Zeckendorf representations using negative fibonacci numbers

M W. Bunder

University of Wollongong, mbunder@uow.edu.au

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Zeckendorf representations using negative fibonacci numbers

Abstract
It is well known that every positive integer can be represented uniquely as a sum of distinct, nonconsecutive Fibonacci numbers (see, e.g., Brown [1]). This representation is called the Zeckendorf representation of the positive integer. Other Zeckendorf-type representations where the Fibonacci numbers are not necessarily consecutive are possible. Brown [2] considers one where a maximal number of distinct Fibonacci numbers are used rather than a minimal number.

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ZECKENDORF REPRESENTATIONS USING NEGATIVE FIBONACCI NUMBERS

M. W. Bunder
The University of Wollongong, Wollongong, N.S.W. 2500, Australia
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It is well known that every positive integer can be represented uniquely as a sum of distinct, nonconsecutive Fibonacci numbers (see, e.g., Brown [1]). This representation is called the Zeckendorf representation of the positive integer. Other Zeckendorf-type representations where the Fibonacci numbers are not necessarily consecutive are possible. Brown [2] considers one where a maximal number of distinct Fibonacci numbers are used rather than a minimal number.

We show here that every integer can be represented uniquely as a sum of nonconsecutive Fibonacci numbers \( F_{\ell} \) where \( \ell \leq 0 \) and we specify an algorithm that leads to this representation. We also show that no maximal representation of this form is possible.

We note that for all integers \( \ell \),
\[
F_{-\ell} = (-1)^{i+1} F_i
\]
and
\[
F_{i+1} = F_i + F_{i-1}.
\]

We note further that \( F_0 = 0, F_1, F_3, F_5, \ldots \) are positive and \( F_2, F_4, \ldots \) are negative. Also for \( \ell > 1 \),
\[
|F_{-\ell}| < |F_{-\ell-1}|.
\]

The four lemmas below will show that the algorithm that follows them is effective.

**Lemma 1:** If \( n, k > 0 \) and \( -F_{-2k} \leq n < F_{-2k-1} - 1 \) then, for some \( \ell, k > \ell > 0 \),
\[
-F_{-2k+2\ell-1} \leq n - F_{-2k-1} < -F_{-2k+2\ell+1} < 0.
\]

If \( n = F_{-2k-1} - 1 \), then
\[
n - F_{-2k-1} = -F_{-1}.
\]

**Proof:** If \( -F_{-2k} \leq n < F_{-2k-1} - 1 \), then
\[
1 < F_{-2k-1} - n \leq F_{-2k-1} + F_{-2k},
\]
i.e.,
\[
1 < F_{-2k-1} - n \leq F_{-2k+1} = F_{2k-1}.
\]

Now every integer \( p > 1 \) is in a range \( 0 < F_{2m-3} < p \leq F_{2m-1} \) where \( m \geq 2 \).

We must, if \( p = F_{-2k-1} - n \), then have \( m + \ell = k + 1 \) for some \( \ell > 0 \) and so:
\[
0 < F_{2k-2\ell} - n < F_{-2k-1} - n \leq F_{2k-2\ell+1};
\]
thus,
\[
-F_{-2k+2\ell-1} \leq n - F_{-2k-1} < -F_{-2k+2\ell+1} < 0.
\]

**Lemma 2:** If \( n, k > 0 \) and \( F_{-2k+1} < n \leq F_{-2k} \) then, for some \( \ell, k > \ell > 0 \):
\[
0 \leq -F_{-2k+2\ell+2} < n - F_{-2k+1} \leq -F_{-2k+2\ell}.
\]

**Proof:** If \( F_{-2k+1} < n \leq F_{-2k} \), then
\[
0 < n - F_{-2k+1} \leq F_{-2k} - F_{-2k+1},
\]
so
\[
0 < n - F_{-2k+2} \leq F_{2k-2}.
\]
Now every positive integer \( p \) is in the range

\[
0 \leq F_{2m-4} < p \leq F_{2m-2}
\]

where \( m \geq 2 \).

