from the points either side of the \( i^{th} \) point, the end points being treated as special cases. The correctness of these assumptions was again gauged by evaluating the variance of fit for each set of parameters.

The factors contributing to variations between parameters derived from different measurements of the spatial distributions were assumed to be the following.

(i) Error in the position of the centre of the stack.

(ii) Error in the density.

(iii) Systematic error due to stack boundaries.
(iv) Systematic error due to the presence of higher spatial modes.

(v) Systematic error due to the perturbation of the flux distribution by the counter.

The error in the central position was estimated to be $\pm 0.03$ cm and was checked by treating the central position as a variable in the weighted least squares fit. An error of 0.03 cm represents an error of about 0.2 per cent. in the value of the extrapolated length for the smallest assembly and about 0.02 per cent. for the largest assembly. The error in the density arises from the fact that the stacks were made out of tiles whose densities varied about the mean of the whole sample by about 0.5 per cent. However as the expected root mean square deviation from the mean was only 0.08 per cent. in the smallest assembly, the error in the density was neglected.

In the present series of experiments the major goal was to note any systematic change in the extrapolated length with time in a particular assembly. Hence since care was taken to measure the spatial distributions in the various assemblies in essentially the same way the systematic errors listed above can be neglected. However it has been shown elsewhere (Ritchie (41)) that the change of flux shape due to the detector is negligibly small. It has also been shown (see Section 3.4.1) that since the omission of space points closest to the boundary has negligible effect on the value of the extrapolated length, boundary transients can also be neglected.
In general the errors on any of the parameters were dominated by the errors on the spatial distribution measurement which in turn were dominated by the statistical error on the counts of the movable detector. Hence in most cases the errors quoted in the text are just those calculated by the fitting routine FORFIT. The most important exceptions are the spatial distributions measured at long times (greater than 1.5 ms after the pulse) in the assemblies of buckling $0.0292 \text{ cm}^{-2}$ and $0.0399 \text{ cm}^{-2}$. In these cases the results quoted for the variation of the extrapolated length with time were the mean of several measurements done at different times. It was assumed that the errors due to density variations, uncertainties in the central position etc. appeared as an increased spread of the various independent results about the mean. The error on the final result was calculated from the deviations about the mean in the usual way.

### 3.6 Summary of results

In assemblies with $B^2 > B_*^2$ and in particular $B^2 > B_{\text{exp}}^2$ the spatial distribution can be described at all times from immediately after to $\sim 2.0$ ms after the pulse using the concept of an extrapolated length and buckling in the traditional way. Most importantly the value of the extrapolated length does not change in any dramatic way with time so that the use of a buckling to describe leakage is physically realistic. This justifies at least on experimental grounds the use of a buckling by Conn and Corngold in their analytical work and Ritchie et al. in their numerical work. There is, however, a change in the extrapolated length of $\sim 2$ per cent./1,000 µs in the
time range 1.4 to 2.3 ms in the assembly of buckling 0.0399 cm$^{-2}$
which would imply a change of ~4 per cent./1,000 µs in the buckling
of that assembly and is comparable with the observed change in decay
constant of ~4 per cent./1,000 µs over the same time range.

The second major point is that the decay constants of the
higher spatial modes do not lie on the $\lambda(B^2)$ curve of the fundamental.
When $B^2_x < B^2 < B^2_{\text{exp}}$ the decay constants are markedly higher. When
$B^2 > B^2_{\text{exp}}$ the decay constants of the higher modes are reasonably
consistent with the behaviour of the fundamental. However at long
times the decay constant of $A_{211}$ in two of the assemblies lies well
below the envelope of the $\lambda(B^2)$ curve. This implies that the
asymmetry in these small assemblies persists for a very long time
($\sim 5$ fundamental mean lives) after the initial neutron pulse.

The last point considered to be significant is the
difference between the measured values of the "decay constant" at the
edges of the assembly and the centre in the assembly of buckling
0.0399 cm$^{-2}$ at long times after the pulse. The values either side
of the centre agree from ~1.5 ms after the pulse but from about that
time on till the end of the time range considered the value at the
centre decreases more rapidly. At ~2.2 ms after the pulse the
difference is some 2 per cent. when the error on the individual
measurements is about 0.8 per cent. This difference is barely
significant, compared to the errors, but the continuous trend is
believed to be significant.
4. COMPUTATION OF TIME AND SPACE DEPENDENT SPECTRA

4.1 Description of theory

The basic calculational tool used was the zero dimensional multigroup diffusion theory code TENDS which has been described elsewhere (47) but which will be outlined briefly again.

The code solves the multigroup set of diffusion equations

\[
\frac{1}{v_i} \frac{\partial \phi_i}{\partial t} - D_i \nabla^2 \phi_i + \Sigma_i \phi_i = \sum_{j=1}^{N} \phi_j \Sigma_{j \rightarrow i}
\]  

(4.1)

where the quantities \( v_i, D_i, \Sigma_i \) have the usual significances and are suitably averaged quantities over the energy interval of the \( i^{th} \) group. The quantity \( \Sigma_{j \rightarrow i} \) is the average of the scattering kernel.

The assumption is made that the spatial distribution can be described by a buckling and the leakage by a \( DB^2 \) term. Hence equation 4.1 becomes

\[
\frac{\partial \phi_i}{\partial t} = \sum H_{ij} \phi_j
\]

(4.2)

where \( H \) is the matrix with elements

\[
H_{ij} = v_i \left[ \delta_{ij}(D_i B^2 + \Sigma_i) - \Sigma_{j \rightarrow i} \right].
\]

Now 4.2 can be written in operator notation as

\[
\frac{\partial \phi}{\partial t} = H \phi
\]

(4.3)
where $\phi$ is a vector. It is easily verified that the formal solution of 4.3 is

$$\phi(t) = \begin{cases} 
I - \exp(-tH) & t < P \\
\exp(- (t-P)H) & t > P 
\end{cases} \phi(0)$$

where $\phi(0)$ is the source spectrum and the source is assumed to be a square pulse of length $P$. For small values of $t$, $\exp(-tH)$ was evaluated by expanding the exponential as a Taylor series. For larger values of $t$ the semi-group properties of $H$ were used, namely,

$$\exp\left[(t_1 + t_2)H\right] \phi(0) = \exp t_1 H \exp t_2 H \phi(0).$$

It should be noted that the code did not search for the eigenvalues and eigenfunctions of $H$ and reconstruct the source spectra as a sum of the eigenfunctions but effectively followed the evolution of the energy spectrum through the pulse and at times after the pulse. It can be shown (48) that this treatment is formally the same as an eigenfunction expansion. In particular at large bucklings when the multigroup method is in fact sampling eigenvalues from a continuum and there is little numerical difference between the eigenvalues, the above technique is much more economical of computer time.

Since the leakage is described by a buckling term there is no reason in principle why the code should not calculate the time
dependent energy spectra for the bucklings corresponding to higher Fourier modes. The contributions from each mode can then be added together at each space point, the weight being determined by the value of the space point (e.g. asymmetric modes will obviously have zero weight at the centre of the stack) and by the amplitude of each mode in the initial spatial distribution.

This technique which has been used elsewhere (see for example 49, 50) allowed the energy spectra to be calculated as a function of both space and time after the initial pulse. The flux in each group had the form

\[ \phi_i(t,x,y,z) = \sum_n A_n \phi_{in}(B_n^2 t) \chi_n(x,y,z) \] (4.4)

\[ \chi_n(x,y,z) = \sin \frac{k\pi}{L_x} (x + L_x/2) \sin \frac{\pi}{L_y} (y + L_y/2) \sin \frac{m\pi}{L_z} (z + L_z/2) \]

\[ = \sin B_k (x + L_x/2) \sin B_l (y + L_y/2) \sin B_m (z + L_z/2) \] (4.5)

\[ B_n^2 = B_k^2 + B_l^2 + B_m^2 \]

where the \( A_n \) were determined by the initial spatial distribution and it was assumed that the assembly was a parallelepiped of extrapolated dimensions \( L_x, L_y, L_z \) with the centre situated at the centre of the assembly. The initial distribution was assumed to be a delta
function situated at the centre of that face of the stack nearest
the neutron producing target. The $\phi_{in} (B^2_n, t)$ were derived in
individual TENDS runs and the addition done in a code TIMEX.

The form 4.4 allowed the reaction rate of a detector to be
calculated as a function of both space and time, the reaction rate
having the form

$$R(t, x, y, z) = \sum_i w_i \phi_i (t, x, y, z)$$

where the $w_i$ were group averaged cross sections for the detector.
In particular the reaction rate of a $1/v$ detector could be calculated.
This could be evaluated at the space and at time points identical to
or close to those encountered in the experiment. The resulting time
dependent spatial distributions were then directly comparable with
the experimental data and could be processed in the same way, using
the codes EXPFIT and FORFIT, to yield the same parameters. Any
systematic errors, such as the neglect of the second cosine at early
times, then appeared in both results and the parameters were directly
comparable.

It was also a simple matter to calculate the $1/v$ reaction
rate corresponding, say, to all those neutrons below the Bragg cut-
off. It was then possible to see just what proportion of the
detector response was due to neutrons of various energies.

It is as well to emphasise at this stage the two important
assumptions inherent in the above approach. Using a finite
number of Fourier modes to describe the spatial distribution is
rather like describing the evolution of the neutron population by a small number of discrete time eigenfunctions. If there is a discrete time mode then the time evolution is described well. If there is a continuum then many modes are required. Similarly if the spatial distribution is close to a cosine or a sum of cosines then the use of a few Fourier modes is reasonable. If the spatial distribution is rather different from a cosine many Fourier modes may be required.

The approach also inherently assumes that the neutrons change energy whilst "marking time". The Fourier mode is set up and the neutron energy spectrum in that mode evolves with time but the neutron spatial distribution still remains the same. The approach assumes that the subsequent summing of the different Fourier modes will describe the actual neutron transport through the system. This is reasonable if the neutrons do diffuse through the assembly but many neutrons are created which have mean free paths the same order as the assembly dimensions. For these neutrons the assumption that the neutrons slow down in the one spot is quite invalid. This point will be developed later.

4.2 Data

There are a number of fairly distinctive features of the \( \lambda(B^2) \) curve for polycrystalline moderators that should provide a check on the validity of the cross section set being used to describe the time evolution of the energy spectrum in small pulsed systems. These features will be discussed below. The actual cross section set used
in the present calculation will be described and its ability to predict the features of the $\lambda(B^2)$ curve discussed.

Conn and Corngold (34) have shown that above the limit point $\lambda^*$ there is another limit which they called

$$\lambda_{el} = (v\Sigma_{inel} + vD_{el}B^2)_{v=v_B}$$

where $D_{el}$ is the diffusion coefficient for purely elastically scattered neutrons. This limit is slightly different from the limit proposed by Kothari

$$\lambda_K = (v\Sigma_{inel} + vDB^2)_{v=v_B}$$

where $D$ is the diffusion coefficient of neutrons scattered both elastically and inelastically. However it can be seen from Figure 3 that these limits are not much different and for the present purposes of checking the usefulness of cross section sets will be assumed the same. Conn and Corngold have shown that $\lambda_{el}$ separates the region of pseudodiscrete decay from that of non exponential decay and further that the transition from one mode to the other is quite sharp (noticeable within a $\delta\lambda$ of $\sim 100$ sec$^{-1}$ when $\lambda \sim 4,000$ sec$^{-1}$) and indeed this transition is seen experimentally (31). Hence any cross section set used should be able to predict the line which separates these time regions. Assuming $v_B$ is well known a correctly predicted $\lambda_{el}$ provides a check on the inelastic cross section and the value of the transport mean free path at the Bragg cut-off.
In most common polycrystalline systems the decay of the system is dominated by leakage. For example, in a BeO assembly 30.48 x 30.48 x 30.48 in (B^2 = 0.0292 cm^2) the leakage contributes 92 per cent. of the loss rate in the lowest decay "mode". In particular this is true for assemblies with decay constants below \( \lambda^* \) where a discrete decay mode exists and where a \( \lambda(B^2) \) curve is meaningful. In these assemblies the slope of the \( \lambda(B^2) \) curve is determined almost entirely by the diffusion coefficient which in turn is determined almost entirely by the value of the elastic cross section in the energy range between the Bragg cut-off and about 0.05 eV. Hence any evaluation of the elastic cross section should predict the diffusion coefficient derived from decay constants below \( \lambda^* \). Moreover for systems with decay constants above \( \lambda_{el} \) the rapid change of the "decay constants" with time will be determined by the elastic cross section, in particular the elastic cross section in the region just above the Bragg cut-off. Here again are two more features that the cross section set should predict.

In systems which have apparent decay constants above \( \lambda^* \) the decay constant will change with time until eventually it reaches \( \lambda^* = (v\Sigma_{inel})_{min} \). Now the theory of neutron scattering in polycrystals shows (see for example 15, 46) that this minimum will occur at zero velocity and further that in the region of zero velocity \( \Sigma_{inel} \) is proportional to \( 1/v \). In graphite for example, (51) \( \Sigma_{inel} \) is proportional to \( 1/v \) at energies below \( \sim 10^{-3} \) eV. In BeO experimental information in this energy region is sparse but it seems safe to
assume that by $10^{-4}$ eV the inelastic cross section is proportional
to $1/v$. Hence our cross section set should reflect the $1/v$
behaviour at low velocities.

From the above it can be seen that we have to be able to
calculate the transport cross sections at energies from just above
the Bragg cut-off to about 0.05 eV and ensure that the inelastic
cross section is $1/v$ below about $10^{-4}$ eV.

