Invariant solutions for transient solute transport in saturated and unsaturated soils

Raseelo Joel Moitsheki

University of Wollongong

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Invariant Solutions for Transient Solute Transport in Saturated and Unsaturated Soils

A thesis submitted in fulfilment of the requirements for the award of the degree of

Doctor of Philosophy

from

The University of Wollongong

by

Raseelo Joel Moitsheki, BSc Ed, BSc Hons, MSc, University of North West, South Africa

School of Mathematics and Applied Statistics

2004
Declaration

I, Raseelo Joel Moitsheki, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document or part of it has not been submitted for qualifications at any other academic institution.

..........................................

Raseelo Joel Moitsheki

June, 2004
Dedicated to my grandmother

Ke go leboga go menagane mokwena - o nkgodisitse!
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Abstract

Contaminant transport in soils is described by a system of coupled (soil-water) advection-diffusion equations. This system has few symmetries. However, we first consider the two dimensional partial differential equations (P.D.E.s) describing transport of solutes in saturated soils, i.e. solute transport under steady water flow background. The quest for exact analytic solutions for these equations has continued unabated. The problem is difficult when velocity must be given by the modulus of a potential flow velocity field for incompressible fluids. However, the Laplace preserving transformations from Cartesian to streamline coordinates results in a much simpler form of solute transport equation. Classical symmetry analysis of the transformed equations with point water source and point vortex water flow results in a rich array of classical point symmetries. Exploitation of symmetry properties and other transformation techniques lead to a number of exotic exact analytic solutions.

Next, we examine solutions for the one dimensional solute transport during steady evaporation from a water table. If we classify the symmetry-bearing cases of the coefficient functions within a single solute transport equation, then some of the cases are compatible with special solutions for soil-water flow. This may be viewed as reduction of a system of couple soil-water equations by conditional or nonclassical symmetries. In most known solutions, a trivial uniform background soil-water content is assumed. Here, we find new exact analytic and numerical solutions for
non-reactive solute transport in non-trivial saturated and unsaturated water flow fields. Furthermore, we investigate the case of adsorption-diffusion which has the added complication of both transport equations being nonlinear of Fokker-Planck type. The nonlinear adsorption-diffusion equation (A.D.E.) is transformed into a class of inhomogeneous nonlinear diffusion equations (I.N.D.E.s). Hidden nonlocal symmetries that seem not to be recorded in the literature are systematically constructed by considering an integrated equation obtained using the general integral variable rather than a system of first order P.D.E.s associated with the concentration and the flux of a conservation law. Reductions for the I.N.D.E. to ordinary differential equations (O.D.E.s) are performed and some invariant solutions are constructed. Also, we obtain solutions for adsorbing solutes in saturated and unsaturated soils.
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7 Summary

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A.D.E. Adsorption-diffusion equation.

B.V.P. Boundary value problem.

C.D.E. Convection-dispersion equation.

D.E. Differential equation.

I.N.D.E. Inhomogeneous nonlinear diffusion equation.

I.S.C. Invariant surface condition.

O.D.E. Ordinary differential equation.

P.D.E. Partial differential equation.
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Chapter 1

Introduction

"You are the salt of the earth; but if the salt loses its flavor how shall it be seasoned?..."

Gospel of Matthew 5:13

1.1 Physical motivation

One of the world's most serious environmental problems of the late 20th century is chemical contamination and salinisation of soil. Soil salinisation is caused primarily by natural processes which include mobilization due to rising of water tables, of vast stores of subsurface salts originating from weathering of rock minerals or by the oceanic salts deposits in the landscapes through wind or rain; and secondarily by human induced processes such as agricultural practices, irrigation and land clearing.
Hazardous industrial and agricultural wastes such as fertilizers and pesticides serve as acute contamination sources of groundwater.

The impacts of salinity include the loss of productive land, diminished agricultural production; significant depletion of aquatic ecosystems and restrictions on consumptive use of water, e.g. in affected rivers and networks such as the Murray-Darling basin; loss of native vegetation; damage to public infrastructure and threats to urban households. The prime minister’s science, engineering and innovation council of 1998 reports\(^1\) that an estimated 2.5 million hectares of land in Australia have been affected so far and this figure could multiply by seven in another fifty years. The economic implications of these impacts are obvious. This problem is not unique to Australia. Various forms of salinity have been reported\(^2\) in other parts of the world including United States of America, Canada, Turkey and Iran.

The ability to predict the rate of movement of solutes in soils and the effects of the soil on this rate is useful for many industrial and agricultural purposes. These purposes are principally four-fold; firstly, knowing the amount and the concentration of fertilizer nutrients below the root zone in order to design a management scheme that minimizes fertilizer losses and keeps concentration of certain solutes, e.g. nitrates, within acceptable water quality standards; secondly, to understand

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\(^2\)Victorian auditor-general’s office, Special report number 19: Salinity, 1993.
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the movement in soils of disposed substances such as pesticides, radioactive mate-
rial, industrial waste which is essential to predict their impact on the environment;
thirdly, to devise management practices for irrigation to avoid excessive salt accu-
mulation in soil and finally, to understand the transport of solutes to the roots for
better prediction of the effect of soil on plant nutrition.

The time scales over which groundwater is polluted or salinity establishes itself,
spreads and has environmental effects can be of the order of decades of years. For
the purpose of environmental management it takes too long to experimentally de-
termine the outcomes of agricultural and industrial practices and prediction must
be made by mathematical modelling or by designing small physical models whose
results may be sensibly scaled up to predict outcomes at the field scale. A full theory
of solute transport will require understanding of microscopic transport processes in
fluctuating fluid flow fields in networks of tortuous channels (for a review, see e.g.
Jury, 1988). However, macroscopic transport models, described in terms of P.D.E.s,
will remain important for efficiently predicting solute transport at the field scale.
In practical problems, it is normal to solve the relevant P.D.E.s by approximate
numerical methods. However, we face the serious problem that available numerical
packages have significant disagreements in their prediction of solute dispersion
(Woods et al., 1998). Therefore, exact analytic solutions are very important not only
because they provide insight into the water and solute transport processes but also
because they are needed as validation tests for numerical schemes. Additionally,
exact analytic solutions form the bases for rational approximations and simplifications in terms of readily measured soil water properties and they are useful for testing various inverse techniques used in the estimation of soil hydraulic properties (Broadbridge and White, 1988).

In this thesis we present exact analytic solutions for one and two dimensional P.D.E.s describing transport of nonreactive, inert and adsorbing solutes in saturated and unsaturated soils. We use symmetry techniques as the main tool for construction of these solutions.

1.2 Structure of this thesis

In Chapter 2, we present mathematical modelling for water and solute transport in saturated and unsaturated soils. Throughout this thesis we focus on macroscopic models which are given in terms of P.D.E.s. A brief theory of algebraic techniques for symmetry reduction is provided in chapter 3. In chapter 4, we perform symmetry analysis of equations for transient solute transport in saturated soils. It turns out that a rich array of extra classical point symmetries are admitted for solute transport under radial and point vortex water flow. These symmetries lead to a number of exotic exact analytic solutions for non-radial solute transport on a background of radial water flow and solute transport under point vortex water flow. We classify these exact analytic or invariant solutions by the optimal systems of one dimen-
sional subalgebras. In chapter 5, we present exact analytic and numerical solutions for solute transport in unsaturated soils during steady evaporation from a water table. We choose reasonable functions $\theta(z)$ to describe the volumetric water content profile and then construct the corresponding realistic diffusivity and hydraulic conductivity functions. Classical and nonclassical Lie symmetry techniques are employed and exact analytic solutions are constructed. The nonclassical symmetry solutions are used to validate the method of lines software PDE TWO (Melgaard and Sincovec, 1981), which is then used to construct other relevant solutions. In chapter 6, we discuss modelling of transport of adsorbing solutes. The nonlinear A.D.E. is transformed into an I.N.D.E.. We systematically construct seemingly hidden potential symmetries. Instead of considering the auxiliary system associated with the governing equation, we analyse the related integrated equation obtained using the general integral variable. Reductions of the I.N.D.E. to O.D.E.s are performed and some invariant solutions are constructed. Furthermore, we use a class of integrodifferential equations for population densities to illustrate applications of integral variables. Also, numerical solutions are provided for transport of adsorbing solutes at a constant evaporation rate. Finally, a brief summary is presented in chapter 7.
Chapter 2

Formulation of water and solute transport equations

2.1 Introduction

Flow of water and transport of dissolved chemicals through soils may be described by microscopic or macroscopic models. Recent accounts on the theory of water and solute transport may be found in the literature e.g. Philip (1969), Bear and Verruijt (1987), Jury (1988) and Ségol (1994). A full theory of water and solute transport will require understanding of microscopic transport processes in fluctuating fluid flow fields in networks of tortuous channels (see e.g. Jury, 1988). Nevertheless, macroscopic models, described in terms of P.D.E.s, remain important for efficiently predicting transport at the field scale. Microscopic modelling is based on statistical
distribution of the grain and pore sizes, and requires many complicated measurements. Throughout our study we shall focus on macroscopic models based on local conservation laws (see e.g. Philip, 1969; Ségo, 1994; Wierenga, 1995). Here, the measured quantities are averaged over some region which is large compared to the mean grain and pore sizes. Soil water flow problems are usually considered to be isothermal and the law of conservation of energy is neglected.

2.2 Flow of water in saturated and unsaturated soils

Equations of motion of water in soils are based on Darcy’s law. This law assumes flow in saturated soils in which water occupies the entire soil void or pore spaces. However, most field-scale processes occur in unsaturated conditions and so Darcy’s law can be modified for such flows. Hydraulic properties for unsaturated flows involve complex relationships between variables such as hydraulic conductivity and water content and so on. The problem becomes more difficult when hysteresis is involved. Equations describing water flow are highly nonlinear and recently there has been an ongoing interest in their solutions for applications in soil physics and hydrology (e.g. Philip, 1969; Parlange, 1971, 1972; Clothier et al., 1981a; Broadbridge and White, 1988; Read and Broadbridge, 1996; Baikov et al., 1997).
2.2.1 Darcy’s law

Darcy (1856) experimentally investigated the flow of water in vertical homogeneous sand filters in connection with the fountains of the city of Dijon in France. His law states that the vector flow velocity or volumetric water flux $V [LT^{-1}]$ is proportional to the gradient of the pressure, i.e.

$$V = -\frac{K_s}{\rho_w g} \nabla (p - \rho_w g z), \quad (2.1)$$

where $\rho_w [ML^{-3}]$ is the water density, $g [LT^{-2}]$ is the gravitational acceleration, $z$ is the depth measured positively downward and $K_s [LT^{-1}]$ is a constant for each soil, called the hydraulic conductivity at saturation. The gravitational contribution to the overall pressure is represented by $\rho_w g z$. The generalization of Darcy’s law (Darcy, 1856) in isotropic saturated soils is given by

$$V = -K_s \nabla \Phi, \quad (2.2)$$

where $\Phi[L]$ is the total potential or the hydraulic pressure head which is due to molecular soil-water interaction and gravity, that is

$$\Phi = \Psi(\theta) - z, \quad (2.3)$$

obtained by writing $p = \rho_w g \Psi$ in (2.1). Here, $\Psi [L]$ is the soil water interaction potential and it is always negative in the unsaturated water flow regime. $p$ has dimension $[MLT^{-2}]$. Furthermore, $\Psi$ decreases rapidly as $\theta$ decreases (Moore, 1939:...
Philip, 1969). $K_s$ may be replaced by the hydraulic conductivity tensor $K_s$ for flow in anisotropic soil.

Experimental observations confirm that in unsaturated soils hydraulic conductivity depends on volumetric water content (Childs and Collis-George, 1950). Thus the modified Darcy’s law for flow in unsaturated soils is given by Darcy-Buckingham’s law (Buckingham, 1907)

$$V = -K(\theta)\nabla\Phi,$$

(2.4)

where $\theta$ $[L^3/L^3]$ is volumetric water content. It is observed that hydraulic conductivity $K$ is a strongly increasing and concave function of volumetric water content (Richards, 1931; Moore, 1939; Childs and Collis-George, 1950; Philip, 1967). Darcy’s law is applicable if the flow is laminar (i.e. $Re \leq 10$, where $Re$ is the Reynold’s number), and where soil-water interaction does not result in a change of permeability with a change in hydraulic gradient.

### 2.2.2 Classical Richards equation

Since we are dealing with the macroscopic description to model transport of water in soils then the equation of continuity can be used to model the flow and the conserved quantities are water concentration and flux density (Childs, 1969; Bear and Verruijt, 1987). The equation of continuity for mass conservation is given by

$$\frac{\partial c_w}{\partial t} + \nabla \cdot c_w u = 0,$$

(2.5)
where \( c_w \) \([\text{ML}^{-3}]\) is the gravimetric water concentration, \( t \) \([\text{T}]\) is time and \( u \) \([\text{LT}^{-1}]\) is the water velocity. Substituting \( c_w = \rho_w \theta \) into (2.5) yields

\[
\frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{V} = 0, \tag{2.6}
\]

where \( \rho_w \) \([\text{ML}^{-3}]\) is the density of water and \( \mathbf{V} = \frac{c_w}{\rho_w} \mathbf{u} \) is the volumetric flux density or the Darcian water flux. Under steady saturated water flow, Equation (2.6) reduces to Laplace’s equation

\[
\nabla^2 \Phi = 0. \tag{2.7}
\]

Combining (2.2), (2.3) and (2.6) yields Richards equation (Richards, 1931)

\[
\frac{\partial \theta}{\partial t} = \nabla \cdot [K(\theta) \nabla \Psi] - K'(\theta) \frac{\partial \theta}{\partial z}, \tag{2.8}
\]

or equivalently (2.8) may be written as the nonlinear Fokker-Plank diffusion-convection equation commonly used to describe nonhysteretic flow of water in uniform nonswelling soils, namely

\[
\frac{\partial \theta}{\partial t} = \nabla \cdot [D(\theta) \nabla \theta] - K'(\theta) \frac{\partial \theta}{\partial z}, \tag{2.9}
\]

where \( D(\theta) = K(\theta) \frac{\partial \rho_w}{\partial \theta} \) is soil water diffusivity with dimensions \([\text{L}^2\text{T}^{-1}]\) and \( z \) \([\text{L}]\) is the vertical depth measured positively downward. Diffusivity \( D(\theta) \) is a strongly increasing and concave function of volumetric water content (Buckingham, 1907). Here, the root extraction term \( S(\theta) \) is neglected. The dependence of \( D \) and \( K \) on \( \theta \) renders Equation (2.9) highly nonlinear. Equation (2.9) has a wide range of applications; its linear forms arise in heat conduction from sources moving relative to
Chapter 2: Transport equations

the medium (Carslaw and Jaeger, 1959), diffusion under an external force and in the mathematical probability theory of Markov processes (e.g. Bailey, 1964). Analytical solutions for Equation (2.9) have been constructed for steady infiltration into infinite and semi infinite seepage geometries (e.g. Philip, 1988) and finite geometries of arbitrary shape (e.g. Read and Broadbridge, 1996).

Few analytical solutions to nonlinear Fokker-Planck equations were studied before the early 1970s except for the general nonlinear diffusion equation obtained when the nonlinear convective term is neglected, e.g. in unsaturated water flow in which capillarity is much stronger than gravity (Philip, 1969). The general one dimensional nonlinear diffusion equation is invariant under the Boltzmann scaling symmetry and may be reduced to an O.D.E. with the invariant solutions of the form 

\[ zt^{-1/2} = f(\theta). \]

This leads to a definition of sorptivity constant (Philip, 1969)

\[ S = \int_{\theta_i}^{\theta_0} f(\theta) d\theta. \]

Here, \( \theta_i \) is the initial water content whilst \( \theta_0 \) is the water content at the soil surface in an infiltration problem. Fujita (1952) gave the solutions to the general one dimensional nonlinear diffusion equation for special cases of diffusivity, namely

\[ D(\theta) = a(b - \theta)^{-1}, \quad D(\theta) = a(b - \theta)^{-2}, \quad \text{and} \quad D(\theta) = a(\text{quadratic in } \theta)^{-1}, \]

with \( a \) and \( b \) being the constants. Linearisation of one dimensional Richards equation (2.9) is possible for special cases of diffusivity and hydraulic conductivity; for example assuming the diffusivity to be constant and hydraulic conductivity to be a quadratic...
function of water content leads to a Burgers equation which can be linearised by the Hopf-Cole transformation (Knight, 1973). Also a realistic form of diffusivity $D(\theta) = a(b - \theta)^{-2}$ in a zero gravity flow may be linearised by the Storm transformation (Knight and Philip, 1974). Further special cases which permit exact analytic solutions for the one dimensional forms of Richards equation have been considered over the years (see e.g. Fokas and Yortsos, 1982; Rosen, 1982; Rogers et al., 1983; Broadbridge and White, 1988; White and Broadbridge, 1988; Sander et al., 1988). Lie symmetry classification of Equation (2.9) was carried out using point symmetries (Oron and Rosenau, 1986; Sposito, 1990; Edwards, 1994; Yung et al., 1994) and potential symmetries (Bluman and Kumei, 1989; Katoanga, 1992; Sophocleous, 1996).

Since $\Psi$ is a one to one function of $\theta$, an alternative Richards equation may be written as

$$C(\Psi) \frac{\partial \Psi}{\partial t} = \nabla \cdot [K(\Psi) \nabla \Psi] - K'(\Psi) \frac{\partial \Psi}{\partial z},$$ (2.10)

where $C(\Psi) = \frac{\partial \theta}{\partial \Psi}$ is the specific water capacity. Similar to Equation (2.9) we neglect sources or sinks represented by a nonzero $S(\Psi)$ term.

**Classes of soils**

In the context of Richards equation, the relationships among the volumetric water content, soil-water interaction potential and hydraulic conductivity define hydraulic properties of a soil. Different classes of soils have been identified with different math-
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mathematical functions describing physical characteristics (e.g. Raats, 2001). Linear soils admit a constant diffusivity and hydraulic conductivity which is a linear function of water content. The resulting Richards equation is linear and solvable (Philip, 1969). The Green and Ampt or the so-called delta function soils have the property 

\[ D(\theta) = S^2 \delta(\theta_1 - \theta)/[2(\theta_1 - \theta_0)], \]

where \( \delta \) is a Dirac delta function. The Brooks-Corey power soils have the power law function for the hydraulic coefficient. The Gardner (1958) soil has \( K = K_f e^{a\Phi} \), where \( a \) is the sorptive number, regarded as the reciprocal of the sorptive length scale. The Broadbridge and White versatile nonlinear soils are defined by 

\[ D(\theta) = a(b - \theta)^{-2} \]

and 

\[ K(\theta) = \beta + \gamma(b - \theta) + \lambda/[2(b - \theta)], \]

where \( \beta, \gamma, \lambda \) are constants. The Knight soil has the constant diffusivity and quadratic conductivity. These mathematically idealised soils vary in their ability to represent real soil hydraulic data and in the ease with which they allow Richards equation to be solved either analytically or numerically.

2.3 Solute transport in soils

Water present in the soil and constituting its liquid phase is never chemically pure. To begin with, the water entering the soil as rain or irrigation is itself a solution. Rain water whilst in the atmosphere dissolves atmospheric gases such as carbon dioxide, oxygen and industrially produced gases e.g. oxides of sulphur and nitrogen. Moreover, during its residence, infiltrated water due to rain or irrigation dissolves
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salts stored in the soil. The soil subsurface contains vast stores of salts occurring naturally or from chemicals introduced by human agricultural or industrial practices. Transport of chemicals in soils is due to convection of dissolved substances, molecular or ionic diffusion and dispersion. Other factors affecting solute transport in soils include soil matrix (solid)-solute interaction and decay phenomena. Total solute concentration \( c_{\text{tot}} \) in soil is expressed in terms of the phase contribution,

\[
c_{\text{tot}} = \rho_b c_a + \theta c_l + a c_g,
\]

where \( c_a, c_l, c_g \) \([ML^{-3}]\) are adsorbed, dissolved and gaseous solute concentration respectively. \( \rho_b \) \([ML^{-3}]\) is the bulk density, \( \theta \) is as in Section 2.2.1 and \( a \) \([L^3L^{-3}]\) is the volumetric air content. Mathematical formulations of solute transport process, as in the case of water transport, obey the laws of conservation of mass. Since we consider solute transport to be an isothermal process, laws of conservation of energy are neglected. We restrict our study to transport, through homogeneous soils, of non-reactive solutes which have negligible vapour pressure.

### 2.3.1 Solute transport by convection

Solute flux density is the sum of three components due to convection, molecular diffusion and dispersion. The convective solute flux \( J_c \), transported by convection of soil water, is proportional to the solute concentration \( c \), i.e.

\[
J_c = cV,
\]  

(2.11)
where $V$ is the Darcian water flux discussed in the previous section and $c$ is solute concentration.

### 2.3.2 Solute transport by diffusion

Diffusive transport results from the movement of molecules, which takes place only in that fraction of the pore space filled with water, from higher to lower concentrations. Molecular diffusion is expressed by Fick's law

$$J_{md} = -\theta D_0 \nabla c,$$

(2.12)

where $D_0$ is the diffusion coefficient and $\theta$ is as in previous section. The diffusion coefficient of soil or other porous media is generally less than that of water due to the tortuous nature of the flow path and also since water occupies only a fraction of the soil volume.

### 2.3.3 Hydrodynamic dispersion

The porous nature of soils results from existence of pores, fissures, worm holes and loose stacking of soil material. Different shapes, sizes and orientation of the open spaces and pores results in changes and differences in fluid flow velocity from place to place and from pore to pore. Velocities have been found to be the lowest near the pore wall. Furthermore, velocity variations cause solute to be transported at different rates leading to mixing similar to molecular diffusion at the macroscopic level.
This mechanical spreading which takes into account the tortuous convection of the solution relative to the average convective motion is called hydrodynamic dispersion (see e.g. Bear, 1979). Hydrodynamic dispersion flux $J_{hd}$ can be formulated in a manner analogous to (2.12), except that dispersion coefficient $D_e$ is used instead of a diffusion coefficient (see e.g. Bear, 1979) that is,

$$J_{hd} = -\theta D_e \nabla c. \quad (2.13)$$

The dispersion coefficient $D_e [L^2 T^{-1}]$ is found to be an increasing function of water flow velocity $v$, which is given by $v = |V|/\theta$. Experimental observation reveals that this function is given by a power law, $D_e = D_1 v^m$, where $D_1 > 0$ and $1 < m < 2$ (Salles et al., 1993; Philip, 1994). In the case $m = 1$, $D_1 [L]$ is the dispersivity. In stratified or fractal heterogeneous porous media, microscopic pore velocity is not a dominant dispersion process, in fact it has been shown that field-measured dispersivities are spatial or scale-dependent (Wheatcraft and Tyler, 1988). Based on this fact, Su (1995, 1997) shows that for transport in fractal subsurface flow, the dispersion coefficient is space dependent. This led to some simplification and construction of similarity solutions for Fokker-Plank and Feller-Fokker-Planck equations describing transport of conservative and reactive solutes in fractal heterogeneous porous media.
2.3.4 Governing solute transport equations

Combining (2.11), (2.12) and (2.13) we have the total solute flux density \( J \), i.e.

\[
J = -\theta D_0 \nabla c - \theta D_e \nabla c + \nabla c.
\]  

(2.14)

Since \( D_0 \) and \( D_e \) are microscopically similar these terms are usually added together (see e.g. Bear and Verruijt, 1987), i.e. we may write \( D_\nu(v) = D_0 + D_e(v) \). Here, molecular diffusion is negligible compared to dispersion; hence \( D_\nu(v) \) is approximated by \( D_e(v) \). Combining (2.14) with the mass conservation law

\[
\frac{\partial (c\theta)}{\partial t} + \nabla \cdot J = 0,
\]  

(2.15)

we obtain the convection-dispersion equation (C.D.E.) describing transport of non-
adsorbing, non-volatile solutes, namely;

\[
\frac{\partial (c\theta)}{\partial t} = \nabla \cdot (\theta D_\nu(v) \nabla c) - \nabla \cdot (c \nabla c).
\]  

(2.16)

In most solute transport problems, water flow is adequately modelled by steady state flows rather than transient flows. For two dimensional steady water flows where \( \theta = \theta_s \) with \( \theta_s \) being water content at saturation, Darcy’s law (2.2) and the equation of continuity \( \nabla \cdot \mathbf{V} = 0 \) implies Laplace equation (2.7). In this case Equation (2.16) reduces to

\[
\frac{\partial c}{\partial t} = \nabla \cdot (D_\nu(v) \nabla c) + k \nabla \Phi \cdot \nabla c,
\]  

(2.17)

where \( k = K_s/\theta_s \) and \( v = |k \nabla \Phi| \). Most of existing analytical solutions for Equation (2.16) have assumed a uniform water flow, with Darcian water flux and \( D_\nu(v) \)
Chapter 2: Transport equations

constant (see e.g. van Genuchten and Alves, 1982; Broadbridge et al., 2000). The problem is difficult when $v$ must be the modulus of potential flow velocity field for incompressible fluid. However, the Laplace preserving transformations or the conformal mappings from the Cartesian to streamline coordinates result in a simpler version of Equation (2.17). We re-derive this version in the next steps (For a review, see e.g. Hoopes and Harleman, 1967; Ségol, 1994).

Convection-dispersion equation in streamline coordinates

Let $(x, y) \rightarrow (\phi, \psi)$ be a conformal map, with $\phi = K_s \Phi$ being the velocity potential satisfying $\mathbf{v} = \nabla \phi$ and $\psi$ being a conjugate harmonic stream function satisfying $\phi_x = \psi_y$ and $\phi_y = -\psi_x$. Also $\phi$ and $\psi$ satisfy Laplace equation i.e. $\nabla^2 \phi = 0$ and $\nabla^2 \psi = 0$. Here $\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}$ and $\nabla = \hat{\epsilon}_\phi \frac{\partial}{\partial \phi} + \hat{\epsilon}_\psi \frac{\partial}{\partial \psi}$. It follows that for an arbitrary function $f(\phi, \psi)$

$$
\nabla^2 f = \frac{\partial}{\partial x} \left[ \frac{\partial\phi}{\partial x} \frac{\partial}{\partial \phi} + \frac{\partial\phi}{\partial y} \frac{\partial}{\partial \psi} \right] f + \frac{\partial}{\partial y} \left[ \frac{\partial\phi}{\partial y} \frac{\partial}{\partial \phi} + \frac{\partial\phi}{\partial x} \frac{\partial}{\partial \psi} \right] f,
$$

$$
= \left[ (\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 \right] \left[ \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial \psi^2} \right] + \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial \psi^2},
$$

$$
= v^2 \nabla^2 f,
$$

(2.18)

where $\nabla^2 = \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \psi^2}$ and $v = \| \mathbf{v} \|$. 

$$
\mathbf{v} \cdot \nabla f = \left( \frac{\partial \phi}{\partial x} \right) \left[ \frac{\partial\phi}{\partial x} \frac{\partial}{\partial \phi} + \frac{\partial\phi}{\partial y} \frac{\partial}{\partial \psi} \right] f + \left( \frac{\partial \phi}{\partial y} \right) \left[ \frac{\partial\phi}{\partial y} \frac{\partial}{\partial \phi} + \frac{\partial\phi}{\partial x} \frac{\partial}{\partial \psi} \right] f,
$$

$$
= v^2 \frac{\partial f}{\partial \phi},
$$

(2.19)
and finally

\[ \nabla v \cdot \nabla f = \left( \frac{\partial \phi}{\partial x} \frac{\partial v}{\partial \phi} - \frac{\partial \phi}{\partial y} \frac{\partial v}{\partial \psi} \right) \left( \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial \phi} - \frac{\partial \phi}{\partial y} \frac{\partial f}{\partial \psi} \right) \]

\[ + \left( \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x} \right) \left( \frac{\partial \phi}{\partial y} \frac{\partial f}{\partial \phi} + \frac{\partial \phi}{\partial x} \frac{\partial f}{\partial \phi} \right). \]

\[ = v^2 \nabla v \cdot \nabla f. \quad (2.20) \]

**Corollary:** From Equation (2.18), Laplace equation implies \( \nabla^2 f = 0 \iff \nabla^2 f = 0. \)

Now by (2.18), (2.19) and (2.20) we may write Equation (2.17) as

\[ \frac{\partial c}{\partial t} = v^2 \nabla \cdot [D_\nu(v) \nabla c] + v^2 \frac{\partial c}{\partial \phi}, \quad (2.21) \]

or equivalently

\[ \frac{1}{v^2} \frac{\partial c}{\partial t} = \frac{\partial}{\partial \phi} \left[ D_\nu(v) \frac{\partial c}{\partial \phi} \right] + \frac{\partial}{\partial \psi} \left[ D_\nu(v) \frac{\partial c}{\partial \psi} \right] + \frac{\partial c}{\partial \phi}. \quad (2.22) \]

Note that the convective term is now singular. One advantage of using streamline coordinates is that an irregular computational flow may be mapped to a rectangle. Another advantage is that one of the derivatives in either the streamwise or the cross-stream direction may be negligible compared to the other, affording further simplification.

### 2.3.5 Miscible displacement and breakthrough curves

An experiment is set up, where water is flowing uniformly at steady state through a homogeneous soil column of length \( L \) maintained at a constant volumetric water
content. If a solution varying in composition or concentration of the pre-existing pore solution is introduced into the column and outflow is collected at one end, its composition is seen to change in time as the old solution is displaced and replaced by the new one. If the two solutions readily mix, then the process is called miscible displacement. Otherwise it is immiscible displacement. Moreover the plots of outflowing solution concentration versus time are called breakthrough curves.

Furthermore the time $t_{BT} = L/v$ at which the centre of each solute front, drawn for different values of dispersion coefficient $D_v$ arrive at the outflow end of the column is called breakthrough time. Here, $v$ is the average pore velocity. The pattern of liquid displacement without mixing (or a flow in which dispersion is neglected) is called piston flow. Lastly, the dimensionless parameter which gives a measure of the convective versus the diffusive flow of the solute is called Péclet number defined by $Pe = \frac{vL}{D_v}$.
Chapter 3

Algebraic techniques for

symmetry reductions

3.1 Continuous symmetry groups

The theory and application of continuous symmetry groups were founded 120 years ago by S. Lie (1880). Recent accounts on this theory may be found in many excellent texts such as those of Ibragimov (1985); Olver (1986); Bluman and Kumei (1989); Sposito (1990); Hill (1992); Bluman and Anco (2002). A symmetry group of a system of D.E.s is a group of transformations mapping any arbitrary solution to another solution of the system. Such groups depend on continuous parameters and may be classified as point transformations (point symmetries) if they act on the graph space coordinatised by independent and dependent variables of a system of
differential equations (D.E.s) or more generally, contact or Lie Bäcklund symmetries if they act on the space including all first or higher derivatives of dependent variables respectively.

Given a continuous one-parameter symmetry group, in most practical cases, one may reduce the number of independent variables by one. The most familiar symmetry is the rotational symmetry that enables one to reduce the variables \((x, y)\) to a single radial variable \(r\). For example, consider the nonlinear diffusion equation

\[
\frac{\partial \theta}{\partial t} = \nabla \cdot (D(\theta) \nabla \theta).
\]  

This equation is invariant under the group of plane rotations

\[
x' = x \cos(\epsilon) - y \sin(\epsilon),
\]

\[
y' = x \sin(\epsilon) + y \cos(\epsilon),
\]

\[
\theta' = \theta.
\]

This Lie group of transformations depends continuously on the group parameter \(\epsilon\) which is the rotation angle. The invariants of this transformation group are \(\theta\) and radial coordinate \(r = \sqrt{x^2 + y^2}\). Rotationally invariant solutions satisfy a reduced P.D.E. for \(\theta(r, t)\). The next most common example is the Boltzmann scaling symmetry;

\[
x' = xe^\epsilon,
\]

\[
t' = te^{2\epsilon},
\]
\[ \theta' = \theta, \]

which leaves the nonlinear diffusion equation invariant. The group invariants are \( \theta \) and \( \phi = \frac{x}{\sqrt{t}} \). Invariant solutions satisfy an O.D.E. for \( \theta = f(\phi) \). Philip (1957, 1969) used this form as a starting point for the infiltration series to be used when gravity is not ignored in flow of water in unsaturated soils. The identity transformation, which must belong to any group, is conventionally labeled by \( \epsilon = 0 \) (e.g. for rotation by angle \( \epsilon = 0 \) or scaling by factor \( e^\epsilon = e^0 = 1 \)). As well as the familiar geometric one-parameter groups, there may be additional more complicated symmetry groups that apply only to a special subclasses within the class of governing equations. For example, if \( D(\theta) = \theta^{-4/3} \), then the one dimensional nonlinear diffusion equation is invariant under the non-obvious symmetry group (e.g. Galaktionov et al., 1988)

\[
\begin{align*}
x' &= \frac{x}{1 + \epsilon x}, \\
\theta' &= \theta(1 + \epsilon x)^{-3}, \\
t' &= t.
\end{align*}
\]

We need only to use infinitesimal symmetries which are obtained from transformation laws up to first degree in \( \epsilon \).

