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Pricing puttable convertible bonds with integral equation approaches

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Pricing puttable convertible bonds with integral equation approaches

Song-Ping Zhu ^{*}   Sha Lin [†]   Xiaoping Lu [‡]

Abstract

American-style puttable convertible bonds are often priced with various numerical solutions because the predominant complexity arises from the determination of the two free boundaries together with the bond price. In this paper, two forms of integral equation are derived to price a puttable convertible bond on a single underlying asset. The first form is obtained under the Black-Scholes framework by using an incomplete Fourier transform. However, this integral equation formulation possesses a discontinuity along both free boundaries. An even worse problem is that this representation contains two first-order derivatives of the unknown exercise prices, which demands a higher smoothness of the interpolation functions used in the numerical solution procedure. Thus, a second integral equation formulation is developed based on the first form to overcome those problems. Numerical experiments are conducted to show several interesting properties of puttable convertible bonds.

Keywords. Puttable convertible bond, Fourier transform, integral equation approach

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1 Introduction

A convertible bond (CB) is one of the widely-used hybrid financial instruments. It gives the holder the right to convert a bond into a predetermined number of stocks at any time during the life of the bond, or to hold the bond until maturity to receive the principal payment. Such a conversion right gives the holder the possibility to gain a maximum benefit. But, this particular conversion feature has made the valuation problem more complicated because the optimal conversion boundary needs to be determined as part of the solution of the problem.

The theoretical framework for pricing CBs under the Black-Scholes model was initially proposed by Ingersoll [17] and Brennan & Schwartz [5]. They priced a convertible bond by using contingent claims, in which they took the firm value as the underlying variable. However, the model is not practical since the firm value is not observable in market. In 1986, McConnel & Schwartz [23] proposed a single-factor pricing model for a zero-coupon convertible bond, using stock price as the underlying variable.

Since then, various approaches have been proposed to price convertible bonds. Analytical solutions are only available for CBs with very simply exercises clauses. For example, Nyborg [24] obtained a closed-form solution for a simple convertible bond, which can only be converted at maturity, while Zhu [28] presented a closed-form analytical solution for a convertible bond, which can be converted at any time on or before maturity, using the homotopy analysis method. Recently, Chan & Zhu [10] provided an approximate solution for the price of a convertible bond under the regime-switching model.

On the other hand, numerical approaches are resorted to when CBs with more complex exercise clauses need to be priced. Among them, the finite element approach [2], the finite difference approach [25] and the finite volume approach [30] have been adopted by various authors. In terms of integral equation formulations for pricing CBs, Zhu & Zhang [29] used a decomposition approach to obtain an integral equation formulation for pricing a vanilla

convertible bond without any additional feature such as the putability discussed in this paper.

Apart from the Black-Scholes model, there are other models having been adopted for the evaluation of CBs. For example, Brennan & Schwartz [7] proposed a stochastic interest rate model to price convertible bonds, taking the value of the issuing firm as the underlying state variable. Carayannopoulos [8] priced convertible bonds with a different stochastic interest rate model (the so-called CIR model (Cox-Ingersoll-Ross) [12]), while David & Lischka [13] adopted the Vasicek's model [26]. All these models are based on an assumption that CBs are usually designed for a long time period, during which interest rate itself may be subject to changes. However, such an addition of stochastic nature of interest rate would not be necessary if one only needs to price a CB with short time to expiry. It is certainly not necessary if one aims to develop numerical approaches as their first step. Furthermore, Hung & Wang [16] used the binomial tree model to value the convertible bond, taking the risk of interest rate change as well as the default risk of the issuer into consideration, while Chambers & Lu [9] further extended Hung & Wang's work by allowing correlations among those two stochastic processes.

In addition to model complexity contributing to the pricing of CBs, various added additional rights to either or both the bond issuer and/or the bond holder, may also make the pricing problem more complicated, which demands better numerical solution approaches. For example, call and put features can be added to convertible bonds to form the so-called callable convertible bonds and puttable convertible bonds [1], respectively. A callable convertible bond is a bond in which the issuer has the right to call (repurchase) the bond from the investor for a predetermined call price within a predetermined callable period. The call feature in a convertible bond is in favor to the issuer, as if the underlying price increases significantly beyond the call price, the issuer can call back the bond. As a result, a callable convertible bond should be worth less than that of a vanilla convertible bond. A puttable convertible bond, on the other hand, allows the holder to sell the bond

back to the issuer, prior to maturity, at a price that is specified at the time that the bond is issued. This price is commonly referred to as the put price [22], which is also called the strike or exercise price [23]. Obviously, the put feature benefits the holder of the bond, and hence, a puttable convertible bond trades at a higher price than that of a vanilla convertible bond.

The pricing problem of callable convertible bonds has been studied for many years. For example, Brennan & Schwartz [5] explained in theory how to price such contracts, and provided numerical solutions in their later article [6], Bernini [3] used a binomial tree method to obtain their numerical solution. It is interesting to note that Kifer [19] presented a new derivative security called game options, similar to the callable convertible bond, which was used by Yagi & Sawaki [27] to study callable convertible bonds. There are also many references on puttable convertible bonds in the literature. For example, Nyborg [24] presented the boundary condition of puttable CBs, and checked if the boundary condition is reasonable and correct, while Lvov *et al.* [22] obtained the numerical solution by using Monte Carlo simulations. However, there has not been any integral equation formulation for puttable convertible bonds, which forms the base of the current research.

In this paper, we present two integral equation formulations to analyze a puttable convertible bond under the Black-Scholes model. It should be pointed out that although it is more practical to adopt a stochastic interest rate for convertible bond pricing, we assume a constant interest rate in our formulation. This is because it is more feasible to start with a simpler model when introducing a new solution approach to an already complicated problem with two free boundaries. There are two partial differential equation (PDE) systems governing the price of a puttable convertible bond, as the lifetime of a puttable CB is divided into two intervals by the time when the face value of the bond discounted by the time to expiry equals the predetermined put price. From this critical time, only convertible bond boundary conditions need to be considered since the price of a puttable CB is always greater than the put price during this time period there is no

financial incentive to exercise the put feature. Thus, the PDE system for this part should be the same as that for the vanilla CB presented in [28]. On the other hand, from the beginning of the contract until the critical time, the minimal price of puttable CB would be floored below by the put price, forming a second free boundary. Financially, the bound price is bounded below is because the holder would otherwise sell the bond back to the issuer at the put price with the warranted puttable. As a result, the puttable CB can no longer be treated as a vanilla CB and another PDE system is needed with two free boundary conditions associated with the conversion and put feature, respectively.

In order to obtain the first integral equation formulation, we apply the method of incomplete Fourier transform [11] to both of the two PDE systems. However, the resulting integral equations possess a discontinuity at both of two free boundaries and they contain the first-order derivatives of the unknown free boundaries. These problems could lead to computational difficulties when the numerical results are calculated. To overcome the problems, we derive a second integral equation representation from the first integral representation.

The paper is organized as follows. In Section 2, the PDE systems governing the price of a puttable CB are established to reflect all the unique features associated with conversion and puttable at any time prior to expiry. In Section 3, the first form of integral equation is derived by using the incomplete Fourier transform, which serves as a base to obtain another integral equation representation. In Section 4, we compared our results with the known benchmarks such as the convergent results obtained with the binomial tree method. Numerical examples are presented in Section 5, followed by some concluding remarks given in the last section.

2 The model

In this section, we will establish the PDE systems to price a puttable convertible bond.

Let S be an underlying asset price and we assume that its dynamics follows the stochastic different equation:

$$dS = (r - D_0)Sdt + \sigma SdW_t, \quad (2.1)$$

where W_t is a Brownian motion, σ is the volatility of the underlying asset, r is the risk-free interest rate, and D_0 is the rate of continuous dividend.

Now, consider a puttable convertible bond of maturity T , with face value Z , conversion ratio n and put price M . Let the time to expiry be $\tau = T - t$, there exists a critical value of $\tau = \tau_M$ when the minimum value of the puttable CB (face value discounted by time to expiry) equals the put price, that is, $Ze^{-r\tau_M} = M$, or $\tau_M = -\frac{1}{r} \log \frac{M}{Z}$.

Let $V_1(S, \tau)$ be the value of the puttable CB in the interval $\tau \in [0, \tau_M]$. The price of the CB is always greater than the put price in this time interval, and thus the optimal put exercise price is always equal to zero. As a result, there is no difference between the price of a vanilla CB and that of a puttable CB in this time interval, and it should satisfy the following PDE system (*cf.* Zhu [28]).

$$\left\{ \begin{array}{l} -\frac{\partial V_1}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_1}{\partial S^2} + (r - D_0)S \frac{\partial V_1}{\partial S} - rV_1 = 0, \\ V_1(S, 0) = \max\{nS, Z\}, \\ V_1(S_c(\tau), \tau) = nS_c(\tau), \\ \frac{\partial V_1}{\partial S}(S_c(\tau), \tau) = n, \\ V_1(0, \tau) = Ze^{-r\tau}, \end{array} \right. \quad (2.2)$$

where $S_c(\tau)$ is the optimal conversion boundary, $S \in [0, S_c(\tau)]$ and $\tau \in [0, \tau_M]$.

In the interval $\tau \in [\tau_M, T]$, the price of bond should not fall below the put price, as the holder would otherwise sell the bond back to the issuer at the predetermined put price. Therefore, when $\tau > \tau_M$, the value of the bond is bounded below by the put price, which is an important feature of puttable CBs. In fact there exist an optimal put price $S_p(\tau)$ associated with the puttability, as well as an optimal conversion price $S_c(\tau)$ in this time

interval. The price of the puttable CB, $V_2(S, \tau)$, satisfies the following PDE system:

$$\left\{ \begin{array}{l} -\frac{\partial V_2}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_2}{\partial S^2} + (r - D_0)S \frac{\partial V_2}{\partial S} - rV_2 = 0, \\ V_2(S_c(\tau), \tau) = nS_c(\tau), \\ \frac{\partial V_2}{\partial S}(S_c(\tau), \tau) = n, \\ V_2(S_p(\tau), \tau) = M, \\ \frac{\partial V_2}{\partial S}(S_p(\tau), \tau) = 0, \\ V_2(S, \tau_M^+) = V_1(S, \tau_M^-), \end{array} \right. \quad (2.3)$$

The value of the puttable convertible bond for the lifetime $\tau \in [0, T]$ can be found by solving the two PDE systems (2.2) and (2.3). We start the solution process of the systems by making the following variable transforms

$$x = \log(S), \quad v_1(x, \tau) = V_1(S, \tau), \quad v_2(x, \tau) = V_2(S, \tau).$$

The PDE systems (2.2) and (2.3) become

$$\left\{ \begin{array}{l} -\frac{\partial v_1}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 v_1}{\partial x^2} + (r - D_0 - \frac{1}{2}\sigma^2) \frac{\partial v_1}{\partial x} - rv_1 = 0, \\ v_1(x, 0) = \max\{ne^x, Z\}, \\ v_1(\ln(S_c(\tau)), \tau) = nS_c(\tau), \\ \frac{\partial v_1}{\partial x}(\ln(S_c(\tau)), \tau) = nS_c(\tau), \\ v_1(-\infty, \tau) = Ze^{-r\tau}, \end{array} \right. \quad (2.4)$$

with the domain $x \in (-\infty, \ln(S_c(\tau))]$ and $\tau \in [0, \tau_M]$, and

$$\left\{ \begin{array}{l} -\frac{\partial v_2}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 v_2}{\partial x^2} + (r - D_0 - \frac{1}{2}\sigma^2) \frac{\partial v_2}{\partial x} - rv_2 = 0, \\ v_2(\ln(S_c(\tau)), \tau) = nS_c(\tau), \\ \frac{\partial v_2}{\partial x}(\ln(S_c(\tau)), \tau) = nS_c(\tau), \\ v_2(\ln(S_p(\tau)), \tau) = M, \\ \frac{\partial v_2}{\partial x}(\ln(S_p(\tau)), \tau) = 0, \\ v_2(x, \tau_M^+) = v_1(x, \tau_M^-), \end{array} \right. \quad (2.5)$$

with the domain of x and τ being $[\ln(S_p(\tau)), \ln(S_c(\tau))]$ and $[\tau_M, T]$, respectively. By now, we have derived two dimensionless PDE systems. In next section, the solution techniques to obtain integral equation formulations for Systems (2.4) and (2.5) will be discussed.