Now BeO is a purely coherent scatterer so that above the
Bragg cut-off there is both inelastic and elastic scattering, the
latter of which dominates in the energy region of interest, while
below the Bragg cut-off there is only inelastic scattering. The
coherent elastic term and the elastic part of the transport cross
section have a relatively simple form (52, 53) and are easy to
calculate. The coherent inelastic term is however very messy to
calculate because of the interference terms and, as is usual, the
assumption was made that the inelastic cross section is adequately
reproduced by assuming that BeO is a purely incoherent scatterer.
Again the total incoherent inelastic cross section is messy to
calculate and it is better (54) to compute the total incoherent
cross section and subtract the incoherent elastic term which again
has a simple form. The total cross section and transport cross
section at any energy were then taken to be

$$
\sigma_t = \sigma_{el}^{coh} + \sigma_{t}^{inc} - \sigma_{el}^{inc} + \sigma_a
$$
Figure 17. 17a. Total cross section for BeO used in calculations compared with some experimental values.
17b. Multigroup values of the transport cross section derived from exact and extinction corrected elastic cross section data.
\( \sigma_{tr} = \sigma_{coh}^{t} - [\mu\sigma]_{coh}^{t} + \sigma_{inc}^{t} - [\mu\sigma]_{inc}^{t} - \left\{ \sigma_{coh} - [\mu\sigma]_{coh}^{t} \right\} + \sigma_{a} \)

where for example

\[ \sigma_{coh}^{t}(E) = \int d\mu \sigma_{coh}^{t}(E,\mu) \]

\[ [\mu\sigma]_{coh}^{t} = \int d\mu \mu\sigma_{coh}^{t}(E,\mu) \]

The \( \sigma_{inc}^{t} \) and \( [\mu\sigma]_{inc}^{t} \) were obtained from the code PIXSE (55) as point cross sections and the other scattering terms were evaluated from the simple analytic expressions for them. The details of these and the values used for the lattice parameters of BeO are described by Maher and Trimble (56). The frequency distribution used for BeO was that due to Sinclair (57) and the \( S(\alpha,\beta) \) table for PIXSE was prepared using the code LEAP (58). Figure 17a shows a comparison between calculated value of the scattering cross section and experimental values (59,60,61). It can be seen that above the Bragg cut-off the comparison is very good but below the Bragg cut-off the experimental values are higher than the theoretical values. This point will be discussed later. The quantity \( v\varepsilon_{inel} \) is shown in Figure 17b where it corresponds to the loss rate curve for \( B^2 = 0 \) and where it can be seen that \( v\varepsilon_{inel} \) is sensibly constant at energies below about \( 2 \times 10^{-4} \) eV.

The diffusion coefficient was derived from the transport cross section in the normal way by taking
and the group averaged values of $D_i$ in equation 4.1 derived by averaging the point values of the diffusion coefficient over a flat spectrum. The quantities $\Sigma_{i \rightarrow j}$ were obtained from PIXSE using the group averaging option of that code and the group boundaries of the 41 group set that were used in the TENDS calculations. This set spanned the range $1.125 \text{ eV}$ to $9.36 \times 10^{-6} \text{ eV}$ with 18 groups below the Bragg cut-off and 23 above the Bragg cut-off, the fineness of the group structure justifying the use of a flat spectrum average. The details of the group boundaries are given in Appendix A.

It should be noted at this stage that there are two processes both of which depend on the physical nature of the polycrystalline sample and both of which can modify the magnitude of the scattering cross section in the energy regions of interest. The first of these is primary extinction which is dealt with in some detail by Bacon (62). Rather than considering this effect as the attenuation of the primary neutron beam penetrating the sample it is more illuminating to consider it as multiple reflection of the beam in the sample. Such a process is possible and indeed likely since the reflected beam satisfies the Bragg reflection conditions for the same lattice planes that cause the initial reflection. This multiple scattering will increase the transmission of the neutron beam and produce a decrease in the effective cross section of the sample. The decrease in the cross section will depend on the size of the sample and the reflection
coefficient at the energy under consideration, being most pronounced at energies just above the Bragg cut-off where the reflection coefficients are highest. In a polycrystalline sample there are crystallites of various sizes depending on the way the sample was produced. Hence we can expect the actual elastic scattering cross section of a sample to differ from that of an ideal polycrystal by an amount depending on the average crystallite size in the sample.

The second process will modify the cross section below the Bragg cut-off. As noted above the elastic cross section is zero below the Bragg cut-off for an ideal polycrystalline sample. However the samples used in most pulsed experiments are either sintered or cold pressed both of which processes leave small voids in the sample the size of these voids depending on how close the sample is to the theoretical density. Now a void in a material will diffract (63,64) a neutron beam, the angle of scatter being given approximately by the relation \( \Theta = \lambda / D \). Hence we can expect in a sintered material fairly strongly forward peaked elastic scattering below the Bragg cut-off, the mean angle of scatter increasing as the energy decreases for a fixed void diameter. With voids \( \sim 100 \) \( \AA \) in diameter, which are known (61) to occur in samples of BeO with density between about 94 and 98 per cent. theoretical, the angle of scatter at a neutron energy corresponding to 9 \( \AA \) \( (\sim 10^{-3} \) eV) will be \( \sim 5^\circ \) and easily seen by a transmission measurement of the scattering cross section.

Zhezherun et al. (65) have reported a number of measurements of the scattering cross section of BeO as a function of energy for...
samples with grain sizes ranging from 8 µm to 29 µm. The sample of BeO used in the present set of experiments had an average grain size of 18 µm which fortunately lies within the range examined by Zhezherun. Correction factors for the various Bragg peaks were interpolated from Zhezherun's results and applied to the scattering cross section calculation for an ideal polycrystal.

The problem of estimating the effect of voids is harder since no set of measurements comparable to Zhezherun et al. has been reported on the scattering cross section below the Bragg cut-off for samples of different densities. There is also no real guide from pulsed neutron work. Hence it was decided that this effect would be neglected in preparing the cross section set.

This cross section set, with extinction corrected elastic scattering and using the incoherent approximation for inelastic scattering, has been used (35) in conjunction with the code TENDS to calculate the "decay constants" as a function of time in BeO assemblies over the same buckling range investigated by Rainbow and Ritchie (31). The agreement between calculation and experiment is excellent over the whole buckling range (see Figure 18). Hence it would appear that this data reproduces all the important features of the $\lambda(B^2)$ curve. In particular it should be noted that group averaged diffusion coefficients calculated assuming a perfect polycrystalline material underestimate the slope of the $\lambda(B^2)$ curve (cross section too high) for $\lambda < \lambda^*$ and predict a much smaller spread in the variation of the decay constants when $\lambda > > \lambda_e$ than is seen experimentally.
Figure 18. Comparison of experimental $\lambda(B^2)$ relationship for BeO with various theoretical predictions.
The detailed agreement between the calculated and experimental values of the $\lambda(B^2)$ curve, both below and above $\lambda^*$, would lead us to believe that we have adequately calculated both $\Sigma_{el}$ and $\Sigma_{inel}$ in the energy range of interest. However there are some puzzling features that require comment.

Shown in Figure 18 are the values of $\lambda_K$ corresponding to the extinction corrected multigroup data, uncorrected multigroup data, extinction corrected point data, uncorrected point data, and a value quoted by Rainbow and Ritchie (31) which uses uncorrected data and assumes that $\Sigma_{inel}$ varies as $1/v$ in the region of the first Bragg peak. It can be seen that the multigroup values are markedly higher than the point values. This is to be expected since the averaging process in the evaluation of the group data will reduce the maximum values of the cross section in the elastic peaks. It can also be seen that the values using point data cut the $\lambda(B^2)$ curve in a region well beyond that at which Rainbow and Ritchie observed the onset of non-exponential behaviour. The slope of the line $\lambda_{el}$ is even greater than that of $\lambda_K$ and will consequently cut the $\lambda(B^2)$ curve even further from the transition region. It should be recalled that Conn and Corngold (34) predict a mode switch when the effective decay constant comes within $\sim 100$ sec$^{-1}$ of $\lambda_{el}$. Using the extinction corrected values this would lead us to expect a transition region at a buckling of at least $\sim 0.035$ cm$^{-2}$ which is markedly higher than the observed value of $\sim 0.03$ cm$^{-2}$. The value of $\lambda_K$ which assumes $\Sigma_{inel}$ varies as $1/v$ at the Bragg cut-off does, on the other hand, cut the $\lambda(B^2)$ curve very close to the transition region.
The excellent agreement of the calculations with the experimental results but the poor prediction of the transition region using the values of $\lambda_K$ (or $\lambda_{el}$) indicate that either the transition region is not in fact sensitive to the position of these limits or that the present data predicts the wrong value of $\lambda_K$. The value of $\lambda_K$ which does predict the transition region well assumes $\Sigma_{inel}$ varies as $1/v$ at the Bragg cut-off and consequently has a much lower value of $v\Sigma_{inel}$ ($\sim 2.75 \times 10^3 \, \text{sec}^{-1}$) at the first Bragg peak than the other evaluations which do not show $\Sigma_{inel}$ varying as $1/v$ in that energy region and predict a value of $\sim 3.1 \times 10^3 \, \text{sec}^{-1}$ for $v\Sigma_{inel}$ at the first peak. It is possible that the inelastic cross section is even lower than that predicted by the present cross section data and hence lower still than the experimental measurements.

Tewari and Trikha (66) have done similar time dependent multigroup diffusion theory calculations of the flux in pulsed BeO assemblies. It is of interest to discuss the cross section set they used and compare their predictions of the $\lambda(B^2)$ curve with the present ones. These authors calculated the inelastic cross section in the same way but used a Debye spectrum with $\Theta_D = 855^0\text{K}$ (and $950^0\text{K}$ in a later calculation) to describe the frequency spectrum of BeO instead of that due to Sinclair (57). Their frequency distribution predicts a higher value of the inelastic cross section below the Bragg cut-off than Sinclair's distribution and is more in accord with the experimental results in this region. The increased value of the inelastic cross section below the Bragg cut-off will of course lead to a much
higher value of $\lambda^*$ and Tewari and Trikha quote a value $(3.99 \times 10^3$ sec\(^{-1}\)) with $\Theta_D = 855^\circ K$ and $3.38 \times 10^3$ sec\(^{-1}\) at $\Theta_D = 950^\circ K$ for this parameter. The first of these is significantly higher than the value of $\lambda_{exp}$ of $\sim 3.65$ sec\(^{-1}\) at which Rainbow and Ritchie see the onset of non-exponential behaviour while the second lies slightly below. Tewari and Trikha also predict changes with time of the effective decay constant in assemblies of large buckling and compare these with the results of Rainbow and Ritchie. It should be noted that they calculate the instantaneous decay constant which will be rather different from the quantity derived by Rainbow and Ritchie who fitted an exponential to the decay curves from progressively later times after the pulse while keeping the last point of the decay curve fixed. Such a method will tend to produce lower values of the effective decay constant than the instantaneous decay constant if the decay rate decreases with time. Hence the fact that Tewari and Trikha's values of the decay constant at a buckling of $0.04$ cm\(^{-2}\) lie below those quoted by Rainbow and Ritchie for both $\Theta_D = 855^\circ K$ or $\Theta_D = 950^\circ K$ would indicate that their cross section set underestimates the decay constants at large buckling. This is consistent with the apparent underestimate of the decay constants below $\lambda_{exp}$ and with the fact that they used the elastic scattering from an ideal polycrystal to calculate their diffusion coefficient. Their estimates for $\lambda_K$ using both $\Theta_D = 855^\circ K$ and $\Theta_D = 950^\circ K$ are well above those using the present data set. This means that their cross section set certainly predicts too high a value for the transition region if in fact they
associate the experimental transition region with the point where $\lambda_k$ cuts the $\lambda(B^2)$ curve.

In general the data described in the present work gives much better agreement with the experimental results of Rainbow and Ritchie and using that as a criterion, is more credible than the data of Tewari and Trikha.

4.3 Results

4.3.1 Variation of extrapolation distance with time and buckling

The method of calculation assumes that the leakage is given by a $DB^2$ term with the buckling derived from the prescription in equation 4.5. In all cases the actual dimensions of the assemblies under study were used as input data to the program TIMEX and the extrapolation distance was derived from the earlier time dependent spatial distribution measurements (41). Now since it is assumed that higher spatial modes will obey the same $\lambda(B^2)$ curve as the fundamental, these modes should vanish at long times. The spatial distribution will then be described by the lowest mode. Obviously then the method of calculation cannot predict departures of the spatial distribution from a cosine like shape at long times after the pulse. However since in small assemblies the method does predict changes in the energy spectrum with time during the times of interest (up to 3.0 ms after the pulse) the average mean free path can be calculated as a function of time to see if it changes markedly in any way. If there is such a change it implies a change in the extrapolation distance and hence a change in the extrapolated length for the assembly.
<table>
<thead>
<tr>
<th>Stack Dimensions (x,y,z)</th>
<th>Number of Fourier Modes</th>
<th>Density (gm cm(^{-3}))</th>
<th>Ext. Distance (cm)</th>
<th>((v\Sigma_a)^{-1})</th>
<th>Space position at which flux was evaluated</th>
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</thead>
<tbody>
<tr>
<td>55.88, 60.96, 60.96</td>
<td>19 Modes</td>
<td>2.87</td>
<td>1.44</td>
<td>1.89x10(^2)</td>
<td>-18.0 cm to 18.0 cm by steps of 1.5 cm</td>
</tr>
<tr>
<td></td>
<td>10 in beam direction</td>
<td></td>
<td></td>
<td></td>
<td>-20.0 cm and 20.0 cm</td>
</tr>
<tr>
<td>35.56, 45.72, 45.72</td>
<td>17 Modes</td>
<td>2.868</td>
<td>1.29</td>
<td>1.89x10(^2)</td>
<td>-10.2 cm to 10.2 cm by steps of 1.7 cm</td>
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<tr>
<td></td>
<td>10 in beam direction</td>
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<tr>
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<td>2.865</td>
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<tr>
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<td>2.860</td>
<td>1.3</td>
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<td>-1.0 cm, 0.0 cm, and 1.0 cm</td>
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<td></td>
<td>6 in beam direction</td>
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</table>
Figure 19. Theoretical values of the neutron density averaged transport mean free path as a function of time for assemblies with $B^2 = 0.0292 \text{ cm}^{-2}$ and $0.0399 \text{ cm}^{-2}$. 
It is shown in Appendix B that the most suitable average to use for the transport mean free path is

\[
\overline{\ell}_{tr} = \int_{0}^{\infty} \ell_{tr}(E) n(E) \, dE / \int_{0}^{\infty} n(E) \, dE
\]

where \( n(E) \) = neutron density since the spatial distributions were measured using a 1/v detector which in fact measured the spatial variation of the neutron density.

The variation of \( \overline{\ell}_{tr} \) with time and distance has been calculated for the assemblies of buckling 0.0399 cm\(^{-2}\) and 0.0292 cm\(^{-2}\) since in these the energy spectrum is known (35) to change as a function of time. The details of the number of modes, spatial position, etc., are given in Table 6.