If a P.D.E. is invariant under a point symmetry, one can often determine similarity or invariant solutions which are invariant under some subgroup of the full group admitted by the P.D.E. In some cases reduced equations may not be solvable. If the governing equation is an O.D.E. rather than a P.D.E., reduction of
order rather than reduction of variables, is possible. Classical symmetry methods are useful tools in finding the analytical solutions to differential equations. After specification of initial and boundary value problems, in standard treatments the admitted symmetry group must also leave all the conditions invariant. See however the recent paper by Goard (2003b) in which symmetric solutions are constructed for asymmetric boundary value problems. In our analysis we employ classical Lie point symmetries and extensions of classes of symmetries admitted by systems of D.E.s namely potential (nonlocal) symmetries, equivalence transformations, contact symmetries and nonclassical (conditional) symmetries.

We now outline Lie symmetry methods developed in an effort to find further new analytical solutions. These methods are algorithmic and hence amenable to symbolic computation (e.g. Sherring, 1993; Mansfield, 1993).

3.1.1 Classical Lie point symmetries

Consider $m$th order partial differential equation ($m \geq 2$) in $n$ independent variables $x = (x_1, x_2, ..., x_n)$ and one dependent variable $u$, given by

$$F(x, u, u^{(1)}, ..., u^{(m)}) = 0, \quad (3.2)$$

where $u^{(k)}$, denotes the set of coordinates corresponding to all the $k$th order partial derivatives of $u$ with respect to $x_1, x_2, ..., x_n$. That is, a coordinate in $u^{(k)}$ is denoted
by

\[ u_{j_1, j_2, \ldots, j_k} = \frac{\partial^k u}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_k}}, \]  

(3.3)

with \( j_p = 1, 2, \ldots, n \) and \( p = 1, 2, \ldots, k \). To determine a one-parameter group of transformations we consider the infinitesimal transformations

\begin{align*}
    x_j' &= x_j + \epsilon X_j(x, u) + O(\epsilon^2), \\
    u' &= u + \epsilon U(x, u) + O(\epsilon^2),
\end{align*}

(3.4) (3.5)

which leave (3.2) invariant. The coefficients \( X_j \) (\( j = 1, \ldots, n \)) and \( U \) are the components of the infinitesimal symmetry generator, which is one of the vector fields

\[ \Gamma = X_j(x, u) \frac{\partial}{\partial x_j} + U(x, u) \frac{\partial}{\partial u}, \]

(3.6)

which span the associated Lie algebra. Here, we sum over a repeated index (see e.g. Bluman and Kumei, 1989). Sometimes the components \( X_j \) and \( U \) are referred to simply as the “infinitesimals”. The infinitesimal criterion for invariance of a P.D.E. such as (3.2) is given by

\[ \Gamma^{(m)} F\bigg|_{F=0} = 0, \]

(3.7)

where \( \Gamma^{(m)} \) is the \( m \)th extension or prolongation of the infinitesimal generator \( \Gamma \) in (3.6) given by

\[ \Gamma^{(m)} = \Gamma + U_j(x, u, u^{(1)}) \frac{\partial}{\partial u_j} + \ldots + U_{j_1 j_2 \ldots j_m}(x, u, u^{(1)}, \ldots, u^{(m)}) \frac{\partial}{\partial u_{j_1 j_2 \ldots j_m}}, \quad m = 1, 2, \ldots. \]

(3.8)
Chapter 3: Symmetry techniques for differential equations

Here

\[ U_j = D_j U - (D_j X_k) u_k, \quad j = 1, 2, ..., n; \]  
\[ U_{j_1 j_2 ... j_m} = D_{j_m} (U_{j_1 j_2 ... j_{m-1}}) - (D_{j_m} X_k) u_{j_1 j_2 ... j_{m-1} k}, \]

with \( j_p = 1, 2, ..., n \) for \( p = 1, 2, ..., m, \quad m = 2, 3, ... \) and \( D_j \) being the total \( x_j \) derivative

\[ D_j = \frac{\partial}{\partial x_j} + u_j \frac{\partial}{\partial u} + ... + u_{j_1 j_2 ... j_m} \frac{\partial}{\partial u_{j_1 j_2 ... j_m}}. \]

The invariance condition (3.7) results in an overdetermined linear system of determining equations for the coefficients \( X_j(x, u) \) and \( U(x, u) \), which may be automatically solvable by many computer algebra programs. It may happen that the only solution to the overdetermined system of linear equations is trivial that is, \( X_j(x, u) = U(x, u) = 0 \). When the general solution of the determining equations is nontrivial two cases arise; (a) if the general solution contains a finite number, say \( s \), of essential arbitrary constants then it corresponds to an \( s \)-parameter Lie group of point transformations or an \( s \)-dimensional Lie algebra spanned by the base vectors (3.6); and (b) if the general solution cannot be expressed in terms of a finite number of essential constants, for example when it contains an arbitrary function of independent and/or dependent variables, then it corresponds to an infinite-parameter Lie group of transformations of the infinite-dimensional superposition symmetry generator or simply the infinite symmetry generator. If the coefficients of the governing differential equation are functions of independent and/or dependent variables, then
the vector fields or symmetries admitted by the equation in question when these coefficients are arbitrary span the principal Lie algebra (see e.g. Ibragimov et al., 1991).

In the method of variable reduction by invariants one seeks a compatible invariant solution expressed in the form

$$\omega(\phi_1, \phi_2, ..., \phi_n) = 0;$$

where $\phi_1, \phi_2, ..., \phi_n$ is a complete set of $n$ independent invariants for a one-parameter Lie point transformation group (3.4) and (3.5). The basis for invariants may be constructed by solving the characteristic equations in Pfaffian form

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = ... = \frac{du}{U}. \quad (3.12)$$

In essence, a one-parameter group of transformations is a classical symmetry of (3.2), provided (3.4) and (3.5) leave (3.2) invariant. Moreover an invariant solution of (3.2),

$$u = G(x),$$

must also satisfy the invariant surface condition (I.S.C.)

$$X_j(x, u) \frac{\partial u}{\partial x_j} = U(x, u), \quad (3.13)$$

which follows from

$$\frac{d}{d\epsilon} (u' - G(x')) = 0. \quad (3.14)$$
Note that the invariant surface condition (3.13) is equivalent to the solution of the system (3.12). We illustrate with the next example, the method for construction of an invariant solution for a D.E. such as

$$\frac{\partial^2 F}{\partial \phi^2} - e^{-4\phi} \frac{\partial F}{\partial \gamma} + \left(2 + e^{-2\phi}\right) \frac{\partial F}{\partial \phi} + a^2 F = 0, \quad a \in \mathbb{R},$$

which, among other point symmetries, admits

$$\Lambda_1 = \gamma^2 \frac{\partial}{\partial \gamma} - \frac{\gamma}{2} \frac{\partial}{\partial \phi} + \left(\frac{\gamma e^{-2\phi}}{4} - \frac{e^{-4\phi}}{16} - \frac{\gamma^2}{4} - \frac{\gamma}{2}\right) F \frac{\partial}{\partial F}.$$

The corresponding characteristic equation is

$$\frac{d\gamma}{\gamma^2} = -\frac{2d\phi}{\gamma} = \frac{dF}{\left(\frac{\gamma e^{-2\phi}}{4} - \frac{e^{-4\phi}}{16} - \frac{\gamma^2}{4} - \frac{\gamma}{2}\right) F}.$$

Integrating the first two terms in (3.15) leads to

$$\ln \gamma = -2\phi + \text{constant}$$

and rearranging we obtain the invariant

$$\vartheta = \gamma e^{2\phi}.$$  \hspace{1cm} (3.16)

The first and last term in (3.15) may be rewritten as

$$\frac{dF}{F} = \left(\frac{\gamma e^{-2\phi}}{4} - \frac{e^{-4\phi}}{16} - \frac{\gamma^2}{4} - \frac{\gamma}{2}\right) \frac{d\gamma}{\gamma^2}.$$

After expressing $e^{-2\phi}$ and $e^{-4\phi}$ in terms of the variables $(\vartheta, \gamma)$, we then integrate to obtain the functional form

$$F = \exp \left(\frac{e^{-2\phi}}{4} - \frac{\gamma}{4} - \frac{\ln \gamma}{2} - \frac{e^{-4\phi}}{16\gamma}\right) \times g(\vartheta),$$  \hspace{1cm} (3.17)
wherein, we have re-substituted \( \vartheta \) in terms of the variables \((\gamma, \phi)\) and expressed the constant of integration as a function \( g \) of an invariant \((3.16)\). Note that \( g \) is also an invariant. Substituting \((3.17)\) into the governing equation we observe that \( g \) must satisfy the Euler equation

\[
\vartheta^2 g'' + 2\vartheta g' + \frac{a^2}{4} g = 0.
\]

The solutions of this O.D.E. arise for three cases. Considering only the case \( a^2 > 1 \), we obtain (see also Polyanin and Zaitsev, 1995)

\[
g = \vartheta^{-1/2} \left[ k_1 \sin \left( \mu \log |\vartheta| \right) + k_2 \cos \left( \mu \log |\vartheta| \right) \right],
\]

where \( \mu = \frac{1}{2} \sqrt{1 - a^2} \). Hence in terms of the original variables we obtain an invariant solution

\[
F = \frac{1}{\gamma} \exp \left( \frac{e^{-2\phi}}{4} - \frac{\gamma}{4} - \frac{e^{-4\phi}}{16\gamma} - \phi \right) \times \left[ k_1 \sin \left( \mu \log |\gamma e^{2\vartheta}| \right) + k_2 \cos \left( \mu \log |\gamma e^{2\vartheta}| \right) \right].
\]

### 3.1.2 Contact and higher order symmetries

The group of contact symmetries of a system of D.E.s is conceptualised as the group of invariance transformations acting on the tangent space coordinatised by independent variables, dependent variables and all first order derivatives of the dependent variables. The group of Lie Bäcklund symmetries is the invariance group acting on the jet space coordinatised by all independent and dependent variables, as well as all higher order derivatives of the dependent variables. Introducing the characteristic
function $W$ defined by
\[ W(x, u, u^{(1)}) = X_j u_j - U, \]
on one may construct contact symmetries admitted by a P.D.E. such as (3.2), by considering the infinitesimal transformations
\[
\begin{align*}
x'_j &= x_j + \epsilon X_j (x, u, u^{(1)}) + O(\epsilon^2), \\
u' &= u + \epsilon U (x, u, u^{(1)}) + O(\epsilon^2), \\
u'_j &= u_j + \epsilon U_j^{(1)} (x, u, u^{(1)}) + O(\epsilon^2),
\end{align*}
\]
where
\[
\begin{align*}
X_j &= \frac{\partial W}{\partial u_j}, \\
U &= u_i\frac{\partial W}{\partial u_i} - W, \\
U_j^{(1)} &= -\frac{\partial W}{\partial x_j} - u_j\frac{\partial W}{\partial u},
\end{align*}
\]
are the infinitesimals of the symmetry generator
\[
\Gamma = X_j (x, u, u^{(1)}) \frac{\partial}{\partial x_j} + U (x, u, u^{(1)}) \frac{\partial}{\partial u} + U_j^{(1)} (x, u, u^{(1)}) \frac{\partial}{\partial u_j}.
\]
Here $u^{(1)}$ denotes all first order derivatives of $u$, i.e. a coordinate in $u^{(1)}$ is denoted by $u_j = \frac{\partial u}{\partial x_j}$. Contact symmetries are said to be equivalent to classical point symmetries if the infinitesimals $X_j$ and $U$ are independent of $u^{(1)}$. 
3.1.3 Nonclassical (Conditional) symmetries

The nonclassical symmetry method introduced by Bluman and Cole (1969), seek the invariance of the system of P.D.E.s composed of the given equation such as (3.2) with its invariant surface condition (3.13). That is, the infinitesimal criterion for invariance of P.D.E. such as (3.2) is

$$\Gamma^{(m)} F \bigg|_{F=0,I.S.C.\,(3.13)} = 0, \quad (3.18)$$

where $\Gamma^{(m)}$ is given in (3.8). Unlike in the classical symmetry methods, this invariance condition results in an overdetermined nonlinear system of equations for the infinitesimals $X_j$ and $U$. This nonlinear system of equations is much harder and more complicated to solve. Nearly all computer software algebras to date, are written specifically to calculate classical symmetries. However, one may use an interactive computer software program such as REDUCE (Hearn, 1985) to determine the nonclassical symmetries. The term strictly nonclassical is often used for nonclassical symmetries that are not equivalent to any classical symmetries.

3.1.4 Nonlocal (Potential) symmetries

Symmetry transformations whose infinitesimals are dependent on independent variables, dependent variables and derivatives of dependent variables are classified as local symmetries. On the other hand, if the infinitesimals depend not only on independent and dependent variables but also on the integrals of the dependent variable
then they are referred to as nonlocal symmetries. Examples of local symmetries include point, contact and Lie Bäcklund symmetries whereas, potential symmetries are examples of nonlocal symmetries.

By nonlocal (potential) symmetry methods, it is possible to find further types of analytical solutions that are not obtainable via local symmetries. Furthermore, one may construct solutions for boundary value problems posed for P.D.E.s and linearisation of nonlinear P.D.E.s is also possible via potential symmetry analysis (Bluman and Kumei, 1989).

The technique for finding potential symmetries involves writing a given P.D.E. in a conserved form with respect to some choices of its variables. Consider a scalar \( m \)th order P.D.E. \( R\{x, u\} \) with \( n \) independent variables \( x = (x_1, x_2, ..., x_n) \) and a single dependent variable \( u \) and suppose \( R\{x, u\} \) can be written in a conserved form

\[
D_j f^j(x, u, u^{(1)}, u^{(2)}, ..., u^{(m-1)}) = 0,
\]

where \( D_j \) is the total derivative operator defined in (3.11)

\[
D_j = \frac{\partial}{\partial x_j} + u_j \frac{\partial}{\partial u} + ... + u_{j_1j_2...j_m} \frac{\partial}{\partial u_{j_1j_2...j_m}}.
\]

Now, one may introduce \( m - 1 \) auxiliary dependent variables or the potentials \( v = (v^1, v^2, ..., v^{m-1}) \) and form an auxiliary system \( S\{x, u, v\} \) namely;

\[
f^1(x, u, u^{(1)}, u^{(2)}, ..., u^{(m-1)}) = \frac{\partial}{\partial x_2} v^1;
\]

\[
f^k(x, u, u^{(1)}, u^{(2)}, ..., u^{(m-1)}) = (-1)^{(k-1)} \left( \frac{\partial}{\partial x_{k+1}} v^k + \frac{\partial}{\partial x_{k-1}} v^{k-1} \right), 1 < k < n;
\]
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\[ f^m(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(m-1)}) = (-1)^{n-1} \frac{\partial}{\partial x_{n-1}} v^{n-1}. \]

Since \( R(x, u) \) is embedded in \( S(x, u, v) \), then any arbitrary solution \( (u(x), v(x)) \) of \( S(x, u, v) \) will define a solution \( u(x) \) of \( R(x, u) \). Moreover, to any solution \( u(x) \) of \( R(x, u) \) there corresponds a function \( v(x) \) such that \( (u(x), v(x)) \) defines a solution of \( S(x, u, v) \). Local symmetries admitted by an auxiliary system \( S(x, u, v) \) are realised as nonlocal or potential symmetries admitted by \( R(x, u) \) if and only if the infinitesimals of variables \( (x, u) \) of \( S(x, u, v) \) depend explicitly on the potential variable \( v \). A symmetry group is referred to as a strictly nonlocal (potential) symmetry if it is not equivalent to any local symmetry. Pucci and Saccomandi (1993) provided the necessary conditions for a given governing P.D.E. written in conserved form to admit potential symmetries. Furthermore, a P.D.E. may be written as a system of first order P.D.E.s in more than one inequivalent way (Pucci and Saccomandi, 1993). In some cases potential symmetry-bearing auxiliary systems may be hidden (see e.g. Moitsheki et al., 2003b), hence care must be taken in defining auxiliary systems associated with the governing equations.

3.2 Equivalence transformations

An equivalence transformation is a nondegenerate change of variables of any D.E. into a D.E. of the same form, i.e. equivalence transformation leaves a specific class of equations invariant (Ovsiannikov, 1982). The set of all point transformations de-
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fined by the equivalence transformations forms a group. Calculation of equivalence transformations involves the use of Lie's algorithm, after regarding all arbitrary coefficient functions of the system of governing D.E.s to be variables (see e.g. Akhatov et al., 1987; Ibragimov et al., 1991). These methods are mainly used in the preliminary group classification of classes of D.E.s. Lisle (1992) presented the theoretical foundation of the construction of the equivalence group.

3.3 Optimal Systems

Suppose a P.D.E. of the form (3.2) admits an \( s \)-dimensional Lie algebra \( \mathcal{L}_s \) namely \( \Gamma_1, \Gamma_2, ..., \Gamma_s \). Reduction of independent variables by one is possible using any linear combination of base vectors

\[
\Gamma = a_1 \Gamma_1 + a_2 \Gamma_2 + ... + a_s \Gamma_s. \tag{3.19}
\]

In order to ensure that a minimal complete set of reductions is obtained from symmetries admitted by the governing equation, an optimal system (Ovsiannikov 1982; Olver 1989; Ibragimov et al., 1991) is constructed. An optimal system of a Lie algebra is a set of \( r \)-dimensional subalgebras such that every \( r \)-dimensional subalgebra is equivalent to a unique element of the set under some element of the adjoint representation;

\[
\text{Ad}(\exp(\epsilon \Gamma_i))\Gamma_j = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \text{ad}(\Gamma_i)^n \Gamma_j = \Gamma_j - \epsilon [\Gamma_i, \Gamma_j] + \frac{\epsilon^2}{2} [\Gamma_i, [\Gamma_i, \Gamma_j]] - ... \tag{3.20}
\]
where $[\Gamma_i, \Gamma_j] = \Gamma_i \Gamma_j - \Gamma_j \Gamma_i$ is the commutator of $\Gamma_i$ and $\Gamma_j$. Patera and Winternitz (1977) constructed the optimal system of all one dimensional Lie subalgebras arising from three and four dimensional Lie algebras by comparing the Lie algebra with standard classifications previously evaluated. An alternative method developed by Olver (1989) involves simplifying as much as possible the generator (3.19) by subjecting it to judiciously chosen adjoint transformations. The latter method is illustrated by the following example and will be used in our analysis.

**Example**  
Equation

$$\frac{\partial^2 F}{\partial \phi^2} - e^{-4\phi} \frac{\partial F}{\partial \gamma} + \left(2 + e^{-2\phi}\right) \frac{\partial F}{\partial \phi} + a^2 F = 0,$$  
(3.21)

which arises in solute transport theory admits, beside the infinite symmetry generator, the four dimensional Lie algebra spanned by the base vectors

$$\begin{align*}
\Lambda_1 &= \gamma^2 \frac{\partial}{\partial \gamma} - \frac{\gamma}{2} \frac{\partial}{\partial \phi} + \left(\frac{e^{-2\phi}}{4} - \frac{e^{-4\phi}}{16} - \frac{\gamma^2}{4} - \frac{\gamma}{2}\right) F \frac{\partial}{\partial F}, \\
\Lambda_2 &= \gamma \frac{\partial}{\partial \gamma} - \frac{1}{4} \frac{\partial}{\partial \phi} + \left(\frac{e^{-2\phi}}{8} - \frac{\gamma}{4}\right) F \frac{\partial}{\partial F}, \\
\Lambda_3 &= \frac{\partial}{\partial \gamma}, \\
\Lambda_4 &= F \frac{\partial}{\partial F}.
\end{align*}$$

(3.22)

Reduction of independent variables of Equation (3.21) by one is possible using any linear combination

$$\Lambda = a_1 \Lambda_1 + a_2 \Lambda_2 + a_3 \Lambda_3 + a_4 \Lambda_4,$$  
(3.23)

where $a_1, a_2, ..., a_4$ are real constants. We require to simplify as much as possible the coefficients $a_1, ..., a_4$ by carefully applying the adjoint maps to $\Lambda$. To compute
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the optimal system we first construct the commutator table, given by

<table>
<thead>
<tr>
<th>$[\Lambda_i, \Lambda_j]$</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
<th>$\Lambda_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_1$</td>
<td>0</td>
<td>$-\Lambda_1$</td>
<td>$\frac{1}{4}\Lambda_4 - \Lambda_2$</td>
<td>0</td>
</tr>
<tr>
<td>$\Lambda_2$</td>
<td>$\Lambda_1$</td>
<td>0</td>
<td>$\frac{1}{4}\Lambda_4 - \Lambda_3$</td>
<td>0</td>
</tr>
<tr>
<td>$\Lambda_3$</td>
<td>$\Lambda_2 - \frac{1}{4}\Lambda_4$</td>
<td>$\Lambda_3 - \frac{1}{4}\Lambda_4$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Lambda_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where $[\Lambda_i, \Lambda_j] = \Lambda_i\Lambda_j - \Lambda_j\Lambda_i$.

Using (3.20) in conjunction with the commutator table, for example

$$\text{Ad} (\exp (\epsilon \Lambda_2)) \Lambda_3 = \Lambda_3 - \epsilon \left[ \Lambda_2, \Lambda_3 \right] + \frac{\epsilon^2}{2!} \left[ \Lambda_2, \left[ \Lambda_2, \Lambda_3 \right] \right] - \ldots,$$

$$= \Lambda_3 - \epsilon \left( \frac{1}{4} \Lambda_4 - \Lambda_3 \right) - \frac{\epsilon^2}{2!} \left( \frac{1}{4} \Lambda_4 - \Lambda_3 \right) - \ldots,$$

$$= e^\epsilon \Lambda_3 + \frac{1}{4} \left( 1 - e^\epsilon \right) \Lambda_4,$$  \hspace{1cm} (3.24)

$$\text{Ad} (\exp (\epsilon \Lambda_3)) \Lambda_1 = \Lambda_1 - \epsilon \left[ \Lambda_3, \Lambda_1 \right] + \frac{\epsilon^2}{2!} \left[ \Lambda_3, \left[ \Lambda_3, \Lambda_1 \right] \right] - \ldots,$$

$$= \Lambda_1 - \epsilon \left( \Lambda_2 - \frac{1}{4} \Lambda_4 \right) + \frac{\epsilon^2}{2!} \left( \Lambda_3 - \frac{1}{4} \Lambda_4 \right),$$

$$= \Lambda_1 - \epsilon \Lambda_2 + \frac{\epsilon^2}{2!} \Lambda_3 + \frac{1}{4} \left( \epsilon - \frac{\epsilon^2}{2!} \right) \Lambda_4,$$  \hspace{1cm} (3.25)

$$\text{Ad} (\exp (\epsilon \Lambda_1)) \Lambda_3 = \Lambda_3 - \epsilon \left[ \Lambda_1, \Lambda_3 \right] + \frac{\epsilon^2}{2!} \left[ \Lambda_1, \left[ \Lambda_1, \Lambda_3 \right] \right] - \ldots,$$

$$= \Lambda_3 - \epsilon \left( \frac{1}{4} \Lambda_4 - \Lambda_2 \right) + \frac{\epsilon^2}{2!} \Lambda_1,$$

$$= \Lambda_3 + \epsilon \Lambda_2 - \frac{\epsilon}{4} \Lambda_4 + \frac{\epsilon^2}{2!} \Lambda_1,$$  \hspace{1cm} (3.26)

we construct the adjoint representation table, namely;
Table 3.2: Adjoint representation for the base vectors given in (3.22)

<table>
<thead>
<tr>
<th>Ad</th>
<th>$\Lambda_1$</th>
<th>$\Lambda_2$</th>
<th>$\Lambda_3$</th>
<th>$\Lambda_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_1$</td>
<td>$\Lambda_1$</td>
<td>$\Lambda_2 + \varepsilon \Lambda_1$</td>
<td>$\Lambda_3 + \varepsilon \Lambda_2$</td>
<td>$\Lambda_4$</td>
</tr>
<tr>
<td>$\Lambda_2$</td>
<td>$\varepsilon \Lambda_1 e^{-c}$</td>
<td>$\Lambda_2$</td>
<td>$\varepsilon \Lambda_3 + \frac{1}{4} (1 - \varepsilon) \Lambda_4$</td>
<td>$\Lambda_4$</td>
</tr>
<tr>
<td>$\Lambda_3$</td>
<td>$\Lambda_1 - \varepsilon \Lambda_2 + \frac{\varepsilon^2}{2!} \Lambda_3$</td>
<td>$\Lambda_2 - \varepsilon \Lambda_3 + \frac{\varepsilon^4}{4!} \Lambda_4$</td>
<td>$\Lambda_3$</td>
<td>$\Lambda_4$</td>
</tr>
<tr>
<td>$\Lambda_4$</td>
<td>$\Lambda_1$</td>
<td>$\Lambda_2$</td>
<td>$\Lambda_3$</td>
<td>$\Lambda_4$</td>
</tr>
</tbody>
</table>

where the entry $(i, j)$ indicates $\text{Ad}(\exp(\varepsilon \Lambda_i)) \Lambda_j$.

Starting with a nonzero vector (3.23), with $a_1 \neq 0$ and rescale $\Lambda$ such that $a_1 = 1$.

It follows from the adjoint representation table that acting on $\Lambda$ by $\text{Ad}(\exp(a_2 \Lambda_3))$ one obtains

$$G^{[I]} = \text{Ad}(\exp(a_2 \Lambda_3)) \Lambda = \Lambda_1 + a'_3 \Lambda_3 + a'_4 \Lambda_4,$$

where $a'_3 = a_3 - \frac{a_2^2}{2}$ and $a'_4 = a_4 + \frac{1}{4} (a_2 + \frac{a_2^3}{2})$. Acting on $G^{[I]}$ by $\text{Ad}(\exp(c_1 \Lambda_2))$ one obtains

$$G^{[II]} = \text{Ad}(\exp(c_1 \Lambda_2)) G^{[I]} = e^{-c_1} \Lambda_1 + a'_3 e^{c_1} \Lambda_3 + \left[ a'_4 + \frac{a_3^2}{4} (1 - e^{c_1}) \right] \Lambda_4.$$

We choose $c_1 = \ln \left| 1 + \frac{4a_1 a'_3}{a_3^2} \right|$, with $a'_3 \neq 0$, so that the coefficient of $\Lambda_4$ is eliminated and observe that the remaining expression is a scalar multiple of $\Lambda_1 + a'_3 e^{2c_1} \Lambda_3$.

There are no entries in the adjoint representation table to simplify the coefficient of
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Λ₃ further. Therefore, any one dimensional subalgebra spanned by Λ with \( a₁ ≠ 0 \), if \( a₃' ≠ 0 \) is equivalent to the one spanned by \( Λ₁ + αΛ₃, \ α ∈ ℝ \). Otherwise, Λ is equivalent to a multiple of \( Λ₁ + a₃'Λ₄ \) if \( a₃' = 0 \). Acting by \( \text{Ad}(\exp(c₁Λ₂)) \), one obtains a vector which is a scalar multiple of \( Λ₁ + a₃'e^{c₁}Λ₄ \). There are no entries in the adjoint representation table to simplify the coefficient of \( Λ₄ \) further. However, depending on the sign of \( a₃' \) the coefficient of \( Λ₄ \) can be assigned to either +1, -1 or 0. Therefore, any one dimensional subalgebra spanned by \( a₁ ≠ 0 \), if \( a₃ = 0 \) is equivalent to the one spanned by \( Λ₁ + Λ₄, \ Λ₁ − Λ₄, \ Λ₁ \). Any remaining one dimensional subalgebras are spanned by vectors of the form (3.23) with \( a₁ = 0 \). Now assuming \( a₂ ≠ 0 \) and like previously rescale Λ such that \( a₂ = 1 \). Acting on Λ by \( \text{Ad}(\exp(-4a₄Λ₃)) \) we obtain

\[
G^{[III]} = \text{Ad}(\exp(-4a₄Λ₃))Λ = Λ₂ + a₃''Λ₃,
\]

where \( a₃'' = a₃ + 4a₄ \). There are no entries which can act on \( G^{[III]} \) to simplify the coefficient of \( Λ₃ \). Hence every one dimensional subalgebra spanned by the base vector in (3.23) with \( a₁ = 0 \) is equivalent to the subalgebra spanned by \( Λ₂ + αΛ₃ \) for \( α ∈ ℝ \). The next case \( (a₁ = a₂ = 0) \) is equivalent to \( G^{[IV]} = Λ₃ + a₄Λ₄ \) obtained by choosing \( a₃ ≠ 0 \) and rescaling Λ so that \( a₃ = 1 \). No further simplification is possible. Hence, any one dimensional subalgebra spanned by Λ with \( a₁ = a₂ = 0 \) is equivalent to \( G^{[IV]} = Λ₃ + αΛ₄, \ α ∈ ℝ \). Finally, the remaining case \( (a₁ = a₂ = a₃ = 0) \) is equivalent to \( Λ₄ \). Thus the one dimensional optimal system is

\[
\{Λ₁ + αΛ₃, \ Λ₁ ± Λ₄, \ Λ₁, \ Λ₂ + αΛ₃, \ Λ₃ + αΛ₄, \ Λ₄\}.
\]
If we admit the discrete symmetry \((-\gamma, -\phi, F) \mapsto (\gamma, \phi, F)\) which maps \(\Lambda_1 - \Lambda_4\) to \(\Lambda_1 + \Lambda_4\), then the number of inequivalent subalgebras is reduced by one.

### 3.3.1 Two dimensional subalgebras

Suppose that two distinct base vectors of a Lie algebra admitted by the governing D.E. satisfy the property

\[ [\Gamma_i, \Gamma_j] = \lambda_1 \Gamma_i + \lambda_2 \Gamma_j, \quad \lambda_1, \lambda_2 \in \mathbb{R}; \tag{3.27} \]

where \([\Gamma_i, \Gamma_j]\) is the commutator of \(\Gamma_i\) and \(\Gamma_j\) defined in the previous section. These base vectors span a two dimensional subalgebra \(L_2\) which is said to be abelian or commutative whenever \(\lambda_1 = \lambda_2 = 0\). Moreover, a 2+1 dimensional P.D.E. admitting a non-trivial abelian two dimensional subalgebra may be completely reduced to an O.D.E. by solving successively in any order the system of first order P.D.E.s

\[ \Gamma_i \Omega = 0, \quad \Gamma_j \Omega = 0; \]

where \(\Omega\) is the basis of invariants. Furthermore, following a reduction of an original P.D.E. by \(\Gamma_i\) when (3.27) with \(\lambda_2 = 0\) and \(\lambda_1 \neq 0\) hold (i.e. non-abelian subalgebra), \(\Gamma_j\) in new variables is automatically admitted by the reduced equation. We will refer to \(\Gamma_j\) as an *inherited symmetry*. The reduced equation by \(\Gamma_i\) may admit symmetries which are distinct from the symmetries of the original equation. In more complicated situations, successive reductions can take place by a chain of symmetries for the original P.D.E., provided that set of symmetries has the algebraic structure of a
solvable Lie algebra (e.g. Olver, 1986). Even if the original P.D.E. with a symmetry does not have a solvable Lie symmetry algebra, an additional hidden symmetry may still show up for the reduced P.D.E. In practice, these may be found by successively reapplying symmetry-finding procedures to each reduced equation. In our analysis we may not entirely rely on the existence of inherited symmetries or two dimensional subalgebras when performing reductions of the governing D.E.s, rather elements of the one dimensional optimal systems will also be utilised for symmetry reductions.
Chapter 4

Analytical solutions for two dimensional solute transport with velocity dependent dispersion

4.1 Introduction

The search for exact analytic solutions to convection-dispersion and diffusion equations relevant to problems arising in hydrology and chemical engineering, has continued unabated. These solutions are very important not only because they provide general physical insight into transport processes but also because they are needed as validation tests for numerical schemes.