3 Integral equation formulations of puttable convertible bond

In this section, two forms of integral equations will be presented for pricing a puttable convertible bond. One is obtained by applying the so-called incomplete Fourier transform to the PDE systems directly, and the second one is a further extension of the first one, in order to avoid some potential numerical problems.

3.1 First integral equation formulation of puttable convertible bond

In this subsection, we use the incomplete Fourier transform method to derive an integral equation representation to price a puttable convertible bond. The incomplete Fourier transform is adopted as a result of the presence of free boundaries, which have limited the domain of x to a semi-infinite domain, rather than an infinite domain from $-\infty$ to ∞ , on

which the classical Fourier transform can be applied [11]. Before applying the incomplete Fourier transform to System (2.4), it should be noted that the boundary condition at infinity is non-zero, which can cause problems. Therefore, a simple transform

$$U(x, \tau) = v_1(x, \tau) - Ze^{-r\tau}, \quad (3.1)$$

is introduced, so that System (2.4) can be rewritten as

$$\left\{ \begin{array}{l} -\frac{\partial U}{\partial \tau} + \frac{1}{2}\sigma^2 \frac{\partial^2 U}{\partial x^2} + (r - D_0 - \frac{1}{2}\sigma^2) \frac{\partial U}{\partial x} - rU = 0, \\ U(x, 0) = \max\{ne^x - Z, 0\}, \\ U(\ln(S_c(\tau)), \tau) = nS_c(\tau) - Ze^{-r\tau}, \\ \frac{\partial U}{\partial x}(\ln(S_c(\tau)), \tau) = nS_c(\tau), \\ U(-\infty, \tau) = 0. \end{array} \right. \quad (3.2)$$

Now, define the following incomplete Fourier transform

$$\mathcal{F}\{U(x, \tau)\} = \int_{-\infty}^{\ln(S_c(\tau))} U(x, \tau) e^{i\omega x} dx \triangleq \hat{U}(\omega, \tau), \quad (3.3)$$

and by applying (3.3) to System (3.2), we can obtain the following ordinary differential equation (ODE) system

$$\left\{ \begin{array}{l} \frac{\partial \hat{U}}{\partial \tau}(\omega, \tau) + B(\omega) \hat{U}(\omega, \tau) = f(\omega, \tau), \\ \hat{U}(\omega, 0) = \int_{-\infty}^{\ln(S_c(0))} \max\{ne^x - Z, 0\} e^{i\omega x} dx, \end{array} \right. \quad (3.4)$$

where

$$\begin{aligned} B(\omega) &= \frac{1}{2}\sigma^2\omega^2 + (r - D_0 - \frac{1}{2}\sigma^2)i\omega + r, \\ f(\omega, \tau) &= (nS_c(\tau) - Ze^{-r\tau})e^{i\omega \ln(S_c(\tau))} \left[\frac{S'_c(\tau)}{S_c(\tau)} + r - D_0 - \frac{1}{2}\sigma^2 - \frac{1}{2}\sigma^2 i\omega \right] + \frac{1}{2}\sigma^2 nS_c(\tau) e^{i\omega \ln(S_c(\tau))}. \end{aligned}$$

System (3.4) is a non-homogeneous first-order linear ODE system with an initial condition. The solution of this system is as follows

$$\hat{U}(\omega, \tau) = \hat{U}(\omega, 0)e^{-B(\omega)\tau} + \int_0^\tau f(\omega, \xi)e^{-B(\omega)(\tau-\xi)}d\xi. \quad (3.5)$$

By now, the integral equation formulation in the Fourier space has been derived. In order to obtain the formulation in the original space, we define the following incomplete Fourier inversion transform

$$\mathcal{F}^{-1}\{\hat{U}(\omega, \tau)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}(\omega, \tau)e^{-i\omega x}d\omega. \quad (3.6)$$

The incomplete Fourier inversion transform appears to be the same as the classical one, but there is a difference in the domain of x , in our case, the domain of x is replaced by $(-\infty, \ln(S_c(\tau))]$. Applying this new definition to (3.5) and after some tedious algebraic manipulations (see Appendix A), we obtain

$$\begin{aligned} U(x, \tau) = & \int_{-\infty}^{\ln(S_c(0))} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)\tau+x-u]^2}{2\sigma^2\tau}} \max\{ne^u - Z, 0\}du \\ & + \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\xi)-\ln(S_c(\xi))+x]^2}{2\sigma^2(\tau-\xi)}} \\ & \cdot \{(nS_c(\xi) - Ze^{-r\xi})[\frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2}(r-D_0-\frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - x}{\tau-\xi})]\} \\ & + \frac{1}{2}n\sigma^2 S_c(\xi)\}d\xi. \end{aligned} \quad (3.7)$$

Rewriting the integral equation using the original parameters, we derive an integral equation formulation of System (2.2) as follows

$$\begin{aligned} V_1(S, \tau) = & \int_{-\infty}^{\ln(S_c(0))} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)\tau+\ln(S)-u]^2}{2\sigma^2\tau}} \max\{ne^u - Z, 0\}du \\ & + \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\xi)-\ln(S_c(\xi))+\ln(S)]^2}{2\sigma^2(\tau-\xi)}} \end{aligned}$$

$$\begin{aligned} & \cdot \{ (nS_c(\xi) - Ze^{-r\xi}) \left[\frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2}(r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - \ln(S)}{\tau - \xi}) \right] \\ & + \frac{1}{2}n\sigma^2 S_c(\xi) \} d\xi + Ze^{-r\tau}. \end{aligned} \quad (3.8)$$

We have just presented the first part of pricing a puttable convertible bond, and the next step is to solve System (2.5). It should be emphasized that there are actually two free boundaries that need to be determined at the same time when we try to find the solution of this particular system. Therefore, we need to introduce another definition of the incomplete Fourier transform as

$$\mathcal{F}\{v_2(x, \tau)\} = \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} v_2(x, \tau) e^{i\omega x} dx \triangleq \hat{v}_2(\omega, \tau). \quad (3.9)$$

Substituting Equation (3.9) into System (2.5) yields the following ODE system

$$\begin{cases} \frac{\partial \hat{v}_2}{\partial \tau}(\omega, \tau) + B(\omega) \hat{v}_2(\omega, \tau) = g(\omega, \tau), \\ \hat{v}_2(\omega, \tau_M) = \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} v_1(x, \tau_M) e^{i\omega x} dx, \end{cases} \quad (3.10)$$

where

$$\begin{aligned} g(\omega, \tau) &= nS_c(\tau) e^{i\omega \ln(S_c(\tau))} \left[\frac{S'_c(\tau)}{S_c(\tau)} - \frac{1}{2}\sigma^2 i\omega + r - D_0 \right] \\ &\quad - M e^{i\omega \ln(S_p(\tau))} \left[\frac{S'_p(\tau)}{S_p(\tau)} - \frac{1}{2}\sigma^2 i\omega + r - D_0 - \frac{1}{2}\sigma^2 \right]. \end{aligned}$$

System (3.10) is again a non-homogeneous first-order linear ODE system, the solution of which can be derived as

$$\hat{v}_2(\omega, \tau) = \hat{v}_2(\omega, \tau_M) e^{-B(\omega)(\tau - \tau_M)} + \int_0^{\tau - \tau_M} g(\omega, \tau_M + \xi) e^{-B(\omega)(\tau - \tau_M - \xi)} d\xi. \quad (3.11)$$

To obtain the integral equation formulation in the original space, we apply the Fourier

inversion transform to Equation (3.11) (the details are left in Appendix B). Finally, we arrive at

$$\begin{aligned}
V_2(S, \tau) = & \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S)-u]^2}{2\sigma^2(\tau-\tau_M)}} du \\
& + \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M+\xi)e^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S)-\ln(S_c(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\
& \cdot \left\{ \frac{S'_c(\tau_M+\xi)}{S_c(\tau_M+\xi)} + \frac{1}{2} \left[r - D_0 + \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\tau_M+\xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] \right\} d\xi \\
& - \int_0^{\tau-\tau_M} \frac{Me^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S)-\ln(S_p(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\
& \cdot \left\{ \frac{S'_p(\tau_M+\xi)}{S_p(\tau_M+\xi)} + \frac{1}{2} \left[r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_p(\tau_M+\xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] \right\} d\xi. \quad (3.12)
\end{aligned}$$

Equation (3.8) and Equation (3.12) could be used, for $\tau \in [0, \tau_M]$ and $\tau \in [\tau_M, T]$, respectively, to determine the value of puttable CBs. However, both (3.8) and (3.12) involve the optimal conversion price and the optimal put price, $S_c(\tau)$ and $S_p(\tau)$, which still remain unknown. Fortunately, we can derive three integral equations for the boundary using the free boundaries conditions

$$\begin{aligned}
\frac{nS_c(\tau)}{2} = & \int_{-\infty}^{\ln(S_c(0))} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)\tau+\ln(S_c(\tau))-u]^2}{2\sigma^2\tau}} \max\{ne^u - Z, 0\} du \\
& + \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\xi)-\ln(S_c(\xi))+\ln(S_c(\tau))]^2}{2\sigma^2(\tau-\xi)}} \\
& \cdot \left\{ (nS_c(\xi) - Ze^{-r\xi}) \left[\frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2} \left(r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - \ln(S_c(\tau))}{\tau - \xi} \right) \right] \right. \\
& \left. + \frac{1}{2} n\sigma^2 S_c(\xi) \right\} d\xi + \frac{1}{2} Ze^{-r\tau}, \quad (3.13)
\end{aligned}$$

$$\begin{aligned}
\frac{nS_c(\tau)}{2} = & \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S_c(\tau))-u]^2}{2\sigma^2(\tau-\tau_M)}} du \\
& + \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M+\xi)e^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S_c(\tau))-\ln(S_c(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}}
\end{aligned}$$

$$\begin{aligned}
& \cdot \left\{ \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 + \frac{1}{2} \sigma^2 + \frac{\ln(S_c(\tau_M + \xi)) - \ln(S_c(\tau))}{\tau - \tau_M - \xi} \right] \right\} d\xi \\
& - \int_0^{\tau - \tau_M} \frac{M e^{-r(\tau - \tau_M - \xi)}}{\sigma \sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S_c(\tau)) - \ln(S_p(\tau_M + \xi))]^2}{2\sigma^2(\tau - \tau_M - \xi)}} \\
& \cdot \left\{ \frac{S'_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 - \frac{1}{2} \sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - \ln(S_c(\tau))}{\tau - \tau_M - \xi} \right] \right\} d\xi, \quad (3.14)
\end{aligned}$$

$$\begin{aligned}
\frac{M}{2} &= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S_p(\tau)) - u]^2}{2\sigma^2(\tau - \tau_M)}} du \\
&+ \int_0^{\tau - \tau_M} \frac{n S_c(\tau_M + \xi) e^{-r(\tau - \tau_M - \xi)}}{\sigma \sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau)) - \ln(S_c(\tau_M + \xi))]^2}{2\sigma^2(\tau - \tau_M - \xi)}} \\
&\cdot \left\{ \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 + \frac{1}{2} \sigma^2 + \frac{\ln(S_c(\tau_M + \xi)) - \ln(S_p(\tau))}{\tau - \tau_M - \xi} \right] \right\} d\xi \\
&- \int_0^{\tau - \tau_M} \frac{M e^{-r(\tau - \tau_M - \xi)}}{\sigma \sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau)) - \ln(S_p(\tau_M + \xi))]^2}{2\sigma^2(\tau - \tau_M - \xi)}} \\
&\cdot \left\{ \frac{S'_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 - \frac{1}{2} \sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - \ln(S_p(\tau))}{\tau - \tau_M - \xi} \right] \right\} d\xi. \quad (3.15)
\end{aligned}$$

It should be noted that there is a factor of $1/2$ on the left hand side of Equations (3.13), (3.14) and (3.15), which arises by performing the incomplete Fourier transform. Actually, it can be viewed as the complete Fourier transform of a discontinuous function, and thus the corresponding Fourier inversion converges to the midpoint of the discontinuity [14]. Sometimes such discontinuity can lead to problems when numerical experiments are conducted. An even worse problem is that both of these two integral equation formulations contain the first-order derivative of the optimal exercise prices which can lead to large numerical errors due to the infinite slope associated with these derivative functions at expiry for the optimal conversion price and at threshold value of the time to expiry for the optimal put price. To overcome these shortfalls, we propose another integral equation formulation in the next subsection.