The variation of \( \overline{\ell}_{tr} \) with time can be seen in Figures 19a and 19b for these two assemblies. It can be seen that both curves show similar trends from about 250 \( \mu \)s to just greater than 1.0 ms after the pulse, with \( \overline{\ell}_{tr}(-8.5) < \overline{\ell}_{tr}(0) < \overline{\ell}_{tr}(8.5) \). For a period just after 1.0 ms the values of \( \overline{\ell}_{tr} \) at the three points are sensibly the same and exhibit a steady increase with time. At longer times the values diverge with \( \overline{\ell}_{tr}(-8.5) > \overline{\ell}_{tr}(0) > \overline{\ell}_{tr}(8.5) \). This divergence is very small for the stack of smaller buckling and in the range 1.0-3.0 ms the three values barely differ. It is much more marked in the case of the larger buckling and indeed the values of \( \overline{\ell}_{tr} \) at the three positions are approximately equal for a period of only some 300 \( \mu \)s between about 1,100 \( \mu \)s and 1,400 \( \mu \)s after the pulse. The discussion
of this divergence and the physical explanation of the variation of $\bar{L}_{tr}$ with distance at long times will be deferred for the moment and discussed in Section 4.3.4.

The initial convergence of $L_{tr}$ at the three positions is again consistent with the vanishing of the higher spatial modes necessary to describe an initially asymmetric distribution. If we consider for the moment the behaviour of $L_{tr}$ with time at the centre position which is the mean of that at position 8.5 cm and -8.5 cm we can see that the value increases by about 2.2 per cent./1,000 µs at a buckling of 0.0399 cm$^{-2}$ and 0.55 per cent./1,000 µs on a buckling of 0.0292 cm$^{-2}$. This was the order of change in the extrapolated length noted experimentally. However the increase in the average mean free path would appear as an increase in the extrapolation distance and the corresponding percentage increase in the extrapolated length would be much smaller. Both these assemblies had physical dimensions of nominally 30.48 cm in the direction in which the distributions were measured. We would therefore expect the percentage change in the extrapolated length to be

$$\frac{\Delta L}{L} = 2\frac{\Delta \epsilon}{\epsilon} \cdot \frac{\epsilon}{L} \sim 0.2 \frac{\Delta \epsilon}{\epsilon} = 0.2 \frac{\Delta L_{tr}}{L_{tr}}$$

This leads to an increase of 0.5 per cent./1,000 µs in the extrapolated length at a buckling of 0.0399 cm$^{-2}$ which is much smaller than the experimentally observed change of 2 per cent./1,000 µs. The calculated change in the extrapolated length at a buckling of 0.0292 cm$^{-2}$ would be correspondingly smaller.
Figure 20. 20a. Comparison of experimental and theoretical values of the extrapolation distance as a function of buckling $B^2 < B_{th}^2$.  
20b. Average mean free path as a function of buckling for various forms of the scattering cross section.
It is now of interest to see how the calculated values of the average transport mean free path compare with measured (41) values of the extrapolation distance for buckling below $B^2$. The comparison can be seen in Figure 20a where all the parameters have been normalised to a density of $2.96 \text{ gm cm}^{-3}$. It can be seen that the variation of the calculated extrapolation distance with buckling is in accord with the trend of the measured values and both are in disagreement with the theoretical trend predicted by Williams (46).

Moreover the extrapolation distances calculated using the extinction corrected data agree more closely with the experimental result than those calculated using the exact elastic cross section data. This is further justification for using extinction corrected data to describe scattering processes in this particular sample of BeO.

In this calculation the program TENDS was not used as the quantity desired was an average over the asymptotic energy spectrum, which does exist in assemblies with $B^2 < B^2$, and not one over the time dependent energy spectrum. The program used was TEV which could find the eigenvalues and eigenfunctions of the matrix $H$ in equation 4.2, the lowest eigenfunction of which represents the asymptotic energy spectrum. These calculations were done with a 37 group set which differed slightly from the 41 group set used for the more detailed calculations on the assemblies of larger buckling. The 37 group set is given in Appendix C where it can be seen that the 41 group set contains an extra 3 group below $10^{-5}$ eV and an extra group in the region of the first Bragg peak.
Figure 21. 21a. The energy spectrum in assemblies with $B^2 < B_{crit}^2$.
21b. The energy spectra for different forms of the scattering cross section.
It is obviously of interest to see why the average mean free path does decrease with buckling and under what conditions Williams' prediction of an increase with buckling for this parameter in polycrystalline moderators is correct.

Figure 21a shows the energy spectrum of the asymptotic mode for two values of buckling in the range \( B^2 < B^2 \) using the 37 group set. It can be seen that as the buckling increases the peak in the energy distribution in the region of the first Bragg peak and the dip below the Bragg cut-off become more and more pronounced. We might be tempted to think that the decrease in the average mean free path with buckling was due to the increased flux of neutrons having the small mean free path associated with the first Bragg peak. If however the same calculation is carried out when the transport cross section above the Bragg cut-off is assumed constant the transport mean free path averaged over the neutron density still decreases with buckling as can be seen in Figure 20b. A similar technique can be used to remove the possibility that the decrease in the average mean free path is due to the increased neutron density at low energies well below the Bragg cut-off. Curve 2 in Figure 20b shows the result of a 25 group calculation where only the first three groups below the Bragg cut-off have been included. The value of \( \ell_{tr} \) still decreases with increasing buckling. The reason then must lie in the ever decreasing neutron density just below the Bragg cut-off as the buckling increases. As the buckling increases the weight associated with the long mean free path neutrons in the averaging of the mean free path
decreases and consequently the result is weighted towards the shorter mean free path neutrons.

Figure 21a also shows that the energy spectrum does not peak at the peak of the Maxwellian distribution (0.025 eV) but at 0.015 eV which corresponds to the energy at which the scattering mean free path in BeO has a minimum. This and the discontinuity at the Bragg cut-off underline the marked differences between the energy spectra in pulsed polycrystalline assemblies and the Maxwellian distribution of energies even at comparatively small values of buckling. The spectra are a far cry from the original concept of a Maxwellian distribution slightly altered by the preferential leakage of neutrons at particular energies. It should be recalled that this concept led to the technique of expanding the energy part of the flux distribution as a sum of orthogonal functions (Laguerre or Hermite) the lowest order of which represented a Maxwellian distribution. In particular it was assumed that a two term expansion was sufficiently good to account for most of the perturbation and would predict adequately, trends seen in the experimental results. This was the technique employed by Williams who used a two term Laguerre expansion. Such an expansion cannot match the considerable "bite" out of the energy spectrum just below the Bragg cut-off and consequently gives the mean free path associated with these neutrons too much weight in the averaging process, so predicting an increase instead of a decrease in the extrapolation distance with buckling. An increase of the average transport mean free path with buckling can be achieved...
by making the transport cross section above the Bragg cut-off constant at ten barns and constant below the cut-off at five barns as can be seen in Figure 20b.

The spectra for the various models used above can be seen for a buckling of 0.02 cm\(^{-2}\) in Figure 21b.

4.3.2 Variation of decay constant with position at large buckling

Once the time dependent energy spectra for the bucklings of 0.0292 cm\(^{-2}\) and 0.0399 cm\(^{-2}\) had been computed for the various Fourier modes it was comparatively simple to obtain the 1/v reaction rate as a function of time and distance. These reaction rates were then processed using EXPFIT in exactly the manner used in the experiment and the decay constant as a function of time at various positions in the assemblies calculated.

Figures 11c and 12b show the theoretical decay constant as a function of time at the positions 8.5, 0, and -8.5 cm for the assemblies of buckling 0.0292 cm\(^{-2}\) and 0.0399 cm\(^{-2}\). It can be seen that in both figures the decay constants differ quite markedly at early times (\(\sim 1.0\) ms) but the difference decreases with time until they are very close together at about 1.7 ms in the case of the larger buckling and about 2.0 ms in the cases of the smaller buckling. Thereafter the values diverge. In the case of the assembly of buckling 0.0399 cm\(^{-2}\) the divergence is quite large as can be seen in Figure 12b but that for the assembly of buckling 0.0292 cm\(^{-2}\) is quite small and not worth plotting on the scale of Figure 11c. This behaviour is mindful of that of the average mean free path as a function of position and time.
and, as in that case we will discuss the behaviour up to ~2.0 ms here but defer the discussion of the decay constant at later times until Section 4.3.4.

In both cases the behaviour up to about 1.5 ms is consistent with higher asymmetric modes decaying with time to leave the decay constant independent of position. However a comparison of Figures 11c and 12b with the experimental results shown in Figures 11a and 12a would indicate that the theory greatly overestimates the magnitude of these higher asymmetric modes. In both assemblies the measured decay constants at the edges of the stack show nothing like the disparity at ~1.0 ms after the pulse predicted by the theory. Moreover in neither assembly does the decay constant measured at the position furthest from the source show the tendency to increase with time predicted by the theory.

The general trend of the change of decay constant with time in the assembly of buckling 0.0399 cm\(^{-2}\) is predicted fairly well as can be seen in Figures 12a and 12b. This is to be expected since the present calculational techniques have already (35) been used with considerable success to predict the change of "fundamental decay constant" with time measured by Rainbow and Ritchie.

4.3.3 Amplitude and decay constants of higher spatial modes

In the last section we saw that the theory apparently overestimated the amplitude of the higher modes in the assemblies of buckling 0.0399 cm\(^{-2}\) and 0.0292 cm\(^{-2}\). Furthermore in Section 3.4.3 we saw that the experimental values of the decay constants for the
Figure 22. Theoretical prediction of the time dependence of the fundamental and 2,1,1 mode in assemblies with $B^2 = 0.0292\, \text{cm}^{-2}$ and $0.0399\, \text{cm}^{-2}$. 
'211' spatial mode were higher than the corresponding points on the $\lambda(B^2)$ curve of the fundamental. It is therefore of interest to compare as far as possible the decay curve of the measured 211 mode with the calculated decay curve and to compare the ratio of amplitudes of the higher mode to the amplitudes of the fundamental at the end of the neutron pulse derived from the theoretical and experimental results.

The analysis of the theoretically produced data again followed the pattern developed before; the $1/v$ reaction rate was calculated as a function of time and position in assemblies having the same physical dimensions as those used in the experiment and analysed using the code FORFIT to evaluate the higher spatial modes as a function of time. Any systematic errors arising due to the neglect of even higher spatial modes would then appear in the results of both theory and experiment and any parameters derived from these results should be directly comparable.

Figure 22 shows the amplitude of the 211 mode in the assemblies of buckling $0.0292 \text{ cm}^{-2}$ and $0.0399 \text{ cm}^{-2}$. It can be seen that the slope of the theoretical decay curves is greater than that of the experimental ones throughout the whole time region. This is to be expected from Figure 16 where it can be seen that the decay constants of the higher modes in the assemblies of large buckling tend to lie below those for the fundamental. Now TENDS treats the higher modes just like fundamental modes of the same buckling and hence will overestimate the decay constant. This should be compared with the behaviour in the region $B^2_{*} < B^2 < B^2_{\text{exp}}$ where the experimental values of the decay constants of the higher modes lie above those of the fundamental.
TABLE 7 - Comparison of theoretical and experimental values
of $A_s / A_1$ at times immediately after the pulse

<table>
<thead>
<tr>
<th>Stack Dimensions $x, y, z$ (cm)</th>
<th>Buckling (cm$^{-2}$)</th>
<th>$A_s / A_1$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>55.88, 60.96, 60.96</td>
<td>0.0083</td>
<td>1.48</td>
<td>0.46</td>
</tr>
<tr>
<td>35.56, 45.72, 45.72</td>
<td>0.0164</td>
<td>1.08</td>
<td>—</td>
</tr>
<tr>
<td>30.48, 30.48, 30.48</td>
<td>0.0292</td>
<td>0.86</td>
<td>0.13</td>
</tr>
<tr>
<td>30.48, 20.32, 30.48</td>
<td>0.0399</td>
<td>0.78</td>
<td>0.078</td>
</tr>
<tr>
<td>58.42, 60.96, 60.96</td>
<td>0.0076</td>
<td>2.0</td>
<td>1.92</td>
</tr>
</tbody>
</table>

500 µs long initial pulse.
Error on experimental ratios 10 per cent. or better.

Note: Private communication from M.T. Rainbow (see text)
It can also be seen from Figures 22 and 14 that the theory predicts a proportionately higher asymmetric mode component than experiment at about 1.5 ms after the pulse in these two assemblies. This would explain why the theory predicts a different behaviour for the decay constant at positions 8.5 cm and -8.5 cm from that seen in the experimental measurements.

It is of interest to note that although theory overestimates the decay constant of the higher spatial modes in these small assemblies the amplitude of this mode is still proportionately too high at ~1.5 ms after the pulse compared to the experimental result. This means that the amplitude of the higher mode just at the end of the neutron pulse must be proportionately much higher than experiment. The ratio of the amplitude of the first asymmetric mode to the fundamental $A_s/A_1$ has been evaluated at the end of the pulse for these two small assemblies and also for a number of larger assemblies and the results tabulated in Table 7. The technique used was to extrapolate graphically the decay curves back to the time corresponding to the end of the pulse and read off the amplitudes. In order to obtain a reasonable comparison between theory and experiment the theoretical decay curves were evaluated over the same time range as that accessible to experiment.

It can be seen that in all cases theory predicts considerably more asymmetric mode component than is actually seen in the experiment, the discrepancy increasing with increasing buckling. At the smallest buckling there is a factor of just over three between the ratio $A_s/A_1$.
Figure 23. Calculation of the spectrum in the assembly with $B^2=0.0399 \text{ cm}^{-2}$ at early times after the pulse and at different positions in the assembly.
predicted by theory and that derived from experiment and at the largest buckling there is a factor of ten difference. It is unlikely that the ratios will be in error by more than 10 per cent, so that this discrepancy is very significant.

Also shown in Table 7 are the results of a similar measurement carried out by Rainbow (67) in an assembly nominally 58.42 x 60.96 x 60.96 which is little different from the largest assembly of 55.88 x 60.96 x 60.96 studied in the present series of experiments. In Rainbow's experiments the initial neutron pulse was only 20 μs long instead of the 500 μs used in the present set of experiments and it can be seen that there is considerably better agreement between theory and experiment. Indeed theory would appear to predict the experimental results within the experimental errors. The method of calculation was the same and the experimental technique little different so that the difference would seem to lie in the much shorter pulse length. This point will also be discussed in Chapter 5.

