University of Wollongong

Research Online

Faculty of Engineering and Information
Sciences - Papers: Part A

Faculty of Engineering and Information
Sciences

1-1-1993

Systems of illative combinatory logic complete for first order propositional and predicate calculus

Henk Barendregt
Catholic University Nijmegen

Martin Bunder
University of Wollongong, mbunder@uow.edu.au

Wil Dekkers
Catholic University Nijmegen

Follow this and additional works at: <https://ro.uow.edu.au/eispapers>

Part of the [Engineering Commons](#), and the [Science and Technology Studies Commons](#)

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au

Systems of illative combinatory logic complete for first order propositional and predicate calculus

Abstract

Illative combinatory logic consists of the theory of combinators or lambda calculus extended by extra constants (and corresponding axioms and rules) intended to capture inference. The paper considers systems of illative combinatory logic that are sound for first-order propositional and predicate calculus. The interpretation from ordinary logic into the illative systems can be done in two ways: following the propositions-as-types paradigm, in which derivations become combinators or, in a more direct way, in which derivations are not translated. Both translations are closely related in a canonical way. The two direct translations turn out to be complete. The paper fulfills the program of Church [1932], [1933] and Curry [1930] to base logic on a consistent system of A-terms or combinators. Hitherto this program had failed because systems of ICL were either too weak (to provide a sound interpretation) or too strong (sometimes even inconsistent).

Keywords

propositional, order, first, calculus, complete, predicate, logic, combinatory, illative, systems

Disciplines

Engineering | Science and Technology Studies

Publication Details

Barendregt, H., Bunder, M. & Dekkers, W. (1993). Systems of illative combinatory logic complete for first order propositional and predicate calculus. *The Journal of Symbolic Logic*, 58 (3), 769-788.

Systems of Illative Combinatory Logic Complete for First-Order Propositional and Predicate Calculus

Author(s): Henk Barendregt, Martin Bunder and Wil Dekkers

Source: *The Journal of Symbolic Logic*, Vol. 58, No. 3 (Sep., 1993), pp. 769-788

Published by: [Association for Symbolic Logic](#)

Stable URL: <http://www.jstor.org/stable/2275096>

Accessed: 29/01/2014 20:09

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Association for Symbolic Logic is collaborating with JSTOR to digitize, preserve and extend access to *The Journal of Symbolic Logic*.

<http://www.jstor.org>

SYSTEMS OF ILLATIVE COMBINATORY LOGIC COMPLETE FOR FIRST-ORDER PROPOSITIONAL AND PREDICATE CALCULUS

HENK BARENDREGT, MARTIN BUNDER, AND WIL DEKKERS

Abstract. Illative combinatory logic consists of the theory of combinators or lambda calculus extended by extra constants (and corresponding axioms and rules) intended to capture inference. The paper considers systems of illative combinatory logic that are sound for first-order propositional and predicate calculus. The interpretation from ordinary logic into the illative systems can be done in two ways: following the propositions-as-types paradigm, in which derivations become combinators or, in a more direct way, in which derivations are not translated. Both translations are closely related in a canonical way. The two direct translations turn out to be complete. The paper fulfills the program of Church [1932], [1933] and Curry [1930] to base logic on a consistent system of λ -terms or combinators. Hitherto this program had failed because systems of ICL were either too weak (to provide a sound interpretation) or too strong (sometimes even inconsistent).

§1. Introduction. The theory of combinators (Curry et al. [1958], [1972]) and the lambda calculus (Church [1941], Barendregt [1984]) are theories that successfully analyze the notion of effective computability. However, the original founders of these subjects, Curry and Church, also had aimed to provide a basis for logic (and thereby mathematics). Formal systems intended to achieve this are given in Church [1932], [1933] and Curry [1930], [1931], [1932], [1933], [1934a], [1934b], [1935]. Unfortunately, it was shown in Kleene and Rosser [1935] that these systems are inconsistent. In Curry [1942c] the inconsistency of Curry [1934] was simplified. This derivation, now known as “Curry’s paradox”, is akin to the Russell paradox but requires no properties of negation. It can be written in only a few lines.

Curry and his school then started a program of defining several systems of illative combinatory logic (ICL) of varying strength, see Curry [1942a]. The goal was to “find stronger and stronger systems which are consistent and weaker and weaker systems which are inconsistent but strong enough to interpret logic, hoping to end up with a consistent system in which logic can be interpreted” (quotation from Curry and Feys [1958; §8S3, p. 276]).

Following this methodology, Bunder [1969], [1973], [1974] introduced restrictions on the rules of the illative constants so that first-order propositional and

Received May 5, 1992.

This research was supported by the Australian Research Council Grant A 68930230.

© 1993, Association for Symbolic Logic
0022-4812/93/5803-0001/\$03.00

predicate calculus can be interpreted in the resulting systems. Bunder [1983a] also allows much of set theory. In all these systems the usual derivation of Curry's paradox is blocked, but the consistency of these systems remains an open question. That the question is not academic was shown in Bunder [1976] and [1983a], where related illative systems were proved to be inconsistent.

In the rest of this section we give a short introduction to illative combinatory logic by showing the early inconsistent system of Curry [1934]. In §2 we introduce systems slightly weaker than the ones in Bunder [1973], [1974] but strong enough to interpret logic. We derive roughly the following soundness result

$$\Delta \vdash_L A \Rightarrow [\Delta] \vdash_C [A],$$

where L represents propositional or predicate logic and $[-]$ one of two possible translations of each system into an ICL system C (there will then be 4 such C 's). Of the interpretations one is the propositions-as-types interpretation due to Curry, Howard, and de Bruijn; the other is a more direct interpretation. Finally, in §2 we show that the two interpretations are canonically related.

In §3 we derive completeness results for 2 of the 4 systems of ICL. These, again roughly, take the following form

$$[\Delta] \vdash_C [A] \Rightarrow \Delta \vdash_L A.$$

This completeness result implies the consistency of the ICL's involved.

Illative combinatory logic. Now we will present a simple system $\mathcal{I}$ of illative combinatory logic in order to explain the general idea. The system is strong enough to represent the $\{\supset, \forall\}$ fragment of first-order intuitionistic predicate calculus.

The intuition behind the system $\mathcal{I}$ is as follows. Terms are type-free lambda terms extended by some extra constants. A term X is considered to have an assertive value. A term XZ can be seen as a statement saying “ Z is of type X ” or “ $Z \in X$ ” or “ Z satisfies the predicate X ”. The term “ $\lambda\zeta.X$ ” corresponds to the class $\{\zeta \mid X\}$. There is a term Ξ such that the statement ‘ ΞXY ’ is interpreted as “ $X \subseteq Y$ ” or “ $(\forall x \in X) Yx$ ”. Using this Ξ one can define implication and quantification.

1.1. DEFINITION. The system $\mathcal{I}$ is defined as follows.

(i) T , the set of *terms* of $\mathcal{I}$, is given by the following abstract grammar:

$$T = V \mid \Xi \mid TT \mid \lambda V.T.$$

Here V is the syntactical category of variables and Ξ is a constant. We also write

$$T = \Lambda(\Xi),$$

since T is obtained from the set Λ of type-free lambda terms by adding the constant Ξ .

(ii) On T the usual notion of $\beta\eta$ -reduction is given by the contraction rules

$$\begin{aligned} (\lambda x.M)N &\rightarrow M[x := N], \\ \lambda x.Mx &\rightarrow M \quad \text{if } x \notin \text{FV}(M). \end{aligned}$$

Here $\text{FV}(M)$ is the set of free variables of M . The resulting (more step) $\beta\eta$ -reduction and $\beta\eta$ -convertibility relation are denoted by $\rightarrow$ and $=$. Syntactic equality is denoted by $\equiv$.

- (iii) A statement of $\mathcal{J}$ is just an element of T . A basis is a set of statements.
 (iv) Let Γ be a basis, and let X be a statement; then X is *derivable* from Γ , notation $\Gamma \vdash X$, if $\Gamma \vdash X$ can be produced by the natural deduction system in Table 1.

TABLE 1

$X \in \Gamma \Rightarrow \Gamma \vdash X,$
$\Gamma \vdash X, X = Y \Rightarrow \Gamma \vdash Y,$
$\Gamma \vdash \mathbf{E}XY, \Gamma \vdash XZ \Rightarrow \Gamma \vdash YZ,$
$\Gamma, Xx \vdash Yx, x \notin \text{FV}(\Gamma, X, Y) \Rightarrow \Gamma \vdash \mathbf{E}XY.$

In the last rule x is some variable. The system is based on $\beta\eta$ -conversion. Therefore, this last rule could be replaced by

$$\Gamma, Xx \vdash Y, x \notin \text{FV}(\Gamma) \Rightarrow \Gamma \vdash \mathbf{E}(\lambda x.X)(\lambda x.Y).$$

1.2. DEFINITION. For $X, Y \in T$ write

- (i) $X \supset Y \equiv \mathbf{E}(\mathbf{K}X)(\mathbf{K}Y),$
 (ii) $\forall u \in X.Y \equiv \mathbf{E}X(\lambda u.Y).$

1.3. PROPOSITION. *The following holds for the system $\mathcal{J}$.*

- (i) $\Gamma \vdash X \supset Y, \Gamma \vdash X \Rightarrow \Gamma \vdash Y.$
 (ii) $\Gamma, X \vdash Y \Rightarrow \Gamma \vdash X \supset Y.$
 (iii) $\Gamma \vdash \forall u \in X.Y, \Gamma \vdash Xt \Rightarrow \Gamma \vdash Y[u := t].$
 (iv) $\Gamma, Xu \vdash Y, u \notin \text{FV}(\Gamma, X) \Rightarrow \Gamma \vdash \forall u \in X.Y.$

Now it is possible to interpret the $\{\supset, \forall\}$ fragment of first-order intuitionistic predicate logic into $\mathcal{J}$. For example, a sentence like

$$\forall x(Rx \supset Rx)$$

holding in a universe A is translated as the statement

$$(\forall x \in A.Rx \supset Rx)$$

which is $\mathbf{E}A(\lambda x.\mathbf{E}(\mathbf{K}(Rx))(\mathbf{K}(Rx)))$ and is provable in $\mathcal{J}$.

Unfortunately, the interpretation is not complete (i.e., if the translation of a formula φ is provable in $\mathcal{J}$, then φ itself is provable in logic) because the system $\mathcal{J}$ is not consistent; every statement X (i.e., every term) can be derived in $\mathcal{J}$ (from the empty basis).

