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Abstract

Let S be a subsemigroup of an abelian torsion-free group G. If S is a positive cone of G, then all C*-algebras generated by faithful isometrical non-unitary representations of S are canon- ically isomorphic. Proved by Murphy, this statement generalized the well-known theorems of Coburn and Douglas. In this note we prove the reverse.

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ISOMETRIC REPRESENTATIONS OF TOTALLY ORDERED SEMIGROUPS

M.A.AUKHADIEV AND V.H.TEPOYAN

Abstract. Let S be a subsemigroup of an abelian torsion-free group G. If S is a positive cone of G, then all C*-algebras generated by faithful isometrical non-unitary representations of S are canonically isomorphic. Proved by Murphy, this statement generalized the well-known theorems of Coburn and Douglas. In this note we prove the reverse. If all C*-algebras generated by faithful isometrical non-unitary representations of S are canonically isomorphic, then S is a positive cone of G. Also we consider $G = \mathbb{Z} \times \mathbb{Z}$ and prove that if S induces total order on G, then there exist at least two unitarily not equivalent irreducible isometrical representation of S. And if the order is lexicographical-product order, then all such representations are unitarily equivalent.

1. Introduction and preliminaries

Within this paper S is a subsemigroup of an additive abelian torsion-free group G with zero. S induces a partial order on G: $a \prec b$ if there exists $c \in S$ such that $a + c = b$. Semigroup S induces full order on G, i.e. for any $a, b \in S$ either $a \prec b$ or $b \prec a$, if $G = S \cup (-S)$ and $S \cap (-S) = \{0\}$. In this case write $S = G^+$ – *a positive cone* of G. Each semigroup S, which doesn't contain groups, is contained in some positive cone G^+. This follows from the axiom of choice.

Let G be an abelian totally ordered group and S – subsemigroup of G^+, which doesn't contain groups. We denote by Δ_S a set of unitary equivalence classes of faithful irreducible non-unitary isometrical representations of semigroup S. For $V \in \Delta_S$ define S_V as a semigroup generated by operators V_a and V_b^*, where $a, b \in S$ and $V_a = V(a)$.

An *inverse semigroup* P is a semigroup, such that each element x has a unique *inverse* element x^*, which satisfies the following:

$$xx^*x = x, \; x^*xx^* = x^* \tag{1}$$

Definition 1.1. *We call the representation $V \in \Delta_S$ inverse, if S_V is an inverse semigroup.*

In the well-known work [2] Coburn proved that all isometric representations of semigroup $\mathbb{N}$ generate canonically isomorphic C^*-algebras.

Key words and phrases. totally ordered semigroup, group, inverse semigroup, regular representation, isometric representation.

The same was proved by Douglas [3] for positive cones in $\mathbb{R}$ and by Murphy [8] for positive cones of abelian totally ordered groups. In section 2 we show that every semigroup S has at least one inverse representation. Therefore all faithful isometric representations of positive cone are inverse.

S.A.Grigoryan assumed that all representations in Δ_S are inverse if and only if S is a totally ordered semigroup, i.e. S is a positive cone of some group. We prove this hypothesis in section 2.

In section 3 we prove that if S induces full archimedian order on $\mathbb{Z} \times \mathbb{Z}$, then it has at least two unitarily not equivalent irreducible isometric representations. In case S induces a total lexicographical-product order, all such representations are unitarily equivalent.

2. Inverse representations

Regular isometric representation is a map $V : S \to B(l^2(S))$, $a \mapsto V_a$, defined as follows:

$$(V_a f)(b) = \begin{cases} f(c), & \text{if } b = a + c \text{ for some } c \in S; \\ 0, & \text{otherwise} \end{cases}$$

C^*-algebra generated by regular isometric representation of semigroup S is called a *reduced semigroup C^*-algebra*, denoted by $C^*_{red}(S)$ [6].

A finite product of operators of the form V_a and V_b^*, $a, b \in S$ is called a *monomial. An index of monomial* $W = V_{a_1} V_{a_2}^* V_{a_3} ... V_{a_n}^*$ is an element of group $\Gamma = S - S$, equal to

$$\text{ind}W = (a_2 + a_4 + ... + a_n) - (a_1 + a_3 + ... + a_{n-1}),$$

when n is even [4]. For odd n we have:

$$W = V_{a_1} V_{a_2}^* V_{a_3} ... V_{a_n},$$

$$\text{ind}W = (a_2 + a_4 + ... + a_{n-1}) - (a_1 + a_3 + ... + a_n).$$

It is clear that

$$\text{ind}(W_1 \cdot W_2) = \text{ind}W_1 + \text{ind}W_2.$$

Due to definition, monomials form a semigroup, which we denote by S_V.

