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# Classical versions of BCI, BCK and BCIW logics

Martin W. Bunder

*University of Wollongong, mbunder@uow.edu.au*

John K. Slaney

*Australian National University*

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# Classical versions of BCI, BCK and BCIW logics

## **Abstract**

The question is, is there a formula  $X$ , independent of B,C,K1, I and W that creates distinct subclassical logics BCIX,BCKX and BCIWX, while BCKWX is the full classical implicational logic TV?

## **Keywords**

logics, bciw, bck, classical, bci, versions

## **Disciplines**

Engineering | Science and Technology Studies

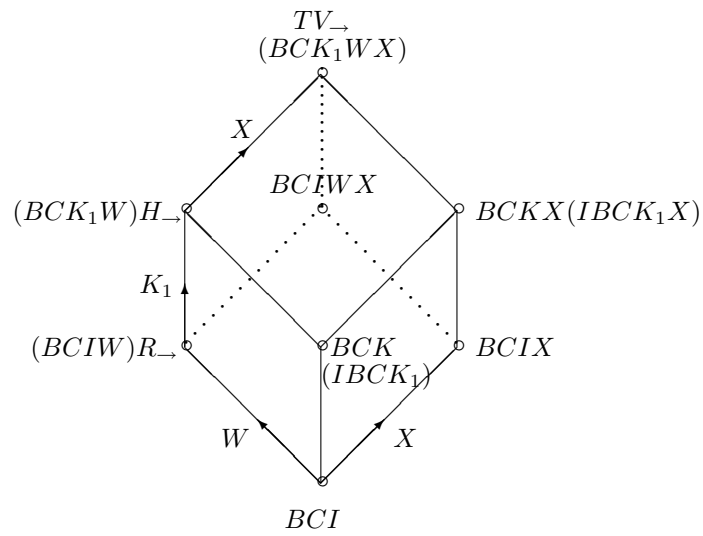
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John K. Slaney  
 Martin W. Bunder

## CLASSICAL VERSIONS OF *BCI*, *BCK* AND *BCIW* LOGICS

Karpenko in [2] raises an interesting problem which can be represented in the diagram below.



Each of the corners of the cube is to represent a distinct system of implicative logic based on some of the axioms:

- $I$  :  $p \rightarrow p$
- $B$  :  $(q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
- $C$  :  $(p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r))$
- $W$  :  $(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)$
- $K_1$  :  $(p \rightarrow q) \rightarrow (r \rightarrow (p \rightarrow q))$
- $X$  : ?

and the rules modus ponens and substitution.

The axioms shown on the cube for the logic  $BCI$  and the relevance logic  $R \rightarrow (BCIW)$  are well known to be independent. Karpenko shows that  $I, B, C$  and  $K_1$  and  $B, C, K_1$  and  $W$  are independent axioms for  $BCK$  logic and intuitionistic implicational logic ( $H_{\rightarrow}$ ) respectively.

The question is, is there a formula  $X$ , independent of  $B, C, K_1, I$  and  $W$  that creates distinct subclassical logics  $BCIX, BCKX$  and  $BCIWX$ , while  $BCKWX$  is the full classical implicational logic  $TV_{\rightarrow}$ ? In [1] Karpenko considers various candidates which do not meet all of the requirements. Since then however he has, in [3], found such an  $X$  (which we will call  $X_k$ ):

$$X_k : (p \rightarrow ((q \rightarrow q) \rightarrow p)) \rightarrow (((p \rightarrow q) \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow p)).$$

He has also extended the work so that he now has an alternative to  $C$  and one to  $K_1$  which are independent of each other and of  $B, W, X_k$  as well as  $I$ .

Independently the present authors arrived at another version of  $X$ . We show here that:

$$X : (((p \rightarrow q) \rightarrow q) \rightarrow p) \rightarrow (((((q \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow r) \rightarrow r)$$

meets the requirements. We also show that our  $BCKX$  and  $BCIWX$  have  $BCKX_k$  and  $BCIWX_k$  respectively as proper subsystems. Our  $X$  is not provable in  $BCIX_k$ , but whether  $BCIX_k$  is a subsystem of our  $BCIX$  is still open.