1.4. PROPOSITION (Curry's paradox). *Let X be a statement of $\mathcal{J}$. Then $\vdash X$.*

PROOF. Let X be given. Take

$$Y \equiv (\lambda y.(yy) \supset X)(\lambda y.(yy) \supset X).$$

Then $Y = Y \supset X$. Therefore, the following derivation shows that $\vdash X$.

$$\begin{aligned} Y &\vdash Y; \\ Y &\vdash Y \supset X, \quad \text{since } Y = Y \supset X; \\ Y &\vdash X, \quad \text{by 1.3(i);} \end{aligned}$$

$$\begin{aligned}
&\vdash Y \supset X, && \text{by 1.3(ii);} \\
&\vdash Y, && \text{since } Y \supset X = Y; \\
&\vdash X, && \text{by 1.3(i).} \quad \square
\end{aligned}$$

Note that the derivation of X is related to the proof of the theorem of Löb [1955].

In the next section the illative system $\mathcal{I}$ will be formulated more carefully so that the system becomes consistent and in fact complete over ordinary logic.

§2. Sound interpretations of logics in ICL's. In the introduction we stated that logic can be interpreted in two ways in ICL's. In fact, this can be done both for the propositional and predicate calculus, so there will be four related illative systems. The interpretation will be done for the $\{\supset\}$ (respectively $\{\supset, \forall\}$) fragment of intuitionistic logic. This is the most essential part of logic and the direct interpretation ($[-]$ below) can be extended to include the logical operators $\neg$, $\&$, $\vee$, and $\exists$. In second-order logic these operators are definable from $\supset$, $\forall$, so both our interpretations can be extended into sound (and probably complete) interpretations of second-order logical calculi.

Now we display the two logical calculi that will be interpreted.

2.1. DEFINITION. Let **PROP** be the $\supset$ fragment of intuitionistic propositional logic determined as follows.

(i) The set of formulas of **PROP**, notation $\mathbb{F}_{\text{PROP}}$, is defined by the following abstract syntax:

$$\mathbb{F}_{\text{PROP}} = \mathbb{V} \mid \mathbb{F}_{\text{PROP}} \supset \mathbb{F}_{\text{PROP}}.$$

Here $\mathbb{V}$ is a set of propositional variables.

(ii) Let $\Gamma \subseteq \mathbb{F}_{\text{PROP}}$ and $\varphi \in \mathbb{F}_{\text{PROP}}$. Then $\Gamma \vdash_{\text{PROP}} \varphi$ is defined by the system of natural deduction in Table 2.

TABLE 2 **PROP**.

$ \begin{aligned} &\varphi \in \Gamma \Rightarrow \Gamma \vdash \varphi; \\ &\Gamma \vdash \varphi \supset \psi, \Gamma \vdash \varphi \Rightarrow \Gamma \vdash \psi; \\ &\Gamma, \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \supset \psi. \end{aligned} $
--

2.2. DEFINITION. Let **PRED** be the $\{\supset, \forall\}$ fragment of first-order many-sorted intuitionistic predicate calculus of a given signature s .

Below as an example, we will treat a version of **PRED** with s the signature of the structure

$$\langle \mathbf{A}_1, \mathbf{A}_2, \mathbf{f}, \mathbf{g}, \mathbf{P}, \mathbf{a} \rangle$$

with

$$\begin{aligned}
&\mathbf{A}_1, \mathbf{A}_2 \quad \text{nonempty sets;} \\
&\mathbf{f}: \mathbf{A}_1 \rightarrow \mathbf{A}_1 \quad \text{a unary function;}
\end{aligned}$$

$$\begin{aligned} g: A_1 \rightarrow A_2 \rightarrow A_1 & \text{ a binary function;} \\ P \subseteq A_1 & \text{ a unary relation;} \\ a \in A_1 & \text{ a constant.} \end{aligned}$$

(All results also hold for arbitrary signatures.)

(i) The set of terms of **PRED**, notation $\mathbb{T}_{\text{PRED}}$, is defined by the following abstract syntax:

$$\begin{aligned} \mathbb{T}_{\text{PRED}} &= \mathbb{T}_{A_1} \mid \mathbb{T}_{A_2}, \\ \mathbb{T}_{A_1} &= \mathbb{V}^{A_1} \mid a \mid f \mathbb{T}_{A_1} \mid g \mathbb{T}_{A_1} \mathbb{T}_{A_2}, \\ \mathbb{T}_{A_2} &= \mathbb{V}^{A_2}. \end{aligned}$$

(ii) The set of formulas of **PRED**, notation $\mathbb{F}_{\text{PRED}}$, is defined by the following abstract syntax:

$$\mathbb{F}_{\text{PRED}} = P \mathbb{T}_{A_1} \mid \mathbb{F}_{\text{PRED}} \supset \mathbb{F}_{\text{PRED}} \mid \forall \mathbb{V}^{A_i} \mathbb{F}_{\text{PRED}}.$$

(iii) $\Gamma \vdash_{\text{PRED}} \varphi$ is axiomatised by the system of natural deduction in Table 3.

TABLE 3 **PRED**.

$\varphi \in \Gamma \Rightarrow \Gamma \vdash \varphi;$
$\Gamma \vdash \varphi \supset \psi, \Gamma \vdash \varphi \Rightarrow \Gamma \vdash \psi;$
$\Gamma, \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \supset \psi;$
$\Gamma \vdash \forall x^{A_i} \varphi, t \in \mathbb{T}_{A_i} \Rightarrow \Gamma \vdash \varphi[x^{A_i} := t];$
$\Gamma \vdash \varphi, x^{A_i} \notin \text{FV}(\Gamma) \Rightarrow \Gamma \vdash \forall x^{A_i} \varphi.$

Now the systems **PROP** and **PRED** will be interpreted in ICL's. In order to block the proof of the Curry paradox, Bunder [1969], [1973], [1974] modified the system $\mathcal{I}$ by restricting the $\mathcal{E}$ -introduction rule and adding some other axioms and a rule. The resulting system $\mathcal{I}_0$ was strong enough to provide sound interpretations of **PROP** and **PRED**, while the proof of the Curry paradox was blocked. However, the problems of the completeness of the interpretation and even of the consistency of $\mathcal{I}_0$ remained open. (The system $\mathcal{I}_0$ will be described later.)

We will give modified versions of $\mathcal{I}_0$ in which the logics can be embedded in a sound way by two kinds of embeddings. The first kind is “direct”, and the second kind is according to the “propositions-as-types” and “proofs-as-terms” paradigm, see Barendregt [1992; §5.1, §5.4]. As there are two logical systems, **PROP** and **PRED**, there will be four systems of ICL. These systems are called $\mathcal{I}\mathbf{P}$, $\mathcal{I}\mathbf{E}$, $\mathcal{I}\mathbf{F}$, and $\mathcal{I}\mathbf{G}$ respectively. Their use for the two kinds of interpretation is as follows. Let $[]^1$ be the direct and $[]^2$ the propositions-as-types translation. Then Table 4 (see next page) shows the systems of ICL that are used for the two translations of **PROP** and **PRED**.

TABLE 4

	$[]^1$	$[]^2$
PROP	$\mathcal{I}\mathbf{P}$	$\mathcal{I}\mathbf{F}$
PRED	$\mathcal{I}\mathbf{E}$	$\mathcal{I}\mathbf{G}$

For example

$$[]^2: \text{PRED} \rightarrow \mathcal{I}\mathbf{G}.$$

The four systems ICL will be described now, and moreover, their relative strengths will be compared.

2.3. DEFINITION. Let $T = \Lambda(\mathbf{\Xi}, \mathbf{L})$ be the set of type-free lambda terms extended by the extra constants $\mathbf{\Xi}$ and $\mathbf{L}$.

(i) Define the following terms in T .

$$\mathbf{P} \equiv \lambda xy. \mathbf{\Xi}(\mathbf{K}x)(\mathbf{K}y),$$

$$\mathbf{F} \equiv \lambda xyz. \mathbf{\Xi}x(y \circ z),$$

$$\mathbf{G} \equiv \lambda xyz. \mathbf{\Xi}x(\mathbf{S}yz),$$

$$\mathbf{H} \equiv \mathbf{L} \circ \mathbf{K},$$

where $\mathbf{K} \equiv \lambda pq.p$, $M \circ N \equiv \lambda x.M(Nx)$, and $\mathbf{S} \equiv \lambda pqr.pr(qr)$.

Write $X \supset Y$ for $\mathbf{P}XY$.

(ii) Define the following four systems of illative combinatory logic $\mathcal{I}\mathbf{P}$, $\mathcal{I}\mathbf{E}$, $\mathcal{I}\mathbf{F}$, and $\mathcal{I}\mathbf{G}$. All four systems have as rules those given in Table 5.

TABLE 5 All systems.

$X \in \Gamma \Rightarrow \Gamma \vdash X;$
$\Gamma \vdash X, X =_{\beta\eta} Y \Rightarrow \Gamma \vdash Y.$

The four systems have the specific rules given in Tables 6–9.

TABLE 6 $\mathcal{I}\mathbf{P}$.

P_e	$\Gamma \vdash X \supset Y, \Gamma \vdash X \Rightarrow \Gamma \vdash Y;$
P_i	$\Gamma, Xx \vdash Y, \Gamma \vdash \mathbf{H}X \Rightarrow \Gamma \vdash X \supset Y;$
P_H	$\Gamma, Xx \vdash \mathbf{H}Y, \Gamma \vdash \mathbf{H}X \Rightarrow \Gamma \vdash \mathbf{H}(X \supset Y).$

TABLE 7 $\mathcal{I}\mathbf{E}$.

E_e	$\Gamma \vdash \mathbf{\Xi}XY, \Gamma \vdash XV \Rightarrow \Gamma \vdash YV;$
E_i	$\Gamma, Xx \vdash Yx, \Gamma \vdash \mathbf{L}X, x \notin \text{FV}(\Gamma, X, Y) \Rightarrow \Gamma \vdash \mathbf{\Xi}XY;$
E_H	$\Gamma, Xx \vdash \mathbf{H}(Yx), \Gamma \vdash \mathbf{L}X, x \notin \text{FV}(\Gamma, X, Y) \Rightarrow \Gamma \vdash \mathbf{H}(\mathbf{\Xi}XY).$

TABLE 8 $\mathcal{IF}$.