Solute dispersion is complicated even at the macroscopic level because the dis-
persion coefficient increases with fluid velocity, which in general is varying in space and time. The fluid velocity vector field cannot be an arbitrary smooth function of space and time; it must conform to the established laws of fluid flow in porous media. Although passive scalar transport in solvent-conducting porous media has been intensively studied by many people for many years, realistic exactly solvable models with spatially varying dispersion coefficient are very rare. Perhaps the most notable effort in this direction has been that of Moench (1989), who obtained the Laplace transform for solute concentration during transport from an injection well to a pumped withdrawal well after approximating the flow as being radial towards the withdrawal well. In this case, the Laplace transform must be inverted numerically but this has been achieved with demonstrated accuracy (Moench, 1991). Most other good approximate and exact analytic results have similarly focussed on radial transport in two or three dimensions (Hoopes and Harleman, 1967; Eldor and Dagan, 1972; Tang and Babu, 1979; Hsieh, 1986; Novakowski, 1992; Fry et al., 1993). Simple one dimensional, axisymmetric or spherically symmetric geometries are most insightful in our analysis of solute dispersion. However, in these simplest cases the transport process is automatically represented as a partial differential equation in two dependent variables, including the time variable $t$ and only one space variable $r$. Such equations do not provide a genuine test for numerical simulations of two dimensional flows. Zoppou and Knight (1998) made some progress in this direction by producing the point source solution for dispersion in a background of hyperbolic
water streamlines bounded by a wedge. The main drawback of this solution is that it required a special form of anisotropy in the velocity dependence of the dispersion tensor. In this chapter we provide exact analytic axisymmetric solutions for instantaneous point source for dispersion during radial flow in porous media. We construct exact two dimensional solutions to velocity-dependent isotropic dispersion that are not axially symmetric and also we construct exact analytic solutions for solute transport under a vortex water flow. We select a form of the solute transport equation (2.22), that has a particularly large symmetry group of invariance transformations, depending continuously on a number of real parameters.

4.2 Axisymmetric solution: J R Philip solution

We now extend J R Philip’s instantaneous point source solution for dispersion during radial flow in porous media. In such flow, one may regard dispersivity as depending on the space variable rather than on Péclet number (Philip, 1994). Also, both convective velocity and diffusivity depend on space variables. It is often assumed that dispersivity is proportional to the Péclet number, \( Pe \). However, Salles et al. (1993) suggest that dispersivity is proportional to \( (Pe)^n \) with the index \( n \approx 1.6 \). De Gennes (1983) proposed that \( n = 2 \) for flows in unsaturated media where the connectivity of the liquid phase is poor. Furthermore, \( n = 2 \) is in agreement with Taylor’s theory of dispersion by fluctuations of a fluid velocity (Taylor, 1953). The
convection-diffusion equation for radial flow from a point source is given by

\[
\frac{\partial c_*}{\partial t_*} = \frac{1}{r_* \partial r_*} \left( Dr_* \frac{\partial c_*}{\partial r_*} \right) - \frac{v_1}{r_* \partial r_*},
\]

(4.1)

where $c_*$ is concentration, normalised with respect to some standard concentration; $t_*$ is dimensionless physical time and $r_*$ is the physical radial coordinate. $D$ is the diffusivity depending on Péclet number, in particular $D \propto (Pe)^n$, where $n > 0$. $v_1$ \([L^2T^{-1}]\) is a constant. The flow velocity across the circle of constant radius $r_*$ is given by $v_1/r_*$. Equation (4.1) becomes

\[
\frac{\partial c_*}{\partial t_*} = \frac{\partial}{\partial \sigma_*} \left( 2D \sigma_* \frac{\partial c_*}{\partial \sigma_*} \right) - v_1 \frac{\partial c_*}{\partial \sigma_*},
\]

(4.2)

under the transformation

\[
\sigma_* = \frac{1}{2} r_*^2.
\]

(4.3)

Since $D \propto (Pe)^n \propto r_*^{-n} \propto \sigma_*^{-n/2}$, it turns out that (Philip, 1994)

\[
D = \frac{1}{2} D_1 \sigma_*^{-n/2},
\]

(4.4)

where $D_1$ \([L^{2+n} T^{-1}]\) is a constant. The dimensionless variables

\[
\sigma = \left( \frac{v_1}{D_1} \right)^{n/2} \sigma_* \quad \text{and} \quad T = \frac{v_1^{1+2/n}}{D_1^{2/n}} t_*.
\]

(4.5)

reduce equation (4.2) to

\[
\frac{\partial C}{\partial T} = \frac{\partial}{\partial \sigma} \left( \sigma_1^{-n/2} \frac{\partial C}{\partial \sigma} \right) - \frac{\partial C}{\partial \sigma}.
\]

(4.6)
Here, the normalised $C = c/Q$, where given the physical instantaneous source strength $Q$, $Q$ is the dimensionless instantaneous source strength given by

\[ Q = \left( \frac{v_t}{D_1} \right)^{2/n} Q. \tag{4.7} \]

For $n = 2$ equation (4.6) reduces to

\[ \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial \sigma^2} - \frac{\partial C}{\partial \sigma} \tag{4.8} \]

and the relevant boundary conditions for instantaneous point source flow are

\[ 0 \leq \sigma \leq \infty \quad T = 0 \quad C = \delta(\sigma) \tag{4.9} \]

and

\[ T > 0 \quad \sigma = 0 \quad C = \frac{\partial C}{\partial \sigma}, \tag{4.10} \]

where $\delta(\cdot)$ in (4.9) is the Dirac delta function and condition (4.10) asserts that there is no solute flux flow passing through $\sigma = 0$ for $T > 0$. The solution to Equation (4.8) subject to the boundary conditions (4.9) and (4.10) is given by (Philip, 1994)

\[ C(\sigma, T) = \frac{1}{\sqrt{\pi T}} \exp \left[ -\frac{(\sigma - T)^2}{4T} \right] - \frac{e^\sigma}{2} \text{erfc} \left( \frac{\sigma + T}{2\sqrt{T}} \right). \tag{4.11} \]

We now map solution (4.11) into solution of Equation (2.22) which is given in terms of dimensionless variables by

\[ \frac{1}{v^2} \frac{\partial C}{\partial T} = \frac{\partial}{\partial \phi} \left[ D_\nu(v) \frac{\partial C}{\partial \phi} \right] + \frac{\partial}{\partial \psi} \left[ D_\nu(v) \frac{\partial C}{\partial \psi} \right] + \frac{\partial C}{\partial \phi}. \tag{4.12} \]

Here, $D_\nu$ is the velocity-dependent dispersion coefficient given by the power law

$D_\nu(v) = v^p$, where $1 < p < 2$ (see Section 2.3.3); $T = t/t_s$ and $C = c/c_s$ are
normalised time and concentration respectively. We set \( p = 2 \). Now let

\[ v = \frac{1}{r_*} \quad \Rightarrow \quad r_* = \frac{1}{v}. \]

Thus from (4.3) we obtain

\[ \sigma = \frac{1}{2v^2}. \]

Also from complex potential for a radial flow of source strength \( Q = 1 \),

\[ \log r_* = -\phi \quad \Rightarrow \quad r_* = e^{-\phi}. \]

Therefore

\[ \sigma = \frac{1}{2} e^{-2\phi} \quad \text{and} \quad v = e^\phi. \]

Substituting into (4.11) we obtain

\[ C(\phi, T) = \frac{1}{\sqrt{\pi T}} \exp \left[ -\frac{(e^{-2\phi} - 2T)^2}{16T} \right] - \frac{1}{2} \exp \left( \frac{e^{-2\phi}}{2} \right) \text{erfc} \left( \frac{e^{-2\phi} + 2T}{4\sqrt{T}} \right), \quad (4.13) \]

which is in fact an instantaneous point source solution to Equation (4.12). We note that the solutions (4.11) and (4.13) depend only on time and radius.

### 4.3 Classical symmetry analysis of solute transport equations

Ultimately, we wish to obtain exact solutions to the system of Equations (2.7) and (2.17) namely;

\[ \nabla^2 \Phi = 0 \]
and

\[
\frac{\partial c}{\partial t} = \nabla \cdot (D_v(v)c) + k \nabla \Phi \cdot \nabla c.
\]

However, if we look for classical Lie point symmetries of the entire system (2.7) and (2.17), by considering the infinitesimal transformations (Olver, 1986; Bluman and Kumei, 1989)

\[
\begin{align*}
x' &= x + \epsilon \xi^1(x,y,t,c) + O(\epsilon^2), \\
y' &= y + \epsilon \xi^2(x,y,t,c) + O(\epsilon^2), \\
t' &= t + \epsilon \tau (x,y,t,c) + O(\epsilon^2), \\
c' &= c + \epsilon \eta (x,y,t,c) + O(\epsilon^2),
\end{align*}
\]

with \( \xi^1, \xi^2, \tau, \eta \) being the infinitesimals of the symmetry generator

\[
\Gamma = \tau \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial c},
\]

then we will find nothing more than rescaling of \( c \), translation in \( t \), and translations and rotations in \( (x,y) \). These are the only conformal maps that leave not only (2.7) but (2.17) invariant. Nevertheless, we may hope to find symmetries that leave the single Equation (2.17) invariant when \( \Phi(x,y) \) is a special solution of Laplace’s equation. This may lead to useful reductions and solutions of (2.17) even if \( \Phi(x,y) \) itself is not an invariant solution of Laplace’s equation. For this purpose, we could carry out a symmetry classification of the single Equation (2.17), treating \( \Phi(x,y) \) as a free coefficient function. Given the class of functions \( \Phi(x,y) \) that lead to extra symmetries, we could later select from these, solutions of Laplace’s equation. In
order to generate and solve the symmetry determining relations, we have used the freely available program DIMSYM (Sherring, 1993), that is written as a subprogram for the computer algebra package REDUCE (Hearn, 1985). The only point symmetries for the general Equation (2.17) are combinations of translations in $t$, rescaling of $c$ and linear superposition. The symmetry operators are linear combinations of $\Gamma_1 = \frac{\partial}{\partial t}$, $\Gamma_2 = c \frac{\partial}{\partial c}$, and $\Gamma_\infty = h(x,y,t) \frac{\partial}{\partial c}$, where $h(x,y,t)$ is any particular solution of (2.17). The output of DIMSYM indicates special algebraic and differential equations among the free functions $D_\nu(v)$ and $\Phi(x,y)$, which, if satisfied, may lead to additional special symmetries. In fact, we have found that DIMSYM more easily finds special symmetric cases when the general C.D.E. is expressed in terms of streamline coordinates, as in Equation (2.22) namely:

$$
\frac{1}{v^2} \frac{\partial c}{\partial t} = \frac{\partial}{\partial \phi} \left[ D_\nu(v) \frac{\partial c}{\partial \phi} \right] + \frac{\partial}{\partial \psi} \left[ D_\nu(v) \frac{\partial c}{\partial \psi} \right] + \frac{\partial c}{\partial \phi}.
$$

Not surprisingly, even when the pore velocity is non-uniform, many special symmetries arise when $D_\nu$ is constant. Even this simpler case is directly applicable for modelling convection and molecular diffusion, or as a first approximation to dispersion. This case was studied more extensively in an earlier work by Broadbridge et al. (2000). Not all symmetric cases have yet been determined, but useful additional symmetries certainly occur when the water velocity is radial or when it represents strained flow along hyperbolic streamlines bounded by a wedge. From an arbitrary initial condition, we showed how to construct exact solute concentration profiles
in terms of Laguerre polynomials, modified Bessel functions and confluent hyper-
geometric functions when the water flow had hyperbolic streamlines bounded by a
wedge. For the case of hyperbolic strained flow, additional symmetries do not occur
for any velocity-dependent dispersion coefficient $D_v(v)$. For radial water flow, with
power-law dispersion coefficient $D_v(v) = v^p$, additional symmetries occur only for
the cases $p = 0$, $p = -2$ and $p = 2$. Furthermore, extra symmetries exist for a point
vortex flow (Broadbridge et al., 2000) and we show that this is true for the case
$p = 0$.

4.4 Classical symmetry reductions for solute trans-
port with $D_v(v) = v^2$ under radial water flow
background

For saturated radial water flows from a line source of strength $Q$, in terms of the
radial coordinate $r$, the Darcian flux is $V = Q/r$, and the pore velocity is $v = V/\theta_s$,
for which the velocity potential is $\phi = -(Q/\theta_s) \log r$ and the stream function is
$\psi = -(Q/\theta_s) \arctan(y/x)$. In this case, Equation (2.17) takes the form

$$\frac{\partial c}{\partial t} = \nabla \cdot \left[ D_1 \frac{(Q/\theta_s)^p}{r^p} \nabla c \right] + \frac{(Q/\theta_s)}{r} \frac{\partial c}{\partial r}. \quad (4.15)$$

Note that the gradient operator here is not simply radial as we are allowing so-
lute concentration to depend on the polar angle. Equation (4.15) may be non-
dimensionalised and rescaled so that all coefficients of proportionality are unity. Consider dimensionless quantities $C = c/c_s$, $T = t/t_s$ and $(X,Y,R) = (x,y,r)/l_s$, where s-subscripted parameters represent suitable concentration, time and length scales. The unique choice of time scale $t_s$ and length scale $l_s$ that will normalize Equation (4.15) is

$$t_s = D_{1}^{2/p} \left( \frac{q}{\theta_s} \right)^{1-2/p},$$

$$l_s = D_{1}^{1/p} \left( \frac{q}{\theta_s} \right)^{1-1/p}.$$

Then Equation (4.15) rescales to

$$\frac{\partial C}{\partial T} = \nabla \cdot \left[ \frac{1}{R^p} \nabla C \right] + \frac{1}{R} \frac{\partial C}{\partial R}. \quad (4.16)$$

We may write an equivalent dimensionless equation as in (4.12) and construct its classical point symmetries by considering the infinitesimal transformations (Olver, 1986; Bluman and Kumei, 1989)

$$T' = T + \epsilon \tau(T, \phi, \psi, C) + O(\epsilon^2),$$

$$\psi' = \psi + \epsilon \xi^1(T, \phi, \psi, C) + O(\epsilon^2),$$

$$\phi' = \phi + \epsilon \xi^2(T, \phi, \psi, C) + O(\epsilon^2),$$

$$C' = C + \epsilon \eta(T, \phi, \psi, C) + O(\epsilon^2), \quad (4.17)$$

where $\xi^1$, $\xi^2$, $\tau$, $\eta$ are the infinitesimals of the symmetry generator

$$\Gamma = \tau \frac{\partial}{\partial T} + \xi^1 \frac{\partial}{\partial \psi} + \xi^2 \frac{\partial}{\partial \phi} + \eta \frac{\partial}{\partial C}.$$ 

For the remainder of this chapter, appropriate infinitesimal transformations will be considered when constructing symmetries admitted by distinct P.D.E.s. We set
Chapter 4: Two dimensional solute transport in saturated soils

\( p = 2 \), which is in accord with Taylor’s (1953) theory of dispersion by fluctuations of a fluid velocity field and it seems to be a reasonable model for dispersion in porous media (de Gennes, 1986; Philip, 1994). For the relevant normalised point water source, \( \phi = - \log R \), \( \psi \) is simply the clockwise polar angle coordinate \(-\arctan(Y/X)\) and \( v = e^\phi \). In this case, besides the generic symmetries \( \Gamma_1 = \frac{\partial}{\partial T}, \Gamma_2 = C \frac{\partial}{\partial C} \) and \( \Gamma_\infty = h(T, \phi, \psi, C) \frac{\partial}{\partial C} \), with \( h \) being any arbitrary solution of (4.12), Equation (4.12) has three additional independent symmetries

\[
\Gamma_3 = -\left( \frac{T^2}{4} + \frac{T}{2} - \frac{T}{4} e^{-2\phi} + \frac{1}{16} e^{-4\phi} \right) C \frac{\partial}{\partial C} + T \frac{\partial}{\partial T} - \frac{T}{2} \frac{\partial}{\partial \phi},
\]

\[
\Gamma_4 = \left( -\frac{T}{4} + \frac{1}{8} e^{-2\phi} \right) C \frac{\partial}{\partial C} + T \frac{\partial}{\partial T} - \frac{T}{4} \frac{\partial}{\partial \phi} \quad \text{and}
\]

\[
\Gamma_5 = \frac{\partial}{\partial \psi}.
\]

### 4.4.1 Optimal system

Symmetry reductions of Equation (4.12) are possible using any linear combination given by

\[
\Gamma = a_1 \Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3 + a_4 \Gamma_4 + a_5 \Gamma_5.
\]

However, some of these symmetries are equivalent by a change of variable that also preserves the governing equation. In order to ensure a complete but minimal set of inequivalent reductions for Equation (4.12) we construct a one dimensional optimal system (Ovsiannikov, 1982; Olver, 1986). We employ Olver’s method which involves simplifying (4.18) as much as possible by carefully chosen adjoint transformations.
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We begin by setting up the commutator table (Table 4.1) and constructing the adjoint representation in the manner discussed in Section 3.3.

Table 4.1: Commutator table of the generators admitted by (4.12) with \( D(v) = v^2 \)

<table>
<thead>
<tr>
<th>([\Gamma_i, \Gamma_j])</th>
<th>(\Gamma_1)</th>
<th>(\Gamma_2)</th>
<th>(\Gamma_3)</th>
<th>(\Gamma_4)</th>
<th>(\Gamma_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Gamma_1)</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{4}\Gamma_2 - \Gamma_4)</td>
<td>(\Gamma_1 - \frac{1}{4}\Gamma_2)</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_2)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_3)</td>
<td>(\Gamma_4 - \frac{1}{4}\Gamma_2)</td>
<td>0</td>
<td>0</td>
<td>(-\Gamma_3)</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_4)</td>
<td>(\frac{1}{4}\Gamma_2 - \Gamma_1)</td>
<td>0</td>
<td>(\Gamma_3)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_5)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where \([\Gamma_i, \Gamma_j]\) = \(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i\).

Using the formula (3.20) in conjunction with the Table 4.1, for example

\[
\text{Ad}(\exp(\epsilon \Gamma_1)) \Gamma_3 = \Gamma_3 - \epsilon [\Gamma_1, \Gamma_3] + \frac{\epsilon^2}{2!} [\Gamma_1, [\Gamma_1, \Gamma_3]] - \ldots,
\]

\[
= \Gamma_3 - \frac{\epsilon}{4} \Gamma_2 + \epsilon \Gamma_4 + \frac{\epsilon^2}{2!} \left( -\Gamma_1 + \frac{1}{4} \Gamma_2 \right),
\]

\[
= \Gamma_3 + \epsilon \Gamma_4 - \frac{\epsilon^2}{2!} \Gamma_1 - \frac{1}{4} \left( \epsilon - \frac{\epsilon^2}{2!} \right) \Gamma_2,
\] (4.19)

\[
\text{Ad}(\exp(\epsilon \Gamma_4)) \Gamma_1 = \Gamma_1 - \epsilon [\Gamma_4, \Gamma_1] + \frac{\epsilon^2}{2!} [\Gamma_4, [\Gamma_4, \Gamma_1]] - \ldots,
\]

\[
= \Gamma_1 - \epsilon \left( \frac{1}{4} \Gamma_2 - \Gamma_1 \right) + \frac{\epsilon^2}{2!} \left( \Gamma_1 - \frac{1}{4} \Gamma_2 \right) - \ldots,
\]

\[
= e^\epsilon \Gamma_1 + \frac{1}{4} (1 - e^\epsilon) \Gamma_2,
\] (4.20)

we construct the adjoint representation table (Table 4.2):
Table 4.2: Adjoint representation of the generators admitted by (4.12)

<table>
<thead>
<tr>
<th>Adj</th>
<th>$\Gamma_1$</th>
<th>$\Gamma_2$</th>
<th>$\Gamma_3$</th>
<th>$\Gamma_4$</th>
<th>$\Gamma_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1$</td>
<td>$\Gamma_1$</td>
<td>$\Gamma_2$</td>
<td>$\Gamma_3 + \epsilon\Gamma_4 - \frac{\epsilon^2}{2!}\Gamma_1$</td>
<td>$\Gamma_4 - \epsilon\Gamma_1$</td>
<td>$\Gamma_5$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$-\frac{1}{4} (\epsilon - \frac{\epsilon^2}{2!}) \Gamma_2$</td>
<td>$+\frac{\epsilon}{4} \Gamma_2$</td>
<td></td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>$\Gamma_1$</td>
<td>$\Gamma_2$</td>
<td>$\Gamma_3$</td>
<td>$\Gamma_4$</td>
<td>$\Gamma_5$</td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>$\Gamma_1 - \frac{\epsilon^2}{2!}\Gamma_3 - \epsilon\left(\Gamma_4 - \frac{1}{4}\Gamma_2\right)$</td>
<td>$\Gamma_2$</td>
<td>$\Gamma_3$</td>
<td>$\Gamma_4 + \epsilon\Gamma_3$</td>
<td>$\Gamma_5$</td>
</tr>
<tr>
<td>$\Gamma_4$</td>
<td>$e^\epsilon\Gamma_1 + \frac{1}{4}(1 - e^\epsilon)\Gamma_2$</td>
<td>$\Gamma_2$</td>
<td>$e^{-\epsilon}\Gamma_3$</td>
<td>$\Gamma_4$</td>
<td>$\Gamma_5$</td>
</tr>
<tr>
<td>$\Gamma_5$</td>
<td>$\Gamma_1$</td>
<td>$\Gamma_2$</td>
<td>$\Gamma_3$</td>
<td>$\Gamma_4$</td>
<td>$\Gamma_5$</td>
</tr>
</tbody>
</table>

where the entry $(i, j)$ indicates $\text{Ad}(\exp(\epsilon\Gamma_i))\Gamma_j$.

Given a nonzero vector

$$\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4 + a_5\Gamma_5,$$

we seek to simplify as many of the coefficients $a_i$ as possible through judicious applications of adjoint maps to $\Gamma$. From Table 4.2, acting on $\Gamma$ by $\text{Ad}(\exp(\alpha\Gamma_1))$ yields

$$\Gamma^{[\alpha]} = \left(a_1 - \alpha a_4 - \frac{\alpha^2}{2!} \alpha^2\right) \Gamma_1 + \left(a_2 - \alpha \left(\frac{a_3}{4} - \frac{a_4}{4}\right) + \frac{\alpha^3}{8} \alpha^2\right) \Gamma_2$$

$$+ a_3\Gamma_3 + (a_4 + \alpha a_3)\Gamma_4 + a_5\Gamma_5.$$

(4.22)

Acting on $\Gamma^{[\alpha]}$ by $\text{Ad}(\exp(\beta\Gamma_3))$ produces

$$\Gamma^{[\alpha][\beta]} = \left(a_1 - \alpha a_4 - \frac{\alpha^2}{2!} \alpha^2\right) \Gamma_1 + \left\{a_2 - \alpha \left(\frac{a_3}{4} - \frac{a_4}{4}\right) + \frac{a_4}{8} \alpha^2 + \frac{\beta}{4} \left(a_1 - \alpha a_4 - \frac{\alpha^2}{2!} \alpha^2\right)\right\} \Gamma_2$$

$$+ \left\{a_3 + \beta (a_4 + \alpha a_3) - \frac{\beta^2}{2!} \left(a_1 - \alpha a_4 - \frac{\alpha^2}{2!} \alpha^2\right)\right\} \Gamma_3.$$
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\[ + \left\{ a_4 + \alpha a_3 - \beta \left( a_1 - \alpha a_4 - \frac{a_3}{2!} \alpha^2 \right) \right\} \Gamma_4 + a_5 \Gamma_3. \]  

(4.23)

In the simplification process we concentrate on the coefficients \( \tilde{a}_i, i = 1, 3, 4 \). namely:

\[ \tilde{a}_1 = a_1 - \alpha a_4 - \frac{a_3}{2!} \alpha^2, \]  

(4.24)

\[ \tilde{a}_3 = a_3 + \beta \left( a_4 + \alpha a_3 \right) - \frac{\beta^2}{2!} \left( a_1 - \alpha a_4 - \frac{a_3}{2!} \alpha^2 \right), \]  

(4.25)

\[ \tilde{a}_4 = a_4 + \alpha a_3 - \beta \left( a_1 - \alpha a_4 - \frac{a_3}{2!} \alpha^2 \right). \]  

(4.26)

Let \( \eta = a_4^2 + 2a_1a_3 \). Depending on the sign of \( \eta \), three cases arise. Wherever they appear \( a \) and \( b \) are arbitrary constants.

**case 1** \( \eta > 0 \) and let \( \alpha \) be a root of \( \frac{a_3}{2!} \alpha^2 + a_4 \alpha - a_1 = 0 \), so that \( \tilde{a}_1 = 0 \). We set

\[ \beta = -\frac{a_3}{\sqrt{a_4^2 + 2a_1a_3}}, \]  

(4.27)

so that \( \tilde{a}_3 = 0 \). Note that \( \tilde{a}_4 = \sqrt{a_4^2 + 2a_1a_3} = \sqrt{\eta} \neq 0 \). Hence \( \Gamma \) is equivalent to a scalar multiple of

\[ \Gamma^{III} = \tilde{a}_2 \Gamma_2 + \sqrt{a_4^2 + 2a_1a_3} \Gamma_4 + a_5 \Gamma_5, \]  

(4.28)

where the coefficient of \( \Gamma_4 \) has been rescaled to one. There are no entries in Table 4.2 to simplify \( \Gamma^{III} \) further. Therefore, any one dimensional subalgebra spanned by \( \Gamma \) if \( \eta > 0 \), is equivalent to the one spanned by a scalar multiple of \( \Gamma_4 + a\Gamma_2 + b\Gamma_5 \). \( a, b \in \mathbb{R} \).

**case 2** \( \eta < 0 \). Let \( \alpha = 0 \) so that \( \tilde{a}_1 = a_1 \) and choose

\[ \beta = \frac{a_4}{a_1}, \quad a_1 \neq 0. \]  

(4.29)
Equation (4.29) implies $\tilde{a}_4 = 0$. Note that $\tilde{a}_3 = a_3 + \frac{a_4^2}{2a_1} \neq 0$ (otherwise if $\tilde{a}_3 = 0$, then $\eta = 0$ but this case will be considered in the last steps). Hence $\Gamma$ is equivalent to a scalar multiple of

\[ \Gamma^{[II]} = \Gamma_1 + \tilde{a}_2 \Gamma_2 + \tilde{a}_3 \Gamma_3 + a_5 \Gamma_5. \quad (4.30) \]

wherein $a_1$ has been rescaled to unity, since $a_1 \neq 0$. Acting on $\Gamma^{[II]}$ by $\text{Ad} \left( \exp \left( c_1 \Gamma_4 \right) \right)$ yields

\[ \Gamma^{[III]} = \text{Ad} \left( \exp \left( c_1 \Gamma_4 \right) \right) \Gamma^{[II]} = e^{c_1} \Gamma_1 + \left( \tilde{a}_2 + \frac{1-e^{c_1}}{4} \right) \Gamma_2 + \tilde{a}_3 e^{-c_1} \Gamma_3 + a_5 \Gamma_5. \quad (4.31) \]

We choose $c_1 = \ln |1 + 4\tilde{a}_2|$ so that the coefficient of $\Gamma_2$ is eliminated and observe that the remaining expression is a scalar multiple of

\[ \Gamma^{[IV]} = \Gamma_1 + \tilde{a}_3' \Gamma_3 + \tilde{a}_5' \Gamma_5, \quad (4.32) \]

where $\tilde{a}_3' = \tilde{a}_3 e^{-2c_1}$ and $\tilde{a}_5' = a_5 e^{-c_1}$. There are no entries in Table 4.2 to simplify (4.32) further. Hence if $\eta < 0$, then $\Gamma$ is equivalent to the scalar multiple of $\Gamma_1 + \varepsilon \Gamma_3 + b \Gamma_5$, $\varepsilon \neq 0$, $b \in \mathbb{R}$.

**Case 3** $\eta = 0$. Here two subcases arise.

**Subcase 1** if all the coefficients $a_1, a_3, a_4$ vanish i.e. $a_1 = a_3 = a_4 = 0$, then $\Gamma^{[III]} = \tilde{a}_2 \Gamma_2 + a_5 \Gamma_5$. There are no entries in Table 4.2 for this case to be further simplified, thus $\Gamma^{[III]} = \{ \Gamma_2, \Gamma_5, a \Gamma_2 \}$.

**Subcase 2** Suppose that not all $a_1, a_3, a_4$ vanish. Let $\alpha = 0$ and $\beta = a_4/a_1$ then
\( \tilde{a}_1 = a_1; a_4 = a_3 = 0. \) Now rescale \( a_1 \) to 1 so that

\[
\Gamma^[[II]] = \Gamma_1 + \tilde{a}_2 \Gamma_2 + a_5 \Gamma_5. \tag{4.33}
\]

Acting on \( \Gamma^[[II]] \) by \( \text{Ad}(\exp(c_1 \Gamma_4)) \) yields

\[
\Gamma^[[III]] = \text{Ad}(\exp(c_1 \Gamma_4)) \Gamma^[[II]] = e^{c_1} \Gamma_1 + \left( \frac{1-e^{c_1}}{4} + \tilde{a}_2 \right) \Gamma_2 + a_5 \Gamma_5. \tag{4.34}
\]

We choose \( c_1 = \ln |1 + 4\tilde{a}_2| \) so that the coefficient of \( \Gamma_2 \) is eliminated and observe that the remaining expression is a scalar multiple of \( \Gamma_1 + a_5 e^{-c_1} \Gamma_5 \). There are no entries in Table 4.2 to further simplify the coefficient of \( \Gamma_5 \). Thus \( \Gamma \) is equivalent to a scalar multiple of

\[
\Gamma^[[III]] = \Gamma_1 + a \Gamma_5. \tag{4.35}
\]

The one dimensional optimal system is therefore

\[
\{ \Gamma_1 + \varepsilon \Gamma_3 + b \Gamma_5, \, \Gamma_5 + a \Gamma_2, \, \Gamma_4 + a \Gamma_2 + b \Gamma_5, \, \Gamma_1 + a \Gamma_5, \, \Gamma_2, \, \Gamma_5 \}. 
\]

Reductions by these elements are given in Table 4.3.