3.2 Second integral equation formulation for puttable convertible bond

As pointed out in the previous subsection, the integral representations (3.8) and (3.12) and the integral equations (3.13), (3.14) and (3.15) are not ideal to be used for computing the value of a puttable convertible bond and its optimal boundaries, since they all contain first-order derivatives of the optimal exercise prices. So we derive the second integral representation as an extension from the first one. While we shall leave the details of the derivation in Appendix C and Appendix D, the main results are summarized here

$$\begin{aligned}
V_1(S, \tau) &= \int_0^\tau nSD_0 e^{-D_0(\tau-\xi)} \mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(\tau-\xi) + \ln(S) - \ln(S_c(\xi))}{\sigma\sqrt{\tau-\xi}}\right) d\xi \\
&+ nSe^{-D_0\tau} \mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)\tau + \ln(S) - \ln(S_c(0))}{\sigma\sqrt{\tau}}\right) \\
&- Ze^{-r\tau} \mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)\tau + \ln(S) - \ln(S_c(0))}{\sigma\sqrt{\tau}}\right) + Ze^{-r\tau}, \tag{3.16}
\end{aligned}$$

and

$$\begin{aligned}
V_2(S, \tau) &= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M) + \ln(S)-u]^2}{2\sigma^2(\tau-\tau_M)}} du \\
&+ \int_0^{\tau-\tau_M} nD_0Se^{-D_0(\tau-\tau_M-\xi)} \\
&\quad \cdot \mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi) + \ln(S) - \ln(S_c(\tau_M+\xi))}{\sigma\sqrt{\tau-\tau_M-\xi}}\right) d\xi \\
&- \int_0^{\tau-\tau_M} rMe^{-r(\tau-\tau_M-\xi)} \\
&\quad \cdot \mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi) + \ln(S) - \ln(S_p(\tau_M+\xi))}{\sigma\sqrt{\tau-\tau_M-\xi}}\right) d\xi \\
&+ nSe^{-D_0(\tau-\tau_M)} \mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(\tau-\tau_M) + \ln(S) - \ln(S_c(\tau_M))}{\sigma\sqrt{\tau-\tau_M}}\right) \\
&- Me^{-r(\tau-\tau_M)} \mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M) + \ln(S) - \ln(S_p(\tau_M))}{\sigma\sqrt{\tau-\tau_M}}\right) + M. \tag{3.17}
\end{aligned}$$

To determine the price of a puttable convertible bond, $V_1(S, \tau)$ and $V_2(S, \tau)$, the two free boundaries, S_p and S_c , need to be computed first from the following three integral equations constructed from substituting (3.16) and (3.17) into the boundaries conditions of Systems (2.2) and (2.3):

$$\begin{aligned}
nS_c(\tau) &= \int_0^\tau nS_c(\tau) D_0 e^{-D_0(\tau-\xi)} \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \xi) + \ln(S_c(\tau)) - \ln(S_c(\xi))}{\sigma\sqrt{\tau - \xi}}\right) d\xi \\
&+ nS_c(\tau) e^{-D_0\tau} \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)\tau + \ln(S_c(\tau)) - \ln(S_c(0))}{\sigma\sqrt{\tau}}\right) \\
&- Z e^{-r\tau} \mathcal{N}\left(\frac{(r - D_0 - \frac{1}{2}\sigma^2)\tau + \ln(S_c(\tau)) - \ln(S_c(0))}{\sigma\sqrt{\tau}}\right) + Z e^{-r\tau}, \tag{3.18}
\end{aligned}$$

$$\begin{aligned}
nS_c(\tau) &= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S_c(\tau))-u]^2}{2\sigma^2(\tau-\tau_M)}} du \\
&+ \int_0^{\tau-\tau_M} nD_0 S_c(\tau) e^{-D_0(\tau-\tau_M-\xi)} \\
&\quad \cdot \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S_c(\tau)) - \ln(S_c(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}\right) d\xi \\
&- \int_0^{\tau-\tau_M} rM e^{-r(\tau-\tau_M-\xi)} \\
&\quad \cdot \mathcal{N}\left(\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S_c(\tau)) - \ln(S_p(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}\right) d\xi \\
&+ nS_c(\tau) e^{-D_0(\tau-\tau_M)} \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S_c(\tau)) - \ln(S_c(\tau_M))}{\sigma\sqrt{\tau - \tau_M}}\right) \\
&- M e^{-r(\tau-\tau_M)} \mathcal{N}\left(\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S_c(\tau)) - \ln(S_p(\tau_M))}{\sigma\sqrt{\tau - \tau_M}}\right) + M, \tag{3.19}
\end{aligned}$$

$$\begin{aligned}
M &= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S_p(\tau))-u]^2}{2\sigma^2(\tau-\tau_M)}} du \\
&+ \int_0^{\tau-\tau_M} nD_0 S_p(\tau) e^{-D_0(\tau-\tau_M-\xi)} \\
&\quad \cdot \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S_p(\tau)) - \ln(S_c(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}\right) d\xi
\end{aligned}$$

$$\begin{aligned}
& - \int_0^{\tau-\tau_M} r M e^{-r(\tau-\tau_M-\xi)} \\
& \quad \cdot \mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi) + \ln(S_p(\tau)) - \ln(S_p(\tau_M+\xi))}{\sigma\sqrt{\tau-\tau_M-\xi}}\right) d\xi \\
& + n S_p(\tau) e^{-D_0(\tau-\tau_M)} \mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(\tau-\tau_M) + \ln(S_p(\tau)) - \ln(S_c(\tau_M))}{\sigma\sqrt{\tau-\tau_M}}\right) \\
& - M e^{-r(\tau-\tau_M)} \mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M) + \ln(S_p(\tau)) - \ln(S_p(\tau_M))}{\sigma\sqrt{\tau-\tau_M}}\right) + M.
\end{aligned} \tag{3.20}$$

By now, the integral equation formulations for pricing a puttable convertible bond have been presented. The solutions of the integral equations (3.18), (3.19) and (3.20) would give rise to the optimal boundaries, which can be plugged into the integral representations (3.16) and (3.17) to calculate the bond price. However, the integral equations are highly non-linear that a numerical method is needed to obtain their solutions. Therefore, in the next section, the numerical implementation of the solution procedure for pricing a puttable convertible bond will be presented.

4 The numerical implementation

In the following, we will provide an outline of our numerical scheme and its validation.

The major task in obtaining the numerical solutions of the integral equations is to find the values of free boundaries from Equations (3.18), (3.19) and (3.20). Once the free boundaries are known, we only need to numerically integrate (3.16) and (3.17) to obtain the bond prices. Our solution procedure for the free boundaries is as follows:

First, Equation (3.18) is used to obtain the values of the function $S_c(\tau)$ when $\tau \in [0, \tau_M]$. In this process, we discretize uniformly the time interval

$$0 = s_1 < s_2 < \cdots < s_N = \tau_M \quad \text{where} \quad s_i = (i-1) * \tau_M / (N-1),$$

and thus we obtain a set of non-linear algebraic equations for $S_c(s_i)$ (denoted by $S_c^{(i)}$) for $i = 1, 2, \dots, N$,

$$\begin{aligned}
nS_c^{(i)} = & \sum_{k=1}^{i-1} nS_c^{(i)} D_0 e^{-D_0(s_i-s_k)} \mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(s_i-s_k) + \ln(S_c^{(i)}) - \ln(S_c^{(k)})}{\sigma\sqrt{s_i-s_k}}\right)(s_{k+1}-s_k) \\
& - nS_c^{(i)} D_0 e^{-D_0(s_i-s_1)} \mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(s_i-s_1) + \ln(S_c^{(i)}) - \ln(S_c^{(1)})}{\sigma\sqrt{s_i-s_1}}\right) \cdot \frac{s_2-s_1}{2} \\
& + nS_c^{(i)} D_0 \cdot \frac{s_i-s_{i-1}}{4} + nS_c^{(i)} e^{-D_0 s_i} \mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)s_i + \ln(S_c^{(i)}) - \ln(S_c^{(1)})}{\sigma\sqrt{s_i}}\right) \\
& - Z e^{-r s_i} \mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)s_i + \ln(S_c^{(i)}) - \ln(S_c^{(1)})}{\sigma\sqrt{s_i}}\right) + Z e^{-r s_i}.
\end{aligned}$$

Since the terminal value of the free boundary, $S_c^{(1)} = \frac{Z}{n}$, is known, we can calculate $S_c^{(i)}$ for $i = 2, 3, \dots, N$ recursively with a MATLAB built-in root finding function (**fsolve**). It should be noted that the integral term here and those in other places are replaced by summations using a standard quadrature rule, the trapezoidal rule. The above procedure is similar to the one used in [20].

Equations (3.19) and (3.20) are discretized and solved simultaneously to obtain the values of functions $S_c(\tau)$ and $S_p(\tau)$ in the interval $\tau \in [\tau_M, T]$ by using the same method mentioned above. Instead of providing lengthy discretized equations, here we give brief outlines only. The time interval $\tau \in [\tau_M, T]$ is again divided uniformly into $L - 1$ time intervals: $[h_1, h_2], [h_2, h_3] \dots [h_{L-1}, h_L]$, where $h_1 = \tau_M$ and $h_L = T$. The discretized free boundaries $S_c(h_i)$ and $S_p(h_i)$ are denoted by $S_c^{(i)}$ and $S_p^{(i)}$, respectively, for $i = 1, 2, \dots, L$. Since the value of the function $S_p(\tau)$ at $\tau = \tau_M$ is equal to 0, we have $S_p^{(1)} = 0$. In addition, $S_c^{(1)}$ should be the same as that of $S_c^{(N)}$ obtained in $[0, \tau_M]$. Knowing $S_c^{(1)}$ and $S_p^{(1)}$, we calculate $S_c^{(i)}$ and $S_p^{(i)}$ for $i = 2, 3, \dots, L$, recursively using another MATLAB built-in root finding function (**lsqnonlin**) as this is a two dimensional problem.

Once the values of the functions $S_c(\tau)$ and $S_p(\tau)$ are obtained, the value of the bond can be straightforwardly computed through Equations (3.16) and (3.17).

We are now ready to validate our numerical scheme. Unless otherwise stated, parameters used are listed below (the same parameter setting will be used in the next section):

- Face value $Z = 100$,
- Conversion ratio $n = 1$,
- Maturity $T = 1$ (year),
- Risk-free annual interest rate $r = 0.1$,
- Rate of continuous dividend payment $D_0 = 0.07$,
- Volatility $\sigma = 0.4$,
- The put price $M = 95$.

Under these parameters, the critical value of time to expiry $\tau_M = 0.5129$ (year).

We choose to use the results calculated from the binomial tree method as the benchmark to validate our numerical scheme. Prior to benchmarking the numerical results obtained from the integral equation approach against the benchmark, numerical experiments are conducted in order to make sure that the benchmark itself obtained from the binomial tree method displays convergency. This is indeed verified as our numerical test results show that the convergence of the binomial tree method match with those reported in the literature, *i.e.*, convergence has been established, as shown in Table 1, when the length of time interval is reduced to $1/10000$ [18, 21]. Therefore, the binomial-tree results with 10,000 time-steps will be used as the benchmark in the remaining numerical experiments.

Now, we are ready to carry out numerical experiments to benchmark the accuracy and efficiency of our integral equation approach against the binomial tree method. From Table 2, one can see that all of the results from the integral equation approach at different N (the number of time intervals) agree very well with the benchmark results with maximum

Table 1: Convergency test of the Binomial tree method

Puttable Convertible bond price at $t = 0$				
S	N=1,000	N=5,000	N=8,000	N=10,000
100	107.6894	107.6905	107.6906	107.6906
110	114.4379	114.4371	114.4369	114.4366
120	122.1746	122.1736	122.1736	122.1736
130	130.7636	130.7629	130.7629	130.7629

relative error within the order of 10^{-4} . It is observed, from the CPU time listed in Table 2, that the integral equation approach is much more efficient than the binomial tree method. In addition, it should be noted that the time consumed in the integral equation approach includes the computation of the free boundaries, whereas the much longer time spent in the binomial tree method is only limited to producing the bond price, which makes the computational speed of the integral equation approach even more impressive. To further illustrate the accuracy of the integral equation method, the optimal boundaries obtained by the integral equation method are compared with those obtained by the binomial tree method in Table 3. Again, the results from the two approaches agree very well with the maximum relative error less than 10^{-2} . The benchmark tests clearly demonstrated the accuracy and efficiency of our integral equation method.