F_e	$\Gamma \vdash \mathbf{F}XYZ, \Gamma \vdash XV \Rightarrow \Gamma \vdash Y(ZV);$
F_i	$\Gamma, Xx \vdash Y(Zx), \Gamma \vdash \mathbf{L}X, x \notin \text{FV}(\Gamma, X, Y, Z) \Rightarrow \Gamma \vdash \mathbf{F}XYZ;$
F_L	$\Gamma, Xx \vdash \mathbf{L}Y, \Gamma \vdash \mathbf{L}X, x \notin \text{FV}(\Gamma, X, Y) \Rightarrow \Gamma \vdash \mathbf{L}(\mathbf{F}XY).$

TABLE 9 $\mathcal{IG}$.

G_e	$\Gamma \vdash \mathbf{G}XYZ, \Gamma \vdash XV \Rightarrow \Gamma \vdash YV(ZV);$
G_i	$\Gamma, Xx \vdash Yx(Zx), \Gamma \vdash \mathbf{L}X, x \notin \text{FV}(\Gamma, X, Y, Z) \Rightarrow \Gamma \vdash \mathbf{G}XYZ;$
G_L	$\Gamma, Xx \vdash \mathbf{L}(Yx), \Gamma \vdash \mathbf{L}X, x \notin \text{FV}(\Gamma, X, Y) \Rightarrow \Gamma \vdash \mathbf{L}(\mathbf{G}XY).$

To get a taste for what will follow, we give some examples of interpretations of tautologies in the ICL's.

2.4. EXAMPLES. (i) The formula $p \supset p$ of PROP is translated as $p \supset p$ in $\mathcal{IP}$. The fact that $p \supset p$ is indeed a wff of PROP is expressed in $\mathcal{IP}$ as $\mathbf{H}p \vdash \mathbf{H}(p \supset p)$, which should be interpreted as “if p is a proposition, then so is $p \supset p$ ”. So $\mathbf{H}$ functions as the class of propositions. It was used in Curry [1942a], Bunder [1969], and others to block the derivation of Curry's paradox. Aczel [1980] uses $\mathbf{H}$ as in Bunder [1969] and in $\mathcal{IP}$.

The fact that $p \supset p$ is derivable in PROP is interpreted in $\mathcal{IP}$ as $\mathbf{H}p \vdash p \supset p$. This should be interpreted as “if p is a proposition, then $p \supset p$ is derivable”.

(ii) The same formula $p \supset p$ is translated in $\mathcal{IF}$ as $\mathbf{F}pp$. The fact that $p \supset p$ is a wff of PROP is expressed in $\mathcal{IF}$ by $\mathbf{L}p \vdash \mathbf{L}(\mathbf{F}pp)$ which should be interpreted as “if p is a type, then $\mathbf{F}pp$ is a type”. The type $\mathbf{F}pp$ is intuitively the function space type $p \rightarrow p$. The fact that $p \supset p$ is derivable in PROP is interpreted in $\mathcal{IF}$ by making the type $\mathbf{F}pp$ “inhabited” by the expression $\lambda y.y$ (formulas-as-types and terms-as-derivations interpretation)

$$\mathbf{L}p \vdash \mathbf{F}pp(\lambda y.y).$$

(iii) Similarly, consider the formula $\forall x^A(\mathbf{P}x \supset \mathbf{P}x)$ of PRED. Interpreted in $\mathcal{IE}$ this becomes

$$\mathbf{L}A, \mathbf{F}A\mathbf{H}P \vdash \mathbf{H}(\mathbf{E}A(\lambda x.Px \supset Px)).$$

The intuitive meaning of $\mathbf{F}A\mathbf{H}P$ is “ P is of type $A \rightarrow \mathbf{H}$ ”, that is, P is a map from A into the propositions and, hence, a predicate on A . (In generalised type systems the basis $\mathbf{L}A, \mathbf{F}A\mathbf{H}P$ would be written as the context $A : *^s, P : A \rightarrow *^p$, see Barendregt [1992], especially the systems λPRED and λP .) The fact that $\forall x^A(\mathbf{P}x \supset \mathbf{P}x)$ is derivable in PRED becomes

$$\mathbf{L}A, \mathbf{F}A\mathbf{H}P \vdash \mathbf{E}A(\lambda x.Px \supset Px),$$

which is derivable in $\mathcal{IE}$.

(iv) The formula $\forall x^A(\mathbf{P}x \supset \mathbf{P}x)$ in PRED translated in $\mathcal{IG}$ becomes

$$\mathbf{L}A, \mathbf{F}A\mathbf{L}P \vdash \mathbf{L}(\mathbf{G}A(\lambda x.Px \supset Px)),$$

which should be interpreted as “if A is a type and P is in $A \rightarrow \mathbf{L}$, then $\mathbf{GA}(\lambda x.Px \supset Px)$ is a type”. In the PTS language of $\lambda\mathbf{P}$ this is

$$A : *, P : A \rightarrow * \vdash (\prod x : A. Px \rightarrow Px) : *.$$

The fact that $\forall x^A (Px \supset Px)$ is a tautology is interpreted in $\mathcal{J}\mathbf{G}$ by the inhabitation of the type $\mathbf{GA}(\lambda x.Px \supset Px)$.

$$\mathbf{LA}, \mathbf{FALP} \vdash \mathbf{GA}(\lambda x.Px \supset Px)(\lambda x.\lambda y.y).$$

2.5. Notes. (i) $\mathcal{J}_0$ is essentially $\mathcal{J}\mathbf{E}$ plus the following:

$$\vdash \mathbf{EIH}, \quad \vdash \mathbf{LA}_1, \quad \vdash \mathbf{LA}_2, \quad \text{and} \quad \vdash \mathbf{LH}.$$

By the axiom “ $\vdash \mathbf{LH}$ ” one can interpret second-order propositional and predicate logic. For example, by rule $\mathbf{E}_i$ one gets

$$\vdash \mathbf{EH}(\lambda p.p \rightarrow p) \quad (\equiv \forall p \in \mathbf{H}. p \rightarrow p).$$

So one can quantify over propositions. By “ $\vdash \mathbf{EIH}$ ” one can derive $p \vdash \mathbf{Hp}$ and even $p \vdash \mathbf{H(Hp)}$. Here one has a mixture of statements in the language and in the meta-language. This gives problems in the study of completeness and maybe it is better to skip this axiom “ $\vdash \mathbf{EIH}$ ”, as we have done in the systems $\mathcal{J}\mathbf{P}$, $\mathcal{J}\mathbf{E}$, $\mathcal{J}\mathbf{F}$, and $\mathcal{J}\mathbf{G}$.

(ii) In the ICL’s not only tautologies can be derived but also so-called syntactical conditions. For example, in signature s one has $\mathbf{f(a)} \in \mathbb{T}_{A_1}$ and $\mathbf{P(f(a))} \in \mathbb{F}_{\text{PRED}}$. These translate as $\mathbf{FA}_1 A_1 f, A_1 a \vdash A_1(fa)$ and $\mathbf{FA}_1 A_1 f, A_1 a, \mathbf{FA}_1 \mathbf{HP} \vdash \mathbf{H(P(fa))}$, respectively.

The following lemma is useful for determining the relative strength of the four systems.

2.6. LEMMA. For all $X, Y \in \mathbf{T}$ one has the following in $\Lambda(\mathbf{E}, \mathbf{L})$:

$$(i) \quad \mathbf{F(KX)(KY)} = \mathbf{K(PXY)},$$

$$(ii) \quad \mathbf{GX(K \circ Y)} = \mathbf{K(EXY)},$$

$$(iii) \quad \mathbf{FXY} = \mathbf{GX(KY)}.$$

PROOF. (i) $\mathbf{F(KX)(KY)} = \lambda z. \mathbf{E(KX)((KY) \circ z)} = \lambda z. \mathbf{E(KX)(KY)} = \mathbf{K(PXY)}$, since $(\mathbf{KY}) \circ z = \lambda x. \mathbf{KY}(zx) = \lambda x. Y = \mathbf{KY}$.

(ii) $\mathbf{GX(K \circ Y)} = \lambda z. \mathbf{EX(S(K \circ Y)z)} = \lambda z. \mathbf{EXY} = \mathbf{K(EXY)}$, since $\mathbf{S(K \circ Y)z} = \lambda c. (\mathbf{K \circ Y})c(zc) = \lambda c. Yc = Y$.

(iii) $\mathbf{FXY} = \lambda z. \mathbf{EX(Y \circ z)} = \lambda z. \mathbf{EX(S(KY)z)} = \mathbf{GX(KY)}$, since $\mathbf{S(KY)z} = \lambda v. \mathbf{KY}v(zv) = \lambda v. Y(zv) = Y \circ z$. $\square$

2.7. PROPOSITION. The systems $\mathcal{J}\mathbf{P}$, $\mathcal{J}\mathbf{E}$, $\mathcal{J}\mathbf{F}$, and $\mathcal{J}\mathbf{G}$ are related as follows:

where $\rightarrow$ denotes nondecreasing strength, i.e., $s_1 \rightarrow s_2$ means that for all Γ, X

$$\Gamma \vdash_{s_1} X \Rightarrow \Gamma \vdash_{s_2} X.$$

PROOF. We will show that if $s_1 \rightarrow s_2$ in the diagram then every rule of s_1 can be derived in s_2 .

(i) Case $\mathcal{I}\mathbf{P} \rightarrow \mathcal{I}\mathbf{E}$. The rules P_e , P_i , and P_h follow from respectively Ξ_e , Ξ_i , and Ξ_h by the substitutions of $\mathbf{K}X$ for X and $\mathbf{K}Y$ for Y .

(ii) Case $\mathcal{I}\mathbf{P} \rightarrow \mathcal{I}\mathbf{F}$. Then P_e , P_i , and P_h follow from F_e , F_i , and F_L by the substitutions of $\mathbf{K}X$ for X and $\mathbf{K}Y$ for Y and Lemma 2.6(i).

(iii) Case $\mathcal{I}\mathbf{F} \rightarrow \mathcal{I}\mathbf{G}$. Now F_e , F_i , and F_L follow from G_e , G_i , and G_L by the substitution of $\mathbf{K}Y$ for Y and Lemma 2.6(iii).

(iv) Case $\mathcal{I}\mathbf{E} \rightarrow \mathcal{I}\mathbf{G}$. Then Ξ_e , Ξ_i , and Ξ_h follow from G_e , G_i , and G_L by the substitution of $\mathbf{K} \circ Y$ for Y and Lemma 2.6(ii). $\square$

Now we will show formally how the logics PROP and PRED can be interpreted in the illative systems. We start with PROP.

