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Publication Details

Bunder, M. W. (1998). Cancellation laws for BCI-algebra, atoms and p-semisimple BCI-algebras. *Scientiae Mathematicae Japonicae*, 1 (1), 19-22.

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Abstract

We derive cancellation laws for BCI-algebras and for p-semisimple BCI-algebras, show that the set of all atoms of a BCI-algebra is a p-semisimple BCI-algebra and that in a p-semisimple BCI-algebra \wedge and $=$ are the same.

Keywords

p, semisimple, algebras, atoms, cancellation, algebra, bci, laws

Disciplines

Engineering | Science and Technology Studies

Publication Details

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**CANCELLATION LAWS FOR BCI-ALGEBRA, ATOMS AND
P-SEMISIMPLE BCI-ALGEBRAS**

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Received July 8, 1997

ABSTRACT. We derive cancellation laws for *BCI*-algebras and for *p*-semisimple *BCI*-algebras, show that the set of all atoms of a *BCI*-algebra is a *p* semisimple *BCI*-algebra and that in a *p*-semisimple *BCI*-algebra \leq and $=$ are the same.

1. Introduction. *BCI*-algebras, first introduced by Iséki in [1], can be defined as follows:

Definition 1 An algebra $\langle X; *, 0 \rangle$ of type $(2, 0)$ is a *BCI*-algebra if for all $x, y, z \in X$.

- BCI*-1 $(x * y) * (x * z) \leq z * y$;
- BCI*-2 $x * (x * y) \leq y$;
- BCI*-3 $x \leq x$;
- BCI*-4 $x \leq y$ and $y \leq x$ imply $x = y$;
- BCI*-5 $x \leq y$ iff $x * y = 0$.

The following well known properties of *BCI*-algebras are used below.

- (1) $(x * y) * z = (x * z) * y$
- (2) $0 * (x * y) = (0 * x) * (0 * y)$
- (3) $x * 0 = x$
- (4) $x * (x * (x * y)) = x * y$
- (5) $x * x = 0$
- (6) $x \leq 0 \Rightarrow x = 0$.

2. A Cancellation law for BCI-Algebras.

Theorem 1 If $\langle X; *, 0 \rangle$ is a *BCI*-algebra and $x, y, z \in X$ then:

- (i) $x * y \leq x * z \Rightarrow 0 * y = 0 * z$;
- (ii) $y * x \leq z * x \Rightarrow 0 * y = 0 * z$.

Proof (i) If $x * y \leq x * z$, by *BCI*-5,

$$(x * y) * (x * z) = 0$$

and so by *BCI*-1 and *BCI*-5,

$$0 * (z * y) = 0 \tag{a}$$

and by (2),

$$(0 * z) * (0 * y) = 0.$$

1991 *Mathematics Subject Classification.* 06F35.

Key words and phrases. BCI-algebras.

Hence by *BCI-5*

$$0 * z \leq 0 * y.$$

We now apply the same cancellation procedure to this as we did to $x * y \leq x * z$, this time “cancelling” the 0 to give:

$$0 * y \leq 0 * z$$

$$\therefore 0 * y = 0 * z.$$

(ii) If $y * x \leq z * x$, by *BCI-5*,

$$(y * x) * (z * x) = 0.$$

BCI-1 and (1) give

$$((y * x) * (z * x)) * (y * z) = 0$$

so

$$0 * (y * z) = 0 \tag{-b)}$$

giving, as above,

$$0 * y \leq 0 * z.$$

As in (i) this gives $0 * y = 0 * z$.

Corollary If $\langle X; *, 0 \rangle$ is a *BCI*-algebra and $x, y, z \in X$ then

$$(i) \quad x * y = x * z \quad \Rightarrow \quad 0 * y = 0 * z$$

$$(ii) \quad y * x = z * x \quad \Rightarrow \quad 0 * y = 0 * z.$$

We have two further properties resulting from the above cancellation laws:

Theorem 2 If $\langle X; *, 0 \rangle$ is a *BCI*-algebra and $x, y, z \in X$ then:

$$(i) \quad x \leq x * z \quad \Rightarrow \quad 0 \leq z$$

$$(ii) \quad x * y \leq x \quad \Rightarrow \quad 0 \leq y.$$

Proof (i) If $x \leq x * z$, by (3) $x * 0 \leq x * z$ and so by Theorem 1 (i) $0 * z = 0 * 0$. This gives $0 * z = 0$ ie $0 \leq z$.

(ii) If $x * y \leq x$, by (3), $x * y \leq x * 0$ and so by Theorem 1 (ii) $0 * y = 0 * 0 = 0$, so $0 \leq y$.

3. P-Semisimple Algebras. These were introduced by Lei and Xi in [2] as follows:

Definition 2 A *BCI*-algebra $\langle X; *, 0 \rangle$ is p-semisimple if

$$(\forall x \in X)(0 * x = 0 \quad \Rightarrow \quad x = 0).$$

In these algebras we find that \leq becomes the same as $=$.

Theorem 3 If $\langle X; *, 0 \rangle$ is a p-semisimple *BCI*-algebra and $x, y \in X$ then if $x \leq y$ also $x = y$.