Lemma 2.1. *The regular isometric representation of S is inverse.*

Proof. Consider a family $\{e_a\}_{a \in S}$ of elements in $l^2(S)$ such that $e_a(b) = \delta_{a,b}$. This is a natural orthonormal basis in $l^2(S)$. Every monomial W in S_V satisfies the following:

$$We_b = e_{b-d} \quad \text{or} \quad 0, \quad \text{where} \quad d = \text{ind}W.$$

Note that WW^* and W^*W are monomials also, besides

$$\text{ind}(W \cdot W^*) = \text{ind}(W^* \cdot W).$$

By virtue of Lemma 2.2 in [5], WW^* and W^*W are orthogonal projections. This implies immediately that $W = WW^*W$ and $W^* = W^*WW^*$. Therefore, an inverse element for W is W^*.

□

Lemma 2.2. *There exists at least one noninverse representation in Δ_S for a semigroup $S \subsetneq G^+$.*

Proof. Take a regular representation V of S in $B(l^2(S))$, $a \mapsto V_a$. Since S is not equal to G^+, there exist incomparable elements $c, d \in S$, i.e. $c - d \notin S$ and $d - c \notin S$. Consider function $g_{c,d} = \frac{e_c + e_d}{\sqrt{2}}$ in $l^2(S)$. Denote by H a Hilbert space generated by linear span of $\{V_a g_{c,d}\}_{a \in S}$. Note that $V_a g_{c,d} = g_{c+a,d+a}$. Define representation $\widetilde{V}$ of semigroup S on H, $a \mapsto \widetilde{V_a}$, by setting $\widetilde{V_a} = V_a P$, where $P : l^2(S) \to H$ is a projection on H.

This representation is faithful isometric due to its definition.

Let us show that

$$\widetilde{V_c}\widetilde{V_c^*}\widetilde{V_d}\widetilde{V_d^*} \neq \widetilde{V_d}\widetilde{V_d^*}\widetilde{V_c}\widetilde{V_c^*}. \tag{2}$$

Consider $\widetilde{V_d^*} g_{2c,c+d}$ and find such elements $x \in S$ that

$$(\widetilde{V_d^*} g_{2c,c+d}, g_{c+a,d+a}) = 0.$$

To this end, calculate

$$\begin{aligned} (\widetilde{V_d^*} g_{2c,c+d}, g_{c+a,d+a}) &= (g_{2c,c+d}, g_{c+d+a,2d+a}) = \\ &= (\tfrac{e_{2c}+e_{c+d}}{\sqrt{2}}, \tfrac{e_{c+d+a}+e_{2d+a}}{\sqrt{2}}) = \\ = \tfrac{1}{2}((e_{2c}, e_{c+d+a}) &+ (e_{2c}, e_{2d+a}) + (e_{c+d}, e_{c+d+a}) + (e_{c+d}, e_{2d+a})). \end{aligned} \tag{3}$$

First and last summands are equal to zero, since c and d are incomparabe. Therefore the scale product $(\widetilde{V_d^*} g_{2c,c+d}, g_{c+a,d+a})$ is not equal to zero if and only if either $a = 0$ or $a = 2c - 2d$. Note that element $2c - 2d$ may not be contained in semigroup S. Despite this fact we continue the proof assuming $2c - 2d \in S$. One can easily see that without this assumption the proof is trivial.

Denote by H_0 a Hilbert space in H generated by elements of the following set

$$\{g_{c+a,d+a} |\ a \neq 0, a \neq 2c - 2d\}$$

Repeating the same arguments as above one can show that $g_{c,d}$ and $g_{3c-d,2c-d}$ are mutually orthogonal, and both are orthogonal to H_0. Consequently, $codim H_0 = 2$ and the elements $g_{c,d}$ and $g_{3c-d,2c-d}$ form an orthonormal basis in $H_0^\perp \subset H$. Thus,