The overestimate of the higher modes in these small polycrystalline assemblies can cause quite marked differences in the energy spectra. This can be seen in Figure 23 where the spectra have been evaluated at very short times after the pulse. Because of the very high decay rates in these assemblies it is experimentally difficult because of lack of neutron intensity to measure the spectra at times much later than these after the pulse. Hence it is obvious that if the spectra are measured at distances much removed from the centre, the spectra, particularly in the region of the first Bragg peak, are
sensitive to this distance from the centre and the diffusion theory codes will not give good agreement with experiment. Even if the measurements are carried out at the centre, it can be seen from Figure 23 that the effect of the higher cosine modes is sufficient to modify the spectrum from that due to the lowest Fourier mode and again the predictions of diffusion theory codes are suspect. It is of course possible to measure the spatial distributions in the assembly at the times at which the spectra are measured and use the amplitudes of the higher Fourier modes derived from such a measurement to synthesise the energy spectrum. However this method is not very satisfying as the calculational technique takes on a semi-empirical appearance. Furthermore, it is not really possible for a theoretician, remote from the experiment, to be able to compute the spectra unless the experimenter gives details of the time dependent spatial distributions. The apparent dependence on pulse length of the ratio of the amplitudes of the higher modes to the fundamental further complicates the problem.

4.3.4 Asymmetry in small assemblies at long times

Let us now return to the problem of the increase of asymmetry with time apparent in the assembly of buckling 0.0399 cm\(^{-2}\) and to a lesser extent in the assembly of buckling 0.0292 cm\(^{-2}\).

It can be seen from both Figures 12b and 19a that the values of the "decay constant" and the average value of the transport mean free path evaluated on the positions 8.5 and \(-8.5\) cm from the centre of the assembly of buckling 0.0399 cm\(^{-2}\) become increasingly different.
Figure 24. The variation with time of the decay constants of the higher spatial modes in the assembly with $B^2=0.0399 \text{ cm}^{-2}$. 

<table>
<thead>
<tr>
<th>Mode</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>111</td>
<td>2,553</td>
<td>2,472</td>
<td>2,446</td>
<td>2,433</td>
<td>2,426</td>
</tr>
<tr>
<td>211</td>
<td>2,550</td>
<td>2,474</td>
<td>2,450</td>
<td>2,438</td>
<td>2,431</td>
</tr>
<tr>
<td>311</td>
<td>2,555</td>
<td>2,483</td>
<td>2,460</td>
<td>2,449</td>
<td>2,442</td>
</tr>
<tr>
<td>411</td>
<td>2,567</td>
<td>2,497</td>
<td>2,475</td>
<td>2,465</td>
<td>2,459</td>
</tr>
</tbody>
</table>
from each other at times greater than about 1.5 ms after the pulse. This comes as a surprise since the normal picture of the space-time distributions in these assemblies is that immediately after the pulse the spatial distribution will be markedly asymmetric if the source is placed at one face of the assembly as it was in these experiments. At later times the asymmetry will decrease until about 1.0 to 2.0 ms after the pulse it will have vanished altogether. Here apparently if the theory is to be believed the asymmetry first decreases and then increases.

A clue to this behaviour comes from Figure 24 which shows the instantaneous decay constant for the lowest Fourier mode and a number of the higher Fourier modes for the assembly of buckling $0.0399 \text{ cm}^{-2}$. It can be seen that these decay constants are characterised by a short plateau at early times (0 to about 5 ms) after the pulse when the value changes noticeably with time, a region of rapid change, then a region of slow change which continues past the times shown in Figure 24. The plateau region becomes less and less marked with higher buckling and the onset of the rapid change occurs earlier. It can be seen that at early times after the pulse the instantaneous decay constants increase with the increasing buckling but that from about 1.5 ms there is a switch to a state in which some of the higher modes decay less rapidly than the modes of nominally lower buckling and in particular some decay less rapidly than the fundamental. At sufficiently long times after the pulse the system once again returns to the state where higher modes decay faster than modes of lower buckling.
Figure 25. The energy spectra at 1ms and 20ms in the assembly with buckling 0.0399 cm$^{-2}$. 
We should recall that in these small assemblies with $\lambda > \lambda^*$ the energy spectrum will evolve in such a way that the decay constant will get closer and closer to the minimum loss rate $\lambda^*$ and the energy spectra will be dominated by the low energy neutrons which have this minimum loss rate. Initially however some neutrons will be trapped in the Bragg peaks and these neutrons will dominate the energy spectrum. The length of time that the super-Bragg neutrons dominate will depend on the buckling; in assemblies of comparatively small buckling, $B^2 < B^2 < B^2_{\text{exp}}$, super-Bragg neutrons dominate for $\sim 5 \text{ ms}$ and we get the pseudodiscrete region but in assemblies of large buckling, $B^2 > B^2_{\text{exp}}$, these neutrons dominate for only about a millisecond. This then is the explanation of the shape of the decay constants in Figure 24; the plateau region corresponds to the time when the super-Bragg neutrons dominate and the slow change with time at long times corresponds to domination by far sub-Bragg neutrons (energies $\sim 10^{-4} \text{ eV or less}$). These two regions are illustrated by the energy spectra shown in Figure 25.

The mean time between collision for the super-Bragg neutrons is about $6 \mu\text{s}$ whereas the mean time between collisions for the far sub-Bragg neutrons is $1/\lambda^* = 450 \mu\text{s}$. It is obvious then that an asymmetric distribution dominated by super-Bragg neutrons will become symmetric quite rapidly but one dominated by far sub-Bragg neutrons will become symmetric on a time scale about 70 times as long. This then is physically what is happening. Just after the pulse the distribution in space and energy is dominated by the super-Bragg
neutron and any reaction rate or other energy averaged quantities will tend to become symmetric. However at later times as the far sub-Bragg neutrons which have not had time to become symmetric begin to dominate, the asymmetry of energy averaged parameters becomes more obvious. Eventually this asymmetry should also disappear.

Another way of looking at the phenomenon is to consider it as a slowing down process which is in fact what it is. The source at one face of the slab injects a pulse of fast neutrons into the slab. These propagate through the moderator and slow down. The neutrons first reach super-Bragg energies and because they are partially trapped there and can diffuse through the stack for some time much greater than the mean time between collisions (mean life in the stack $\sim 220 \mu s$) they will almost reach a symmetric distribution. However they will leave a "trail" of far sub-Bragg neutrons behind them in this process. When the super-Bragg neutrons disappear to leave the far sub-Bragg neutrons these have still to propagate and diffuse throughout the assembly, a process which will take a very long time.

It is of interest to estimate the time scale needed to obtain almost complete symmetry. Note that the term "almost" is as much as can be used since this is a neutron slowing down process and there is no discrete mode which will describe the diffusion of neutrons throughout the assembly. Hence there will always be some memory of where the source was. We have seen above that the time scale for scattering of far sub-Bragg neutrons is about 70 times greater than
that of super-Bragg neutrons. If we assume that symmetry can be achieved in a small assembly with a discrete mode in about 2.0 ms then it will be achieved in one dominated eventually by sub-Bragg neutrons in about 140 ms.

In these assemblies the decay rate is such that the count rate changes by about 1 decade in 500 µs. Hence to see the symmetry being established we would have to count through some 280 decades. At the 140 ms mark we would want \( \sim 1 \) count/sec to obtain measurable counting statistics in a 100 µs wide channel after, say, a 100 hour run. This means a neutron flux of about \( 10^2 \) n cm\(^{-2}\) sec\(^{-1}\) and a density of about \( 5 \times 10^{-4} \) n cm\(^{-3}\). Hence we would require an initial neutron density of some \( 5 \times 10^{27} \) n cm\(^{-3}\). It is of interest to compare this figure with a density of \( \sim 10^{38} \) n cm\(^{-3}\) in a neutron star. Here we have an experiment which at first seems not unreasonable but which rapidly becomes physically impossible to carry out.

4.3.5 Summary

The first point is that the theory does predict an increase in the extrapolation distance with time in the assemblies of largest buckling. However, this increase is about a factor of four too small to explain the increase in extrapolated length noted for these assemblies in the experimental measurements. However for assemblies with \( B^2 < B^* \) the present theory apparently predicts rather well the decrease of extrapolation distance with buckling noted in earlier experiments in BeO.

The theory did not predict the same dependence of decay constant on position and time noted in the experimental results.
general the behaviour at short times indicates too high an asymmetric mode component and the behaviour at long times was quite different.

The third point is that the present theory predicts too large an asymmetric mode (211 mode) component at all times and in particular just at the end of the neutron pulse. The discrepancy between theory and experiment increases with buckling, the two being a factor of ten different at a buckling of 0.0399 cm$^{-2}$. This effect seems to depend on the length of the pulse since experimental measurements done using a 20 µs long pulse produces amplitudes for the higher modes in reasonable agreement with the theory. This could prove embarrassing when trying to predict the energy spectra in small assemblies at early times after the pulse.

The last point and in some way the most interesting is the prediction by the theory of an increase in the asymmetry in the assemblies of large buckling at times greater than about 1.5 ms. This is related to the increased domination of the energy spectrum at these times by the far sub-Bragg neutrons which dominate the energy distribution at long times. These neutrons take a long time to diffuse through the assembly and take a long time to reach a symmetric distribution. Moreover initially they are masked by the super-Bragg neutrons which diffuse rapidly and set up a distribution which is almost symmetric. It is important to note that this problem of asymmetry at long times is rather academic since to see symmetry actually established would require a neutron source far more intense than is physically possible.
5. DISCUSSION OF THEORETICAL AND EXPERIMENTAL RESULTS

5.1 Variation of the extrapolated length with time

It can be seen from the results of Chapters 3 and 4 (Sections 3.4.1 and 4.3.1) that the multigroup diffusion theory used, grossly underestimates the increase in the extrapolated length measured experimentally. It is worth recalling at this stage the philosophy behind the theoretical and experimental comparisons, in particular the use of Fourier mode analysis.

Once a time dependent spatial distribution has been measured, then, provided sufficient spatial points have been chosen and the counting statistics are good, the estimation of time dependent Fourier components is a well defined process. This of course stems from the fact that Fourier modes form a complete set of functions in any closed interval. Now the results of Section 3.4.1 showed that when a set of Fourier modes was fitted to the spatial distributions the "wavelength" associated with the distribution showed a steady increase with time in assemblies of large buckling but remained constant with time for small bucklings. Since the second cosine had negligible amplitude at these times the result means simply that the symmetric part of the distribution preserves its shape with time at small bucklings but becomes progressively raised in the wings at large bucklings.

The problem is now to compare this properly with theory. The form of the diffusion equation in a homogeneous system is such that it provides no coupling between the various Fourier modes.
This is because the $Q \cdot \text{grad}$ operator of the transport equation which does allow for coupling between the modes is replaced by the operator $-Dv^2$. Hence if the physical situation does require coupling between the various Fourier modes, diffusion theory cannot adequately describe it unless from some other calculation we have found the proportion of the Fourier modes which must be added together at progressively later times after the pulse. In this case we could possibly use the $\phi(E, t, B^2_n)$ from the diffusion theory calculation. But a calculation to find the proportion of the Fourier component as a function of time would almost certainly be a transport calculation which would provide the flux as a function of space, time and energy anyway.

If we look at the integral form of the transport equation then we can see that in small polycrystalline systems it is reasonable to expect coupling between the various Fourier modes. In other words we expect a situation which cannot adequately be described by a diffusion equation. The integral form of the transport equation (68) at times after the end of the pulse is,

$$f(\mathbf{r}, v, \Omega, t) = \int ds \, dv' \, d\Omega' \, \exp(-\Sigma s) \, f(\mathbf{r} - s\Omega, v', \Omega', t - s/v) \Sigma(v' \rightarrow v, \Omega' \rightarrow \Omega)$$

(5.1)

Physically this equation says that the angular flux of neutrons at position $\mathbf{r}$ and time $t$ is the sum of those neutrons which were scattered into the angle $\Omega$ from $\Omega'$ and into the velocity $v$ from $v'$ at
a distance $s$ from $r$ and at a time $s/v$ prior to $t$ and which travelled to $r$ without a collision. In many systems the cross section $\Sigma$ is so large that the term $\exp(-\Sigma s)$ is vanishingly small for quite small values of $s$ and the angular flux at $r$ is made up of contributions from only a small volume above $r$. If this is true then the expression for the angular flux inside the integral term of equation 5.1 can be expanded about $r$ and $t$ and if third order terms are neglected the diffusion equation can be obtained.