2.8. DEFINITION. Let r be a closed term in $\mathcal{A}(\mathcal{E}, \mathbf{L})$. Two maps (for $i = 1, 2$)

$$[-]_r^i: \mathbb{F}_{\text{PROP}} \rightarrow \mathcal{A}(\mathcal{E}, \mathbf{L})$$

and two maps

$$\Gamma_r^i: \mathbb{F}_{\text{PROP}} \rightarrow \text{illative contexts}$$

are defined by Table 10. (Note that these illative contexts are effectively grammatical conditions on the variables (propositional, individual) that appear in a proposition.)

TABLE 10

φ	$[\varphi]_r^1$	$\Gamma_r^1(\varphi)$	$[\varphi]_r^2$	$\Gamma_r^2(\varphi)$
p	rp	$\mathbf{H}(rp)$	rp	$\mathbf{L}(rp)$
$\psi \supset \chi$	$[\psi]_r^1 \supset [\chi]_r^1$	$\Gamma_r^1(\psi), \Gamma_r^1(\chi)$	$\mathbf{F}[\psi]_r^2 [\chi]_r^2$	$\Gamma_r^2(\psi), \Gamma_r^2(\chi)$

The r in the above definition and in 2.12 can be replaced by $\mathbf{I}$ (i.e., omitted). However, in 2.15 we use it to derive a relation between the two interpretations.

2.9. LEMMA. Let $\varphi \in \mathbb{F}_{\text{PROP}}$ and let $\{p_1, \dots, p_n\}$ be the set of (free) propositional variables in φ . Then

(i) $\Gamma_r^1(\varphi) = \{\mathbf{H}(rp_1), \dots, \mathbf{H}(rp_n)\}$.

(ii) $\Gamma_r^2(\varphi) = \{\mathbf{L}(rp_1), \dots, \mathbf{L}(rp_n)\}$.

(iii) $\mathbf{H}\varphi, \mathbf{H}\psi \vdash_{\mathcal{I}\mathbf{P}} \mathbf{H}(\varphi \supset \psi)$.

(iv) $\mathbf{L}\varphi, \mathbf{L}\psi \vdash_{\mathcal{I}\mathbf{F}} \mathbf{L}(\mathbf{F}\varphi\psi)$.

(v) $\Gamma_r^1(\varphi) \vdash_{\mathcal{I}\mathbf{P}} \mathbf{H}[\varphi]_r^1$.

(vi) $\Gamma_r^2(\varphi) \vdash_{\mathcal{I}\mathbf{F}} \mathbf{L}[\varphi]_r^2$.

PROOF. (i), (ii) follow by induction on the length of φ .

(iii), (iv) follow by P_h and F_L .

(v) follows by (i) and (iii).

(vi) follows by (ii) and (iv). $\square$

2.10. DEFINITION. Let $\mathcal{A} \subseteq \mathbb{F}_{\text{PROP}}$.

(i) $[\mathcal{A}]_r^1 = \{[\varphi]_r^1 \mid \varphi \in \mathcal{A}\}$.

(ii) $[\mathcal{A}]_r^2 = \{[\varphi]_r^2 x_\varphi \mid \varphi \in \mathcal{A}\}$ with x_φ a fresh variable chosen uniquely for φ .

(iii) $\Gamma_r^i(\Delta) = \{\Gamma_r^i(\varphi) \mid \varphi \in \Delta\}$.

(iv) $\Gamma_r^i(\Delta, \varphi) = \Gamma_r^i(\Delta), \Gamma_r^i(\varphi)$.

If Δ is a set of assumptions in a deduction in **PROP** or **PRED**, then $[\Delta]_r^i$ is the set of translated assumptions. Note that $[\varphi]_r^2$ in a sense represents a class. Each $[\varphi]_r^2 x_\varphi$ then represents the condition that $[\varphi]_r^2$ is inhabited, corresponding to the fact that φ is assumed to be true. The $\Gamma_r^i(\Delta)$ are grammatical conditions required for the variables of Δ .

In the proof of the following proposition there is an unexpected difficulty in showing the soundness of modus ponens. The difficulty can be avoided by a trick, which however, does not work for **PRED** as we will see and explain.

2.11. PROPOSITION (soundness of the interpretations for **PROP**). *Let $\Delta \cup \{\varphi\} \subseteq \mathbb{F}_{\text{PROP}}$. Then one has the following for all closed r .*

(i) $\Delta \vdash_{\text{PROP}} \varphi \Rightarrow [\Delta]_r^1, \Gamma_r^1(\Delta, \varphi) \vdash_{\mathcal{JF}} [\varphi]_r^1$.

(ii) $\Delta \vdash_{\text{PROP}} \varphi \Rightarrow \exists M \in A[[\Delta]_r^2, \Gamma_r^2(\Delta, \varphi) \vdash_{\mathcal{JF}} [\varphi]_r^2 M]$.

PROOF. (i) By induction on the derivation of $\Delta \vdash_{\text{PROP}} \varphi$ in **PROP**.

If $\varphi \in \Delta$, then the result holds by the first rule for $\vdash$ in $\mathcal{JF}$.

If $\Delta \vdash \varphi$ is a direct consequence of $\Delta \vdash \psi \supset \varphi$ and $\Delta \vdash \psi$ then the induction hypothesis (IH) implies (leaving out the super- and subscripts)

$$[\Delta], \Gamma(\Delta, \psi \supset \varphi) \vdash [\psi] \supset [\varphi],$$

$$[\Delta], \Gamma(\Delta, \psi) \vdash [\psi].$$

Therefore, by rule P_e one has $[\Delta], \Gamma(\Delta, \varphi), \Gamma(\psi) \vdash [\varphi]$. If $\{q_1, \dots, q_m\}$ is the set of propositional variables occurring in ψ but not in φ or Δ and p is a propositional variable occurring in φ , then we have

$$[\Delta], \Gamma(\Delta, \varphi), \mathbf{H}(rq_1), \dots, \mathbf{H}(rq_m) \vdash [\varphi],$$

where $\mathbf{H}(rp) \in \Gamma(\Delta, \varphi)$. Substituting p for each of $q_1, \dots, q_m$, we obtain

$$[\Delta], \Gamma(\Delta, \varphi) \vdash [\varphi].$$

If $\Delta \vdash \varphi$ is $\Delta \vdash \psi \supset \chi$ and is a direct consequence of $\Delta, \psi \vdash \chi$, then by the IH one has

$$[\Delta], [\psi], \Gamma(\Delta, \psi, \chi) \vdash [\chi].$$

By Lemma 2.9(v)

$$\Gamma(\psi) \vdash \mathbf{H}[\psi].$$

Hence, we have $[\Delta], \Gamma(\Delta), \Gamma(\psi), \Gamma(\chi) \vdash [\psi \supset \chi]$. So by the definition of Γ

$$[\Delta], \Gamma(\Delta), \Gamma(\psi \supset \chi) \vdash [\psi \supset \chi].$$

(ii) Same as for (i) except that for every $p \in \Delta \cup \{\varphi\}$ in the derivation $[p]_r^i$ will have attached a variable x_p and every compound proposition a compound term. For example, if $\varphi \in \Delta$, then $[\Delta]_r^2 \vdash [\varphi]_r^2 x_\varphi$ and in the modus ponens case if

$$[\Delta], \Gamma(\Delta, \psi \supset \varphi) \vdash_{\mathcal{JF}} (\mathbf{F}[\psi][\varphi])M$$

and

$$[\Delta], \Gamma(\Delta, \psi) \vdash_{\mathcal{JF}} [\psi]N,$$

then

$$[\Delta], \Gamma(\Delta, \psi, \psi \supset \varphi) \vdash_{\mathcal{F}} [\varphi](MN).$$

□

2.12. DEFINITION. (i) $\mathcal{A}^s(\mathcal{E}, \mathbf{L})$ is $\mathcal{A}(\mathcal{E}, \mathbf{L})$ extended by the extra constants A_1, A_2, P, f, g, a associated with the signature s of the many-sorted structure of our example. Because we are going to interpret many-sorted predicate logic with sorts $\mathbf{A}_1, \mathbf{A}_2$, it is useful to have among the free variables of the λ -calculus infinite sets $\mathcal{V}_1, \mathcal{V}_2$, with $\mathcal{V}_i = \{x_i, y_i, z_i, \dots\}$. x, y, z denote arbitrary variables.

(ii) Let r be a closed term in $\mathcal{A}^s(\mathcal{E}, \mathbf{L})$. Two maps (for $i = 1, 2$)

$$[-]_r^i: \mathbb{F}_{\text{PRED}} \rightarrow \mathcal{A}^s(\mathcal{E}, \mathbf{L})$$

and a map

$$\Gamma: \mathbb{F}_{\text{PRED}} \rightarrow \text{illative contexts}$$

are defined by Tables 11 and 12.

TABLE 11

t	$[t]_r^i$	$\Gamma(t)$
$x^{\mathbf{A}_j}$	x_j	$A_j x_j$
a	a	$\emptyset$
fs	$f[s]_r^i$	$\Gamma(s)$
gst	$g[s]_r^i[t]_r^i$	$\Gamma(s), \Gamma(t)$

TABLE 12

φ	$[\varphi]_r^1$	$[\varphi]_r^2$	$\Gamma(\varphi)$
Pt	$r(P[t]_r^1)$	$r(P[t]_r^2)$	$\Gamma(t)$
$\psi \supset \chi$	$[\psi]_r^1 \supset [\chi]_r^1$	$\mathbf{F}[\psi]_r^2[\chi]_r^2$	$\Gamma(\psi), \Gamma(\chi)$
$\forall x^{\mathbf{A}_i} \psi$	$\mathbf{E}A_i(\lambda x_i \cdot [\psi]_r^1)$	$\mathbf{GA}_i(\lambda x_i \cdot [\psi]_r^2)$	$\Gamma(\psi) - \{A_i x_i\}$

(iii)

$$\Gamma_{r,s}^1 = \langle \mathbf{L}A_1, \mathbf{L}A_2, \mathbf{F}A_1 A_1 f, \mathbf{F}A_1(\mathbf{F}A_2 A_1)g, \mathbf{F}A_1 \mathbf{H}(r \circ P), A_1 a \rangle,$$

$$\Gamma_{r,s}^2 = \langle \mathbf{L}A_1, \mathbf{L}A_2, \mathbf{F}A_1 A_1 f, \mathbf{F}A_1(\mathbf{F}A_2 A_1)g, \mathbf{F}A_1 \mathbf{L}(r \circ P), A_1 a \rangle,$$

and

$$\Gamma_{r,s}^{i,+} = \Gamma_{r,s}^i \cup \{A_2 x_2\} \text{ where } x_2 \in \mathcal{V}_2 \text{ is some variable.}$$

The definitions $\Gamma_{r,s}^i$ and $\Gamma_{r,s}^{i,+}$ of course refer to our example of a many-sorted predicate calculus with signature s .

