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Classical versions of BCI, BCK and BCIW logics

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Abstract
The question is, is there a formula X, independent of B,C,K1, I and W that creates distinct subclassical logics BCIX,BCKX and BCIWX, while BCKWX is the full classical implicational logic TV?

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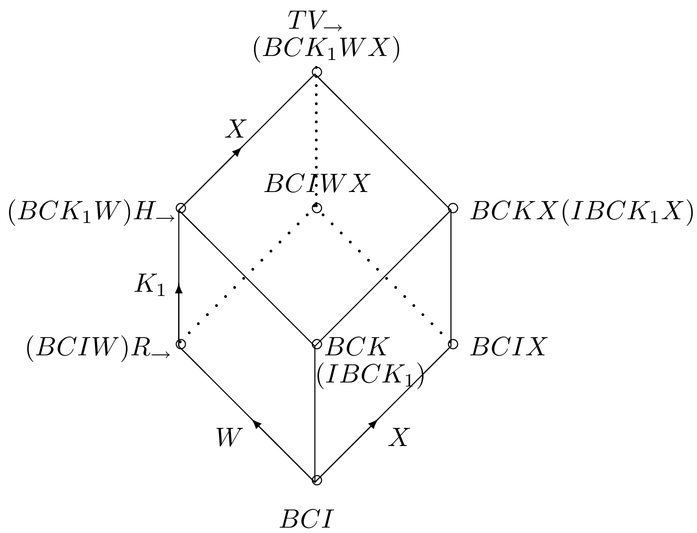
CLASSICAL VERSIONS OF BCI, BCK AND $BCIW$ LOGICS

Karpenko in [2] raises an interesting problem which can be represented in the diagram below.



Each of the corners of the cube is to represent a distinct system of implicational logic based on some of the axioms:

$$
\begin{array}{rcl}
I & : & p \rightarrow p \\
B & : & (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)) \\
C & : & (p \rightarrow (q \rightarrow r)) \rightarrow (q \rightarrow (p \rightarrow r)) \\
W & : & (p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q) \\
K_1 & : & (p \rightarrow q) \rightarrow (r \rightarrow (p \rightarrow q)) \\
X & : & ?
\end{array}
$$

and the rules modus ponens and substitution.

The axioms shown on the cube for the logic BCI and the relevance logic $R \rightarrow (BCIW)$ are well known to be independent. Karpenko shows that I, B, C and K_1 and B, C, K_1 and W are independent axioms for BCK logic and intuitionistic implicational logic ($H_{\rightarrow}$) respectively.

The question is, is there a formula X, independent of B, C, K_1, I and W that creates distinct subclassical logics $BCIX, BCKX$ and $BCIWX$, while $BCKWX$ is the full classical implicational logic $TV_{\rightarrow}$? In [1] Karpenko considers various candidates which do not meet all of the requirements. Since then however he has, in [3], found such an X (which we will call X_k):

$$X_k : (p \rightarrow ((q \rightarrow q) \rightarrow p)) \rightarrow (((p \rightarrow q) \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow p)).$$

He has also extended the work so that he now has an alternative to C and one to K_1 which are independent of each other and of B, W, X_k as well as I.

Independently the present authors arrived at another version of X. We show here that:

$$X : ((((p \rightarrow q) \rightarrow q) \rightarrow p) \rightarrow r) \rightarrow (((((q \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow r) \rightarrow r)$$

meets the requirements. We also show that our $BCKX$ and $BCIWX$ have $BCKX_k$ and $BCIWX_k$ respectively as proper subsystems. Our X is not provable in $BCIX_k$, but whether $BCIX_k$ is a subsystem of our $BCIX$ is still open.

Other interesting open questions are:

(1) Is there an infinite number of distinct systems $BCIX_i, BCKX_i$ and $BCIWX_i$?

(2) Is there a weakest and stronger system $BCIX_i, BCKX_i$ or $BCIWX_i$?

Our X is due to Meyer and Parks [4], who proposed it as the independent axioms for the system $RM_{\rightarrow}$. $BCIWX$ is in fact equivalent to $RM_{\rightarrow}$.