Now because there is a Bragg cut-off in polycrystalline scatterers and a consequent decrease in the scattering cross section there is a marked increase in the mean free path. In BeO the change of mean free path is from about 1.5 cm above the Bragg cut-off to about 30 cm below the Bragg cut-off. It is obvious then that for neutrons with energies below the Bragg cut-off the term $\exp(-\Sigma s)$ is not small for quite large values of $s$ and that the flux of sub-Bragg neutrons at a point $r$ will be made up of contributions from a considerable volume about $r$. In particular if we consider the smaller assemblies used in this investigation which had dimensions of about 30 cms we can see that the sub-Bragg flux at one edge of the assembly will be made up of contributions from the other edge of the assembly. We can interpret this by saying either that there will be a considerable degree of coupling between the various Fourier components or that the system must be described by a transport equation.
It is possible to get some feel for the possible shape of the spatial distribution by considering the following rather crude model. Let us consider two groups of neutrons, a "fast" group with short mean free path and a "slow" group with long mean free path. Let us assume that the neutrons are confined to travel in one dimension (the rod model of Wing 69) and that there is no scattering from the slow group to the fast group. Neutron transport in the rod is then described, at times after the end of the initial pulse, by the equations,

\[
\begin{align*}
\frac{1}{v_s} \frac{\partial \mu_s}{\partial t} + \frac{\partial \mu_s}{\partial x} + \Sigma^s \mu_s &= \frac{1}{2} \Sigma_{\text{inel}}^f (T+W) + \frac{1}{2} \Sigma_{\text{el}}^s (\mu+\nu) \\
\frac{1}{v_s} \frac{\partial \nu_s}{\partial t} - \frac{\partial \nu_s}{\partial x} + \Sigma^s \nu_s &= \frac{1}{2} \Sigma_{\text{inel}}^f (T+W) + \frac{1}{2} \Sigma_{\text{el}}^s (\mu+\nu) \\
\frac{1}{v_f} \frac{\partial T_f}{\partial t} + \frac{\partial T_f}{\partial x} + \Sigma^f T_f &= \frac{1}{2} \Sigma_{\text{el}}^f (T+W) \\
\frac{1}{v_f} \frac{\partial W_f}{\partial t} - \frac{\partial W_f}{\partial x} + \Sigma^f T_f &= \frac{1}{2} \Sigma_{\text{el}}^f (T+W)
\end{align*}
\]

(5.2)

where \( \mu \) and \( T \) represent the fluxes of slow and fast neutrons respectively travelling to the right and \( \nu \) and \( W \) represent the corresponding left travelling fluxes. The subscripts \( f \) and \( s \) refer to the fast and slow groups in a fairly obvious notation.
Now let us assume that the flux in each group is decaying exponentially with time and the time behaviour is the same in each group. This is a reasonable assumption for a limited period of time in view of the effective decay constants which exist in small systems and which do not change markedly over periods of about 500 $\mu$s. The equation (5.2) then takes the form

$$\left[ \frac{\partial}{\partial x} + \frac{1}{2} \Sigma^S - \frac{1}{2} \Sigma^S_{el} - \lambda/v_s \right] \mu - \frac{1}{2} \Sigma^S_{el} \nu - \frac{1}{2} \Sigma^f_{inel} (T+W) = 0$$

$$- \frac{1}{2} \Sigma^S_{el} \mu + \left[ - \frac{\partial}{\partial x} + \frac{1}{2} \Sigma^S - \frac{1}{2} \Sigma^S_{el} - \lambda/v_s \right] \nu - \frac{1}{2} \Sigma^f_{inel} (T+W) = 0$$

$$\left[ \frac{\partial}{\partial x} + \frac{1}{2} \Sigma^f - \frac{1}{2} \Sigma^f_{el} - \lambda/v_f \right] T - \frac{1}{2} \Sigma^f_{el} W = 0$$

$$- \frac{1}{2} \Sigma^f_{el} T + \left[ - \frac{\partial}{\partial x} + \frac{1}{2} \Sigma^f - \frac{1}{2} \Sigma^f_{el} - \lambda/v_f \right] W = 0$$

(5.3)

It is shown in Appendix D that the solution for the scaler flux in the fast group has the form

$$T(x,t) + W(x,t) = \sum_{n=1}^{\infty} 2A_n \sin B_n L \cos B_n x \exp -\lambda t \quad n \text{ odd}$$

$$= \sum_{n=1}^{\infty} 2A_n \cos B_n L \sin B_n x \exp -\lambda t \quad n \text{ even}$$

(5.4)

$$B_n \sim n\pi/2L$$
where the boundary conditions are

\[ T(-L, t) = 0 = W(L, t) \]

and assume no incoming neutron flux at the boundaries. The rod has length 2L with the origin at the centre of the rod. It is also shown that if \( \lambda > v_s \Sigma_s \), which is certainly true for the effective decay constant in assemblies with \( B^2 > B^2_{\text{exp}} \) in about the first two milliseconds after the pulse, the solution for the scaler flux in the slow group for the lowest mode is

\[
\mu(x, t) + \nu(x, t) = \left\{ - C_s^c \cos Bx + C_s^c \cosh Kx \right\} \exp^{-\lambda t}
\]

\[
C_s^c = 2A_1 \frac{\gamma(-\rho/2, -\lambda)}{(K^2 + B^2)} \sin BL
\]

\[
C_s^c = A_1 B \frac{\gamma(\frac{\lambda + \rho/2}{2})}{\sigma(K^2 + B^2)} \left[ \frac{\sinh KL + (K \cosh KL + \lambda \sinh KL) 2/\rho}{K \cosh 2KL + \lambda \sinh 2KL} \right]
\]

where the various constants are defined in Appendix D. Again it is assumed that there is no incoming neutron flux at the boundary i.e.

\[ \mu(-L, t) = 0 = \nu(L, t). \]

Now the form of the solution 5.4 is almost identical to that obtained by using the diffusion approximation and extrapolated dimension boundary conditions. However the solution for the slow group contains a diffusion theory like part and a part containing hyperbolic functions, which will be referred to as the transport part of the solution. The expression for the total flux will also have this form. Now the
diffusion theory like part will fall off as a cosine towards the edge of the stack while the transport part which reflects the existence of long mean free path neutrons, increases towards the outer edge. It is shown in Appendix D that $K \sim 0.055 \text{ cm}^{-1}$ if $B^2 \sim 10^{-2} \text{ cm}^{-2}$ and

$$\frac{\Sigma f}{\Sigma \text{inel}} \sim 30$$

$$\frac{v_s}{v_f} \sim 3$$

$$\Sigma f_{e1} \sim 1 \text{ cm}^{-1}$$

$$v_f \sim 2.2 \times 10^5 \text{ cm/sec}$$

all of which are typical of one group close to thermal energy and one group just below the Bragg cut-off. Now in the context of a rod, $B^2 \sim 10^{-2} \text{ cm}^{-2}$ implies a total length of $\sim 30 \text{ cm}$. Hence the transport part of the solution will not change much in amplitude over the length of the rod.

It is further shown in Appendix D that if $2A_1$ is unity the contribution at the origin from the diffusion theory part will be $\sim 0.92$ and from the transport like part $\sim 0.23$. Halfway towards the edge of the assembly the respective contributions will be $\sim 0.67$ and $\sim 0.25$. Hence the resulting spatial distribution will be raised in the wings compared to that from a purely diffusion theory solution which is in qualitative agreement with the measured spatial distributions in the small assemblies. The model is too crude to expect quantitative agreement and because of the $\exp{-\lambda t}$ ansatz cannot be expected to reproduce the time dependent change in the shape of the
It does seem likely that the multigroup version of equation 5.2 with more generalised up and down scatter is worthy of more attention.

It was pointed out in Section 3.4.1 that the increasing value of the extrapolated length would, if applied to all the dimensions, produce a decrease in the decay constant in the assembly of buckling 0.0399 cm$^{-2}$ which is comparable with that seen experimentally. This would seem to indicate that buckling used in the diffusion theory calculations should be time dependent. However it should be recalled that in earlier (41) measurements of the space-time distribution in BeO assemblies the extrapolation distance was found to be noticeably buckling dependent and quite different from 0.71 $\mathcal{L}_{\text{tr}}$ ($\mathcal{L}_{\text{tr}} = 3\mathcal{D}$) values. This contrasted with the observation that the value of the extrapolation length, giving the best fit to the $\lambda(B^2)$ curve, was very close to 0.71 $\mathcal{L}_{\text{tr}}$. Furthermore the effective decay constants of assemblies with $B^2 > B_{\text{exp}}^2$ and their change with time can be evaluated very well using a buckling that does not change with time. Hence it would appear that there is a buckling which describes the spatial distribution and one which describes leakage, the latter of which can be derived from the dimensions of the assembly in the usual way.

5.2 Decay constants and amplitudes of higher modes

We saw from Section 4.3.3 that the diffusion theory code grossly overestimated $A_s$, the amplitude of the first asymmetric mode, in all the assemblies, and that the disparity between theory and experiment increased with increasing buckling. The considerable
discrepancy, of about a factor ten in $A_s/A_1$, is somewhat surprising in view of the very good quantitative agreement (within a few percent.) between the effective decay constants computed theoretically and those measured experimentally. It is perhaps even more surprising in view of the good agreement between the theoretical and experimental values of a number of spatial modes found when a short ($\sim 20 \mu s$) initial pulse of neutrons was used in an assembly 60.96 x 60.96 x 58.42. In the same way the lack of agreement between the $\lambda(B^2)$ curve for the fundamental and that for the 211 mode is surprising since the excellent agreement between experimental values of effective decay constants for $B^2 > B^2_{\text{exp}}$ and the corresponding theoretical values would seem to imply that a buckling is a reasonable parameter to use in describing leakage in small pulsed systems.

The integral form of the transport equation once more gives us a clue to the physical processes which might give rise to the observed results. We have already seen that in pulsed polycrystalline assemblies there are neutrons which have mean free paths of about 30 cm. Now let us for a moment consider the space-time distribution in an assembly which has dimensions of about 60 cm and in which there is a markedly asymmetric distribution. Let us assume that some neutrons have undergone a scatter at a position halfway between the source position and the centre of the stack, the position which corresponds to the peak of the first asymmetric mode. A fraction of these neutrons will be scattered below the Bragg cut-off and of this fraction a proportion will travel in a direction towards the source while a proportion will travel down the stack away from the source. Since
the mean free path of these neutrons is about 30 cm a very large fraction (60 per cent.) of those travelling towards the source will escape whereas only 22 per cent. of those travelling away from the source escape and of these some 4 per cent. actually suffer another collision in a 3 cm interval about the position +15.0 cm. This position corresponds to the peak of the asymmetric distribution on the other side of the origin. Hence we can see that these long mean free path neutrons can in one fell swoop transfer neutrons from one side of the stack to the other. Now since the distribution is assumed to be asymmetric the contribution from positive to negative x positions will be much less. Hence here is a mechanism which will rapidly produce symmetry in an initially asymmetric distribution.

The time scale of this process must also be considered. The sub-Bragg neutrons are quite slow having a maximum velocity of $7.5 \times 10^4$ cm/sec in BeO and will take some 400 $\mu$s to travel the 30 cm from -15.0 cm to +15.0 cm. We can see that not only do these sub-Bragg neutrons transport neutrons from one side of the stack to the other in one collision but the neutrons appearing on the side away from the source originate from a spatial distribution which existed 400 $\mu$s earlier. With a 500 $\mu$s long initial neutron pulse the flux at the position +15.0 cm will be made up of neutrons which have essentially diffused there and neutrons which have undergone one scatter at the position -15.0 cm some 100 $\mu$s after the initial pulse when the spatial distribution was highly asymmetric. With a 20 $\mu$s long pulse these sub-Bragg neutrons have just not had time to travel to the point of interest by the time the pulse ends. Hence for a short pulse, the
space-time distributions should be adequately described by the diffusive part of the flux and hence by the diffusion equation.

In the larger assemblies this process will continue after the pulse has finished and we can expect the asymmetry will disappear more rapidly than expected. In particular the decay constant for the 211 mode will be higher than expected. This process could even explain the apparent increase in decay constant with time seen in the 50.8 x 45.72 x 45.72 cm stack.

Now if this picture is correct we must also look at those neutrons which are left after the scatter that produced the sub-Bragg neutrons. The neutrons left will have energies above the Bragg cut-off where, as a consequence of the short mean free path the loss rate is low (see Figure 1). There will therefore be a tendency to concentrate sub-Bragg neutrons on the source side of the distribution. This tendency will be aggravated if the dimensions of the stack are such that a sub-Bragg neutron is not likely to suffer a collision in the system when it travels away from the source. This is really a further manifestation of the trapping effect in the Bragg peaks. Since these trapped neutrons have a low loss rate we have the situation that sub-Bragg neutrons will tend to symmetrise an initially asymmetric distribution faster than expected but any asymmetry left will persist longer than expected because it contains a higher proportion of low loss rate neutrons. These two effects will work in opposite directions and the resulting ratio \( \frac{A_s}{A_1} \) will depend on the pulse length and the stack dimensions, and possibly on the initial neutron energy distribution in the source. It is of interest to note in this connection
that the asymmetric mode in the stack of dimensions 35.56 x 45.72 x 45.72 (B^2 = 0.0164 cm^-2) is only 0.2 per cent. of the fundamental at 1380 μs after the pulse (theory predicts 3 per cent.) whereas in the 30.48 x 30.48 x 30.48 stack (B^2 = 0.0292 cm^-2) the asymmetric mode is about 1 per cent. at this time.

For the smaller assemblies, where the dimensions are less than, or the order of, the sub-Bragg mean free path, we would expect the asymmetry to contain a high proportion of trapped neutrons. With smaller assemblies and longer times we would expect the asymmetric mode to be composed predominantly of trapped neutrons and the decay constant of the asymmetric mode to be the loss rate for these neutrons. This would explain why, for the assemblies of large buckling, the effective decay constant of the asymmetric mode lies below the effective decay constant for the fundamental. It would also lead us to expect that the λ(B^2) relationship of the decay constant of the 211 mode evaluated at times sufficiently long after the initial pulse, would just be that of the line λ_K which represents the loss rate of neutrons that are predominantly elastically scattered. It can be seen from Figure 16 that the two such decay constants which could be evaluated at long times lie on the lowest evaluation of λ_K which also cuts the λ(B^2) curve of the fundamental close to the transition point noted by Rainbow and Ritchie. The major difference between the various determinations of λ_K shown in Figure 16 is the value assumed for the inelastic cross section in the region of the first Bragg peak, the lowest evaluation of λ_K corresponding to the lowest value assumed for the inelastic cross section. The fact that the two decay constants
and the transition point all lie close to the lowest evaluation of $\lambda_k$ could indicate that the value of the inelastic cross section in the region of the first Bragg peak is even lower than that assumed in the present calculations and consequently much lower than that assumed by Tewari and Trikha.

5.3 Long lived asymmetry in small systems

In Section 4.3.4 we saw that the diffusion theory code TIMEX predicted an asymmetry in small systems which would persist for long times (possibly up to 140 ms after the initial pulse). We were able to show that within the context of a diffusion theory calculation this asymmetry was quite reasonable and did in fact represent the propagation of the sub-Bragg neutrons through the assembly. As was pointed out earlier the calculation assumes that the neutrons slow down whilst "marking time" and that each Fourier mode preserves its identity as time progresses. In the last two sections we have seen that this assumption is unreasonable since the long mean free path neutrons will transport neutrons from one side of the stack to the other and cause considerable mixing of the Fourier modes. However in these small assemblies at long times there is a very low population of large mean free path neutrons (see for example Figure 25) and once again there is a region where diffusion theory might be expected to hold.