Proof If $x \leq y$, $x * y = 0$ by *BCI-5*. Also by (5), $x * y = x * x$, so by the corollary to Theorem 1, $0 * y = 0 * x$.

As $(0 * x) * (0 * x) = 0$, we have $(0 * y) * (0 * x) = 0$ and by (2), $0 * (y * x) = 0$.

As *BCI*-algebras are closed under $*$, $y * x \in X$, so if the algebra is p-semisimple, $y * x = 0$.

By *BCI-4*, $x = y$.

Our cancellation laws can now be strengthened.

Theorem 4 If $\langle X; *, 0 \rangle$ is a p-semisimple *BCI*-algebra and $x, y, z \in X$ then:

$$(i) \quad x * y \leq x * z \quad \Rightarrow \quad y = z;$$

$$(ii) \quad y * x \leq z * x \quad \Rightarrow \quad y = z.$$

Proof (i) If $x * y \leq x * z$, by Theorem 1(i) we get $0 * z = 0 * y$ and so $(0 * z) * (0 * y) = 0$. By (2) this gives $0 * (z * y) = 0$, so if the algebra is p-semisimple we have $z * y = 0$ i.e. $z \leq y$. The result then follows from Theorem 3.

(ii) Similar.

Corollary If $\langle X; *, 0 \rangle$ is a p-semisimple *BCI*-algebra and $x, y, z \in X$ then

- (i) $x * y = x * z \Rightarrow y = z$;
- (ii) $y * x = z * x \Rightarrow y = z$.

4. Atoms. Meng and Xin in [5] introduced the notion of atom and the class of all atoms of a *BCI*-algebra.

Definition 3 An element of a *BCI*-algebra $\langle X; *, 0 \rangle$ is an atom if

$$(\forall z \in X)(z * a = 0 \Rightarrow z = a)$$

Definition 4 $L(X) = \{x \in X \mid a \text{ is an atom of } X\}$

Meng and Xin prove in [5]:

Theorem 5 If $\langle X; *, 0 \rangle$ is a *BCI*-algebra then

- (i) a is an atom iff $a = 0 * (0 * a)$;
 - (ii) $(\forall x \in X) 0 * x \in L(X)$.
- ((ii) also follows from (4) and (i).)

The following simple representation of $L(X)$ results:

Theorem 6 $L(X) = \{0 * x \mid x \in X\}$.

Meng and Xin prove that $L(X)$ is a *BCI*-algebra. The following result of Lei and Xi [2]:

Theorem 7 If $\langle X; *, 0 \rangle$ is a *BCI*-algebra then X is p-semisimple iff

$$(\forall x \in X) 0 * (0 * x) = x.$$

and Theorem 5(i) give us:

Theorem 8 If $\langle X; *, 0 \rangle$ is a *BCI*-algebra $\langle L(X); *, 0 \rangle$ is a p-semisimple *BCI*-algebra.

A final result on $L(X)$ is the following:

Theorem 9 If $\langle X; *, 0 \rangle$ is a *BCI*-algebra then $L(L(X)) = L(X)$.

Proof By Theorem 6,

$$\begin{aligned} L(L(X)) &= \{0 * x \mid x \in L(X)\} \\ &= \{0 * (0 * y) \mid y \in X\} \end{aligned}$$

Similarly

$$L(L(L(X))) = \{0 * (0 * (0 * z)) \mid z \in X\},$$

so by (4)

$$L(L(L(X))) = L(X).$$

Hence as $L(L(L(X))) \subseteq L(L(X)) \subseteq L(X)$ we have $L(L(X)) = L(X)$.

5. Powers. In [2] Lei and Xi define a new operation $+$ by:

Definition 5 $x + y = x * (0 * y)$

and show that if $\langle X; *, 0 \rangle$ is a p-semisimple *BCI*-algebra then $\langle X, + \rangle$ is an abelian group.

In [3] Meng and Wei use the same operation to define powers of elements by:

$$\begin{aligned} x^1 &= x \\ x^{n+1} &= x * (0 * x^n), \end{aligned}$$

(though mx instead of x^m might have been in better keeping with $+$).

The following are new properties of this form of exponentiation:

Theorem 10 If x is an element of a *BCI*-algebra $\langle X; *, 0 \rangle$ then:

- (i) $(0 * x)^n = 0 * x^n$;
- (ii) $(0 * x)^n = (\dots((0 * x) * x)\dots) * x$

(where there are n x s on the right hand side).

Proof (i) By induction on n .

$n = 1$ - obvious.

Assuming (i) for n ,

$$\begin{aligned}
 (0 * x)^{n+1} &= (0 * x) * (0 * (0 * x)^n) \\
 &= (0 * x) * (0 * (0 * x^n)) && \text{-(c)} \\
 &= 0 * (x * (0 * x^n)) && \text{by (2)} \\
 &= 0 * x^{n+1}
 \end{aligned}$$

(ii) By induction on n .

$n = 1$ - obvious.

Assuming (ii) for n , by (c) above, (1) and (4):

$$\begin{aligned}
 (0 * x)^{n+1} &= (0 * (0 * (0 * x^n))) * x \\
 &= (0 * x^n) * x \\
 &= (0 * x)^n * x && \text{by (i)} \\
 &= (\dots((0 * x) * x)\dots) * x.
 \end{aligned}$$

as required.

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