### 4.4.2 Invariant non-radial solutions

Consider the \( \Gamma_3 \)-invariant solutions of the form

\[
C = \exp \left( -\frac{T}{4} - \frac{\log T}{2} + \frac{e^{-2\phi}}{4} - \frac{e^{-4\phi}}{16T} \right) \times F(\rho, \gamma),
\]

with \( F \) satisfying the P.D.E.

\[
4\rho^2 \frac{\partial^2 F}{\partial \rho^2} + 8\rho \frac{\partial F}{\partial \rho} + \frac{\partial^2 F}{\partial \gamma^2} = 0, \quad \text{where} \quad \rho = T e^{2\phi}, \, \gamma = \psi. \tag{4.36}
\]
Since

\[ [\Gamma_3, \Gamma_4] = -\Gamma_3, \]

the reduced equation inherits the symmetry \( \Gamma_4 \), which now takes the form

\[ \Gamma_4 = \rho \frac{\partial}{\partial \rho} + \frac{F}{\partial F}, \]

leading to the reduction

\[ F = \rho g(\gamma), \quad \text{with} \quad g'' + 8g = 0. \]

In terms of the original variables, this leads to the solution

\[
C = k_3 + k_4 \cos(2\sqrt{2}\psi + k_5)\sqrt{T}R^{-2} \times \exp\left(-\frac{T}{4} + \frac{R^2}{4} - \frac{R^4}{16T}\right), \quad (4.37)
\]

where \( k_3, k_4, k_5 \in \mathbb{R} \). Given \( \frac{\pi}{2} < \omega < \pi \), with \( \omega = k_5 - 2^{3/2}\psi \), the rate of mass transport across \( R = R_0 \) is

\[
\frac{dM}{dt} = \int_{\omega_0}^{\omega_1} \left(-D(v) \frac{\partial C}{\partial R} + VC\right) R \, d\omega
= 2^{-3/2}k_4 T^{1/2} \left(\frac{1}{2} - \frac{1}{4T} - \frac{2}{R^4}\right) \exp\left(\frac{R^2}{4} - \frac{T}{4} - \frac{R^4}{16T}\right)
+ 2^{-5/2}k_3 \pi. \quad (4.38)
\]

For convenience of interpretation, we have neglected the analogous solutions wherein sine functions replace cosine functions, and we have added the constant solution \( k_3 \). The solution (4.37) has the concentration boundary condition \( C = k_3 \) at \( \psi = 2^{-3/2}(k_5 - \frac{\pi}{2}) \) and the zero flux boundary condition \( J \cdot n = 0 \), where \( n \) is the outward (circumferential) normal vector at \( \psi = 2^{-3/2}(k_5 - \pi) \). Solution (4.37) is depicted
schematically in Figure 4.1 whilst rate of mass transport (4.38) is plotted against $T$ in Figure 4.2. Liquid at the lower radial boundary is maintained at concentration $k_3$; for example this may be the equilibrium saturated concentration where the liquid contacts a salt block. The liquid and the solute are contained by a barrier at the upper radial boundary. The concentration is initially at the uniform equilibrium value. For some time, water with a lower concentration of solute flows in from the origin, flushing the interior and reducing its solute concentration. After some time, the inflowing water again becomes saturated with solute and the interior again approaches its initial concentration. At each point, the concentration reaches its minimum value at time

$$T = 1 + \sqrt{1 + \frac{R^4}{4}}.$$  

Unfortunately, it is common for symmetry solutions not to have easily interpretable boundary conditions because usually they are very special solutions with few parameters that can be adjusted to satisfy boundary conditions. Sometimes, the number of free parameters may be greatly increased because the first reduced equation happens to be a standard constant-coefficient linear equation with many solutions obtainable by linear transforms or series methods. For example, the P.D.E. obtained by reduction under $\Gamma_3$ transforms to the negative Helmholtz equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial v^2} - V = 0,$$  

(4.39)
under the transformation

\[ F = U(\chi, \nu)e^{-\chi}; \quad \rho = e^{2\chi}; \quad \psi = \nu; \quad V = \frac{\partial U}{\partial \nu}. \]

Equation (4.39) has many available solutions since it has been intensively studied in several physical applications including quasilinear steady unsaturated flow (e.g. Philip, 1985b; Waechter and Philip, 1985). If we impose zero-concentration initial conditions and zero normal flux boundary conditions on a wedge,

\[ C = 0 \text{ at } T = 0, \]

\[ \mathbf{J} \cdot \mathbf{n} = 0 \text{ at } \psi = \psi_0, \psi_1, \]

these transform to

\[ V \to 0, \quad \chi \to -\infty; \quad (4.40) \]

\[ V = 0, \quad \nu = \psi_0, \psi_1. \quad (4.41) \]

Without loss of generality, we take \( \psi_0 = 0 \). By separation of variables within the self adjoint Equation (4.39), we obtain a general Fourier series solution. Expressed in the original variables, this solution is,

for \( R > T^{1/2} \),

\[ C = T^{-1/2} \exp \left( -\frac{T}{4} + \frac{R^2}{4} - \frac{R^4}{16T} \right) \times \sum_{n=0}^{\infty} A_n (TR^{-2})^{\alpha_n} \cos(n\pi \psi/\psi_1), \quad (4.42) \]

and for \( R < T^{1/2} \),

\[ C = T^{-1/2} \exp \left( -\frac{T}{4} + \frac{R^2}{4} - \frac{R^4}{16T} \right) \times \sum_{n=0}^{\infty} A_n (TR^{-2})^{-1-\alpha_n} \cos(n\pi \psi/\psi_1). \quad (4.43) \]
where
\[ \alpha_n = \frac{1}{2} \left[ \sqrt{(n\pi/\psi - 1)^2 + 1} - 1 \right] \]
and \( A_n \) are the \( nth \) harmonic fluctuations in \( \theta \). Figure 4.4 is a polar plot of this solution for solute contained in a right angled wedge (\( \psi_1 = \pi/2 \)) and with a step profile at \( T=1 \),

\[ C(\psi,1) = \begin{cases} 
1.0 & \text{for } \psi > \pi/4; \\
0.8 & \text{for } \psi < \pi/4.
\end{cases} \]

The dimensionless total solute content is

\[ \int_0^\infty \int_0^{\psi_1} C(R,\psi,T)R \, dR \, d\psi = A_0\pi \left[ 1 + \text{erf} \left( \frac{\sqrt{T}}{2} \right) \right]. \]

This shows that an amount \( A_0\pi \) is deposited instantaneously at the origin and that an equal amount is injected continuously over time, with decreasing supply rate. Note that total solute content does not depend on \( A_n \) with \( n > 0 \). These coefficients have no effect on total solute content or on mean solute transport rate across a circular arc \( R = \text{constant} \) and \( 0 < \psi < \psi_1 \). However, they dictate the variability of concentration and flux on polar angle. Figure 4.3 depicts solute flux against \( x_p \).

Although the source is isotropic with respect to water flux, it is not isotropic with respect to solute flux. Notionally, the continuous source represent a discharge pipe that is covered with a filter of variable strength.
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Figure 4.1: Schematic representation of solute streamlines for a solution given by Equation (4.37).
Figure 4.2: Rate of mass transport given in (4.38) across $R = R_0$. Parameters used: $k_3 = 1$, $R_0 = 3$, $k_4 = 0.5$. 
Figure 4.3: Solute flux for concentration given in (4.42) for values $n = 0$, $n = 1$ and $n = 2$ respectively. Parameters used: $T = 0.2$ and $R = 1$. 
Figure 4.4: Polar plot for analytic solution of solute concentration given by Equation (4.42) showing concentration as a function of polar angle for given values of $R$ and $T$. The step function represents the imposed condition at $(R,T) = (1,1)$. The other curve shows smoothing at a later time; $(R,T) = (1.2,1.2)$. 
Table 4.3: Classical symmetry reductions of Equation (4.12) with $D_v(v) = v^2$

<table>
<thead>
<tr>
<th>$\tilde{\Gamma}[I]$</th>
<th>$\Gamma_1 + a\Gamma_5$</th>
<th>Reduced equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\Gamma}[II]$</td>
<td>$\Gamma_1 + a\Gamma_5 + b\Gamma_2 + a_3\Gamma_5$</td>
<td>$\gamma \frac{\partial^2 F}{\partial \gamma^2} + \gamma \left(1 - \frac{b_2}{\rho}\right) \frac{\partial F}{\partial \gamma} + 16b_2^2 \rho^2 \frac{\partial^2 F}{\partial \rho^2} + b_2^2 (24 \rho - 1) \frac{\partial F}{\partial \rho}$ with $C = \exp \left(-\frac{T}{4} + \frac{e^{-2\phi}}{4} + a \ln T\right) \times F(\rho, \gamma)$, $\rho = Te^{4\phi}$ and $\gamma = Te^{-\psi/b}$</td>
</tr>
<tr>
<td>$\tilde{\Gamma}[III]$</td>
<td>$\Gamma_1 + \varepsilon \Gamma_3 + b\Gamma_5$</td>
<td>$\frac{\partial^2 F}{\partial \gamma^2} - \frac{b_2^2}{\rho^2} \frac{\partial F}{\partial \gamma} + 4b_2^2 \rho^2 \frac{\partial^2 F}{\partial \rho^2} + 8b_2^2 \rho \frac{\partial F}{\partial \rho} + \left(\frac{b_2^2}{16\rho^4} - \frac{b_2^2}{4\rho^2}\right) F = 0$, with $\varepsilon = 1$, $b \neq 0$ and $\gamma = \tan^{-1}(T) - \frac{\nu}{b}$.</td>
</tr>
<tr>
<td>$\tilde{\Gamma}[IV]$</td>
<td>$\Gamma_5 + a\Gamma_2$</td>
<td>$\frac{\partial^2 F}{\partial \phi^2} - e^{-4\phi} \frac{\partial F}{\partial \phi} + \left(2 + e^{-2\phi}\right) \frac{\partial F}{\partial \phi} + a^2 F = 0$, with $C = e^{a\psi} \times F(\phi, \gamma)$, $\gamma = T$.</td>
</tr>
</tbody>
</table>
4.4.3 Further invariant solutions

Equation (4.36), other than the infinite symmetry generator $S_\infty = h(\rho, \gamma) \frac{\partial}{\partial F}$ with $h(\rho, \gamma)$ being any solution to Equation (4.36), admits a four dimensional Lie algebra spanned by the base vectors

\[ S_1 = -\rho \frac{\partial}{\partial \rho}, \]
\[ S_2 = 2F\gamma \frac{\partial}{\partial F} + \log \rho \frac{\partial}{\partial \gamma} - 4\rho \gamma \frac{\partial}{\partial \rho}, \]
\[ S_3 = -2\rho \frac{\partial}{\partial \rho} + F \frac{\partial}{\partial F}, \]
\[ S_4 = \frac{\partial}{\partial \gamma}. \]

Using the method outlined in Chapter 3, we obtain the one dimensional optimal system;

\[ \{S_1 + \alpha S_2, S_2, S_3 + \alpha S_4, S_4, S_1 + \alpha S_3\}, \quad (4.44) \]

where $\alpha$ is an arbitrary constant. Reductions of Equation (4.36) by elements of the set of optimal system (4.44) are listed in Table 4.4, wherein $J_\nu$ and $Y_\nu$ are Bessel functions of the first and second kind respectively (Abramowitz and Stegun, 1965) with the indicated order and without loss of generality we have taken $\alpha = 1$. 
Table 4.4: Reductions of Equation (4.36) and exact analytic solutions to (4.12)

<table>
<thead>
<tr>
<th>$S_i$</th>
<th>Reduced O.D.E.</th>
<th>Invariant solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_2$</td>
<td>$16\vartheta g'' + 16 g' - g = 0,$ with $F = \frac{1}{\sqrt{p}} g(\vartheta)$, $\vartheta = 4\gamma^2 + (\log \rho)^2$.</td>
<td>$g = k_1 J_0 \left( \frac{i}{2} \sqrt{\vartheta} \right) + k_2 Y_0 \left( \frac{i}{2} \sqrt{\vartheta} \right),$ and so $C = \frac{R}{\sqrt{T}} \left{ k_1 J_0 \left( \frac{i}{2} \sqrt{4\psi^2 + (\log (TR^{-2}))^2} \right) + k_2 Y_0 \left( \frac{i}{2} \sqrt{4\psi^2 + (\log (TR^{-2}))^2} \right) \right} \times \exp \left{ -\frac{T}{4} - \log T + \frac{R^2}{4} - \frac{R^4}{16T} \right}.$</td>
</tr>
<tr>
<td>$S_3 + S_4$</td>
<td>$2\vartheta^2 g'' + 2\vartheta g' - g = 0,$ with $F = \frac{1}{\sqrt{p}} g(\vartheta)$, $\vartheta = \sqrt{p} \ e^\gamma$.</td>
<td>$g = k_1 (\vartheta)^{1/\sqrt{2}} + k_2 (\vartheta)^{1/\sqrt{2}},$ and so $C = \frac{R}{\sqrt{T}} \left{ k_1 (\frac{Te^\psi}{R})^{1/\sqrt{2}} + k_2 (\frac{Te^\psi}{R})^{1/\sqrt{2}} \right} \times \exp \left{ -\frac{T}{4} - \log T + \frac{R^2}{4} - \frac{R^4}{16T} \right}.$</td>
</tr>
<tr>
<td>$S_1 + S_2$</td>
<td>$(4\vartheta + 1)^2 g'' + (4\vartheta + 1)g' - \vartheta g = 0,$ with $F = \frac{\psi}{\sqrt{p}} g(\vartheta)$, $\vartheta = \frac{1}{4} \sin^{-1} \left( \frac{2 \log \rho}{\sqrt{4\vartheta + 1}} \right),$ $\vartheta = 4\gamma^2 + 2\gamma + (\log \rho)^2.$</td>
<td>$g = (4\vartheta + 1)^{3/8} \left{ k_1 J_0 \left( \frac{\psi}{4} \right) + n \left( \frac{\psi}{16} \right) \right},$ hence $C = \frac{R^{3/8}}{\sqrt{T}} \left{ k_1 J_0 \left( \frac{\psi}{4} \right) + n \left( \frac{\psi}{16} \right) \right} \times \exp \left{ -\frac{T}{4} - \log T + \frac{R^2}{4} - \frac{R^4}{16T} - \frac{1}{4} \sin^{-1} \left( \frac{2 \log (TR^{-2})}{\sqrt{\psi}} \right) \right},$ where $\psi = 16\psi^2 + 8\psi + 4 \log (TR^{-2}) + 1.$</td>
</tr>
<tr>
<td>$S_1 + S_3$</td>
<td>$g'' - \frac{8}{5} g = 0,$ with $F = \rho^{-1/3} g(\gamma).$</td>
<td>$g = k_1 \sinh \left( \frac{2\sqrt{2} \gamma}{3} \right) + k_2 \cosh \left( \frac{2\sqrt{2} \gamma}{3} \right),$ $C = \exp \left( \frac{R^2}{4} - \frac{R^4}{16T} - \frac{T}{4} \right) \times \sqrt{\frac{R^2}{T^2}} \times \left{ k_1 \sinh \left( \frac{2\sqrt{2} \psi}{3} \right) + k_2 \cosh \left( \frac{2\sqrt{2} \psi}{3} \right) \right}.$</td>
</tr>
</tbody>
</table>

Following reduction by $\tilde{\Gamma}^{IV}$ listed in Table 4.3, the reduced P.D.E. namely:

$$\frac{\partial^2 F}{\partial \varphi^2} - e^{-4\varphi} \frac{\partial F}{\partial \gamma} + \left( 2 + e^{-2\varphi} \right) \frac{\partial F}{\partial \varphi} + a^2 F = 0,$$
admits, beside the infinite symmetry generator $\Lambda_\infty = h(\phi, \gamma) \frac{\partial}{\partial F}$ with $h(\phi, \gamma)$ being any arbitrary solution to the reduced P.D.E. in question, the four dimensional Lie algebra spanned by the base vectors,

\[
\begin{align*}
\Lambda_1 &= \gamma^2 \frac{\partial}{\partial \gamma} - \frac{\gamma}{2} \frac{\partial}{\partial \phi} + \left( \frac{\gamma e^{-2\phi}}{4} - \frac{e^{-4\phi}}{16} - \frac{\gamma^2}{4} - \frac{\gamma}{2} \right) F \frac{\partial}{\partial F}, \\
\Lambda_2 &= \gamma \frac{\partial}{\partial \gamma} - \frac{1}{4} \frac{\partial}{\partial \phi} + \left( \frac{e^{-2\phi}}{8} - \frac{\gamma}{4} \right) F \frac{\partial}{\partial F}, \\
\Lambda_3 &= \frac{\partial}{\partial \gamma}, \\
\Lambda_4 &= F \frac{\partial}{\partial F}.
\end{align*}
\]

The one dimensional optimal system is

\[
\{ \Lambda_1 + \alpha \Lambda_3, \Lambda_1 \pm \Lambda_4, \Lambda_1, \Lambda_2 + \alpha \Lambda_3, \Lambda_3 + \alpha \Lambda_4, \Lambda_4 \}, \tag{4.45}
\]

where $\alpha$ is an arbitrary constant (see example in Section 3.3). In the following examples we present reductions and invariant solutions by elements of the optimal system (4.45).

**Example i**

Reduction by $\Lambda_3 + \alpha \Lambda_4$ leads to the functional form

\[
F = e^{\alpha \gamma} g(\phi),
\]

with $g$ satisfying the O.D.E.

\[
g'' + \left( 2 + e^{-2\phi} \right) g' + \left( a^2 - \alpha e^{-4\phi} \right) g = 0. \tag{4.46}
\]

Under the transformation $\xi = e^{-2\phi}$, $g(\phi) = g(\xi)$ we obtain,

\[
\xi^2 g'' - \frac{\xi^2}{2} g' + \left( \frac{a^2}{4} - \frac{\alpha \xi^2}{4} \right) g = 0. \tag{4.47}
\]
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Equation (4.47) becomes

\[ zw'' + \left(-\frac{z}{2} + 2k\right)w' - \left(\frac{k}{2} + \frac{\alpha z}{4}\right)w = 0. \quad (4.48) \]

under the transformations \( z = \xi, \ w = g\xi^{-k} \) and has a solution

\[ w = e^{z/4}z^{1/2-k} \left[ k_1 J_{-1/2+k} \left(\frac{i\sqrt{4\alpha + 1}}{4} z\right) \right. \]
\[ + \left. k_2 Y_{-1/2+k} \left(\frac{i\sqrt{4\alpha + 1}}{4} z\right) \right], \quad (4.49) \]

with \( k \) being a root of the quadratic \( k^2 - k - \frac{\alpha}{4} = 0 \). \( J_\nu \) and \( Y_\nu \) are the Bessel functions of first and second kind respectively, with order \( \nu \) (Abramowitz and Stegun, 1965).

In terms of the original variables we obtain

\[ C = \exp \left(\frac{R^2}{4} + \alpha T + a\psi + \ln R\right) \times \left[ k_1 J_{-1/2+k} \left(\frac{i\sqrt{4\alpha + 1}}{4} R^2\right) \right. \]
\[ + \left. k_2 Y_{-1/2+k} \left(\frac{i\sqrt{4\alpha + 1}}{4} R^2\right) \right] . \]

**Example ii**

Reduction by \( \Lambda_1 + \alpha \Lambda_3 \) with \( \alpha = 1 \), leads to the functional form

\[ F = \exp \left(\frac{e^{-2\vartheta}}{4} - \frac{\gamma e^{-4\vartheta}}{16(\gamma^2 + 1)} - \frac{\gamma}{4} + \frac{\tan^{-1}(\gamma)}{4} - \frac{\ln(\gamma^2 + 1)}{4}\right) \times g(\vartheta), \]

where \( \vartheta = \sqrt{\gamma^2 + 1} e^{2\varphi} \) and \( g \) satisfies the O.D.E.

\[ 4\vartheta^2 g'' + 8\vartheta g' + \left( a^2 - \frac{1}{4}\vartheta^2 + \frac{1}{16}\vartheta^4 \right) g = 0. \quad (4.50) \]

Equation (4.50) transforms to

\[ \xi w'' + \left(2k + \frac{1}{2}\right)w' + \left(\xi - \frac{1}{256} - \frac{1}{64}\right)w = 0, \]
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under $\xi = \vartheta^{-2}$, \( w(\xi) = g\xi^{-k} \) where \( k \) is a root of the quadratic \( 4k^2 - 2k + \frac{a^2}{4} = 0 \).

Hence we obtain a solution

\[
w = \xi^{-1/4-k} \left[ k_1 M_{p,q} \left( \frac{i\xi}{8} \right) + k_2 W_{p,q} \left( \frac{i\xi}{8} \right) \right],
\]

where \( p = \frac{i}{8}, q = -\frac{1}{4} + k \), and \( M_{\mu,\nu}(x) \) and \( W_{\mu,\nu}(x) \) are the Whittaker's functions.

These functions may be represented as the hypergeometric confluent or the KummerM and KummerU functions (see p505 Abramowitz and Stegun, 1965). In terms of the original variables we have

\[
C = \exp \left( \frac{R^2}{4} - \frac{TR^4}{16(T^2+1)} - \frac{T}{4} + \frac{\tan^{-1}(T)}{4} - \frac{\ln(T^2+1)}{4} + a \tan^{-1}(Y/X) \right) \\
\times \left( \frac{\frac{1}{T^2+1}}{R} \right) \times \left[ k_1 M_{p,q} \left( \frac{iR^4}{8(T^2+1)^2} \right) + k_2 W_{p,q} \left( \frac{iR^4}{8(T^2+1)^2} \right) \right].
\]

**Example iii**

Reduction by \( \Lambda_2 + \alpha \Lambda_3 \) leads to the functional form

\[
F = \exp \left( \frac{e^{-2\vartheta}}{4} - \frac{\gamma}{4} + \frac{\alpha \ln(\gamma + \alpha)}{4} \right) \times g(\vartheta),
\]

where \( \vartheta = (\gamma + \alpha) e^{4\vartheta} \) and \( g \) satisfies the O.D.E.

\[
\vartheta^2 g'' + \left( \frac{3\vartheta}{2} - \frac{1}{16} \right) g' + \left( -\frac{\alpha}{64\vartheta} + \frac{\alpha^2}{16} \right) g = 0. \tag{4.51}
\]

Under the transformation \( \xi = \vartheta^{-1}, \ w = g\xi^{-k} \), Equation (4.51) becomes

\[
\xi w'' + \left( \frac{\xi}{16} + 2k - \frac{1}{2} \right) w' + \left( \frac{k}{16} - \frac{\alpha}{64} \right) w = 0,
\]

where \( k \) is a root of the quadratic \( k^2 - k + \frac{a^2}{16} = 0 \). Thus we obtain

\[
w = e^{-\xi/32} \xi^{-k+1/4} \left[ k_1 M_{p,q} \left( \frac{\xi}{16} \right) + k_2 W_{p,q} \left( \frac{\xi}{16} \right) \right].
\]
where \( p = \frac{1-a}{4} \) and \( q = k - \frac{3}{16} \). In terms of original variables we have

\[
C = \exp \left( \frac{R^2}{4} - \frac{T}{4} + \frac{a \ln(T+a)}{4} - \frac{R^4}{32(T+a)} + a \psi \right) \times \left( \frac{R}{\sqrt{T+a}} \right) \\
\times \left[ k_1 M_{p,q} \left( \frac{R^4}{16(T+a)} \right) + k_2 W_{p,q} \left( \frac{R^4}{16(T+a)} \right) \right].
\]

**Example iv**

Reduction by \( \Lambda_1 \) leads to the functional form

\[
F = \exp \left( \frac{-e^{2\phi}}{4} - \frac{\gamma}{4} - \frac{\ln \gamma}{2} - \frac{e^{-4\phi}}{16\gamma} \right) \times g(\psi),
\]

where \( \psi = \gamma e^{2\phi} \) and \( g \) satisfies the Euler equation

\[
\psi^2 g'' + 2\psi g' + \frac{a^2}{4} g = 0. \tag{4.52}
\]

Solutions to Equation (4.52) may be constructed for three cases: \( a^2 > 1 \), \( a^2 = 1 \), \( a^2 < 1 \).

**case 1** if \( a^2 > 1 \), then

\[
g = \psi^{-1/2} \left[ k_1 \sin \left( \mu \log |\psi| \right) + k_2 \cos \left( \mu \log |\psi| \right) \right],
\]

where \( \mu = \frac{1}{2} \sqrt{|1-a^2|} \). Hence in the original variables we have

\[
C = \frac{2}{T} \exp \left( -\frac{T}{4} + \frac{R^2}{4} - \frac{R^4}{16T} + a \tan^{-1}(Y/X) \right) \\
\times \left[ k_1 \sin \left( \mu \log \left| \frac{T}{R^2} \right| \right) + k_2 \cos \left( \mu \log \left| \frac{T}{R^2} \right| \right) \right].
\]

**case 2** if \( a^2 = 1 \), then

\[
g = |\psi|^{-1/2}(k_1 + k_2 \log |\psi|)
\]
and hence in terms of original variables we have

$$C = \frac{R}{T} \exp \left( \frac{R^2}{4} - \frac{T}{4} - \frac{R^4}{16T} + a \tan^{-1}(Y/X) \right) \times \left( k_1 + k_2 \log \left| \frac{T}{R^2} \right| \right).$$

**case 3** if $a^2 < 1$, then

$$g = |\theta|^{-1/2} \left( k_1 |\theta|^\mu + k + 2|\theta|^{-\mu} \right).$$

Thus in terms of the original variables we obtain

$$C = \frac{R}{T} \left[ k_1 (TR^{-2})^{1/2} \sqrt{1-a^2} + k_2 (TR^{-2})^{-1/2} \sqrt{1-a^2} \right] \times \exp \left( -\frac{T}{4} + \frac{e^{-2\phi}}{4} - \frac{e^{-4\phi}}{16T} + a \tan^{-1}(Y/X) \right).$$

(4.53)

We observe that three subcases in which solution (4.53) may be rewritten arise:

**subcase (a)** if $a^2 = 0$, then the solution (4.53) is independent of the polar angle $\tan^{-1}(Y/X)$, and takes the form

$$C = \frac{R}{T} \left[ k_1 (TR^{-2})^{1/2} + k_2 (TR^{-2})^{-1/2} \right] \times \exp \left( \frac{R^2}{4} - \frac{T}{4} - \frac{R^4}{16T} \right).$$

**subcase (b)** $a^2 \in (0, 1)$.

Let $a^2 = 3/4$, then (4.53) reduces to

$$C = \frac{R}{T} \left[ k_1 (TR^{-2})^{1/4} + k_2 (TR^{-2})^{-1/4} \right] \times \exp \left( -\frac{T}{4} + \frac{R^2}{4} - \frac{R^4}{16T} + \frac{\sqrt{3} \tan^{-1}(Y/X)}{2} \right).$$

**subcase (c)** $a^2 < 0$.

Let $a^2 = -3$, then (4.53) becomes a linear combination of real and complex solutions.
We shall only consider the real solution given by

\[ C = \frac{R}{T} \left[ k_1 \left( TR^{-2} \right) + k_2 \left( TR^{-2} \right)^{-1} \right] \times \exp \left( -\frac{T}{4} + \frac{R^2}{4} - \frac{R^4}{16T} \right) \times \cos(\sqrt{3} \tan^{-1}(Y/X)). \]  

(4.54)

The solution (4.54) satisfies the no flux flow boundary conditions \( \mathbf{J} \cdot \mathbf{n} \), where \( \mathbf{n} \) is the outward normal vector, at \( \tan^{-1}(Y/X) = 0, \frac{\pi}{\sqrt{3}} \) and the initial condition \( C \to 0 \) as \( T \to 0 \). Furthermore, \( C \to \infty \) as \( R \to 0 \), implying that solute is continuously supplied at the origin.

Example v

Reduction by \( \Lambda_1 + \Lambda_4 \) leads to a functional form

\[ F = \exp \left( \frac{e^{-2\phi}}{4} - \frac{e^{-4\phi}}{16\gamma} - \frac{\gamma}{2} - \frac{1}{\gamma} \right) \times g(\vartheta), \]

with \( g \) satisfying the O.D.E.

\[ \vartheta^2 g'' + 2\vartheta g' + \left( \frac{a^2}{4} - \frac{1}{4\vartheta^2} \right) g = 0, \]  

(4.55)

where \( \vartheta = \gamma e^{2\phi} \). Equation (4.55) becomes

\[ \xi w'' + 2kw' - \frac{\xi}{4} w = 0, \]

under the transformations \( \xi = \vartheta^{-1}, \ w(\xi) = g\xi^{-k} \), with \( k \) being a root of the quadratic \( k^2 - k + \frac{a^2}{4} = 0 \). Hence we have

\[ w = \xi^{1/2-k} \left\{ k_1 I_{k-1/2} \left( \frac{\xi}{2} \right) + k_2 K_{k-1/2} \left( \frac{\xi}{2} \right) \right\} \]
and in terms of the original variables we obtain

\[
C = \exp \left\{ \frac{R^2}{4} - \frac{R^4}{16T} - \frac{T}{4} - \frac{1}{T} + a\gamma \right\} \times \frac{R}{T} \left[ k_1 I_{k-1/2} \left( \frac{R^2}{2T} \right) + k_2 K_{k-1/2} \left( \frac{R^2}{2T} \right) \right].
\]

On the other hand, except stated otherwise, all the reduced P.D.E.s listed in Table 4.3 admit trivial point symmetries namely; translation of \( \gamma \) and scaling in \( F \). Reduction by linear combination of these symmetries is given in Table 4.5.

<table>
<thead>
<tr>
<th>Using symmetry ( \Gamma[i] )</th>
<th>Reduced O.D.E.s by linear combination of ( F \frac{\partial}{\partial F} ) and ( \frac{\partial}{\partial \r} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma[i] )</td>
<td>( g'' + \left( 2 + e^{-2\phi} \right) g' + \left( \frac{\alpha^2}{\sigma^2} - \alpha e^{-4\phi} \right) g = 0 ), with ( F = e^{\alpha \gamma} g(\phi) ), ( \alpha \in \mathbb{R} )</td>
</tr>
<tr>
<td>( \Gamma[i] )</td>
<td>Reduced P.D.E. admits no point symmetries.</td>
</tr>
<tr>
<td>( \Gamma[i] )</td>
<td>( \rho^2 g'' + 2\rho g' + \left( \frac{1}{4\kappa^2 \alpha^2} - \frac{\alpha+4}{16\rho^2} + \frac{1}{64\rho^4} \right) g = 0 ), with ( F = e^{\gamma/\alpha} g(\rho) ), ( \alpha \in \mathbb{R} )</td>
</tr>
</tbody>
</table>

We now list further exact analytic solutions for the C.D.E. (4.12) in the following examples. For further calculations see e.g. the handbooks by Kamke (1959), Murphy (1960) or Polyanin and Zaitsev (1995). In the next examples \( M_{\mu,\nu}(x) \) and \( W_{\mu,\nu}(x) \) are Whittaker functions, and \( J_{\nu} \) and \( Y_{\nu} \) are the Bessel functions (Abramowitz and Stegun, 1965). Reductions by \( \Gamma[i] \) lead to no invariant solutions.
Example i

Reduction by $\tilde{T}^{[II]}$ resulted in an O.D.E.

$$\rho^2 g'' + 2 \rho g' + \left( \frac{1}{4b^2 \alpha^2} - \frac{\alpha + 4}{16 \rho^2} + \frac{1}{64 \rho^4} \right) g = 0,$$

which has solution

$$g = \sqrt{\rho} \left[ k_1 M_{\frac{\alpha + 4}{8 \omega}} \frac{\sqrt{\beta^2 + 1}}{4 \omega} \left( \frac{i}{8 \rho^2} \right) + k_2 W_{\frac{\alpha + 4}{8 \omega}} \frac{\sqrt{\beta^2 + 1}}{4 \omega} \left( \frac{i}{8 \rho^2} \right) \right].$$

Thus in terms of the original variables we obtain

$$C = \frac{1}{R} \times \exp \left[ \left( \frac{\alpha + 4}{4 \omega} \right) \tan^{-1}(T) - \frac{\psi}{\omega} - \frac{T}{4} - \frac{R^2}{4} \right]$$

$$\times \left[ k_1 M_{\frac{\alpha + 4}{8 \omega}} \frac{\sqrt{\beta^2 + 1}}{4 \omega} \left( \frac{i R^4}{8(1+T^2)} \right) + k_2 W_{\frac{\alpha + 4}{8 \omega}} \frac{\sqrt{\beta^2 + 1}}{4 \omega} \left( \frac{i R^4}{8(1+T^2)} \right) \right].$$

Example ii

Following reduction by $\tilde{T}^{[I]}$ we obtained the O.D.E.

$$g'' + \left( 2 + e^{-2\phi} \right) g' + \left( \frac{\alpha^2}{a^2} - \alpha e^{-4\phi} \right) g = 0,$$

which transforms to

$$4 \xi w'' + (8k - 2 \xi) w' + (\alpha \xi - 2k) w = 0,$$

under $\xi = e^{-2\phi}$, $w(\xi) = g\xi^{-k}$, where $k$ is a root of the quadratic $k^2 - k + \frac{\alpha^2}{4a^2} = 0$.

Thus we obtain

$$w = e^{\xi/4} \xi^{1/2-k} \left\{ k_1 J_{k-\frac{1}{2}} \left( \frac{\sqrt{4\alpha - 1}}{4} \xi \right) + k_2 Y_{k-\frac{1}{2}} \left( \frac{\sqrt{4\alpha - 1}}{4} \xi \right) \right\}$$

and so in terms of the original variables we have

$$C = \frac{1}{R} \exp \left( \alpha \left( T - \frac{\psi}{a} \right) + \frac{1}{4R^2} \right) \times \left\{ k_1 J_{k-\frac{1}{2}} \left( \frac{\sqrt{4\alpha - 1}}{4R^2} \right) + k_2 Y_{k-\frac{1}{2}} \left( \frac{\sqrt{4\alpha - 1}}{4R^2} \right) \right\}.$$
4.5 Classical symmetry reductions for solute transport with $D_\nu(v) = 1$ under radial water flow background

In this section, we consider transport of solute under a uniform radial water flow described by the C.D.E.,

$$\frac{1}{v^2} \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial \phi^2} + \frac{\partial^2 C}{\partial \psi^2} + \frac{\partial C}{\partial \phi}. \quad (4.56)$$

Saturated flow of this nature is discussed in Section 4.3.1 except that here we consider a constant dispersion coefficient. We consider solute transport under steady saturated water flow. In this case other than the generic $\Gamma_1$, $\Gamma_2$ and $\Gamma_\infty$, Equation (4.56) admits three additional independent symmetries, namely;

$$\Gamma_3 = -\left( \frac{T}{2} + \frac{e^{-2\phi}}{4} \right) C \frac{\partial}{\partial C} - T \frac{\partial}{\partial \phi} + T^2 \frac{\partial}{\partial T},$$

$$\Gamma_4 = 2T \frac{\partial}{\partial T} - \frac{\partial}{\partial \phi} \quad \text{and} \quad \Gamma_5 = \frac{\partial}{\partial \psi}.$$

The one dimensional optimal system is

$$\{ \Gamma_1 \pm \Gamma_2 \pm \Gamma_5, \Gamma_1 + \Gamma_2, \Gamma_1 + \Gamma_5, \Gamma_1, a\Gamma_2 + \Gamma_4 + b\Gamma_5, \Gamma_1 + \Gamma_3 + a\Gamma_2 + b\Gamma_5, \Gamma_2 + a\Gamma_5, \Gamma_5, \Gamma_2 \}.$$

If we admit the discrete symmetry which maps $\Gamma_1 + \Gamma_2 + \Gamma_5$ into $\Gamma_1 - \Gamma_2 - \Gamma_5$, then the number of elements of the optimal system reduces by one. The reductions by these elements are listed in Table 4.6.
### Table 4.6: Symmetry reductions for Equation (4.56).