It is essential to add $A_2 x_2$ to $\Gamma_{r,s}^i$ (and if required, similarly, for other sorts) to avoid the problem of possibly empty domains. It would be natural that

$$\Delta \vdash_{\text{PRED}} \varphi \Rightarrow \Gamma_{r,s}^1, [\Delta]_r^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{E}} [\varphi]_r^1.$$

However, this is not true. The similar problem for PTS's (see Barendregt [1992]) was first noted by E. Barendsen [1989]. The point is that in ordinary (minimal, intuitionistic, or classical) logic it is always assumed that the universes $\mathbf{A}_1, \mathbf{A}_2, \dots$ of the structure are supposed to be nonempty. For example,

$$(\forall x^{\mathbf{A}}(Px \rightarrow Q)) \rightarrow (\forall x^{\mathbf{A}}Px) \rightarrow Q$$

is provable in PRED, but only valid in structures with $\mathbf{A} \neq \emptyset$. In so-called *free logic* one also allows structures with empty domains. This logic has been axiomatised by Peremans [1949] and Mostowski [1951]. What is unexpected is that the problem turns up in the case of modus ponens (cf. the proof of Proposition 2.14).

2.13. LEMMA. Let $\varphi \in \mathbb{F}_{\text{PRED}}$.

(i) If $t \in \mathbb{T}_{\mathbf{A}_j}$, then in $\mathcal{J}\mathbf{E}$ or $\mathcal{J}\mathbf{G}$ one has $\Gamma(t), \Gamma_{r,s}^i \vdash A_j[t]_r^i$.

(ii) $\Gamma_{r,s}^1, \Gamma(\varphi) \vdash_{\mathcal{J}\mathbf{E}} \mathbf{H}[\varphi]_r^1$.

(iii) $\Gamma_{r,s}^2, \Gamma(\varphi) \vdash_{\mathcal{J}\mathbf{G}} \mathbf{L}[\varphi]_r^2$.

PROOF. (i) By induction on the length of t , using the statements $A_1a, \mathbf{FA}_1A_1f$, and $\mathbf{FA}_1(\mathbf{FA}_2A_1)g$ in $\Gamma_{r,s}^i$.

(ii) If $\varphi = \mathbf{P}t$ where $t \in \mathbb{T}_{\mathbf{A}_1}$, then by $\Gamma_{r,s}^1 \vdash \mathbf{FA}_1\mathbf{H}(r \circ P)$, (i), and $\Gamma(\varphi) = \Gamma(t)$ we have $\Gamma_{r,s}^1, \Gamma(\varphi) \vdash_{\mathcal{J}\mathbf{E}} \mathbf{H}(r(P[t]))$ as required. The remaining cases are as in Lemma 2.9(v).

(iii) As in (ii) and Lemma 2.9(vi). $\square$

2.14. PROPOSITION (soundness of the interpretations for PRED). Let $\Delta \cup \{\varphi\} \subseteq \mathbb{F}_{\text{PRED}}$; then the following hold for all closed r .

(i) $\Delta \vdash_{\text{PRED}} \varphi \Rightarrow \Gamma_{r,s}^{1,+}, [\Delta]_r^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{J}\mathbf{E}} [\varphi]_r^1$.

(ii) $\Delta \vdash_{\text{PRED}} \varphi \Rightarrow \Gamma_{r,s}^{2,+}, [\Delta]_r^2, \Gamma(\Delta, \varphi) \vdash_{\mathcal{J}\mathbf{G}} [\varphi]_r^2 M$ for some M .

PROOF. (i) The induction on the proof of $\Delta \vdash_{\text{PRED}} \varphi$ is as in the proof of Proposition 2.11(i). When, in the case of modus ponens, terms A_1x_1 for variables $x_1 \in \text{FV}(\psi) - \text{FV}(\varphi)$ need to be removed from the left of the $\vdash$ we replace x_1 by a . If terms A_2x_i for $x_i \in \text{FV}(\psi) - \text{FV}(\varphi)$, we replace x_i by x_2 and note that $A_2x_2 \in \Gamma_{r,s}^{1,+}$.

(ii) The induction on the proof of $\Delta \vdash_{\text{PRED}} \varphi$ is as in the proof of Proposition 2.11(ii) with $\mathbf{G}[\psi](\lambda x. [\varphi])$ instead of $\mathbf{F}[\psi][\varphi]$. Variables may need to be replaced as in (i).

2.15. PROPOSITION (the relation between the two interpretations). (i) For $\varphi \in \mathbb{F}_{\text{PROP}}$ one has $\mathbf{K}[\varphi]_r^1 = [\varphi]_{\mathbf{K} \circ r}^2$.

(ii) For $\varphi \in \mathbb{F}_{\text{PRED}}$ one has $\mathbf{K}[\varphi]_r^1 = [\varphi]_{\mathbf{K} \circ r}^2$.

PROOF. (i) By induction on the length of φ .

$$\mathbf{K}[p]_r^1 = \mathbf{K}(rp) = [p]_{\mathbf{K} \circ r}^2,$$

$$\mathbf{K}[\varphi \supset \psi]_r^1 = \mathbf{K}([\varphi]_r^1 \supset [\psi]_r^1) = \mathbf{F}(\mathbf{K}[\varphi]_r^1)(\mathbf{K}[\psi]_r^1),$$

by Lemma 2.6(i), so by the IH

$$\mathbf{K}[\varphi \supset \psi]_r^1 = \mathbf{F}([\varphi]_{\mathbf{K} \circ r}^2)([\psi]_{\mathbf{K} \circ r}^2) = [\varphi \supset \psi]_{\mathbf{K} \circ r}^2.$$

(ii) By induction on the length of φ as in (i) but also using Lemma 2.6(ii). $\square$

§3. Completeness of two of the interpretations. In this section we derive completeness for the interpretations $[\]^1: \text{PROP} \rightarrow \mathcal{J}\mathbf{P}$ and $[\]^1: \text{PRED} \rightarrow \mathcal{J}\mathbf{E}$. We con-

jecture completeness for the interpretations $[]^2: \text{PROP} \rightarrow \mathcal{J}\mathbf{F}$ and $[]^2: \text{PRED} \rightarrow \mathcal{J}\mathbf{G}$, but we have not been able to prove it.¹

We start with the proof of the completeness for $\mathcal{J}\mathbf{E}$ relative to PRED . This occupies subsections 3.1–3.11. The proof for $\mathcal{J}\mathbf{P}$ relative to PROP in 3.12–3.14 proceeds in a similar way but is much easier.

Completeness for $\mathcal{J}\mathbf{E}$ relative to PRED . We will show

$$\forall r[\Gamma_{r,s}^{1,+}, [\Delta]_r^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{J}\mathbf{E}} [\varphi]_r^1] \Rightarrow \Delta \vdash_{\text{PRED}} \varphi.$$

Here the signature s and the context $\Gamma_{r,s}^{1,+}$ are as in 2.2 and 2.12; again the result can easily be generalised to other signatures.

It is sufficient to show

$$\Gamma_{r,s}^{1,+}, [\Delta]_r^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{J}\mathbf{E}} [\varphi]_r^1 \Rightarrow \Delta \vdash_{\text{PRED}} \varphi$$

for a special r . We choose $r \equiv \mathbf{I}$, i.e., we omit r and we prove

$$\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{J}\mathbf{E}} [\varphi]^1 \Rightarrow \Delta \vdash_{\text{PRED}} \varphi,$$

where the definitions of $\Gamma_s^{1,+}$ and $[-]^1$ are obtained from 2.12 by everywhere omitting r .

The proof goes in two steps. First we define a *grammar* in order to analyze the terms M such that $\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{J}\mathbf{E}} M$. Then the completeness is shown by means of this analysis. Instead of $\vdash_{\mathcal{J}\mathbf{E}}$ we shall mostly write $\vdash$.

3.1. REMARK. That it is not obvious that completeness holds is because not only translations of tautologies can be derived in the ICL's, but also syntactical statements. Let $\Gamma = \Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi)$. Then we can derive in $\mathcal{J}\mathbf{E}$ sequents of the form

$$\Gamma \vdash [\varphi],$$

where $[\varphi]$ is (the translation of) a logical formula, but also

$$\Gamma \vdash \mathbf{H}(Pt),$$

where $\mathbf{H}(Pt)$ corresponds to the syntactical statement $\mathbf{Pt} \in \mathbb{F}_{\text{PRED}}$ in the meta-language. In

$$\Gamma \vdash \mathbf{L}A_i$$

$\mathbf{L}A_i$ corresponds to the fact that A_i is one of the sets in the signature s . Even a mixture is possible

$$\mathbf{H}p \vdash p \supset \mathbf{H}p.$$

Using the grammar it will be shown that such mixed statements do not interfere with the logic. The translations of logical formulas will form a class $\bar{\mathcal{P}}$ (propositions) in our grammar and the other statements a class $\bar{\mathcal{G}}$ (grammatical conditions).

¹After the paper had been sent to the journal we succeeded in proving completeness for $[]^2: \text{PROP} \rightarrow \mathcal{J}\mathbf{F}$, but completeness for $[]^2: \text{PRED} \rightarrow \mathcal{J}\mathbf{G}$ is still open.

3.2. DEFINITION (grammar for derivable statements in $\mathcal{I}\mathcal{E}$).

$$\begin{aligned}
\mathcal{T} &= \mathcal{T}_1 \mid \mathcal{T}_2, \\
\mathcal{T}_1 &= \mathcal{V}_1 \mid a \mid f\mathcal{T}_1 \mid g\mathcal{T}_1\mathcal{T}_2, \\
\mathcal{T}_2 &= \mathcal{V}_2, \\
\mathcal{P} &= P\mathcal{T}_1 \mid \mathfrak{E}A_i(\lambda x_i. \mathcal{P}) \mid \mathfrak{E}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{P}), \\
\mathcal{G} &= \mathbf{L}A_i \mid A_i\mathcal{T}_i \mid \mathfrak{E}A_i(\lambda x_i. \mathcal{G}) \mid \mathfrak{E}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{G}) \mid \mathbf{L}(\mathbf{K}\mathcal{P}), \\
\mathcal{O} &= \mathcal{G} \mid \mathcal{P}, \\
\bar{\mathcal{T}}_i &= \{M \mid \exists N \in \mathcal{T}_i \mid N =_{\beta\eta} M\}, \\
\bar{\mathcal{P}} &= \{M \mid \exists N \in \mathcal{P} \mid N =_{\beta\eta} M\}, \\
\bar{\mathcal{G}} &= \{M \mid \exists N \in \mathcal{G} \mid N =_{\beta\eta} M\}, \\
\bar{\mathcal{O}} &= \bar{\mathcal{P}} \cup \bar{\mathcal{G}}.
\end{aligned}$$

3.3. REMARKS. (i) All elements of $\mathcal{O}$ are in $\beta\eta$ -normal form if we read $\mathbf{K}M$ as $\lambda y.M$. So all elements of $\bar{\mathcal{O}}$ have a (unique) $\beta\eta$ -normal form.