The credibility of long term asymmetry in the assemblies of larger buckling will then depend on how much asymmetry is left by the trapped neutrons and how much this feeds directly into the flux at $\sim 10^{-4}$ eV. If there are a significant number of neutrons which slow down to $10^{-4}$ eV through collisions at say $10^{-3}$ eV then once again there
will be considerable coupling between the various Fourier modes and the diffusion theory assumption will be inadequate. In this case we must not expect to see the long term asymmetry predicted by the diffusion theory code. If however the majority of neutrons which are scattered from $\sim 1.5 \times 10^{-2}$ eV to $10^{-4}$ eV do so in one collision then the vast majority of neutrons in the assembly will have mean free paths the order of 1.5 - 4.0 cm and there will be little coupling between the modes. In this case the diffusion theory approach will be reasonable and we could expect to see the propagation of very low energy neutrons through the assembly.
The experimental results show that at long times (~2.0 ms after the pulse) there is no marked change of the decay constant with position in assemblies with $B^2 > B^2_*$. In the assembly of largest buckling ($B^2 = 0.0399$ cm$^{-2}$) the effective decay constants at the points +8.5 cm were the same, within their errors, 2,235 $\mu$s after the pulse but were some 2.3 per cent higher than the effective decay constant at the centre, the errors being ~0.8 per cent. This result is consistent with the earlier measurements of Rainbow and Ritchie who noted the same order of change. The present measurements, however, were done with displacements in the direction of the incoming neutron beam and should have been more sensitive to changes of the decay constant with position. The results imply that in BeO the time dependent decay constants measured in assemblies with $B^2$ much greater than $B^2_*$ over the time range of about 1.0 to 3.0 ms can be considered characteristic of the assembly within confidence limits of about 3 per cent. It means that the results of different experiments in the high buckling region should agree within these limits if the polycrystalline samples are comparable and that theoretical calculations should be able to predict the experimental results to within this order of accuracy.

The experimental results also show that the value of the extrapolated length is constant with time for assemblies with $B^2 < B^2_*$ but that there is a small increase with time in assemblies with $B^2 > B^2_*$. The change was $(2.0\pm0.5)$ per cent./1,000 $\mu$s at
$B^2 = 0.0399 \text{ cm}^{-2}$ and only $(0.5 \pm 0.25)$ per cent./1,000 μs at $B^2 = 0.0292 \text{ cm}^{-2}$ in the time ranges 1.4 ms to 2.3 ms and 1.4 ms to 2.9 ms respectively. Now it should be noted that spatial distribution measurements in assemblies with $B^2$ less than $B^2_*$ yield extrapolated lengths greater than those which give the best fit to the $\lambda(B^2)$ curve. It should also be noted that numerical calculations, using the conventional prescription for the buckling and assuming it to be time independent show excellent agreement with the measured effective decay constants. Hence it seems likely that we must think in terms of two bucklings, one to describe the spatial distribution and one to describe leakage. With this in mind and noting the relatively small changes with time in the extrapolated lengths at large buckling we must conclude that the use of a buckling term in the manner of Conn and Corngold or, coupled in a $DB^2$ term, in the manner of Ritchie et al. and Tewari and Trikha, is reasonable in determining the time dependence of the energy spectrum in pulsed polycrystalline assemblies.

It is possible that a buckling which decreased with time could be used in a diffusion theory calculation. At first sight the decreasing values of buckling will predict decay constants which decrease with time in an even more pronounced way than calculations using a fixed value of buckling. However the smaller values of buckling will produce less rapid changes in the energy spectrum and hence in the decay constant in the 1.0 to 3.0 ms time range. This might be sufficient to compensate for the changes with time intro-
duced by the time dependent buckling. The time dependent bucklings would have to be introduced into such a calculation in a phenomenological manner based on experiments in a sample of the material under study.

The diffusion theory calculations used in the present work were unable to predict correctly the change of extrapolated length with time or the change in the effective decay constant with distance in assemblies with $B^2 > B^2_*$. The calculations underestimated the change in the extrapolated length. They could predict these changes only by changes in the spectrum with time and hence changes in the average transport mean free path used in deriving the extrapolation distance. Since the same kind of calculation successfully predicted the experimentally observed changes in the extrapolation distance with buckling below $B^2_*$ and since the present calculations predict correctly the changes of effective decay constant with time which depend on changes in the spectrum, it seems reasonable to conclude that the changes in the extrapolation length with time are due to changes in the shape of the flux. A crude two group model based on neutron transport in a rod shows that if the slow group has a long mean free path ($\sim 30$ cm) and the time dependence is assumed to be determined by the absorption and leakage of the fast group which has a short mean free path ($\sim 1.0$ cm) the spatial distribution is made up of a diffusion theory like part and a part which is raised in the wings of the distribution. Such a distribution will be flatter than that derived from diffusion theory and will give rise to a larger
extrapolated length. It is possible that further work based on the multigroup form of this model would predict time dependent changes in the flux comparable with those seen in the experiment.

The experimental results also show that the decay constant of the first higher spatial mode (the 211 mode) does not fit on the \( \lambda(B^2) \) curve of the fundamental for \( B^2 > B^* \). As the assumption that the higher modes do follow the same \( \lambda(B^2) \) curve as the fundamental is the cornerstone in using time dependent diffusion theory to predict time-space-energy distributions, it is obvious that such a theory cannot predict such distributions for small polycrystalline assemblies. Again the reason seems to lie in the existence of long mean free path neutrons which can transport neutrons from one side of the stack to the other in only one collision. This process rapidly reduces any asymmetry in the distribution and in the smaller assemblies leads to a concentration of short mean free path neutrons in that part of the flux which describes the asymmetry. Hence any asymmetry is much smaller than expected on the basis of diffusion theory and in the smaller assemblies persists much longer than expected.

Monte Carlo calculations of the space-time-energy distribution in these small assemblies would obviously be of interest. The time region in which super-Bragg neutrons dominate (up to \(~5.0\) ms after the pulse) appears particularly suited to this type of calculation because the mean life of neutrons in these systems is only some 240 \( \mu \)s which represents about 40 collisions at thermal energies. Now a normal Monte Carlo code can investigate \(~10^4\) collisions/min
so that a 40 minute run would yield $10^4$ histories on average. Several such runs should build up a fairly clear picture of the space-time-energy distribution during the first 5 ms or so after the pulse.

It can be seen that cross section data is still a problem in any efforts to compare detailed theoretical calculations with the results of pulsed experiments in BeO assemblies. That used in the present calculations gives excellent agreement with the $\lambda(B^2)$ curve and good agreement with the variation of the extrapolation distance with buckling. The data used by Tewari and Trikha underestimates the points of the $\lambda(B^2)$ curve but not as badly as the present inelastic cross section data used in conjunction with the transport cross section uncorrected for extinction. The data of Tewari and Trikha also gives a value of $\lambda^*$ greater than the $\lambda_{\text{exp}}$ of Rainbow and Ritchie and a $\lambda_{\text{K}}$ well above the region where these authors observed markedly non-exponential behaviour. Hence it would appear that the data used in the present calculations is better than that of Tewari and Trikha. However, the position of the transition region from the pseudo-exponential to the non-exponential mode of decay, and the values of the decay constant of the 211 spatial mode in the smaller assemblies cast some doubt on the credibility of the inelastic scattering cross sections used in the present data set. The work of Conn and Corngold (34) would lead us to expect that the transition region should be close to the point where the line $\lambda_{\text{el}}$ (or $\lambda_{\text{K}}$) crosses the $\lambda(B^2)$ curve of the fundamental mode of decay. The arguments presented in Section 5.2 would lead us to expect the decay constant
of the 211 mode evaluated at long times in assemblies with $B^2 > B^2_{\text{exp}}$ to have the same buckling dependence as $\lambda_K$. It has been shown (see Figure 16) that the transition region and those decay constants which could be measured at long times all agree with an evaluation of $\lambda_K$ which assumes a lower value (by $\sim 15$ per cent.) of the inelastic cross section in the region of the first Bragg peak than that used in the present calculations and consequently much lower than that used by Tewari and Trikha.

The crux of the data problem lies in a correct estimate of the inelastic cross section since the elastic part of the cross section is fairly well defined by the slope of the $\lambda(B^2)$ curve in the region $B^2 < B^*_K$. Some measurements in BeO of the total cross section at energies below the Bragg cut-off, taking special care to account for scattering from voids, dislocations, small grains etc., would prove very useful since this would determine the inelastic cross section in that energy region.

Lastly the long term asymmetry which was predicted by the diffusion theory calculations in these small systems, and which appeared to be associated with the slowing down and propagation of very low energy neutrons ($\sim 10^{-4}$ eV and less) seems to be an interesting problem. It should however be stressed that it is purely an academic problem as the neutron intensities required to carry out the experiments necessary to check the calculations are impractically high.

The overall conclusion is that the time dependent energy spectrum in these small polycrystalline assemblies is fairly well understood and can be calculated confidently, except for uncertainties
in the data, at times and positions where higher modes can be neglected. The space dependence of the flux cannot be calculated very well at all. This reflects the fact that the emphasis, over the last ten years or so, has been on the thermalisation problem and the production of reliable thermal neutron scattering data. The streaming term in the Boltzmann equation has been ignored or approximated in rather crude ways. It would seem reasonable to expect that over the next ten years or so much more effort will be devoted to handling this part of the transport equation.
REFERENCES

REFERENCES (continued)


REFERENCES (continued)


REFERENCES (continued)


REFERENCES (continued)


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## APPENDIX A

Group boundaries and scattering matrix of 41 group data set

### Group boundaries

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**Group averaged microscopic transport cross section**

The array below contains the group averaged values (in barns) of the microscopic transport cross section where the values above the Bragg cut-off (groups one to twenty-three) have been corrected for extinction in the BeO crystallites as described in Section 4.2. The first entry corresponds to the group one and the other entries, reading from left to right, correspond to groups two, three ... up to forty-one.

\[
\begin{array}{cccccccccc}
0.757 & 1.065 & 1.391 & 1.677 & 2.027 & 2.441 & 2.980 & 3.622 \\
21.677 & & & & & & & \\
\end{array}
\]

**The scattering kernel**

The scattering kernel used in the calculations is listed on the next three pages. The first 41 entries in the matrix of numbers corresponds to the group averaged microscopic cross sections for scattering from the \(i^{th}\) group to the first group \((\Sigma_{i \rightarrow 1})\). The second set of 41 entries gives \(\Sigma_{i \rightarrow 2}\) and so on.
| $E$ | $t_{01}$ | $t_{02}$ | $t_{03}$ | $t_{04}$ | $t_{10}$ | $t_{11}$ | $t_{12}$ | $t_{13}$ | $t_{14}$ | $t_{20}$ | $t_{21}$ | $t_{22}$ | $t_{23}$ | $t_{24}$ | $t_{30}$ | $t_{31}$ | $t_{32}$ | $t_{33}$ | $t_{34}$ | $t_{40}$ | $t_{41}$ | $t_{42}$ | $t_{43}$ | $t_{44}$ |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.0 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 | 1.0000E+00 |
### 41-GROUP FE C SCATTERING MATRIX

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These values represent scattering matrix elements for a 41-group FE model in natural language.
APPENDIX B

Justification of the use of the neutron density averaged mean free path in estimating the extrapolation distance in pulsed assemblies

The energy dependent diffusion equation in a homogeneous system at times after the end of the pulse is

$$\frac{1}{v} \frac{\partial \phi}{\partial t} - D \nabla^2 \phi + \Sigma_a \phi + \Sigma_s \phi = \int \Sigma_s(E' \rightarrow E) \phi(r, E', t) \, dE'$$  \hspace{1cm} (B1)

where $\phi(r, E, t)$ is the neutron flux.

In any experimental determinations of the time dependent spatial distribution, the quantity actually measured is a reaction rate of the form

$$R(t, r) = \int \phi(E, r, t) \, w(E) \, dE$$  \hspace{1cm} (B2)

where $w(E)$ is the response function for the detector. In the present measurement the detectors used were 1/4 inch BF$_3$ detectors which had an efficiency of about 0.3 per cent. for thermal neutrons. It is reasonable then to assume that the response function $w(E)$ is proportional to the cross section of the reaction used in these detectors. This varies at $1/v$ in the energy region of interest. Equation (B2) can then be written as

$$R(t, r) = \alpha \int \phi(E, r, t) \, (1/v) \, dE$$

$$= \alpha \int n(E, r, t) \, dE$$

$$= \alpha \, N(r, t)$$  \hspace{1cm} (B3)
where \( n(\vec{r},E,t) \) is the neutron density and \( \alpha \) depends on the geometry of the detector. Equation (B3) means that the time and space dependence of the count rate from the detectors is proportional to the total neutron density.

If we now put \( \phi = n\nu \) in Equation (B1) and integrate over energy we obtain

\[
\frac{\partial N}{\partial t} - \overline{\nu D} \nabla^2 N + \overline{\nu \Sigma_a} N = 0 \tag{B4}
\]

where \( N(\vec{r},t) = \int dE n(E,\vec{r},t) \)

\[
\overline{\nu D} = \int dE \nu D(E) n(E,\vec{r},t) \int dE n(E,\vec{r},t)
\]

\[
\overline{\nu \Sigma_a} = \int dE \nu \Sigma_a(E) n(E,\vec{r},t) \int dE n(E,\vec{r},t)
\]

and we have used the fact that

\[
\Sigma_s(E) = \int \Sigma_s(E \rightarrow E') dE'.
\]

It follows from Equation (B3) that the solution to (B4) describes precisely the quantity measured experimentally. We can solve (B4) using standard techniques provided we know the boundary conditions for the total neutron density \( N(\vec{r},t) \).

At times after the end of the pulse no neutrons enter the assembly from outside. In diffusion theory this means the current entering the assembly at any face is zero i.e.

\[
\phi(\vec{r},E,t) + \frac{2}{3} \overline{\mathcal{L}_{tr}}(E) \text{grad} \phi(\vec{r},E,t) = 0
\]

i.e.

\[
n(\vec{r},E,t) + \frac{2}{3} \overline{\mathcal{L}_{tr}}(E) \text{grad} n(\vec{r},E,t) = 0
\]
\[
\dot{N}(r, t) + \frac{2}{3} \mathcal{L}_{tr} \nabla N(r, t) = 0 \tag{B5}
\]

where \(\mathcal{L}_{tr} = \int dE \mathcal{L}_{tr}(E) n(E, r, t) / \int n(E, r, t) \, dE \tag{B6}\)

Equation B(5) represents appropriate boundary conditions which will determine those solutions of Equation (B4) that describe the space and time dependence of the count rate from the detector at times after the end of the pulse.