Table 2: Accuracy and efficiency test of IE method

Puttable convertible bond price at $t = 0$				
S	Benchmark N=10,000	IE N=1,000	IE N=2,000	IE N=3,000
100	107.6906	107.6895	107.6873	107.6866
105	110.9301	110.9307	110.9291	110.9286
110	114.4366	114.4381	114.4367	114.4363
115	118.1907	118.1923	118.1911	118.1907
120	122.1736	122.1756	122.1745	122.1741
max. relative error	-	3.53×10^{-5}	3.06×10^{-5}	3.73×10^{-5}
Time (second)	2486.1521	21.2070	46.6737	64.8526

In the following section, the number of time intervals in solving our integral equations

Table 3: Accuracy and efficiency test of IE method

Optimal boundaries				
τ	S_c Benchmark N=10,000	S_c IE N=2,000	S_p Benchmark N=10,000	S_p IE N=2,000
0.1021	123.9848	124.2248	-	-
0.2560	133.0856	133.3356	-	-
0.4098	138.0200	138.2800	-	-
0.5612	141.0831	141.3731	58.4431	58.3931
0.7073	143.0447	143.3247	63.4205	63.8305
0.8534	144.3288	144.6088	66.6847	67.2047
max. relative error	-	2.06×10^{-3}	-	7.80×10^{-3}

is set to be 2000 to achieve a balance between accuracy and efficiency. In addition, all of our calculations in this paper are done on a PC with the following specifications: Intel(R) Core(TM), i7-4790 CPU@3.60GHz 3.60 GHz, and 16.0 GB of RAM.

5 Examples and discussions

In this section, numerical examples are provided to illustrate various properties of puttable convertible bonds, and the difference between vanilla and puttable CBs is also demonstrated.

Figure 1 shows both the optimal conversion price and the optimal put price with respect to the time to expiry. It can be seen that both the optimal conversion price and the optimal put price are the monotonically increasing functions of the time to expiry, $\tau = T - t$. And as the conversion ratio becomes larger, the optimal exercise curves become flatter. Naturally, both the optimal conversion price and the optimal put price vary inversely with the conversion ratio, and the optimal conversion prices at expiry are the strike price divided by the conversion ratio. In fact, these properties are the same as those of the vanilla convertible bond. For puttable CBs, it should be observed that there is only one free boundary, the optimal conversion boundary, during $[0, \tau_M]$, since the value of the optimal

Figure 1: The value of the optimal boundaries for three different conversion ratios

put boundary is equal to zero in this time interval. When the time to expiry is greater than the critical value of the time to expiry, τ_M , the optimal put boundary “appears” due to the existence of the “put” feature. It is observed that when time to expiry is closer to zero, the optimal conversion price decreases quickly to the value of the strike price divided by the conversion ratio, and that as time to expiry approaches τ_M , the optimal put price drops rapidly to zero. The large slope of $S_c(\tau)$ and that of $S_p(\tau)$ near $\tau = 0$ and $\tau = \tau_M$, respectively, are similar to the behavior of the optimal exercise price near expiry [15].

Depicted in Figure 2 are the price curves of the puttable CB versus the underlying asset value, S , at times $t = 0.0000$, $t = 0.2435$, $t = 0.4871$, $t = 0.7435$. We observe that the slope of the price curves is zero when the underlying asset is worthless, increasing slowly at first at lower underlying asset price, and eventually all curves become tangent to the payoff line. This observation indicates that the first-order partial derivative of the bond price with respect to the underlying asset price is between 0 and the conversion ratio n , that is $0 \leq \frac{\partial V}{\partial S} \leq n$. It can be seen that the bond price remains almost unchanged when the underlying

Figure 2: The price of the puttable CB at four different time moments

asset price is low, the greater the time, the higher the bond price. However, when the underlying asset price increases to a certain extent, a completely different phenomenon can be observed: the bond price becomes lower as time increases.

Figure 3 displays the value of a puttable CB and its vanilla counterpart. It can be seen that the value of the puttable convertible bond is higher than that of the vanilla one. This certainly makes sense since the holder of a puttable convertible bond has an additional right to sell the bond back to the issuer, and thus the holder should be expected to pay an extra amount as a “premium”. It is interesting to observe that such a premium decreases as the underlying asset price becomes higher, and when the price of the underlying asset is very high, this premium is almost equal to zero. In other words, the price of the puttable convertible bond and that of the vanilla counterpart are almost equivalent to each other when the price of the underlying asset is very high. This can be easily explained since when the price of the underlying asset is high, there is no financial incentive for the holder to sell the bond back to the issuer. In this case, the puttable convertible bond can almost

Figure 3: The price of the puttable and vanilla CBs at the same time

be replaced by the vanilla convertible bond. In contrast, the puttable convertible bond is worth more when the price of the underlying asset is low.

Figure 4 show the effects of the volatility on the bond price as well as its optimal conversion price and optimal put price. In particular, exhibited in Figure 4(a) are the bond prices corresponding to three different volatility values. When the stock price is very low, the bond prices are insensitive to the variation of volatility, since when the stock price is low, the bond price remains almost unchanged and are equal when S is zero. Moreover, the value of the puttable CB is a monotonically increasing function of volatility. This is reasonable because when the volatility becomes larger, there is a higher risk, which will lead to a higher price. On the other hand, from Figure 4(b), it is easy to note that both the optimal conversion price and the optimal put price are the increasing functions of the time to expiry. Another interesting phenomenon is that a higher volatility will lead to a higher optimal conversion price while it will lead to a less optimal put price.

In Figure 5, we show how the price of a puttable convertible bond and both its optimal

(a) The price of puttable CBs for three different volatilities

(b) Optimal boundaries prices for three different volatilities

Figure 4: Effects of volatility on bond price and optimal boundaries

(a) The price of puttable CBs for three different risk-free interest rates

(b) Optimal boundaries prices for three different risk-free interest rates

Figure 5: Effects of risk-free interest rate on bond price and optimal boundaries

conversion price and optimal put price change with the risk-free interest rate. Comparing the bond price as well as its two free boundaries shown in this figure with those shown in Figure 4, one can observe that the risk-free interest rate has quite a different influence than volatility does. If we increase the risk-free interest rate, the bond price will decrease, this can be easily explained since when the risk-free interest rate is higher, it gives more incentive for investors to leave their money in a risk-free environment than buying a risky bond, resulting a lower CB price as displayed in Figure 5(a). On the other hand, in Figure 5(b) opposite trends for the two sets of free boundaries are observed, the optimal conversion price is a decreasing function of the risk-free interest rate while the optimal put price is an increasing function.

6 Conclusion

In this paper, the pricing problem of a puttable convertible bond on a single underlying asset with constant dividend is considered, two integral equation formulations are presented for the first time. The integral equations are solved numerically to obtain the two free boundaries, and the bond price is then calculated from the integral representations in their respective domains. Numerical examples are provided to show some interesting properties of puttable convertible bonds, subject to different values of the volatility and the interest rate.

It should be remarked here that the current approach can be extended to solving the pricing problem for callable convertible bonds, in which case there exists at most one free boundary.

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7 Appendix

Appendix A

In this appendix, we will present the detail of the computation process from System (3.2) to Equation (3.7). Under the definition of (3.3), we can transform System (3.2) to System (3.4), and here are several important processes. Since the incomplete Fourier transform

operator is linear, we can obtain the following equation:

$$-\mathcal{F}\left\{\frac{\partial U}{\partial \tau}\right\} + \frac{1}{2}\sigma^2\mathcal{F}\left\{\frac{\partial^2 U}{\partial x^2}\right\} + (r - D_0 - \frac{1}{2})\mathcal{F}\left\{\frac{\partial U}{\partial x}\right\} - r\mathcal{F}\{U\} = 0. \quad (\text{A.1})$$

Now, we compute every term of Equation (A.1):

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial U}{\partial \tau}\right\} &= \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial U}{\partial \tau}(x, \tau) e^{i\omega x} dx \\ &= \frac{\partial}{\partial \tau} \left[\int_{-\infty}^{\ln(S_c(\tau))} U(x, \tau) e^{i\omega x} dx \right] - \frac{S'_c(\tau)}{S_c(\tau)} U(\ln(S_c(\tau)), \tau) e^{i\omega \ln(S_c(\tau))} \\ &= \frac{\partial \hat{U}}{\partial \tau}(\omega, \tau) - \frac{S'_c(\tau)}{S_c(\tau)} (nS_c(\tau) - Ze^{-r\tau}) e^{i\omega \ln(S_c(\tau))}, \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial U}{\partial x}\right\} &= \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial U}{\partial x}(x, \tau) e^{i\omega x} dx \\ &= \int_{-\infty}^{\ln(S_c(\tau))} e^{i\omega x} dU(x, \tau) \\ &= e^{i\omega x} U(x, \tau) \Big|_{-\infty}^{\ln(S_c(\tau))} - i\omega \int_{-\infty}^{\ln(S_c(\tau))} e^{i\omega x} U(x, \tau) dx \\ &= e^{i\omega \ln(S_c(\tau))} [nS_c(\tau) - Ze^{-r\tau}] - i\omega \hat{U}(\omega, \tau), \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial^2 U}{\partial x^2}\right\} &= \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial^2 U}{\partial x^2}(x, \tau) e^{i\omega x} dx \\ &= \int_{-\infty}^{\ln(S_c(\tau))} e^{i\omega x} d\frac{\partial U}{\partial x}(x, \omega) \\ &= e^{i\omega x} \frac{\partial U}{\partial x}(x, \tau) \Big|_{-\infty}^{\ln(S_c(\tau))} - i\omega \int_{-\infty}^{\ln(S_c(\tau))} \frac{\partial U}{\partial x}(x, \tau) e^{i\omega x} dx \\ &= nS_c(\tau) e^{i\omega \ln(S_c(\tau))} - i\omega [e^{i\omega \ln(S_c(\tau))} (nS_c(\tau) - Ze^{-r\tau}) - i\omega \hat{U}(\omega, \tau)] \\ &= (1 - i\omega) nS_c(\tau) e^{i\omega \ln(S_c(\tau))} + i\omega Ze^{-r\tau} e^{i\omega \ln(S_c(\tau))} - \omega^2 \hat{U}(\omega, \tau). \end{aligned} \quad (\text{A.4})$$

Apply these three results to Equation (A.1), we can obtain the following equation:

$$\begin{aligned} &\frac{\partial \hat{U}}{\partial \tau}(\omega, \tau) + \left[\frac{1}{2}\sigma^2\omega^2 + (r - D_0 - \frac{1}{2}\sigma^2)i\omega + r \right] \hat{U}(\omega, \tau) \\ &= (nS_c(\tau) - Ze^{-r\tau}) e^{i\omega \ln(S_c(\tau))} \left[\frac{S'_c(\tau)}{S_c(\tau)} + (r - D_0 - \frac{1}{2}\sigma^2) - \frac{1}{2}\sigma^2 i\omega \right] \\ &\quad + \frac{1}{2}\sigma^2 e^{i\omega \ln(S_c(\tau))} nS_c(\tau). \end{aligned} \quad (\text{A.5})$$

Therefore, System (3.4) can be derived directly. Using the technique of the solution of ODE system, we can write the solution of System (3.4)

$$\hat{U}(w, \tau) = \hat{U}(\omega, 0)e^{-B(\omega)\tau} + \int_0^\tau f(\omega, \xi)e^{-B(\omega)(\tau-\xi)}d\xi. \quad (\text{A.6})$$

Now, the integral equation formulation in the Fourier space has been presented. To obtain the integral equation formulation in the original space, the Fourier Inversion transform should be applied to Equation (A.6). And then we obtain

$$\begin{aligned} U(x, \tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, \tau) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, 0) e^{-B(\omega)\tau} d\omega \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \int_0^\tau f(\omega, \xi) e^{-B(\omega)(\tau-\xi)} d\xi d\omega \\ &\triangleq I_1 + I_2. \end{aligned} \quad (\text{A.7})$$

Now, we compute I_1 and I_2 respectively, and I_1 first.

