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## Abstract

This paper gives a simple, elegant statement of the condensed detachment rule that is independent of most general unifiers and proves that this is equivalent to the longer, more usual, formulation.

## Keywords

detachment, condensed, simplified, form

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# A Simplified Form of Condensed Detachment

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**Abstract.** This paper gives a simple, elegant statement of the condensed detachment rule that is independent of most general unifiers and proves that this is equivalent to the longer, more usual, formulation.

**Key words:** Condensed detachment, unification.

## 1. Introduction

The idea of *condensed detachment* is that, given premises $\alpha \rightarrow \beta$ and $\gamma$, we can conclude $\delta$ where $\delta$ is the most general result that can be obtained by using a substitution instance $\xi$ of $\gamma$ as a minor premise with the substitution instance $\xi \rightarrow \delta$ of $\alpha \rightarrow \beta$ as major premise in modus powers.

Condensed detachment was first introduced by C.A. Meredith (see Prior (1955)) and is a simple form of Robinson's unification and resolution (Robinson (1965)).

An accurate statement of condensed detachment has, to date, required a definition of most general unifier (*m.g.u*). Also the simplest form of the rule, used by some, was shown to be inadequate by Hindley, who has added appropriate restrictions (see Hindley and D. Meredith (1990)).

The present note gives a simple, elegant statement of condensed detachment, independent of *m.g.u.s*, that is more in keeping with the above "idea". It also may be useful in metamathematical investigations of logical systems based on the rule.

Our version of the rule is shown to be equivalent to Hindley's.

## 2. Most general unifoers and condensed detachment

A *unifier* of a pair of formulas $\alpha$ and $\beta$ with no variables in common, is a substitution $\sigma$ such that

$$\sigma(\alpha) = \sigma(\beta)\,.$$

$\sigma(\alpha)$ is then a *unification* of $\alpha$ and $\beta$.
$\sigma$ is a *most general unifier* (*m.g.u.*) of $\alpha$ and $\beta$ and $\sigma(\alpha)$ is a *most general unification*

(*m.g.u*) of $\alpha$ and $\beta$ if every other unification of $\alpha$ and $\beta$ is a substitution instance of $\sigma(\alpha)$.

A substitution $\sigma$ is said to be *alphabetic*, relative to a formula $\alpha$, if it replaces all or some of the variables in $\alpha$ by variables distinct from each other and from any variables in $\alpha$ that are not replaced. Note that an alphabetic substitution $\sigma$, relative to $\alpha$, always has an inverse $\sigma^{-1}$ such that $\sigma^{-1}\big(\sigma(\alpha)\big) = \alpha$. If $\sigma$ is alphabetic relative to $\alpha$ we call $\sigma(\alpha)$ an *alphabetic variant* of $\alpha$.

We can now state Hindley's version of the condensed detachment rule:

**RULE D** Premises: any ordered pair of formulas $\xi, \gamma$
Conclusion: a formula called $D\xi\gamma$ constructed as follows:

If $\xi$ is a variable $a$, define $Da\gamma \equiv a$.
If $\xi \equiv \alpha \rightarrow \beta$, then change $\gamma$ to an alphabetic variant $\gamma'$ with no variables in common with $\alpha$, compute an *mgu* $\sigma$ of $\{\alpha, \gamma'\}$; next change $\sigma$ to an alphabetic variant $\sigma^*$ such that no new variables[1] in $\sigma^*(\alpha)$ occur in $\beta$, then define

$$D(\alpha \rightarrow \beta)\gamma = \sigma^*(\beta)\,.$$

The Unification Theorem of Robinson (1965) proves that if $\alpha$ and $\gamma$ have a unifier then they have an *m.g.u.* Hindley (1990) shows that $D(\alpha \rightarrow \beta)\gamma$, if defined, is unique except for alphabetic variants.

## 3. A new rule equivalent to D

We now state the new rule, called Rule $D_1$ for the time being.

**Rule $D_1$** Premises: any ordered pair of formulas $\xi, \gamma$
Conclusion: a formula called $D_1\xi\gamma$ constructed as follows:

If $\xi$ is a variable, define $D_1 a\gamma = a$.
If $\xi \equiv \alpha \rightarrow \beta$, define $D_1(\alpha \rightarrow \beta)\gamma \equiv \sigma_1(\beta)$, if there are substitutions $\sigma_1$ and $\sigma_2$ such that:

$$\sigma_1(\alpha) = \sigma_2(\gamma) \tag{1}$$

given (1), the number of variable occurrences in $\sigma_1(\beta)$ is minimal (2)

given (1) and (2), the number of distinct variables in $\sigma_1(\beta)$ is maximal. (3)

THEOREM 1 If $D(\alpha \rightarrow \beta)\gamma$ or $D_1(\alpha \rightarrow \beta)\gamma$ is defined then both are defined and are identical except for alphabetic variants.