We must, if \( p = n - F_{-2k+1} \), then have \( m + k = k + 1 \) for some \( k, k > k > 0 \), and so

\[
0 \leq F_{2k-2k+1} - 2 < n - F_{-2k+1} \leq F_{2k-2k-1}
\]

i.e.,

\[
0 \leq F_{-2k+2k+2} - 2 < n - F_{-2k+1} \leq F_{-2k+2k+1}
\]

**Lemma 3:** If \( n < 0, k > 0 \), and \( 1 + F_{-2k} < n \leq F_{-2k+1} \) then, for some \( k, k > k > 0 \),

\[
0 \leq F_{2k+2k+2} - 2 < n - F_{-2k} \leq F_{-2k+2k+1}
\]

If \( n = F_{-2k} + 1 \),

\[
n - F_{-2k} = F_{-1}.
\]

**Proof:** If \( 1 + F_{-2k} < n \leq F_{-2k+1} \), then

\[
1 < n - F_{-2k} \leq F_{-2k+2} = F_{-2k+2}
\]

and as in the proof of Lemma 2,

\[
0 \leq F_{2k-2k+1} - 2 < n - F_{-2k} \leq F_{2k-2k} \quad \text{for some} \ k, k > k > 0;
\]

thus,

\[
0 \leq F_{-2k+2k+2} - 2 < n - F_{-2k} \leq F_{-2k+2k+1}
\]

**Lemma 4:** If \( n < 0, k > 0 \), and \( -F_{-2k-1} \leq n < F_{-2k-1} \) then, for some \( k, k > k > 0 \),

\[
-F_{-2k+2k+1} - 2 < n - F_{-2k} \leq F_{-2k+2k+1} < 0.
\]

If \( n = F_{-2k} - 1 \),

\[
n - F_{-2k} = F_{-2}.
\]

**Proof:** If \( -F_{-2k-1} \leq n < F_{-2k-1} \), then

\[
1 < F_{-2k} - n \leq F_{-2k} + F_{-2k-1} = F_{-2k+1},
\]

so

\[
1 < F_{-2k} - n \leq F_{-2k-1}
\]

and, as in the proof of Lemma 1,

\[
0 < F_{2k-2k+1} - 2 < F_{-2k} - n \leq F_{2k-2k+1} \quad \text{where} \ k > k \geq 1,
\]

i.e.,

\[
-F_{2k+2k+1} - 2 < n - F_{-2k} \leq -F_{-2k+2k+1} < 0.
\]

**Algorithm Z:** This algorithm produces, for a given integer, the promised sum of Fibonacci numbers.

1. If \( n = F_{-i} \) for some \( i \), then stop.
2. If \( n > 0 \) and for \( k > 0 \), \( F_{2k} < n < F_{2k+1} \), i.e., \( -F_{-2k} < n < F_{-2k-1} \), write \( n = F_{-2k-1} + (n - F_{-2k-1}) \), and apply this algorithm to \( n - F_{-2k-1} \), giving the next term in the sum.
3. If \( n > 0 \) and for \( k > 0 \), \( F_{2k-1} < n < F_{2k} \), i.e., \( F_{-2k+1} < n < -F_{-2k} \), write \( n = F_{-2k+1} + (n - F_{-2k+1}) \), and apply this algorithm to \( n - F_{-2k+1} \), giving the next term in the sum.
4. If \( n < 0 \) and for \( k > 0 \), \( F_{2k-1} < -n < F_{2k} \), i.e., \( F_{-2k} < n < -F_{-2k+1} \), write \( n = F_{-2k} + (n - F_{-2k}) \), and apply this algorithm to \( n - F_{-2k} \), giving the next term in the sum.
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(5) If \( n < 0 \) and for \( k > 0 \), \(-F_{2k} < -n < F_{2k+1}\), i.e., \(-F_{2k-1} < n < F_{-2k}\), write 
\( n = F_{-2k} + (n - F_{-2k}) \), and apply this algorithm to \( n - F_{-2k} \), giving the 
next term in the sum.

The algorithm terminates when, eventually,

\[ n = F_{-t_1} - F_{-t_2} - \cdots - F_{-t_m} = F_{-t_{m+1}}. \]

Lemma 5: Algorithm Z produces a representation of any nonzero integer \( n \) as a 
sum of Fibonacci numbers \( F_i \) where \( i \leq 0 \) and any two of the \( i \)'s differ by at 
least 2.