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, 0) e^{-B(\omega)\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, 0) \cdot e^{-[\frac{1}{2}\sigma^2\omega^2 + (r-D_0 - \frac{1}{2}\sigma^2)i\omega + r]\tau} d\omega \\ &\triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, 0) \cdot G(\omega, \tau) d\omega, \end{aligned} \quad (\text{A.8})$$

where $G(\omega, \tau) = e^{-[\frac{1}{2}\sigma^2\omega^2 + (r-D_0 - \frac{1}{2}\sigma^2)i\omega + r]\tau}$.

In order to use the Convolution theorem [4] to obtain the value of I_1 , we should obtain the Fourier Inversion transform of $G(\omega, \tau)$ first. Define

$$\begin{aligned} g(x, \tau) &\triangleq \mathcal{F}^{-1}\{G(\omega, \tau)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} G(\omega, \tau) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} e^{-[\frac{1}{2}\sigma^2\omega^2 + (r-D_0 - \frac{1}{2}\sigma^2)i\omega + r]\tau} d\omega \end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-r\tau}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\tau\omega^2 - [(r-D_0-\frac{1}{2}\sigma^2)\tau+x]i\omega} d\omega \\
&= \frac{e^{-r\tau}}{2\pi} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)\tau+x]^2}{2\sigma^2\tau}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2\tau[\omega+i\frac{(r-D_0-\frac{1}{2}\sigma^2)\tau+x}{\sigma^2\tau}]^2} d\omega \\
&= \frac{e^{-r\tau}}{\sqrt{2\pi\tau\sigma}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)\tau+x]^2}{2\sigma^2\tau}}.
\end{aligned} \tag{A.9}$$

Therefore, I_1 can be expressed

$$\begin{aligned}
I_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{U}(\omega, 0) \cdot G(\omega, \tau) d\omega \\
&= U(x, 0) * g(x, \tau) \\
&= \int_{-\infty}^{\ln(S_c(0))} \max\{ne^u - Z, 0\} \frac{e^{-r\tau}}{\sqrt{2\pi\tau\sigma}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)\tau+x-u]^2}{2\sigma^2\tau}} du.
\end{aligned} \tag{A.10}$$

Then, we compute I_2

$$\begin{aligned}
I_2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \int_0^{\tau} f(\omega, \xi) e^{-B(\omega)(\tau-\xi)} d\xi d\omega \\
&= \int_0^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} f(\omega, \xi) e^{-B(\omega)(\tau-\xi)} d\omega d\xi \\
&= \int_0^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x - [\frac{1}{2}\sigma^2\omega^2 + (r-D_0-\frac{1}{2}\sigma^2)i\omega + r](\tau-\xi)} e^{i\omega \ln(S_c(\xi))} \\
&\quad \cdot \{(nS_c(\xi) - Ze^{-r\xi})[\frac{S'_c(\xi)}{S_c(\xi)} + (r - D_0 - \frac{1}{2}\sigma^2) - \frac{1}{2}\sigma^2 i\omega] \\
&\quad + \frac{1}{2}\sigma^2 nS_c(\xi)\} d\omega d\xi \\
&\triangleq \int_0^{\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x - [\frac{1}{2}\sigma^2\omega^2 + (r-D_0-\frac{1}{2}\sigma^2)i\omega + r](\tau-\xi)} e^{i\omega \ln(S_c(\xi))} \\
&\quad \cdot \{f_1(\xi) - f_2(\xi)\omega\} d\omega d\xi,
\end{aligned} \tag{A.11}$$

where

$$\begin{aligned}
f_1(\xi) &= (nS_c(\xi) - Ze^{-r\xi})[\frac{S'_c(\xi)}{S_c(\xi)} + (r - D_0 - \frac{1}{2}\sigma^2)] + \frac{1}{2}\sigma^2 nS_c(\xi), \\
f_2(\xi) &= \frac{1}{2}\sigma^2 i(nS_c(\xi) - Ze^{-r\xi}).
\end{aligned}$$

Now, we compute I_2 :

$$\begin{aligned}
I_2 &= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{2\pi} \int_{-\infty}^\infty e^{-\frac{1}{2}\sigma^2(\tau-\xi)\omega^2 - [(r-D_0-\frac{1}{2}\sigma^2)(\tau-\xi)+x-\ln(S_c(\xi))]i\omega} \cdot \{f_1(\xi) - f_2(\xi)\omega\} d\omega d\xi \\
&= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}\sigma} \cdot e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\xi)+x-\ln(S_c(\xi))]^2}{2\sigma^2(\tau-\xi)}} \cdot \{(nS_c(\xi) - Ze^{-r\xi}) \\
&\quad \cdot [\frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2}(r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - x}{\tau - \xi})] + \frac{1}{2}n\sigma^2 S_c(\xi)\} d\xi, \tag{A.12}
\end{aligned}$$

where the following equation is used

$$\int_{-\infty}^\infty e^{-p\omega^2 - q\omega} \omega^n d\omega = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{\frac{q^2}{4p}}.$$

Combining I_1 and I_2 , $U(x, \tau)$ can be expressed as

$$\begin{aligned}
U(x, \tau) &= \int_{-\infty}^{\ln(S_c(0))} \max\{ne^u - Z, 0\} \frac{e^{-r\tau}}{\sqrt{2\pi\tau}\sigma} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)\tau+x-u]^2}{2\sigma^2\tau}} du \\
&\quad + \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}\sigma} \cdot e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\xi)+x-\ln(S_c(\xi))]^2}{2\sigma^2(\tau-\xi)}} \\
&\quad \cdot \{(nS_c(\xi) - Ze^{-r\xi})[\frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2}(r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - x}{\tau - \xi})] \\
&\quad + \frac{1}{2}n\sigma^2 S_c(\xi)\} d\xi, \tag{A.13}
\end{aligned}$$

which gives Equation (3.7).

Appendix B

In this appendix, we will present the detail of the computation process from System (2.5) to Equation (3.12). First, we apply the incomplete Fourier transform (3.9) to System (2.5). Since incomplete Fourier transform operator is linear, we can obtain the following equation:

$$-\mathcal{F}\left\{\frac{\partial v_2}{\partial \tau}\right\} + \frac{1}{2}\sigma^2 \mathcal{F}\left\{\frac{\partial^2 v_2}{\partial x^2}\right\} + (r - D_0 - \frac{1}{2}\sigma^2) \mathcal{F}\left\{\frac{\partial v_2}{\partial x}\right\} - r \mathcal{F}\{v_2\} = 0. \tag{B.1}$$

Now, we compute every term of Equation (B.1):

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial v_2}{\partial \tau}(x, \tau)\right\} &= \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} \frac{\partial v_2}{\partial \tau}(x, \tau) e^{i\omega x} dx \\
&= \frac{\partial}{\partial \tau} \left[\int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} v_2(x, \tau) e^{i\omega x} dx \right] - \frac{S'_c(\tau)}{S_c(\tau)} n S_c(\tau) e^{i\omega \ln(S_c(\tau))} + \frac{S'_p(\tau)}{S_p(\tau)} M e^{i\omega \ln(S_p(\tau))} \\
&= \frac{\partial \hat{v}_2}{\partial \tau}(\omega, \tau) - \frac{S'_c(\tau)}{S_c(\tau)} n S_c(\tau) e^{i\omega \ln(S_c(\tau))} + \frac{S'_p(\tau)}{S_p(\tau)} M e^{i\omega \ln(S_p(\tau))}, \tag{B.2}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial v_2}{\partial x}(x, \tau)\right\} &= \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} \frac{\partial v_2}{\partial x}(x, \tau) e^{i\omega x} dx \\
&= \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} e^{i\omega x} dv_2(x, \tau) \\
&= \frac{\partial v_2}{\partial x}(x, \tau) e^{i\omega x} \Big|_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} - i\omega \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} v_2(x, \tau) e^{i\omega x} dx \\
&= e^{i\omega \ln(S_c(\tau))} n S_c(\tau) - e^{i\omega \ln(S_p(\tau))} M - i\omega \hat{v}_2(\omega, \tau), \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}\left\{\frac{\partial^2 v_2}{\partial x^2}(x, \tau)\right\} &= \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} \frac{\partial^2 v_2}{\partial x^2}(x, \tau) e^{i\omega x} dx \\
&= \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} e^{i\omega x} d \frac{\partial v_2}{\partial x}(x, \tau) \\
&= e^{i\omega x} \frac{\partial v_2}{\partial x}(x, \tau) \Big|_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} - i\omega \int_{\ln(S_p(\tau))}^{\ln(S_c(\tau))} e^{i\omega x} \frac{\partial v_2}{\partial x}(x, \tau) dx \\
&= e^{i\omega \ln(S_c(\tau))} n S_c(\tau) - i\omega [e^{i\omega \ln(S_c(\tau))} n S_c(\tau) - e^{i\omega \ln(S_p(\tau))} M - i\omega \hat{v}_2(\omega, \tau)] \\
&= e^{i\omega \ln(S_c(\tau))} [n S_c(\tau) - i\omega n S_c(\tau) + i\omega M] - \omega^2 \hat{v}_2(\omega, \tau). \tag{B.4}
\end{aligned}$$

Substituting these three expressions into Equation (B.1), the following equation can be obtained:

$$\begin{aligned}
&\frac{\partial \hat{v}_2}{\partial \tau}(\omega, \tau) + \left[\frac{1}{2} \sigma^2 \omega^2 + (r - D_0 - \frac{1}{2} \sigma^2) i\omega + r \right] \hat{v}_2(\omega, \tau) \\
&= n S_c(\tau) e^{i\omega \ln(S_c(\tau))} \left[\frac{S'_c(\tau)}{S_c(\tau)} - \frac{1}{2} \sigma^2 i\omega + r - D_0 \right] \\
&\quad - M e^{i\omega \ln(S_p(\tau))} \left[\frac{S'_p(\tau)}{S_p(\tau)} - \frac{1}{2} \sigma^2 i\omega + r - D_0 - \frac{1}{2} \sigma^2 \right]. \tag{B.5}
\end{aligned}$$

Therefore, the ODE System (2.5) is derived directly. Using the technique of the solution of the ODE system, we obtain

$$\hat{v}_2(\omega, \tau) = \hat{v}_1(\omega, \tau_M) e^{-B(\omega)(\tau - \tau_M)} + \int_0^{\tau - \tau_M} f(\omega, \tau_M + \xi) e^{-B(\omega)(\tau - \tau_M - \xi)} d\xi, \quad (\text{B.6})$$

which gives the integral equation formulation in the Fourier space. We apply the Fourier inversion transform to the last equation to obtain the integral equation formulation $v_2(x, \tau)$ in the original space,

$$\begin{aligned} v_2(x, \tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{v}_2(\omega, \tau) e^{-i\omega x} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{v}_1(\omega, \tau_M) e^{-B(\omega)(\tau - \tau_M)} d\omega \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \int_0^{\tau - \tau_M} f(\omega, \tau_M + \xi) e^{-B(\omega)(\tau - \tau_M - \xi)} d\xi d\omega \\ &\triangleq I_1 + I_2. \end{aligned} \quad (\text{B.7})$$

We compute I_1 first.