*Proof* (i) Assume that $D(\alpha \rightarrow \beta)\gamma$ is defined. Let $\sigma_0(\gamma)$ be an alphabetic variant of $\gamma$ such that $\sigma_0(\gamma)$ has no variables in common with $\alpha \rightarrow \beta$. Let $\sigma$ be an *m.g.u.* of $\alpha$ and $\sigma_0(\gamma)$, then

$$\sigma(\alpha) = \sigma\big(\sigma_0(\gamma)\big) .$$

Now change $\sigma$ to $\sigma^*$ so that no new variables in $\sigma^*(\alpha)$ occur in $\beta$.

Let $\sigma_1 = \sigma^*$ and $\sigma^2 = \sigma^* \circ \sigma_0$,[2] then

$$\sigma_1(\alpha) = \sigma_2(\gamma) . \tag{1}$$

If (2) or (3) fails, i.e. if the number of variable occurrences in $\sigma_1(\beta)$ is not minimal or the number of distinct variables in $\sigma_1(\beta)$ is not maximal, there are substitutions $\sigma_1'$, $\sigma_2'$ and $\sigma_3$ such that:

$$\sigma_1'(\alpha) = \sigma_2'(\gamma) ,$$

and

$$\sigma_3\big(\sigma_1'(\beta)\big) = \sigma_1(\beta) ,$$

where $\sigma_3$ is not alphabetic (i.e. $\sigma_1'(\beta)$ has fewer variable occurrences than $\sigma_1(\beta)$, or the same number, but fewer distinct variable occurrences).

Now let[3]

$$\sigma_4 = \sigma_1' \cup (\sigma_2' \circ \sigma_0^{-1}) ;$$

then as $\alpha$ and $\sigma_0(\gamma)$ have no variables in common, it follows from

$$\sigma_1'(\alpha) = \sigma_2'\Big(\sigma_0^{-1}\big(\sigma_0(\gamma)\big)\Big)$$

that

$$\sigma_4(\alpha) = \sigma_4\big(\sigma_0(\gamma)\big) .$$

Also $\beta$ and $\sigma_0(\gamma)$ have no variables in common, so

$$\sigma_4(\beta) = \sigma_1'(\beta)$$

and so

$$\sigma_1 = \sigma_3 \circ \sigma_4 ,$$

where $\sigma_2$ is not alphabetic.

Thus $\sigma_1 = \sigma^*$ is not a most general unifier and neither is $\sigma$. This is a contradiction and so (2) and (3) hold and so

$$D_1(\alpha \to \beta)\gamma = \sigma_1(\beta) = \sigma^*(\beta) = D(\alpha \to \beta)\gamma .$$

(ii) Assume that $D_1(\alpha \to \beta)\gamma$ is defined using substitutions $\sigma_1$ and $\sigma_2$. Let $\sigma_1^*$ be the restriction of $\sigma_1$ to variables in $\alpha \to \beta$ and $\sigma_2^*$ the restriction to variables in $\gamma$; then $D_1(\alpha \to \beta)\gamma$ is also (identically) defined using $\sigma_1^*$ and $\sigma_2^*$.

Now define $\sigma_0(\gamma)$ as above.

$D_1(\alpha \to \beta)\big(\sigma_0(\gamma)\big)$ is then defined using $\sigma_1^*$ and $\sigma_2^* \circ \sigma_0^{-1}$ and, as $\alpha \to \beta$ and $\sigma_0(\gamma)$ have no variables in common, these substitutions replace disjoint sets of variables.

Hence if we let

$$\sigma = \sigma_1 \cup (\sigma_2 \circ \sigma_0^{-1})$$

we have

$$\sigma(\alpha) = \sigma_1(\alpha)$$

and

$$\sigma\big(\sigma_0(\gamma)\big) = \sigma_2(\gamma) = \sigma(\alpha)\,.$$

Thus $\sigma$ is a unifier of $\alpha$ and $\sigma_0(\gamma)$.