From Equation (B6) it can be seen that the appropriate average to take for the transport mean free path is the average over the neutron density. The factor \(\frac{2}{3}\) should be replaced by the more accurate 0.71 which can be derived from transport theory.
## APPENDIX C

Group boundaries for 37 group data set

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<td>6.208x10^{-3}</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The two group transport equation in a rod can be written in the following form at times after the end of the initial pulse:

\[
\begin{bmatrix}
\frac{\partial}{\partial x} + \Lambda_t & -\rho/2 & -\gamma/2 & -\gamma/2 \\
-\rho/2 & \frac{\partial}{\partial x} + \Lambda_t & -\gamma/2 & -\gamma/2 \\
-\xi/2 & -\xi/2 & \frac{\partial}{\partial x} + \chi_t & -\sigma/2 \\
-\xi/2 & -\xi/2 & -\sigma/2 & \frac{\partial}{\partial x} + \chi_t
\end{bmatrix}
\begin{bmatrix}
\mu(x,t) \\
\nu(x,t) \\
T(x,t) \\
W(x,t)
\end{bmatrix} = 0
\]

where
\[\mu = \text{slow flux travelling to the right}\]
\[\nu = \text{slow flux travelling to the left}\]
\[T = \text{fast flux travelling to the right}\]
\[W = \text{fast flux travelling to the left}\]

\[\Lambda_t = \Sigma_s - \Sigma_{e1}/2 + \frac{1}{v_s} \frac{\partial}{\partial t}\]

\[\chi_t = \Sigma_f - \Sigma_{e1}/2 + \frac{1}{v_f} \frac{\partial}{\partial t}\]

\[\rho = \Sigma_{e1}\]
\[\sigma = \Sigma_{e1}\]
\[\xi = \Sigma_{s_{inel}}\]
\[\gamma = \Sigma_{f_{inel}}\]

and the indices \(f\) and \(s\) refer to the fast and slow fluxes respectively. The scattering is assumed to be isotropic in both groups.
Now let us assume that
\[
\begin{align*}
\mu(x,t) & = \mu(x) \exp^{-\lambda t} \\
\nu(x,t) & = \nu(x) \exp^{-\lambda t} \\
T(x,t) & = T(x) \exp^{-\lambda t} \\
W(x,t) & = W(x) \exp^{-\lambda t}
\end{align*}
\]

and there is no inelastic scatter in the slow group.

The equation (D1) now becomes

\[
\begin{bmatrix}
\frac{d}{dx} + \chi & -\rho/2 & -\gamma/2 & -\gamma/2 \\
-\rho/2 & -\frac{d}{dx} + \chi & -\gamma/2 & -\gamma/2 \\
0 & 0 & \frac{d}{dx} + \chi & -\sigma/2 \\
0 & 0 & -\sigma/2 & -\frac{d}{dx} + \chi
\end{bmatrix}
\begin{bmatrix}
\mu(x) \\
\nu(x) \\
T(x) \\
W(x)
\end{bmatrix}
= 0
\]

\[
\chi = \sum^s -\rho/2 - \lambda/\nu_s \\
\chi = \sum^f -\sigma/2 - \lambda/\nu_f
\]

It is obvious that we can first of all solve the reduced set of equations

\[
\begin{bmatrix}
\frac{d}{dx} + \chi & -\sigma/2 \\
-\sigma/2 & -\frac{d}{dx} + \chi
\end{bmatrix}
\begin{bmatrix}
T(x) \\
W(x)
\end{bmatrix}
= 0
\]
The solutions to these will be of the form

\[ T(x) = \sum_{n=1}^{2} b_{3n} \exp B_n x \]  

\[ W(x) = \sum_{n=1}^{2} b_{4n} \exp B_n x \]

with \( B_1 \) and \( B_2 \) the roots of

\[ B^2 - \chi^2 + \frac{\sigma^2}{4} = 0 \]  

Hence

\[ B_1 = -B_2 = B = \sqrt{\chi^2 - \frac{\sigma^2}{4}} \]

Now \( T(x) \) and \( W(x) \) satisfy equations (D4)

\[ \therefore b_{43} = b_{33} \frac{2}{\sigma} (B + \chi) \]

\[ b_{44} = -b_{34} \frac{2}{\sigma} (B - \chi) \]

\[ \therefore T = b_{33} \exp Bx + b_{34} \exp -Bx \]

\[ W = \frac{2}{\sigma} \left[ (B + \chi) b_{33} \exp Bx - (B - \chi) b_{34} \exp -Bx \right] \]

Now we will assume that there is no incoming flux in either group at the boundaries
\[ i.e. \ T(x) = 0 \quad \text{at} \ x = -L \]

\[ W(x) = 0 \text{ at } x = L \]

where the rod is assumed to be 2L long and the origin is at the centre of the rod.

\[
\begin{bmatrix}
\exp(-BL) & \exp(BL) \\
(B + \chi) \exp(BL) & -(B - \chi) \exp(-BL)
\end{bmatrix}
\begin{bmatrix}
b_{33} \\
b_{34}
\end{bmatrix} = 0 \tag{D9}
\]

For non trivial solutions of \(b_{33}\) and \(b_{34}\) the determinant of the matrix must be zero.

\[
\begin{align*}
\therefore (B + \chi) \exp(2BL) + (B - \chi) \exp(-2BL) &= 0 \\
\end{align*} \tag{D10}
\]

\[ i.e. \ \tanh(2BL) = -\frac{B}{\chi} \tag{D11} \]

and if \(B\) is imaginary

\[ \tan(2BL) = -\frac{B}{\chi} \tag{D12} \]

From D10 and D9

\[ b_{34} = -b_{33} \exp(-2BL) \]

\[ T(x) = b_{33} \exp(-BL) \left[ \exp(B(x+L)) - \exp(-B(x+L)) \right] \]

\[ = C_1 \left[ \exp(B(x+L)) - \exp(-B(x+L)) \right] \tag{D13} \]
\[ W(x) = b_{33} \frac{2(B + \chi)}{\sigma} \exp BL \left[ \exp B(x-L) - \exp -B(x-L) \right] \]

\[ = b_{33} \exp -BL \frac{2(B + \chi)}{\sigma} \exp 2BL \left[ \exp B(x-L) - \exp -B(x-L) \right] \]

\[ = C_1 \frac{2(B + \chi)}{\sigma} \exp 2BL \left[ \exp B(x-L) - \exp -B(x-L) \right] \quad (D14) \]

Consider \( Z = (B + \chi) \exp 2BL \)

If \( B \) is real, \( Z \) is real.

If \( B \) is imaginary,

\[ \bar{Z} = (-B + \chi) \exp -2BL \]

\[ = (B + \chi) \exp 2BL \] from D10

\[ \leadsto \text{Z is real for } B \text{ real or imaginary.} \]

Hence if \( T(x) \) is real, \( W(x) \) is real.

We can now find solutions for \( \mu(x) \) and \( \nu(x) \). The first two terms of equation (D3) can be rewritten as

\[
\begin{bmatrix}
\frac{\partial}{\partial x} + \chi & -\rho/2 \\
-\rho/2 & -\frac{\partial}{\partial x} + \chi
\end{bmatrix}
\begin{bmatrix}
\mu \\
\nu
\end{bmatrix}
= \frac{7}{2}
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
T \\
W
\end{bmatrix}
\]

(D15)
The complementary function is

\[
\begin{bmatrix}
\frac{\partial}{\partial x} + \lambda & -\rho/2 \\
-\rho/2 & -\frac{\partial}{\partial x} + \lambda
\end{bmatrix}
\begin{bmatrix}
\mu \\
\nu
\end{bmatrix} = 0
\]

the solutions of which are of the form

\[
\begin{align*}
\mu(x) &= b_{11} \exp Kx + b_{12} \exp -Kx \\
\nu(x) &= b_{21} \exp Kx + b_{22} \exp -Kx
\end{align*}
\]

\( (D16) \)

\[ K^2 - \lambda^2 + \rho^2/4 = 0 \]

\( (D17) \)

We now have to find a particular integral for D15. We can write D15 in matrix notation in the following way

\[
A(\otimes)U = HV
\]

\( (D18) \)

where \( U \) is the vector \( \begin{bmatrix} u \\ \nu \end{bmatrix} \) and \( V \) the vector \( \begin{bmatrix} T \\ W \end{bmatrix} \).

A particular integral of equation (D18) is (70)

\[
U = A^{-1}(\otimes) HV
\]

\[
= \frac{7/2}{(-2^2 + \lambda^2 - \rho^2/4)} \begin{bmatrix}
-\lambda + \rho/2 \\
\rho/2
\end{bmatrix}
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\begin{bmatrix} T \\ W \end{bmatrix}
\]
\[
\begin{align*}
\frac{\gamma/2}{(-B^2 + \Lambda^2 - \rho^2/4)} &= \frac{\gamma^2}{(-B^2 + \Lambda^2 - \rho^2/4)} \left[ C_1 \left( \exp B(x+L) - \exp-B(x+L) \right) \right. \\
& \quad - \left. C_2 \left( \exp B(x-L) - \exp-B(x-L) \right) \right] \\
\mu(x) &= b_{11} \exp Kx + b_{12} \exp-Kx \\
\varphi(x) &= b_{21} \exp Kx + b_{22} \exp-Kx 
\end{align*}
\]
where \( C_2 = C_1 \frac{2 \exp 2BL}{\sigma} (B + \alpha) \)

\[
b_{21} = b_{11} \frac{2}{\rho^2} (K + \Lambda)
\]

\[
b_{22} = - b_{12} \frac{2}{\rho} (K - \Lambda)
\]

We can now find \( b_{11} \) and \( b_{12} \) in terms of \( C_1 \) by using the fact that

\[
\mu(x) = 0 \quad x = -L
\]

\[
\nu(x) = 0 \quad x = L
\]

if we assume that there are no incoming slow neutrons at the boundaries.

Using these boundary conditions

\[
b_{11} \bar{p} + b_{12} p + C_1 \frac{\gamma}{2 \left(-B^2 + \Lambda^2 - \rho^2/4\right)} x
\]

\[
x \left[ -2B - \frac{2B}{\sigma} q^2 (q^2 + q^2)(B + \alpha) + \frac{2q^2 (B + \alpha)(\Lambda + \rho/2)(q^2 - q^2)}{\sigma} \right] = 0
\]

\[
b_{11} \frac{2}{\rho^2} (K + \Lambda) \bar{p} - b_{12} \frac{2}{\rho} (K - \Lambda) p + C_1 \frac{\gamma}{2 \left(-B^2 + \Lambda^2 - \rho^2/4\right)} x
\]

\[
x \left[ B \left(q^2 - q^2\right) + (\Lambda + \rho/2)(q^2 - q^2) + 4\frac{q^2 B(B + \alpha)}{\sigma} \right] = 0
\]
where \( p = \exp \ KL, \ \bar{p} = \exp-KL \)
\( q = \exp \ BL, \ \bar{q} = \exp'-BL \)

We can solve these equations for \( b_{11} \) and \( b_{12} \) if

\[ (K-A)\bar{p}^2 + (K+A)p^2 \neq 0 \]

Now from D10 it follows that

\[ B(q^2 + \bar{q}^2) = \chi(q^2 - \bar{q}^2) \]

and from D7 that

\[ B^2 - \chi^2 = -\sigma^2/4 \]

Using these equations (D21) and (D22) reduce to

\[ b_{11} \bar{p} + b_{12} p + C_1 \frac{B 7}{(-B^2 + \alpha^2 - \rho^2/4)} \left[ -1 + \frac{2(\Lambda + \rho/2 - \chi)}{\sigma} \right] = 0 \quad (D23) \]

\[ b_{11} \frac{2}{\rho} (K+\Lambda) \bar{p} - b_{12} \frac{2}{\rho} (K-\Lambda) \bar{p} \]

\[ - \frac{2q^2(B+\chi)}{\sigma} C_1 \frac{B 7}{(-B^2 + \alpha^2 - \sigma^2/4)} \left[ -1 + \frac{2(\Lambda + \rho/2 - \chi)}{\sigma} \right] = 0 \quad (D24) \]

From equations (D23) and (D24)

\[ b_{11} = C_1 \frac{B 7 \cdot \rho}{(B^2 - \Lambda^2 + \sigma^2/4)} \frac{[\Lambda + \rho/2 - (\chi + \sigma/2)]}{[K(p^2 + \rho^2) + \Lambda(p^2 - \rho^2)]} \frac{2}{\sigma} x \]

\[ x \ \exp KL \left[ \frac{K-\Lambda}{\rho} \bar{p}^2 - \frac{(B+\chi)}{\sigma} q^2 \right] \quad (D25) \]
\[
\begin{align*}
\frac{b_{12}}{C} &= \frac{B y \varphi}{(B^2 - \Lambda^2 + \sigma^2)} \left[ \frac{\Lambda + \varphi}{2 - (\Lambda + \sigma/2)}\right] 2 \sigma x \\
x \exp -KL \left[ \frac{K+\Lambda}{\varphi} p^2 + \frac{(B+\chi)}{\sigma} q^2 \right] \\
\end{align*}
\]