(ii) $\bar{\mathcal{P}}$ and $\bar{\mathcal{G}}$ do not exhaust the possible theorems of $\mathcal{I}\mathcal{E}$, e.g.,

$$\mathbf{H}(\mathbf{H}p) \vdash \mathbf{H}(\mathbf{H}p \supset p), \mathbf{H}p, \mathbf{H}(\mathbf{H}p) \vdash \mathbf{H}(p \supset \mathbf{H}p).$$

3.4. NOTATION. Let $M \in \bar{\mathcal{O}}$ with normal form N , and let u be a variable. Then we write $u \notin_{\beta\eta} \text{FV}(M)$ for $u \notin \text{FV}(N)$.

Now in 3.5–3.10 we state some technical results that are needed in the completeness proof. The main proposition is 3.10, stating that only terms in $\bar{\mathcal{O}}$ can be derived from $\Gamma_s^{1,+}$; this gives the required analysis.

3.5. LEMMA. (i) Let $\mathcal{W} = \mathcal{T}_1, \mathcal{T}_2, \mathcal{P}, \mathcal{G}$, or $\mathcal{O}$. Then

$$w \in \mathcal{W}, t_i \in \mathcal{T}_i, x_i \in \mathcal{V}_i \Rightarrow w[x_1 := t_1, x_2 := t_2] \in \mathcal{W}.$$

(ii) Let $\bar{\mathcal{W}} = \bar{\mathcal{T}}_1, \bar{\mathcal{T}}_2, \bar{\mathcal{P}}, \bar{\mathcal{G}}$, or $\bar{\mathcal{O}}$. Then

$$w \in \bar{\mathcal{W}}, t_i \in \bar{\mathcal{T}}_i, x_i \in \mathcal{V}_i \Rightarrow w[x_1 := t_1, x_2 := t_2] \in \bar{\mathcal{W}}.$$

PROOF. (i) By a simple induction.

(ii) From (i) and

$$M_1 =_{\beta\eta} M_2, N_1 =_{\beta\eta} N_2 \Rightarrow M_1[x := N_1] =_{\beta\eta} M_2[x := N_2]. \quad \square$$

3.6. LEMMA. Let $cX_1 \cdots X_n =_{\beta\eta} M$ for some $M \in \mathcal{O}$ and some constant c . Then $n \in \{1, 2\}$ and $M \equiv cY_1 \cdots Y_n$ with $Y_i =_{\beta\eta} X_i$.

PROOF. By Church-Rosser and the fact that all elements of $\mathcal{O}$ are in $\beta\eta$ -normal form. $\square$

3.7. LEMMA. (i) $\Gamma_s^{1,+} \subset \bar{\mathcal{G}}$.

(ii) $\mathcal{G} \cap \mathcal{P} = \emptyset$.

(iii) $\bar{\mathcal{G}} \cap \bar{\mathcal{P}} = \emptyset$.

PROOF. (i) $\mathbf{F}A_1A_1f = \mathfrak{E}A_1(\lambda x_1. A_1(fx_1)) \in \mathcal{G}$ because $A_1(fx_1) \in \mathcal{G}$. Moreover,

$$\mathbf{F}A_1(\mathbf{F}A_2A_1)g = \mathfrak{E}A_1(\lambda x_1. \mathfrak{E}A_2(\lambda x_2. A_1(gx_1x_2))) \in \mathcal{G}.$$

Finally, $\mathbf{F}A_1\mathbf{H}P = \mathfrak{E}A_1(\lambda x_1. \mathbf{H}(Px_1)) \in \mathcal{G}$ because $\mathbf{H}(Px_1) \in \mathcal{G}$.

(ii) By an easy induction.

(iii) From (ii) by Church-Rosser and the fact that the elements of $\mathcal{O} = \mathcal{G} \cup \mathcal{P}$ are in $\beta\eta$ -normal form. $\square$

3.8. LEMMA. (i) $[-]^1: \text{PRED} \rightarrow \Lambda^s(\Xi, \mathbf{L})$ induces bijections

$$\begin{aligned} [-]^1: \mathbb{T}_{\mathbf{A}_1} &\rightarrow \mathcal{T}_1, \\ [-]^1: \mathbb{T}_{\mathbf{A}_2} &\rightarrow \mathcal{T}_2, \\ [-]^1: \mathbb{F}_{\text{PRED}} &\rightarrow \mathcal{P}. \end{aligned}$$

(ii) If $\varphi \in \mathbb{F}_{\text{PRED}}$, then $[\varphi]^1 \in \mathcal{P}$ and $\Gamma(\varphi) \subset \mathcal{G}$.

(iii) If $\Delta \subset \mathbb{F}_{\text{PRED}}$ and $\varphi \in \mathbb{F}_{\text{PRED}}$, then $\Gamma_s^{1,+} \cup [\Delta]^1 \cup \Gamma(\Delta, \varphi) \subset \bar{\mathcal{O}}$.

PROOF. (i) and (ii) by easy inductions.

(iii) from (ii) and 3.7(i). □

3.9. LEMMA. If $\varphi \in \mathbb{F}_{\text{PRED}}$, $x^{A_i} \in \mathbb{V}^{A_i}$, $t_{A_i} \in \mathbb{T}_{\mathbf{A}_i}$, $x_i \in \mathcal{V}_i$, then

$$[\varphi]^1[x_i := [t_{A_i}]^1] \equiv [\varphi[x^{A_i} := t_{A_i}]]^1.$$

PROOF. By an easy induction. □

3.10. PROPOSITION.

$$\Gamma \vdash M, \Gamma \subset \bar{\mathcal{O}} \Rightarrow M \in \bar{\mathcal{O}}.$$

PROOF. We use induction loading and show

$$(*) \quad \Gamma \vdash M, \Gamma \subset \bar{\mathcal{O}} \ \& \ u \notin_{\beta\eta} \text{FV}(\Gamma) \Rightarrow M \in \bar{\mathcal{O}} \ \& \ u \notin_{\beta\eta} \text{FV}(M)$$

by induction on the derivation of $\Gamma \vdash M$. We only consider the three Ξ -rules; the other two rules are easy.

Case Ξ_e . $\Gamma \vdash M$ is $\Gamma \vdash YV$ as a direct consequence of $\Gamma \vdash \Xi XY$, $\Gamma \vdash XV$.

By the IH one has $\Xi XY \in \bar{\mathcal{O}}$, $u \notin_{\beta\eta} \text{FV}(\Xi XY)$ & $XV \in \bar{\mathcal{O}}$, $u \notin_{\beta\eta} \text{FV}(XV)$. We distinguish two cases according to the form of ΞXY , using Lemma 3.6.

Subcase $\Xi_e(a)$. $X = A_i$, $Y = \lambda x_i.O$ with $O \in \mathcal{O}$. Now $A_iV \in \bar{\mathcal{O}}$ & $u \notin_{\beta\eta} \text{FV}(A_iV)$. So $V = t_i$ where $t_i \in \mathcal{T}_i$ and $u \notin \text{FV}(t_i)$. Hence, by Lemma 3.5(i)

$$M = (\lambda x_i.O)t_i = O[x_i := t_i] \in \bar{\mathcal{O}}$$

and

$$u \notin_{\beta\eta} \text{FV}(O[x_i := t_i]),$$

because $u \notin \text{FV}(\lambda x_i.O) \cup \text{FV}(t_i)$.

Subcase $\Xi_e(b)$. $X = \mathbf{K}p$, $Y = \mathbf{K}O$, with $p \in \mathcal{P}$, $O \in \mathcal{O}$. Now $YV = O$, where $u \notin \text{FV}(O)$.

Case Ξ_i . $\Gamma \vdash M$ is $\Gamma \vdash \Xi XY$ as direct consequence of $\Gamma \vdash \mathbf{L}X$, $\Gamma, Xx \vdash Yx$ with $x \notin \text{FV}(\Gamma, X, Y)$.

By the IH one has $\mathbf{L}X \in \bar{\mathcal{O}}$ and $u \notin_{\beta\eta} \text{FV}(\mathbf{L}X)$. We distinguish two cases according to the form of X .

Subcase $\Xi_i(a)$. $X = A_i$. Now x is any variable, so we may assume that $x \in \mathcal{V}_i$, $u \neq x$. Then $A_ix \in \bar{\mathcal{O}}$ and $u \notin \text{FV}(A_ix)$; hence, by the IH one has

$$Yx \in \bar{\mathcal{O}} \quad \text{and} \quad u \notin_{\beta\eta} \text{FV}(Yx).$$

Let $Yx = O \in \mathcal{O}$. Then $M = \Xi A_i(\lambda x.O)$, where $u \notin_{\beta\eta} \text{FV}(\lambda x.O)$.

Subcase $\Xi_i(b)$. $X = \mathbf{K}p$. Then $\Gamma, p \vdash Yx$. One has $x \notin \text{FV}(\Gamma, p)$, $u \notin_{\beta\eta} \text{FV}(\Gamma, p)$ because $u \notin_{\beta\eta} \text{FV}(\mathbf{L}X)$. So by the IH one has

$$Yx = O \in \mathcal{O}, \quad \text{where } x \notin \text{FV}(O), \ u \notin \text{FV}(O).$$

Hence, $Y = \mathbf{K}O$ and $M = \Xi(\mathbf{K}p)(\mathbf{K}O)$, where $u \notin \text{FV}(pO)$.