Proof: If after the application of (2), \( n - F_{-2k-1} \neq F_{-j} \) for any \( j \), we have, by 
Lemma 1:

\[-F_{-2k+2l} - 1 < n - F_{-2k-1} < F_{2k+2l} + 1 < 0, \text{ where } l > 0.\]

By applying (4) or (5), the algorithm next considers \( n - F_{-2k-1} - F_{-2k+2l} \).
If after (3), \( n - F_{-2k+1} \neq F_{-j} \), by Lemma 2:

\[ 0 < -F_{-2k+2l+2} < n - F_{-2k+1} < F_{-2k+2l}, \text{ where } l > 0.\]

By (2) or (3), the algorithm next considers \( n - F_{-2k+1} - F_{-2k+2l+1} \).
If after (4), \( n - F_{-2k} \neq F_{-j} \), by Lemma 3:

\[ 0 < -F_{-2k+2l+2} < n - F_{-2k} < F_{-2k+2l}, \text{ where } l > 0.\]

By (2) or (3) the algorithm next considers \( n - F_{-2k} - F_{-2k+2l+1} \).
If after (5), \( n - F_{-2k} \neq F_{-j} \), by Lemma 4:

\[-F_{-2k+2l+1} < n - F_{-2k} < F_{-2k+2l+1} < 0, \text{ where } l > 0.\]

By (4) and (5), the algorithm next considers \( n - F_{-2k} - F_{-2k+2l}.\)
Thus, if the first stage of the algorithm produces \( n - F_{-i} (i < 0) \), the 
second produces \( n - F_{-i} = F_{-i+p} \), where \( p \geq 2 \) and \( i + p < 0.\)
The same applies to later stages of the algorithm which therefore produces 
Fibonacci numbers with subscripts at least two apart.

The next two lemmas are required to prove the uniqueness of this representa-
tion.

Lemma 6: (i) \( \sum_{i=1}^{k} F_{-2i} = 1 - F_{-2k-1} \);

(ii) \( \sum_{i=1}^{k} F_{-2i+1} = F_{-2k} \);

(iii) \( \sum_{i=1}^{k} F_{-i} = 1 - F_{-k+1} \).

Proof: The proof is simple and is therefore omitted here.

Lemma 7: If \( i_1 > i_2 > \cdots > i_h > 0 \) and, for \( 2 < j \leq h \), \( i_{j-1} - i_{j+1} \geq 2, \)

\[-F_{-i_1+1} \leq \sum_{k=1}^{h} F_{-i_k} \leq -F_{-i_1} - 1 \text{ if } i_1 \text{ is odd, and} \]

\[-F_{-i_1} - 1 \leq \sum_{k=1}^{h} F_{-i_k} \leq -F_{-i_1} + 1 \text{ if } i_1 \text{ is even.} \]

Proof: If \( i_1 \) is odd, by Lemma 6:

\[ F_{-i_1} + F_{-i_1+3} + F_{-i_1+5} + \cdots + F_{-2} \leq \sum_{k=1}^{h} F_{-i_k} \leq F_{-i_1} + F_{-i_1+2} + \cdots + F_{-1} \]

\[ F_{-i_1} + 1 - F_{-i_1+2} \leq \sum_{k=1}^{h} F_{-i_k} \leq -F_{-i_1} - 1, \]
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so,

\[-F_{-i_1 + 1} < 1 - F_{-i_1 + 1} \leq \sum_{k=1}^{h} F_{-i_k} < -F_{-i_1 - 1}.\]

If \(i_1\) is even, by Lemma 6:

\[F_{-i_1} + F_{-i_1 + 2} + \cdots + F_{-2} \leq \sum_{k=1}^{h} F_{-i_k} \leq F_{-i_1} + F_{-i_1 + 3} + \cdots + F_{-3} + F_{-1}\]

\[1 - F_{-i_1 - 1} \leq \sum_{k=1}^{h} F_{-i_k} \leq -F_{-i_1 + 1}.\]

**Theorem 1**: Algorithm Z expresses every integer \(n\) as a unique sum of a minimal number of distinct Fibonacci numbers \(F_i\), where \(i \leq 0\).