Other interesting open questions are:

- (1) Is there an infinite number of distinct systems  $BCIX_i, BCKX_i$  and  $BCIWX_i$ ?
- (2) Is there a weakest and stronger system  $BCIX_i, BCKX_i$  or  $BCIWX_i$ ?

Our  $X$  is due to Meyer and Parks [4], who proposed it as the independent axioms for the system  $RM_{\rightarrow}$ .  $BCIWX$  is in fact equivalent to  $RM_{\rightarrow}$ .

Our results are expressed as the following theorems:

**THEOREM 1.**  $H_{\rightarrow} + X = TV_{\rightarrow}$

**PROOF.**  $I, C \vdash ((p \rightarrow r) \rightarrow r) \rightarrow [(((p \rightarrow r) \rightarrow r) \rightarrow p) \rightarrow p]$ ,  
so  $I, C, B, X \vdash ((p \rightarrow r) \rightarrow r) \rightarrow [(((r \rightarrow p) \rightarrow p) \rightarrow r) \rightarrow p]$   
and  $I, C, B, X, W \vdash (((((p \rightarrow q) \rightarrow p) \rightarrow p) \rightarrow (p \rightarrow q)) \rightarrow p) \rightarrow p$   
(with  $p \rightarrow q/r$ ).



$$BCKW \neq BCIWX, \quad BCIW \neq BCIWX, \quad BCKX \neq BCKW, \\ BCIW \neq BCKW, \quad BCIW \neq BCIW$$

The matrices in (ii) and (iii), as well as a discussion on  $RM_{\rightarrow}$ , appear in Anderson and Belnap [1].

**THEOREM 3.**  $X$  is not provable in  $BCIWX_k$  or  $BCKX_k$  and so not in  $BCIX_k$ .

**PROOF.**

(i) All theorems of  $BCIWX_k$  satisfy the following matrix (generated by MaGIC [6]), where 1, 2 and 3 are designated values.

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	1	0	3
2	0	0	2	3
3	0	0	0	3

Our  $X$  has value 0 when  $p = 2$ ,  $q = 1$  and  $r = 0$ .

(ii) All theorems of  $BCKX_k$  satisfy the following matrix (generated by MaGIC [6]), where the designated value is 3.

$\rightarrow$	0	1	2	3
0	3	2	3	3
1	2	3	3	3
2	2	2	3	3
3	0	1	2	3

Our  $X$  has value 2 when  $p = 1$ ,  $q = 0$  and  $r = 2$ .

**THEOREM 4.**  $X_k$  is provable in  $BCKX$  and in  $BCIWX$ .

**PROOF.**

(i) By $I$ and $C$ ,	$(p \rightarrow q) \rightarrow q \vdash (((p \rightarrow q) \rightarrow q) \rightarrow p) \rightarrow p.$
By $K$	$\vdash p \rightarrow ((q \rightarrow p) \rightarrow p),$
so by $B$ and $C$ ,	$\vdash (((q \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow (p \rightarrow q).$
Therefore	$q \rightarrow p, (p \rightarrow q) \rightarrow q \vdash (((q \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow p.$
Then using (1) and $X$	$q \rightarrow p, (p \rightarrow q) \rightarrow q \vdash p,$
hence	$\vdash ((p \rightarrow q) \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow p)$

and  $\vdash X_k$  follows by  $K$ .

(ii) The system  $BCIWX$  is the system  $RM \rightarrow$  of Anderson and Belnap [1]. R. K. Meyer shows in [1] that a formula  $Y$  is provable in  $RM \rightarrow$  iff it has only nonnegative valuations  $v(Y)$ , where  $v$  is defined over the integers as follows:

$$\begin{aligned} v(p \rightarrow q) &= \min(-v(p), v(q)) \text{ if } v(p) > v(q) \\ &= \max(-v(p), v(q)) \text{ if } v(p) \leq v(q). \\ \text{as } v(X_k) &= \max(|v(p)|, |v(q)|), X_k \text{ is provable in } BCIWX. \end{aligned}$$

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Automated Reasoning Project  
Australian National University  
Cannberra ACT 0200  
Australia

Department of Mathematics  
University of Wollongong  
Wollongong NSW 2522  
Australia