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Functional forms and reduced equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1 + \Gamma_2 + \Gamma_5$</td>
<td>$C = e^T F(\phi, \rho)$, where $\rho = T - \psi$ and $F$ satisfies $\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \rho^2} + \frac{\partial F}{\partial \phi} - e^{-2\phi}\frac{\partial F}{\partial \rho} - e^{-2\phi} F = 0$.</td>
</tr>
<tr>
<td>$\Gamma_1 + \Gamma_2$</td>
<td>$C = e^T F(\phi, \psi)$ and $F$ satisfies $\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \rho^2} + \frac{\partial F}{\partial \phi} - e^{-2\phi} F = 0$.</td>
</tr>
<tr>
<td>$\Gamma_1 + \Gamma_5$</td>
<td>$C = F(\phi, \rho)$, where $\rho = T - \psi$ and $F$ satisfies $\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \rho^2} + \frac{\partial F}{\partial \phi} - e^{-2\phi}\frac{\partial F}{\partial \rho} = 0$.</td>
</tr>
<tr>
<td>$a\Gamma_2 + \Gamma_4 + b\Gamma_5$</td>
<td>$C = T_a^{\alpha/2} F(\rho, \gamma)$, where $\rho = \sqrt{T}e^\phi$, $\gamma = \sqrt{T}e^{-\psi/b}$ and $F$ satisfies $\gamma^2 \frac{\partial^2 F}{\partial \gamma^2} + \left(\gamma - \frac{b^2}{2\rho^2}\right) \frac{\partial F}{\partial \gamma} + b^2 \rho^2 \frac{\partial^2 F}{\partial \rho^2} + \left(2b^2 \rho - \frac{b^2}{2\rho^2}\right) \frac{\partial F}{\partial \rho} + \frac{b^2}{2\rho^2} F = 0$.</td>
</tr>
<tr>
<td>$\Gamma_2 + a\Gamma_5$</td>
<td>$C = e^{\psi/a} F(T, \phi)$, where $F$ satisfies $\frac{\partial^2 F}{\partial T^2} - \frac{\partial^2 F}{\partial \phi^2} + \frac{\partial F}{\partial \phi} + \frac{1}{a^2} F$.</td>
</tr>
<tr>
<td>$\Gamma_1 + \Gamma_3 + a\Gamma_2 + b\Gamma_5$</td>
<td>$C = \exp\left(\frac{a}{\sqrt{T}} T - \frac{\ln(1+T^2)}{4} - \frac{T e^{-2\phi}}{4(1+T^2)}\right) \times F(\gamma, \rho)$, where $\rho = \sqrt{1+T^2} e^\phi$, $\gamma = \tan^{-1} T - \frac{\psi}{b}$ and $F$ satisfies $\frac{\partial^2 F}{\partial \gamma^2} - \frac{\partial^2 F}{\partial \rho^2} + b^2 \rho^2 \frac{\partial^2 F}{\partial \rho^2} + \frac{2b^2 \rho \frac{\partial F}{\partial \rho} + \left(\frac{b^2}{4\rho^2} - \frac{a^2}{\rho^2}\right)}{F = 0$.</td>
</tr>
</tbody>
</table>

### 4.5.1 Invariant solutions

Consider the $\Gamma_3$ -invariant solution of the form

$$C = \frac{1}{\sqrt{T}} \exp\left(-\frac{e^{-2\phi}}{4T}\right) \times F(\rho, \psi),$$
where $\rho = T e^\phi$ and $F$ satisfies the P.D.E.

$$\rho^2 \frac{\partial^2 F}{\partial \rho^2} + 2\rho \frac{\partial F}{\partial \rho} + \frac{\partial^2 F}{\partial \psi^2} = 0. \quad (4.57)$$

Equation (4.57) is transformable into modified Helmholtz equation (see Section 4.4.2). Here, we avoid further analysis of the transformed Helmholtz equation.

Also, classical symmetry analysis results in similar results obtain in Section 4.3.3 for Equation (4.36). However, we note that since $T_3$ and $T_4$ satisfy the property (3.22), then Equation (4.57) inherits $T_4$ which now takes the form

$$T_4 = \frac{\rho}{2} \frac{\partial}{\partial \rho} + F \frac{\partial}{\partial F}$$

and leads to reduction

$$F = \rho^2 g(\psi),$$

where $g$ satisfies the O.D.E.

$$g'' + 6g = 0.$$  

In terms of original variables, wherein the sine function has been replaced by the cosine function, we obtain

$$C = \frac{T^{3/2}}{R^2} \exp \left( -\frac{R^2}{4T} \right) \times \left( k_1 \cos \left( \sqrt{6} \psi + k_2 \right) \right), \quad (4.58)$$

where $k_1$ and $k_2$ are constants. The solution (4.58) satisfies the no flux flow boundary conditions $J \cdot n$, where $n$ is the outward normal vector, at $\psi = \frac{k_2}{\sqrt{6}}$, $\frac{\pi - k_2}{\sqrt{6}}$ and the initial condition $C \to 0$ as $T \to 0$. Furthermore, $C \to \infty$ as $R \to 0$, implying that solute is continuously supplied at the origin and travels along the radii.
The reduced equation obtained using $\Gamma_2 + a\Gamma_3$ in Table 4.6 admits, beside the infinite symmetry generator, the four dimensional Lie algebra spanned by the base vectors

$$\begin{align*}
\chi_1 &= T^2 \frac{\partial}{\partial T} - T \frac{\partial}{\partial \phi} - \left( \frac{e^{-2\phi}}{4} + \frac{T}{2} \right) F \frac{\partial}{\partial F}, \\
\chi_2 &= 2T \frac{\partial}{\partial T} - \frac{\partial}{\partial \phi}, \\
\chi_3 &= \frac{\partial}{\partial T}, \\
\chi_4 &= F \frac{\partial}{\partial F}.
\end{align*}$$

The one dimensional optimal system is

$$\{ \alpha \chi_2 + \chi_4, \chi_1 \pm \chi_3, \chi_4 \chi_3, \chi_4 \},$$

where $\alpha$ is an arbitrary constant. If we admit the discrete symmetry $(-T, -\phi, F) \mapsto (T, \phi, F)$ which maps $\chi_1 - \chi_3$ to $\chi_1 + \chi_3$ and $\chi_4 - \chi_3$ to $\chi_4 + \chi_3$, then the number of elements of the optimal system reduces by two. Reductions using these elements are given in Table 4.7 wherein, $M_{\nu, \mu}(x)$ and $W_{\nu, \mu}(x)$ are the Whittaker functions, and $J_{\nu}$ and $Y_{\nu}$ are the Bessel functions of order $\nu$ (Abramowitz and Stegun, 1965).

We now list the remaining results in the following examples. Methods of finding exact analytic solutions for the O.D.E.s may be found in texts such as those of Kamke (1959), Murphy (1960) and, Polyanin and Zaitsev (1995) or O.D.E.s may be solved using Maple. Wherever they appear, $J_{\nu}(x)$ and $Y_{\nu}(x)$ are the Bessel functions of order $\nu$, and $I_{\nu}(x)$ and $K_{\nu}(x)$ are the modified Bessel functions (see e.g.
Table 4.7: Further symmetry reductions and exact solutions for Equation (4.56).

<table>
<thead>
<tr>
<th>( \chi_i )</th>
<th>Reduced O.D.E.</th>
<th>Invariant solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_1 )</td>
<td>( \rho^2 g'' + 2\rho g' - \frac{1}{a^2} g = 0 ), with ( F = \frac{1}{\sqrt{T}} \exp \left(-\frac{e^{-2\phi}}{4T}\right) g(\rho) ) and ( \rho = Te^\phi ).</td>
<td>( g = \frac{1}{\sqrt{T}} \left( k_1</td>
</tr>
<tr>
<td>( \alpha \chi_2 + \chi_4 )</td>
<td>( \rho^2 g'' + \left(2\rho + \frac{1}{2\rho^2}\right) g' ) with ( F = \sqrt{T} g(\rho), \rho = \sqrt{T} e^\phi ).</td>
<td>( C = \sqrt{T} \exp \left(-\frac{R^2}{4T} + \frac{\psi}{a}\right) \times \left( k_1</td>
</tr>
<tr>
<td>( \chi_1 + \chi_3 )</td>
<td>( \rho^2 g'' + 2\rho g' + \left(\frac{1}{4\rho^4} - \frac{1}{a^2}\rho^2\right) g = 0 ), with ( F = \frac{\exp \left(-\frac{e^{-2\phi}}{4\rho^4+4}\right)}{\sqrt{T^2+1}} g(\rho) ), ( \rho = \sqrt{T^2+1} e^\phi ).</td>
<td>( g = e^{1/8\rho^2} \rho^{1/2} \left( k_1 M_{-1/4,k-1/4} \left( \frac{R^2}{4T^2} \right) + k_2 W_{-1/4,k-1/4} \left( \frac{R^2}{4T^2} \right) \right) )</td>
</tr>
<tr>
<td>( \chi_4 + \chi_3 )</td>
<td>( g'' + g' + \left(\frac{1}{a^2} - e^{-2\phi}\right) g = 0 ), with ( F = e^\phi g(\phi) ).</td>
<td>( C = \frac{\sqrt{T}}{R} \exp \left(-\frac{R^2}{4T^2} + \frac{\psi}{a}\right) \times \left( k_1 M_{-1/4,k-1/4} \left( \frac{iR^2}{2(T^2+1)} \right) + k_2 W_{-1/4,k-1/4} \left( \frac{iR^2}{2(T^2+1)} \right) \right) ) with ( k = 0, -2 ).</td>
</tr>
</tbody>
</table>
Example i

Following reduction by $\Gamma_1 + \Gamma_2 + \Gamma_3$, the reduced P.D.E. admits, beside the infinite symmetry generator, $Z_1 = -2\frac{\partial}{\partial \rho}$ and $Z_2 = F\frac{\partial}{\partial F} - \frac{\partial}{\partial \rho}$. $Z_2$ leads to a functional form

$$F = e^{-\rho}g(\phi),$$

where $g$ satisfies the O.D.E.

$$g'' + g' + \left(2 - e^{-2\phi}\right) g = 0,$$

which has a general solution

$$g = e^{-\phi/2} \left[k_1 J_\nu(-i e^{-\phi}) + k_2 Y_\nu(-i e^{-\phi})\right],$$

where $\nu = \frac{i\sqrt{\alpha}}{2}$. Hence, in terms of the original variables we obtain

$$C = e^{\psi}R^{1/2} \left[k_1 J_\nu(-iR) + k_2 Y_\nu(-iR)\right].$$

Example ii

Following reduction by $a\Gamma_2 + \Gamma_4 + b\Gamma_5$, the reduced equation admits, beside the infinite symmetry generator, scaling in $F$ and translation of $\gamma$, the linear combination of which leads to the functional form

$$F = e^\gamma g(\rho), \quad \alpha \in \mathbb{R},$$

where $g$ satisfies the O.D.E.

$$\rho^2 g'' + \left(2\rho - \frac{1}{2\rho^2}\right) g' + \left[\frac{\alpha^2}{b^2} + \left(\frac{a - 1}{2\rho^2}\right)\right] g = 0.$$
The solutions of this linear equation are not familiar but they could be constructed by power series methods.

**Example iii**

Following reduction by $\Gamma_1 + \Gamma_2$, the reduced equation admits, beside the infinite symmetry generator, scaling in $F$ and translation of $\psi$, the linear combination of which leads to the functional form

$$F = e^{\psi/\alpha} g(\phi), \quad \alpha \in \mathbb{R},$$

where $g$ satisfies the O.D.E.

$$g'' + g' + \left(\frac{1}{\alpha^2} - e^{-2\phi}\right) g = 0,$$

which has solution

$$g = e^{-\phi/2} \left[ k_1 I_\nu \left( e^{-\phi} \right) + k_2 K_\nu \left( e^{-\phi} \right) \right],$$

where $\nu = \frac{\sqrt{\alpha^2 - 4}}{2\alpha}$ and so in terms of the original variables we obtain

$$C = R^{1/2} \exp \left( T + \frac{\psi}{\alpha} \right) \times \left[ k_1 I_\nu (R) + k_2 K_\nu (R) \right].$$

**Example iv**

Following reduction by $\Gamma_1 + \Gamma_5$, the reduced equation admits, beside the infinite symmetry generator, scaling in $F$ and translation of $\rho$, the linear combination of which leads to the functional form

$$F = e^{-\alpha \rho} g(\phi), \quad \alpha \in \mathbb{R},$$
where \( g \) satisfies the O.D.E.

\[
g'' + g' + \left( \alpha^2 + \alpha e^{-2\phi} \right) g = 0,
\]

which has solution

\[
g = e^{-\phi/2} \left[ k_1 J_\nu \left( \alpha^{1/2} e^{-\phi} \right) + k_2 Y_\nu \left( \alpha^{1/2} e^{-\phi} \right) \right],
\]

where \( \nu = -\frac{\sqrt{1-4\alpha^2}}{2} \) and so in terms of the original variables we have

\[
C = R^{1/2} \exp(-\alpha(T - \psi)) \times \left[ k_1 J_\nu \left( \alpha^{1/2} R \right) + k_2 Y_\nu \left( \alpha^{1/2} R \right) \right].
\]

**Example v**

Following reduction by \( \Gamma_1 + \Gamma_3 + a\Gamma_2 + b\Gamma_5 \), the reduced P.D.E. admits, beside the infinite symmetry generator, scaling in \( F \) and translation of \( \gamma \), the linear combination of which leads to the functional form

\[
F = e^{\gamma/\alpha} g(\rho), \quad \alpha \in \mathbb{R},
\]

where \( g \) satisfies the O.D.E.

\[
\rho^2 g'' + 2\rho g' + \left( \frac{1}{4\rho^4} - \frac{1 + a\alpha}{\alpha \rho^2} + \frac{1}{\alpha^2 b^2} \right) g = 0.
\]

Without loss of any generality we take \( a = b = \alpha = 1 \) and obtain a solution

\[
g = \sqrt{\rho} \left\{ k_1 M_{i,i\sqrt{3}/4} \left( \frac{i}{2\rho^2} \right) + k_2 W_{i,i\sqrt{3}/4} \left( \frac{i}{2\rho^2} \right) \right\}.
\]

Hence in terms of the original variables we have

\[
C = \frac{1}{\sqrt{R}} \exp \left( 2\tan^{-1} T - \frac{T e^{-2\phi}}{4(1+T^2)} - \psi \right) \times \left\{ k_1 M_{i,i\sqrt{3}/4} \left( \frac{iR^2}{2(1+T^2)} \right) + k_2 W_{i,i\sqrt{3}/4} \left( \frac{iR^2}{2(1+T^2)} \right) \right\}.
\]
4.6 Classical symmetry reductions for the Fokker-Planck equation with $D_v(v) = v^{-2}$ under radial water flow background

Numerous empirical equations have been proposed to model the observed relationship between shear rate, shear stress and viscosity for fluids. Many industrial processes involve fluids which do not have a linear relationship between shear stress and shear rate and their viscosities are functions of shear rate. These non-Newtonian fluids can often be represented by the power law model.

The relationship between average velocity and average viscosity suggests similar relationships between dispersivity and average velocity. As in Poiseville flow through a capillary tube in a saturated porous medium, the mean velocity should be proportional to the mean shear rate. Saturated radial water flow is discussed in Section 4.4 except that here the dispersion coefficient is given by the power law $v^{-2}$, then beside the generic symmetries $\Gamma_1$, $\Gamma_2$ and $\Gamma_\infty$, Equation (4.12) admits two additional independent symmetries

$$\Gamma_3 = -\psi C \frac{\partial}{\partial C} + 2T \frac{\partial}{\partial \psi} \quad \text{and} \quad \Gamma_4 = \frac{\partial}{\partial \psi}.$$

The one dimensional optimal system is

$$\{\Gamma_3 + a\Gamma_1, \ a\Gamma_2 + \Gamma_1, \ \Gamma_4, \ \Gamma_2\}.$$
with $a$ being an arbitrary constant. We list reduction by the elements of this optimal system in Table 4.8.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Reduced equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_3 + a\Gamma_1$</td>
<td>$F_{\rho\rho} + F_{\phi\phi} = \left(2 - e^{-2\phi}\right) F_\phi + \frac{6}{a} F$, where $C = \exp \left(\frac{2T^3}{3a} - \frac{4T}{a}\right)$ and $\rho = \frac{T^2}{a} - \psi$.</td>
</tr>
<tr>
<td>$a \neq 0$</td>
<td></td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>$F_T = F_{\phi\phi} + \left(e^{-2\phi} - 2\right) F_\phi - \frac{1}{2T} F$, where $C = \exp \left(-\frac{\psi^2}{4T}\right) F(\phi, T)$.</td>
</tr>
<tr>
<td>$a = 0$</td>
<td></td>
</tr>
<tr>
<td>$a\Gamma_2 + \Gamma_1$</td>
<td>$F_{\psi\psi} + F_{\phi\phi} = \left(2 - e^{-2\phi}\right) F_\phi + aF$, where $C = e^{\alpha T} F(\phi, \psi)$.</td>
</tr>
<tr>
<td>$a \neq 0$</td>
<td></td>
</tr>
</tbody>
</table>

### 4.6.1 Invariant solutions

The reduced P.D.E. by $\Gamma_3$ in Table 4.8 admits point symmetries

\[ \mathcal{I}_1 = -\frac{F}{2} \frac{\partial}{\partial F} + T \frac{\partial}{\partial T}, \]
\[ \mathcal{I}_2 = F \frac{\partial}{\partial F}, \]
\[ \mathcal{I}_\infty = h(\phi, T) \frac{\partial}{\partial F}. \]
where $h$ is any arbitrary solution of the reduced equation. The following are examples of some exact analytic solutions.

**Example i**

Reduction by $I_1$ leads to the functional form

$$F = \frac{1}{\sqrt{T}} g(\phi),$$

where

$$g'' + \left(e^{-2\phi} - 2\right) g' = 0. \quad (4.59)$$

Letting $\xi = e^\phi$, then (4.59) becomes

$$\xi^2 g''(\xi) + \left(\frac{1}{\xi} - \xi\right) g'(\xi) = 0,$$

which integrates twice to

$$g = k_1 \left\{-\frac{2e^u}{3u^{3/2}} + \frac{2}{3} \int \frac{e^u}{u^{3/2}} du\right\} + k_2,$$

with $k_1$ and $k_2$ being the integration constants and $u = \frac{\xi^2}{2}$. Evaluating the integral twice by parts, we obtain in terms of the original variables a solution

$$C = \frac{1}{\sqrt{T}} \exp\left(-\frac{\psi^2}{4T}\right) \times \left\{ k_1 \left[-\frac{2}{3} \sqrt{\frac{2}{\pi}} \exp\left(\frac{R^2}{2}\right) \times \left(\frac{1}{R^3} - \frac{1}{R}\right) - \text{erf}\left(\frac{R}{\sqrt{2}}\right)\right] + k_2\right\}. $$

**Example ii**

Linear combination of $I_1$ and $I_2$ leads to the functional form

$$F = \frac{e^{\alpha T}}{\sqrt{T}} g(\phi),$$
where

$$g'' + \left( e^{-2\phi} - 2 \right) g' - ag = 0. \quad (4.60)$$

Equation (4.60) becomes the Kummer or degenerate hypergeometric equation

$$zy'' + [2(k + 1) - z]y' - ky = 0,$$

under the transformations $y(z) = (2z)^{-k}g, \quad 2z = e^{-2\phi}$, with $k$ being a root of the quadratic equation $4k^2 + 4k + a = 0$. We therefore obtain,

$$y = e^{z/2}z^{-1/2-k} \left\{ k_1 \left[ (2k + z)I_{k+1/2} \left( -\frac{z}{2} \right) + zI_{k-1/2} \left( -\frac{z}{2} \right) \right] + k_2 \left[ -(2k + z)K_{k+1/2} \left( -\frac{z}{2} \right) + zK_{k-1/2} \left( -\frac{z}{2} \right) \right] \right\}.$$

In terms of the original variables we have

$$C = \frac{2k^{1/2}}{\sqrt{T_R}} \exp \left( aT + \frac{R^2}{4} - \frac{\phi^2}{4T} \right) \times \left\{ k_1 \left[ (2k + \frac{R^2}{2})I_{k+1/2} \left( -\frac{R^2}{4} \right) + \frac{R^2}{2}I_{k-1/2} \left( -\frac{R^2}{4} \right) \right] + k_2 \left[ -(2k + \frac{R^2}{2})K_{k+1/2} \left( -\frac{R^2}{4} \right) + \frac{R^2}{2}K_{k-1/2} \left( -\frac{R^2}{4} \right) \right] \right\},$$

where $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions of first and third kind of order $\nu$ (Abramowitz and Stegun, 1965). Following reduction by $\Gamma_3 + a\Gamma_1$, the reduced P.D.E. admits no symmetries, hence no further exact solution were obtained. Also, the reduced P.D.E. obtained using $a\Gamma_2 + \Gamma_1$ admits translation in $\psi$ and scaling of $F$, the linear combination of which leads to the equation which is transformable to the degenerate hypergeometric equation as in example ii above.
4.7 Classical symmetry reductions for solute transport under point vortex water flow background

We now consider solute transport under point vortex water flow which is conjugate to the previously studied case of the point source flow and is amenable to symmetry analysis (Broadbridge et al., 2000). The flow considered here is irrotational everywhere with an exception at the origin and the radial component of velocity is zero whilst the Darcian flux is given by \( V = \kappa/(2\pi r) \). Furthermore, the pore velocity is \( v = V/\theta_s \), for which the velocity potential is \( \phi = -(\kappa/2\pi\theta_s) \arctan(y/x) \) and the stream function is \( \psi = (\kappa/2\pi\theta_s) \log r \). We note that the solute transport equation under point vortex water flow admits extra point symmetries only for the case \( p = 0 \) in (4.12). For relevant normalised point vortex water flow, \( \psi = \log R \), \( \phi = -\arctan(Y/X) = -\omega \) and \( v = e^{-\psi} \). Beside the generic symmetries \( \Gamma_1, \Gamma_2 \) and \( \Gamma_\infty \), Equation (4.56) admits three additional independent symmetries

\[
\Gamma_3 = -\left(e^{2\psi} + 4T\right)C \frac{\partial}{\partial C} + 4T \frac{\partial}{\partial \psi} + 4T^2 \frac{\partial}{\partial T},
\]

\[
\Gamma_4 = 2T \frac{\partial}{\partial T} + \frac{\partial}{\partial \psi} \quad \text{and}
\]

\[
\Gamma_5 = \frac{\partial}{\partial \phi}.
\]

The one dimensional optimal system is

\[
\{\Gamma_1 \pm \Gamma_2 \pm \Gamma_5, \, \Gamma_1 + \Gamma_2, \, \Gamma_1 + \Gamma_5, \, \Gamma_1, \, a\Gamma_2 + \Gamma_4 + b\Gamma_5, \, \Gamma_1 + \Gamma_3 + a\Gamma_2 + b\Gamma_5, \, \Gamma_2 + a\Gamma_5, \, \Gamma_5, \, \Gamma_2\}
\]
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and the reductions by these elements are listed in Table 4.9.

Table 4.9: Symmetry reductions for solute transport under point vortex water flow

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Reduced equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1 + \Gamma_2 + \Gamma_5$</td>
<td>$C = e^T F(\psi, \rho)$, with $\rho = \phi - T$ and $F$ satisfies</td>
</tr>
<tr>
<td></td>
<td>$\frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \psi^2} + \frac{\partial F}{\partial \rho} + e^{2\psi} \left( \frac{\partial F}{\partial \rho} - F \right) = 0.$</td>
</tr>
<tr>
<td>$\Gamma_1 + \Gamma_2$</td>
<td>$C = e^T F(\phi, \psi)$ and $F$ satisfies</td>
</tr>
<tr>
<td></td>
<td>$e^{2\psi} F = \frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial \psi^2} + \frac{\partial F}{\partial \phi}.$</td>
</tr>
<tr>
<td>$\Gamma_1 + \Gamma_5$</td>
<td>$C = F(\psi, \rho)$, with $\rho = \phi - T$ and $F$ satisfies</td>
</tr>
<tr>
<td></td>
<td>$\frac{\partial^2 F}{\partial \psi^2} + \frac{\partial^2 F}{\partial \rho^2} + \left( 1 + e^{2\psi} \right) \frac{\partial F}{\partial \rho} = 0.$</td>
</tr>
<tr>
<td>$a\Gamma_2 + \Gamma_4 + b\Gamma_5$</td>
<td>$C = Ta^{1/2} F(\rho, \gamma)$</td>
</tr>
<tr>
<td>$b \neq 0$</td>
<td>with $\rho = Te^{-2\psi}$, $\gamma = Te^{-2\phi/b}$ and $F$ satisfies</td>
</tr>
<tr>
<td></td>
<td>$4\gamma^2 \frac{\partial^2 F}{\partial \gamma^2} - \left( \frac{b}{\rho} + 2b - 4 \right) \gamma \frac{\partial F}{\partial \gamma} + 4b^2 \rho^2 \frac{\partial^2 F}{\partial \rho^2} + (4\rho - 1)b^2 \frac{\partial^2 F}{\partial \rho} = 0.$</td>
</tr>
<tr>
<td>$\Gamma_2 + a\Gamma_5$</td>
<td>$C = e^{\phi/a} F(\psi, T)$, where $F$ satisfies</td>
</tr>
<tr>
<td>$a \neq 0$</td>
<td>$e^{2\psi} \frac{\partial F}{\partial T} = \frac{\partial^2 F}{\partial \phi^2} + \left( \frac{1}{a^2} + \frac{1}{a} \right) F.$</td>
</tr>
<tr>
<td>$\Gamma_1 + \Gamma_3 + a\Gamma_2 + b\Gamma_5$</td>
<td>$C = \exp \left( \frac{a \tan^{-1}(2T)}{2} - \frac{Te^{2\psi}}{1+4T^2} - \frac{\ln(1+4T^2)}{2} \right) \times F(\gamma, \rho)$, with</td>
</tr>
<tr>
<td>$b \neq 0$</td>
<td>$\gamma = \sqrt{1 + 4T^2} e^{-\psi}$ and $\rho = \frac{\tan^{-1}(2T)}{2} - \frac{\phi}{b}$, where $F$ satisfies</td>
</tr>
<tr>
<td></td>
<td>$\gamma^2 \frac{\partial^2 F}{\partial \gamma^2} + \gamma \frac{\partial F}{\partial \gamma} + \frac{b}{a^2} \frac{\partial^2 F}{\partial \rho^2} - \left( \frac{1}{\gamma^2} + \frac{1}{b} \right) \frac{\partial F}{\partial \rho} + \left( \frac{1}{\gamma^2} - \frac{a}{\gamma^2} \right) F = 0.$</td>
</tr>
</tbody>
</table>
4.7.1 Invariant solutions

Consider the $\Gamma_3$-invariant solution of the form

$$C = \frac{1}{T} \exp \left( -\frac{e^{2\psi}}{4T} \right) F(\rho, \phi),$$

where $\rho = Te^{-\psi}$ and $F$ satisfies the P.D.E.

$$\frac{\partial^2 F}{\partial \phi^2} + \rho^2 \frac{\partial^2 F}{\partial \rho^2} + \rho \frac{\partial F}{\partial \rho} + \frac{\partial F}{\partial \phi} = 0. \quad (4.61)$$

Since $\Gamma_3$ and $\Gamma_4$ span a non-abelian two dimensional subalgebra, then the reduced Equation (4.61) inherits $\Gamma_4$ which now takes the form

$$\Gamma_4 = \frac{\rho}{2} \frac{\partial}{\partial \rho} + F \frac{\partial}{\partial F}$$

and leads to the reduction

$$F = \rho^2 g(\phi),$$

where $g$ satisfies the equation of damping oscillation

$$g'' + g' + 4g = 0.$$

In terms of original variables we obtain solution

$$C = \frac{T}{R^2} \exp \left( \frac{\omega}{2} - \frac{R^2}{4T} \right) \times \left\{ k_1 \cos \left( \frac{\sqrt{15}}{2} \omega + k_2 \right) \right\} \quad (4.62)$$

and the total flux within a wedge, $\frac{\pi}{\sqrt{15}} - k_3 \leq \omega \leq \frac{3\pi}{\sqrt{15}} - k_3$, is

$$V_{tot} = \int_{\frac{\pi}{\sqrt{15}} - k_3}^{\frac{3\pi}{\sqrt{15}} - k_3} -R^2 \frac{\partial C}{\partial R} d\omega = \frac{k_3}{1+2\sqrt{15}} \times \left[ 2T \frac{R^2}{2} + \frac{1}{2} \right] \times \left\{ \exp \left( \frac{3\pi}{2\sqrt{15}} - \frac{k_3}{2} + k_2 - \frac{R^2}{4T} \right) \right.

\times \cos \left( \frac{3\pi}{2} - \frac{\sqrt{15}k_3}{2} + k_2 \right) - \left. \exp \left( \frac{\pi}{2\sqrt{15}} - \frac{k_3}{2} + k_2 - \frac{R^2}{4T} \right) \right.

\times \cos \left( \frac{\pi}{2} - \frac{\sqrt{15}k_3}{2} + k_2 \right) \right\}.$$
where convective flux is zero. For convenience of interpretation, we have neglected the solution wherein sine functions replace cosine functions. The solution (4.62) has impervious boundary condition, \( J \cdot n = 0 \), where \( n \) is the outward normal vector at \( \omega = \frac{\pi}{\sqrt{15}} - k_3 \) and \( \omega = \frac{3\pi}{\sqrt{15}} - k_3 \). From our solution, total flux tends to infinity as \( R \) tends to 0 and this implies that solute is continuously supplied at the origin. Also, total flux is zero at \( T = 0 \) and \( R \to \infty \). Figure 4.5 and 4.6 are plots of total flux against radius and time respectively.

![Graphical representation of change in solute flux with radius.](image)

Figure 4.5: Graphical representation of change in solute flux with radius.

Similar to equations in Section (4.3.2), Equation (4.61) may be transformed into
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Figure 4.6: Graphical representation of change in solute flux with time.

The modified Helmholtz equation

\[
\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} - \frac{1}{4} S = 0, \tag{4.63}
\]

under the transformation

\[ F = S(x, y)e^{-x/2}; \quad \text{and} \quad \rho = e^\psi. \]

At this stage we omit further analysis of Equation (4.63).

The reduced equation obtained using \( \Gamma_2 + a\Gamma_5 \) in Table 4.9, wherein without loss of generality we have let \( a = 1 \), namely;

\[
\frac{\partial^2 F}{\partial \psi^2} - e^{2\psi} \frac{\partial F}{\partial T} + 2F = 0. \tag{4.64}
\]
admits, other than the infinite symmetry generator, a four dimensional Lie algebra spanned by the base vectors

\[
\begin{align*}
\Xi_1 &= 4T \frac{\partial}{\partial \phi} + 4T^2 \frac{\partial}{\partial T} - \left( e^{2\psi} + 4T \right) F \frac{\partial}{\partial F}, \\
\Xi_2 &= \frac{\partial}{\partial \phi} + 2T \frac{\partial}{\partial T}, \\
\Xi_3 &= \frac{\partial}{\partial T}, \\
\Xi_4 &= F \frac{\partial}{\partial F}.
\end{align*}
\] (4.65)

Note that the same point symmetries given in (4.65) are obtainable for any values of \( \alpha \). The one dimensional optimal system is

\[
\{ \alpha \Xi_2 + \Xi_4, \xi_1 \pm \xi_3, \Xi_1, \Xi_4 \pm \Xi_3, \Xi_4 \},
\]

where \( \alpha \) is an arbitrary constant. If we admit the discrete symmetry which maps \( \Xi_1 - \Xi_3 \) to \( \Xi_1 + \Xi_3 \) and \( \Xi_4 - \Xi_3 \) to \( \Xi_4 + \Xi_3 \), then the number of elements of the optimal system reduces by two. Reductions using these elements are given in Table 4.10 wherein, \( M_{a,b}(x) \) and \( W_{a,b}(x) \) are the Whittaker functions; \( J_\nu \) and \( Y_\nu \) are the Bessel functions of order \( \nu \) (Abramowitz and Stegun, 1965).