Our results are expressed as the following theorems:

THEOREM 1. $H_{\rightarrow} + X = TV_{\rightarrow}$

PROOF. $I, C \vdash ((p \rightarrow r) \rightarrow r) \rightarrow [(((p \rightarrow r) \rightarrow r) \rightarrow p) \rightarrow p]$,
so $I, C, B, X \vdash ((p \rightarrow r) \rightarrow r) \rightarrow [((((r \rightarrow p) \rightarrow p) \rightarrow r) \rightarrow p) \rightarrow p]$
and $I, C, B, X, W \vdash (((((p \rightarrow q) \rightarrow p) \rightarrow p) \rightarrow (p \rightarrow q)) \rightarrow p) \rightarrow p$
(with $p \rightarrow q/r$).

$$B, C \vdash (p \to v) \to [((p \to s) \to p) \to ((v \to s) \to p)],$$

so
$$K, B, C \vdash ((p \to s) \to p) \to (((u \to p) \to s) \to p)$$
(with $u \to p/v$)

and
$$I, C, B, X, W, K \vdash [(p \to (p \to q)) \to p] \to p$$
(with $p \to q/s, (p \to q) \to p/u$).

Finally
$$W, B \vdash ((p \to q) \to p) \to ((p \to (p \to q)) \to p)$$

so
$$I, C, B, X, W, K \vdash ((p \to q) \to p) \to p.$$

The above proof was discovered using the automatic theorem prover SCOTT (see [5]).

This formula strengthens intuitionistic implicational logic to classical implicational logic.

THEOREM 2.

(i) $BCKWX \neq BCKW$

(ii) $BCIWX \neq BCK_1WX$

(iii) $BCK_1X \neq BCK_1WX$

PROOF.

(i) It follows from Theorem 1 that X is a classical but not an intuitionistic tautology. It is therefore not derivable in BCK_1W.

(ii) It is easy to show that every theorem of $BCIWX$ logic has value 1 or 2 under the given matrix, but K_1 does not.

$\to$	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2

(iii) It is easy to show that every theorem of BCK_1X has value 2 under the given matrix but that W does not.

$\to$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

Note that other inequalities come directly from these. For example from (i)

$$BCKW \neq BCIWX, \quad BCIW \neq BCIWX, \quad BCKX \neq BCKW,$$
$$BCIX \neq BCKW, \quad BCIX \neq BCIW$$

The matrices in (ii) and (iii), as well as a discussion on $RM_{\rightarrow}$, appear in Anderson and Belnap [1].

THEOREM 3. X is not provable in $BCIWX_k$ or $BCKX_k$ and so not in $BCIX_k$.

PROOF.

(i) All theorems of $BCIWX_k$ satisfy the following matrix (generated by MaGIC [6]), where 1, 2 and 3 are designated values.

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	1	0	3
2	0	0	2	3
3	0	0	0	3

Our X has value 0 when $p = 2,\ q = 1$ and $r = 0$.

(ii) All theorems of $BCKX_k$ satisfy the following matrix (generated by MaGIC [6]), where the designated value is 3.

$\rightarrow$	0	1	2	3
0	3	2	3	3
1	2	3	3	3
2	2	2	3	3
3	0	1	2	3

Our X has value 2 when $p = 1,\ q = 0$ and $r = 2$.

THEOREM 4. X_k is provable in $BCKX$ and in $BCIWX$.

PROOF.

(i) By I and C,	$(p \rightarrow q) \rightarrow q \vdash (((p \rightarrow q) \rightarrow q) \rightarrow p) \rightarrow p.$
By K	$\vdash p \rightarrow ((q \rightarrow p) \rightarrow p),$
so by B and C,	$\vdash (((q \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow (p \rightarrow q).$
Therefore	$q \rightarrow p, (p \rightarrow q) \rightarrow q \vdash (((q \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow p.$
Then using (1) and X	$q \rightarrow p, (p \rightarrow q) \rightarrow q \vdash p,$
hence	$\vdash ((p \rightarrow q) \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow p)$

and $\vdash X_k$ follows by K.

(ii) The system $BCIWX$ is the system $RM \rightarrow$ of Anderson and Belnap [1]. R. K. Meyer shows in [1] that a formula Y is provable in $RM \rightarrow$ iff it has only nonnegative valuations $v(Y)$, where v is defined over the integers as follows:

$$\begin{array}{rrcl} & v(p \rightarrow q) & = & \min(-v(p), v(q)) \;\text{ if } v(p) > v(q) \\ & & = & \max(-v(p), v(q)) \;\text{ if } v(p) \leq v(q). \\ \text{as} & v(X_k) & = & \max(\mid v(p) \mid, \mid v(q) \mid), \; X_k \;\text{ is provable in } BCIWX. \end{array}$$

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