Case Ξ_H . $\Gamma \vdash M$ is $\Gamma \vdash \mathbf{H}(\Xi XY)$ as a direct consequence of $\Gamma \vdash \mathbf{L}X$,

$$\Gamma, Xx \vdash \mathbf{H}(Yx) \quad \text{with } x \notin \text{FV}(\Gamma, X, Y).$$

The proof is similar to the proof for case Ξ_i . We now get $\mathbf{H}(Yx) \in \bar{\mathcal{O}}$; hence, $Yx \in \bar{\mathcal{P}}$ and

$$M = \mathbf{H}(\Xi A_i(\lambda x_i.p)) \quad \text{with } u \notin \text{FV}(\lambda x_i.p) \text{ in case } \Xi_H(a),$$

$$M = \mathbf{H}(\Xi(\mathbf{K}p_1)(\mathbf{K}p_2)) \quad \text{with } u \notin \text{FV}(p_1p_2) \text{ in case } \Xi_H(b). \quad \square$$

3.11. PROPOSITION (completeness for $\mathcal{S}\Xi$ relative to PRED).

$$\Gamma_s^{1,+}, [\Delta]^1, \Gamma(\Delta, \varphi) \vdash_{\mathcal{S}\Xi} [\varphi]^1 \Rightarrow \Delta \vdash_{\text{PRED}} \varphi.$$

PROOF. $\Gamma_s^{1,+} \subset \bar{\mathcal{G}}$ by Lemma 3.7(i) and $\Gamma(\Delta, \varphi) \subset \bar{\mathcal{G}}$ by Lemma 3.8(ii). Hence, it is sufficient to prove

$$(**) \quad \Gamma_g, [\Delta]^1 \vdash_{\mathcal{S}\Xi} M, \Gamma_g \subset \bar{\mathcal{G}}, M = [\varphi]^1 \Rightarrow \Delta \vdash_{\text{PRED}} \varphi.$$

Write $\Gamma = \Gamma_g, [\Delta]^1$. Then $\Gamma \subset \bar{\mathcal{O}}$; hence, $M \in \bar{\mathcal{O}}$ by Proposition 3.10. The proof of (**) goes by induction.

Case 1. $\Gamma \vdash M$ because $M \in \Gamma$.

$M = [\varphi]^1 \in \mathcal{P}$ by Lemma 3.8(i), so as $\mathcal{P} \subset \bar{\mathcal{P}}$ and $\bar{\mathcal{P}} \cap \bar{\mathcal{G}} = \emptyset$, $M \in [\Delta]^1$. As the elements of $[\Delta]^1$ are in nf as are those of $\mathcal{P}$, one has $[\varphi]^1 \in [\Delta]^1$ and by Lemma 3.8(i) $\varphi \in \Delta$. Hence, $\Delta \vdash_{\text{PRED}} \varphi$.

Case 2. $\Gamma \vdash M$ is a direct consequence of $\Gamma \vdash N$ and $M = N$.

Now $N = M = [\varphi]^1$ and by the IH for N one has $\Delta \vdash_{\text{PRED}} \varphi$.

Case Ξ_e . $\Gamma \vdash M$ is $\Gamma \vdash YV$ as a direct consequence of $\Gamma \vdash \Xi XY$, $\Gamma \vdash XV$.

As $\Xi XY \in \mathcal{O}$ by Proposition 3.10, we need consider only 4 cases.

Subcase $\Xi_e(a)$. $X = A_i$, $Y = \lambda x_i.p$. Now $\Gamma \vdash A_iV \in \bar{\mathcal{O}}$ by Proposition 3.10. Therefore, $V = t_i = [t_{A_i}]^1$. Since $p \in \mathcal{P}$, we can write $p = [\psi]^1$ by 3.8(i). Therefore, $[\varphi]^1 = YV = (\lambda x_i.p)[t_{A_i}]^1 = [\psi]^1[x_i := [t_{A_i}]^1] = [\psi[x^{A_i} := t_{A_i}]]^1$. So

$$[\varphi]^1 \equiv [\psi[x^{A_i} := t_{A_i}]]^1.$$

Hence, $\varphi \equiv \psi[x^{A_i} := t_{A_i}]$. $\Xi XY = \Xi A_i(\lambda x_i.p) = [\forall x^{A_i}\psi]^1$. By the IH one has $\Delta \vdash_{\text{PRED}} \forall x^{A_i}\psi$. So $\Delta \vdash_{\text{PRED}} \psi[x^{A_i} := t_{A_i}] \equiv \varphi$.

Subcase $\Xi_e(b)$. $X = \mathbf{K}p_1$, $Y = \mathbf{K}p_2$. Then $[\varphi]^1 = M = YV = p_2 \in \mathcal{P}$, and by 3.8(i), we can write $p_1 = [\varphi_1]^1$. Therefore, $\Xi XY = \Xi(\mathbf{K}p_1)(\mathbf{K}p_2) = [\varphi_1 \supset \varphi]^1$. By the IH one has

$$\Delta \vdash_{\text{PRED}} \varphi_1 \supset \varphi.$$

Also, $XV = p_1 = [\varphi_1]^1$, so by the IH one has

$$\Delta \vdash_{\text{PRED}} \varphi_1.$$

Therefore, it follows by modus ponens that

$$\Delta \vdash_{\text{PRED}} \varphi.$$

Subcase $\Xi_e(c)$. $X = A_i$, $Y = \lambda x_i.g$. Since $\Gamma \vdash A_iV \in \bar{\mathcal{O}}$, one has $V \in \bar{\mathcal{T}}_i$. Therefore, $M = YV = g[x_i := V] \in \bar{\mathcal{G}}$. So $M \neq [\varphi]^1$ for all φ by Lemma 3.7(iii).

Contradiction.

Subcase $\Xi_e(d)$. $X = \mathbf{K}p$, $Y = \mathbf{K}g$. Now $M = YV = g$, so $M \neq [\varphi]^1$ for all φ . Contradiction.

Case Ξ_i . $\Gamma \vdash M$ is $\Gamma \vdash \Xi XY$ as a direct consequence of $\Gamma \vdash \mathbf{L}X$, $\Gamma, Xx \vdash Yx$ with $x \notin \text{FV}(\Gamma, X, Y)$.

As $\Xi XY = [\varphi]^1 \in \mathcal{P}$ we need consider only 2 cases.

Subcase $\Xi_i(a)$. $X = A_i$, $Y = \lambda x_i.p$. Let $p = [\psi]^1$. Then $M = [\forall x^{\mathbf{A}_i}\psi]^1$. Now x is any variable, so we may assume $x \in \mathcal{V}_i$. Then $Xx = \mathbf{A}_i x \in \bar{\mathcal{G}}$.

As $\Gamma, Xx \vdash Yx$ one has $\Gamma, Xx \vdash [\psi]^1$. So $\Delta \vdash_{\text{PRED}} \psi$ by the IH. Now x does not occur in Γ , so

$$\Delta \vdash_{\text{PRED}} \forall x^{\mathbf{A}_i} \psi.$$

Subcase $\Xi_i(b)$. $X = \mathbf{K}p_1$, $Y = \mathbf{K}p_2$. Let $p_1 = [\varphi_1]^1$, $p_2 = [\varphi_2]^1$. Then $M = [\varphi_1 \supset \varphi_2]^1$. Now $\Gamma, Xx \vdash Yx$ is $\Gamma, p_1 \vdash p_2$. So by the IH one has $\Delta, \varphi_1 \vdash_{\text{PRED}} \varphi_2$. Hence,

$$\Delta \vdash_{\text{PRED}} \varphi_1 \supset \varphi_2.$$

Case Ξ_H . $\Gamma \vdash M$ is $\Gamma \vdash \mathbf{H}(\Xi XY)$ as a direct consequence of $\Gamma \vdash \mathbf{L}X$, $\Gamma, Xx \vdash \mathbf{H}(Yx)$ with $x \notin \text{FV}(\Gamma, X, Y)$.

This case is not applicable because $M \notin \bar{\mathcal{P}}$. □

Completeness for $\mathcal{I}\mathbf{P}$ relative to PROP. The proof of this completeness follows the same pattern as the proof of the completeness for $\mathcal{I}\mathbf{E}$ relative to PRED, but it is easier. As in that proof it is sufficient to take $r \equiv \mathbf{I}$, i.e., we omit r .

3.12. DEFINITION (grammar for derivable statements for $\mathcal{I}\mathbf{P}$).

$$\begin{aligned} \mathcal{P} &= \mathcal{V} \mid \mathcal{P} \supset \mathcal{P}, \\ \mathcal{G} &= \mathbf{H}\mathcal{P} \mid \mathcal{P} \supset \mathcal{G}, \\ \mathcal{O} &= \mathcal{G} \mid \mathcal{P}. \end{aligned}$$

$\bar{\mathcal{P}}$, $\bar{\mathcal{G}}$, and $\bar{\mathcal{O}}$ are then defined as in Definition 3.2.

3.13. PROPOSITION.

$$\Gamma \vdash_{\mathcal{I}\mathbf{P}} M, \Gamma \subset \bar{\mathcal{O}} \Rightarrow M \in \bar{\mathcal{O}}.$$

PROOF. By induction on the derivation of $\Gamma \vdash_{\mathcal{I}\mathbf{P}} M$. The various cases correspond to the two initial cases and cases $\Xi_e(b)$, $\Xi_i(b)$, and $\Xi_H(b)$ of the proof of Proposition 3.10. □

3.14. PROPOSITION (completeness for $\mathcal{I}\mathbf{P}$ relative to PROP).

$$[\Delta]^1, \Gamma^1(\Delta, \varphi) \vdash_{\mathcal{I}\mathbf{P}} [\varphi]^1 \Rightarrow \Delta \vdash_{\text{PROP}} \varphi.$$

PROOF. This consists of the initial cases and cases $\Xi_e(b)$, $\Xi_e(d)$, $\Xi_i(b)$ of the proof of Proposition 3.11 and the Ξ_H case with $\mathbf{K}p_1$ for X and $\mathbf{K}p_2$ for Y . As before $[-]^1$ is 1-1 and $\mathcal{G} \cap \mathcal{P} = \emptyset$ and if $X \in \mathcal{O}$ then $\Gamma^1(X) \subset \mathcal{G}$. □

§4. Remarks and open problems.

4.1. REMARKS. (i) The systems $\mathcal{I}\mathbf{P}$, $\mathcal{I}\mathbf{E}$, $\mathcal{I}\mathbf{F}$, and $\mathcal{I}\mathbf{G}$ are based on $\beta\eta$ -conversion. It is possible to work with variants of these systems based on β -conversion only. Change the rules for $\mathcal{I}\mathbf{E}$ as in Table 13 (see next page) and similarly

for $\mathcal{J}\mathbf{P}$, $\mathcal{J}\mathbf{F}$, and $\mathcal{J}\mathbf{G}$. Then in the proof of the completeness for $\mathcal{J}\mathbf{E}$ relative to PRED only minor changes need to be made, like replacing A_i , X , Xx , and $\beta\eta$ by $\lambda x_i. A_i x_i$, $\lambda x. X$, X , and β , respectively.