Now from D19, D20, D25 and D26

\[
\begin{align*}
\mu(x) &= C_1 C_{11} \left[ \frac{(K-\Lambda)}{\varphi} \exp K(x-L) + \frac{(K+\Lambda)}{\varphi} \exp -K(x-L) \right] \\
- C_1 C_{11} \frac{(B+\chi)}{\sigma} \frac{q^2}{2} \left[ \exp K(x+L) - \exp -K(x+L) \right] \\
- C_1 \frac{\gamma}{2(B^2 - \Lambda^2 + \varphi^2/4)} x \\
x \left[ (\Lambda+\varphi/2) \left[ \exp B(x+L) - \exp -B(x+L) \right] - B \left[ \exp B(x+L) + \exp -B(x+L) \right] \right] \\
- C_1 \frac{\gamma}{2q^2 (B+\chi) \sigma} x \\
x \left[ (\Lambda+\varphi/2) \left[ \exp B(x-L) - \exp -B(x-L) \right] - B \left[ \exp B(x-L) + \exp -B(x-L) \right] \right]
\end{align*}
\]
\[ \nu(x) = C_1 C_{11} \frac{2(K^2 - \lambda^2)}{\rho^2} \left[ \exp K(x-L) - \exp K(x-L) \right] \]

\[- C_1 C_{11} \frac{2q^2(B+\chi)}{\sigma \rho} \left[ (K+\lambda) \exp K(x+L) + (K-\lambda) \exp K(x+L) \right] \]

\[- C_1 \frac{\gamma}{2(B^2 - \lambda^2 + \rho^2 /4)} \frac{2q^2(B+\chi)}{\sigma} x \]

\[ x \left[ \left( A + \rho/2 \right) \left[ \exp B(x+L) - \exp B(x+L) \right] + B \left[ \exp B(x+L) + \exp B(x+L) \right] \right] \]

\[- C_1 \frac{\gamma}{2(B^2 - \lambda^2 + \rho^2 /4)} \frac{2q^2(B+\chi)}{\sigma} x \]

\[ x \left[ \left( A + \rho/2 \right) \left[ \exp B(x-L) - \exp B(x-L) \right] + B \left[ \exp B(x-L) + \exp B(x-L) \right] \right] \]

where
\[ C_{11} = \frac{B \rho L}{\left( B^2 - \lambda^2 + \rho^2 /4 \right)} \frac{\left[ (A + \rho/2) - (\chi + \sigma/2) \right]}{\left[ K(p^2 + p^2) + \lambda(p^2 - p^2) \right]} \cdot \frac{2}{\sigma} \]

The precise form of the solution for \( T(x), W(x), \mu(x) \) and \( \nu(x) \) will depend on whether the parameters \( B \) and \( K \) are real or imaginary. Let us consider first of all equation (D7) which defines \( B \) in terms of \( \lambda \).

\[ B = \sqrt{\chi^2 - \sigma^2 /4} = \sqrt{(\chi - \sigma /2) (\chi + \sigma /2)} \]
But \[ \chi = \Sigma_f - \sigma/2 - \lambda/v_f \]

\[ = \Sigma_a^f + \eta + \sigma - \sigma/2 - \lambda/v_f \]

\[ = \Sigma_a^f + \eta - \lambda/v_f + \sigma/2 \]

\[ \therefore B = \sqrt{(\Sigma_a^f + \eta - \lambda/v_f) (\Sigma_a^f + \eta + \sigma - \lambda/v_f)} \]

Now \( v_f(\Sigma_a^f + \gamma) \) is the removal rate from the first group due to absorption and inelastic scatter. The decay constant will exceed this by an amount depending on the leakage from the system. \( v_f(\Sigma_a + \gamma + \sigma) \) is the reaction rate in the fast group and the decay constant will be less than this unless every scatter leads to leakage out of the system. Hence the product inside the square root is negative (or zero) and \( B \) is imaginary. The allowed values of \( B \) will be given by the transcendental equation (D12)

\[ \tan 2BL = -\frac{B}{\chi} \]

For small values of \( B/\chi \) the allowed values of \( B \) will be given approximately by

\[ 2B_n L \approx n\pi \]

\[ B_n = \frac{n\pi}{2L} \]

These define spatial modes for the fast group that are very like the spatial modes which arise in diffusion theory when the usual boundary conditions of zero flux at the extrapolated dimensions are assumed.
We have now found the possible values of $B$ under the assumption that the time dependence is of the form given by equation (D2). From equation (D7) we can find the value of $\lambda_n$ associated with each of the $B_n$ spatial modes. We now have to find the equivalent values of $K$. From equation (D17)

$$K = \sqrt{\lambda^2 - \rho^2/4}$$

$$= \sqrt{\left(\Lambda - \rho/2\right) \left(\Lambda + \rho/2\right)}$$

$$= \sqrt{\left(\Sigma_a^S - \lambda/v_s\right) \left(\Sigma_a^S + \rho - \lambda/v_s\right)}$$

$$= \sqrt{\frac{1}{v_s} \left(v_s \Sigma_a^S - \lambda\right) \left(\Sigma_a^S + \rho \right) v_s - \lambda}$$ \hspace{1cm} (D29)

If we assume that $\Sigma_a$ varies as $1/v$ then for $B^2 > 0$

$$\lambda > v_s \Sigma_a^S$$

since it follows from equation (D7) that $\lambda$ is given to first order in $B^2$ by

$$\lambda = v_f \Sigma_a^f + v_f \gamma + v_f B^2 / \sigma$$

It should be noted that this is the same form expected from diffusion in a rod. If we further assume that

$$\lambda > v_s \rho + v_s \Sigma_a$$

which is just the $\lambda > \lambda^*$ criterion then it follows from equation (D29) that $K$ is real.
Equation (D29) then takes the form

\[ K = \sqrt{\frac{1}{v_s}} \left( \lambda - \nu_s \sum_a^S \left( \lambda - \nu_s (\rho + \sum_a^S) \right) \right) \]

from which it can be seen that as \( \lambda \) increases \( K \) increases. But \( \lambda \) increases as \( B \) increases. Hence \( K \) increases as \( B \) increases.

We can now rewrite the form of the solutions as

\[ T_n(x) = A_n^1 \sin B_n(x+L) \quad \text{(D30)} \]

\[ W_n(x) = A_n^2 \frac{2q^2 (\pm B + \chi)}{\sigma} \sin B_n(x-L) \quad \text{(D31)} \]

\[ \mu_n(x) = C_n \left[ \frac{(K_n - A_n)}{\rho} \exp K_n(x-L) + \frac{(K_n + A_n)}{\rho} \exp -K_n(x-L) \right] \]

\[- C_n \left[ \frac{(iB_n + \chi)}{\sigma} 2q^2 \sinh K_n(x+L) \right] \quad \text{(D32)} \]

\[ + \frac{A_n^1 \gamma}{2(K_n^2 + B_n^2)} \left[ (\pm \rho/2) \sin B_n(x+L) - B_n \cos B_n(x+L) \right] \]

\[ + \frac{A_n \gamma}{2(K_n^2 + B_n^2)} \frac{2q^2 (\pm B_n + iB_n)}{\sigma} \left[ (\pm \rho/2) \sin B_n(x-L) - B_n \cos B_n(x-L) \right] \]
\[ \psi_n(x) = -C_{11}^n C_{11}^n \sinh K_n(x-L) \]

\[ = C_{11}^n C_{11}^n \frac{2q_n^2(\xi_n + iB_n)}{\sigma \rho} \left[ (K_n + A_n) \exp K_n(x+L) + (K_n - A_n) \exp -K_n(x+L) \right] \]

\[ + \frac{A_n}{2(K_n^2 + B_n^2)^n} \frac{2q_n^2(\xi_n + iB_n)}{\sigma} \left[ (A_n + \rho/2) \sin B_n(x+L) + B_n \cos B_n(x+L) \right] \]

\[ + \frac{A_n}{2(K_n^2 + B_n^2)^n} \frac{2q_n^2(\xi_n + iB_n)}{\sigma} \left[ (A_n + \rho/2) \sin B_n(x-L) + B_n \cos B_n(x-L) \right] \]

where \( q_n = \exp iB_n L \) and \( A_n = 2iC_n \)

We now need to consider the quantity

\[ \frac{2q_n^2(\xi_n + iB_n)}{\sigma} \]

From equations (D7) and (D12) it can be shown that

\[ \frac{2q_n^2(\xi_n + iB_n)}{\sigma} = -1 \quad n \text{ odd} \]

\[ = 1 \quad n \text{ even} \]

Using this it follows that the space dependence of the vector fluxes in the \( n^{\text{th}} \) mode can be written as
\[ T_n(x) = A_n \sin B_n (x+L) \]

\[ W_n(x) = \mp A_n \sin B_n (x-L) \]

\[ \mu_n(x) = \frac{c_n}{\rho} \left[ 2K_n \cosh K_n (x-L) - 2A_n \sinh K_n (x-L) \right] \]

\[ \pm c_n C_n^{n_{11}} \sinh K_n (x+L) \]

\[ + \frac{A_n \gamma}{2 \left( K_n^2 + B_n^2 \right)} \left[ (A_n + \rho/2) \sin B_n (x+L) + B_n \cos B_n (x+L) \right] \]

\[ \mp \frac{A_n \gamma}{2 \left( K_n^2 + B_n^2 \right)} \left[ (A_n + \rho/2) \sin B_n (x-L) + B_n \cos B_n (x-L) \right] \]

\[ \nu_n(x) = -c_n C_n^{n_{11}} \sinh K_n (x-L) \]

\[ + \frac{c_n}{\rho} \left[ 2K_n \cosh K_n (x-L) + 2A_n \sinh K_n (x-L) \right] \]

\[ + \frac{A_n \gamma}{2 \left( K_n^2 + B_n^2 \right)} \left[ (A_n + \rho/2) \sin B_n (x+L) + B_n \cos B_n (x+L) \right] \]

\[ - \frac{A_n \gamma}{2 \left( K_n^2 + B_n^2 \right)} \left[ (A_n + \rho/2) \sin B_n (x-L) + B_n \cos B_n (x-L) \right] \]

where the upper sign is used for \( n \) odd and the lower for \( n \) even.
The scaler fluxes in the fast and slow groups are

\[ T_n(x) + W_n(x) = 2A_n \cos B_n x \sin B_n L \quad n \text{ odd} \]

\[ = 2A_n \sin B_n x \cos B_n L \quad n \text{ even} \]  

(D34)

\[ \mu_n(x) + \nu_n(x) = 2A_n^\eta \left( \frac{\Lambda_n + \phi/2}{\sqrt{K_n^2 + B_n^2}} \right) \sin B_n L \cos B_n x \]

\[ + 2C_{11}^n \left\{ \sinh K_n L + \frac{2}{\sigma} \left[ K_n \cosh K_n L + \Lambda_n \sinh K_n L \right] \right\} \cosh K_n x \]

\[ + 2C_{11}^n \left\{ \cosh K_n L - \frac{2}{\sigma} \left[ K_n \sinh K_n L + \Lambda_n \cosh K_n L \right] \right\} \sinh K_n x \]

(D35)

In particular the fundamental mode for the slow group will have a spatial dependence of the form

\[ \mu(x) + \nu(x) = 2A_1^\eta \left( \frac{\Lambda + \phi/2}{\sqrt{K^2 + B^2}} \right) \sin B L \cos B x \]

\[ - \frac{A_1 B}{\sigma(K^2 + B^2)} \left[ \frac{(\Lambda + \phi/2) - (\chi + \sigma/2)}{K \cosh 2KL + \Lambda \sinh 2KL} \right] x \]

\[ \times \left\{ \sinh KL + \frac{2}{\sigma} \left[ K \cosh KL + \Lambda \sinh KL \right] \right\} \cosh K x \]

(D36)
In order to gauge the importance of the various terms in equation (D36) we must make some assumptions about the order of magnitude of the scattering cross sections. It is reasonable to assume that $\Sigma_{e1}^f$ is much greater than both $\Sigma_{e1}^s$ and $\Sigma_{ine1}^f$ if we are trying to describe scattering in a polycrystalline system with one group just above and one group just below the Bragg cut-off. From this it follows that

$$\chi + \sigma/2 \gg \Lambda + \rho/2$$

and in fact $\chi + \sigma/2 \sim \sigma$

It is also fairly easy to show that

$$\frac{\left[ \sin KL + \frac{2}{B} (K \cosh KL + \Lambda \sinh Kx) \right]}{\left[ K \cosh 2KL + \Lambda \sinh 2KL \right]} \sim \frac{2}{\rho} \text{ for } B^2 \sim 10^{-2}$$

Hence the coefficient of $\cosh Kx$ is $\sim 2A_1 \frac{\eta}{\sigma} \frac{B}{K^2 + B^2}$

If we assume that $v_s(\Sigma_{a}^s + \rho) = v_f(\Sigma_{a}^f + \rho)$ then

$$\Lambda + \rho/2 \sim - \frac{v_f}{v_s} \frac{B^2}{\sigma}$$

and the coefficient of the $\cos Bx$ term is given approximately by

$$-2A_1 \frac{\eta}{\sigma} \frac{v_f}{v_s} \frac{B^2}{K^2 + B^2}$$
The coefficient of the cos Bx in the fast scaler flux is

\[ 2A_1 \sin BL \sim 2A_1 \]

If we assume that \( \gamma/\sigma = \frac{1}{30} \)

\[ \frac{v_f}{v_s} = 3 \]

\[ \sigma = 1 \text{ cm}^{-1} \]

\[ v_f = 2.2 \times 10^5 \text{ cm/sec} \]

\[ B^2 = 10^{-2} \text{ cm}^{-2} \]

Then \( \gamma = 3 \times 10^{-2} \text{ cm}^{-1} \)

\[ K = 5.5 \times 10^{-2} \text{ cm}^{-1} \]

The coefficient \( C^s_{ch} \) of cosh Kx in the slow flux is

\[ 2A_1 \left(3 \times 10^{-2}\right) \frac{1}{1.3 \times 10^{-1}} \]

i.e. \( C^f_{ch} = 2A_1 \frac{2.3}{10^{-1}} \)

while the coefficient \( C^s_c \) of cos Bx in the slow flux is

\[ -2A_1 \left(\frac{3}{30}\right) \cdot \frac{1}{1.3} \]

i.e. \( C^s_c = -2A_1 \frac{7.7}{10^{-2}} \)
We can now estimate the contribution from the diffusion theory like part (cos Bx term) and the transport theory part (cosh Kx term) in the total flux at various positions. The value of $B^2 = 10^{-2}$ cm$^{-2}$ implies that $L \sim 15.4$ cm. It should be recalled that the outermost point in the spatial distributions measured in the smallest assemblies was at about $L_x/2$ where $L_x$ was the extrapolated length. This would correspond in our present model to $x \sim 7.2$. At this point

$$\cos Bx \sim 1/\sqrt{2}$$

$$\cosh Kx \sim 1.07$$

and we get, assuming $2A_1$ unity -

<table>
<thead>
<tr>
<th>Position</th>
<th>Contribution to Total Flux</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Diffusion Theory Like Part</td>
</tr>
<tr>
<td>0</td>
<td>0.92</td>
</tr>
<tr>
<td>7.2</td>
<td>0.67</td>
</tr>
</tbody>
</table>

It can be seen that the transport theory contribution is quite significant throughout the region in which the spatial distributions were measured and would predict an increase in the wings of the distribution compared to that predicted purely by diffusion theory.