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{v}_1(\omega, \tau_M) e^{-B(\omega)(\tau - \tau_M)} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{v}_1(\omega, \tau_M) e^{-[\frac{1}{2}\sigma^2\omega^2 + (r - D_0 - \frac{1}{2}\sigma^2)i\omega + r](\tau - \tau_M)} d\omega \\ &\triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{v}_1(\omega, \tau_M) \cdot G(\omega, \tau) d\omega, \end{aligned} \quad (\text{B.8})$$

where $G(\omega, \tau) = e^{-[\frac{1}{2}\sigma^2\omega^2 + (r - D_0 - \frac{1}{2}\sigma^2)i\omega + r](\tau - \tau_M)}$. To use the Convolution theorem here, we need to obtain the inverse Fourier transform of $G(\omega, \tau)$. Define

$$\begin{aligned} g(x, \tau) &\triangleq G(\omega, \tau) \\ &= \mathcal{F}^{-1}\{G(\omega, \tau)\} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} e^{-[\frac{1}{2}\sigma^2\omega^2 + (r - D_0 - \frac{1}{2}\sigma^2)i\omega + r](\tau - \tau_M)} d\omega \end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-r(\tau-\tau_M)}}{2\pi} \cdot e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+x]^2}{2\sigma^2(\tau-\tau_M)}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2(\tau-\tau_M)[\omega+i\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+x}{\sigma^2(\tau-\tau_M)}]^2} d\omega \\
&= \frac{e^{-r(\tau-\tau_M)}}{\sqrt{2\pi(\tau-\tau_M)}\sigma} \cdot e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+x]^2}{2\sigma^2(\tau-\tau_M)}}.
\end{aligned} \tag{B.9}$$

Then

$$\begin{aligned}
I_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \hat{v}_1(\omega, \tau_M) \cdot G(\omega, \tau) d\omega \\
&= v_1(x, \tau) * g(x, \tau) \\
&= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} v_1(u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sqrt{2\pi(\tau-\tau_M)}\sigma} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+x-u]^2}{2\sigma^2(\tau-\tau_M)}} du.
\end{aligned} \tag{B.10}$$

Now, we compute I_2

$$\begin{aligned}
I_2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} \int_0^{\tau-\tau_M} f(\omega, \tau_M + \xi) e^{-B(\omega)(\tau-\tau_M-\xi)} d\xi d\omega \\
&= \int_0^{\tau-\tau_M} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x} f(\omega, \tau_M + \xi) e^{-B(\omega)(\tau-\tau_M-\xi)} d\omega d\xi \\
&= \int_0^{\tau-\tau_M} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega x - [\frac{1}{2}\sigma^2\omega^2 + (r-D_0-\frac{1}{2}\sigma^2)i\omega + r](\tau-\tau_M-\xi)} \\
&\quad \cdot \{nS_c(\tau_M + \xi) e^{i\omega \ln(S_c(\tau_M+\xi))} [\frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} - \frac{1}{2}\sigma^2 i\omega + r - D_0] \\
&\quad - M e^{i\omega \ln(S_p(\tau_M+\xi))} [\frac{S'_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} - \frac{1}{2}\sigma^2 i\omega + r - D_0 - \frac{1}{2}\sigma^2]\} d\omega d\xi \\
&\triangleq \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M + \xi) e^{-r(\tau-\tau_M-\xi)}}{2\pi} \\
&\quad \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2(\tau-\tau_M-\xi)\omega^2 - [(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+x-\ln(S_c(\tau_M+\xi))]i\omega} \\
&\quad \cdot \{f_1(\xi) - f_2(\xi)\omega\} d\omega d\xi \\
&\quad - \int_0^{\tau-\tau_M} \frac{M e^{-r(\tau-\tau_M-\xi)}}{2\pi} \\
&\quad \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2(\tau-\tau_M-\xi)\omega^2 - [(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+x-\ln(S_p(\tau_M+\xi))]i\omega} \\
&\quad \cdot \{f_3(\xi) - f_4(\xi)\omega\} d\omega d\xi,
\end{aligned} \tag{B.11}$$

where

$$\begin{aligned} f_1(\xi) &= \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + r - D_0, & f_2(\xi) &= \frac{1}{2}\sigma^2 i, \\ f_3(\xi) &= \frac{S'_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + r - D_0 - \frac{1}{2}\sigma^2, & f_4(\xi) &= \frac{1}{2}\sigma^2 i. \end{aligned}$$

So, I_2 can be computed as

$$\begin{aligned} I_2 &= \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau-\tau_M-\xi)}}{2\pi} \\ &\quad \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2(\tau-\tau_M-\xi)\omega^2 - [(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi) + x - \ln(S_c(\tau_M + \xi))]i\omega} \\ &\quad \cdot \{f_1(\xi) - f_2(\xi)\omega\} d\omega d\xi \\ &\quad - \int_0^{\tau-\tau_M} \frac{Me^{-r(\tau-\tau_M-\xi)}}{2\pi} \\ &\quad \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2(\tau-\tau_M-\xi)\omega^2 - [(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi) + x - \ln(S_p(\tau_M + \xi))]i\omega} \\ &\quad \cdot \{f_3(\xi) - f_4(\xi)\omega\} d\omega d\xi \\ &= \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau-\tau_M-\xi)}}{\sqrt{2\pi(\tau-\tau_M-\xi)}\sigma} \cdot e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi) + x - \ln(S_c(\tau_M + \xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\ &\quad \cdot \left\{ \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 + \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\tau_M + \xi)) - x}{\tau - \tau_M - \xi} \right] \right\} d\xi \\ &\quad - \int_0^{\tau-\tau_M} \frac{Me^{-r(\tau-\tau_M-\xi)}}{\sqrt{2\pi(\tau-\tau_M-\xi)}\sigma} \cdot e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi) + x - \ln(S_p(\tau_M + \xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\ &\quad \cdot \left\{ \frac{S'_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - x}{\tau - \tau_M - \xi} \right] \right\} d\xi, \quad (\text{B.12}) \end{aligned}$$

where the follow equation is used:

$$\int_{-\infty}^{\infty} e^{-p\omega^2 - q\omega} \omega^n d\omega = (-1)^n \sqrt{\frac{\pi}{p}} \frac{\partial^n}{\partial q^n} e^{\frac{q^2}{4p}}.$$

Combining I_1 and I_2 , the expression of $v_2(x, \tau)$ is displayed as

$$v_2(x, \tau) = \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sqrt{2\pi(\tau-\tau_M)}\sigma} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M) + x - u]^2}{2\sigma^2(\tau-\tau_M)}} du$$

$$\begin{aligned}
& + \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau-\tau_M-\xi)}}{\sqrt{2\pi(\tau-\tau_M-\xi)}\sigma} \cdot e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+x-\ln(S_c(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\
& \cdot \left\{ \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 + \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\tau_M + \xi)) - x}{\tau - \tau_M - \xi} \right] \right\} d\xi \\
& - \int_0^{\tau-\tau_M} \frac{Me^{-r(\tau-\tau_M-\xi)}}{\sqrt{2\pi(\tau-\tau_M-\xi)}\sigma} \cdot e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+x-\ln(S_p(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\
& \cdot \left\{ \frac{S'_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - x}{\tau - \tau_M - \xi} \right] \right\} d\xi. \quad (\text{B.13})
\end{aligned}$$

Appendix C

Now, we present the detail of computing Equation (3.16) from Equation (3.8). First, we rewrite Equation (3.8) as follows

$$\begin{aligned}
V_1(S, \tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}\sigma} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\xi)-\ln(S_c(\xi))+\ln(S)]^2}{2\sigma^2(\tau-\xi)}} \\
&\cdot \left\{ (nS_c(\xi) - Ze^{-r\xi}) \left[\frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2} \left(r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - \ln(S)}{\tau - \xi} \right) \right] \right. \\
&\left. + \frac{1}{2} n\sigma^2 S_c(\xi) \right\} d\xi + Ze^{-r\tau}. \quad (\text{C.1})
\end{aligned}$$

It should be noted that the first term of Equation (3.8) is missing. Actually, it is always zero, since $S_c(0) = \frac{Z}{n}$. Now, we define

$$\begin{aligned}
h(S, \xi) &\triangleq \frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)]^2}{2\sigma^2(\tau - \xi)} \\
&= \frac{1}{2(\tau - \xi)} \left[\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)}{\sigma} \right]^2 \\
&= \frac{1}{2(\tau - \xi)} \left[\frac{(r - D_0 - \frac{1}{2}\sigma^2)\tau + \ln(S)}{\sigma} - \frac{(r - D_0 - \frac{1}{2}\sigma^2)\xi + \ln(S_c(\xi))}{\sigma} \right]^2 \\
&\triangleq \frac{1}{2(\tau - \xi)} [y - P(\xi)]^2, \quad (\text{C.2})
\end{aligned}$$

where $y = \frac{(r - D_0 - \frac{1}{2}\sigma^2)\tau + \ln(S)}{\sigma}$ and $P(\xi) = \frac{(r - D_0 - \frac{1}{2}\sigma^2)\xi + \ln(S_c(\xi))}{\sigma}$. It can be found that $P'(\xi) = \frac{1}{\sigma} \left[\frac{S'_c(\xi)}{S_c(\xi)} + r - D_0 - \frac{1}{2}\sigma^2 \right]$. Therefore, we apply it to Equation (C.1)

and obtain

$$\begin{aligned}
V_1(S, \tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}\sigma} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\xi)-\ln(S_c(\xi))+\ln(S)]^2}{2\sigma^2(\tau-\xi)}} \\
&\quad \cdot \{(nS_c(\xi) - Ze^{-r\xi})[\frac{S'_c(\xi)}{S_c(\xi)} + \frac{1}{2}(r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\xi)) - \ln(S)}{\tau - \xi})] \\
&\quad + \frac{1}{2}n\sigma^2 S_c(\xi)\} d\xi + Ze^{-r\tau} \\
&= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}\sigma} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \\
&\quad \cdot \{(nS_c(\xi) - Ze^{-r\xi})[\frac{S'_c(\xi)}{S_c(\xi)} + r - D_0 - \frac{1}{2}\sigma^2 - \frac{1}{2}(r - D_0 - \frac{1}{2}\sigma^2 - \frac{\ln(S_c(\xi)) - \ln(S)}{\tau - \xi})] \\
&\quad + \frac{1}{2}n\sigma^2 S_c(\xi)\} d\xi + Ze^{-r\tau} \\
&= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \\
&\quad \cdot \{(nS_c(\xi) - Ze^{-r\xi})[P'(\xi) - \frac{1}{2\sigma}(r - D_0 - \frac{1}{2}\sigma^2 - \frac{\ln(S_c(\xi)) - \ln(S)}{\tau - \xi})] \\
&\quad + \frac{1}{2}n\sigma S_c(\xi)\} d\xi + Ze^{-r\tau} \\
&= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \\
&\quad \cdot \{(nS_c(\xi) - Ze^{-r\xi})[P'(\xi) - \frac{1}{2(\tau-\xi)}(\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \xi) - \ln(S_c(\xi)) + \ln(S)}{\sigma})] \\
&\quad + \frac{1}{2}n\sigma S_c(\xi)\} d\xi + Ze^{-r\tau} \\
&= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot \{(nS_c(\xi) - Ze^{-r\xi})[P'(\xi) - \frac{y - P(\xi)}{2(\tau-\xi)}] \\
&\quad + \frac{1}{2}n\sigma S_c(\xi)\} d\xi + Ze^{-r\tau} \\
&= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot nS_c(\xi) [\frac{\sigma}{2} + P'(\xi) - \frac{y - P(\xi)}{2(\tau-\xi)}] d\xi \\
&\quad - \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot Ze^{-r\xi} [P'(\xi) - \frac{y - P(\xi)}{2(\tau-\xi)}] d\xi + Ze^{-r\tau} \\
&\triangleq R_1(S, \tau) - ZR_2(S, \tau) + Ze^{-r\tau}, \tag{C.3}
\end{aligned}$$

where

$$\begin{aligned}
R_1(S, \tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot nS_c(\xi) \left[\frac{\sigma}{2} + P'(\xi) - \frac{y-P(\xi)}{2(\tau-\xi)} \right] d\xi, \\
R_2(S, \tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot e^{-r\xi} \left[P'(\xi) - \frac{y-P(\xi)}{2(\tau-\xi)} \right] d\xi.
\end{aligned}$$

Now, we compute $R_1(S, \tau)$ and $R_2(S, \tau)$, respectively.