**Proof**: If \(n = 0\), \(n = F_0\).

If \(n \neq 0\), by Lemma 5 the algorithm produces a sum of the form

\[n = \sum_{k=1}^{h} F_{-i_k}, \text{ where } i_k \geq i_{k+1} + 2.\]

If the representation were not unique or not minimal, we would also have

\[n = \sum_{k=1}^{m} F_{-j_k}, \text{ where } j_k \geq j_{k+1} + 2, \text{ and possibly } m < h.\]

Let \(-i_{p}\) and \(-j_{p}\) be the first of these subscripts, if any, that are distinct and assume \(i_{p} > j_{p}\). Then

\[n - F_{-i_1} - \cdots - F_{-(i_{p} - 1)} = \sum_{k=p}^{h} F_{-i_k} = \sum_{k=p}^{m} F_{-j_k}.\]

If \(i_{p}\) and \(j_{p}\) are odd, then, by Lemma 7,

\[\sum_{k=p}^{h} F_{-i_k} > -F_{-i_p + 1} \text{ and } -F_{-j_p - 1} \geq \sum_{k=p}^{m} F_{-j_k}.\]

Also, \(i_{p} - 2 \geq j_{p}\), and so \(-F_{-i_p + 1} \geq -F_{-j_p - 1}\), which is impossible.

If \(i_{p}\) is odd and \(j_{p}\) is even, then

\[\sum_{k=p}^{h} F_{-i_k} \text{ is positive and } \sum_{k=p}^{m} F_{-j_k} \text{ is negative}\]

by Lemma 7.

Similarly, if \(i_{p}\) is even and \(j_{p}\) is odd, then

\[\sum_{k=p}^{h} F_{-i_k} \text{ is negative and } \sum_{k=p}^{m} F_{-j_k} \text{ is positive}\]

by Lemma 7.

If \(i_{p}\) and \(j_{p}\) are both even, then \(i_{p} - 2 \geq j_{p}\), and by Lemma 7,

\[\sum_{k=p}^{h} F_{-i_k} \leq -F_{i_p} + 1 \text{ and } -F_{i_p} - 1 < \sum_{k=p}^{m} F_{-j_k}\]

and also

\[F_{i_p} + 1 \leq F_{-j_{p} - 1},\]

which is impossible.

Thus, for \(1 \leq k \leq m\), \(i_k = j_k\).

If \(m < h\), we have by the above:

\[n = \sum_{k=1}^{m} F_{-i_k} = \sum_{k=1}^{h} F_{-i_k},\]

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so
\[ \sum_{k=m+1}^{h} F_{-i_k} = 0. \]

If \( h > m + 1 \), then by Lemma 7, if \( i_{m+1} \) is odd, \( -F_{-i_{m+1}} + 1 \) < 0, and if \( i_{m+1} \) is even, then \( 0 \leq -F_{-i_{m+1}} + 1 \), both of which are impossible.

If \( h = m + 1 \), then \( F_{-i_h} = 0 \), which is impossible because \( i_h \neq 0 \).

Therefore, the representation of \( n \) is unique and minimal.

As any representation of a number \( n \) as a sum of Fibonacci numbers
\[ \sum_{k=1}^{h} F_{-i_k}, \text{ where } i_1 > i_2 > \cdots > i_h > 0, \]
can be changed to
\[ \sum_{k=1}^{h-1} F_{-i_k} + F_{-i_h-1} + F_{-i_h-2}, \]
it is clear that there can be no maximal number of Fibonacci numbers in a given sum.

References


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Author and Title Index for
The Fibonacci Quarterly

Currently, Dr. Charles K. Cook of the University of South Carolina at Sumter is working on an AUTHOR index, TITLE index and PROBLEM index for The Fibonacci Quarterly. In fact, the three indices are already completed. We hope to publish these indices in 1993 which is the 30th anniversary of The Fibonacci Quarterly. Dr. Cook and I feel that it would be very helpful if the publication of the indices also had AMS classification numbers for all articles published in The Fibonacci Quarterly. We would deeply appreciate it if all authors of articles published in The Fibonacci Quarterly would take a few minutes of their time and send a list of articles with primary and secondary classification numbers to

PROFESSOR CHARLES K. COOK
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH CAROLINA AT SUMTER
1 LOUISE CIRCLE
SUMTER, S.C. 29150

The Editor
Gerald E. Bergum

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