If $\sigma$ is not an *m.g.u.* of $\alpha$ and $\sigma_0(\gamma)$ there are substitutions $\sigma_3$ and $\sigma_4$, with $\sigma_4$ not trivial, such that:

$$\sigma_3(\alpha) = \sigma_3\big(\sigma_0(\gamma)\big)$$

and

$$\sigma_4 \circ \sigma_3 = \sigma\,.$$

Let $\sigma_{31}$ be the sub-substitution of $\sigma_3$ that applies to the variables to which $\sigma_1$ applies and $\sigma_{32}$ that which applies to the same variables as $|\sigma_2 \circ \sigma_0^{-1}$. Then as these sets of variables are disjoint we have

$$\sigma_{31}(\alpha) = \sigma_{32}\big(\sigma_0(\gamma)\big)$$

and

$$\sigma_4 \circ \sigma_{31} = \sigma_1\,.$$

However, as $\sigma_4$ is not trivial, applying $\sigma_4$ will increase the number of variable occurrences in $\sigma_{31}(\alpha)$ or decrease the number of distinct variables. This is impossible in view of clauses (2) and (3) of $D_1$.

Therefore $\sigma$ is an m.g.u.

Now we show that an alphabetic variant of our $\sigma$ is in fact the $\sigma^*$ in the definition of Rule D.

If the new variables of $\sigma(\alpha)$ that also occur in $\beta$ are $b_1, ..., b_n$ and these arise by substitution for $a_1, ..., a_m$, let

$$\sigma(a_i) = \delta_i\,;$$

then at least some $b_j$s will be in each $\delta_i$.

Let

$$\sigma(b_i) = \xi_i$$

(here we include $\xi_i = b_i$ if $\sigma$ does not change $b_i$).

There are then 2 subcases.

(a) There is a $b_i$ in a $\delta_j$ and also in a $\xi_k$.
Let $\sigma'$ be an alphabetic variant of $\sigma$ with a distinct variable $b'_i$, that does not appear in $\alpha \to \beta, \gamma, \sigma(\alpha \to \beta)$ or $\sigma(\gamma)$, for each $b_i$ in each $\delta_k$. Then

$$\begin{aligned}\sigma'(\alpha) &= ([b'_1/b_1, ..., b'_n/b_n]\,\sigma)(\alpha)\\ &= [b'_1/b_1, ..., b'_n/b_n]\,\sigma(\alpha)\end{aligned}$$

and $b_i$ is not in $\alpha$, and

$$\begin{aligned}\sigma'\big(\sigma_0(\gamma)\big) &= ([b'_1/b_1, ..., b'_n/b_n]\,\sigma)\big(\sigma_0(\gamma)\big)\\ &= [b'_1/b_1, ..., b'_n/b_n]\,\sigma\big(\sigma_0(\gamma)\big)\end{aligned}$$

as $b_i$ is not in $\sigma_0(\gamma)$.
Therefore

$$\sigma'(\alpha) = \sigma'\big(\sigma_0(\gamma)\big)\ .$$

So $\sigma'$ is a unification of $\alpha$ and $\sigma_0(\gamma)$.
Also every other unification is a substitution instance of $\sigma(\alpha)$ and so of $\sigma'(\alpha)$. Hence $\sigma'$ is an m.g.u. of $\alpha$ and $\sigma_0(\gamma)$.
However, the number of distinct variables of $\sigma'(\beta)$ has increased from that of $\sigma(\beta)$, as in addition to $b_1, ..., b_n$ at least one $b'_i$ is added. This is impossible by clause (2) of Rule $D_1$.

(b) No $b_i$ in a $\delta_j$ is also in a $\xi_k$.
This time the above alphabetic variant $\sigma'$ of $\sigma$ is still an m.g.u., but no new variables $b'_i$ of $\sigma'(\alpha)$ are in $\beta$, so $D(\alpha \to \beta)\gamma = \sigma'(\beta)$, which is an alphabetic variant of $\sigma(\beta) = \sigma_1(\beta) = D_1(\alpha \to \beta)\gamma$.

## Notes

[1] By this we mean variables appearing in $\sigma^*(\alpha)$ but not in $\alpha$.
[2] Composition is defined in the usual functional way so that $(\sigma \circ \sigma')(\delta) = \sigma\left(\sigma'(\delta)\right)$.
[3] $(\sigma \cup \sigma')(\delta)$ is defined to represent the substitutions $\sigma$ and $\sigma'$ applied simultaneously to disjoint sets of variables in $\delta$.

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