$$\begin{aligned}
R_1(S, \tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot nS_c(\xi) \left[\frac{\sigma}{2} + P'(\xi) - \frac{y-P(\xi)}{2(\tau-\xi)} \right] d\xi \\
&= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} e^{(y-P(\xi))\sigma + \frac{\sigma^2(\tau-\xi)}{2}} \cdot nS_c(\xi) \left[\frac{\frac{\sigma}{2} + P'(\xi) - \frac{y-P(\xi)}{2(\tau-\xi)}}{\sqrt{\tau-\xi}} \right] d\xi \\
&= \int_0^\tau \frac{e^{-r(\tau-\xi) + (y-P(\xi))\sigma + \frac{\sigma^2(\tau-\xi)}{2}}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \\
&\quad \cdot nS_c(\xi) \left\{ \frac{\sigma(\tau-\xi) + 2P'(\xi)(\tau-\xi) - y + P(\xi)}{2(\tau-\xi)\sqrt{\tau-\xi}} \right\} d\xi \\
&= \int_0^\tau \frac{e^{-r(\tau-\xi) + (r-D_0 - \frac{1}{2}\sigma^2)(\tau-\xi) - \ln(S_c(\xi)) + \ln(S) + \frac{\sigma^2(\tau-\xi)}{2}}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \\
&\quad \cdot nS_c(\xi) \left\{ \frac{P(\xi) - y - \sigma(\tau-\xi) + 2P'(\xi)(\tau-\xi) - 2\sigma(\tau-\xi)}{2(\tau-\xi)\sqrt{\tau-\xi}} \right\} d\xi \\
&= \int_0^\tau \frac{e^{-D_0(\tau-\xi) - \ln(S_c(\xi)) + \ln(S)}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \\
&\quad \cdot nS_c(\xi) \left\{ \frac{[P(\xi) - y - \sigma(\tau-\xi)] + [P'(\xi) - \sigma] \cdot 2(\tau-\xi)}{2(\sqrt{\tau-\xi})^2\sqrt{\tau-\xi}} \right\} d\xi \\
&= \int_0^\tau \frac{S}{S_c(\xi)} \cdot \frac{e^{-D_0(\tau-\xi)}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \\
&\quad \cdot nS_c(\xi) \left\{ \frac{\frac{1}{2\sqrt{\tau-\xi}} [P(\xi) - y - \sigma(\tau-\xi)] + (P'(\xi) + \sigma)\sqrt{\tau-\xi}}{(\sqrt{\tau-\xi})^2} \right\} d\xi \\
&= \int_0^\tau \frac{nS e^{-D_0(\tau-\xi)}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)+\sigma(\tau-\xi)]^2}{2(\tau-\xi)}} \cdot \frac{\partial}{\partial \xi} \left[\frac{P(\xi) - y - \sigma(\tau-\xi)}{\sqrt{\tau-\xi}} \right] d\xi \\
&= \int_0^\tau -nS e^{-D_0(\tau-\xi)} \frac{\partial}{\partial \xi} \mathcal{N}\left(\frac{y-P(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi. \tag{C.4} \\
R_2(S, \tau) &= \int_0^\tau \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi(\tau-\xi)}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot e^{-r\xi} \left[P'(\xi) - \frac{y-P(\xi)}{2(\tau-\xi)} \right] d\xi \\
&= \int_0^\tau \frac{e^{-r\tau}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot \left[\frac{2P'(\xi)(\tau-\xi) - y + P(\xi)}{2(\tau-\xi)\sqrt{\tau-\xi}} \right] d\xi
\end{aligned}$$

$$\begin{aligned}
&= \int_0^\tau \frac{e^{-r\tau}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot \left[\frac{P'(\xi) \cdot 2(\tau-\xi) + (P(\xi) - y)}{2(\tau-\xi)\sqrt{\tau-\xi}} \right] d\xi \\
&= \int_0^\tau \frac{e^{-r\tau}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot \left[\frac{P'(\xi)\sqrt{\tau-\xi} + \frac{P(\xi)-y}{2\sqrt{\tau-\xi}}}{(\sqrt{\tau-\xi})^2} \right] d\xi \\
&= \int_0^\tau -\frac{e^{-r\tau}}{\sqrt{2\pi}} e^{-\frac{[y-P(\xi)]^2}{2(\tau-\xi)}} \cdot \frac{\partial}{\partial \xi} \left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}} \right) d\xi \\
&= \int_0^\tau -e^{-r\tau} \frac{\partial}{\partial \xi} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right) d\xi. \tag{C.5}
\end{aligned}$$

Thus, we substitute the expression of $R_1(S, \tau)$ and $R_2(S, \tau)$ into Equation (C.3). Equation (3.16) can be obtained

$$\begin{aligned}
V_1(S, \tau) &= R_1(S, \tau) - ZR_2(S, \tau) + Ze^{-r\tau} \\
&= \int_0^\tau -nSe^{-D_0(\tau-\xi)} \frac{\partial}{\partial \xi} \mathcal{N}\left(\frac{y-P(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi \\
&\quad + Z \int_0^\tau e^{-r\tau} \frac{\partial}{\partial \xi} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right) d\xi + Ze^{-r\tau} \\
&= \int_0^\tau -nSe^{-D_0(\tau-\xi)} d\mathcal{N}\left(\frac{y-P(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) \\
&\quad + Z \int_0^\tau e^{-r\tau} d\mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right) + Ze^{-r\tau} \\
&= -nSe^{-D_0(\tau-\xi)} \mathcal{N}\left(\frac{y-P(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) \Big|_{\xi=\tau} \\
&\quad + nSe^{-D_0(\tau-\xi)} \mathcal{N}\left(\frac{y-P(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) \Big|_{\xi=0} \\
&\quad + \int_0^\tau nD_0Se^{-D_0(\tau-\xi)} \mathcal{N}\left(\frac{y-P(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi \\
&\quad + Ze^{-r\tau} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right) \Big|_{\xi=\tau} - Ze^{-r\tau} \mathcal{N}\left(\frac{y-P(\xi)}{\sqrt{\tau-\xi}}\right) \Big|_{\xi=0} + Ze^{-r\tau} \\
&= -nS \mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\xi))}{\sigma\sqrt{\tau-\xi}}\right) \Big|_{\xi=\tau} \\
&\quad + nSe^{-D_0\tau} \mathcal{N}\left(\frac{(r-D_0 + \frac{1}{2}\sigma^2)\tau - \ln S_c(0) + \ln(S)}{\sigma\sqrt{\tau}}\right) \\
&\quad + \int_0^\tau nD_0Se^{-D_0(\tau-\xi)} \mathcal{N}\left(\frac{y-P(\xi) + \sigma(\tau-\xi)}{\sqrt{\tau-\xi}}\right) d\xi \\
&\quad + Ze^{-r\tau} \mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\xi))}{\sigma\sqrt{\tau-\xi}}\right) \Big|_{\xi=\tau}
\end{aligned}$$

$$\begin{aligned}
& -Ze^{-r\tau}\mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)\tau-\ln(S_c(0))+\ln(S)}{\sigma\sqrt{\tau}}\right)+Ze^{-r\tau} \\
= & nSe^{-D_0\tau}\mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)\tau-\ln S_c(0)+\ln(S)}{\sigma\sqrt{\tau}}\right) \\
& -Ze^{-r\tau}\mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)\tau-\ln(S_c(0))+\ln(S)}{\sigma\sqrt{\tau}}\right) \\
& +\int_0^\tau nD_0Se^{-D_0(\tau-\xi)} \\
& \cdot\mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(\tau-\xi)-\ln(S_c(\xi))+\ln(S)}{\sigma\sqrt{\tau-\xi}}\right)d\xi \\
& -\mathbf{1}_{S=S_c(\tau)}(nS-Ze^{-r\tau})+Ze^{-r\tau}, \tag{C.6}
\end{aligned}$$

where

$$\mathbf{1}_{S=S_c(\tau)} = \begin{cases} \frac{1}{2} & S = S_c(\tau), \\ 0 & S < S_c(\tau). \end{cases} \tag{C.7}$$

Appendix D

Now, we give the detail of computing Equation (3.17) from Equation (3.12), using the same method mentioned in Appendix C. First, we rewrite Equation (3.12) as follows

$$\begin{aligned}
V_2(S, \tau) &= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S)-u]^2}{2\sigma^2(\tau-\tau_M)}} du \\
&+ \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M+\xi)e^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S)-\ln(S_c(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\
&\cdot \left\{ \frac{S'_c(\tau_M+\xi)}{S_c(\tau_M+\xi)} + \frac{1}{2} \left[r-D_0 + \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\tau_M+\xi))-\ln(S)}{\tau-\tau_M-\xi} \right] \right\} d\xi \\
&- \int_0^{\tau-\tau_M} \frac{Me^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S)-\ln(S_p(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\
&\cdot \left\{ \frac{S'_p(\tau_M+\xi)}{S_p(\tau_M+\xi)} + \frac{1}{2} \left[r-D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_p(\tau_M+\xi))-\ln(S)}{\tau-\tau_M-\xi} \right] \right\} d\xi \\
&\triangleq \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S)-u]^2}{2\sigma^2(\tau-\tau_M)}} du \\
&+ R_1(S, \tau) - R_2(S, \tau), \tag{D.1}
\end{aligned}$$

where

$$\begin{aligned}
R_1(S, \tau) &= \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S)-\ln(S_c(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\
&\quad \cdot \left\{ \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 + \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] \right\} d\xi, \\
R_2(S, \tau) &= \int_0^{\tau-\tau_M} \frac{Me^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S)-\ln(S_p(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\
&\quad \cdot \left\{ \frac{S'_p(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] \right\} d\xi.
\end{aligned}$$

Now, we compute $R_1(S, \tau)$ first. Define

$$\begin{aligned}
h_1(S, \xi) &\triangleq \frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi))]^2}{2\sigma^2(\tau - \tau_M - \xi)} \\
&= \frac{1}{2(\tau - \tau_M - \xi)} \left[\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi))}{\sigma} \right]^2 \\
&= \frac{1}{2(\tau - \tau_M - \xi)} \left[\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S)}{\sigma} \right. \\
&\quad \left. - \frac{(r - D_0 - \frac{1}{2}\sigma^2)\xi + \ln(S_c(\tau_M + \xi))}{\sigma} \right]^2 \\
&\triangleq \frac{1}{2(\tau - \tau_M - \xi)} [y_1 - P_1(\xi)]^2, \tag{D.2}
\end{aligned}$$

where $y_1 = \frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S)}{\sigma}$ and $P_1(\xi) = \frac{(r - D_0 - \frac{1}{2}\sigma^2)\xi + \ln(S_c(\tau_M + \xi))}{\sigma}$.

It should be noted that $P'_1(\xi) = \frac{\frac{S'_c(\tau_M+\xi)}{S_c(\tau_M+\xi)} + r - D_0 - \frac{1}{2}\sigma^2}{\sigma}$. Therefore, we can obtain

$$\begin{aligned}
R_1(S, \xi) &= \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S)-\ln(S_c(\tau_M+\xi))]^2}{2\sigma^2(\tau-\tau_M-\xi)}} \\
&\quad \cdot \left\{ \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 + \frac{1}{2}\sigma^2 + \frac{\ln(S_c(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] \right\} d\xi \\
&= \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[y_1-P_1(\xi)]^2}{2(\tau-\tau_M-\xi)}} \\
&\quad \cdot \left\{ \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + r - D_0 + \frac{1}{2}\sigma^2 - \frac{1}{2} \left[r - D_0 + \frac{1}{2}\sigma^2 - \frac{\ln(S_c(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] \right\} d\xi \\
&= \int_0^{\tau-\tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau-\tau_M-\xi)}}{\sigma\sqrt{2\pi(\tau-\tau_M-\xi)}} e^{-\frac{[y_1-P_1(\xi)]^2}{2(\tau-\tau_M-\xi)}}
\end{aligned}$$