$$H = H_0 \oplus \mathbb{C} g_{c,d} \oplus \mathbb{C} g_{3c-d,2c-d},$$

and from equation (3) we have

$$V_d^* g_{2c,c+d} = \frac{1}{2}(g_{c,d} + g_{3c-2d,2c-d}).$$

For futher be noted, the assumption $2c-2d \in S$ implies that $2d-2c$ is not contained in semigroup S. Otherwise G^+ would contain non-trivial group, which is impossible. Therefore, due to symmetry we get

$$V_c^* g_{c+d,2d} = \frac{1}{2} g_{c,d}.$$

Thus,

$$\widetilde{V_c}\widetilde{V_c^*}\widetilde{V_d}\widetilde{V_d^*} g_{2c,c+d} = \frac{1}{2}\widetilde{V_c}\widetilde{V_c^*}\widetilde{V_d}(g_{c,d} + g_{3c-2d,2c-d}) =$$
$$= \frac{1}{2}(\widetilde{V_c}\widetilde{V_c^*} g_{c+d,2d} + \widetilde{V_c}\widetilde{V_c^*} g_{3c-d,2c}) =$$
$$= \frac{1}{4} V_c g_{c,d} + \frac{1}{2} V_c g_{2c-d,c} = \frac{1}{4} g_{2c,c+d} + \frac{1}{2} g_{3c-d,2c}.$$

On the other hand,

$$\widetilde{V_d}\widetilde{V_d^*}\widetilde{V_c}\widetilde{V_c^*} g_{2c,c+d} = \widetilde{V_d}\widetilde{V_d^*} g_{2c,c+d} = \frac{1}{2} g_{c+d,2d} + \frac{1}{2} g_{3c-d,2c}.$$

Consequently, we get inequality (2)

□

Theorem 2.1. *The following properties of semigroup S are equivalent*

1. $S = G^+$;
2. *all representations in Δ_S are canonically isomorphic;*
3. *all representations in Δ_S are inverse;*
4. *for any representation V in Δ_S and for any $a, b \in S$ the following equality is satisfied*

$$V_a V_a^* V_b V_b^* = V_b V_b^* V_a V_a^*.$$

Proof. (1)⇒(2) was proved by Murphy [8].

Let us show implication (2)⇒(3). Suppose all representations in Δ_S are canonically isomorphic and $S \subset G^+$. Consider representation $V : S \to l^2(G^+)$, $a \mapsto V_a$, defined by

$$V_a e_b = e_{a+b},$$

where $\{e_a\}_{a \in G^+}$ is an orthonormal basis in $l^2(G^+)$. For any $a, b, c \in S$ if $a \prec b$ or $a = b$ we have $V_a^* V_b e_c = e_{c+b-a}$. Since all elements in G^+ are pairwise comparable, we have two cases. If $a \prec b$, then operator $V_a^* V_b$ is isometric, otherwise ($b \prec a$) operator $(V_a^* V_b)^*$ is isometric. Consequently, semigroup S_V is inverse.

Implication (3)⇒(4) concerns only inverse semigroups, and it was proved in [1].

Lemma 2.2 implies (4)⇒(1).

□

Corollary 2.1. *The C^*-algebras $C^*(S)$ and $C^*_{red}(S)$ are isomorphic if and only if S is totally ordered, where $C^*(S)$ is a universal enveloping C^*-algebra, generated by all isometric representations of semigroup S* [9].

In particular case $S = \mathbb{Z}^+$ this statement implies that the algebras $C^*(\mathbb{Z}^+)$ and $C^*_{red}(\mathbb{Z}^+)$ are isomorphic. This result was proved by Coburn in his well-known work [2]. As an example of the converse to this statement take $S = \mathbb{Z}^+ \setminus \{1\}$. Due to Corollary 2.1, the algebras $C^*(S)$ and $C^*_{red}(S)$ are not isomorphic. The same was shown in [7], and this case was studied in details in [10].

3. C^*-ALGEBRAS GENERATED BY TOTALLY ORDERED SEMIGROUP IN $\mathbb{Z} \times \mathbb{Z}$

Consider group $G = \mathbb{Z} \times \mathbb{Z}$. Total order on G is equivalent to straight line dividing it into two parts. It implies two cases. The first case: the line meets the point $(0,0)$ and doesn't meet any integers. Such line is characterized by equation $x + \alpha y = 0$, where α is irrational. The second case: the line meets integers, i.e. $x + \alpha y = 0$, for rational α. The order induced by the first line is archimedian. In the second case we may consider $G = S \cup (-S)$, where $S = \{(n,m) \in \mathbb{Z} \times \mathbb{Z} \mid m > 0 \text{ or } (n,0),\ n \geq 0\}$ $(S = G^+)$. In this case the order cannot be archimedian, since we have $(-1,1) < (0,1)$ together with $n \cdot (-1,1) < (0,1)$ for any $n > 0$.