Equation (4.61) admits, beside the infinite symmetry generator, the four dimensional Lie algebra spanned by the base vectors

\[
\begin{align*}
\mathcal{K}_1 &= 2 \log \rho \frac{\partial}{\partial \phi} - 2 \phi \rho \frac{\partial}{\partial \rho} - \log \rho F \frac{\partial}{\partial F}, \\
\mathcal{K}_2 &= -\rho \frac{\partial}{\partial \rho}, \\
\mathcal{K}_3 &= F \frac{\partial}{\partial F}, \\
\mathcal{K}_4 &= 2 \frac{\partial}{\partial \phi} - F \frac{\partial}{\partial F}.
\end{align*}
\]
Table 4.10: Reductions of Equation (4.64) and exact solutions to (4.56)

<table>
<thead>
<tr>
<th>Ξ_i</th>
<th>Reduced O.D.E.</th>
<th>Invariant solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ξ_1</td>
<td>$\rho^2 g'' + \rho g' + g = 0,$</td>
<td>$g = k_1 \sin[\log(\rho)] + k_2 \cos[\log(\rho)],$</td>
</tr>
<tr>
<td></td>
<td>with $F = \frac{1}{\rho} \exp\left(-\frac{\rho^2}{4T}\right) g(\rho)$</td>
<td>$C = \frac{1}{T} \exp\left(-\frac{R^2}{4T} - \omega\right) \times \left{k_1 \sin\left[\log\left(\frac{R}{T}\right)\right] + k_2 \cos\left[\log\left(\frac{R}{T}\right)\right]\right}.$</td>
</tr>
<tr>
<td></td>
<td>and $\rho = Te^{-\psi}$</td>
<td></td>
</tr>
<tr>
<td>$\alpha \Xi_2 + \Xi_4$</td>
<td>$4\rho^2 g'' + (4\rho - 1)g'$</td>
<td>$g = e^{-1/8\rho} \rho^{1/2} \left[k_1 M_{-\frac{a+1}{2a},k} \left(\frac{1}{4\rho}\right) + k_2 W_{-\frac{a+1}{2a},k} \left(\frac{1}{4\rho}\right)\right]$</td>
</tr>
<tr>
<td>$\alpha \neq 0$</td>
<td>$+ \left(2 - \frac{1}{2\alpha}\right) g = 0$</td>
<td>and so $C = \frac{T^\alpha+1/2}{R} \exp\left(-\frac{\rho}{a} - \frac{R^2}{8T}\right)$</td>
</tr>
<tr>
<td></td>
<td>with $F = T^{1/2a} g(\rho)$, $\rho = Te^{-\psi}$</td>
<td>$\times \left[k_1 M_{-\frac{a+1}{2a},k} \left(\frac{R^2}{4T}\right) + k_2 W_{-\frac{a+1}{2a},k} \left(\frac{R^2}{4T}\right)\right]$ with $k$ given by $k = \pm \frac{1}{\sqrt{2}}$.</td>
</tr>
<tr>
<td>Ξ_1 + Ξ_3</td>
<td>$\rho^2 g'' + \rho g' + \left(1 + \frac{1}{\rho^4}\right) g = 0,$</td>
<td>$g = k_1 J_k \left(\frac{1}{2\rho^4}\right) + k_2 Y_k \left(\frac{1}{2\rho^4}\right)$</td>
</tr>
<tr>
<td></td>
<td>with $F = \frac{\exp\left(-\frac{\rho^2}{4T^2+1}\right)}{\sqrt{4T^2+1}} g(\rho)$</td>
<td>and so $C = \frac{1}{\sqrt{4T^2+1}} \exp\left(-\frac{R^2}{4T^2+1} + \frac{\rho}{a}\right) \left{k_1 J_k \left(\frac{R^2}{8T^2+2}\right) + k_2 Y_k \left(\frac{R^2}{8T^2+2}\right)\right}$ with $k$ given by $k = \pm \frac{1}{2}$.</td>
</tr>
<tr>
<td></td>
<td>and $\rho = \sqrt{4T^2+1} e^{-\psi}$</td>
<td></td>
</tr>
<tr>
<td>Ξ_4 + Ξ_3</td>
<td>$g'' + \left(2 - e^{2\psi}\right) g = 0,$</td>
<td>$g = k_1 J_{i\sqrt{2}} \left(i e^{\psi}\right) + k_2 Y_{i\sqrt{2}} \left(i e^{\psi}\right)$</td>
</tr>
<tr>
<td></td>
<td>$F = e^T g(\psi).$</td>
<td>and so $C = \exp\left(T + \frac{\rho}{a}\right) \times \left[k_1 J_{i\sqrt{2}} \left(i e^{\psi}\right) + k_2 Y_{i\sqrt{2}} \left(i e^{\psi}\right)\right].$</td>
</tr>
</tbody>
</table>
The one dimensional optimal system is

\[ \{ C_1 + \alpha C_3, \quad C_4 \pm C_3, \quad C_4, \quad C_2 + \alpha C_3, \quad C_2 \} , \]

where \( \alpha \) is an arbitrary constant. If we admit the discrete symmetry that maps \( C_4 + C_3 \) to \( C_4 - C_3 \), then the number elements of the optimal system reduce by one.

In the next examples, we construct invariant solution using members of the optimal system (4.66).

**Example i**

\( C_1 + \alpha C_3 \) leads to the functional form

\[ F = \exp \left[ -\frac{\phi}{2} + \frac{\alpha}{2} \sin^{-1} \left( \frac{\phi}{\sqrt{\phi^2 + (\log \rho)^2}} \right) \right] g(\psi), \]

where \( \psi = \frac{\phi^2 + (\log \rho)^2}{2} \) and \( g \) satisfies the O.D.E.

\[ \frac{\psi^2}{2} g'' + \psi g' + \left( \frac{\alpha}{16} - \frac{\psi}{8} \right) g = 0. \]

Thus we have

\[ g = k_1 J_{\frac{i \nu \alpha}{2}} \left( i \frac{\sqrt{\psi}}{2} \right) + k_2 Y_{\frac{i \nu \alpha}{2}} \left( i \frac{\sqrt{\psi}}{2} \right) \]

and in terms of the original variables we obtain

\[ C = \frac{1}{T} \exp \left\{ -\frac{\phi}{2} - \frac{R^2}{4T} + \frac{\alpha}{2} \sin^{-1} \left( \frac{\phi}{\sqrt{\phi^2 + (\log(TR^{-1}))^2}} \right) \right\} \]

\[ \times \left\{ k_1 J_{\frac{i \nu \alpha}{2}} \left( i \frac{\sqrt{\phi^2 + (\log(TR^{-1}))^2}}{2} \right) + k_2 Y_{\frac{i \nu \alpha}{2}} \left( i \frac{\sqrt{\phi^2 + (\log(TR^{-1}))^2}}{2} \right) \right\} , \]

where \( J_{\nu} \) and \( Y_{\nu} \) are the Bessel functions.
Example ii

$K_2 + \alpha K_3$ leads to the functional form

$$F = \rho^{-\alpha} g(\phi),$$

where $g$ satisfies the O.D.E.

$$g'' + g' + (\alpha^2 - \alpha)g = 0.$$

Two cases arise ($\alpha = 1$, $\alpha \neq 1$).

**case 1** if $\alpha = 1$ then

$$g = k_1 + k_2 e^{-\phi}$$

and so in terms of the original variables we have

$$C = \frac{R}{T^2} \exp \left( -\frac{R^2}{4T} \right) \times \left( k_1 + k_2 e^{-\phi} \right).$$

**case 2** for $\alpha \neq 1$, three subcases arise

**subcase (a)** if $\alpha^2 - \alpha < \frac{1}{4}$, then

$$g = k_1 \exp \left( \frac{\sqrt{1-4(\alpha^2-\alpha)}-1}{2} \phi \right) + k_2 \exp \left( -\frac{\sqrt{1-4(\alpha^2-\alpha)}-1}{2} \phi \right)$$

and so in terms of the original variables we obtain

$$C = \frac{R^\alpha}{T^{\alpha+1}} \exp \left( -\frac{R^2}{4T} \right) \times \left\{ k_1 \exp \left( \frac{\sqrt{1-4(\alpha^2-\alpha)}-1}{2} \phi \right) + k_2 \exp \left( -\frac{\sqrt{1-4(\alpha^2-\alpha)}-1}{2} \phi \right) \right\}.$$

**subcase (b)** if $\alpha^2 - \alpha > \frac{1}{4}$ then

$$g = \exp \left( -\frac{\phi}{2} \right) \times \left[ k_1 \sin \left( \frac{\sqrt{1-4(\alpha^2-\alpha)}\phi}{2} \right) + k_2 \cos \left( \frac{\sqrt{1-4(\alpha^2-\alpha)}\phi}{2} \right) \right]$$
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and so in terms of the original variables we obtain

\[
C = \frac{R^\alpha}{T^{\alpha+1}} \exp \left( -\frac{R^2}{4T} - \frac{\phi}{2} \right) \times \left\{ k_1 \sin \left( \frac{\sqrt{1-4(\alpha^2-\alpha)\phi}}{2} \right) \right. \\
\left. + k_2 \cos \left( \frac{\sqrt{1-4(\alpha^2-\alpha)\phi}}{2} \right) \right\}. 
\]  

(4.67)

**Subcase (c)** if \( \alpha^2 - \alpha = \frac{1}{4} \) then

\[
g = \exp \left( -\frac{\phi}{2} \right) (k_1 + k_2 \phi)
\]

and hence in the original variables we obtain

\[
C = \frac{R^\alpha}{T^{\alpha+1}} \exp \left( -\frac{R^2}{4T} - \frac{\phi}{2} \right) \times (k_1 + k_2 \phi).
\]

**Example iii**

\( \mathcal{K}_4 - \mathcal{K}_3 \) leads to the functional form

\[
F = e^\phi g(\rho),
\]

where \( g \) satisfies the Euler equation

\[
\rho^2 g'' + \rho g' + 2g = 0.
\]

Thus we have

\[
g = k_1 \sin \left( \sqrt{2} \log |\rho| \right) + k_2 \cos \left( \sqrt{2} \log |\rho| \right)
\]

and so in term of the original variables we obtain

\[
C = \frac{1}{T} \exp \left( -\frac{R^2}{4T} - \tan^{-1} \left( \frac{Y}{X} \right) \right) \times \left\{ k_1 \sin \left( \sqrt{2} \log \left| \frac{T}{R} \right| \right) + k_2 \cos \left( \sqrt{2} \log \left| \frac{T}{R} \right| \right) \right\}.
\]
Finally we analyse the remaining equations in Table 4.9 and present the invariant solutions in the next examples.

**Example iv**

Following reduction by $aT_2 + T_4 + bT_5$, the reduced equation admits, beside the infinite symmetry generator, the scaling of $F$ and $\gamma$, the linear combination of which leads to the functional form

$$F = \gamma^x g(\rho), \quad \alpha \in \mathbb{R},$$

where $g$ satisfies the O.D.E.

$$\rho^2 g'' + \left( \rho - \frac{1}{4} \right) g' + \left( \frac{\alpha^2}{b^2} - \frac{\alpha}{2b} - \frac{\alpha}{4b\rho} \right) g = 0,$$

which becomes

$$\xi w'' + \left( 2k + 1 - \frac{\xi}{4} \right) w' - \left( \frac{kb + \alpha}{4b} \right) w = 0,$$

under the transformations $\xi = \rho^{-1}$, $w = g\xi^{-k}$, with $k$ given by $k = \sqrt{\frac{\alpha}{2b} - \frac{\alpha^2}{b^2}}$. Thus we obtain

$$w = e^{\xi/2} \xi^{-1/2-k} \left[ k_1 M_{p,q} \left( \frac{\xi}{4} \right) + k_2 W_{p,q} \left( \frac{\xi}{4} \right) \right],$$

where $p = \frac{b-2\alpha}{2b}$, $p = k$. Hence in terms of the original variables, we have

$$C = \frac{T^{(3\alpha+1)/2}}{R} \exp \left( \frac{R^2}{8T} - \frac{2\alpha}{b} \right) \times \left[ k_1 M_{p,q} \left( \frac{R^2}{4T} \right) + k_1 M_{p,q} \left( \frac{R^2}{4T} \right) \right].$$

**Example v**

Following reduction by $\Gamma_1 + \Gamma_2 + \Gamma_5$, the reduced equation admits, beside the infinite
symmetry generator, \( \mathcal{V}_1 = -F \frac{\partial}{\partial F} + 2 \frac{\partial}{\partial \phi} \) and \( \mathcal{V}_2 = F \frac{\partial}{\partial F} \). \( \mathcal{V}_1 \) leads to the functional form

\[
F = e^{-\phi/2} g(\psi),
\]

where \( g \) satisfies the O.D.E.

\[
g'' - \left( \frac{3e^{2\psi}}{2} + \frac{1}{4} \right) g = 0,
\]

which becomes

\[
4zw'' + 6w' - \frac{3}{2} w = 0,
\]

under the transformation \( z = e^{2\psi} \), \( w = z^{-1/4} \) and so we obtain

\[
w = z^{-1/4} \left\{ k_1 \sin \left( i \sqrt{\frac{3}{2}} z^{1/2} \right) + k_2 \cos \left( i \sqrt{\frac{3}{2}} z^{1/2} \right) \right\}.
\]

Thus in terms of the original variables we obtain

\[
C = e^{(3T-\phi)/2} \times \left\{ k_1 \sin \left( i \sqrt{\frac{3}{2}} R \right) + k_2 \cos \left( i \sqrt{\frac{3}{2}} R \right) \right\}.
\]

**Example vi**

Following reduction by \( \Gamma_1 + \Gamma_2 \), the reduced equation admits, beside the infinite symmetry generator, the scaling of \( F \) and translation of \( \phi \). The linear combination of which leads to the functional form

\[
F = e^{\alpha \phi} g(\psi), \quad \alpha \in \mathbb{R},
\]

where \( g \) satisfies the O.D.E.

\[
g'' + \left( \alpha^2 + \alpha - e^{2\psi} \right) g = 0.
\]
Thus

\[ g = k_1 I_\nu \left( e^\psi \right) + k_2 K_\nu \left( e^\psi \right), \]

where \( \nu = i\sqrt{\alpha^2 + \alpha} \). Hence, in terms of the original variables we obtain

\[ C = \exp(T + \alpha\phi) \times [k_1 I_\nu(R) + k_2 K_\nu(R)]. \]

**Example vii**

Following reduction by \( \Gamma_1 + \Gamma_5 \), the reduced equation admits, beside the infinite symmetry generator, the scaling of \( F \) and translation of \( \rho \). The linear combination of which leads to the functional form

\[ F = e^{a\phi} g(\psi), \quad \alpha \in \mathbb{R}, \]

where \( g \) satisfies the O.D.E.

\[ g'' + \left( \alpha^2 + \alpha + e^{-2\psi} \right) g = 0. \]

This O.D.E. may be solved as in example v and hence in terms of the original variables we obtain

\[ C = e^{a(\phi - T)} \times [k_1 J_\nu(R) + k_2 Y_\nu(R)], \]

where \( \nu = -i\sqrt{\alpha^2 + \alpha} \).

**Example viii**

Following reduction by \( \Gamma_1 + \Gamma_3 + a\Gamma_2 + b\Gamma_5 \), the reduced equation admits, beside
the infinite symmetry generator, the scaling of $F$ and translation of $\rho$. The linear combination of which leads to the functional form

$$F = e^{\rho/\alpha} g(\gamma), \quad \alpha \in \mathbb{R},$$

where $g$ satisfies the O.D.E.

$$\gamma^2 g'' + \gamma g' - \left( \frac{1}{\gamma^4} + \frac{2}{\gamma^2} \right) g = 0,$$

wherein, without loss of any generality, we have taken $a = \alpha = b = 1$. Thus

$$g = \exp \left( \frac{1}{2\gamma^2} \right) \times \left\{ k_1 + k_2 \text{Ei} \left( 1, \frac{1}{\gamma^2} \right) \right\}.$$

where $\text{Ei}(n, x)$ is the exponential integral function (Abramowitz and Stegun, 1965).

Hence in terms of the original variables we obtain

$$C = \frac{1}{\sqrt{1 + 4T^2}} \exp \left( \tan^{-1}(2T) + \frac{(1 - T) R^2}{1 + 4T^2} - \phi \right) \times \left\{ k_1 + k_2 \text{Ei} \left( 1, \frac{R^2}{1 + 4T^2} \right) \right\}.$$

### 4.8 Concluding remarks

The symmetry analysis has produced a rich array of variable reductions and exact analytic solutions for non-radial solute transport on a background of radial and point vortex water flow. As far as we are aware, these are the only known solutions for non-radial two dimensional isotropic velocity-dependent dispersion. However, we know that the outcomes fall a little short of the ideals of exact solutions for key nonlinear boundary value problems with direct testable practical implications.
In symmetry analysis of complicated partial differential equations, it is unusual that the boundary conditions are directly interpretable. In the solutions that we have displayed in Section 4.3.2, we maintained constant concentration boundary conditions or zero-flux boundary conditions that are indeed interpretable. However, the solute injection rates at the source must be special functions of time. These may provide some insight on the effects of varying water velocity in dispersion but they are more likely to be important as bench tests for two dimensional numerical schemes.

While the solute transport equations are linear, they have highly variable coefficients. We have found that this has been more troublesome than the nonlinearity in symmetry classification of the general nonlinear Richards equation for unsaturated water transport (Oron and Rosenau, 1986; Sposito, 1990; Edwards, 1994; Edwards and Broadbridge, 1994; Yung et al., 1994; Edwards and Broadbridge, 1995; Baikov et al., 1997). However, the scope for symmetry analysis of linear P.D.E.s such as the solute transport analysis has greatly broadened following the initially surprising results of Broadbridge and Arrigo (1999) that every solution of any second or higher order linear P.D.E. is invariant under some classical Lie symmetry. This gives us hope that we may be able to incorporate boundary conditions from the outset of symmetry analysis.

Preliminary group classification using equivalence transformations for solute transport Equation (4.12) has proved to be impossible. It turned out that Equation (4.12)
admits a six dimensional but trivial equivalence algebra.

The formulation of the dispersion problem with \( p = 1 \) in radial two dimensional flows goes back to Hoopes and Harleman (1967) (see also Philip, 1994). The solutions in this case are given in terms of Airy functions and inversion in exact closed form is not possible. We observed that when \( p = 1 \), the only point symmetries for solute transport under radial water flow are scaling in \( C \), translation in \( T \) and \( v \). and the infinite dimensional superposition symmetry generator. In the case of a point vortex water flow, \( p = 0 \) is the only case for which solute transport Equation (4.12) admits extra point symmetries.
Chapter 5

Solutions for transient solute transport in unsaturated soils during steady evaporation from a water table

5.1 Introduction

A search for exact analytic transient solutions for water flow and solute transport equations continues to be of scientific interest. For example, exact analytic solutions for solute transport in unsaturated soil have been considered by Nachabe et al. (1994) in which the Broadbridge and White nonlinear model was used to solve the
Richards equation for vertical water flow under a constant infiltration rate. The method of characteristics was then used to determine the location of the solute front. Other notable contributors include van Genuchten and Alves (1982) who constructed a compendium of solutions for the one dimensional constant-coefficient approximations to the convective-dispersion equation. Perhaps the most notable challenge in the search for exact analytic solutions is that both water flow and solute transport in unsaturated soils contribute to the transient phenomena (Nachabe et al., 1994). However if obtained, exact analytic solutions could be used as tools to verify the accuracy of numerical schemes.

The simplest transient solute transport problem worth considering, involves transient solute dispersion on a steady one dimensional saline water flow background, with evaporation at the soil surface. Although the evaporation rate may be small, a significant quantity of salts may accumulate over a large time scale. Gardner (1958) derived several soil hydraulic models which allow exact solution of some steady-state unsaturated flow problems and approximate solutions of some transient problems in one dimension. However, these solutions for volumetric water content would then appear in the solute transport equation as complicated non-constant coefficients. It would then be unlikely to find exact solutions, in simple analytic form, for the coupled solute transport equation. Therefore, we adopt an alternative inverse approach to this problem. We find suitable functions $\theta(z)$ that are physically reasonable water content profiles and which lead to extra symmetry in the coupled solute dispersion
equation. We then construct the realistic nonlinear soil hydraulic conductivity $K'(\theta)$ and diffusivity $D(\theta)$ functions that permit the solution $\theta(z)$, and we are able to construct exact transient solutions to the convection-dispersion equation for solute transport. This has previously been achieved only for uniform water concentration with the very simplest of background water flows.

In practical problems, it is normal to determine approximate analytical solutions for relevant P.D.E.s (e.g. Warrick et al., 1971; Smiles et al., 1978; De Smedt and Wierenga, 1978; Elrick et al., 1987). Elrick et al. (1987) claimed that approximate analytical solutions bypass the more complicated exact analytic solutions and numerical techniques. De Smedt and Wierenga (1994) presented approximate analytical solutions for solute flow during infiltration and redistribution. They assumed that the volumetric water content was dependent on time and independent of the depth down to the transition zone of the solute penetration depth. After this simplification, knowledge of the relationship between hydraulic conductivity and volumetric water content and between volumetric water content and soil water interaction potential was not required. However, in unsaturated water flow the essential properties are the hydraulic conductivity as a function of volumetric water content or soil water interaction potential, and the relationship between volumetric water content and soil water interaction potential.
5.2 Governing equations

In order to solve the equation governing solute transport in soils, one needs to first find the volumetric water content of the soil as a function of space and time. This is achieved by solving the nonlinear Fokker-Planck diffusion-convection equation describing the water flow in porous media. In one dimension the water flow Equation (2.9) is given by

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( D(\theta) \frac{\partial \theta}{\partial z} \right) - \frac{dK}{d\theta} \frac{\partial \theta}{\partial z},$$  \hspace{1cm} (5.1)

which is a version of the familiar Richards equation. Here, the nonlinear soil water diffusivity is $D(\theta)$, hydraulic conductivity $K(\theta)$, $t$, $z$, $\theta$ and $\Psi$ are all described in chapter 2. Analytical models have been developed to predict water movement in the unsaturated zone (e.g. Broadbridge and White, 1988; Sander et al., 1988; Philip, 1992). The possibility of exact solutions of Equation (5.1) requires special choices for the models $K(\theta)$ and $D(\theta)$.

Movement of non-volatile, inert solutes in unsaturated soils is described by the convection-dispersion equation (C.D.E.)

$$\frac{\partial \theta c}{\partial t} = \frac{\partial}{\partial z} \left( \theta D_v(v) \frac{\partial c}{\partial z} \right) - \frac{\partial (qc)}{\partial z},$$  \hspace{1cm} (5.2)

where $c$ is the concentration of the solute in solution, $q$ is the Darcian water flux, $\theta$, $t$ and $z$ are as in Equation (5.1). $D_v(v)$ is the dispersion coefficient depending on pore water velocity $v = \frac{|q|}{\theta}$ and it includes both the molecular diffusion and
the mechanical dispersion (Bear, 1979). $D_v(v)$ has frequently been observed to be linearly proportional to the pore water velocity (Biggar and Nielsen, 1967; Bear, 1979; Anderson, 1979). We assume $D_v(v) = D_1 v$, where $D_1$ is the dispersivity. Dispersivity has a significant impact on the migration process and is a natural candidate for the spatial characteristic length. It ranges from about 0.5 cm for laboratory-scale displacement experiments involving disturbed recompacted soils to about 10 cm or more for field scale experiments (Nielsen et al., 1986). For steady-state water flow and using equation of continuity, Equation (5.2) reduces to

$$\theta(z) \frac{\partial c}{\partial t} = \frac{\partial}{\partial z} \left( D_1 |q| \frac{\partial c}{\partial z} \right) - q \frac{\partial c}{\partial z}. \quad (5.3)$$

With application to evaporation from a water table at a constant Darcian water flux $q = -R$, Equation (5.3) then becomes

$$\theta(z) \frac{\partial c}{\partial t} = D_1 R \frac{\partial^2 c}{\partial z^2} + R \frac{\partial c}{\partial z}. \quad (5.4)$$

### 5.3 Steady water flow models and hydraulic properties

Before considering the soil hydraulic properties, we first choose an appropriate function for the volumetric water content profile. For simplicity we consider a steady one dimensional profile that may represent steady evaporation from a water table.
We choose
\[ \theta(z) = \theta_s + \beta \left(1 - e^{(d-z)/l}\right), \] (5.5)
with \( \beta \) and \( l \) constants, \( d \) being the depth to the water table and \( \theta_s \) being the volumetric water content at saturation. In terms of scaled dimensionless variables,
\[ \Theta = 1 + B(1 - e^{(D-Z)/L}), \] (5.6)
where \( B = \beta/\theta_s, \ L = l/\lambda_s, \ D = d/\lambda_s \) and \( Z = z/\lambda_s \). Here, \( \lambda_s \) is a macroscopic sorptive length scale, see e.g. (White and Sully, 1987). In Figure 5.1, we plot the water content profile (5.6) for selected values of \( B, D \) and \( L \).

Using dimensionless variables; \( \Psi_* = \Psi/\lambda_s \) and \( K_* = K/K_s \), \( K_s \) being the hydraulic conductivity at saturation, the steady-state version of Equation (5.1) may be written
\[ \frac{\partial}{\partial Z} \left( K_*(\Theta) \frac{\partial \Psi_*}{\partial Z} \right) - \frac{dK_*}{d\Theta} \frac{\partial \Theta}{\partial Z} \]
or equivalently,
\[ \frac{\partial}{\partial Z} \left( K_*(\Psi_*) \frac{\partial \Psi_*}{\partial Z} \right) - \frac{dK_*}{d\Psi_*} \frac{\partial \Psi_*}{\partial Z}, \]
which integrates to
\[ K_* \left( \frac{d\Psi_*}{dZ} - 1 \right) = R_*, \] (5.7)
where \( R_* = R/K_s \), \( R \) being the constant evaporation rate so that \(-R\) is the uniform volumetric water flux. Gardner (1958) solved Equation (5.7) for various hydraulic models including the power law and the exponential \( K = K_s e^{\alpha \Psi} \). Here, \( \alpha \) is the
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Figure 5.1: Graphical representation of change in volumetric water content with depth. Parameters used: $D = 2$, $B = 0.55$ and $L = 2$.

sorptive number, regarded as the reciprocal of the sorptive length scale. In the current analysis, we aim to solve the coupled solute transport equation. Therefore, we devise an inverse method in which the function $K_\ast(\Theta)$ is deduced after specifying a relatively simple water content profile. The specification of a simple water content profile allows us to construct some special solutions of the coupled convection-dispersion equation for solute transport.

Since $K_\ast = 1$ at $\Theta = 1$, it follows from (5.7) that
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\[ \frac{d\Psi_*}{d\Theta} \bigg|_{\Theta=1} = \frac{L(1 + R_*)}{B}. \] (5.8)

Assuming \( K'(\Theta) \geq 0 \) and \( K''(\Theta) \geq 0 \), we have the necessary relationships

\[ h(\Theta) \leq g(\Theta) \] (5.9)

and

\[ h' + h^2 - \frac{g' + gh}{R_*} \geq 0, \] (5.10)

where

\[ h(\Theta) = \frac{\Psi''(\Theta)}{\Psi'(\Theta)} \]

and

\[ g(\Theta) = \frac{L}{1 + B - \Theta}. \]

We are now free to choose any function \( h(\Theta) \). If \( h(\Theta) = -\frac{2}{\Theta} \) then

\[ \Psi_*(\Theta) = \frac{L(1 + R_*)}{B} \left( \frac{\Theta - 1}{\Theta} \right). \] (5.11)

Clearly water content is a single valued function of soil water interaction potential and here, we neglect the effects of hysteresis (see e.g. Bear and Verruij, 1987).

Thus from (5.7) we obtain

\[ K_*(\Theta) = \frac{BR_*\Theta^2}{(1 + R_*)(1 + B - \Theta) - B\Theta^2} \] (5.12)
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Figure 5.2: Graphical representation of $K_\ast(\Theta)$ in (5.12). Parameters used: $R_\ast = 1$ and $B = 0.55$.

and so

$$D_\ast(\Theta) = \frac{L R_\ast (1 + R_\ast)}{(1 + R_\ast)(1 + B - \Theta) - B\Theta^2}. \quad (5.13)$$

The graphs of $K_\ast(\Theta)$ and $D_\ast(\Theta)$ are depicted in Figure 5.2 and Figure 5.3 respectively. They show a general trend similar to that observed in laboratory measurements (see e.g. Buckingham, 1907; Richards, 1931; Moore, 1939; Philip, 1969). On the other hand, if we consider $\Psi_\ast$ as the independent variable then

$$K_\ast(\Psi_\ast) = \frac{R_\ast B L^2 (1 + R_\ast)}{(1 + B)[L(1 + R_\ast) - B\Psi_\ast]^2 - F}. \quad (5.14)$$
Figure 5.3: Graphical representation of $D_*(\Theta)$ in (5.13). Parameters used: $B = 0.55$ and $R_* = 1$.

where,

$$F = (L(1 + R_*))^2 - BL(1 + R_*)\Psi_* + BL^2(1 + R_*).$$

Graphical representation of $K_*(\Psi_*)$ is depicted in Figure 5.4. If the water profile is given as in Equation (5.6), then the soil must have hydraulic properties given in (5.12), (5.13) and (5.14).
5.4 Classical symmetry analysis

In this section, we consider Equation (5.4) with $\theta$ given in (5.5). In terms of scaled dimensionless variables we have

$$\left[1 + B \left(1 - e^{(D-Z)/L}\right)\right] \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial Z^2} + \frac{\partial C}{\partial Z},$$  

(5.15)

where $C = c/c_s$ and $T = t/t_s$ with $c_s$ being the suitable concentration and $t_s = \theta_s D_1/R$.

In the initial symmetry analysis of Equation (5.4), in which $\theta(z)$ is an arbitrary
function, the symmetry finding package DIMSYM (Sherring, 1993) under the algebraic manipulation software REDUCE (Hearn, 1991) finds only trivial symmetries, namely, translations in $t$, scaling of $c$ and the infinite dimensional symmetry for linear superposition. These types of simple symmetries that are shared by all forms of the governing equation, are often referred to as the principal Lie algebra for the equation in question (see e.g. Ibragimov, 1995). No extra symmetries were obtained in the classical symmetry analysis of Equation (5.15). However a reduction of (5.15) is possible using a linear combination of the translation in $T$ and the scaling in $C$. Furthermore, DIMSYM reports that extra finite symmetries may be obtained if seventy seven coefficient functions and derivatives (not listed here) are linearly dependent. Full classification of these cases would be a major task and we have not completed it. We note that extra symmetries could be obtained for the cases $\theta = \theta e^{-d \zeta}$, $\theta = \text{constant}$ and $\theta = \theta \left( \frac{\pi + \gamma}{d + \gamma} \right)^{-1}$.

### 5.5 Classical symmetry reductions

Eventually we wish to solve Equation (5.15) subject to the initial and boundary conditions

$$C = 0, \quad T = 0, \quad 0 \leq Z \leq D; \quad (5.16)$$

$$C = 1, \quad Z = D, \quad T \geq 0; \quad (5.17)$$

$$C + \frac{\partial C}{\partial Z} = 0, \quad Z = 0, \quad T \geq 0. \quad (5.18)$$
Condition (5.17) corresponds to the store of solute at fixed concentration at the underlying ground water reservoir. Condition (5.18) corresponds to the assumption that the solute is not carried through the soil surface with the evaporating water, rather it accumulates here. The Laplace transform of Equation (5.15) subject to the conditions (5.16) through (5.18) resulted in the Laplace transform that could not be inverted explicitly. Furthermore, we examined the linear operator obtained from Equation (5.4) and found that it could not commute with its adjoint, and thus spectral theory of normal operators would not be applicable. However, some time dependent solutions for Equation (5.15) could be constructed by assuming

\[ C_s - C = e^{-\lambda T} F(Z), \]

where \( C_s \) is the steady state concentration. In fact (5.19) is the functional form obtained from invariance under linear combination of translation in \( T \) and scaling in \( C \) admitted by Equation (5.15). Equation (5.19) could be obtained by separation of variables. In a steady state, solute transport Equation (5.15) reduces to an O.D.E.

\[ C''_s(Z) + C'_s(Z) = 0. \]

The steady solution satisfying the prescribed boundary condition (5.17) and (5.18) is given by

\[ C_s = e^{D - z}. \]

Therefore if solute concentration approaches steady state during a prolonged period of evaporation, the long-term concentration at the surface will be \( C_0 = e^D \). For
example, for a sandy loam sorptive length 0.15 m with the water table at 1.5 m depth, the salt concentration will be amplified by a factor \(e^{10}\), around 22000. In practice, in arid regions with an artesian basin, the concentration of salt will often reach saturation level well below the soil surface and salt will be precipitated. However, we will see later that a typical time scale for the approach to steady state will be tens of years, as observed in arid zones that are artificially irrigated.