$$\begin{aligned}
& \cdot \left\{ \frac{S'_c(\tau_M + \xi)}{S_c(\tau_M + \xi)} + r - D_0 - \frac{1}{2}\sigma^2 - \frac{1}{2} \left[r - D_0 - \frac{1}{2}\sigma^2 - \frac{\ln(S_c(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] + \frac{\sigma^2}{2} \right\} d\xi \\
&= \int_0^{\tau - \tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[y_1 - P_1(\xi)]^2}{2(\tau - \tau_M - \xi)}} \cdot \left[P'_1(\xi) - \frac{y_1 - P_1(\xi)}{2(\tau - \tau_M - \xi)} + \frac{\sigma}{2} \right] d\xi \\
&= \int_0^{\tau - \tau_M} \frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} e^{-\frac{[y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)]^2}{2(\tau - \tau_M - \xi)}} \cdot e^{\sigma(y_1 - P_1(\xi)) + \frac{\sigma^2(\tau - \tau_M - \xi)}{2}} \\
&\quad \cdot \left\{ \frac{2P'_1(\xi)(\tau - \tau_M - \xi) - y_1 + P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{2(\tau - \tau_M - \xi)\sqrt{\tau - \tau_M - \xi}} \right\} d\xi \\
&= \int_0^{\tau - \tau_M} -\frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi) + \sigma(y_1 - P_1(\xi)) + \frac{\sigma^2(\tau - \tau_M - \xi)}{2}}}{\sqrt{2\pi}} e^{-\frac{[y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)]^2}{2(\tau - \tau_M - \xi)}} \\
&\quad \cdot \left\{ \frac{[y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)] - [P'_1(\xi) + \sigma] \cdot 2(\tau - \tau_M - \xi)}{2(\tau - \tau_M - \xi)\sqrt{\tau - \tau_M - \xi}} \right\} d\xi \\
&= \int_0^{\tau - \tau_M} -\frac{nS_c(\tau_M + \xi)e^{-r(\tau - \tau_M - \xi) + (r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi)) + \frac{\sigma^2(\tau - \tau_M - \xi)}{2}}}{\sqrt{2\pi}} \\
&\quad \cdot e^{-\frac{[y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)]^2}{2(\tau - \tau_M - \xi)}} \cdot \left\{ \frac{[y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)] - [P'_1(\xi) + \sigma] \cdot 2(\tau - \tau_M - \xi)}{2(\tau - \tau_M - \xi)\sqrt{\tau - \tau_M - \xi}} \right\} d\xi \\
&= \int_0^{\tau - \tau_M} -\frac{S}{S_c(\tau_M + \xi)} \cdot \frac{nS_c(\tau_M + \xi)e^{-D_0(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot e^{-\frac{[y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)]^2}{2(\tau - \tau_M - \xi)}} \\
&\quad \cdot \left\{ \frac{\frac{1}{2\sqrt{\tau - \tau_M - \xi}}[y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)] - [P'_1(\xi) + \sigma] \cdot \sqrt{\tau - \tau_M - \xi}}{(\sqrt{\tau - \tau_M - \xi})^2} \right\} d\xi \\
&= \int_0^{\tau - \tau_M} -\frac{nSe^{-D_0(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} \cdot e^{-\frac{[y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)]^2}{2(\tau - \tau_M - \xi)}} \cdot \frac{\partial}{\partial \xi} \left(\frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi \\
&= \int_0^{\tau - \tau_M} -nSe^{-D_0(\tau - \tau_M - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{N} \left(\frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi. \tag{D.3}
\end{aligned}$$

Now, we compute $R_2(S, \tau)$. Define

$$\begin{aligned}
h_2(S, \xi) &\triangleq \frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))]^2}{2\sigma^2(\tau - \tau_M - \xi)} \\
&= \frac{1}{2(\tau - \tau_M - \xi)} \left[\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma} \right]^2 \\
&= \frac{1}{2(\tau - \tau_M - \xi)} \left[\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S)}{\sigma} \right. \\
&\quad \left. - \frac{(r - D_0 - \frac{1}{2}\sigma^2)\xi + \ln(S_p(\tau_M + \xi))}{\sigma} \right]^2 \\
&\triangleq \frac{1}{2(\tau - \tau_M - \xi)} [y_2 - P_2(\xi)]^2, \tag{D.4}
\end{aligned}$$

where $y_2 = \frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S)}{\sigma}$ and $P_2(\xi) = \frac{(r - D_0 - \frac{1}{2}\sigma^2)\xi + \ln(S_p(\tau_M + \xi))}{\sigma}$.
It should be noted that $P_2'(\xi) = \frac{\frac{S_p'(\tau_M + \xi)}{S_p(\tau_M + \xi)} + r - D_0 - \frac{1}{2}\sigma^2}{\sigma}$. Therefore, we can obtain

$$\begin{aligned}
R_2(S, \xi) &= \int_0^{\tau - \tau_M} \frac{M e^{-r(\tau - \tau_M - \xi)}}{\sigma \sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))]^2}{2\sigma^2(\tau - \tau_M - \xi)}} \\
&\quad \cdot \left\{ \frac{S_p'(\tau_M + \xi)}{S_p(\tau_M + \xi)} + \frac{1}{2} \left[r - D_0 - \frac{1}{2}\sigma^2 + \frac{\ln(S_p(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] \right\} d\xi \\
&= \int_0^{\tau - \tau_M} \frac{M e^{-r(\tau - \tau_M - \xi)}}{\sigma \sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[y_2 - P_2(\xi)]^2}{2(\tau - \tau_M - \xi)}} \\
&\quad \cdot \left\{ \frac{S_p'(\tau_M + \xi)}{S_p(\tau_M + \xi)} + r - D_0 - \frac{1}{2}\sigma^2 - \frac{1}{2} \left[r - D_0 - \frac{1}{2}\sigma^2 - \frac{\ln(S_p(\tau_M + \xi)) - \ln(S)}{\tau - \tau_M - \xi} \right] \right\} d\xi \\
&= \int_0^{\tau - \tau_M} \frac{M e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi(\tau - \tau_M - \xi)}} e^{-\frac{[y_2 - P_2(\xi)]^2}{2(\tau - \tau_M - \xi)}} \cdot \left[P_2'(\xi) - \frac{y_2 - P_2(\xi)}{2(\tau - \tau_M - \xi)} \right] d\xi \\
&= \int_0^{\tau - \tau_M} \frac{M e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} e^{-\frac{[y_2 - P_2(\xi)]^2}{2(\tau - \tau_M - \xi)}} \cdot \left\{ \frac{2P_2'(\xi)(\tau - \tau_M - \xi) - y_2 + P_2(\xi)}{2(\tau - \tau_M - \xi)\sqrt{\tau - \tau_M - \xi}} \right\} d\xi \\
&= \int_0^{\tau - \tau_M} -\frac{M e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} e^{-\frac{[y_2 - P_2(\xi)]^2}{2(\tau - \tau_M - \xi)}} \cdot \left\{ \frac{[y_2 - P_2(\xi)] \frac{1}{2\sqrt{\tau - \tau_M - \xi}} - P_2'(\xi)\sqrt{\tau - \tau_M - \xi}}{\sqrt{\tau - \tau_M - \xi}^2} \right\} d\xi \\
&= \int_0^{\tau - \tau_M} -\frac{M e^{-r(\tau - \tau_M - \xi)}}{\sqrt{2\pi}} e^{-\frac{[y_2 - P_2(\xi)]^2}{2(\tau - \tau_M - \xi)}} \cdot \frac{\partial}{\partial \xi} \left(\frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi \\
&= \int_0^{\tau - \tau_M} -M e^{-r(\tau - \tau_M - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{N} \left(\frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi. \tag{D.5}
\end{aligned}$$

Now, we substitute the expression of $R_1(S, \tau)$ and $R_2(S, \tau)$ into $V_2(S, \tau)$, we obtain

$$\begin{aligned}
V_2(S, \tau) &= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S) - u]^2}{2\sigma^2(\tau - \tau_M)}} du \\
&\quad + R_1(S, \tau) - R_2(S, \tau) \\
&= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S) - u]^2}{2\sigma^2(\tau - \tau_M)}} du \\
&\quad - \int_0^{\tau - \tau_M} n S e^{-D_0(\tau - \tau_M - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{N} \left(\frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi \\
&\quad + \int_0^{\tau - \tau_M} M e^{-r(\tau - \tau_M - \xi)} \cdot \frac{\partial}{\partial \xi} \mathcal{N} \left(\frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}} \right) d\xi \\
&= \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau - \tau_M)}}{\sigma \sqrt{2\pi(\tau - \tau_M)}} e^{-\frac{[(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S) - u]^2}{2\sigma^2(\tau - \tau_M)}} du
\end{aligned}$$

$$\begin{aligned}
& - \int_0^{\tau-\tau_M} nS e^{-D_0(\tau-\tau_M-\xi)} \cdot d\mathcal{N}\left(\frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}}\right) \\
& + \int_0^{\tau-\tau_M} M e^{-r(\tau-\tau_M-\xi)} \cdot d\mathcal{N}\left(\frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}}\right) \\
= & \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S)-u]^2}{2\sigma^2(\tau-\tau_M)}} du \\
& - nS e^{-D_0(\tau-\tau_M-\xi)} \cdot \mathcal{N}\left(\frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}}\right)\Big|_{\xi=\tau-\tau_M} \\
& + nS e^{-D_0(\tau-\tau_M-\xi)} \cdot \mathcal{N}\left(\frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}}\right)\Big|_{\xi=0} \\
& + \int_0^{\tau-\tau_M} nD_0 S e^{-D_0(\tau-\tau_M-\xi)} \mathcal{N}\left(\frac{y_1 - P_1(\xi) + \sigma(\tau - \tau_M - \xi)}{\sqrt{\tau - \tau_M - \xi}}\right) d\xi \\
& + M e^{-r(\tau-\tau_M-\xi)} \cdot \mathcal{N}\left(\frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}}\right)\Big|_{\xi=\tau-\tau_M} \\
& - M e^{-r(\tau-\tau_M-\xi)} \cdot \mathcal{N}\left(\frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}}\right)\Big|_{\xi=0} \\
& - \int_0^{\tau-\tau_M} r M e^{-r(\tau-\tau_M-\xi)} \cdot \mathcal{N}\left(\frac{y_2 - P_2(\xi)}{\sqrt{\tau - \tau_M - \xi}}\right) d\xi \\
= & \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S)-u]^2}{2\sigma^2(\tau-\tau_M)}} du \\
& - nS \mathcal{N}\left(\frac{\ln(S) - \ln(S_c(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}\right)\Big|_{\xi=\tau-\tau_M} \\
& + nS e^{-D_0(\tau-\tau_M)} \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S) - \ln(S_c(\tau_M))}{\sigma\sqrt{\tau - \tau_M}}\right) \\
& + \int_0^{\tau-\tau_M} nD_0 S e^{-D_0(\tau-\tau_M-\xi)} \\
& \quad \mathcal{N}\left(\frac{(r - D_0 + \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_c(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}\right) d\xi \\
& + M \mathcal{N}\left(\frac{\ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}\right)\Big|_{\xi=\tau-\tau_M} \\
& - M e^{-r(\tau-\tau_M)} \mathcal{N}\left(\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M) + \ln(S) - \ln(S_p(\tau_M))}{\sigma\sqrt{\tau - \tau_M}}\right) \\
& - \int_0^{\tau-\tau_M} r M e^{-r(\tau-\tau_M-\xi)} \\
& \quad \mathcal{N}\left(\frac{(r - D_0 - \frac{1}{2}\sigma^2)(\tau - \tau_M - \xi) + \ln(S) - \ln(S_p(\tau_M + \xi))}{\sigma\sqrt{\tau - \tau_M - \xi}}\right) d\xi \\
= & \int_{\ln(S_p(\tau_M))}^{\ln(S_c(\tau_M))} V_1(e^u, \tau_M) \frac{e^{-r(\tau-\tau_M)}}{\sigma\sqrt{2\pi(\tau-\tau_M)}} e^{-\frac{[(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S)-u]^2}{2\sigma^2(\tau-\tau_M)}} du
\end{aligned}$$

$$\begin{aligned}
& +nSe^{-D_0(\tau-\tau_M)}\mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S)-\ln(S_c(\tau_M))}{\sigma\sqrt{\tau-\tau_M}}\right) \\
& -nS\mathbf{1}_{S=S_c(\tau)}+\int_0^{\tau-\tau_M}nD_0Se^{-D_0(\tau-\tau_M-\xi)} \\
& \quad \cdot\mathcal{N}\left(\frac{(r-D_0+\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S)-\ln(S_c(\tau_M+\xi))}{\sigma\sqrt{\tau-\tau_M-\xi}}\right)d\xi \\
& -Me^{-r(\tau-\tau_M)}\mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M)+\ln(S)-\ln(S_p(\tau_M))}{\sigma\sqrt{\tau-\tau_M}}\right) \\
& +M\cdot\mathbf{1}_{S=S_p(\tau)}-\int_0^{\tau-\tau_M}rMe^{-r(\tau-\tau_M-\xi)} \\
& \quad \cdot\mathcal{N}\left(\frac{(r-D_0-\frac{1}{2}\sigma^2)(\tau-\tau_M-\xi)+\ln(S)-\ln(S_p(\tau_M+\xi))}{\sigma\sqrt{\tau-\tau_M-\xi}}\right)d\xi, \quad (\text{D.6})
\end{aligned}$$

where

$$\mathbf{1}_{S=S_c(\tau)}=\begin{cases} \frac{1}{2} & S=S_c(\tau), \\ 0 & S<S_c(\tau), \end{cases} \quad (\text{D.7})$$

$$\mathbf{1}_{S=S_p(\tau)}=\begin{cases} \frac{1}{2} & S=S_p(\tau), \\ 1 & S>S_p(\tau). \end{cases} \quad (\text{D.8})$$