Theorem 3.1.

(1) *If G^+ induces total archimedian order on G, then $card\Delta_S > 1$;*
(2) *If G^+ induces lexicographical-product order, then $card\Delta_S = 1$.*

Proof. (1) Suppose G^+ induces total archimedian order on G. Without loss of generality, we may assume that $G^+ \subset \mathbb{R}^+$. Therefore $\overline{G^+} = \mathbb{R}^+$. Let us give a new representation of semigroup G^+.

Consider the Hardi space H^2. By the help of inner singular function $exp\{\frac{1+e^{i\theta}}{1-e^{i\theta}}\}$ define nonunitary faithful isometric representation of the semigroup $\mathbb{R}^+$ in $B(H^2)$, $t \mapsto V_t$, by the following equation:

$$(V_t g)(e^{i\theta}) = exp(t\frac{1+e^{i\theta}}{1-e^{i\theta}})g(e^{i\theta}).$$

One can easily verify that V_t is an isometric operator on H^2. Let us show that this representation is not uquivalent to regular representation.

In case of regular representation W there exists element e_0 such that $W_t e_0 \perp e_0$ for any $t \in G^+$. It is sufficient to show that H^2 does not contain element g, such that $V_t g \perp g$ for any $t \in G^+$. Indeed, suppose that there exists such element g. Then we have

$$0 = (V_t g, g) = \frac{1}{2\pi}\int_{S^1} exp(t\frac{1+e^{i\theta}}{1-e^{i\theta}})g(e^{i\theta})\overline{g(e^{i\theta})}d\mu(\theta) = \qquad (4)$$
$$= \frac{1}{2\pi}\int_{S^1} exp(t\frac{1+e^{i\theta}}{1-e^{i\theta}})d\mu(\theta).$$

If $t \to 0$, the right-hand side of (4) converges to 1, which leads to a contradiction. Thus, representations V and W are not equivalent.

Now let us prove the second part of the theorem, (2).

The group of transformations of iteger lattice $\mathbb{Z} \times \mathbb{Z}$ is a group $SL(2, \mathbb{Z})$. For any pair of lexicographical-product orders on $\mathbb{Z} \times \mathbb{Z}$ there exists an element of $SL(2, \mathbb{Z})$, which transforms the first one to the second one. Therefore, without loss of generality, we may consider that S is equal to the following semigroup:

$$\{(n, m) \in \mathbb{Z} \times \mathbb{Z} \mid m > 0 \ \text{ or } (n, 0),\ n \geq 0\}.$$

Take representation $V : S \to B(H)$ in Δ_S. Since operator $V_{(1,0)}$ is isometric and not unitary, there exists $h_0 \in H$ such that $V^*_{(1,0)} h_0 = 0$. Since $V_{(0,1)} = V_{(1,0)} V_{(-1,1)}$, we have

$$V^*_{(0,1)} h_0 = V^*_{(-1,1)} V^*_{(1,0)} h_0 = 0. \tag{5}$$

Therefore, h_0 is an initial vector for operators $V_{(0,1)}$ and $V_{(1,0)}$. Consequently, it is initial for any $V_{(n,m)}$, where $(n, m) \in S$.

Consider Hilbert space H_1, generated by linear span of the set

$$\{V_{(n,m)} h_0,\ (n, m) \in S\}$$

Equation (5) implies that the family $\{V_{(n,m)} h_0,\ (n, m) \in S\}$ forms an orthonormal basis in H_1, and

$$V^*_{(k,l)} V_{(n,m)} h_0 = V_{(a,b)} h_0 \ \text{ or } 0.$$

Therefore, H_1 is an invariant subspace for C^*-algebra $C^*_{red}(S)$. Since representation V is irreducible, we have $H_1 = H$.

Consequently, the family of vectors $e_{n,m} = V_{(n,m)} h_0$, for $(n, m) \in S$, forms an orthonormal basis of H. This implies immediately $H \cong l^2(S)$. □

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