We now solve for the transient \(C_1 = C_s - C\) with \(\lambda > 0\). We observe that \(F\) satisfies

\[
F''(Z) + F'(Z) + \left(\lambda (1 + B) - \lambda B e^{(D-Z)/L}\right) F(Z) = 0. \tag{5.22}
\]

Under the transformation \(\zeta = 2L\sqrt{\lambda} B e^{D/L} e^{-Z/2L}\) Equation (5.22) becomes

\[
\zeta^2 \frac{d^2 F}{d\zeta^2} + (1 - 2L) \zeta \frac{dF}{d\zeta} + \left(4\lambda L^2 (1 + B) - \zeta^2\right) F = 0. \tag{5.23}
\]

Hence,

\[
C_1 = C_s - C = e^{-\lambda T} \zeta^L \left\{a_1 I_\nu(\zeta) + a_2 K_\nu(\zeta)\right\}, \tag{5.24}
\]

where \(I_\nu\) and \(K_\nu\) are modified Bessel functions of the first and the third kind (Watson, 1958; Abramowitz and Stegun, 1965) of the order

\[
\nu = 2L \sqrt{1 - 4\lambda (1 + B)},
\]

and \(a_1\) and \(a_2\) are constants. This exact solution, however, does not satisfy the prescribed boundary conditions. In the next section, we will find if Equation (5.4) has nonclassical symmetry reductions that yield other solutions.
5.6 Nonclassical symmetry analysis

We now investigate the possibility of nonclassical symmetries admitted by Equation (5.4) with arbitrary \( \theta(z) \). By the nonclassical method, it is possible to find further types of explicit solutions by the same reduction technique that is commonly used in the classical method. Moreover, there exist P.D.E.s which possess symmetry reductions not obtainable via classical Lie group methods (Clarkson and Kruskal, 1989; Arrigo et al., 1994; Clarkson and Mansfield, 1994; Goard and Broadbridge, 1996). We consider the general equation given in scaled dimensionless variables, namely;

\[
\Theta(Z) \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial Z^2} + \frac{\partial C}{\partial Z}. \tag{5.25}
\]

The nonclassical method, introduced by Bluman and Cole (1969), seeks the invariance of the system of P.D.E.s composed of the given equation such as (5.25) with its I.S.C.

\[
\tau(T, Z, C) \frac{\partial C}{\partial T} + \xi(T, Z, C) \frac{\partial C}{\partial Z} = \eta(T, Z, C), \tag{5.26}
\]

where the coefficients \( \tau, \xi \) and \( \eta \) are the infinitesimals of the transformations

\[
\begin{align*}
T' &= T + \epsilon \tau(T, Z, C) + O(\epsilon^2), \\
Z' &= Z + \epsilon \xi(T, Z, C) + O(\epsilon^2), \\
C' &= C + \epsilon \eta(T, Z, C) + O(\epsilon^2).
\end{align*} \tag{5.27}
\]

\( \epsilon \) is a real parameter (see Sections 3.1.1 and 3.1.2). If \( \tau \neq 0 \), then by dividing by \( \tau \) one may assume without loss of generality that \( \tau = 1 \) so that (5.26) is rescaled. This
choice eases the complexity of the calculations and the effect on the group action is simply rescaling of the group parameter (see e.g. Mansfield, 1999). Similarly calculations may be simplified by assuming \( \tau = 0 \), which imply that the group action is translation of time. If we demand the invariance of Equation (5.25) subject to the constraints of the I.S.C. with \( \tau = 1 \), we obtain

\[
\xi = \xi(T, Z), \quad (5.28) \\
\eta = A(T, Z) C + B(T, Z), \quad (5.29)
\]

where,

\[
\begin{align*}
2A_Z + \Theta(Z)\xi_T - \xi_{2Z} + 2\Theta(Z)\xi_Z\xi + \xi_Z + \Theta'(Z)\xi^2 &= 0, \\
\Theta(Z)A_T - A_{2Z} - A_Z + 2\Theta(Z)\xi_Z A + \Theta'(Z)\xi A &= 0, \\
\Theta(Z)B_T - B_{2Z} - B_Z + 2\Theta(Z)\xi_Z B + \Theta'(Z)\xi B &= 0.
\end{align*}
\]

(5.30)

The prime denotes differentiation with respect to the indicated argument. Assuming \( \xi(T, Z) \) to be a nonzero constant we thus obtain nonclassical symmetries admitted by Equation (5.25) for special cases of \( \Theta(Z) \) and these are summarized in Table 5.1 below. The cases listed in Table 5.1 are realistic and moreover, we are able to derive hydraulic properties that resemble those of real soils. \( \xi(T, Z) = 0 \) leads to the recovery of the classical symmetries or the so-called principal Lie algebra admitted by Equation (5.25). A complete solution to this system of nonclassical determining relations is yet to be achieved.
5.7 Nonclassical symmetry reductions and numerical solutions

A symmetry reduction enables a simplification in the governing equation, with the aim of finding an exact analytic solution. In the standard approach to symmetry analysis of boundary value problems, the admitted infinitesimal generator must also leave all the boundary and initial conditions invariant. We consider examples of nonclassical symmetry reductions:

**Example i** Equation

\[
\left( \frac{\left( \frac{1}{2} + A_1 \right) + \left( A_1 - \frac{1}{2} \right) e^{2A_1(Z+c_1)}}{1 - e^{2A_1(Z+c_1)}} \right) \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial Z^2} + \frac{\partial C}{\partial Z},
\]

(5.31)

adopts a nonclassical symmetry generator

\[
\Gamma = \frac{\partial}{\partial T} + 2 \frac{\partial}{\partial Z} - 2 \left( \frac{\left( \frac{1}{2} + A_1 \right) + \left( A_1 - \frac{1}{2} \right) e^{2A_1(Z+c_1)}}{1 - e^{2A_1(Z+c_1)}} \right) C \frac{\partial}{\partial C}.
\]
which leads to a functional form

\[ C = \left[ \exp \left( (A_1 - \frac{1}{2}) (Z + c_1) \right) - \exp \left( - \left( \frac{1}{2} + A_1 \right) (Z + c_1) \right) \right] \]

\[ \times F(Z - 2T), \]

where \( F \) satisfies the O.D.E.

\[ F'' + F' + \left( A_1^2 - \frac{1}{4} \right) F = 0. \]

Thus we obtain solutions

**case (a) \( A_1^2 = \frac{1}{2} \)**

\[ C = \left[ \exp \left( (A_1 - \frac{1}{2}) (Z + c_1) \right) - \exp \left( - \left( \frac{1}{2} + A_1 \right) (Z + c_1) \right) \right] \]

\[ \times \exp \left( T - \frac{Z}{2} \right) \times (k_1(Z - 2T) + k_2). \]

**case (b) \( A_1^2 < \frac{1}{2} \)**

\[ C = \left[ \exp \left( (A_1 - \frac{1}{2}) (Z + c_1) \right) - \exp \left( - \left( \frac{1}{2} + A_1 \right) (Z + c_1) \right) \right] \]

\[ \times \left( k_1 \exp \left( \left( \frac{\sqrt{2 - 4A_1^2}}{2} \right) (Z - 2T) \right) \right. \]

\[ + k_2 \exp \left( \left( - \frac{\sqrt{2 - 4A_1^2}}{2} \right) (Z - 2T) \right). \]

**case (c) \( A_1^2 > \frac{1}{2} \)**

\[ C = \left[ \exp \left( (A_1 - \frac{1}{2}) (Z + c_1) \right) - \exp \left( - \left( \frac{1}{2} + A_1 \right) (Z + c_1) \right) \right] \]

\[ \times \exp \left( T - \frac{Z}{2} \right) \times \left( k_1 \sin \left( \frac{\sqrt{4A_1^2 - 2}}{2} (Z - 2T) \right) \right. \]

\[ + k_2 \cos \left( \frac{\sqrt{4A_1^2 - 2}}{2} (Z - 2T) \right). \]
Example ii  Equation

\[
\left( A_1 \tan[A_1(Z + c_2)] + \frac{1}{2} \right) \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial Z^2} + \frac{\partial C}{\partial Z},
\]

admits a nonclassical symmetry generator

\[
\Gamma = \frac{\partial}{\partial T} + 2 \frac{\partial}{\partial Z} - 2 \left( A_1 \tan[A_1(Z + c_2)] + \frac{1}{2} \right) C \frac{\partial}{\partial C},
\]

which leads to a functional form

\[
C = e^{-T} \cos[A_1(Z + c_2)] \times F(Z - 2T),
\]

where \( F \) satisfies the O.D.E.

\[
F'' + 2 F' + \left( \frac{1}{2} - A_1^2 \right) F = 0.
\]

Thus we obtain solutions

case (a)  \( A_1^2 = -\frac{1}{2} \)

\[
C = e^{-T} \cos[A_1(Z + c_2)] \times \exp(2T - Z) \times (k_1(Z - 2T) + k_2).
\]

case (b)  \( A_1^2 > -\frac{1}{2} \)

\[
C = e^{-T} \cos[A_1(Z + c_2)] \times \left( k_1 \exp \left( \left( \frac{\sqrt{2 + 4A_1^2} - 2}{2} \right)(Z - 2T) \right) \right.
\]
\[
+ k_2 \exp \left( \left( \frac{-\sqrt{2 + 4A_1^2} - 2}{2} \right)(Z - 2T) \right) \).
\]

case (c)  \( A_1^2 < -\frac{1}{2} \)

\[
C = e^{-T} \cos[A_1(Z + c_2)] \times \exp(-2T + Z) \times \left( k_1 \sin \left( \frac{\sqrt{4A_1^2} - 2}{2}(Z - 2T) \right) \right.
\]
\[
+ k_2 \cos \left( \frac{\sqrt{4A_1^2} - 2}{2}(Z - 2T) \right) \).
Example iii  Equation

\[
\left(\frac{1}{2} - \frac{1}{Z + c_3}\right) \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial Z^2} + \frac{\partial C}{\partial Z},
\]

(5.33)

admits a nonclassical symmetry generator

\[
\Gamma = \frac{\partial}{\partial T} + 2 \frac{\partial}{\partial Z} - 2 \left(\frac{1}{2} - \frac{1}{Z + c_3}\right) C \frac{\partial}{\partial C},
\]

which leads to a functional form

\[
C = e^{-T}(Z + c_2) \times F(Z - 2T),
\]

where \(F\) satisfies the O.D.E.

\[
F'' + 2F' + \frac{1}{2}F = 0.
\]

Thus we obtain the solution

\[
C = e^{-T}(Z + c_2) \times \left( k_1 \exp\left(\frac{\sqrt{2}}{2}(Z - 2T)\right) \right. \\
\left. + k_2 \exp\left(-\frac{\sqrt{2}}{2}(Z - 2T)\right) \right).
\]

We have provided nonclassical symmetry reductions and hence exact analytical solutions which are not obtainable via the classical symmetry analysis. However, the obtained solutions do not satisfy the imposed boundary conditions (5.16), (5.17) and (5.18) with \(D = 2\), but could be used to test the accuracy of numerical approximation schemes such as the PDETWO program (Melgaard and Sincovec, 1981). This software reduces the partial differential equation to a system of ordinary differential
equations using the method of lines with a continuous time variable and a discrete spatial discretization. The resulting system of ordinary differential equations is then solved by the Runge-Kutta method, incorporating Gear’s method for stiff systems when necessary. As we are using Equation (5.32) only to validate the numerical software PDETWO, we can make any convenient choice of the constant $k_1$ and $k_2$. We therefore choose $k_1 = -153.46$ and $k_2 = -156.012$. Also, we choose $A_1 = -0.01$ and $c_1 = 5$ to obtain a realistic non negative $\theta$ function. We then solve Equation (5.33) subject to the conditions

$$C = -0.0042k_1e^{(0.414-0.414T)} - 0.0042k_2e^{(-2.414+2.414T)}, \quad Z = 2$$

and

$$C + \frac{\partial C}{\partial Z} = -0.0075k_1e^{-0.414T} + 0.0042k_2e^{2.414T}, \quad Z = 0.$$

The results of this are shown in Figure 5.5 in which the numerical solution agrees well with the corresponding exact analytic solution. Since confidence in the capability of PDETWO is established, we thus use this software to determine the numerical solutions. Figure 5.6 depicts the numerical solution for Equation (5.15) subject to the initial and boundary conditions (5.16) through (5.18).

5.8 Results and discussion

The Fokker-Planck diffusion-convection equation and the C.D.E. describing a steady water flow and solute transport in soils respectively, are simultaneously solved. In
solving the diffusion-convection equation, we first assumed a realistic function of \( z \) to relate the volumetric water content and constructed hydraulic properties which resemble those of real soils. We note that \( \psi_*(\Theta) \) in (5.11) does not show the inflection point. However, we are able to obtain exact analytic solutions for the C.D.E. using the smooth function \( \Theta(Z) \) in (5.6). The only two point symmetries admitted by the C.D.E. with a nontrivial volumetric water content background led to an invariant solution which could be obtained by the method of separation of variables. It is notable that Equation (5.4) may be transformed into a diffusion equation. However, this does not simplify the boundary value problem.
Figure 5.6: Numerical solution for Equation (5.15) subject to conditions (5.16) to (5.18). $B = 0.55$, $L = 2$ and $D = 2$. $C_s$ is the steady state solution given in (5.21) and the volumetric water content in (5.6).

We have taken the infinitesimal $\xi$ to be a nonzero constant in order to solve the system of nonlinear determining equations arising from nonclassical symmetry analysis of the C.D.E. (5.25) with arbitrary volumetric water content. Some nonclassical symmetries are obtained for special cases of volumetric water content. A full nonclassical symmetry classification for this equation would be a major task if not impossible. Choosing the realistic water content profiles from the obtained special symmetric cases, we could again reconstruct the corresponding realistic soil
hydraulic properties. In these special cases, arbitrary constants were chosen conve-
niently to avoid the unphysical negative $\theta(Z)$ function. Since this function divides
the diffusion term, we could have time reversed diffusion which is unstable, instead
of a smoothing process. We obtained some nonclassical reductions which could not
be obtained via the classical techniques. We however, observed that these reductions
are not compatible with the prescribed boundary condition but were used to test the
accuracy of the numerical package PDETWO. Figure 5.6 shows solute concentration
build-up at the surface of the soil as time evolves. This concentration approaches
the steady state concentration which could be obtained exactly in (5.21). Also from
Figure 5.6, it is seen that the time taken to approach steady state may be of the
order of $\frac{10\theta_D}{R}$. For example, for a mean evaporation rate of 0.3 cm per day, over a
water table of depth 4 m covered by soil with porosity 0.4, this time is around 15
years.

5.9 Concluding remarks

Solutions for the coupled solute transport equations are presented. We employed
the inverse method to solve the nonlinear Richards equation. These solutions then
appeared in the solute dispersion equations as non constant coefficients. Symmetry
analysis of the solute dispersion equation led to exact analytic solutions which were
then used to test the accuracy of the numerical software PDETWO.
From our analytic and numerical solutions, under the prolonged periods of evaporation in arid regions over an artesian basin, the salt content is likely to reach saturation level, with precipitation occurring beneath the surface. This could occur even if the underlying ground water had a low salt concentration. The precipitation front would be an interesting subject for future mathematical modelling.
Chapter 6

On classes of nonlinear P.D.E.s: adsorption-, inhomogeneous- and integrodifferential-diffusion equations

6.1 Introduction

The transport of chemicals in soils is affected by various chemical and physical processes including adsorption and desorption, advection, hydrodynamic dispersion and diffusion, as well as chemical and biological transformations (Jury and Flühler, 1972; Brusseau, 1994). Sorption greatly influences the mobility and spreading of contami-
nants. Adsorption temporarily removes some of the solute from the mobile solution phase and reduces its transport rate. The resulting diffusion-adsorption equation has the added complication of solute transport being nonlinear of Fokker-Planck type. These types of equations arise in various physical phenomena and have proved to be more difficult to be analytically solved than their linear counterparts. However, over the years some techniques have been developed in an effort to linearise and hence to solve these equations; for example, the Hopf-Cole transformation which maps Burgers' equation into a linear diffusion equation, the Storm transformation (Storm, 1951), the Veins (see e.g. Ames, 1965; Munier et al., 1981; Broadbridge, 1988) and Rosen methods (Rosen, 1982). Furthermore, a more systematic approach to linearise and solve nonlinear equations originally developed by S. Lie (Lie, 1880) has been extended to complicated nonlinear equations. The group theoretic methods include Lie Bäcklund symmetries (see e.g. Broadbridge, 1988) and potential symmetries (see e.g. Bluman and Kumei, 1989).

Using the methods of Munier et al. (1981) we are able to transform some nonlinear A.D.E.s into linearisable equations. In particular using other transformations, we are able to map a class of the A.D.E.s into a special case of a class of the I.N.D.E.s

$$ f(x) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( g(x) u^n \frac{\partial u}{\partial x} \right). $$ (6.1)

Potential symmetry analysis for the I.N.D.E. (6.1) has been carried out by a few authors including Khater et al. (2002) and Sophocleous (2000, 2003a). This equation
is of utmost interest in mathematical physics. Special cases describing physical situations involving slow and fast diffusion where \( n < 0 \) and \( n > 0 \) respectively have been studied (see e.g. Crank, 1979). Symmetries presented in Khater et al. (2002) have been previously obtained in Bluman and Kumei (1989) and Sophocleous (2000). We observe that although equation (6.1) with special case \( n = -2 \), \( f(x) = 1 \) and \( g(x) = x^2 \), seems not to admit potential symmetries, it is indeed transformable to potential-symmetry-admitting Storm equation (Broadbridge, 1987) and may be mapped into a constant coefficient equation of the same form by local or nonlocal transformations (Sophocleous, 2003b). This contradicts the reports by Khater et al., (2002) and Sophocleous (2003a) that potential symmetries are nonexistent for this special case.

It is common when constructing potential symmetries to first express the governing P.D.E. written in conserved form as a system of first order P.D.E.s. Pucci and Saccomandi (1993) provided the necessary conditions for a P.D.E. to admit potential symmetries. Furthermore, a P.D.E. may be expressed in an auxiliary system in more than one inequivalent ways. For example (Pucci and Saccomandi, 1993), the linear wave equation

\[
\frac{\partial^2 u}{\partial t^2} = x \frac{\partial^2 u}{\partial x^2},
\]

may be expressed in two distinct systems of first order P.D.E.s, namely

\[
\frac{\partial \phi}{\partial x} = \frac{1}{x} \frac{\partial u}{\partial t}, \quad \frac{\partial \phi}{\partial t} = \frac{\partial u}{\partial x}.
\]
or

\[ \frac{\partial \phi}{\partial x} = \frac{\partial u}{\partial t}, \quad \frac{\partial \phi}{\partial t} = x \frac{\partial u}{\partial x} - u, \]

(6.4)

where \( \phi \) is the potential variable. Note that the systems (6.3) and (6.4) are not equivalent. It therefore remains a question of which system should be considered as an auxiliary system when seeking potential symmetries. In this example, system (6.3) possesses point symmetries which induce potential symmetries for the wave equation (6.2), whereas system (6.4) does not. In some cases, auxiliary systems possessing point symmetries which induce potential symmetries admitted by the governing equation may be hidden. Hence in this chapter, we avoid analysing the auxiliary system associated with the governing P.D.E. We systematically construct the seemingly hidden potential symmetries by first assuming a general integral variable and then considering an integrated form of the governing equation.

### 6.2 Nonlinear adsorption-diffusion equations

As in the previous chapters, we concentrate on the macroscopic deterministic models based on local conservation laws (see e.g. Ségol, 1994; Wierenga, 1995). Combining the equation of continuity for mass conservation

\[ \frac{\partial(\theta(t, z) c)}{\partial t} + \frac{\partial J}{\partial z} = 0, \]
Chapter 6: Nonlinear diffusion equations

Together with solute flux density

\[ J = -\theta(t, z) D_\nu(v) \frac{\partial c_f}{\partial z} + q c_f, \]

we obtain the A.D.E. given by

\[ \frac{\partial (\theta(t, z)c)}{\partial t} = \frac{\partial}{\partial z} \left[ D_\nu(v)\theta(t, z) \frac{\partial c_f}{\partial z} - q c_f \right], \quad (6.5) \]

where \( c = c_a + c_f \) is the total concentration of solute, \( c_f \) is the concentration of free molecules, \( c_a \) is the concentration of the adsorbed molecules, \( q, \theta(t, z), t, z \) and \( D_\nu(v) \) are described in Chapter 2. As the adsorption-desorption equilibrium adsorption process is bimolecular and desorption process is monomolecular (see e.g. Rosen, 1982) then the equilibrium condition is

\[ \frac{c_f c}{c_a} = \kappa. \quad (6.6) \]

Since \( c = c_a + c_f \), the local free concentration is given by

\[ c_f = \frac{c}{1 + \kappa^{-1} c}, \quad (6.7) \]

where \( \kappa \) is the equilibrium constant (Rosen, 1982). For steady water flow and using the equation for continuity, we obtain the nonlinear A.D.E.

\[ \frac{\theta(z)}{(1 - \kappa^{-1} c_f)^2} \frac{\partial c_f}{\partial t} = \frac{\partial}{\partial z} \left( D_1|q| \frac{\partial c_f}{\partial z} \right) - q \frac{\partial c_f}{\partial z}. \quad (6.8) \]

With application to evaporation from a water table at a constant Darcian water flux \( q = -R \), so that \( R \) is the evaporation rate, equation (6.8) then becomes

\[ \frac{\theta(z)}{(1 - \kappa^{-1} c_f)^2} \frac{\partial c_f}{\partial t} = D_1 R \frac{\partial^2 c_f}{\partial z^2} + R \frac{\partial c_f}{\partial z}. \quad (6.9) \]
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In terms of scaled dimensionless variables we write Equation (6.9) as

\[
\frac{\Theta(Z)}{(1 - C)^2} \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial Z^2} + \frac{\partial C}{\partial Z},
\]  

(6.10)

where \( C = c_f/c_s \), \( T = t/t_s \), \( Z = z/\lambda_s \), \( \Theta = \theta/\theta_s \), with \( c_s \) being concentration at saturation, \( c_s = \kappa \), \( D_1 = \lambda_s \) and \( t_s = \theta_s D_1/R \). \( \theta_s \) is the water content at saturation and \( \lambda_s \) is a macroscopic sorptive length scale (White and Sully, 1987).

Assuming a store of a fixed free concentration \( c_0 \), at the underlying groundwater reservoir \( (z = d) \) and no flow at the soil surface \( (z = 0) \), then the boundary conditions for the locally free concentration, in scaled dimensionless variables, are given by

\[
C = 1, \quad Z = D;
\]  

(6.11)

\[
C + \frac{\partial C}{\partial Z} = 0, \quad Z = 0,
\]  

(6.12)

where \( D = d/\lambda_s \) is the dimensionless depth to a water table. Moreover, we impose the initial zero free concentration condition;

\[
C = 0, \quad T = 0.
\]  

(6.13)

6.3 Linearisable models

In this section we determine all function \( \Theta(Z) \) for which a class of equations (6.10) is linearisable. Given \( \mu = 1 - C \), let a change of space variable and rescaling be
defined by (Munier et al., 1981)

\[ \mu(Z,T) = g(Z)v(\dot{z},T), \quad \dot{z} = D(Z). \]  

(6.14)

Substituting into Equation (6.10) we obtain

\[ \frac{\partial v}{\partial T} = \vartheta(Z)gv^3(\ddot{g} + \dot{g}) + \vartheta(Z)gv^2(2\dot{g}\dot{D} + g\dot{D} + g\ddot{D})\frac{\partial v}{\partial \dot{z}} + \vartheta(Z)g^2v^2\dot{D}^2\frac{\partial^2 v}{\partial \dot{z}^2}, \]  

(6.15)

where the dot denotes ordinary derivative of a given function and \( \vartheta(Z) = \frac{1}{\Theta(Z)} \). We investigate three cases.

**case 1** If we require

\[ \ddot{g} + \dot{g} = 0, \]

\[ 2\dot{g}\dot{D} + g\dot{D} + g\ddot{D} = 0, \]

\[ \vartheta(Z)g^2\dot{D}^2 = 1, \]

to hold, then we obtain a non-linear diffusion equation

\[ \frac{\partial v}{\partial T} = v^2\frac{\partial^2 v}{\partial \dot{z}^2}, \]  

(6.16)

whenever \( \vartheta(Z) \) satisfies the O.D.E.

\[ 2\ddot{\vartheta} - 2\dot{\vartheta} - \frac{\dot{\vartheta}^2}{\vartheta} = 0. \]  

(6.17)

That is, \( \vartheta \) must be given by

(a) \( \vartheta = a_1 \)

or

(b) \( \vartheta = (a_1e^z + a_2)^2 \),
where \( a_1, a_2 \) are constants. Linearisation of the equation describing solute transport in saturated soils i.e. for \( \vartheta = \theta_s \), will be discussed in Section 6.6.1. Moreover,

\[
g = e^{-Z} \vartheta^{1/2}
\]

and

\[
D = \int \frac{dZ}{e^{-Z} \vartheta^{1/2}} = \dot{z}.
\]

case 2 If we require

\[
\ddot{g} + \dot{g} = 0,
\]

\[
2\dot{g} \dot{D} + g \ddot{D} + g \dot{D} = \frac{1}{\vartheta g},
\]

\[
\vartheta(Z)g^2 \dot{D}^2 = 1,
\]

to hold, then

\[
g = a_1 + a_2 e^{-Z},
\]

\[
D = \int \frac{dZ}{(a_1 + a_2 e^{Z}) \vartheta^{1/2}} = \dot{z}
\]

and

\[
\vartheta = (\alpha_1 e^Z + \alpha_2)^2,
\]

with \( \alpha_1 = \frac{a_1 a_3 - 1}{a_2}, \alpha_2 = a_2 a_3 \) and \( a_1, a_2, a_3 \) being constants. In this case we obtain

\[
\frac{\partial v}{\partial T} = v^2 \frac{\partial^2 v}{\partial \dot{z}^2} + v^2 \frac{\partial^2 v}{\partial z^2}.
\]

Equation (6.18) may be transformed into a nonlinear diffusion equation

\[
\frac{\partial W}{\partial T} = W^2 \frac{\partial^2 W}{\partial x^2},
\]
by letting $x = e^z$ and $W = ve^z$. Thus both Equations (6.16) and (6.18) may be linearised by the Storm transformations. These equations are amenable to potential symmetries (see e.g. Bluman and Kumei, 1989). The analysis of a nonlinear diffusion equation with an extra coefficient being a function of $\dot{z}$ will be provided in Section 6.4. A solution to Equation (6.16) may be constructed from any solution to the linear diffusion equation using Vein's method (Ames, 1965; Munier et al., 1981).

**case 3** If the following statements

\[ \begin{align*}
\ddot{g} + \dot{g} &= 0, \\
2\dot{g}\ddot{D} + g\ddot{D} + \dot{g}\dot{D} &= 0,
\end{align*} \]

\[ g^2\dot{D}^2 = 1, \]

are true, then we obtain the nonlinear diffusion equation with an arbitrary $H(\dot{z}) = \varphi(Z)$ function;

\[ \frac{\partial v}{\partial T} = H(\dot{z})v^2 \frac{\partial^2 v}{\partial \dot{z}^2}. \quad (6.19) \]

Here

\[ g = a_1 e^{-z}, \]

and

\[ D = \frac{1}{a_1} e^z = \dot{z} \]

with $a_1$ being a constant. The boundary conditions (6.11), (6.12) and (6.13) transform to

\[ v = 0, \quad \dot{z} = \frac{1}{a_1} e^\nu; \quad (6.20) \]
Munier et al (1981) proved that Equation (6.19) may be transformed to a linear equation only when a nontrivial $H(\hat{z})$ takes the form

$$H(\hat{z}) = (c_1 \hat{z} + c_2)^2,$$

where $c_1$ and $c_2$ are constants. Applying the transformation $v = 1/\omega$ suggested by Munier et al (1981) Equation (6.19) then becomes

$$\frac{\partial \omega}{\partial T} = (c_1 \hat{z} + c_2)^2 \frac{\partial}{\partial \hat{z}} \left( \omega^{-2} \frac{\partial \omega}{\partial \hat{z}} \right).$$

Exact solutions of the solvable model given when $c_1 = 0$ in (6.24) have been constructed and applied to unsaturated flow by Knight and Philip (1974). In addition, exact solutions to (6.24) with $c_5 = 0$ had been developed by Fujita (1952). Let $y = (c_1 \hat{z} + c_2)^{-1}$, then after a rescaling and translation of $\omega$ and $y$ or allowing a linear transformation of $\omega$ and $y$, we may write the related conservation equation as

$$\frac{\partial \omega}{\partial T} = \frac{\partial}{\partial y} \left[ \left( \frac{c_3 y + c_6}{c_3 \omega + c_4} \right)^2 \frac{\partial \omega}{\partial y} \right],$$

where $c_i$, $i = 1, \ldots, 6$ are constants. Equation (6.25) is a nonlinear diffusion equation with diffusivity $D(\omega, y) = \left( \frac{c_3 y + c_6}{c_3 \omega + c_4} \right)^2$. This may be viewed as a description of a zero gravity flow of water in heterogeneous medium. Equation (6.25) may be obtained by assuming $\frac{\partial y}{\partial z} = 1$ in a one dimensional form of Richards equation, where
\( \Psi \) is the moisture potential and \( z \) is depth measured positively downward (Broadbridge, 1988).

Let \( \xi = c_5^{-1} \ln(c_5y + c_6) \) (Broadbridge, 1987); then (6.25) transforms to nonlinear Fokker-Planck equation

\[
\frac{\partial \omega}{\partial T} = \frac{\partial}{\partial \xi} \left[ \left( c_3\omega + c_4 \right)^{-2} \frac{\partial \omega}{\partial \xi} \right] - \frac{d}{d\omega} \left[ \frac{c_5/c_3}{c_3\omega + c_4} \right] \frac{\partial \omega}{\partial \xi}. \tag{6.26}
\]

Equation (6.26) may be linearised by a special Lie Bäcklund symmetry group (Fokas and Yortsos, 1982) and by potential symmetries (Bluman and Kumei, 1989; Katoanga, 1992; Sophocleous, 1996). Moreover, Fokas and Yortsos (1982), Rosen (1982) gave the exact solution for equation (6.26) subject to the boundary condition

\[-(c_3\omega + c_4)^{-2} \frac{\partial \omega}{\partial \xi} + \frac{c_5/c_3}{c_3\omega + c_4} = R \quad \xi = 0,
\]

where \( R \) is a constant. Inclusion of nonlinear convection term in Equation (6.25) results in a linearisable model, for example (Rogers and Broadbridge, 1988)

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left[ \frac{1 + cx}{(a + b\theta)^2} \frac{\partial \theta}{\partial x} + \frac{c}{2b(a + b\theta)} \right], \tag{6.27}
\]

may be reduced to the canonical Storm equation

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial y} \left[ \frac{1}{(a + b\theta)^2} \frac{\partial \theta}{\partial y} \right], \tag{6.28}
\]

by change of variables

\[ y = \frac{2}{c} \left[ (1 + cx)^{\frac{3}{2}} - 1 \right], \quad c \neq 0. \]
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The reduction was then used therein to solve the nonlinear moving boundary problem,

\[ \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left[ \frac{1+c x}{(a+b \theta)^2} \frac{\partial \theta}{\partial x} + \frac{c}{2b(a+b \theta)} \right], \quad 0 < x < X(t); \]

\[ \frac{1+c x}{(a+b \theta)^2} \frac{\partial \theta}{\partial x} = U(t), \quad \text{on} \quad x = 0, \quad t > 0; \]

\[ \frac{1+c x}{(a+b \theta)^2} \frac{\partial \theta}{\partial x} = \alpha X(t) \quad \text{on} \quad x = X(t), \quad X(0) = 0. \]

A Lie-Backlund approach to this problem with subsequent application of a reciprocal transformation to solve a boundary value problem was adopted in Broadbridge and Rogers (1993).