TABLE 13 $\mathcal{J}\mathbf{E}$.

	$X \in \Gamma$	$\Rightarrow \Gamma \vdash X;$
	$\Gamma \vdash X, X =_\beta Y$	$\Rightarrow \Gamma \vdash Y;$
$\mathcal{E}_e$	$\Gamma \vdash \mathbf{E}(\lambda x. X)(\lambda x. Y), \Gamma \vdash (\lambda x. X)V$	$\Rightarrow \Gamma \vdash (\lambda x. Y)V;$
$\mathcal{E}_i$	$\Gamma, X \vdash Y, \Gamma \vdash \mathbf{L}(\lambda x. X), x \notin \text{FV}(\Gamma)$	$\Rightarrow \Gamma \vdash \mathbf{E}(\lambda x. X)(\lambda x. Y);$
$\mathcal{E}_H$	$\Gamma, X \vdash \mathbf{H}Y, \Gamma \vdash \mathbf{L}(\lambda x. X), x \notin \text{FV}(\Gamma)$	$\Rightarrow \Gamma \vdash \mathbf{H}(\mathbf{E}(\lambda x. X)(\lambda x. Y)).$

Similar changes should be made in the proof of the completeness for $\mathcal{J}\mathbf{P}$ relative to PROP . The reader is invited to verify all details.

(ii) The additional primitive $\mathbf{L}$ added to $\Lambda(\mathbf{E})$ was not strictly necessary. We could have used the definition $\mathbf{L} = \mathbf{WE}$ and simplified our grammar for derivable statements in $\mathcal{J}\mathbf{E}$ in Definition 3.2 in the following way:

$$\begin{aligned}\mathcal{P} &= P\mathcal{T}_1 \mid \mathbf{E}A_i(\lambda x_i. \mathcal{P}) \mid \mathbf{E}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{P}), \\ \mathcal{G} &= A_i\mathcal{T}_i \mid \mathbf{E}A_i(\lambda x_i. \mathcal{G}) \mid \mathbf{E}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{G}).\end{aligned}$$

Note that now $\mathbf{L}(\mathbf{K}\mathcal{P}) = \mathbf{E}(\mathbf{K}\mathcal{P})(\mathbf{K}\mathcal{P})$, so $\mathbf{L}(\mathbf{K}\mathcal{P})$ shifts from $\mathcal{G}$ to $\mathcal{P}$! Similarly for $\mathcal{J}\mathbf{P}$, $\mathbf{H}$ can be defined as $\mathbf{WP}$ as was done in Curry [1942a].

(iii) In the work of Seldin and others $\mathbf{L}$ is defined as $\mathbf{FEH}$, where $\mathbf{E}$ is a universal class. Under this definition $\mathbf{H}\rho$ and $\mathbf{L}(\mathbf{K}\rho)$ are interderivable, but our proof of Proposition 3.10 fails.

(iv) The title of the paper refers to combinatory logic, but the systems used are based on lambda calculus throughout. The illative systems could have been based on combinatory logic using an appropriate bracket abstraction algorithm.

(v) For historical and other remarks concerning the combinators $\mathbf{E}$, $\mathbf{P}$, $\mathbf{F}$, and $\mathbf{G}$, see Hindley and Seldin [1986; Chapter 17 for $\mathbf{E}$ and $\mathbf{P}$, Chapter 13 and Chapter 15 for $\mathbf{F}$, and Chapter 16 §§C, D for $\mathbf{G}$].

4.2. *Open problems.* The following is a list of open problems.

(i) Is the interpretation $[]^2$: $\text{PROP} \rightarrow \mathcal{J}\mathbf{F}$ complete? Is the interpretation $[]^2$: $\text{PRED} \rightarrow \mathcal{J}\mathbf{G}$ complete?

(ii) Is $\mathcal{J}\mathbf{F}$ a conservative extension of $\mathcal{J}\mathbf{P}$ and $\mathcal{J}\mathbf{G}$ a conservative extension of $\mathcal{J}\mathbf{E}$?

(iii) Adding as axiom $\mathbf{LH}$ to $\mathcal{J}\mathbf{E}$ one can interpret second-order propositional and predicate logic. Is this interpretation complete?

(iv) Is the extension $\mathcal{J}_0$ of $\mathcal{J}\mathbf{E}$ in 2.5 complete? Are similar extensions of $\mathcal{J}\mathbf{F}$ and $\mathcal{J}\mathbf{G}$ complete?

4.3. **REMARK.** The system $\mathcal{J}_0$ is consistent. This can be seen in the following way. Let

$$\begin{aligned}\mathcal{G} &= \mathcal{V} \mid \mathbf{L}A_i \mid \mathbf{LH} \mid \mathbf{EIH} \mid \mathbf{E}A_i(\lambda y. \mathcal{G}) \mid \mathbf{EH}(\lambda y. \mathcal{G}) \mid \mathbf{E}(\mathbf{K}\mathcal{G})(\mathbf{K}\mathcal{G}) \mid A_i\mathcal{G} \mid \mathbf{H}\mathcal{G}, \\ \bar{\mathcal{G}} &= \{M \mid \exists N \in \mathcal{G} \mid N =_{\beta\eta} M\}.\end{aligned}$$

Then one can prove

$$\Gamma \vdash M, \Gamma \subset \bar{\mathcal{G}} \Rightarrow M \in \bar{\mathcal{G}}.$$

So if $\vdash M$, then M has a normal form. Hence, the system is consistent, because $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$ cannot be derived. This weak consistency result was proved by a similar method in Bunder [1983b].

REFERENCES

- P. ACZEL [1980], *Frege structures and the notions of proposition, truth and set*, *The Kleene symposium* (J. Barwise, H. J. Keisler, and K. Kunen, editors), North-Holland, Amsterdam, pp. 31–59.
- H. P. BARENDREGT [1984], *The lambda calculus, its syntax and semantics*, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam.
- [1992], *Lambda calculi with types*, *Handbook of logic in computer science*, Vol. II (S. Abramski, D. M. Gabbay, and T. S. E. Maibaum, editors), Oxford University Press, London and New York.
- E. BARENDSSEN [1989], *Representation of logic, data types and recursive functions in typed lambda calculi*, Masters Thesis, Department of Computer Science, Catholic University, Nijmegen.
- M. W. BUNDER [1969], *Set theory based on combinatory logic*, Ph.D. thesis, Department of Mathematics, University of Amsterdam.
- [1973], *A deduction theorem for restricted generality*, *Notre Dame Journal of Formal Logic*, vol. 14, pp. 341–346.
- [1974], *Propositional and predicate calculus based on combinatory logic*, *Notre Dame Journal of Formal Logic*, vol. 15, pp. 25–32.
- [1976], *The inconsistency of F^*_1* , this JOURNAL, vol. 41, pp. 467–468.
- [1983], *Predicate calculus of arbitrarily high finite order*, *Archiv für Math Logik und Grundlagenforschung*, vol. 23, pp. 1–10.
- [1983a], *A one axiom set theory based on higher order predicate calculus*, *Archiv für Math Logik und Grundlagenforschung*, vol. 23, pp. 99–107.
- [1983b], *A weak absolute consistency proof for some systems of illative combinatory logic*, this JOURNAL, vol. 48, pp. 771–776.
- A. CHURCH [1932, 33], *A set of postulates for the foundation of logic*, *Annals of Mathematics* (2), vol. 33, pp. 346–366 and vol. 34, pp. 839–864.
- [1941], *The calculi of lambda conversion*, Princeton University Press, Princeton.
- H. B. CURRY [1930], *Grundlagen der kombinatorischen Logik*, *American Journal of Mathematics*, vol. 52, pp. 509–536; pp. 789–834. (Inaugural dissertation)
- [1931], *The universal quantifier in combinatory logic*, *Annals of Mathematics*, vol. 32, pp. 154–180.
- [1932], *Some additions to the theory of combinators*, *American Journal of Mathematics*, vol. 54, pp. 551–558.
- [1933], *Apparent variables from the standpoint of combinatory logic*, *Annals of Mathematics*, vol. 34, pp. 381–404.
- [1934a], *Functionality in combinatory logic*, *Proceedings of the National Academy of Sciences of the United States of America*, vol. 20, pp. 584–590.
- [1934b], *Some properties of equality and implication in combinatory logic*, *Annals of Mathematics*, vol. 35, pp. 849–850.
- [1935], *Foundations of the theory of abstract sets from the standpoint of combinatory logic*, *Bulletin of the American Mathematical Society*, vol. 40, p. 654. (Abstract).
- [1936], *First properties of functionality in combinatory logic*, *Tôhoku Mathematical Journal*, vol. 41, pp. 371–401.
- [1942a], *Some advances in the combinatory theory of quantification*, *Proceedings of the National Academy of Sciences of the United States of America*, vol. 28, pp. 564–569.
- [1942b], *The combinatory foundations of mathematical logic*, this JOURNAL, vol. 7, pp. 49–64.
- [1942c], *The inconsistency of certain formal logics*, this JOURNAL, vol. 7, pp. 115–117.
- H. B. CURRY and R. FEYS [1958], *Combinatory logic*, Vol. I, North-Holland, Amsterdam.

H. B. CURRY, J. R. HINDLEY and J. P. SELDIN [1972], *Combinatory logic*, Vol. 2, North-Holland, Amsterdam.

J. R. HINDLEY and J. P. SELDIN [1986], *Introduction to combinators and λ -calculus*, Cambridge University Press, London and New York.

S. KLEENE and J. B. ROSSER [1935], *The inconsistency of certain formal logics*, *Annals of Mathematics* (2), vol. 36, pp. 630–636.

M. LÖB [1955], *A solution of a problem of Henkin*, this JOURNAL, vol. 20, pp. 115–118.

A. MOSTOWSKI [1951], *On the rules of proof in the pure functional calculus of first order*, this JOURNAL, vol. 16, pp. 107–111.

W. PEREMANS [1949], *Een opmerking over intuitionistische logica*, Report ZW-16, CWI, Kruislaan 413, 1098 SJ Amsterdam.

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE

CATHOLIC UNIVERSITY

NIJMEGEN, THE NETHERLANDS

E-mail: henk@cs.kun.nl

E-mail: wil@cs.kun.nl

FACULTY OF INFORMATICS, DEPARTMENT OF MATHEMATICS

UNIVERSITY OF WOLLONGONG

NSW AUSTRALIA

E-mail: Martin_Bunder@info-gw.uow.edu.au

Sorting: The first address is Professor Barendregt's and Professor Dekkers's; the second address is Professor Bunder's.