6.4 Nonlocal symmetry analysis of the I.N.D.E.

In this section we attempt to classify Equation (6.19) via potential or nonlocal symmetry analysis. The transformation

\[ v^{-1} = u, \quad y = \int \frac{1}{H(z)} \, dz, \]

allows us to write Equation (6.19) in conserved form

\[ \frac{\partial u}{\partial T} = \frac{\partial}{\partial y} \left( u^{-2} G(y) \frac{\partial u}{\partial y} \right), \quad (6.29) \]

where \( G(y) = \frac{\partial u}{\partial z} \). Equation (6.29) may naturally be split into an auxiliary system

\[ \frac{\partial \phi}{\partial y} = u, \quad \frac{\partial \phi}{\partial T} = u^{-2} G(y) \frac{\partial u}{\partial y}; \quad (6.30) \]
where $\phi$ is the potential variable. Construction of nonlocal or potential symmetries admitted by a P.D.E. such as (6.29) involves determining the infinitesimal transformations

$$
\begin{align*}
T' &= T + \epsilon \tau(T, y, u, \phi) + O(\epsilon^2), \\
y' &= y + \epsilon Y(T, y, u, \phi) + O(\epsilon^2), \\
u' &= u + \epsilon U(T, y, u, \phi) + O(\epsilon^2), \\
\phi' &= \phi + \epsilon V(T, y, u, \phi) + O(\epsilon^2),
\end{align*}
$$

(6.31)

with $\tau$, $Y$, $U$ and $V$ being the infinitesimals of the symmetry generator

$$
\Gamma = \tau \frac{\partial}{\partial T} + Y \frac{\partial}{\partial y} + U \frac{\partial}{\partial u} + V \frac{\partial}{\partial \phi},
$$

admitted by the auxiliary system such as (6.30). Classical symmetries of the system (6.30) induced nonlocal or potential symmetries admitted by (6.29) if and only if at least one of the infinitesimals $\tau$, $Y$, $U$ depend explicitly on $\phi$ (see e.g. Bluman and Kumei, 1989).

Potential symmetry analysis of the auxiliary system (6.30) using DIMSYM (Shering, 1993) reconfirms the results obtained for the cases $G = constant$, and the power law $G(y) = y^4$ (Bluman and Kumei, 1989; Katoanga, 1992; Sophocleous, 1996, 2000, 2003a; Khater et al., 2002). The second equation in the system (6.30) may further be split into another auxiliary system since it may be written in a conserved form (see e.g. Sophocleous, 2003a). We refrain from this step and from expressing Equation (6.29) as a system of first order partial differential equations or auxiliary system.
(6.30), but rather show that given the power law
\[ G(y) = y^2, \] (6.32)
Equation (6.29) admits extra potential symmetries which are constructed by considering the integrated form of the governing equation derived using the general integral variable. Note that potential symmetries are invisible for the system (6.30) with the power law (6.32). Instead of considering (6.30), we let the general integral variable be given by
\[ \phi = \int k(y) u(y, T) dy + J(T), \] (6.33)
where \( J(T) \) is a constant of integration. This choice provides a wider freedom with the kernel \( k \). Substituting (6.33) into Equation (6.29) we obtain a third order P.D.E.
\[ \frac{\phi_y T \phi_y^2}{(yk)^2} = \frac{2}{yk} \left( 1 - \frac{y \phi_{yy}}{\phi_y} \right) (k \phi_{yy} - k' \phi_y) + \phi_{yyy} - \frac{k'' \phi_y}{k}. \] (6.34)
To determine classical point symmetries admitted by Equation (6.34), we consider the infinitesimal transformations (6.31), (6.31)_2 and (6.31)_4, where at this point the infinitesimals of the symmetry generator
\[ \Gamma = \tau \frac{\partial}{\partial T} + Y \frac{\partial}{\partial y} + V \frac{\partial}{\partial \phi}, \]
namely; \( \tau, Y, V \) are independent of \( u \). Classical point symmetries of Equation (6.34) yield nonlocal or potential symmetries of Equation (6.29) (see e.g. Bluman and Kumei, 1989) provided at least one of the infinitesimals of the variables \( (y, T) \) depends explicitly on \( \phi \). In the classical symmetry analysis of Equation (6.34)
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with arbitrary $k$, DIMSYM (Sherring, 1993) reports that the admitted principal Lie algebra is spanned by the base vectors

$$\Gamma_1 = \phi \frac{\partial}{\partial \phi} + 2T \frac{\partial}{\partial T}; \quad \Gamma_2 = \frac{\partial}{\partial T} \quad \text{and} \quad \Gamma_{1\infty} = h(T) \frac{\partial}{\partial \phi}. \quad (6.35)$$

It remains to find all possible functions $k$ for which Equation (6.34) admits extra point symmetries. DIMSYM reports that extra symmetries may be obtained if $k$ is a solution of a first order O.D.E.

$$yk' + k = 0; \quad (6.36)$$

or a third order nonlinear O.D.E.

$$\left(y^3 kk' + y^2 k^2\right) k''' + \left(y^3 (k')^2 - 2y^3 k k'' - 3y^2 k k' + 4yk^2\right) k''$$

$$+ \left(2y^2 (k')^2 - 4yk k' + 2k^2\right) k' = 0; \quad (6.37)$$

or a fourth order nonlinear O.D.E.

$$\left(y^5 k^2 (k')^2 + 2y^4 k^3 k' + y^3 k^4\right) k^{(iv)} + \left(7y^2 k^4 + 5y^3 k^3 k' + y^4 k^2 (k')^2\right)$$

$$+ 3y^5 k (k')^3 - 6y^4 k^3 k'' - 6y^5 k^2 k' k'' \right) k''' + \left(29y^3 k^2 (k')^2 - 25y^2 k^3 k' - 7y^4 k (k')^3\right)$$

$$+ 10yk^4 + y^5 (k')^4 - 23y^3 k^3 k'' + 8y^4 k^2 k' k'' - 5y^5 k (k')^2 k'' + 6y^5 k^2 (k'')^2\right) k''$$

$$+ \left(2k^4 - 12yk^3 k' + 20y^2 k^2 (k')^2 - 12y^3 k (k')^3 + 2y^4 (k')^4\right) k' = 0. \quad (6.38)$$

From Equation (6.36) we observe the possibility $k = \frac{1}{y}$, where we have taken the constant of integration to be unity. In this case Equation (6.29) integrates completely
to

$$\phi_T = \frac{\phi_{yy}}{\phi_y^2} + w(T), \quad (6.39)$$

where $w(T)$ is a constant of integration which, without loss of generality, will herein be equated to zero. DIMSYM finds extra point symmetries admitted by Equation (6.39) namely;

$$
\begin{align*}
\Gamma_3 &= \frac{\partial}{\partial \phi}; & \Gamma_4 &= y \frac{\partial}{\partial y}; & \Gamma_5 &= 2T \frac{\partial}{\partial \phi} - \phi y \frac{\partial}{\partial y}; \\
\Gamma_6 &= 4\phi T \frac{\partial}{\partial \phi} - y \left( \phi^2 + 2T \right) \frac{\partial}{\partial y} + 4T^2 \frac{\partial}{\partial T} \quad \text{and} \quad \Gamma_{2\infty} = S(\phi, T) \frac{\partial}{\partial \phi}, \quad (6.40)
\end{align*}
$$

where $S$ is a solution of a linear diffusion equation

$$S_{\phi\phi} - S_T = 0.$$  

$\Gamma_5$ and $\Gamma_6$ induce nonlocal or potential symmetries admitted by Equation (6.29) and they seem not to be recorded in the literature. A straightforward calculation shows that Equation (6.39) with $w(T) = 0$ is equivalent to the auxiliary system

$$
\begin{align*}
\phi_y &= \frac{u}{y}; & \phi_T &= \frac{y u}{u^2 \partial y} - \frac{1}{u}. \quad (6.41)
\end{align*}
$$

We refer to (6.41) as a 'hidden auxiliary system'. Both Equations (6.37) and (6.38) admit the solutions $k = \frac{1}{y}$ and $k = \text{constant}$. Moreover, Equation (6.37) has a solution $k = \frac{(a_1 + a_2 y)^a}{y}$ with $a_3 \neq 0$. In fact, Equation (6.38) is a differential consequence of Equation (6.37) and hence no further special cases of $k$ for which Equation (6.34) admits extra potential symmetries exist.
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Now applying this method to a more general P.D.E. (6.29), where $G(y)$ is arbitrary and the term $u^{-2}$ is replaced by $u^n$, we obtain results which are already known (Bluman and Kumei, 1989; Katoanga, 1992; Khater et al., 2002; Sophocleous, 2000, 2003a) and recover the results obtained in (6.35) and (6.40) for $G(y) = y^2$. Details of these calculations are herein omitted. The compatibility of $\phi_T = M(T, y, u, u_y)$ and $\phi_y = N(T, y, u, u_T)$ on solutions of Equation (6.29) given (6.32). results in a more general auxiliary system (Goard, 2003a).

6.5 Nonlocal symmetry reductions for the I.N.D.E.

A wide range of reductions for Equation (6.39) is possible using any linear combination of the finite symmetries given in (6.35) and (6.40). In order to obtain minimal set of reductions we construct the one dimensional optimal system given by

$$\{\Gamma_1 + a\Gamma_4, \pm \Gamma_2 \pm \Gamma_3 + \Gamma_6, \Gamma_2 + \Gamma_6, \Gamma_3 + \Gamma_6, \Gamma_4 \pm \Gamma_5, \Gamma_4\},$$

where $a$ is an arbitrary constant. If we admit the discrete symmetries, then the number of elements in this set reduces by three. In the next examples we list the canonical invariants and the reduced O.D.E.s associated with each of these symmetries. We are able to construct a variety of analytical solutions for Equation (6.39). These solutions may be mapped into solutions for Equation (6.29).

**Example i** Reduction by $\Gamma' = \Gamma_1 + a\Gamma_4$. 
\( \Gamma' \) leads to a functional form

\[ \phi = \sqrt{T} F(\rho) \quad \text{with} \quad \rho = \frac{y}{\sqrt{T}}, \]

wherein, without loss of any generality, we have taken \( a = 1 \) and \( F \) satisfies the O.D.E.

\[ F'' - \frac{1}{2} F(F')^2 + \frac{1}{2} \rho (F')^3 = 0, \]

admitting a nontrivial solution

\[ a_1 \left[ \exp \left( -\frac{F}{4} \right) + \frac{\sqrt{\pi}}{2} F \operatorname{erf} \left( \frac{F}{2} \right) \right] + a_2 F = \frac{y}{\sqrt{T}}, \]

where \( a_1 \) and \( a_2 \) are arbitrary constants.

**Example ii** Reduction by \( \Gamma'' = \Gamma_2 + \Gamma_3 + \Gamma_6 \).

\( \Gamma'' \) leads to a functional form

\[ \ln y = -\frac{T(\phi - T)^2}{4T^2 + 1} - \frac{\phi}{2} + \frac{T}{4} + \frac{\arctan(2T)}{8} - \frac{\ln(4T^2 + 1)}{4} + F(\rho), \]

where \( \rho = \frac{\phi - T}{\sqrt{4T^2 + 1}} \) and \( F \) satisfies the equation

\[ F'' + (F')^2 + \rho^2 - \frac{1}{4} = 0. \]

Thus we obtain a solution to Equation (6.39), namely;

\[ \frac{T(\phi - T)^2}{4T^2 + 1} + \frac{\phi - T}{2} + \frac{T}{4} - \frac{\arctan(2T)}{8} = \ln \left[ \frac{\sqrt{\Delta(\phi, T)}}{y\sqrt{4T^2 + 1}} \right], \]

where

\[ \Delta(\phi, T) = \frac{64}{M_{1+\frac{1}{16}, \frac{1}{4}} \left( \frac{(\phi - T)^2}{4T^2 + 1} \right) W_{\frac{1}{16}, \frac{1}{4}} \left( \frac{(\phi - T)^2}{4T^2 + 1} \right) a_2 + a_1 M_{1+\frac{1}{16}, \frac{1}{4}} \left( \frac{\phi - T}{4T^2 + 1} \right) i + A}{M_{1+\frac{1}{16}, \frac{1}{4}} \left( \frac{(\phi - T)^2}{4T^2 + 1} \right) W_{\frac{1}{16}, \frac{1}{4}} \left( \frac{(\phi - T)^2}{4T^2 + 1} \right) a_2 + a_1 M_{1+\frac{1}{16}, \frac{1}{4}} \left( \frac{\phi - T}{4T^2 + 1} \right) i + A}. \]
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\[ A = 12 M_{1+\frac{1}{16}, \frac{1}{4}} \left( \frac{(\phi-T)^2}{4T^2+1} \right) W_{\frac{1}{16}, \frac{1}{4}} \left( \frac{(\phi-T)^2}{4T^2+1} \right) \]

\[ + 16 M_{1+\frac{1}{16}, \frac{1}{4}} \left( \frac{(\phi-T)^2}{4T^2+1} \right) W_{\frac{1}{16}, \frac{1}{4}} \left( \frac{(\phi-T)^2}{4T^2+1} \right), \]

with \( M_{\kappa,\mu} \) and \( W_{\kappa,\mu} \) being the Whittaker functions (Abramowitz and Stegun, 1965).

**Example iii** Reduction by \( \Gamma^{IV} = \Gamma_2 + \Gamma_6 \).

\( \Gamma^{IV} \) leads to a functional form

\[ \ln y = -\frac{\phi^2 T}{4T^2 + 1} - \frac{\ln(4T^2 + 1)}{4} + F(\rho), \quad \text{with} \quad \rho = \frac{\phi}{\sqrt{4T^2 + 1}}, \]

where \( F \) satisfies the equation

\[ F'' + (F')^2 + \rho^2 = 0. \]

Thus we obtain a solution to Equation (6.39), namely;

\[ \frac{\phi^2 T}{4T^2 + 1} = \ln \left\{ \frac{a_1 J_{\nu} \left( \frac{\phi^2}{8T^2 + 2} \right) + a_2 J_{\nu} \left( \frac{\phi^2}{8T^2 + 2} \right)}{y J_{\nu} \left( \frac{\phi^2}{8T^2 + 2} \right) - J_{\nu} \left( \frac{\phi^2}{8T^2 + 2} \right) Y_{\nu} \left( \frac{\phi^2}{8T^2 + 2} \right)} \right\}, \]

where \( a_1 \) and \( a_2 \) are arbitrary constants, and \( J_{\nu} \) and \( Y_{\nu} \) are Bessel functions of first kind of order \( \nu \) (Abramowitz and Stegun, 1965).

**Example iv** Reduction by \( \Gamma^{V} = \Gamma_3 + \Gamma_6 \).

\( \Gamma^{V} \) leads to a functional form

\[ \ln y = -\frac{24 T^2 \phi^2 + 12 T \phi + 1}{96 T^3} - \ln \sqrt{T} + F(\rho), \]

where

\[ \rho = \frac{8 T \phi + 1}{8 T^2} \]
and $F$ satisfies the equation

$$F'' + (F')^2 - \frac{\rho}{8} = 0.$$  

Thus we obtain a solution to Equation (6.39), namely;

$$\frac{24T^2\phi^2 + 12T\phi + 1}{96T^3} = \ln \left[ \frac{2(a_1Ai(\frac{Rt+1}{16T^2}) + a_2Bi(\frac{Rt+1}{16T^2}))}{y\sqrt{T}(Ai(\frac{Rt+1}{16T^2})Bi(1\frac{Rt+1}{16T^2}) - Ai(1\frac{Rt+1}{16T^2})Bi(\frac{Rt+1}{16T^2}))} \right].$$

with $Ai$ and $Bi$ being the Airy functions (Abramowitz and Stegun, 1965) and $a_1$ and $a_2$ are arbitrary constants. $Ai(1,x)$ and $Bi(1,x)$ denote the first derivative of $Ai$ and $Bi$ with respect to $x$ respectively.

**Example v** Reduction by $\Gamma^{VI} = \Gamma_6$.

$\Gamma^{[VI]}$ leads to a functional form

$$\ln y = -\frac{\phi^2}{4T} - \frac{\ln T}{2} + F \left( \frac{\phi}{T} \right), \quad (6.42)$$

where $F$ satisfies the equation

$$F'' + (F')^2 = 0.$$  

Thus for nonconstant $F$ a solution to Equation (6.39) takes the form

$$\frac{\phi^2}{4T} = \ln \left[ \frac{\phi + a_1T}{y\sqrt{T^3}} \right] + a_2,$$

where $a_1$ and $a_2$ are arbitrary constants, and for $F = constant$ say $a_1$, then Equation (6.42) becomes

$$\ln y = -\frac{\phi^2}{4T} - \frac{\ln T}{2} + a_1.$$
and so Equation (6.29) admits a solution

\[ u = \pm \sqrt{\frac{2T}{a_1 - 2\ln y - \ln T}}. \]

**Example vi**  
Reduction by \( \Gamma^{VII} = \Gamma_4 + \Gamma_5 \).

\( \Gamma^{VII} \) leads to a functional form

\[ \ln y = \frac{1}{2} \left( \phi - \frac{\phi^2}{2} \right) + F(T), \]

where \( F \) satisfies the equation

\[ F'(T) = \frac{1}{4T^2} - \frac{1}{2T}. \]

And so,

\[ \phi = a_2 \pm \sqrt{4a_1 T - 4T\ln y - 2T\ln T}, \]

is a solution to Equation (6.39). Hence Equation (6.29) admits a solution

\[ u = \pm \sqrt{\frac{2T}{a_1 - 2\ln y - \ln T}}, \]

where \( a_1 \) and \( a_2 \) are arbitrary constants.

### 6.6 Nonlinear integrodifferential-diffusion equations

In this section we consider a class of nonlinear integrodifferential diffusion equation

of the form

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial y} \left[ f(v)u - h(v)u \right] - g'(v)u. \]  

(6.43)
where
\[ v = \int_0^y u \, dy + J(t) = \int_{j(t)}^y u \, dy. \]

Equation (6.43) is solved here merely to illustrate how these methods may be applied to integrodifferential equations. Interpretation of the solutions for Equation (6.43) will not be considered here. Equation (6.43) models a population density of a species in a region \( y > j(t) \) whose death rate \( g'(v) \) depends on the total upstream population. For example, this is the case if the species generates air or water pollution, and there is a prevailing wind or current direction. Due to the pollution, the random mobility of the species may be a function of the upstream population (either increase at higher amounts of \( v \) due to increased agitation, or decrease at larger \( v \) due to increased sickness), so diffusivity \( f(v) \) depends on \( v \). In addition, diffusivity is a decreasing function of local population which impedes individual mobility. Due to the threat of upstream polluters there is a systematic unidirectional avoidance migration at velocity \( h(v) \) of the local population towards "greener" pastures. In this case, \( h \) must be negative.

We note that Equation (6.43) with arbitrary \( f(v) \), \( g(v) \) and \( h(v) \) is invariant under space and time translations only. Furthermore, no extra point symmetries were obtained for the cases flagged by DIMSYM (Sherring, 1993) except when the functions \( f(v) \), \( g(v) \), \( h(v) \) are all constants.

Now under hodograph transformation, \( v_t y_t y_v = -1 \), \( y_v = \frac{1}{v_v} \), Equation (6.43) is...
There exists a point transformation of the form (Bluman and Kumei, 1989)

\[ X = X(t,v), \]
\[ T = r(t,v), \]
\[ C = H(t,v)y, \]

such that Equation (6.44) is transformable to a canonical form

\[ \frac{\partial^2 \zeta}{\partial \chi^2} - \frac{\partial \zeta}{\partial \tau} + \omega(\chi, \tau)\zeta = 0, \tag{6.46} \]

where \( \omega \) is a function of \( \chi \) and \( \tau \). Substituting (6.45) into (6.46) and comparing the obtained coefficients with those of (6.44) when \( h = 0 \), then the transformations must be given by

\[ \chi = \int \frac{dv}{\sqrt{q f(v)}} + C_1, \quad q f > 0, \]
\[ \tau = \frac{t + C_2}{q}, \tag{6.47} \]
\[ \zeta = n(t) \exp \left( \frac{1}{2} \int S(v) \, dv \right) y, \]

where \( q, C_1 \) and \( C_2 \) are constants, \( n(t) \) is an arbitrary function of \( t \) and \( S(v) = \frac{g(v)}{f(v)} - \frac{f'(v)}{2f(v)} \). Thus

\[ \omega = -q \left\{ \frac{f'(v)S(v)}{2} + \frac{f(v)S'(v)}{2} + \frac{f(v)S(v)^2}{4} - \frac{n'(t)}{n(t)} \right\}, \tag{6.48} \]
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where the prime denotes differentiation with respect to the indicated argument.

Assuming \( n(t) \) to be a constant, say \( C_3 \), then by virtue of (6.47) \( \omega \) becomes a function of \( \chi \) only and so Equation (6.46) admits separation of variables in \( \tau \) and \( \chi \). Furthermore, if \( \omega \) is a quadratic of the form

\[
\omega = a_0 + a_1\chi + a_2\chi^2,
\]

then (6.44) is transformable into a linear P.D.E. with constant coefficient (Bluman and Kumei, 1989). That is, provided

\[
-q \left\{ \frac{f'(v)S(v)}{2} + \frac{f(v)S'(v)}{2} + \frac{f(v)S(v)^2}{4} - \frac{n'(t)}{n(t)} \right\} = a_0 + a_1 \left( \int \frac{dv}{\sqrt{af(v)}} + C_1 \right) + a_2 \left( \int \frac{dv}{\sqrt{af(v)}} + C_1 \right)^2.
\]

Goard (1997) studied Equation (6.44) with a specific form of \( g(v) \) i.e. \( g(v) = f'(v) \).

Although in their analysis Equation (6.44) was derived from a different nonlinear diffusion equation, it was shown to be transformable to classical linear diffusion equation with constant coefficients when \( n'(t) = C_1 = C_2 = 0 \), \( \tau = t \), \( \omega = a_0 \) and \( f(v) = v^2 \) and \( v^{4/3} \).

Example i

Given \( f(v) = \alpha \) (const) and \( g(v) = C_4v \), then under the transformations

\[
\chi = \frac{v}{\sqrt{q\alpha}} + C_1, \\
\tau = \frac{t + C_2}{q}, \\
\zeta = n(t) \exp \left( \frac{C_4v^2}{4\alpha} \right) y,
\]

(6.49)
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Equation (6.44) takes the canonical form (6.46), where

$$\omega(\chi, \tau) = -q \left\{ \frac{C_4}{2} + \frac{(C_4 v)^2}{4\alpha} - \frac{n'(t)}{n(t)} \right\}.$$  

Using method of separation of variables we obtain a general solution to (6.46):

$$\zeta = \frac{n(t)e^t}{\sqrt{v}} \left\{ k_1 M_{-\frac{a}{4}\sqrt{v^2}} \left( \sqrt{bv^2} \right) + k_2 W_{-\frac{a}{4}\sqrt{v^2}} \left( \sqrt{bv^2} \right) \right\},$$

where $a = \kappa + \frac{ca}{2}$, $b = \frac{c^2}{4\alpha}$, $k_1$ and $k_2$ are constants, $M_{\nu,\mu}(x)$ and $W_{\nu,\mu}(x)$ are the Whittaker functions (Abramowitz and Stegun, 1965).

Example ii: given $\omega = 0$ with $n'(t) = 0$, i.e.

$$\frac{f(v)S(v)^2}{4} = -\frac{1}{2} \frac{d}{dv} (f(v)S(v)), \quad (6.50)$$

then under (6.47), Equation (6.44) transforms to the heat equation

$$\frac{\partial \zeta}{\partial \tau} = \frac{\partial^2 \zeta}{\partial \chi^2}. \quad (6.51)$$

We are free to make some choices for $f$ and $g$ such that (6.50) is satisfied. For example, if we choose $f = \alpha$ (const) then $g$ must be given by $g = \frac{2\alpha}{v}$, so that under the transformations

$$\chi = \frac{v}{\sqrt{q\alpha}} + C_1, \quad \tau = \frac{t + C_2}{q}, \quad (6.52)$$

$$\zeta = C_3 vy,$$

Equation (6.44) becomes Equation (6.51). Solutions of Equation (6.51) are well known and we shall herein avoid further analysis of it.
6.7 Solutions for solute transport under steady saturated water flow background

6.7.1 Rosen method

Most of existing exact analytic solutions have assumed a uniform water flow (Philip, 1994). In this section we deal with a steady saturated water flow. This water flow regime satisfies $\theta(z) = \theta_s$, where $\theta_s$ is the water content at saturation. In this case Equation (6.10) reduces to a linearisable equation, namely;

$$\frac{\theta_s}{(1-C)^2} \frac{\partial C}{\partial T} = \frac{\partial^2 C}{\partial Z^2} + \frac{\partial C}{\partial Z}. \quad (6.53)$$

Under the transformation $\mu = 1 - C$, with $k = 1/\theta_s$, Equation (6.53) becomes

$$\frac{\partial \mu}{\partial T} = k \mu^2 \frac{\partial^2 \mu}{\partial Z^2} + k \mu \frac{\partial \mu}{\partial Z}. \quad (6.54)$$

There exists a single-valued positive function $\rho = \rho(\chi, T)$ with $\chi = \chi(Z, T)$ such that

$$Z = \ln \rho. \quad (6.55)$$

Furthermore

$$\mu^{-1} = \frac{\partial \chi}{\partial Z} \quad (6.56)$$

and hence

$$\mu = \frac{1}{\rho} \frac{\partial \rho}{\partial \chi}. \quad (6.57)$$
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(Rosen, 1982). Equation (6.54) reduces to a linear diffusion equation

\[ \frac{\partial \rho}{\partial T} = k \frac{\partial^2 \rho}{\partial \chi^2}, \]  

(6.58)

under the transformation (6.55) through to (6.57). For \(-\infty < \chi < \infty\) and \(\rho = f(\chi,0)\) the general solution to (6.58) is given by (Carslaw and Jaeger, 1946)

\[ \rho(\chi, T) = \frac{1}{2\sqrt{\pi \gamma}} \int_{-\infty}^{\infty} e^{-\gamma(s)^2/4\gamma T} f(s,0) \, ds. \]  

(6.59)

6.8 Numerical solutions for solute transport under steady unsaturated water flow background

In this section we seek numerical solutions, using a computer software PDETWO (Melgaard and Sincovec, 1981), for the adsorption-diffusion Equation (6.10) with a non-trivial water content simulating evaporation from a water table. That is, we consider a realistic \(\theta\) given in scaled dimensionless variables (Moitsheki et al., 2003a),

\[ \Theta(Z) = 1 + B(1 - e^{(Z-2)/L}), \]  

(6.60)

where \(B\) and \(L\) are constants and \(D\) being the dimensionless depth to a water table.

The accuracy of PDETWO has been established for a single nonlinear parabolic equation (Broadbridge, 1985). Figure 6.1 depicts the numerical solution for the A.D.E. (6.10) subject to the boundary and initial conditions (6.11), (6.12) and (6.13). Here, we have chosen parameters to be \(B = 0.55\), \(D = 1\) and \(L = 2\).
Figure 6.1: Numerical solution for the B.V.P. (6.10) through (6.13) with Θ given in (6.60), for values $T = 1, 5, 10$ respectively.

### 6.9 Concluding remarks

Integral symmetries admitted by the I.N.D.E. (6.29) have been obtained for the special case $G(y) = y^2$ by considering the integrated form obtained using the general potential variable (Moitsheki et al., 2003b). Also, invariant solutions have been constructed using potential symmetries. However, care must be taken in defining auxiliary systems. The method used in Section 6.4 raises a question of whether it could be applied to P.D.E.s which cannot be expressed in conserved forms or to
those that cannot be transformed into simpler ones.

The transformed boundary value problem (6.19) through (6.22) has proved to be difficult to solve because we have the Neumann condition at one end and Dirichlet condition at the other. Furthermore, classical point symmetry analysis revealed that Equation (6.10) admits a trivial principal Lie algebra, namely translation in $T$ and a linear combination of scaling of $T$ and $C$, and translation of $C$. Extra point symmetries can be obtained only for a constant and exponential $\theta$. These cases for $\theta$ are not realistic and further analysis is avoided. Hence we sought numerical solutions for transport of adsorbing solutes under practically important water flow background.
Chapter 7

Summary

Symmetry techniques have been used as our main tool to construct exact analytic solutions for P.D.E.s describing water and solute transport in porous media. These methods produced a compendium of exotic exact analytic solutions for one and two dimensional solute transport equations.

Classical Lie point symmetries analysis of the equations for two dimensional solute transport in soil, results in a rich class of symmetries for a number of practically important background water flow fields. For transient solute diffusion-convection, steady wedge flows (Broadbridge et al., 2000) and radial water flows (Broadbridge et al., 2002) allow symmetry reductions that are compatible with meaningful boundary conditions. Assuming a velocity-dependent dispersion coefficient given in terms of the power law \( D_v(v) = v^p \), where \( p \) is experimentally observed to take values \( 1 \leq p \leq 2 \), we observed that under radial water flow solute transport equations ad-
mit extra classical point symmetries whenever $p = 0, 2, -2$. The case $p = 2$ which is in accord with Taylor’s theory (Taylor, 1953) and seems to be a reasonable model (de Gennes, 1983; Philip, 1994), resulted in a rich array of classical point symmetries. The radial dispersion problem has distinction that it is probably the simplest case for which the dispersion coefficient is a function of a spatially varying velocity field and it is important in the study of solute transport from injected wells (Hsieh, 1986). Following reductions by non-abelian two dimensional subalgebras and by the elements of sets of the one dimensional optimal systems (Ovsiannikov, 1982; Olver, 1986), we are able to construct new exact analytic solutions for non-radial solute transport. These solutions not only depend on time and radius but also on the polar angle. Furthermore, we obtain new solutions for the case $p = 0$ which implies constant dispersion coefficient. This case was considered for solute transport under other water flow backgrounds (van Genuchten and Alves, 1982; Broadbridge et al., 2000). For the case $p = 1$, the solute transport equation admits trivial classical point symmetries which led to no further exact analytic solutions and all the contact symmetries obtained were equivalent to classical point symmetries. Also, further solutions were obtained for the case $p = -2$. Remarkably some of the obtained exact analytic solutions satisfy meaningful boundary conditions.

The equations for solute transport under point vortex water flow are amenable to symmetry analysis (Broadbridge et al., 2000). We observed that under this water flow extra symmetries are obtained only when $p = 0$, that is when the dispersion
coefficient is a constant. Again, construction of new exact solutions satisfying reasonable boundary conditions is possible. Our solutions could represent the spread of detergent supplied at the centre of a washing machine.

The one dimensional problem of unsteady transport of non-reactive, inert solutes has proven to be difficult, with the transport equation admitting trivial classical point symmetries. However, the nonclassical symmetry classification revealed some realistic cases for the water content and exact analytic solutions for the solute transport equations, which were used to validate numerical software PDETWO (Melgaard and Sincovec, 1981). This software was then used to construct further numerical solutions for a practically important water flow background. From our exact analytic and numerical solutions, we observed that under prolonged periods of time, solute concentration is likely to build-up to a comparatively high level at the surface of the soil as water evaporates. The exact steady state solution shows that this concentration may be amplified by a factor of more than 700% at the surface of the soil (see Moitsheki et al., 2003a). The accumulation rate or precipitation front would be an interesting subject for future mathematical modelling.

Transport of adsorbing solutes in one dimension added a complication of both transport equations being nonlinear. However, we are able to transform the nonlinear A.D.E.s into linearisable equations using Munier et al. (1981) transformations. Furthermore, the A.D.E. was transformed into a special case of the I.N.D.E.. Nonlocal or potential symmetry analysis of the general I.N.D.E. was carried out by a
few authors including Khater et al. (2002) and Sophocleous (2000; 2003a) wherein it was reported that I.N.D.E. (6.29) with \( G(y) \) given in (6.32) does not admit potential symmetries. However, in his latest paper on this problem, Sophocleous (2003b) observed that for this case the I.N.D.E. may be transformed into a potential symmetry bearing constant coefficient I.N.D.E.. It is also known that given the power law (6.32), the I.N.D.E. (6.29) may be transformed into the Storm equation (Broadbridge, 1987) which admits potential symmetries as well (see e.g. Katoanga, 1992). As far as we know, potential symmetries for the I.N.D.E. (6.29) have not been recovered. We have therefore systematically constructed these seemingly hidden potential symmetries by considering the integrated equation obtained by using the more general integral variable. The method used in Section 6.4 raises a question of whether it could be applied to P.D.E.s which cannot be expressed in conserved forms or to those that cannot be transformed into simpler ones. Lastly, as an example to illustrate applications of integral variable ideas, we considered a class of one dimensional nonlinear integrodifferential diffusion equations modelling population density and showed that it may be linearised using hodograph transformations. Numerical solutions are constructed for the equation describing transport of adsorbing solutes subject to meaningful boundary conditions.
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Publications of the author

Parts of this work have been submitted and/or published in international journals and part of a book chapter as follows:

