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Stress distributions beneath heaps and rat-hole problems for granular materials

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Stress Distributions Beneath Heaps and Rat-Hole Problems for Granular Materials

A thesis submitted in fulfillment of the requirements for the award of the degree of

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from

University of Wollongong

by

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School of Mathematics and Applied Statistics

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This thesis is submitted to the University of Wollongong, and has not been submitted for a degree to any other University or Institution.

Grant Matthew Cox

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Abstract

Granular materials are widely used throughout the world in many industries, and problems such as stable obstructions or uncontrolled flooding that occur impact significantly on the economies of such industries. Using the proper continuum mechanical theory of granular materials, two such problems are examined in this thesis. Firstly, the phenomenon of stable vertical cylindrical cavities known as rat-holes in stockpiles and hoppers that impede the flow of the granular material through the reclaim hole is examined. Secondly, the stress distribution at the base of a two and three-dimensional sand-pile is considered in search of the peak vertical pressure, which may not be located directly beneath the vertex of the sand-pile.

The rat-hole problem is well known but is not properly understood, and existing theory is unsatisfactory, in that it is believed not to properly incorporate actual material properties. Here the classical rat-hole theory of Jenike and his coworkers is re-examined, with a view to examining the validity of the so-called “stable rat-hole equation”, which is widely used in practice. Jenike’s original theory assumes a symmetrical stress distribution which is independent of height. However in practice, rat-holes tend to exhibit some tapering with height, and here the stress profiles corresponding to a symmetrical but slightly tapered circular cavity are determined. Existing theory for rat-holes applies only to the Coulomb-Mohr yield function. Here for an existing rat-hole, and assuming a shear-index granular material, the limiting stress profiles are determined which extends existing theory to a wider constitutive law.
The determination of the horizontal and vertical force distributions at the base of a sand-pile is by now a famous problem in granular theory. In 1981 it was suggested from experimental work that the peak vertical force at the base does not occur directly beneath the vertex of the pile, but at some intermediate point so that there is a ring of maximum vertical pressure. Practising engineers have some reservation on this result and numerous discrete theoretical and computational models of granular sand-piles have been proposed to explain this phenomenon, with varying degrees of success. Here for two and three-dimensional sand-piles the horizontal and vertical force distributions are estimated using the Jenike solutions for converging hoppers. For a two-dimensional sand-pile, a formal exact parametric solution is presented for the special case of the angle of internal friction equal to ninety degrees. Next, a sand-pile that is not entirely at yield is proposed, which has an inner dead region and an outer yield region. From this model a solution is determined which is not unique but does predict the dip in the vertical force as suggested from experimental work.

Finally, the exact parametric solution for the two-dimensional sand-pile problem for an angle of internal friction equal to ninety degrees is exploited to solve the flow of granular materials in the presence of gravity through a converging wedge shaped hopper, which are used in many industrial situations. This is the only known exact solution of these important equations which involves two arbitrary constants.
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Chapter 1

Introduction

1.1 Background

Granular materials are commonplace in everyday life. They are found in the kitchen cupboard in the form of sugar and coffee, at the beach where waves continually reshape the sand dunes, and in the workplace where numerous industries handle granular materials of all shapes and sizes. This wide diversity of occurrence in everyday life has given importance to the study and understanding of the behaviour of granular materials. For many industries, and indeed the economies of some countries such as Australia, being able to accurately predict the behaviour of granular materials under different conditions is vital to their survival and continued growth.

Granular materials are generally made up of two parts, solid particles and a fluid environment in which the solid particles are immersed. Depending upon the relative proportions of the two parts, the fluid environment may or may not be important. Accordingly, granular material behaviour spans the entire spectrum ranging from
essentially fluid behaviour, but which is slightly perturbed by the presence of a small number of particles, through to granular-solid-like behaviour for closely packed arrangements. Consequently, granular materials constitute an intermediate state between solids and fluids and can behave like either, depending on the bulk density of the solids particles $\rho$, that is the mass of solid particles per unit volume. Thus, for some critical value $\rho_c$, fluid-like behaviour occurs for $\rho < \rho_c$ and granular-solid-like behaviour occurs for $\rho > \rho_c$. When density fluctuations occur throughout the material, unusual physical phenomenon can occur. In addition, when examining a given flow behaviour, the question as to whether it is the fluid environment or the solid particles that dominate the problem, needs to be considered. The two problems considered in this thesis are essentially static problems where dynamic flow does not occur. This means that the solid particles dominate the problem and the influence of the fluid environment is not critical.

The mechanical properties of granular materials are characterised by considerable strength in compression and little or no strength in tension. Their static behaviour is dominated by inter-particle friction while their dynamic behaviour hinges on their capacity to expand the void space, that is to dilate. This means that in reality, granular materials cannot flow without dilation and consequent decrease in the density $\rho$. However, unlike a fluid, the pressure in a granular material does not increase with depth. The weight of the material in a container is taken up by inter-particle friction and by the particle-wall friction, so that the pressure beneath a high column of granular material is frequently so small as to be negligible. For this reason, huge hoppers readily block up at the outlet because the weight of the
material above has little or no effect and the pressure at the outlet is essentially zero. In 1773, Coulomb [18] proposed that the resistance to slip in a granular material is comprised of two physical processes, namely a cohesive strength represented by the parameter \( c \) and inter-particle friction represented by \( \mu = \tan \delta \), where \( \delta \) is referred to as the angle of internal friction. Coulomb postulated that slip occurs when the value of shear stress exceeds the combined resistance arising from these two physical effects. This postulate is the foundation upon which the classical continuum theory of granular materials, which is used throughout this thesis, is based upon.

A well known example of a static friction phenomena is that all conical sand-piles possess the same angle of repose \( \theta_r \), irrespective of their size. By considering the equilibrium of a surface particle, the angle \( \theta_r \) is determined from the equation

\[
mg \sin \theta_r = c + mg \cos \theta_r \tan \delta, \tag{1.1}
\]

where \( \theta_r \) is taken to be the angle between the slope of the sand-pile and the horizontal base, \( m \) denotes the mass of a single particle, and of course, \( g \) denotes acceleration due to gravity.

Examples of unusual dynamic phenomena include density waves in the outflow of hoppers (Baxter et al [9]), segregation arising from the penetration of fine particles through the void space (Williams [56]), heap formation and the formation of convection cells under vibration (Clément et al [17] and Gallas et al [23]) and of course better known phenomena such as landslides and avalanches. Segregation, heap formation and convection cells are all related to the sifting of small particles through the void space and this process is considerably enhanced by vibration. For
example, it is well known for material transported by conveyor belts to separate the
particles, with the fine particles underneath and the coarse particles on top (Gallas
et al [24]). Convection cells arise because fine particles tend to be more frictional and
after penetrating to the bottom are compacted by the heavier particles. On vertical
vibration, this compaction prevents downward movement of the heavier particles,
but the upwards movements allows the smaller particles to run in under the heavier
particles. Consequently the smaller particles are locked into position, and the heavier
particles rise forming convection cells. Strangely enough, the denser the heavier
particles are, the easier it is to get them to rise under vibration (Williams [56]).

Because granular materials exhibit such diverse and complex behaviour, they
have attracted as much attention from physicists as from mathematicians and en-
gineers. Bideau and Hansen [10] found that particle-particle interactions are short
range, inelastic with energy dissipation and depend on the history of contact, size
and shape of the particles, and this configuration constitutes one of the most com-
plicated, many body systems found in nature. They provide a model for complex
phenomena such as earthquakes, forest fires, fluctuations in the stock market and
the change in the weather, and have given rise to the notions of “self-organised crit-
icality” and “theories for everything” (see Mehta and Barker [44] and Jaeger and

The question arises as to how to create an effective mathematical model of granu-
lar behaviour which is capable of successfully predicting macroscopic flow patterns in
terms of physically meaningful and measurable material parameters. Real granular
materials are not readily defined, in that real materials are not regular arrange-
ments of spherical particles, nor is their flow behaviour readily explained and there is no single generally accepted theoretical formulation. This is because the physical characteristics (particle size, porosity, packing, etc) of different materials give rise to behaviour which is so complex and variable that a single model is unlikely to account for the behaviour of all granular materials under all practical or experimental conditions. Mathematical models range from traditional continuum mechanics (Spencer [52]) to statistical mechanics (Kim and Woodcock [41] and Rietema [46]) to molecular dynamics modelling (Allen and Tildesley [2]) and cellular automata modelling (Savage [47]) and various hybridizations of these basic approaches. All theories make some speculation at the microscopic level as to the basic interaction between adjacent particles and since the underlying physics is uncertain and not well understood, this is the weak link which is common to all existing theories. Even for an isolated system comprising only two frictional particles which can collide and roll around each other, the physics is by no means properly understood.

1.2 Basic equations

Throughout this thesis the classical continuum theory of granular materials is used to examine the formation of stable vertical cylindrical tunnels called rat-holes, and in determining the horizontal and vertical stress distributions at the base of a sand-pile in order to explain an observed dip in the experimental stress profile. At the heart of this theory is a quantity denoted by \( \sigma \), which is the stress at a point in the
material and, in simple terms, is defined by

$$\sigma = \frac{F}{A},$$  \hspace{1cm} (1.2)\]

where $F$ is a force acting on the body of the material and $A$ is the cross-sectional area of which the force is acting upon. For a more in depth understanding of stress, Selvadurai [49] states that stresses within a body of material, or region, are the result of external forces applying on the body, or region. However, the measure of stress at a point within the body, or region, should be independent of the internal structure of the body. This idea of independence leads to the notion of vector components of stress at a point within the material that lie in the direction of the appropriate unit vectors. From Figure 1.1 it is clear that there are nine components of stress, which in the usual Cartesian coordinates $(x, y, z)$ are denoted by $\sigma_{xx}, \sigma_{xy},$ etc. These stress components are expressed in such a manner that the first suffix refers to the
direction of the normal to the surface on which the stress vector acts, and the second suffix refers to the particular component of the stress vector. Thus for example, \( \sigma_{xy} \) corresponds to the component in the \( y \) direction of the stress vector acting on the surface whose outward normal is in the \( z \) direction. Also, from Figure 1.1 it should be clear that the quantities \( \sigma_{xx}, \sigma_{yy} \) and \( \sigma_{zz} \) are normal components of stress while the remaining quantities \( \sigma_{xy}, \sigma_{yz}, \) etc. are shear components of stress. It should be noted that the stress components are symmetrical, which means that \( \sigma_{xy} = \sigma_{yx}, \sigma_{xz} = \sigma_{zx} \) and \( \sigma_{yz} = \sigma_{zy} \). Hence, there are only six unique components of stress.

Now, the classical continuum theory of granular materials is based on the assumptions that the stresses within the body are at equilibrium and at the point of yield. This means that the stresses must satisfy the equilibrium equations, which in the usual Cartesian coordinates \((x, y, z)\) are

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho f_x,
\]

\[
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \rho f_y,
\]

\[
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho f_z,
\]

where \( \rho \) is the density of the granular material and \( f_x, f_y, \) and \( f_z \) are forces that act on the granular body as shown in Figure 1.2. However, in some problems it is more convenient to work in a different set of coordinates. Indeed as the problems considered in this thesis are either wedge or cylindrical shaped sand-piles, then it is ideal to express the equilibrium equations in polar cylindrical coordinates \((r, \theta, z)\).
Figure 1.2: Forces on an rectangular element of granular material.

To do this, assume the usual cylindrical polar substitutions

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \quad (1.4) \]

where \( \theta \) is the angle between the positive \( r \) and \( x \) axes. Then from Hunter [31], shows that

\[
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{xz} & \sigma_{yz} & \sigma_{zz}
\end{bmatrix} = A \begin{bmatrix}
\sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\
\sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\
\sigma_{rz} & \sigma_{\theta z} & \sigma_{zz}
\end{bmatrix} A^T. \quad (1.5)
\]

where

\[
A = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}. \quad (1.6)
\]
This means that the equilibrium equations in cylindrical polar coordinates become

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r} = \rho f_r,
\]

\[
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{2 \sigma_{r\theta}}{r} = \rho f_\theta,
\]

where \( f_r \) and \( f_\theta \) are the appropriate combination of \( f_x \) and \( f_y \). The equilibrium equations (1.7) can be simplified further upon noting that both the wedge-shaped sand-pile and the cylindrical sand-pile have a symmetry around an axis, namely the \( z \) axis for the wedge-shaped sand-pile when the origin of the coordinate axis is set at the vertex of the sand-pile (see Figure 5.2, page 92), and the \( \theta \) axis for the cylindrical sand-pile. This gives rise to the assumption that the stress distribution is symmetric around the appropriate axes, and for the cylindrical sand-pile the equilibrium equations become

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r} = \rho f_r,
\]

\[
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = \rho f_z.
\]

A similar reduction can be done for the wedge shaped sand-pile.

As previously stated, the classical continuum theory of granular materials assumes that the material is at the point of yield. This assumption is incorporated into the mathematical model by assuming that the stresses within the granular body satisfy a yield condition which defines how the material yields. There are many such yield conditions (see Hunter [31] for examples), however, within this thesis only the
Coulomb-Mohr yield condition is considered, namely

$$|\tau| = c - \sigma^* \tan \delta,$$  \hspace{1cm} (1.9)

where $c$ is the cohesion and $\sigma^*$ and $\tau$ denote the normal and tangential components of compressive traction, which throughout this thesis are assumed to be positive in tension. This is the usual convention in continuum mechanics that positive forces are assumed to produce positive extensions. From the Coulomb-Mohr yield condition, the Mohr Circle diagram in Figure 1.3 is deduced. This enables the stresses to be expressed in the standard decomposition

$$\sigma_{rr} = -p + q \cos 2\psi, \quad \sigma_{zz} = -p - q \cos 2\psi, \quad \sigma_{rz} = q \sin 2\psi.$$  \hspace{1cm} (1.10)

where $p$ and $q$ are defined by

$$p = -\frac{1}{2}(\sigma_{rr} + \sigma_{zz}), \quad q = \frac{1}{2}\left\{(\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2\right\}^{1/2}.$$  \hspace{1cm} (1.11)
and $\psi$ is given by

$$\tan 2\psi = \frac{2\sigma_{rz}}{(\sigma_{rr} - \sigma_{zz})},$$

(1.12)

where physically $\psi$ is the angle between the maximum principal stress axis and the radial direction, in the direction of increasing $\theta$.

In Figure 1.3, the quantities denoted by $\sigma_I$ and $\sigma_{III}$ are the maximum and minimum principal stresses respectively, such that

$$\sigma_I \geq \sigma_{II} \geq \sigma_{III},$$

(1.13)

where $\sigma_{II}$ denotes the intermediate principal stress. The principal stresses are eigenvalues of the stress matrix, and therefore for an axially symmetric stress distribution around the $\theta$ axis, the principal stresses must satisfy

$$\begin{vmatrix}
\sigma_{rr} - \mu & \sigma_{rz} & 0 \\
\sigma_{rz} & \sigma_{zz} - \mu & 0 \\
0 & 0 & \sigma_{\theta\theta} - \mu
\end{vmatrix} = 0,$$

(1.14)

where $\mu$ denotes a principal stress and the intermediate principal stress $\sigma_{II}$ is assumed to coincide with the hoop stress $\sigma_{\theta\theta}$. Solving (1.14) for $\mu$, gives

$$\sigma_I = \frac{1}{2} \left\{ (\sigma_{rr} + \sigma_{zz}) + \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} \right\},$$

(1.15)

$$\sigma_{III} = \frac{1}{2} \left\{ (\sigma_{rr} + \sigma_{zz}) - \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} \right\}.$$

Therefore, following Shield [50] it can be shown that upon using (1.15), the Coulomb-Mohr yield function becomes

$$\sigma_I - \sigma_{III} = 2c \cos \delta - (\sigma_I + \sigma_{III}) \sin \delta,$$

(1.16)
Figure 1.4: The yield pyramid surface in principal stress space \((\sigma_1, \sigma_2, \sigma_3)\).

which upon solving explicitly for \(\sigma_I\) gives

\[
\sigma_I = 2c \left( \frac{1 - \beta}{1 + \beta} \right)^{1/2} + \left( \frac{1 - \beta}{1 + \beta} \right) \sigma_{III},
\]

(1.17)

where again noting that \(\beta = \sin \delta\). Also from Figure 1.3, it should be noted that \(f_c\) denotes the unconfined yield strength which is defined by \(\sigma_I = 0\) when \(\sigma_{III} = -f_c\), so that from the Mohr Circle diagram, or directly from (1.17), the unconfined yield strength can be expressed as

\[
f_c = 2c \left( \frac{1 + \beta}{1 - \beta} \right)^{1/2},
\]

(1.18)

where \(\beta = \sin \delta\) and \(\delta\) is the angle of internal friction. Now, equation (1.17) gives rise to the concept of a yield pyramid which describes the possible states of stress at which the granular material may yield, as shown in Figure 1.4. The yield pyramid is usually represented in three-dimensional principal stress space with a rectangular Cartesian coordinate \((\sigma_1, \sigma_2, \sigma_3)\) denoting a typical point, and the seven possible
Table 1.1: The seven possible plastic regimes for axially symmetric deformations.

<table>
<thead>
<tr>
<th>Plastic regime</th>
<th>Stress state</th>
<th>Yield function</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>$\sigma_1 = \sigma_2 &gt; \sigma_\phi$</td>
<td>$(1 + \sin \delta)\sigma_1 = 2c \cos \delta + (1 - \sin \delta)\sigma_\phi$</td>
</tr>
<tr>
<td>E</td>
<td>$\sigma_\phi &gt; \sigma_1 = \sigma_2$</td>
<td>$(1 + \sin \delta)\sigma_\phi = 2c \cos \delta + (1 - \sin \delta)\sigma_2$</td>
</tr>
<tr>
<td>AB</td>
<td>$\sigma_1 &gt; \sigma_2 &gt; \sigma_\phi$</td>
<td>$(1 + \sin \delta)\sigma_1 = 2c \cos \delta + (1 - \sin \delta)\sigma_\phi$</td>
</tr>
<tr>
<td>EF</td>
<td>$\sigma_\phi &gt; \sigma_1 &gt; \sigma_2$</td>
<td>$(1 + \sin \delta)\sigma_\phi = 2c \cos \delta + (1 - \sin \delta)\sigma_2$</td>
</tr>
<tr>
<td>A</td>
<td>$\sigma_1 &gt; \sigma_\phi = \sigma_2$</td>
<td>$(1 + \sin \delta)\sigma_1 = 2c \cos \delta + (1 - \sin \delta)\sigma_\phi$</td>
</tr>
<tr>
<td>F</td>
<td>$\sigma_\phi = \sigma_1 &gt; \sigma_2$</td>
<td>$(1 + \sin \delta)\sigma_\phi = 2c \cos \delta + (1 - \sin \delta)\sigma_2$</td>
</tr>
<tr>
<td>AF</td>
<td>$\sigma_1 &gt; \sigma_\phi &gt; \sigma_2$</td>
<td>$(1 + \sin \delta)\sigma_1 = 2c \cos \delta + (1 - \sin \delta)\sigma_2$</td>
</tr>
</tbody>
</table>

plastic regimes available for axially symmetric stress states arise from the general plane $\sigma_3 = \text{constant}$. Points on the varying hexagon represent all the possible plastic principal stress states given by (1.17) and are given in tabular form in Table 1.1.

Therefore, from (1.15), (1.17) and (1.18) the stress decomposition (1.10) can be rewritten in terms of just two unknown variables $\sigma$ and $\psi$, namely

$$
\sigma_{rr} = \sigma(\beta \cos 2\psi - 1) + (1 - \beta) \frac{f_c}{2\beta},
$$

$$
\sigma_{zz} = -\sigma(\beta \cos 2\psi + 1) + (1 - \beta) \frac{f_c}{2\beta}, \tag{1.19}
$$

$$
\sigma_{rz} = \beta \sigma \sin 2\psi,
$$

where $\sigma$ is defined from the Mohr Circle diagram (Figure 1.3). If (1.19) is substituted into the axially symmetric equilibrium equations (1.8), then two differential equations are determined for $\sigma$ and $\psi$, noting that the hoop stress $\sigma_\phi$ can be ex-
pressed in terms of $\sigma$ and $\psi$ using Table 1.1 and (1.15). These differential equations are examined in the following Chapters.

### 1.3 Overview of thesis

In this thesis, two important problems in granular materials using the classical mechanical continuum theory for granular materials are considered and are outlined below.

In Chapter 2, the formation of stable and almost vertical cylindrical cavities in stockpiles and hoppers, called "rat-holes", is examined. Figure 1.5 shows a large stockpile silo containing a stable rat-hole which is impeding the process of reclaiming the granular material from the silo. Figures 1.6, 1.7 and 1.8 show the stable rat-hole within the silo. Figure 1.7 is of particular interest in that it shows a stable and an unstable rat-hole within the same silo, noting that the silo has more then one reclaim hole. The unstable rat-hole has collapsed and the granular material is able to be reclaimed from that reclaim hole, whereas the stable rat-hole prevents any of the granular material to be reclaimed. Here, these rat-holes are examined using the classical rat-hole theory of Jenike [34] and Jenike and Yen [37]. It is shown that the so-called "stable rat-hole equation", which is widely used in practice, is incorrect due to an approximation made by Jenike and his coworkers being invalid. For certain plastic regimes, new exact analytical solutions are determined for two special cases of the angle of internal friction and one of these exact results may be used as the basis of an approximate solution for small angles of internal friction. It should
Figure 1.5: A large stockpile silo containing a stable rat-hole.

Figure 1.6: A stable rat-hole within the silo.
Figure 1.7: A stable and an unstable rat-hole within the silo, where the unstable rat-hole has collapsed.

Figure 1.8: A picture of just the stable rat-hole.
be noted that Jenike's original theory assumes a symmetrical stress distribution which is independent of height. However, in practice rat-holes tend to exhibit some tapering with height. Therefore in Chapter 3, the formation of rat-holes that have stress profiles corresponding to a symmetrical but slightly tapered circular cavity are examined. Stress distributions are found which are a perturbation of those arising from classical theory, and separable solutions involving exponential functions in height are used to "mimic" a slightly tapered rat-hole. Some numerical examples are presented and departures from classical theory are shown graphically.

In Chapter 4, for an existing rat-hole and assuming a shear-index granular material, the limiting stress profiles are determined. It should be noted that existing theory only applies to the Coulomb-Mohr yield function. A shear-index material is one for which failure due to frictional slip between particles occurs when the shear and normal components of stress $\tau$ and $\sigma$ satisfy the Warren Spring equation

$$\left(\frac{|\tau|}{c}\right)^n = 1 - \frac{\sigma}{t},$$

(1.20)

where $c$, $t$, and $n$ are positive constants which are referred to as the cohesion, tensile strength and shear-index respectively, and the known experimental values of the shear-index indicate that for certain materials $n$ lies between the values 1 and 2. The value $n = 1$ corresponds to the standard Coulomb-Mohr yield function while the value $n = 2$ permits some further analytical investigation, and stress profiles for these two values constitute bounds for those shear-index materials for which $1 < n < 2$.

In Chapter 5, the determination of the horizontal and vertical force distributions
at the base of a sand-pile is examined and which has become a famous problem in granular theory. In 1981 it was suggested from experimental work of Smid and Novosad [51] that the peak vertical force at the base does not occur directly beneath the vertex of the pile, but instead at some intermediate point so that there is a ring of maximum vertical pressure. Practising engineers have some reservations on this result, and since that time, numerous discrete theoretical and computational models of granular sand-piles have been proposed to explain this phenomenon, with varying degrees of success. Here, for two-dimensional sand-piles, the horizontal and vertical force distributions are estimated using the proper continuum mechanical theory of granular materials. For an infinite sand-pile the force distributions at a certain height are determined, and it is argued that these forces should approximate those for a sand-pile of finite height resting on a horizontal surface. The classical Jenike radial flow solutions for granular flow in a converging wedge are adapted and exploited in determining a solution. Numerical results indicate that for realistic angles of internal friction there is no solution satisfying all the necessary boundary conditions. However, a formal exact parametric solution is presented for the special case of an angle of internal friction equal to ninety degrees, which coincides with the full numerical solution. The special exact solution is a bona fide solution of the stated problem but does not predict the dip in the vertical force which is suggested from experimental work. An alternative formulation is proposed using a slightly more general form of the stresses in the sand-pile and a numerical solution for the special case of an angle of internal friction equal to the angle of repose is determined. However, this solution exhibits a non-zero stress $\sigma_{r0}$ at the centre of the pile and is
therefore non-physical. Finally, a sand-pile that is not entirely at yield is proposed, but which has an inner dead region and an outer yield region. From this model a solution is determined, which is not unique, but does predict the dip in the vertical force as suggested from experimental work.

In Chapter 6, the horizontal and vertical force distributions for three-dimensional sand-piles are estimated in a similar manner as for those for the two-dimensional sand-piles in Chapter 5. Again, it is shown that for realistic angles of internal friction, there is a solution such that the sand-pile is not entirely at yield, but has an inner dead region and an outer yield region. From this model a solution is determined that does predict the dip in vertical force as suggested from experimental work.

Finally, in Chapter 7 the special exact solution for the horizontal and vertical force distributions within a two-dimensional sand-pile for an angle of internal friction equal to ninety degrees is exploited to solve the problem of granular material falling under gravity through a converging wedge. This is the only known exact solution of this important problem involving two arbitrary constants and a full numerical solution is shown to coincide with the special exact solution. It should be noted here that the accuracy of the mathematical results within Chapter 7, and indeed the entire thesis, have been verified using MAPLE, which is a high powered symbolic algebraic computer package.
Chapter 2

Cylindrical cavities and classical rat-hole theory

2.1 Introduction

The formation of stable circular and almost vertical holes in stockpiles and hoppers is a significant practical problem. This frequently occurring phenomena is sometimes referred to as “piping”, and the holes themselves are known as “rat-holes”. In the mineral and mining industry, once a rat-hole forms in a stockpile, it tends to remain there because the material around the hole dries out and sets as a solid material. Practising engineers believe that the classical rat-hole theory enunciated by Jenike [34] and Jenike and Yen [37, 38] does not accurately reflect actual material behaviour. In addition, there is a view that classical theory does not account for the correct physics of the problem because the apparent material cohesion arises from water pressure capillarity in a thin layer around the hole (see Gudehus [26]). Despite this
latter point of view, the classical theory is re-examined here and in particular, the validity of the Jenike stable rat-hole equation (2.6) is examined, which is widely used by engineers in many granular material industries (see for example, McBride [43]).

Typically, a stockpile rat-hole appears as indicated in Figure 2.1, where $\theta$ denotes the angle of repose, $\delta^*$ is approximately the angle of internal friction $\delta$, and $\alpha$ denotes a small angle. From Figure 2.1, the question arises as to what is the difference between the angle of repose and the angle of internal friction. Indeed, equation (1.1) shows that for a cohesionless granular material, the angle of internal friction and the angle of repose are the same. However for cohesive granular materials, it is clear that the angle of repose and the angle of internal friction do differ. This difference comes directly from assuming the Coulomb-Mohr yield condition (1.9), which introduces the distinction between the angles.
Chapter 2: Cylindrical cavities and classical rat-hole theory

For the idealised situation of a vertical circular rat-hole ($\alpha = 0$) and with the axis as shown in Figure 2.1, the mathematical problem is to solve the equilibrium equations

$$\frac{d\sigma_{rz}}{dr} + \frac{\sigma_{rz}}{r} = \rho g, \quad \frac{d\sigma_{rr}}{dr} + \frac{(\sigma_{rr} - \sigma_{r\phi})}{r} = 0,$$

subject to the boundary conditions that the surface of the hole of radius $r_0$ is stress free

$$\sigma_{rz} = \sigma_{rr} = 0 \quad \text{for} \quad r = r_0.$$  \hspace{1cm} (2.2)

where $\rho$ is the bulk density of the material, $g$ is the acceleration due to gravity. $\sigma_{rr}, \sigma_{rz}$, etc. denote the stresses in a cylindrical polar coordinate system $(r, \phi, z)$, and which following Jenike [34] are assumed to be independent of $\phi$ and $z$. In addition, the material is assumed to satisfy the Coulomb-Mohr yield condition

$$|\tau| = c - \sigma^* \tan \delta,$$  \hspace{1cm} (2.3)

where $c$ is the cohesion and $\sigma^*$ and $\tau$ denote the normal and tangential components of compressive traction, which here is assumed to be positive in tension. Namely, the usual convention in continuum mechanics that positive forces are assumed to produce positive extensions is adopted.

Following Jenike [34] and Jenike and Yen [37, 38], the first equation of (2.1) trivially integrates to give

$$\sigma_{rz} = \frac{\rho g}{2} \left( r - \frac{r_0^2}{r} \right).$$  \hspace{1cm} (2.4)

and on introducing the stress-angle $\psi$ which is defined by (2.10), Jenike [34] and
Jenike and Yen [37, 38] make use of (2.4) to deduce

\[ \frac{d\psi}{dr} = \frac{\sin 2\psi}{\cos 2\psi - \beta} \left\{ \frac{(1 - \beta \cos 2\psi)r}{(r^2 - r_0^2)} - \frac{(1 + \beta)}{2r} \right\}, \tag{2.5} \]

where \( \beta \) denotes \( \sin \delta \), and which for the plastic regime \( A \) is subject to the condition \( \psi = 0 \) at \( r = r_0 \). Now on making use of (2.13)_3 and (2.18)_1 with (2.4) and (2.5) and using \( \ell' \) Hopital’s rule for \( \beta \neq 1 \), then the following result may be deduced, namely

\[ \lim_{r \to r_0} \left( \frac{d\psi}{dr} \right) = \frac{\rho g}{f_c}, \tag{2.6} \]

where \( f_c \) denotes the unconfined yield strength defined by (2.12). In principle then, (2.5) may be solved subject to \( \psi \) zero at \( r = r_0 \), and on equating the gradient of \( \psi \) at \( r = r_0 \) to \( \rho g/f_c \), the rat-hole radius \( r_0 \) may be supposedly determined. Moreover, for \( \frac{1}{3} < \beta < \frac{1}{2} \) the classical theory then proposes that any rat-hole of radius \( R_0 \), where \( R_0 < r_0 \) is stable and this criteria is known as the Jenike stable rat-hole condition (see Jenike and Yen [37], page 20). Further, for \( \frac{\pi}{6} < \delta < \frac{\pi}{2} \) (namely, \( \frac{1}{2} < \beta < 1 \)) Jenike and Yen [37, 38] determine \( d\psi/dr \) at the “constant” \( \psi \) and \( r \) values for which both the denominator and the expression in the curly brackets of (2.5) both vanish. Thus from \( \cos 2\psi = \beta \), \( r/r_0 = (2\beta - 1)^{-1/2} \), \( \ell' \) Hopital’s rule and solving a quadratic, gives rise to

\[ \left( \frac{d\psi}{dr} \right)_{r=r_0} = \frac{(2\beta - 1)^{1/2}}{4r_0} \left( \frac{1 + \beta}{1 - \beta} \right)^{1/2} \left\{ -\beta \pm (\beta^2 + 8\beta - 4)^{1/2} \right\}, \tag{2.7} \]

and the question arises as to how useful either estimate of the gradient actually is. In this Chapter, it is showed that both estimates are completely inaccurate and moreover that equation (2.6) is not an equation for the determination of \( r_0 \), but rather is a mathematical identity applying to any solution of (2.5) such that \( \psi = 0 \) at \( r = r_0 \).
In the following Section the necessary basic equations for the two plastic regimes A and F are presented, and in terms of the cohesion \(c\), the two basic equations for \(\psi\) and \(\sigma\) which are defined by (2.10) (see also Figure 1.3) are formulated. In the two subsequent Sections the special results applying for \(A\) and \(F\) respectively are detailed, including exact solutions applying for \(\beta\) zero and \(\beta\) unity and an approximate solution valid for small \(\beta\), noting however that the case \(\beta = 1\) has a well-defined mathematical meaning, but such materials do not occur in practice. In Section 2.5 the five other plastic regimes are considered and it is showed that results for regimes \(B\) and \(AB\) are similar to regime \(A\), results for regimes \(E\) and \(EF\) are similar to regime \(F\), and for the plastic regime \(AF\) it is not possible to directly determine differential equations in terms of \(\psi\) and \(\sigma\) defined by (2.10) because in this case \(\sigma_{\phi\phi}\) is indeterminant in terms of the maximum and minimum principal stresses. In Section 2.6, a simple Runge-Kutta scheme is used to determine a numerical solution which is related to the various exact and approximate solutions given in previous Sections. Finally, it is should be noted that other available analytical methods can be found in Drescher [21].

2.2 Basic equations for plastic regimes \(A\) and \(F\)

Following the notation adopted in Hill and Wu [27], it is assumed that the three algebraic maximum, intermediate and minimum principal stresses are denoted by \(\sigma_I, \sigma_{II},\) and \(\sigma_{III}\) respectively, such that the Coulomb-Mohr yield condition becomes

\[
\sigma_I = 2c \left(\frac{1 - \beta}{1 + \beta}\right)^{1/2} + \left(\frac{1 - \beta}{1 + \beta}\right) \sigma_{III}.
\]  

(2.8)
where again $c$ denotes the cohesion and $\beta = \sin \delta$ where $\delta$ is the angle of internal friction. The principal stresses are the eigenvalues of the stress matrix and therefore

$$\sigma_1 = \frac{1}{2} \left\{ (\sigma_{rr} + \sigma_{zz}) + \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} \right\}.$$  

$$\sigma_{III} = \frac{1}{2} \left\{ (\sigma_{rr} + \sigma_{zz}) - \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} \right\}.$$  

along with the intermediate principal stress $\sigma_{II}$ coinciding with the hoop stress $\sigma_{\phi\phi}$.

Now on introducing $\sigma$ and $\psi$ as shown in the Mohr Circle diagram (see Figure 1.3), gives

$$\sigma_{rr} - \sigma_{zz} = 2\beta \sigma \cos 2\psi, \quad \sigma_{rz} = \beta \sigma \sin 2\psi.$$  

while from (2.8) and (2.9) it may be deduced that

$$\beta(\sigma_{rr} + \sigma_{zz}) + \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} = (1 - \beta)f_c.$$  

where $f_c$ denotes the unconfined yield strength which is defined by $\sigma_I = 0$ when $\sigma_{III} = -f_c$, so that (2.8) gives

$$f_c = 2c \left( \frac{1 + \beta}{1 - \beta} \right)^{1/2}.$$  

Now, from (2.10) and (2.11) it may be deduced that

$$\sigma_{rr} = \sigma(\beta \cos 2\psi - 1) + (1 - \beta)\frac{f_c}{2\beta},$$

$$\sigma_{zz} = -\sigma(\beta \cos 2\psi + 1) + (1 - \beta)\frac{f_c}{2\beta},$$  

$$\sigma_{rz} = \beta \sigma \sin 2\psi.$$
Moreover from (2.13), (2.9) becomes

\[ \sigma_I = -(1 - \beta)\sigma + (1 - \beta)\frac{f_c}{2\beta}, \quad \sigma_{III} = -(1 + \beta)\sigma + (1 - \beta)\frac{f_c}{2\beta}. \] (2.14)

The yield condition (2.8) is usually represented by a pyramid surface in three-dimensional principal stress space with a rectangular Cartesian frame of reference \((\sigma_1, \sigma_2, \sigma_3)\) denoting a typical point (see Figure 1.4), and the seven possible plastic regimes available for axially symmetric stress states arise from the general plane \(\sigma_3 = \text{constant}\). Points on the varying hexagon represent all possible plastic principal stress states given by (2.8). These stress states are given in tabular form in Table 1.1. For further details see either Hill and Wu [27] or Cox, Eason and Hopkins [19]. It is clear from Table 1.1 that for plastic regimes \(A\) and \(F\)

\[ \begin{align*}
A: \quad & \sigma_I = \left(\frac{1 - \beta}{1 + \beta}\right) (f_c + \sigma_{\phi\phi}) \quad (\sigma_I > \sigma_{\phi\phi} = \sigma_{III}), \\
F: \quad & \sigma_{\phi\phi} = \left(\frac{1 - \beta}{1 + \beta}\right) (f_c + \sigma_{III}) \quad (\sigma_I = \sigma_{\phi\phi} > \sigma_{III}),
\end{align*} \] (2.15)

so that for the plastic regime \(A\), the hoop stress \(\sigma_{\phi\phi}\) is given by

\[ \sigma_{\phi\phi} = -(1 + \beta)\sigma + (1 - \beta)\frac{f_c}{2\beta}, \] (2.16)

while for the plastic regime \(F\),

\[ \sigma_{\phi\phi} = -(1 - \beta)\sigma + (1 - \beta)\frac{f_c}{2\beta}. \] (2.17)

Now from the stress free boundary conditions (2.2), and relations (2.10)\(_2\) and (2.13)\(_1\), the following possibilities at \(r = r_0\) arise. For \(\beta \neq 0\) or 1, the boundary
Chapter 2: Cylindrical cavities and classical rat-hole theory

conditions become

\[ \psi = 0, \quad \sigma = \frac{f_c}{2\beta}, \quad (\beta \neq 0, 1) \]

\( (2.18) \)

\[ \psi = \frac{\pi}{2}, \quad \sigma = \frac{f_c}{2\beta} \left( \frac{1 - \beta}{1 + \beta} \right), \quad (\beta \neq 0) \]

while for \( \beta \) tending to zero they are

\[ \psi = 0, \quad \sigma = \frac{f_c}{2\beta}, \quad (\beta \to 0) \]

\( (2.19) \)

\[ \psi = \frac{\pi}{2}, \quad \sigma = \frac{f_c}{2\beta}, \quad (\beta \to 0) \]

and for \( \beta = 1 \) gives

\[ \psi = 0, \quad \sigma = \text{anything}, \quad (\beta = 1) \]

\( (2.20) \)

\[ \psi = \text{anything}, \quad \sigma = 0. \quad (\beta = 1) \]

Further, from the differential equation \((2.1)_2\) and the relation which is obtained from \((2.4)\) and \((2.13)_3\), namely

\[ \sin 2\psi = \lambda \left( r - \frac{r_0^2}{r} \right), \]

\( (2.21) \)

where the constant \( \lambda \) denotes \( \rho g / 2\beta \), the following differential equations may be deduced

\[ \frac{d\psi}{dr} = \frac{\sin 2\psi}{(\cos 2\psi - \beta)} \left\{ \frac{(1 - \beta \cos 2\psi)r}{(r^2 - r_0^2) \frac{1 + \varepsilon \beta}{2r}} - (1 + \varepsilon \beta) \right\}, \]

\[ (2.22) \]

\[ \frac{d\sigma}{dr} \left\{ \left[ \sigma^2 - \lambda^2 \left( r - \frac{r_0^2}{r} \right) \right]^{1/2} - \beta \sigma \right\} - \frac{\beta \sigma}{r} \left\{ \varepsilon \left[ \sigma^2 - \lambda^2 \left( r - \frac{r_0^2}{r} \right) \right]^{1/2} + \sigma \right\} = -2\lambda^2 \left( r - \frac{r_0^2}{r} \right). \]
where $\varepsilon = +1$ for the plastic regime $A$ and $\varepsilon = -1$ for the plastic regime $F$. In view of the relation (2.21), for each plastic regime, these two differential equations are not independent. In the following two Sections some simple exact and approximate analytical solutions of these equations which have not been given previously are presented. The exact solutions apply for the special cases of $\beta$ tending to zero and $\beta$ unity, where $\beta$ denotes $\sin \delta$. The special case $\beta = 0$ may be viewed as a limiting situation of the yield condition (2.3), that is $|\tau| = c$. The solutions presented in the following Section are exact solutions of (2.22), except that for $\beta$ zero, the solution is a limiting case and appropriate care must be made in terms of the interpretation of such formulae.

![Image](image.png)

**2.3 Exact and approximate solutions for the plastic regime $A$**

In this Section for the plastic regime $A$, the special solutions of (2.22) subject to the conditions (2.18) – (2.20) are derived. Firstly, some transformations of the $\psi$ equation, namely (2.5) or (2.22), with $\varepsilon = +1$, are made. On making the transformation

$$u = (1 + \cos 2\psi)^{-1}, \quad (2.23)$$

it is found that equation (2.5) becomes

$$[1 - (1 + \beta)u] \frac{du}{dr} = \frac{(2u - 1)}{r(r^2 - r_0^2)} \left[ (1 + \beta)(r^2 + r_0^2)u - 23r^2 \right], \quad (2.24)$$
which is an Abel equation of the second kind (see for example Murphy [45], page 25). Now, on making the further transformation

$$u = \frac{1}{2} + \frac{v}{2(1 + \beta)(r - r_0^2/r)^2}, \quad (2.25)$$

equation (2.24) eventually becomes

$$\xi' \left\{ \frac{[v - (1 - \beta)\xi^2]}{4} \frac{dv}{d\xi} + \xi v \right\} = 2\beta \xi v, \quad (2.26)$$

where primes denotes differentiation with respect to $r$ and $\xi$ denotes $(r - r_0^2/r)$. From (2.26) it is clear that the special cases $\beta = 0$ and $\beta = 1$ can be readily integrated. In addition, equation (2.26) is linear in $\beta$ and therefore suitable to deduce an approximate solution of the form

$$v(r) = v_0(r) + \beta V_0(r) + O(\beta^2). \quad (2.27)$$

If $v_{\text{approx}}$ denotes any estimate for $v$, then from (2.23) and (2.25) it may be deduced that an approximation for $\psi$ is given by

$$\tan^2 \psi = \frac{v_{\text{approx}}}{(1 + \beta)(r - r_0^2/r)^2}. \quad (2.28)$$

### 2.3.1 Exact and approximate solutions for small $\beta$

From (2.26) and (2.27), differential equations for $v_0$ and $V_0$ are determined, namely

$$\frac{1}{4} \left( v_0 - \xi^2 \right) \frac{dv_0}{d\xi} + \xi v_0 = 0, \quad (2.29)$$

$$\xi' \left\{ \frac{1}{4} \left( V_0 + \xi^2 \right) \frac{dV_0}{d\xi} + \frac{1}{4} \left( v_0 - \xi^2 \right) \frac{dV_0}{d\xi} + \xi V_0 \right\} = 2\xi v_0.$$
and the first equation may be readily solved to yield

\[ v_0(\xi) = C - \xi^2 \pm \left( C^2 - 2C\xi^2 \right)^{1/2}, \quad (2.30) \]

where \( C \) denotes the constant of integration, and it is observed that the solution remains valid only for \( \xi \leq (C/2)^{1/2} \). Outside this range the solution \( v_0 \equiv 0 \) is adopted. Now as described in Appendix A, on making use of (2.30) the following expression for the solution of \((2.29)_2\) may be eventually deduced

\[ V_0(\xi) = -\frac{\left( C^2 - 2C\xi^2 \right)^{1/2} - C}{2C \left( C^2 - 2C\xi^2 \right)^{1/2}} \left[ \left( C^2 - 2C\xi^2 \right)^{1/2} + \frac{C}{2} \ln \left\{ \xi^2 + \left( \xi^2 + 4r_0^2 \right)^{1/2} \right\} \right] \]

\[ + 2(C^2 + 8Cr_0^2)^{1/2} \left\{ F(\gamma, \nu) - E(\gamma, \nu) \right\} + 2C \left( C^2 - 2C\xi^2 \right)^{1/2} \left( \xi^2 + 4r_0^2 \right)^{-1/2}, \quad (2.31) \]

where \( F(\gamma, \nu) \) and \( E(\gamma, \nu) \) are elliptic functions of the first and second kind respectively and are defined in Appendix A, with \( \gamma \) and \( \nu \) defined by

\[ \gamma = \sin^{-1} \left\{ \frac{\xi}{C} \left( \frac{C^2 + 8Cr_0^2}{\xi^2 + 4r_0^2} \right)^{1/2} \right\}, \quad \nu = \frac{C}{(C^2 + 8Cr_0^2)^{1/2}}, \quad (2.32) \]

where the minus sign has been assumed in (2.30). Note that the solution (2.31) also remains valid only for \( \xi < (C/2)^{1/2} \) and outside this range, the solution of \( V_0 \equiv 0 \) is adopted, which is a bona fide solution of \((2.29)_2\) provided \( v_0 \equiv 0 \) in this region.

Now either making use of (2.25) and (2.30), or by direct integration of (2.24) with \( \beta \) zero, the following exact solution applying for zero angle of internal friction may be deduced

\[ \cos^2 \psi = \frac{1}{2} \left\{ 1 + \left( 1 - k^2\xi^2 \right)^{1/2} \right\}, \quad (2.33) \]

where \( k \) is related to \( C \) by the equation \( k^2 = 2/C \). It is observed from (2.33) that the two possible conditions at \( r = r_0, \psi = 0 \) or \( \psi = \pi/2 \) arising in (2.19), are both
feasible and correspond to the plus and minus signs respectively. Further, upon
examining the relation (2.21), gives

\[
\sigma = \frac{\rho g}{2\beta} \frac{\xi}{\left[\left\{1 \pm (1 - k^2\xi^2)^{1/2}\right\} \left\{1 \mp (1 - k^2\xi^2)^{1/2}\right\}\right]^{1/2}},
\]

from which \( \sigma \) constant may be deduced and given by

\[
\sigma = \frac{\rho g}{2\beta k}, \tag{2.34}
\]

and therefore from both the conditions (2.19) at \( r = r_0 \), the constant \( k = \rho g/f_c \).

Clearly, this exact solution of the differential equation (2.5) for the case \( \beta \) zero can
be interpreted as an asymptotic solution which may be made rigorous by simply
rescaling \( \sigma \) with respect to \( \beta \). This special case is instructive because it makes
it clear that the arbitrary constant \( k \) (or \( C \)) is not necessarily determined by the
condition on \( \psi \) at \( r = r_0 \), but rather by the condition on \( \sigma \) at \( r = r_0 \). Thus, in
summary the exact solution for \( \beta \) tending to zero is given by

\[
\cos^2 \psi = \frac{1}{2} \left\{ 1 \mp \left(1 - \left(\frac{\rho g \xi}{f_c}\right)^2\right)^{1/2}\right\}, \quad \sigma = \frac{f_c}{2\beta}, \tag{2.35}
\]

where the \( \mp \) corresponds to \( \psi = \pi/2 \) or \( \psi = 0 \) respectively. From these results
and the relations (2.13) and (2.16) the following expressions for the stresses can be
determined

\[
\sigma_{rr} = -f_c \sin^2 \psi = -\frac{f_c}{2} \left\{ 1 \pm \left(1 - \left(\frac{\rho g \xi}{f_c}\right)^2\right)^{1/2}\right\},
\]

\[
\sigma_{zz} = -f_c \cos^2 \psi = -\frac{f_c}{2} \left\{ 1 \mp \left(1 - \left(\frac{\rho g \xi}{f_c}\right)^2\right)^{1/2}\right\}, \tag{2.36}
\]

\[
\sigma_{r z} = f_c \sin \psi \cos \psi = \frac{\rho g}{2} \xi,
\]
and \( \sigma_{\phi \psi} = -f_c \). Note that this is a well-defined solution of (2.1) and (2.2) only for the positive case of (2.35), even though the negative case of (2.35) does give rise to \( \psi = \pi/2 \), but does not lead to \( \sigma_{rr} \) vanishing at \( r = r_0 \).

### 2.3.2 Exact solution for \( \beta \) unity

For \( \beta \) unity, (2.26) becomes

\[
\frac{\xi'}{4 \frac{d\xi}{dr}} + \xi v = 2 \xi v,
\]

which upon recalling that \( \xi = (r - r_0^2 / r) \) and assuming \( v \neq 0 \), simplifies considerably to become

\[
\frac{dv}{dr} = \frac{4 \xi^2}{r}.
\]

Now (2.38) integrates easily to give

\[
v(r) = 2 \left( r^2 - \frac{r_0^2}{r^2} - 4r_0^2 \log \left( \frac{r}{r_0} \right) \right),
\]

and note that the constant of integration is zero for boundary condition (2.20).

From (2.23), (2.25) with \( \beta = 1 \) and (2.39), the exact solution of (2.5) for the case \( \beta = 1 \) may be deduced, namely

\[
\cos^2 \psi = \frac{(r - r_0^2 / r)^2}{2 \left[ r^2 - r_0^2 - 2r_0^2 \log \left( r / r_0 \right) \right]},
\]

and (2.21) gives

\[
\sigma = \rho g \frac{[r^2 - r_0^2 - 2r_0^2 \log \left( r / r_0 \right)]}{2 \left[ r^2 - r_0^4 / r^2 - 4r_0^2 \log \left( r / r_0 \right) \right]^{1/2}}.
\]
From (2.13) and (2.16) it is found that in this case

\[
\sigma_{rr} = -2\sigma \sin^2 \psi = -\frac{\rho g}{2} \left\{ r^2 - \frac{r_0^4}{r^2} - 4r_0^2 \log \left( \frac{r}{r_0} \right) \right\}^{1/2},
\]

\[
\sigma_{zz} = -2\sigma \cos^2 \psi = -\frac{\rho g}{2} \left\{ r^2 - \frac{r_0^4}{r^2} - 4r_0^2 \log \left( \frac{r}{r_0} \right) \right\}^{-1/2} \left( r - \frac{r_0^2}{r} \right)^2. \tag{2.42}
\]

\[
\sigma_{rz} = 2\sigma \sin \psi \cos \psi = \frac{\rho g}{2} \left( r - \frac{r_0^2}{r} \right),
\]

and \( \sigma_{\phi \phi} = -2\sigma \), where \( \sigma \) is given by (2.41) and this is a well-defined solution of (2.1) and (2.2). As noted in the numerical results, it is not possible to utilise this exact solution for \( \beta = 1 \) as the basis of an approximate solution valid for \( \beta \) close to unity because the solution characteristics are quite different for \( \beta = 1 \) and for \( \beta \neq 1 \), in the sense that \( \psi \) has an infinite gradient at \( r = r_0 \) for \( \beta = 1 \), while for \( \beta \neq 1 \) the gradient is finite.

### 2.4 Solutions for the plastic regime \( F \)

In this Section the corresponding formulae for the plastic regime \( F \) are given. On making the transformation

\[
u = (1 - \cos 2\psi)^{-1}, \tag{2.43}
\]

equation (2.22) with \( \varepsilon = -1 \) becomes

\[
[1 - (1 - \beta)u] \frac{du}{dr} = \frac{(2u - 1)}{r(r^2 - r_0^2)} \left[ (1 - \beta) \left( r^2 + r_0^2 \right) u + 23r^2 \right]. \tag{2.44}
\]
which is precisely equation (2.24) except that $\beta$ in (2.24) is replaced by $-\beta$. Thus, from the transformation

$$u = \frac{1}{2} + \frac{v}{2(1 - \beta)(r - r_0^2/r)^2}, \quad (\beta \neq 1)$$

(2.45)

the following differential equation for $v(\xi)$ is obtained

$$\xi' \left\{ \frac{v - (1 + \beta)\xi^2}{4} \frac{dv}{d\xi} + \xi v \right\} = -2\beta \xi v,$$

(2.46)

where as before primes denote differentiation with respect to $r$ and $\xi$ denotes $(r - r_0^2/r)$. An approximate solution $v_{\text{approx}}$ of the form (2.27) may be deduced by simply replacing $\beta$ by $-\beta$, and in place of (2.28) it can be shown that for $\beta \neq 1$.

$$\cot^2 \psi = \frac{v_{\text{approx}}}{(1 - \beta)(r - r_0^2/r)^2}.$$  

(2.47)

For $\beta$ tending to zero the approximate solution (2.27) may be used, but with $\beta$ replaced with $-\beta$ and from the results of the previous Section the leading term becomes

$$\sin^2 \psi = \frac{1}{2} \left\{ 1 \pm \left( 1 - \left( \frac{\rho g \xi}{f_c} \right)^2 \right)^{1/2} \right\}, \quad \sigma = \frac{f_c}{2\beta},$$

(2.48)

where here the $\pm$ corresponds to $\psi = \pi/2$ and $\psi = 0$ respectively. From the relations (2.13) and (2.17) the stresses become

$$\sigma_{rr} = -f_c \sin^2 \psi = -\frac{f_c}{2} \left\{ 1 \pm \left( 1 - \left( \frac{\rho g \xi}{f_c} \right)^2 \right)^{1/2} \right\},$$

$$\sigma_{zz} = -f_c \cos^2 \psi = -\frac{f_c}{2} \left\{ 1 \mp \left( 1 - \left( \frac{\rho g \xi}{f_c} \right)^2 \right)^{1/2} \right\},$$

(2.49)

$$\sigma_{rz} = f_c \sin \psi \cos \psi = \frac{\rho g}{2} \xi.$$
and again $\sigma_{\phi\phi} = -f_c$. In this case this is clearly a well-defined solution of (2.1) and (2.2) only for $\psi = 0$, namely taking the minus sign in (2.48).

For $\beta$ unity the transformation (2.45) is invalid, but instead directly from equation (2.43) and (2.44) with $\beta = 1$, $\psi$ is given by

$$\cot^2 \psi = C \left( r^2 - r_0^2 \right)^2. \quad (2.50)$$

where $C$ denotes an arbitrary constant. Further, from (2.21)

$$\sigma = \frac{\rho g}{4} \left[ 1 + C \left( r^2 - r_0^2 \right)^2 \right] \frac{1}{r C^{1/2}}, \quad (2.51)$$

may be deduced. Thus, from (2.13) and (2.17) the stresses become

$$\sigma_{rr} = -2\sigma \sin^2 \psi = -\frac{\rho g}{2r C^{1/2}},$$

$$\sigma_{zz} = -2\sigma \cos^2 \psi = -\frac{\rho g}{2r} C^{1/2} \left( r^2 - r_0^2 \right)^2, \quad (2.52)$$

$$\sigma_{rz} = 2\sigma \sin \psi \cos \psi = \frac{\rho g}{2} \left( r - \frac{r_0^2}{r} \right),$$

along with $\sigma_{\phi\phi} = 0$. It is clear from these expressions that there is no finite value of the arbitrary constant $C$ which produces a solution of (2.1) and (2.2) such that $\sigma_{rr}$ vanishes along $r = r_0$. Thus an approximate solution of the from (2.27) is not meaningful in this context.

### 2.5 Other plastic regimes

In this Section the plastic regimes $AB$ and $B$ are shown to be similar to regime $A$, while $EF$ and $E$ are similar to regime $F$. Finally, the regime $AF$ is examined for
which the hoop stress is unable to be determined in terms of the variables \(\sigma\) and \(\nu\) defined by (2.10), and hence the equilibrium equation (2.1)\(_2\) cannot be solved.

For regimes \(AB\) and \(B\) the hoop stress is always the minimum principal stress. Therefore, from the Mohr Circle diagram (Figure 1.3) the expression for the hoop stress is found to be given by

\[
\sigma_{\Phi\Phi} = -(1 + \beta)\sigma + (1 - \beta) \frac{f_c}{2\beta}. \tag{2.53}
\]

noting that this is the same as for regime \(A\). Now, as the results (2.13) and (2.14) hold for all seven plastic regimes, then from (2.1)\(_2\), (2.13), (2.21), and (2.53) the differential equations (2.22) with \(\varepsilon = +1\) are found to hold for both regimes \(AB\) and \(B\). This suggests that there is little difference between regimes \(A\), \(B\), and \(AB\). The three stress states are defined by

\[
A : \quad \sigma_I > \sigma_{II} = \sigma_{\Phi\Phi},
\]

\[
B : \quad \sigma_I = \sigma_{II} > \sigma_{\Phi\Phi}, \tag{2.54}
\]

\[
AB : \quad \sigma_I > \sigma_{II} > \sigma_{\Phi\Phi},
\]

where \(\sigma_I, \sigma_{II}\), and \(\sigma_{\Phi\Phi}\) are the maximum, intermediate, and minimum principal stresses respectively. From (2.54) it can be seen that the main difference is the location of the intermediate principal stress \(\sigma_{II}\). Note that for regimes \(A\) and \(B\) that the intermediate stress \(\sigma_{II}\) can be determined, whereas for regime \(AB\) it is indeterminant.

For regimes \(EF\) and \(E\) the hoop stress is always the maximum principal stress. Therefore, from the Mohr Circle diagram (Figure 1.3) the expression for the hoop
stress is found to be given by

$$\sigma_{\phi\phi} = -(1 - \beta)\sigma + (1 - \beta)\frac{f_c}{2\beta} \quad (2.55)$$

noting that this is the same as for regime F. Now again, since the results (2.13) and (2.14) hold for all seven plastic regimes, then from (2.1)\(_2\), (2.13), (2.21), and (2.55) the differential equations (2.22) with \(\varepsilon = -1\) are found to hold for both regimes \(EF\) and \(E\). Again this suggests that there is little difference between the regimes \(E\), \(F\), and \(EF\), for which the three stress states are given by

\[
\begin{align*}
F : & \quad \sigma_{\phi\phi} = \sigma_{II} > \sigma_{III}, \\
E : & \quad \sigma_{\phi\phi} > \sigma_{II} = \sigma_{III}, \\
EF : & \quad \sigma_{\phi\phi} > \sigma_{II} > \sigma_{III},
\end{align*}
\]

(2.56)

where \(\sigma_{\phi\phi}, \sigma_{II},\) and \(\sigma_{III}\) are the maximum, intermediate, and minimum principal stresses respectively. From (2.56) it is clear that the difference lies in the location of the intermediate principal stress \(\sigma_{II}\) and note that for regimes \(F\) and \(E\) that the intermediate stress \(\sigma_{II}\) can be determined, whereas for regime \(EF\) it is indeterminant.

The stress state for the plastic regime \(AF\) is given by

\[
AF : \quad \sigma_I > \sigma_{\phi\phi} > \sigma_{III},
\]

(2.57)

and in this case an expression for the hoop stress \(\sigma_{\phi\phi}\) in terms of the variables \(\sigma\) and \(\psi\) defined by (2.10) is unable to be determined and therefore the equilibrium equation (2.1)\(_2\) are unable to be solved.
Figure 2.2: Variation of the numerical $\psi$ (——) with position for two angles of internal friction ($\beta = 0.2$ and $\beta = 0.5$) compared with the analytical result (---) as determined from (2.27) and (2.28).

### 2.6 Numerical results

Figure 2.2 shows the numerically determined variation of $\psi$ (——) with position for two values of the angle of internal friction ($\beta = 0.2$ and $\beta = 0.5$) as compared to the approximate analytical expressions (---) obtained from (2.27) and (2.28). The numerical curves are obtained by solving (2.22) for $\sigma$ with $\varepsilon = +1$ using a fourth order Runge-Kutta numerical scheme together with the condition $\sigma = f_c/2\beta$ at $r = r_0$ and then $\psi$ is determined by (2.21). The constant $r_0$ is taken to be unity and the density is assumed to be $\rho = 0.7$. Note that there is close agreement for small $r$ and that the two results diverge with increasing $r$. Similarly, Figure 2.3 shows the variation of $\sigma$ for both the numerical (——) and approximate (---) results for the same two values of the angle of internal friction. From Figure 2.3 it is observed that
for small $r$ the approximate solution for $\sigma$ does not agree closely with the numerical solution, even though the approximate solution for $\psi$ does. This is because the approximate solution for $\sigma$ was obtained from (2.21) using the approximate solution for $\psi$, and it can be seen that for small $r$, a small change in $\psi$ will cause a large change in $\sigma$. For the special cases of $\beta$ zero and $\beta$ unity the $\psi$ variation with position is shown in Figure 2.4 and note there is exact coincidence with the analytical results obtained from (2.35)\textsubscript{i} and (2.40) respectively and the full numerical solution.

Note that for $\beta = 1$ and only $\beta = 1$ the solution has an infinite gradient at $r = r_0$, and therefore this exact solution cannot be used as the basis of an approximate solution. Note that the infinite gradient for $\beta = 1$ also follows directly from the differential equation (2.5) with (2.21) using the conditions $\psi = 0$ and $\sigma = 0$ at
\[ \frac{d\sigma}{dr} \bigg|_{r=r_0} = \lim_{r \to r_0} \frac{\beta \, 2\sigma}{1 - \beta \, r}, \]  

(2.58)

which is valid provided \( \beta \neq 1 \). If \( \beta = 1 \) then at \( r = r_0 \) there are two possibilities. 

either \( (d\sigma/dr)_{r=r_0} \) is infinite if \( \sigma \neq 0 \) at \( r = r_0 \) or \( (d\sigma/dr)_{r=r_0} \) indeterminant as
both the numerator and denominator vanish if $\sigma = 0$ at $r = r_0$. From the fact that $(d\psi/dr)_{r=r_0}$ is infinite for $\beta = 1$, it follows from (2.21) that $(d\sigma/dr)_{r=r_0}$ is also infinite for $\beta = 1$. Thus, for $\beta = 1$ and any value of $\sigma$ at $r = r_0$, it is found that $(d\sigma/dr)_{r=r_0}$ to be positive infinity. Accordingly from (2.58) it is seen that the solution for $\beta$ unity cannot be used to approximate a solution for $\beta$ close to unity, because their slopes are not close at $r = r_0$.

Figure 2.5 shows the variation of the numerically determined value of $(d\psi/dr)_{r=r_0}$ (---) with $\beta$ for $0 \leq \beta \leq 1$ and compared with the positive estimate obtained from (2.7) (---) for the range $1/2 \leq \beta \leq 1$. There are clearly large discrepancies between the Jenike estimate and the actual values of the gradient. Figure 2.5 also shows
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Figure 2.6: Variation of numerically determined stresses $\sigma_{rr}, \sigma_{rz}, \sigma_{zz}$ and $\sigma_{\phi\phi}$ with position for $\beta = 1/2$.

the variation of the analytical estimate of $(d\psi/dr)_{r=r_0}$ (---) which is obtained from (2.27), (2.28), (2.30) and (2.31) and is compared with the purely numerical result. It is clear that this provides a very accurate approximation for $(d\psi/dr)_{r=r_0}$ in the entire range $0 \leq \beta \leq 1$. This close agreement occurs because for small $r$ the analytical approximation is very close to the purely numerical result and these results diverge for increasing $r$. Finally, Figure 2.6 shows the overall variation of the numerically determined stresses $\sigma_{rr}, \sigma_{rz}, \sigma_{zz}$, and $\sigma_{\phi\phi}$ as a function of position for the angle of internal friction $\delta = \pi/6$ ($\beta = 1/2$).

2.7 Conclusions

Within this Chapter the classical rat-hole theory of Jenike and his coworkers has been re-examined and for the plastic regime $A$ some new analytical results, both ex-
act and approximate, have been determined. The exact results apply to the limiting
angles of internal friction, namely $\delta = 0$ and $\delta = \pi/2$, while the approximate solution
is valid for small angles of internal friction and is obtained by using the exact result
for $\delta$ zero as the leading term. In addition, a fully independent numerical solution
has been determined which has been compared to the approximate analytical result
developed here as well as the estimate of $(d\psi/dr)_{r=r_0}$ used by Jenike. It is concluded
that the numerical results obtained for regime $A$ are accurate and compare well with
the analytical approximation for the various boundary conditions. However, the ap­
proximation made by Jenike and his coworkers does not give accurate results, and in
fact is invalid. It has also been shown that the so-called “stable rat-hole equation”
is simply a mathematical identity valid for any solution of (2.22) with $\varepsilon = +1$ for
which $\psi = 0$ at $r = r_0$.

It has also been noted that from the solution for regime $A$, the solutions for
regimes $B$ and $AB$ can be determined, except that the intermediate principal stress
for regime $AB$ cannot be determined as it is indeterminant in terms of the Mohr
Circle diagram variables $\sigma$ and $\psi$ which are defined by (2.10). Similarly, from the
solution for regime $F$, the solutions for regimes $E$ and $EF$ can be determined, again
with the exception that the intermediate principal stress cannot be determined for
regime $EF$ as it is indeterminant in terms of $\sigma$ and $\psi$. 
Chapter 3

Stress profiles for tapered cylindrical cavities

3.1 Introduction

Stockpiles and hoppers are widely used throughout many mineral and mining industries to store and recover material. From a practical perspective, the desire is to be able to efficiently remove material from the stockpile or hopper at a uniform and uninterrupted rate of flow. Therefore, the occurrence of almost vertical tunnels inside stockpiles or hoppers, which prevents the flow of material, is an unwanted phenomenon, and an understanding of the conditions under which such phenomenon occur is desired. These tunnels are commonly known as “rat-holes” and the process of their formation is referred to as “piping”. Once a rat-hole has formed in a stockpile, the material around the surface of the hole often dries out and sets as a solid material. This makes the removal of rat-holes more difficult, and often they
have to be destroyed manually and the stockpile completely reshaped. Practising engineers believe that the classical rat-hole theory enunciated by Jenike [33] and Jenike and Yen [37] does not accurately reflect actual material behaviour. In the previous Chapter the stable rat-hole equation proposed by Jenike and his coworkers is shown not to be a good approximation of the exact numerical solution. The purpose of this Chapter is to determine the stress distributions for stockpile rat-holes which are slightly tapered, by exploiting the classical stress distributions as the basis for a perturbation scheme. Note that for rat-holes occurring in bins, an approximate analysis, based on the method of "slices", which does incorporate some height variation is provided by Johanson [40]. It is emphasised that for slightly tapered stockpile rat-holes the work presented here constitutes the first rigorous full mathematical analysis of the problem.

Typically, a stockpile rat-hole appears as indicated in Figure 3.1, where $\theta$ denotes the angle of repose, and $\alpha$ and $\gamma$ denote small angles. For the idealised situation of a symmetrical cylindrical rat-hole, with the axis as shown in Figure 3.1, the limiting equilibrium equations become

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{r\phi}}{r} = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = \rho g, \quad (3.1)$$

where $\rho$ is the bulk density of the material, $g$ is the acceleration due to gravity, $\sigma_{rr}, \sigma_{rz}$, etc. denote the stresses in a cylindrical polar coordinate system $(r, \phi, z)$ which are assumed to be independent of $\phi$. In addition, the material is assumed to satisfy the Coulomb-Mohr yield condition

$$|\tau| = c - \sigma^* \tan \delta, \quad (3.2)$$
where $c$ is the cohesion, $\delta$ is the angle of internal friction, and $\sigma^*$ and $\tau$ denote the normal and tangential components of compressive traction, which are assumed to be positive in tension. Namely, the usual convention in continuum mechanics is adopted that positive forces are assumed to produce positive extensions.

In this Chapter, slightly tapered cylindrical cavity profiles such as those depicted in Figure 3.1 are assumed and can be represented by an expression of the form

$$r = r_0 + \varepsilon R(z),$$

where $r_0$ is assumed to be independent of the height $z$, and $\varepsilon$ is a small non-dimensional parameter. The distinction from the classical theory is shown schematically in Figure 3.2. It should be emphasised here that the idea is to look for slightly tapered cylindrical cavities for which the correction terms of order $\varepsilon$ are much smaller than the corresponding terms for a perfectly circular cylindrical vertical cavity. Further, $R(z)$ is a function of $z$ which is assumed to be approximated by an expression...
Figure 3.2: (a). Right circular uniform cylindrical cavity. (b). Cylindrical cavity with height variation.

of the form

\[ R(z) = \sum_{n=1}^{N} R_n e^{-\alpha_n z}, \quad (3.4) \]

for certain constants \( \alpha_n \) and \( R_n \) \((n = 1, 2, ..., N)\). For example, it is shown that the cavity profile shown in Figure 3.1(a) can be adequately approximated by the two term expression

\[ R(z) = R_1 \left(1 + e^{-\alpha z}\right), \quad (3.5) \]

assuming \( \tan \alpha = \varepsilon \). On the other hand it may be shown that the cavity profile shown in Figure 3.1(b) can be approximated by the three term expression

\[ R(z) = R_1 + R_2 e^{-\alpha_2 z} + R_3 e^{-\alpha_3 z}, \quad (3.6) \]

assuming \( \tan \alpha = \varepsilon \) and \( \tan \gamma = K \varepsilon \) for some \( K > 1 \).

Corresponding to a slightly tapered cylindrical cavity of the form (3.3) it is assumed that the non-zero stresses are a small perturbation of those for the classical
theory, namely

\[ \sigma_{rr}(r, z) = \sigma_{r0}(r) + \varepsilon \sigma_{r1}(r, z), \quad \sigma_{rz}(r, z) = \sigma_{z0}(r) + \varepsilon \sigma_{z1}(r, z), \]

\[ \sigma_{zz}(r, z) = \sigma_{z0}(r) + \varepsilon \sigma_{z1}(r, z), \quad \sigma_{\theta\theta}(r, z) = \sigma_{\theta0}(r) + \varepsilon \sigma_{\theta1}(r, z), \]

where \( \varepsilon \) is the small parameter defined by equation (3.3), and the quantities \( \sigma_{r1}, \sigma_{z1}, \) etc. are unknown functions of \( r \) and \( z \). Assume that the stresses (3.7) obey a pressure boundary condition at the cavity wall of the form

\[ \sigma_j = -Pn_j, \]

(3.8)

where \( \sigma_j \) (\( j = 1, 2, 3 \)) denotes the stress vector, \( P \) is the assumed external pressure and \( n_j \) (\( j = 1, 2, 3 \)) denotes the components of the normal vector to the cavity surface. From Figure 3.3, the normal vector to the surface of the sidewall of the unstable rat-hole can be seen to be given by

\[ \mathbf{n} = (-\cos \theta(z), 0, \sin \theta(z)), \]

(3.9)
where $\theta(z)$ is the angle the normal vector $\mathbf{n}$ makes with the $r$ axis. Therefore, on assuming that the external pressure $P$ is zero, (3.8) and (3.9) gives

$$\sigma_r = 0, \quad \sigma_z = 0,$$

(3.10)

and upon noting the fact that $\sigma_{ij} = \sigma_{ij} n^i$, then expanding (3.10) gives

$$-\sigma_{rr}(r_0 + \varepsilon R(z), z) \cos \theta(z) + \sigma_{rz}(r_0 + \varepsilon R(z), z) \sin \theta(z) = 0,$$

(3.11)

$$-\sigma_{rz}(r_0 + \varepsilon R(z), z) \cos \theta(z) + \sigma_{zz}(r_0 + \varepsilon R(z), z) \sin \theta(z) = 0.$$

Now, from Figure 3.3 it may be shown that at the cavity wall

$$\theta(z) = \tan^{-1} \left( \frac{dr}{dz} \right),$$

(3.12)

and expanding (3.11) and (3.12) gives the following conditions

$$\sigma_{rro}(r_0) = 0, \quad \sigma_{zro}(r_0) = 0,$$

(3.13)

$$\sigma_{rzo}(r_0, z) = -R(z) \left( \frac{d\sigma_{rro}}{dr} \right)_{r=r_0}, \quad \sigma_{rzo}(r_0, z) = -R(z) \left( \frac{d\sigma_{zro}}{dr} \right)_{r=r_0} + R'(z) \sigma_{zzo}(r_0),$$

(3.14)

noting that the zeroth order conditions are simply those used in the classical theory.

In the following Section the governing equations for the slightly tapered rat-hole are presented where separable solutions for the stresses are assumed. In the subsequent Section, a second order ordinary differential equation is derived from which the stresses in the slightly tapered rat-hole may be determined. In Section 3.4, various two and three term approximations for $R(z)$ of the form (3.4) are considered, and these are applied to the single and double slightly tapered rat-holes as shown in Figures 3.1(a) and 3.1(b) respectively.
3.2 The governing ordinary differential equations

In this Section the governing ordinary differential equations for a slightly tapered rat-hole are determined. To do this, it is assumed that the stockpile is at equilibrium, and that the rat-hole is on the point of collapse, so that the equilibrium equations (3.1) apply and the stresses are given by (3.7). Next, it is assumed that the unknown functions \( \sigma_{rr}, \sigma_{r\theta}, \sigma_{zz}, \) and \( \sigma_{\phi\phi} \) can be expressed as a sum of separable variable functions, where the \( z \) dependence is uniform for each of the stresses. Thus, assume

\[
\begin{align*}
\sigma_{rr}(r, z) &= \sum_{i=1}^{N} A_i(r) E_i(z), \quad \sigma_{r\theta}(r, z) = \sum_{i=1}^{N} B_i(r) E_i(z), \quad \sigma_{zz}(r, z) = \sum_{i=1}^{N} C_i(r) E_i(z), \quad \sigma_{\phi\phi}(r, z) = \sum_{i=1}^{N} D_i(r) E_i(z).
\end{align*}
\] (3.15)

Then for each \( i = 1, 2, \ldots, N \), it is found from substituting (3.7) and (3.15) into (3.1), that the unknown separable variable functions must satisfy

\[
\frac{dA_i}{dr} E_i(z) + B_i(r) \frac{dE_i}{dz} + \frac{A_i(r) E_i(z) - D_i(r) E_i(z)}{r} = 0,
\] (3.16)

and therefore each \( E_i(z) \) must satisfy an equation of the form

\[
\frac{dE_i}{dz} = -\alpha_i E_i(z),
\] (3.17)

for certain constants \( \alpha_i \). Therefore, solving (3.17) gives \( E_i(z) = e^{-\alpha_i z} \), where the constant of integration has been incorporated into the functions of \( r \). Hence, equation (3.16) becomes

\[
\frac{dA_i}{dr} - \alpha_i B_i(r) + \frac{A_i(r) - D_i(r)}{r} = 0,
\] (3.18)

and similarly, (3.1)\( _2 \) becomes

\[
\frac{dB_i}{dr} - \alpha_i C_i(r) + \frac{B_i(r)}{r} = 0.
\] (3.19)
Now, on assuming that the granular material satisfies the Coulomb-Mohr yield condition defined by (3.2), it is determined in the previous Section that the yield condition becomes

$$\sigma_I = (f_c + \sigma_{III}) \left( \frac{1 - \beta}{1 + \beta} \right),$$

where $\beta = \sin \delta$, $f_c$ is the unconfined yield strength defined by $\sigma_I = 0$ when $\sigma_{III} = -f_c$, where $f_c$ can be written as

$$f_c = 2c \left( \frac{1 + \beta}{1 - \beta} \right)^{1/2},$$

and $\sigma_I, \sigma_{II},$ and $\sigma_{III}$ denote the maximum, intermediate, and minimum principal stresses respectively. Further, it is also assumed that of the seven possible plastic regimes available for axially symmetric stress states, the material is in plastic regime $A$, which means that the stresses satisfy the inequality $\sigma_I > \sigma_{II} = \sigma_{\phi\phi}$. The seven plastic regimes are well known and can be found in tabular form in either Hill and Wu [28], Cox, Eason and Hopkins [19] or Table 1.1. It is clear that a relation between the principal stresses and the stresses in the rat-hole is needed. In the previous Chapter it is shown that the maximum, intermediate and minimum principal stresses for the classical rat-hole are given by

$$\sigma_{I0} = \frac{1}{2} \left\{ (\sigma_{r0} + \sigma_{z0}) + \left[ (\sigma_{r0} - \sigma_{z0})^2 + 4\sigma_{rz0}^2 \right]^{1/2} \right\},$$

$$\sigma_{II0} = \sigma_{\phi\phi0},$$

$$\sigma_{III0} = \frac{1}{2} \left\{ (\sigma_{r0} + \sigma_{z0}) - \left[ (\sigma_{r0} - \sigma_{z0})^2 + 4\sigma_{rz0}^2 \right]^{1/2} \right\}.$$  

(3.20)

In order to determine the principal stresses for the slightly tapered rat-hole, note
that the principal stresses are defined by the eigenvalue equation

\[
\begin{vmatrix}
\sigma_{rr} - \mu & \sigma_{rz} & 0 \\
\sigma_{rz} & \sigma_{zz} - \mu & 0 \\
0 & 0 & \sigma_{\phi\phi} - \mu
\end{vmatrix} = 0,
\]

where \( \mu \) denotes a principal stress for the slightly tapered rat-hole. Next, assume that \( \mu = \mu_0 + \epsilon \mu_1 \) where \( \mu_0 \) denotes a principal stress for the uniform rat-hole, and \( \mu_1 \) is an unknown function of \( r \) and \( z \). Therefore, upon substituting (3.7) and (3.15) into the eigenvalue equation and noting that \( \mu_0 \) satisfies the equation

\[
(\sigma_{\phi\phi} - \mu_0) \left[ (\sigma_{rr0} - \mu_0)(\sigma_{zz0} - \mu_0) - \sigma_{rzo}^2 \right] = 0,
\]

then solving for \( \mu_1 \) obtains the expression

\[
\mu_1 = \left[ (\sigma_{zz0} - \mu_0)(\sigma_{\phi\phi0} - \mu_0) \sum_{i=1}^{N} A_i(r)e^{-\alpha_i z} \right. \\
+ (\sigma_{rr0} - \mu_0)(\sigma_{\phi\phi0} - \mu_0) \sum_{i=1}^{N} C_i(r)e^{-\alpha_i z} \\
+ (\sigma_{rr0} - \mu_0)(\sigma_{zz0} - \mu_0) \sum_{i=1}^{N} D_i(r)e^{-\alpha_i z} \\
\left. -2\sigma_{rzo}(\sigma_{\phi\phi0} - \mu_0) \sum_{i=1}^{N} B_i(r)e^{-\alpha_i z} -\sigma_{rzo}^2 \sum_{i=1}^{N} D_i(r)e^{-\alpha_i z} \right] /
\]

\[
\left[ (\sigma_{rr0} - \mu_0)(\sigma_{zz0} - \mu_0) + (\sigma_{rr0} - \mu_0)(\sigma_{\phi\phi0} - \mu_0) \\
+ (\sigma_{zz0} - \mu_0)(\sigma_{\phi\phi0} - \mu_0) - \sigma_{rzo}^2 \right].
\]
Thus, upon defining
\[ \Delta_0 = \sqrt{(\sigma_{rr0} - \sigma_{zz0})^2 + 4\sigma_{rzo}^2}, \quad \Sigma_0 = \sigma_{rr0} - \sigma_{zz0}. \quad (3.22) \]

and substituting (3.20)\(_1\) into (3.21) for \(\mu_0\), then it can be shown that \(\mu_1\) becomes
\[
\mu_1 = \sum_{i=1}^{N} \frac{e^{-\alpha_i z}}{2\Delta_0} \left[ (\Sigma_0 + \Delta_0)A_i(r) + 4\sigma_{rzo}B_i(r) - (\Sigma_0 - \Delta_0)C_i(r) \right],
\]

and similarly, upon substituting (3.20)\(_3\) into (3.21) gives
\[
\mu_1 = \sum_{i=1}^{N} \frac{e^{-\alpha_i z}}{2\Delta_0} \left[ (-\Sigma_0 + \Delta_0)A_i(r) - 4\sigma_{rzo}B_i(r) + (\Sigma_0 + \Delta_0)C_i(r) \right].
\]

and finally, if \(\mu_0 = \sigma_{\phi\phi_0}\) then it is clear that \(\mu_1 = \sum_{i=1}^{N} D_i(r)e^{-\alpha_i z}\).

Therefore, the maximum, intermediate and minimum principal stresses for the slightly tapered rat-hole are
\[
\sigma_I = \sigma_{I0} + \varepsilon \sum_{i=1}^{N} \frac{e^{-\alpha_i z}}{2\Delta_0} \left[ (\Sigma_0 + \Delta_0)A_i(r) + 4\sigma_{rzo}B_i(r) - (\Sigma_0 - \Delta_0)C_i(r) \right],
\]
\[
\sigma_{II} = \sigma_{II0} + \varepsilon \sum_{i=1}^{N} D_i(r)e^{-\alpha_i z},
\]
\[
\sigma_{III} = \sigma_{III0} + \varepsilon \sum_{i=1}^{N} \frac{e^{-\alpha_i z}}{2\Delta_0} \left[ (-\Sigma_0 + \Delta_0)A_i(r) - 4\sigma_{rzo}B_i(r) + (\Sigma_0 + \Delta_0)C_i(r) \right].
\]

Now, upon substituting (3.23) into the Coulomb-Mohr yield equation, then the stresses for the slightly tapered rat-hole are related by the equation
\[
(\Sigma_0 + \beta\Delta_0)A_i(r) + 4\sigma_{rzo}B_i(r) - (\Sigma_0 - \beta\Delta_0)C_i(r) = 0,
\]
for \( i = 1, \ldots, N \), which from (3.22) becomes

\[
A_i(r) \left[ 1 + \beta \left( 1 + \frac{4\sigma^2_{rz0}}{(\sigma_{rr0} - \sigma_{zz0})^2} \right)^{1/2} \right] + \frac{4\sigma_{rz0}B_i(r)}{(\sigma_{rr0} - \sigma_{zz0})} - C_i(r) \left[ 1 - \beta \left( 1 + \frac{4\sigma^2_{rz0}}{(\sigma_{rr0} - \sigma_{zz0})^2} \right)^{1/2} \right] = 0.
\]

Following Chapter 2, introduce

\[
\tan 2\psi_0 = \frac{2\sigma_{rz0}}{\sigma_{rr0} - \sigma_{zz0}},
\]

so that the Coulomb-Mohr yield condition becomes

\[
A_i(r)(\cos 2\psi_0 + \beta) + 2B_i(r)\sin 2\psi_0 - C_i(r)(\cos 2\psi_0 - \beta) = 0, \quad (3.25)
\]

for \( i = 1, \ldots, N \) and where \( \psi_0 \) is the known function of \( r \) defined by equation (3.24).

To determine the fourth equation, recall that only the plastic regime \( A \) is being considered here, which has the stress relation \( \sigma_i > \sigma_{\phi_1} = \sigma_{III} \). Therefore, for each \( i = 1, 2, \ldots, N \), (3.7) and (3.23) give

\[
A_i(r)(-\Sigma_0 + \Delta_0) - 4\sigma_{rz0}B_i(r) + C_i(r)(\Sigma_0 + \Delta_0) = 2\Delta_0 D_i(r),
\]

and hence, from (3.25) the relation

\[
D_i(r) = \frac{1}{2}(1 + \beta)[A_i(r) + C_i(r)], \quad (3.26)
\]

is obtained.

Therefore, the four equations (3.18), (3.19), (3.25), and (3.26) constitute the four determining equations for the unknown functions \( A_i(r), B_i(r), C_i(r), D_i(r) \) for each \( i = 1, 2, \ldots, N \).
3.3 The differential equation for \( B_i(r) \)

In this Section the four governing equations developed in the previous Section are considered, namely (3.18), (3.19), (3.25), and (3.26), and a second order ordinary differential equation for \( B_i(r) \) is determined by eliminating the other unknowns.

Upon substituting (3.26) into (3.18), which eliminates \( D_i(r) \), gives

\[
\frac{dA_i}{dr} - \alpha_i B_i(r) + \frac{(1 - \beta)A_i(r) - (1 + \beta)C_i(r)}{2r} = 0, \tag{3.27}
\]

and note from (3.19) that

\[
C_i(r) = \frac{1}{\alpha_i} \left[ \frac{dB_i}{dr} + \frac{B_i(r)}{r} \right], \tag{3.28}
\]

and therefore, (3.27) becomes

\[
\frac{dA_i}{dr} + \frac{(1 - \beta)}{2r} A_i(r) = \frac{(1 + \beta)}{2\alpha_i r} \left[ \frac{dB_i}{dr} + \frac{B_i(r)}{r} \right] + \alpha_i B_i(r). \tag{3.29}
\]

If (3.28) is also substituted into (3.25), then

\[
A_i(r) = \frac{s_1(r)}{\alpha_i} \left[ \frac{dB_i}{dr} + \frac{B_i(r)}{r} \right] - s_2(r)B_i(r). \tag{3.30}
\]

where

\[
s_1(r) = \frac{\cos 2\psi_0 - \beta}{\cos 2\psi_0 + \beta}, \quad s_2(r) = \frac{2\sin 2\psi_0}{\cos 2\psi_0 + \beta}. \tag{3.31}
\]

Hence, upon substituting (3.30) into (3.29) gives the second order ordinary differential equation for \( B_i(r) \)

\[
0 = \frac{d^2 B_i}{dr^2} + \left[ \frac{s'_1}{s_1} + \frac{1}{r} - \alpha_i \frac{s'_2}{s_2} + \frac{(1 - \beta)}{2r} - \frac{(1 + \beta)}{2rs_1} \right] \frac{dB_i}{dr}
\]

\[
+ \left[ \frac{s'_1}{rs_1} - \frac{1}{r^2} - \alpha_i \frac{s'_2}{s_2} + \frac{(1 - \beta)}{2r^2} - \alpha_i \frac{s_2(1 - \beta)}{2rs_1} - \frac{(1 + \beta)}{2r^2 s_1} - \frac{\alpha_i^2}{s_1} \right] B_i(r). \tag{3.32}
\]
for \( i = 1, \ldots, N \), and from (3.14), (3.15), and (3.30) the boundary conditions on \( B_i \) become

\[
\sum_{n=1}^{N} e^{-\alpha_n z} B_n(r_0) = -R(z) \left( \frac{d\sigma_{rr_0}}{dr} \right)_{r=r_0} + R'(z)\sigma_{zz_0}(r_0),
\]

(3.33)

\[
\left( \frac{dB_i}{dr} \right)_{r=r_0} = \alpha_i \left( \frac{1+\beta}{1-\beta} \right) A_i(r_0) - \frac{1}{r_0} B_i(r_0),
\]

where \( A_i(r_0) \) is determined from

\[
\sum_{n=1}^{N} e^{-\alpha_n z} A_n(r_0) = -R(z) \left( \frac{d\sigma_{rr_0}}{dr} \right)_{r=r_0} .
\]

(3.34)

In order to simplify matters, make the transformations

\[
r = \eta/\alpha_i, \quad B_i = \alpha_i B_i, \quad A_i = \alpha_i A_i,
\]

(3.35)

so that (3.32) becomes

\[
0 = \frac{d^2B_i}{d\eta^2} + \left[ \frac{s'_1}{s_1} + \frac{1}{\eta} - \frac{s_2}{s_1} + \frac{(1-\beta)}{2\eta} - \frac{(1+\beta)}{2\eta s_1} \right] \frac{dB_i}{d\eta}
\]

\[
+ \left[ \frac{s'_1}{\eta s_1} - \frac{1}{\eta^2} - \frac{s'_2}{s_1} + \frac{(1-\beta)}{2\eta^2} - \frac{s_2}{2\eta s_1} - \frac{(1+\beta)}{2\eta^2 s_1} - \frac{1}{s_1} \right] B_i(\eta),
\]

(3.36)

where \( s_1 \) and \( s_2 \) are now function of \( \eta \), and upon noting (3.4) and expanding, gives the boundary conditions for \( B_i \) at \( \eta = \alpha_i r_0 \) to be

\[
B_i(\alpha_i r_0) = -R_i \left( \frac{d\sigma_{rr_0}}{d\eta} \right)_{\eta=\alpha_i r_0} - R_i \sigma_{zz_0}(\alpha_i r_0),
\]

(3.37)

\[
\left( \frac{dB_i}{d\eta} \right)_{\eta=\alpha_i r_0} = R_i \left( \frac{1+\beta}{1-\beta} \right) \left( \frac{d\sigma_{rr_0}}{d\eta} \right)_{\eta=\alpha_i r_0} - \frac{1}{\alpha_i r_0} B_i(\alpha_i r_0).
\]

and note that each \( B_i \) is considered at different initial values, namely \( \eta = \alpha_i r_0 \).

Thus, (3.36) is a second order differential equation for \( B_i \) with two explicit boundary
conditions at $\eta = \alpha_i r_0$. Further, due to the complexity of the coefficient functions of $B_i, dB_i/d\eta$, and $d^2B_i/d\eta^2$, (3.36) are solved numerically subject to (3.37). This is essential since in general, $s_1$ and $s_2$ can only be determined numerically.

### 3.4 Special cases for $R(z)$

In this Section two possible shapes for the sidewall of the rat-hole are considered where the required shape is approximated using a sum of exponentials of the form of (3.4). Once the unknown constants in (3.4) have been determined, these are then used to solve the system of differential equations defined by (3.36) with the boundary conditions (3.37). Note that there is no unique procedure for this approximation and indeed the procedure adopted here for three terms for the double slightly tapered rat-hole give rise to two possible solutions.

#### 3.4.1 Single slightly tapered rat-hole.

For a single slightly tapered rat-hole as shown in Figure 3.1(a), the equation describing the sidewall of the rat-hole is

$$r = r_0 + z \tan \alpha. \quad (3.38)$$

Assuming a finite height $H_1$ and that the sidewall of the rat-hole can be approximated by a sum of exponentials as defined by (3.4), then (3.38) is approximated with a two term sum of the form

$$r = r_0 + \varepsilon \left( R_1 e^{-\alpha_1 z} + R_2 e^{-\alpha_2 z} \right), \quad (3.39)$$
for certain unknown constants $\alpha_1, \alpha_2, R_1,$ and $R_2$. For simplicity assume $\alpha_1 = 0$ and determine the unknown constants by firstly assuming that (3.38) and (3.39) coincide at the bottom of the rat-hole, namely at $z = 0$, from which $R_2 = -R_1$ is deduced. Secondly, assume that (3.38) and (3.39) coincide at the top of the rat-hole, namely at $z = H_1$, from which

$$H_1 = R_1 \left(1 - e^{-\alpha_2 H_1}\right),$$  

(3.40)

is found, where $\tan \alpha = \varepsilon$ has also been assumed.

Thirdly, assume for each $z$ that the maximum horizontal difference between (3.38) and (3.39) is $\varepsilon$, which gives

$$R_1 \left(1 - e^{-\alpha_2 z}\right) - z - 1 \leq 0,$$

or

$$z - 1 - R_1 \left(1 - e^{-\alpha_2 z}\right) \leq 0,$$

where (3.41)$_1$ is used when the right hand side of (3.39) is larger than the right hand side of (3.38), or in other words, the approximate solution is on the “outside” of the rat-hole, and similarly, (3.41)$_2$ is used when the right hand side of (3.39) is smaller then the right hand side of (3.38), or in other words, the approximate solution is on the “inside” of the rat-hole.

Here, the approximate solution is assumed to be on the outside of the rat-hole, and therefore from (3.41)$_1$ the maximum value occurs when

$$z = \frac{1}{\alpha_2} \ln \alpha_3 R_1,$$

(3.42)

which combined with (3.41)$_1$ gives the relation

$$R_1 \left(1 - \frac{1}{\alpha_2 R_1}\right) - \frac{1}{\alpha_2} \ln \alpha_3 R_1 = 1.$$  

(3.43)
Therefore, (3.40) and (3.43) give rise to two equations for the two unknowns \( R_1 \) and \( \alpha_2 \), and hence \( R(z) \) can be determined.

Note that the transformations (3.35) are only well defined for \( \alpha_i \neq 0 \). For \( \alpha_i = 0 \), the governing equations (3.32) are solved subject to (3.33), from which the boundary conditions become

\[
B_1(r_0) = -R_1 \left( \frac{d\sigma_{rz_0}}{dr} \right)_{r=r_0}, \quad \left( \frac{dB_1}{dr} \right)_{r=r_0} = -\frac{1}{r_0} B_1(r_0). \quad (3.44)
\]

### 3.4.2 Double slightly tapered rat-hole.

For a double slightly tapered rat-hole as shown in Figure 3.1(b), where \( \gamma \) is a small angle such that \( \gamma > \alpha \), denote \( H_1 \) to be the height where the sidewall of the rat-hole changes slope from \( \tan \alpha \) to \( \tan \gamma \), and \( H_2 \) to be the height of the double tapered rat-hole. The equations of the sidewall of the rat-hole are found to be given by

\[
r = r_0 + \epsilon z, \quad \text{for } 0 \leq z \leq H_1,
\]

\[
r = r_0 + \epsilon ((1 - K)H_1 + K z), \quad \text{for } H_1 \leq z \leq H_2,
\]

where \( \tan \alpha = \epsilon \) and \( \tan \gamma = K\epsilon \) has been assumed, for some \( K > 1 \).

Now, assuming that (3.45) can be approximated by a sum of exponentials as defined by (3.4), then for \( \alpha_i = 0 \) and assuming that (3.39) and (3.45) coincide at the bottom of the rat-hole and at the change of slope of the sidewall at \( z = H_1 \), \( R_2 = -R_1 \) is deduced and also that (3.40) holds. Further, assume that (3.39) and (3.45) coincide at the top of the double slightly tapered rat-hole, namely at \( z = H_2 \),
which gives the relation

\[ (1 - K)H_1 + KH_2 = R_1 \left(1 - e^{\alpha_2 H_2}\right), \] (3.46)

and (3.40) and (3.46) determines the unknown constants \( \alpha_2 \) and \( R_1 \). Also note that the appropriate boundary conditions for the double slightly tapered rat-hole with the approximation (3.39) for \( \alpha_1 = 0 \), where \( \alpha_2 \) and \( R_1 \) are determined from (3.40) and (3.46), are given by (3.37) and (3.44).

Now, assume that the sidewall of the double slightly tapered rat-hole as shown in Figure 3.1(b) can be approximated by a three term sum of exponentials as defined by (3.4). Hence, the sidewall of the rat-hole described by (3.45) is approximated by

\[ r = r_0 + \varepsilon \left(R_1 e^{-\alpha_1 z} + R_2 e^{-\alpha_2 z} + R_3 e^{-\alpha_3 z}\right), \] (3.47)

for some unknown constants \( \alpha_i \) and \( R_i \), for \( i = 1, 2, 3 \). Following the previous two term approximations, for simplicity \( \alpha_1 = 0 \) is again assumed and that (3.45) and (3.47) coincide at the bottom of the rat-hole, at the change of slope of the sidewall at \( z = H_1 \), and at the top of the rat-hole, which yield respectively

\[ R_1 + R_2 + R_3 = 0, \]

\[ R_1 + R_2 e^{-\alpha_2 H_1} + R_3 e^{-\alpha_3 H_1} = H_1, \] (3.48)

\[ R_1 + R_2 e^{-\alpha_2 H_2} + R_3 e^{-\alpha_3 H_2} = (1 - K)H_1 + KH_2. \]

Hence, (3.48) gives three equations for five unknowns, and therefore two additional constraints are required between the unknowns. Following Subsection 3.4.1. assume
that for $0 \leq z \leq H_1$, the maximum horizontal difference between (3.45) and (3.47) is $\varepsilon$, so that

$$R_1 + R_2 e^{-\alpha_2 z} + R_3 e^{-\alpha_3 z} - z - 1 \leq 0,$$

or

$$z - 1 - R_1 - R_2 e^{-\alpha_2 z} - R_3 e^{-\alpha_3 z} \leq 0,$$  \hspace{1cm} (3.49)

where (3.49)\textsubscript{1} is used when the solution is on the outside of the rat-hole and (3.49)\textsubscript{2} is used when the solution is on the inside of the rat-hole. If $z_1$ denotes the value of $z$ which gives the maximum value of (3.49), then for both equations, $z_1$ is determined from

$$\alpha_2 R_2 e^{-\alpha_2 z_1} + \alpha_3 R_3 e^{-\alpha_3 z_1} + 1 = 0,$$  \hspace{1cm} (3.50)

which is a transcendental equation for $z_1$. Once $z_1$ is determined then it can be substituted into (3.49) to obtain a new single condition on the unknowns, which depends on whether the solution is on the outside or on the inside of the rat-hole. However, as the values of $\alpha_2, \alpha_3, R_2,$ and $R_3$ are unknown, then this means that $z_1$ is treated as an unknown and (3.50) is included as an additional condition.

Similarly, for $H_1 \leq z \leq H_2$, assume that the maximum horizontal difference between (3.45) and (3.47) is $\varepsilon$, which yields

$$R_1 + R_2 e^{-\alpha_2 z} + R_3 e^{-\alpha_3 z} - (1 - K) H_1 - K z - 1 \leq 0,$$

or

$$K z + (1 - K) H_1 - 1 - R_1 - R_2 e^{-\alpha_2 z} - R_3 e^{-\alpha_3 z} \leq 0,$$  \hspace{1cm} (3.51)

where (3.51)\textsubscript{1} is used when the solution is on the outside of the rat-hole and (3.51)\textsubscript{2} is used when the solution is on the inside of the rat-hole. If $z_2$ denotes the value of $z$
which gives the maximum value of (3.51). Then for both equations, \( z_2 \) is determined from

\[
\alpha_2 R_2 e^{-\alpha_2 z_2} + \alpha_3 R_3 e^{-\alpha_3 z_2} + K = 0, \tag{3.52}
\]

which is also a transcendental equation for \( z_2 \). Once \( z_2 \) is determined it can then be substituted into (3.51) to deduce a new single condition on the unknowns, which depends on whether the solution is on the outside or on the inside of the rat-hole. However, as the values of \( \alpha_2, \alpha_3, R_2, \) and \( R_3 \) are unknown then again this means that \( z_2 \) is treated as an unknown and (3.52) is included as an additional condition.

This means that there are seven equations for seven unknowns which may be solved numerically. The appropriate boundary conditions for the double slightly tapered rat-hole with the approximation (3.47) for \( \alpha_1 = 0 \) are given by (3.37) and (3.44).

### 3.5 Conclusions

For slightly tapered cylindrical vertical cavities, the work presented in this Chapter is the first rigorous mathematical analysis of the limiting equilibrium equations (3.1) for the plastic regime \( A \) to determine an axially symmetric stress distribution which is a perturbation of the classical Jenike solution for a perfectly right circular cylindrical cavity. The perturbations are assumed to be separable functions of \( r \) and \( z \), and it is shown that the only allowable dependence on \( z \) must be exponential. For a slightly tapered rat-hole with profile \( r = r_0 + \varepsilon R(z) \), a second order ordinary differential equation has been solved numerically using boundary conditions arising
Figure 3.4: Single tapered rat-hole with mesh showing approximation on outside of rat-hole which is shown by the shaded area and with \( R(z) \) defined by (3.39).

From the fact that the cavity is stress free. Four solutions for four different shapes of the sidewall of the slightly tapered rat-hole have been determined using a possible, but not a unique set of constraints to determine \( R(z) \) and have evaluated the stress approximations on the plane \( z = 0 \). For all numerical solutions the constant values of \( \rho = 0.7, g = 9.8, \beta = 0.5 \) and \( f_c = 5.2 \) have been assumed.

Figure 3.4 shows the single slightly tapered rat-hole as the shaded area with the mesh boundary showing the approximate solution on the outside of the rat-hole for \( R(z) \) defined by (3.39) with \( \alpha_2 = 0.53, R_1 = 4.55, \) and \( \varepsilon = 1/12 \). Figure 3.5 shows the approximate stresses relative to the classical stresses applying to a right circular cylindrical rat-hole. In particular, \( \sigma_{rr} \) is initially higher but then is below the classical estimate, while \( \sigma_{rz} \) is always below the classical estimate. \( \sigma_{zz} \) starts at the classical estimate and then goes below while \( \sigma_{\phi\phi} \) is initially higher and then
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Figure 3.5: Classical and approximate stresses corresponding to Figure 3.4.

goes below $\sigma_{\phi\phi}$.

Figure 3.6 shows the double slightly tapered rat-hole with the boundary of the shaded area showing the approximate solution on the inside of the rat-hole for $R(z)$ defined by (3.39) with $\alpha_2 = -0.59$, $R_1 = -0.41$, $\varepsilon = 1/12$, and $\gamma = 25$ degrees.

Figure 3.7 shows that the approximate stresses follow the classical stresses. In particular, $\sigma_{rr}$ is very close to the classical estimate, while $\sigma_{rz}$ starts higher, but then asymptotes to the classical estimate. $\sigma_{zz}$ and $\sigma_{\phi\phi}$ are in excess of the classical estimates.

Figure 3.8 shows a double slightly tapered rat-hole with the approximated cavity profile being the shaded area in the lower region and the mesh boundary in the upper region. Here $R(z)$ is defined by (3.47) with $\alpha_2 = -1.15$, $\alpha_3 = 0.70$, $R_1 = 3.80$, $R_2 = 0.004$, $R_3 = -3.806$, $\varepsilon = 1/12$, and $\gamma = 9$ degrees. In this case the approximate
Figure 3.6: Double tapered rat-hole with shaded region showing approximation on inside of rat-hole which is shown by the mesh boundary and with $R(z)$ defined by (3.39).

Figure 3.7: Classical and approximate stresses corresponding to Figure 3.6.
solution starts on the outside of the rat-hole, similar to Figure 3.4, and then goes on the inside of the rat-hole at the change of slope, similar to Figure 3.6. Therefore from Figure 3.9, for the approximate stresses on the plane $z = 0$, it is not surprising to see that they behave in a similar fashion to those for Figure 3.4. Figure 3.10 shows the double slightly tapered rat-hole with the approximate cavity profile being the mesh boundary in the lower region and the shaded area in the upper region. For the approximate solution shown in Figures 3.8 and 3.9, there are two profiles namely that given in Figure 3.8 and that given in Figure 3.10 which is the shaded area in the lower region and the mesh boundary in the upper region. For this shape, $R(z)$ is defined by (3.47) with $\alpha_2 = -0.55$, $\alpha_3 = -1.95$, $R_1 = -0.5144$, $R_2 = 0.5145$, $R_3 = 0.00004$, $\varepsilon = 1/12.$ and $\gamma = 9$ degrees. Here the approximate solution is on
Figure 3.9: Classical and approximate stresses corresponding to Figure 3.8.

Figure 3.10: Double tapered rat-hole with approximation as the shaded area on the inside in the lower region and as the mesh on the outside in the upper region and with \(R(z)\) defined by (3.47).
Figure 3.11: Classical and approximate stresses corresponding to Figure 3.10.

the inside in the lower region, similar to Figure 3.6, and on the outside in the upper region, similar to Figure 3.4. Therefore from 3.11, the approximate stresses on the plane \( z = 0 \) behave in a similar fashion to those shown in Figure 3.6.
Chapter 4

Rat-hole stress profiles for shear-index granular materials

4.1 Introduction

The formation of stable circular and almost vertical cylindrical holes in stockpiles and hoppers, as indicated in Figures 4.1(a) and 4.1(b) respectively, are the cause of significant disruption in many industries such as mining, minerals, grains and chemical industries. This frequently occurring phenomena is referred to as “piping” and the holes themselves are known as “rat-holes”. Often once a rat-hole forms it tends to remain there because the material dries and sets as a solid. The precise conditions under which a rat-hole may form or the conditions prescribing the stability of an existing rat-hole are as yet unresolved issues. These issues are not addressed in this Chapter, but rather the limiting equilibrium stress profiles assuming an existing vertical cylindrical rat-hole in a shear-index granular material are determined. This
Chapter 4: Rat-hole stress profiles for shear-index granular materials

Figure 4.1: (a). Rat-hole occurring in a typical stockpile. (b) Rat-hole causing funnel flow in a hopper.

work extends existing theory for the Coulomb-Mohr yield function.

Shear-index yield functions and certain plane and axially symmetric problems are analysed in detail in Hill and Wu [27, 28]. Failure of powders or granular materials is due to frictional slip between particles, and at yield the magnitude of the shear component of stress $|\tau|$ varies according to the value of the normal component of stress $\sigma$, which are taken to be positive in tension. The locus of the values of $(\sigma, \tau)$ at which permanent deformation or yield occurs is called the yield locus. It can also be defined as the envelope of the Mohr stress circles at yield, because in plasticity the stress state must satisfy the stress equilibrium equations and cannot exceed the yield locus. For certain granular materials, the angle of internal friction is a constant and the yield locus is a straight line and is referred to as the Coulomb-Mohr yield function, thus

$$|\tau| = c - \sigma \tan \delta,$$  \hspace{1cm} (4.1)
where $c$ is a constant, which is called the cohesion of the material. This yield function has been interpreted by Shield [50] in terms of principal stress components to obtain the yield surface for a three dimensional stress field. Shield [50] showed that in principal stress space the yield surface is a right hexagonal pyramid equally inclined to the $\sigma_1, \sigma_2, \sigma_3$ axes, and with its vertex at the point $\sigma_1 = \sigma_2 = \sigma_3 = c \cot \delta$.

In general however, experimental evidence (see Hill and Wu [27] for detailed references) indicates that for most granular materials the angle of internal friction is not constant along the yield locus but decreases for decreasing $\sigma$ from a maximum value $\pi/2$ at the vertex $A$ and typically the corresponding yield locus is as indicated in Figure 4.2. In general therefore, the angle of internal friction is a stress dependent function $\delta(\sigma)$ which is defined incrementally from the equation

$$d\tau = -d\sigma \tan \delta.$$  \hspace{1cm} (4.2)

Here, the yield function sometimes referred to as the Warren Spring equation is
Chapter 4: Rat-hole stress profiles for shear-index granular materials

considered, namely

$$\left(\frac{|\tau|}{c}\right)^n = 1 - \frac{\sigma}{t},$$  \hspace{1cm} (4.3)

where $c$, $t$ and $n$ are positive constants which are referred to as the cohesion, tensile strength and shear index respectively. A number of authors (see Hill and Wu [27]) have performed experiments which confirm the validity of (4.3). Using the Jenike shear tests Farley and Valentin [22] suggest that the cohesion $c$ is usually of the order of twice the tensile strength $t$ and that the shear index $n$ for a particular powder is independent of the bulk density of the compact, and can therefore be used to classify powders according to their flow properties. In addition, Farley and Valentin [22] give simple expressions relating $n$ to the ratio of volume to surface mean diameter and $t$ to the bulk density. The known numerical values of shear index $n$ such as those cited in Hill and Wu [27] all lie between 1 and 2 and Table 4.1 gives the typical values of $n, t, c$ and $\rho$ as determined by Farley and Valentin [22], noting that the units in Table 4.1 are not SI units, but are those used by Farley and Valentin [22].

For a general yield function Hill and Wu [27] show that the yield function in terms of principal stress components is given parametrically by

$$(\sigma_I - \sigma_{III}) \cos \delta = 2f\left[\frac{\sigma_I + \sigma_{III}}{2} + (\sigma_I - \sigma_{III})(\sin \delta)/2\right],$$

$$\tan \delta = -\frac{d}{d\sigma}f\left[\frac{\sigma_I + \sigma_{III}}{2} + (\sigma_I - \sigma_{III})(\sin \delta)/2\right].$$

where the stress dependent angle of internal friction $\delta = \delta(\sigma)$ is the parameter. Thus, for example, if the angle of internal friction is constant and $\tau = f(\sigma)$ is the linear yield condition (4.1), then (4.4) gives the well known Coulomb-Mohr yield
Table 4.1: Typical values of $n$, $t$, $c$ and $\rho$ (Farley and Valentin [22], g denotes 981 dynes).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Granular material} & \text{Shear-index } n & \text{Tensile strength } t & \text{Cohesion } c & \text{Density } \rho \\
\text{(Particle size)} & \text{(g/cm}^2\text{)} & \text{(g/cm}^2\text{)} & \text{(g/cm}^2\text{)} & \text{(gm/cm}^3\text{)} \\
\hline
\text{Alumina} & \text{(+37 } \mu\text{)} & 1.19 & 0.226 & 0.385 & 1.018 \\
& \text{(20-30 } \mu\text{)} & 1.52 & 8.09 & 12.2 & 1.113 \\
& \text{(9 - 15 } \mu\text{)} & 1.40 & 10.55 & 15.2 & 1.176 \\
\text{Zinc dust} & \text{(Standard)} & 1.39 & 2.54 & 5.33 & 3.706 \\
& \text{(Ultra fine)} & 1.86 & 4.89 & 11.7 & 3.238 \\
\text{CaCO}_3 & \text{(< 12 } \mu\text{)} & 1.53 & 2.95 & 8.09 & 0.888 \\
& \text{(12 - 14 } \mu\text{)} & 1.46 & 0.76 & 1.83 & 1.112 \\
\hline
\end{array}
\]

condition

\[
\sigma_I - \sigma_{III} = 2c \cos \delta - (\sigma_I + \sigma_{III}) \sin \delta. \tag{4.5}
\]

In the case of the Warren Spring equation (4.3), the yield function is

\[
f(\sigma) = c \left(1 - \frac{\sigma}{t}\right)^{1/n}, \tag{4.6}
\]

and (4.4) gives the yield condition in parametric form to be

\[
\frac{\sigma_I}{t} = 1 + \frac{\beta c}{t} (\sec \delta - \tan \delta) - \beta^n, \quad \frac{\sigma_{III}}{t} = 1 - \frac{\beta c}{t} (\sec \delta + \tan \delta) - \beta^n. \tag{4.7}
\]

where $\beta$ is a function of $\delta$ defined by

\[
\beta = \left(\frac{nt}{c \tan \delta}\right)^{1/(1-n)}. \tag{4.8}
\]
and note that the special cases \( n = 1 \) and \( n = 2 \) become respectively
\[
\left(1 - \frac{\sigma_I}{t}\right)^{1/2} = \left(1 + \left(\frac{c}{t}\right)^2\right)^{1/2} - \frac{c}{t} \left(1 - \frac{\sigma_{III}}{t}\right)^{1/2} \quad (n = 1),
\]
\[
\left(1 - \frac{\sigma_I}{t}\right)^{1/2} = \left(1 - \frac{\sigma_{II}}{t}\right)^{1/2} - \frac{c}{t} \quad (n = 2).
\]
These relations become more transparent by expressing the parametric solution (4.7) in the form
\[
1 - \frac{\sigma_I}{t} = \frac{1}{\beta^n} \left\{ \left[ \beta^{2n} + \left(\frac{\beta \lambda}{n}\right)^2 \right]^{1/2} - \frac{\beta \lambda}{2} \right\}^2 - (\beta \lambda)^2 \left(\frac{1}{n} - \frac{1}{2}\right)^2,
\]
\[
1 - \frac{\sigma_{III}}{t} = \frac{1}{\beta^n} \left\{ \left[ \beta^{2n} + \left(\frac{\beta \lambda}{n}\right)^2 \right]^{1/2} + \frac{\beta \lambda}{2} \right\}^2 - (\beta \lambda)^2 \left(\frac{1}{n} - \frac{1}{2}\right)^2,
\]
where \( \lambda \) denotes \( c/t \) and from which it is clear that \( n = 1 \) and \( n = 2 \) play special roles and (4.9) may be readily deduced. It appears that \( n = 1 \) and \( n = 2 \) are the only values of \( n \) giving rise to simple analytical yield functions such as (4.9). However, other special values of \( n \) such as \( n = 4/3, 3/2, \) and \( 8/5 \) permit further analytical investigation but the final results are still complicated (see the Appendix of Hill and Wu [27]).

In the following Section the mathematical formulation to determine the limiting equilibrium stresses are provided for a perfectly vertical and circular cylindrical rat-hole for a granular material subject to the yield condition given parametrically by (4.7). The corresponding problem for the Coulomb-Mohr yield function (4.1) is examined in Chapter 1 while in Chapter 2 the stress profiles for slightly tapered vertical circular cavities are determined again assuming (4.1). In the subsequent Section, further analytical details are presented for the cases \( n = 1 \) and \( n = 2 \).
and in the final Section the numerically determined results for general $n$ such that $1 < n < 2$ are shown to be bounded by those for $n = 1$ and $n = 2$ respectively.

### 4.2 Mathematical formulation

For the idealised situation of a vertical circular cylindrical rat-hole with the axis as shown in Figure 4.1, the mathematical problem is to solve the equilibrium equations

\[
\frac{d\sigma_{rz}}{dr} + \frac{\sigma_{rz}}{r} = \rho g, \quad \frac{d\sigma_{rr}}{dr} + \frac{(\sigma_{rr} - \sigma_{\phi\phi})}{r} = 0, \tag{4.11}
\]

subject to the boundary conditions that the surface of the hole of radius $r_0$ is stress free

\[
\sigma_{rz} = \sigma_{rr} = 0 \quad \text{for} \quad r = r_0, \tag{4.12}
\]

where $\rho$ is the bulk density of the material, $g$ is the acceleration due to gravity, $\sigma_{rr}, \sigma_{rz},$ etc. denote the stresses in a cylindrical polar coordinate system $(r, \phi, z)$, and which following Jenike [34] are assumed to be independent of $\phi$ and $z$. In addition, the material is assumed to satisfy the Warren Spring yield condition (4.3), where $\sigma$ and $\tau$ denote the normal and tangential components of compressive traction, which are assumed to be positive in tension. Namely, the usual convention in continuum mechanics is adopted that positive forces are assumed to produce positive extensions.

From (4.11) and (4.12) it is a simple matter to deduce

\[
\sigma_{rz} = \frac{\rho g}{2} \left( r - \frac{r_0^2}{r} \right), \tag{4.13}
\]
while the maximum and minimum principal stresses are given respectively by
\[
\sigma_1 = \frac{1}{2} \left\{ (\sigma_{rr} + \sigma_{zz}) + \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} \right\}.
\]
(4.14)
\[
\sigma_{III} = \frac{1}{2} \left\{ (\sigma_{rr} + \sigma_{zz}) - \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} \right\}.
\]
From the boundary conditions (4.12), (4.14), and noting that for compression \(\sigma_{zz} < 0\), and therefore \(\sqrt{\sigma_{zz}^2}\) is \(-\sigma_{zz}\), then at \(r = r_0\) it is observed that
\[
\sigma_1(r_0) = 0, \quad \sigma_{III}(r_0) = \sigma_{zz}(r_0),
\]
and therefore \(\sigma_{zz}(r_0) = -f_c\), where \(f_c\) is usually referred to as the unconfined yield strength which is defined by \(\sigma_{III} = -f_c\) when \(\sigma_1 = 0\). Moreover, throughout this Chapter the plastic regime conventionally referred to as A (namely one of the Haar-von Karmen regimes) is assumed to apply, and therefore
\[
\sigma_{\phi\phi} = \sigma_{III}.
\]
(4.16)

The above equations, along with the shear-index yield function defined parametrically by (4.7), provide the complete mathematical prescription of the problem. In the following Section the mathematical analysis is extended for the special cases \(n = 1\) and \(n = 2\) for which the yield condition in terms of \(\sigma_1\) and \(\sigma_{III}\) can be given explicitly.

### 4.3 Mathematical analysis for \(n = 1\) and \(n = 2\)

In this Section, some limited mathematical analysis is presented for the two special cases \(n = 1\) and \(n = 2\). For \(n = 1\) this approach differs from that given in Chapter 2.
Now, (4.11)\textsubscript{2} is solved subject to $\sigma_{rr} = 0$ at $r = r_0$ and $\sigma_{\phi\phi}$ is determined from (4.14)\textsubscript{2} and (4.16), thus

$$\sigma_{\phi\phi} = \frac{1}{2} \left\{ (\sigma_{rr} + \sigma_{zz}) - \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} \right\}. \quad (4.17)$$

For $n = 1$ the standard Coulomb-Mohr yield condition becomes (see Chapter 2, page 25)

$$\beta(\sigma_{rr} + \sigma_{zz}) - 2c(1 - \beta^2)^{1/2} = - \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2}. \quad (4.18)$$

where $\beta = \sin \delta$ and $\tan \delta = c/t = \lambda$. Now, on squaring this equation and solving the resulting quadratic in $\sigma_{zz}$, gives

$$\sigma_{zz} = \sigma_{rr} - 2c \left\{ \frac{c}{t} \left( 1 - \frac{\sigma_{rr}}{t} \right) + \left( 1 + \frac{c^2}{t^2} \right)^{1/2} \left[ \left( 1 - \frac{\sigma_{rr}}{t} \right)^2 - \left( \frac{\sigma_{rz}}{c} \right)^2 \right]^{1/2} \right\}, \quad (4.19)$$

where only the negative root has been taken to ensure $\sigma_{zz} < 0$. Clearly, for a real solution the inequality

$$1 - \frac{\sigma_{rr}}{t} > \frac{\sigma_{rz}}{c}, \quad (4.20)$$

is required, noting that $\sigma_{rz}/c > 0$.

Moreover for the case $n = 2$, on writing (4.9)\textsubscript{2} as

$$\left( 1 - \frac{\sigma_{III}}{t} \right)^{1/2} - \left( 1 - \frac{\sigma_{I}}{t} \right)^{1/2} = \frac{c}{t},$$

squaring this equation and solving the resulting quadratic equation, then the following expression for $\sigma_{zz}$ may be deduced, namely

$$\sigma_{zz} = \sigma_{rr} - \frac{c^2}{t} - 2c \left[ 1 - \frac{\sigma_{rr}}{t} - \left( \frac{\sigma_{rz}}{c} \right)^2 \right]^{1/2}. \quad (4.21)$$
Chapter 4: Rat-hole stress profiles for shear-index granular materials

where again only the negative root has been taken to ensure $\sigma_{zz} < 0$. Evidently, for a real solution the inequality

$$1 - \frac{\sigma_{rr}}{t} > \left(\frac{\sigma_{rz}}{c}\right)^2,$$  \hspace{1cm} (4.22)

is required. Now on introducing $u(r)$ defined by

$$u(r) = \lambda + 2 \left[1 - \frac{\sigma_{rr}}{t} - \left(\frac{\sigma_{rz}}{c}\right)^2\right]^{1/2},$$  \hspace{1cm} (4.23)

noting again that $\lambda = c/t$, it is not difficult to show that (4.11)$_2$, (4.17) and (4.21) yield the first order ordinary differential equation

$$(u - \lambda)\frac{du}{dr} + \varepsilon \frac{d\varepsilon}{dr} = \frac{\lambda}{r} \left[u + (u^2 + \varepsilon^2)^{1/2}\right].$$  \hspace{1cm} (4.24)

which for $n = 2$ must be solved numerically subject to the boundary condition

$$u(r_0) = 2 + \lambda,$$  \hspace{1cm} (4.25)

where $\varepsilon(r)$ denotes $\rho g (r - r_0^2/r)/c$. In the following Section the general numerical scheme is presented for $n$ such that $1 < n < 2$ and in the special case $n = 2$, the same result arises from the numerical integration of (4.24) and (4.25).

Although no essential use of the following analysis is made, it may be worthwhile noting that with the transformations

$$s = \log r, \; \; s_0 = \log r_0,$$  \hspace{1cm} (4.26)

the function $\varepsilon(r)$ becomes

$$\varepsilon(s) = \gamma \sinh(s - s_0),$$  \hspace{1cm} (4.27)
where the constant $\gamma$ is given by $\gamma = 2pge^{0}/c$. Further, the differential equation (4.24) transforms to

$$\left(u - \lambda\right)\frac{du}{d\varepsilon} + \varepsilon = \lambda \left\{ \frac{u + (u^{2} + \varepsilon^{2})^{1/2}}{(\varepsilon^{2} + \gamma^{2})^{1/2}} \right\}, \quad (4.28)$$

which still must be solved numerically subject to $u = 2 + \lambda$ when $\varepsilon = 0$. However, it is at least worth noting that the $r$ dependence in (4.24) may be transformed away in this manner.

In the numerical Section below, it is clear that for all $n$ there exists a point $r$ at which the stress profiles no longer exist, and for $n = 1$ and $n = 2$ this corresponds to where the square roots in (4.19) and (4.21) become imaginary. Thus at this point $r$, (4.19) and (4.21) gives directly

$$1 - \frac{\sigma_{rr}}{t} = \frac{\sigma_{rz}}{c} \quad (n = 1), \quad 1 - \frac{\sigma_{rr}}{t} = \left( \frac{\sigma_{rz}}{c} \right)^{2} \quad (n = 2), \quad (4.29)$$

and also, from (4.19) and (4.21) give

$$\sigma_{zz} = \sigma_{rr} - 2\lambda\sigma_{rz} \quad (n = 1), \quad \sigma_{zz} = \sigma_{rr} - \lambda c \quad (n = 2). \quad (4.30)$$

As described in Appendix B these expressions may be utilised to determine the stress values at the point $r$ at which the stress profiles fail to exist. The advantage of the limited mathematical analysis of this Section is that the above formulae, namely (4.29) and (4.30), are immediately apparent, but the question arises as to the appropriate extension for general $n$ such that $1 < n < 2$. Since in the general case further mathematical analysis appears difficult, the question is nontrivial. However, in Appendix B it is established that

$$1 - \frac{\sigma_{rr}}{t} = \left( \frac{\sigma_{rz}}{c} \right)^{n}, \quad \sigma_{zz} = \sigma_{rr} - \frac{2\lambda c}{n} \left( \frac{\sigma_{rz}}{c} \right)^{2-n}. \quad (4.31)$$
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as the appropriate generalisation of (4.29) and (4.30).

4.4 Numerical formulation for general \( n \)

For general \( n \) such that \( 1 < n < 2 \), equations (4.11)_2 and (4.17) show that

\[
\frac{d\sigma_{rr}}{dr} = -\frac{1}{2r} \left\{ (\sigma_{rr} - \sigma_{zz}) + \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} \right\},
\]

(4.32)

must be solved subject to (4.12)_2, where \( \sigma_{rz} \) is defined by (4.13) and \( \sigma_{zz} \) is determined from the Warren Spring yield condition (4.7). Hill and Wu [27] show that (4.7) gives rise to the yield condition

\[
\left[ \frac{n^2}{2\lambda^2(n-1)} \right]^{n/(2-n)} \left\{ B - \left[ B^2 - \frac{(n-1)}{n^2 t^2} A \right]^{1/2} \right\}^{n/(2-n)}
\]

(4.33)

\[
+ \frac{n}{2(n-1)} \left\{ B - \left[ B^2 - \frac{(n-1)}{n^2 t^2} A \right]^{1/2} \right\} = B = 0,
\]

where \( A \) and \( B \) are defined by

\[
A = (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2, \quad B = 1 - \frac{\sigma_{rr} + \sigma_{zz}}{2t}.
\]

(4.34)

In order to numerically solve (4.32), \( \sigma_{rz} \) must be numerically determined from (4.33) at each iteration in the numerical scheme. Note that upon letting \( n = 1 \) in (4.33), the \( n = 1 \) yield equation (4.9)_1 arises after applying \( \ell \) Hopital's rule. Also, upon letting \( n = 2 \) in (4.33) gives rise to the \( n = 2 \) yield equation (4.9)_2, noting that first both sides of (4.33) need to be raised to \( 2 - n \).

Here, a fourth order Runge-Kutta scheme has been used to determine the numerical solution of (4.32) and fsolve, a root finding procedure in MAPLE, is used.
to numerically determine $\sigma_{zz}$ from (4.33) at each iteration. Note that this scheme gives rise to numerical values for which the special cases $n = 1$ and $n = 2$ coincide with those obtained directly. The values of the constants $t, c, \rho$ and the shear-index $n$ used are those obtained by Farley and Valentin [22]. Three specific granular shear-index materials have been considered in this Chapter, namely alumina, standard zinc dust, and precipitated calcium carbonate which have shear-indexes of $n = 1.19, 1.39,$ and $1.53$, respectively. For the sake of comparison and in order to emphasise the dependence on the shear-index $n$, the remaining constants are fixed as those for the standard zinc dust sample, namely $t = 2.54$ g/cm$^2$, $c = 5.33$ g/cm$^2$, and $\rho = 3.7062$ gm/cm$^3$. Farley and Valentin [22] adopt the unusual convention that $g$ in reference to the units of $t$ and $c$ designates 981 dynes.

Figure 4.3 shows the yield function in terms of the variation of the maximum principal stress $\sigma_I$ with the minimum principal stress $\sigma_{III}$ for various values of $n$. It can be see that in the range shown these curves are almost straight lines. The unconfined yield strength $f_c$ which is defined by $\sigma_{III} = -f_c$ when $\sigma_I = 0$ is an important physical property of the material, being the “compressive” stress that an initially unstressed material is able to sustain immediately prior to failure. The variation of this quantity with the shear-index $n$ is shown in Figure 4.4. For $n = 1$ and $n = 2$, explicit values may be deduced, namely

$$f_c = 2c \left[ \frac{c}{t} + \left( 1 + \frac{c^2}{t^2} \right)^{1/2} \right] (n = 1), \quad f_c = c \left( \frac{c}{t} + 2 \right), \quad (n = 2).$$

In Appendix B the following expression is proposed as an appropriate generalisation
Figure 4.3: Shear-index yield condition expressed in terms of the maximum and minimum principal stresses and for various values of $n$.

of (4.35) for general $n$, namely

$$f_c = \frac{2c^2}{n \ell} + 2c \left(1 + \frac{c^2}{\ell^2}\right)^{(2-n)/2n}$$

(4.36)

and this value is shown as a dotted line in Figure 4.4. It should be emphasised that although the expression (4.36) appears to provide a good numerical estimate of $f_c$, it is only an educated guess based on the exact expressions (4.35).

Figures 4.5(a), 4.5(b), 4.5(c) and 4.5(d) show respectively the variations of $\sigma_{rr}, \sigma_{\phi\phi}, \sigma_{zz}$ and $\sigma_I$ for five values of the shear-index $n$, noting that $\sigma_{III}$ coincides with $\sigma_{\phi\phi}$.

Finally, Figures 4.6(a), 4.6(b) and 4.6(c) shows the relative variation of all stresses for the three values $n = 1, 1.53$ and 2 respectively.

It is a matter of common experience that the larger the radius of the rat-hole, the more prone the material is to collapse and the essential problem is to determine the smallest rat-hole radius for which the material is unstable. Accordingly, in
Figure 4.4: Variation of the unconfined yield strength $f_c$ with shear-index $n$ and the approximate analytical expression (4.36) shown as a dashed line.

Figures 4.7, 4.8 and 4.9 the variation of the stresses $\sigma_{rr}, \sigma_{\phi\phi}, \sigma_{zz}$ and $\sigma_I$ are shown for the three values of the shear-index $n = 1, 1.53, \text{and} 2$ respectively. The Figures show conclusively that for each of the three values of $n$ there is a definite rat-hole radius at which there is an abrupt change in the stress patterns, and for each $r_0$ there is a value of $r$ for which the stresses no longer exist. As noted in the previous Section for $n = 1$ and $n = 2$, it may be shown that at this value of $r$ the relations (4.29) and (4.30) apply, while for $1 < n < 2$ the relation (4.31) apply. Figures 4.10 and 4.11 show the variation of $\sigma_{rr}$ and $\sigma_{zz}$ respectively with $n$ for both the exact numerically determined values and those estimates based on (B.1) and the approximate equations (B.10) and (B.11) when $\sigma_I = 0$ and $\sigma_{III} = -f_c$. Note that these estimates provide reasonable overall agreement with the exact numerically determined values.
4.5 Conclusions

Within this Chapter the limiting equilibrium stress profiles within an existing vertical rat-hole in a shear-index granular material have been determined. For values of the shear index of $1 \leq n \leq 2$, the stress profiles show that there exists some point within the material at which they fail to exist. For $n = 1$ and $n = 2$ this point has been determined analytically, whereas for general $n$ the point has only
been determined approximately.

For a shear-index granular material, an exact expression for the unconfined yield strength $f_c$ has been determined for $n = 1$ and $n = 2$. For general $n$, an approximate generalisation has been determined that does coincide with the exact values when $n = 1$ and $n = 2$. 

Figure 4.6: Variation of stresses with position for the three shear-index materials. 
((a) $n = 1$, (b) $n = 1.53$ and (c) $n = 2$).
Figure 4.7: Variation of stresses for various values of the rat-hole radius $r_0$ for the shear-index material $n = 1$. ((a) $\sigma_{rr}$, (b) $\sigma_{\phi\phi}$, (c) $\sigma_{zz}$ and (d) $\sigma_I$).
Figure 4.8: Variation of stresses for various values of the rat-hole radius $r_0$ for the shear-index material $n = 1.53$. ((a) $\sigma_{rr}$, (b) $\sigma_{\phi\phi}$, (c) $\sigma_{zz}$ and (d) $\sigma_I$).
Figure 4.9: Variation of stresses for various values of the rat-hole radius \( r_0 \) for the shear-index material \( n = 2 \). ((a) \( \sigma_{rr} \), (b) \( \sigma_{\phi\phi} \), (c) \( \sigma_{zz} \) and (d) \( \sigma_{I} \)).
Figure 4.10: Numerical variation of $\sigma_{rr}$ with shear-index $n$ when $\sigma_l = 0$ and $\sigma_{III} = -f_c$ compared with the approximate value determined from (B.10).

Figure 4.11: Numerical variation of $\sigma_{zz}$ with shear-index $n$ when $\sigma_l = 0$ and $\sigma_{III} = -f_c$ compared with the approximate value determined from (B.10) and using (B.1) with (B.8)$_1$. 
Chapter 5

Force distributions at the base of two-dimensional sand-piles

5.1 Introduction

Granulated materials in the chemical industries are frequently stored in heaps. In order to evaluate storage conditions, a knowledge of the stress distribution throughout the material is important, and particularly at the base of the heap, since these stresses influence settlement, caking, comminution and overall deterioration of the stored material. Smid and Novosad [51] showed experimentally that the horizontal and vertical force distributions were as shown in Figure 5.1. Of particular importance is the counter-intuitive outcome, that the maximum vertical pressure does not occur directly beneath the sand-pile vertex, but rather at some intermediate point giving rise to a ring of maximum vertical pressure. This result has attracted much attention, both in the popular scientific literature (see for example Watson [54, 55])
Figure 5.1: Horizontal and vertical force distributions as determined experimentally by Smid and Novosad [51] ((a) horizontal and (b) vertical).

and has produced numerous discrete and computational models which attempt to explain this curious phenomenon (see for example Bagster [3, 4, 5], Bagster and Li [7, 8], Bagster and Kirk [6], Brooks and Bagster [12] and Liffman, Chan and Hughes [42]). In this Chapter, following the authors Cantelaube et al [14], Cantelaube and Goddard [15] and Didwania et al [20], a continuum mechanical approach is examined for two-dimensional sand-piles and the solutions first used by Jenike [33, 35, 36] and Johanson [39] are utilized, which describe gravity flows of granular materials in converging wedges and cones. More recently these solutions have been re-examined by Bradley [11] and Spencer and Bradley [53] and for convenience the notation of these latter authors have been adopted. Note however, that gravity in this problem is acting in the opposite direction to their problem and that
In this Chapter, an 'infinite' two-dimensional sand-pile is considered and the coordinate axis is set up at the vertex of the sand-pile as indicated in Figure 5.2, with gravity shown acting in the vertically upwards direction. The basic idea is that the stress distribution in a sand-pile of infinite height is determined and then the horizontal and vertical forces acting along a horizontal plane at a finite height are evaluated. Clearly the problem for a finite sand-pile, resting on a rigid horizontal plane is different to that examined here, but nevertheless the two force distributions would be expected to be similar. As previously stated, initially the solutions for gravity flow of granular materials in converging wedges are used which assumes that the entire sand-pile is at yield. A formal exact parametric solution for the special case of an angle of internal friction equal to ninety degrees is determined, and it is only for this special case that a numerical solution that satisfies all the necessary boundary conditions can be determined for both $\nu$ and $F$ negative. Next,
a slightly more general form of the stresses than that proposed by Jenike [33, 35, 36] is assumed. From this formulation a numerical solution for only the angle of internal friction equal to the angle of repose, which of course is the physically relevant case, can be determined. However, the resulting solution fails to allow zero stress of \( \sigma_{r\theta} \) at the centre of the sand-pile and is therefore non-physical. Finally, a sand-pile which is made up of two regions, an inner dead region and an outer yield region is proposed. From a full numerical solution for the outer yield region, it is found that the solution actually follows a simple well known exact solution of the governing equations for which the stresses are linear in both \( x \) and \( y \). From this observation, a possible stress distribution in the inner dead region is deduced, which is also linear in both \( x \) and \( y \) and satisfies the equilibrium equations, but not the Coulomb-Mohr yield condition, and is not uniquely determined. However, the resulting force distribution has the characteristic M-shape of the experimental curves of Smid and Novosad [51].

It should be noted that the solutions determined within this Chapter only apply for situations where the granular material is cohesionless. It should also be noted that the particular continuum theory of granular materials adopted here ignores the initial arrangement of the individual granular particles within the sand-pile. However, in practice the packing or the manner in which the granular particles are assembled plays an important role in determining the stress distribution within the sand-pile, especially at the base. Despite this, it is expected that the continuum approach adopted here will give some insight into the underlying stress distribution within the sand-pile.

In the following Section for two-dimensional sand-piles the basic equations of
the continuum theory of granular materials are briefly stated and expressions for horizontal and vertical force resultants are given. As previously mentioned the notation of Spencer and Bradley [53] has been adopted, noting again that in this problem gravity acts in the opposite direction and the $x$-axis denotes the vertical direction. In the subsequent Section, a number of additional relations involving boundary values are derived. These additional boundary relations are derived on the assumption that the derivative of the stress angle $\psi$ defined by (5.4) at the surface of the pile remains finite and that both $F$ and $\psi$ are negative. A numerical scheme making use of these additional relations indicates that no solution exists, and therefore the case when this derivative becomes infinite needs to be considered. A formal exact solution of the governing equations for two-dimensional sand piles for the special case of an angle of internal friction equal to ninety degrees is derived and details for this exact solution are presented in Appendix C. It turns out that only for this special case does the formulated problem admit a solution.

In Section 5.4, an alternative formulation to that proposed in Section 5.2 is examined, which produces a non-physical solution in the sense that the stress $\sigma_{r\theta}$ does not have the correct symmetry. In Section 5.5, a sand-pile that has an inner dead region and an outer yield region is considered, and a solution that exhibits the same behaviour as the experimental data shown in Figure 5.1 is determined. In the final Section of this Chapter, numerically determined stress solutions for each of the three models are shown graphically and are compared with the the experimental force distribution curves.
5.2 Basic equations of continuum theory

For quasi-static plane flow, the stress components in a cylindrical polar coordinate system \((r, \theta, z)\) defined as shown in Figure 5.2, satisfy the equilibrium equations

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = -\rho g \cos \theta, \tag{5.1}
\]

\[
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = \rho g \sin \theta,
\]

where \(\rho\) is the density, \(g\) is the acceleration due to gravity and \(\sigma_{rr}, \sigma_{\theta\theta}\) and \(\sigma_{r\theta}\) denote the in-plane physical stress components. Following Spencer and Bradley [53] these components can be expressed in the standard form

\[
\sigma_{rr} = -p + q \cos 2\psi, \quad \sigma_{\theta\theta} = -p - q \cos 2\psi, \quad \sigma_{r\theta} = q \sin 2\psi. \tag{5.2}
\]

where \(p\) and \(q\) are defined by

\[
p = -\frac{1}{2}(\sigma_{rr} + \sigma_{\theta\theta}), \quad q = \left\{\frac{1}{4}(\sigma_{rr} - \sigma_{\theta\theta})^2 + \sigma_{r\theta}^2\right\}^{1/2}, \tag{5.3}
\]

while \(\psi\) is defined by

\[
\tan 2\psi = \frac{2\sigma_{r\theta}}{(\sigma_{rr} - \sigma_{\theta\theta})}, \tag{5.4}
\]

and physically \(\psi\) is the angle between the maximum principal stress axis and the radial direction, in the direction of increasing \(\theta\). For a cohesionless material, the Coulomb-Mohr yield condition takes the form

\[
q = \beta p, \tag{5.5}
\]

where \(\beta = \sin \phi\) and \(\phi\) is a material constant referred to as the angle of internal friction.
Following Jenike [35] and Spencer and Bradley [53], solutions of the form

\[ \psi = \psi(\theta), \quad q = -\rho Gr F(\theta). \]  

(5.6)

are examined and from the above equations the following governing equations may be deduced,

\[
\frac{dF}{d\theta} = \frac{F \sin 2\psi + \beta \sin(2\psi + \theta)}{\beta + \cos 2\psi},
\]

(5.7)

\[
\frac{d\psi}{d\theta} + 1 = \frac{F(\beta^{-1} - \beta) + \cos \theta + \beta \cos(2\nu + \theta)}{2F(\beta + \cos 2\nu)}.
\]

Now for a symmetrical stress distribution and for zero stress along the sand-pile slope, the following conditions arise

\[ \psi(0) = 0, \quad F(\alpha) = 0, \]  

(5.8)

where \( \alpha \) denotes the semi-vertex angle. Observe from the equilibrium equations (5.1) that if \( \sigma_{rr} \) and \( \sigma_{\theta\theta} \) are assumed to be even functions of \( \theta \), then \( \sigma_{r\theta} \) is necessarily an odd function or skew-symmetric and therefore \( \sigma_{r\theta} \) vanishes at the origin, and hence the condition (5.8). Thus, the governing equations (5.7) must be solved subject to (5.8). In general this can only be attempted numerically. Note that for the special case of \( \beta = 1 \) (\( \phi = \pi/2 \)) a possible alternative boundary condition to (5.8) is \( \psi(\alpha) = \pm \pi/2 \). In this special case, this is sufficient to guarantee zero stress on the surface, and this indicates possible non-uniqueness in the decomposition (5.2).

Now, expressions for the horizontal and vertical force distributions acting along a plane \( x = \) constant are evaluated. In cylindrical polar coordinates \((r, \theta, z)\) the normal vector to the plane \( x = \) constant has components

\[
n_r = \cos \theta, \quad n_\theta = -\sin \theta, \quad n_z = 0,
\]

(5.9)
and therefore the in-plane stress vectors $\sigma_r$ and $\sigma_\theta$ are determined from

$$\sigma_r = \sigma_{rr} n_r + \sigma_{r\theta} n_\theta = \cos \theta \sigma_{rr} - \sin \theta \sigma_{r\theta}.$$  \hfill (5.10)

$$\sigma_\theta = \sigma_{r\theta} n_r + \sigma_{\theta\theta} n_\theta = \cos \theta \sigma_{r\theta} - \sin \theta \sigma_{\theta\theta}.$$  \hfill (5.10)

From Figure 5.2 it is clear that the vertical and horizontal stress vectors $\sigma_x$ and $\sigma_y$ are obtained from

$$\sigma_x = \sigma_r \cos \theta - \sigma_\theta \sin \theta, \quad \sigma_y = \sigma_r \sin \theta + \sigma_\theta \cos \theta.$$  \hfill (5.11)

and therefore from (5.10) and (5.11), the expressions

$$\sigma_x = \cos^2 \theta \sigma_{rr} - 2 \sin \theta \cos \theta \sigma_{r\theta} + \sin^2 \theta \sigma_{\theta\theta}.$$  \hfill (5.12)

$$\sigma_y = \sin \theta \cos \theta (\sigma_{rr} - \sigma_{\theta\theta}) + (\cos^2 \theta - \sin^2 \theta) \sigma_{r\theta},$$  \hfill (5.12)

may be deduced. Hence (5.2), (5.5) and (5.12) gives simply

$$\sigma_x = -q \left\{ \beta^{-1} - \cos 2(\psi + \theta) \right\}, \quad \sigma_y = q \sin 2(\psi + \theta).$$  \hfill (5.13)

Along the plane $x = \text{constant} = h$ means that $r = h \sec \theta$ and therefore (5.6) and (5.13) gives

$$\sigma_x = \rho gh \frac{F(\theta)}{\cos \theta} \left\{ \frac{1}{\beta} - \cos 2[\psi(\theta) + \theta] \right\}, \quad \sigma_y = -\rho gh \frac{F(\theta)}{\cos \theta} \sin 2[\psi(\theta) + \theta],$$  \hfill (5.14)

as the required expressions for the vertical and horizontal force distributions. For the three models considered, these force distributions are shown graphically in the final Section.

Finally, note that (5.7) admits the special exact solution

$$\psi(\theta) = -\theta + \psi_0, \quad F(\theta) = -\frac{\beta}{(1-\beta^2)} \{ \cos \theta + \beta \cos (2\psi_0 - \theta) \}.$$  \hfill (5.15)
where $\psi_0$ is a constant, but in general, it is observed that (5.15) can only satisfy one of the boundary conditions of (5.8). In the following Section some additional boundary relations from (5.7) and (5.8) are determined.

5.3 Additional boundary relations

From (5.7) and (5.8) it is possible to derive additional relations involving boundary values. First at $\theta = 0$, from (5.7) and (5.8) gives

$$F'(0) = 0, \quad \psi'(0) = \frac{1}{2} \left\{ \frac{1}{\beta} - 3 + \frac{1}{F(0)} \right\}, \quad (5.16)$$

where the prime throughout the Section denotes differentiation with respect to $\theta$.

Now at $\theta = \alpha$, an assumption needs to be made as to whether $\psi'(\alpha)$ remains finite or not. From the numerous numerical solutions given by Bradley [11] for a different set of assumed boundary conditions, it is clear that this may or may not be the case. The essential point is that $\psi'(\alpha)$ is either finite or infinite. If it is finite then certain additional boundary relations can be deduced and for $\psi(\theta)$ negative, it appears not possible to construct a numerical solution which is entirely consistent with all of the derived boundary relations. Accordingly, this leads to examining the case when $\psi'(\alpha)$ is infinite. However, for completeness the additional boundary relations which apply when $\psi'(\alpha)$ is finite are noted first.

Assuming $\psi'(\alpha)$ is finite and also that $\beta \geq \cos \alpha$ then from (5.7)$_2$ and (5.8)$_2$ it is seen that this can only occur provided

$$\cos \alpha + \beta \cos[2\psi(\alpha) + \alpha] = 0, \quad (5.17)$$
which gives rise to the four distinct roots between \(-\pi \leq \psi \leq \pi\),

\[
\psi(\alpha) = \pm \frac{\pi}{2} - \frac{\alpha}{2} \pm \frac{1}{2} \cos^{-1} \left( \frac{\cos \alpha}{\beta} \right),
\]

(5.18)

where the \(\pm\) signs are independent. For example, the first negative root is

\[
\psi(\alpha) = -\frac{1}{2} \left\{ \pi + \alpha - \cos^{-1} \left( \frac{\cos \alpha}{\beta} \right) \right\}.
\]

(5.19)

and from (5.7)\(_1\) and (5.19) shows that

\[
F'(\alpha) = \frac{\beta}{\sin \alpha - (\beta^2 - \cos^2 \alpha)^{1/2}}.
\]

(5.20)

provided that \(\beta > \cos \alpha\) and note that \(F'(\alpha)\) becomes infinite for \(\beta = 1\). Further, since \(\psi'(\alpha)\) is assumed to be finite, then (5.17) is obtained on the basis that both the numerator and denominator of (5.7)\(_2\) must vanish at \(\theta = \alpha\). This means that \(\ell\)Hopital’s rule may be applied to (5.7)\(_2\), which gives

\[
\psi'(\alpha) + 1 = \frac{F'(\alpha)(\beta^{-1} - \beta) - \sin \alpha - \beta \sin[2\psi'(\alpha) + \alpha][1 + 2\psi'(\alpha)]}{2F'(\alpha)[\beta + \cos 2\psi'(\alpha)]},
\]

(5.21)

and on simplifying this equation using (5.19) and (5.20) and again assuming that \(\beta > \cos \alpha\) then the remarkably simple result

\[
\psi'(\alpha) = -1,
\]

(5.22)

is obtained. Similarly, for

\[
\psi(\alpha) = \frac{1}{2} \left\{ \pi - \alpha + \cos^{-1} \left( \frac{\cos \alpha}{\beta} \right) \right\}.
\]

(5.23)

\(F'(\alpha)\) and \(\psi'(\alpha)\) are given by (5.20) and (5.22) respectively. However, for

\[
\psi'(\alpha) = \frac{1}{2} \left\{ \pm \pi - \alpha - \cos^{-1} \left( \frac{\cos \alpha}{\beta} \right) \right\}.
\]

(5.24)
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$p'(\alpha)$ is given by

\[ F'(\alpha) = \frac{\beta}{\sin \alpha + (\beta^2 - \cos^2 \alpha)^{1/2}} \] \hspace{1cm} (5.25)

provided that $\beta > \cos \alpha$ and $\psi'(\alpha)$ is still given by (5.22).

Alternatively, upon eliminating $F$ from (5.7), the following second order differential equation for $\psi(\theta)$ may be deduced,

\[ (\beta + \cos 2\psi)[\cos \theta + \beta \cos(2\psi + \theta)]\psi'' = 2(\psi' + 1) \left\{ \sin 2\psi[\cos \theta + \beta \cos(2\psi + \theta)]\psi' \right. \\
-2\beta(\beta + \cos 2\psi) \sin(2\psi + \theta) \psi' - (3\beta^2 + 2\beta \cos 2\psi - 1) \sin(2\psi + \theta) \right\}, \]

and from this equation it is clear that one possibility when (5.17) holds is certainly $\psi'(\alpha) = -1$, which agrees with (5.22). However, another possibility arises from

\[ 2\beta(\beta + \cos 2\psi)\psi' + 3\beta^2 + 2\beta \cos 2\psi - 1 = 0, \] \hspace{1cm} (5.27)

and this equation gives rise to

\[ \psi'(\alpha) = -\frac{3}{2} \pm \frac{\sin \alpha}{2(\beta^2 - \cos^2 \alpha)^{1/2}}, \] \hspace{1cm} (5.28)

where the minus sign in (5.28) refers to $\psi(\alpha)$ given by (5.19) and (5.23), and the plus sign to (5.24) respectively. Note that in the derivation of (5.28), the assumption $\beta > \cos \alpha$ must still hold. The essential point is that on the assumption that $\psi'(\alpha)$ remains finite, certain additional constraints on $\psi(\theta)$ are obtained which the numerical solution of (5.26) indicates are not possible to satisfy simultaneously.

Accordingly, $\psi'(\alpha)$ is assumed to be infinite and that it occurs at the first zero of
\( \beta + \cos 2\psi(\alpha) = 0 \). Also note that (5.28) can be derived from (5.21) when \( F'(\alpha) \) is infinite. In addition, observe that the positive case of (5.28) again yields \( \psi'(\alpha) = -1 \) in the special case of \( \beta = 1 \).

As shown in Appendix C, equation (5.26) for the special case of \( \beta = 1 \) admits an exact parametric solution for \( \psi(\theta) \), namely

\[
\tan \theta = \frac{(2\pi)^{1/2} \tan \alpha}{2e^{-\lambda/2}\lambda^{-1/2} + J(\lambda)},
\]

\[
\tan \psi(\theta) = \frac{-J(\lambda)}{(2\pi)^{1/2} \tan \alpha \left( 1 + \frac{\lambda^{1/2}}{2} e^{\lambda/2} J(\lambda) \right) - \left( \frac{\pi \lambda}{2} \right)^{1/2} e^{\lambda/2} \tan \alpha},
\]  

\[
F(\theta) = -\frac{e^{\lambda/2} [2\pi \tan^2 \alpha + J(\lambda)^2]}{4\lambda^{1/2} \left( 2\pi \tan^2 \alpha + [2e^{-\lambda/2}\lambda^{-1/2} + J(\lambda)]^2 \right)},
\]

where \( 0 \leq \lambda \leq \infty \) and the integral \( J(\lambda) \) is defined by

\[
J(\lambda) = (2\pi)^{1/2} \text{erf} (\lambda/2)^{1/2}.
\]  

As described in the final Section of this Chapter, a bona fide solution of the formulated problem only exists for this special angle of internal friction. For all other values of \( \beta \) at least one of the stated conditions is not satisfied. As an aside, note that the condition \( \cos \alpha \leq \beta \) is equivalent to the statement that the angle of repose of the sand-pile is less than or equal to the angle of the internal friction. This in turn is the same condition as the requirement that for a particle at rest on the surface of the sand-pile, that the frictional force \( F \) and the normal reaction \( N \) are such that \( F \leq \mu N \) where \( \mu = \tan \phi \).
5.4 Alternative formulation

An alternative formulation of the problem can be obtained by assuming

\[ \sigma_{rr} = rL(\theta), \quad \sigma_{r\theta} = rM(\theta), \quad \sigma_{\theta\theta} = rN(\theta). \] (5.31)

for certain functions of \( \theta, L(\theta), M(\theta), \) and \( N(\theta). \) From this assumption, (5.1) gives

\[ M' + 2L - N = -\rho g \cos \theta, \quad N' + 3M = \rho g \sin \theta, \] (5.32)

while the yield condition (5.5) becomes

\[ \{ (L - N)^2 + 4M^2 \}^{1/2} = -\beta(L + N). \] (5.33)

or, in other words

\[ (1 - \beta^2)(L^2 + N^2) - 2LN(1 + \beta^2) + 4M^2 = 0. \] (5.34)

Now from (5.33) it is clear that \( L + N < 0 \) is required, while (5.34) is only sensible provided the product \( LN > 0. \) Accordingly, the functions \( L(\theta) \) and \( N(\theta) \) are both negative. The skew-symmetry of \( \sigma_{r\theta} \) and the vanishing of the stress on the surface of the sand-pile means that the differential equations (5.32) must be solved subject to

\[ M(0) = 0, \quad M(\alpha) = 0, \quad N(\alpha) = 0. \] (5.35)

Note the following observations. Firstly, the condition \( M(0) = 0 \) and the differential equation (4.2)\(_2\) are sufficient to ensure \( N'(0) = 0. \) Secondly, for \( \beta \neq 1, \) it is clear from (5.34) that these boundary conditions require in addition that \( L(\alpha) = 0. \)

Now on solving (5.34) as a quadratic gives

\[ L(\theta) = \frac{1}{(1 - \beta^2)} \left\{ (1 + \beta^2)N(\theta) - 2 \left[ \beta^2 N(\theta)^2 - (1 - \beta^2)M(\theta)^2 \right]^{1/2} \right\}. \] (5.36)
where the negative sign has been chosen so that \( L(\theta) \) remains finite in the limit \( \theta \) tending to unity. From (5.32) and (5.36), two nonlinear coupled equations may be deduced for the functions \( M(\theta) \) and \( N(\theta) \), namely

\[
M' + \left( \frac{1 + 3\beta^2}{1 - \beta^2} \right) N = -\rho g \cos \theta + \frac{4}{(1 - \beta^2)} \left[ \beta^2 N^2 - (1 - \beta^2) M^2 \right]^{1/2},
\]

\[
N' + 3M = \rho g \sin \theta,
\]

which are required to be solved subject to the "three" conditions (5.35). Note that for the solution of (5.37) to remain valid, \( M(\theta) \) and \( N(\theta) \) must satisfy \(|M| \leq -N \tan \phi\). where \( N(\theta) \) is known to be negative. As discussed in Section 5.6, the numerical results for \( \beta \neq 1 \) indicate that it is not possible to determine solutions of (5.37) which satisfy all three conditions (5.35), as might be anticipated.

As previously stated, a numerical solution of (5.37) subject to the "three" boundary conditions (5.35) has attempted to be determined, where the three boundary conditions are determined from the assumptions of skew-symmetry of \( \sigma_{r\theta} \) and zero stress on the surface of the sand-pile. In general, it is not possible to determine a solution for this over determined system. However, a numerical solution of (5.37) subject to the two boundary conditions (5.35)_2 and (5.35)_3 may be determined, but only for the special case of the angle of internal friction equal to the angle of repose, that is \( \beta = \cos \alpha \). This is because from (5.37)_1, \(|M| \leq -N \tan \phi\) is required but the numerical results for \( \beta > \cos \alpha \) indicate that this inequality is not satisfied over the entire range \( 0 \leq \theta \leq \alpha \).
5.5 Solution structure incorporating an inner dead region and an outer yield region

In Section 5.3 the entire sand-pile is assumed to be at yield, and (5.7) is solved subject to (5.8). However, a numerical solution is only able to be determined for the special case of $\beta = 1$, which coincides with the exact parametric solution (5.29). In Section 5.4 the entire sand-pile is also assumed to be at yield, but instead of using the standard $\psi(\theta)$ and $F(\theta)$ stress decomposition defined by (5.2), a slightly more general form of the stresses defined by (5.31) is used, and a numerical solution is only determined for the special case of $\beta = \cos \alpha$. However, in this Section it is assumed that the sand-pile has an inner dead region for $0 < \theta < \gamma$, and an outer yield region for $\gamma < \theta < \alpha$, where $\theta = \gamma$ is the boundary between the two regions as shown in Figure 5.2. In the following two Subsections the two regions within the sand-pile are examined.

5.5.1 Outer yield region

In the outer yield region of the sand-pile, the stresses are at equilibrium and also at yield. This means that the stresses satisfy the equilibrium equations (5.1) and the Coulomb-Mohr yield condition (5.5), and therefore the governing equations (5.7). Again, assume that there is zero stress on the slope of the sand-pile which provides the boundary condition for $F(\alpha)$ of (5.8)$_2$, and upon assuming that $\psi'(\alpha)$ is finite shows that $\psi(\alpha)$ is given by (5.19). This gives us two boundary conditions at $\theta = \alpha$ from which (5.7) can be numerically solved backwards towards $\theta = \gamma$. 
In the following Section the numerical solution of (5.7) subject to (5.8) and (5.19) is found, and it remarkably turns out to be the simple exact solution (5.15). where the constant $\psi_0$ is determined from (5.15) and (5.19). This means that the exact form of the solution is known in the outer yield region, and from (5.2), (5.5), (5.6) and (5.15) gives that the cylindrical polar stresses $\sigma_{rr}, \sigma_{\theta\theta}$, and $\sigma_{r\theta}$ are

$$
\sigma_{rr} = -\frac{\rho g [1 - \beta \cos 2(\psi_0 - \theta)] [\cos \theta + \beta \cos(2\psi_0 - \theta)]}{(1 - \beta^2)},
$$

$$
\sigma_{\theta\theta} = -\frac{\rho g [1 + \beta \cos 2(\psi_0 - \theta)] [\cos \theta + \beta \cos(2\psi_0 - \theta)]}{(1 - \beta^2)},
$$

$$
\sigma_{r\theta} = \frac{\rho g \beta \sin 2(\psi_0 - \theta) [\cos \theta + \beta \cos(2\psi_0 - \theta)]}{(1 - \beta^2)},
$$

which from (5.12) or Hunter [31] (page 102), gives

$$
\sigma_{xx} = -\frac{\rho g [1 - \beta \cos 2\psi_0] [(1 + \beta \cos 2\psi_0)x + \beta y \sin 2\psi_0]}{(1 - \beta^2)},
$$

$$
\sigma_{yy} = -\frac{\rho g [1 + \beta \cos 2\psi_0] [(1 + \beta \cos 2\psi_0)x + \beta y \sin 2\psi_0]}{(1 - \beta^2)},
$$

$$
\sigma_{xy} = -\frac{\rho g \beta \sin 2\psi_0 [(1 + \beta \cos 2\psi_0)x + \beta y \sin 2\psi_0]}{(1 - \beta^2)}.
$$

In the following Subsection the stresses are assumed not to be at yield in an inner dead region, but satisfy the equilibrium equations and remain continuous across the boundary at $\theta = \gamma$, where $\gamma$ is yet to be determined.
5.5.2 Inner dead region

In the inner dead region of the sand-pile, the stresses are assumed to satisfy the equilibrium equations, but not the equality of the Coulomb-Mohr yield condition (5.5). Instead, the stresses are not uniquely determined but must satisfy the strict inequality

\[ q < \beta p, \]  

(5.40)

where \( p \) and \( q \) are defined by (5.3). Now, an assumption about the form of the stresses needs to be made. As (5.39) shows that the stresses in Cartesian coordinates in the outer yield region are linear in both \( x \) and \( y \), then the stresses in the inner dead region are also assumed to be linear in both \( x \) and \( y \) so that

\[ \sigma_{xx} = -\rho g(Ax + By), \quad \sigma_{yy} = -\rho g(Cx + Dy), \quad \sigma_{xy} = \rho g(Ex + Gy). \]  

(5.41)

where \( A, B, C, D, E, \) and \( G \) denote constants. Note that the equilibrium equations (5.1) in Cartesian coordinates become

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = -\rho g, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \]  

(5.42)

so that the stresses defined by (5.41) satisfy

\[ E = D, \quad G = A - 1. \]  

(5.43)

Across the boundary \( \theta = \gamma \) between the two regions, both \( \sigma_{r\theta} \) and \( \sigma_{\theta\theta} \) are required.
to remain continuous. Therefore, (5.41), (5.43), and Hunter [31] gives

\[
\sigma_{rr} = -\rho g r \left\{ \left[ A \cos^2 \theta + C \sin^2 \theta - D \sin 2\theta \right] \cos \theta \\
+ \left[ B \cos^2 \theta + D \sin^2 \theta - (A - 1) \sin 2\theta \right] \sin \theta \right\}.
\]

\[
\sigma_{\theta \theta} = -\rho g r \left\{ \left[ A \sin^2 \theta + C \cos^2 \theta + D \sin 2\theta \right] \cos \theta \\
+ \left[ B \sin^2 \theta + D \cos^2 \theta + (A - 1) \sin 2\theta \right] \sin \theta \right\}. \tag{5.44}
\]

\[
\sigma_{r \theta} = \rho g r \left\{ \left[ (A - C) \sin \theta \cos \theta + D \cos 2\theta \right] \cos \theta \\
+ \left[ (B - D) \sin \theta \cos \theta + (A - 1) \cos 2\theta \right] \sin \theta \right\},
\]

and upon equating (5.38)\(_{2}\) with (5.44)\(_{2}\) and (5.38)\(_{3}\) with (5.44)\(_{1}\) at \(\theta = \gamma\), then expressions for \(C\) and \(D\) are determined in terms of \(A\) and \(B\), namely

\[
C = \frac{\left[ \cos \gamma + \beta \cos (2\psi_0 - \gamma) \right] \left[ 1 - 3 \sin^2 \gamma + \beta \cos 2\psi_0 - \beta \sin \gamma \sin (2\psi_0 - \gamma) \right]}{(1 - \beta^2) \cos^3 \gamma} \\
+ 3A \tan^2 \gamma + 2B \tan^3 \gamma - \tan^2 \gamma,
\]

\[
D = \frac{\left[ \cos \gamma + \beta \cos (2\psi_0 - \gamma) \right] \left[ \sin \gamma + \beta \sin (2\psi_0 - \gamma) \right]}{(1 - \beta^2) \cos^2 \gamma} \\
- 2A \tan \gamma - B \tan^2 \gamma + \tan \gamma. \tag{5.45}
\]

Now in order to determine \(A\) and \(B\), assume that \(\sigma_{xx}\) and \(\sigma_{xy}\) are continuous at \(\theta = \gamma\), and hence, comparing (5.39)\(_{1}\) with (5.41)\(_{1}\) gives

\[
B = \frac{\left[ 1 - \beta \cos 2\psi_0 \right] \left[ \cos \gamma + \beta \cos (2\psi_0 - \gamma) \right]}{(1 - \beta^2) \sin \gamma} - A \cot \gamma. \tag{5.46}
\]

and then from comparing (5.39)\(_{3}\) with (5.41)\(_{3}\) at \(\theta = \gamma\), noting (5.43) and (5.45) gives

\[
A = \frac{\left[ 1 - \beta \cos 2\psi_0 \right] \left[ \cos \gamma + \beta \cos (2\psi_0 - \gamma) \right]}{(1 - \beta^2) \cos \gamma} - B \tan \gamma. \tag{5.47}
\]
and note that the two equations (5.46) and (5.47) coincide. In order to ensure that \( \sigma_{r\theta} \) is an odd function or skew-symmetric choose \( B \) such that \( \sigma_{r\theta} = 0 \) at \( \theta = 0 \), which from (5.45) and (5.47) gives

\[
A = \frac{[1 + \beta \cos(2\psi_0)]}{(1 - \beta^2)} \left[ 1 + \frac{\beta \sin(2\psi_0 - \gamma)}{\sin \gamma} \right],
\]

\[
B = \frac{[\sin \gamma - \beta \sin(2\psi_0 + \gamma)] \left[ \cos \gamma + \beta \cos(2\psi_0 - \gamma) \right]}{(1 - \beta^2) \sin^2 \gamma} - \cot \gamma.
\]

\[
C = \frac{[1 + \beta \cos(2\psi_0)]}{(1 - \beta^2)} \left[ 1 + \frac{\beta \cos(2\psi_0 - \gamma)}{\cos \gamma} \right],
\]

and \( D = 0 \). Note that assuming the stresses \( \sigma_{\theta\theta}, \sigma_{r\theta}, \sigma_{xx}, \) and \( \sigma_{xy} \) are continuous across the boundary at \( \theta = \gamma \), ensures that both \( \sigma_{rr} \) and \( \sigma_{yy} \) are also continuous at \( \theta = \gamma \). However, Figure 5.8 shows that while the derivatives of \( \sigma_{\theta\theta} \) and \( \sigma_{r\theta} \) are continuous across the boundary, the derivative of \( \sigma_{rr} \) is discontinuous. Note that if \( B \) is alternatively determined so that the derivative of \( \sigma_{rr} \) is continuous across the boundary at \( \theta = \gamma \), then remarkably the solution given in Section 5.4 is obtained. This means that if all the stresses and their derivatives are assumed to be continuous across the boundary \( \theta = \gamma \) then the full equality of the yield condition (5.5) holds in the dead region. In other words, there is no dead region as the inner region is also at yield. Thus, the derivative of \( \sigma_{rr} \) must be assumed to remain discontinuous across the boundary at \( \theta = \gamma \).

Now, consider how to determine a value of \( \gamma \), which is not unique but instead is such that the stresses satisfy the strict inequality (5.40) throughout the entire inner dead region. Numerical results indicate that if the strict inequality (5.40) is satisfied at \( \theta = 0 \), then it is always satisfied throughout the entire inner dead region.
Chapter 5: Force distributions at the base of 2D sand-piles

3.2

Figure 5.3: Variation of the LHS (—) of the inequality (5.40) versus the RHS (⋯) of the inequality (5.40) for the two values of $\gamma = 0.37$ and $\gamma = 0.39$ showing that the value $\gamma = 0.37$ is too small whereas the value 0.39 is sufficient.

This is shown in Figure 5.3, where for $\gamma = 0.37$ and $\gamma = 0.39$ the LHS (—) of the inequality (5.40) versus the RHS (⋯) of the inequality (5.40) has been plotted. The value $\gamma = 0.37$ is such that the inequality (5.40) is not satisfied at $\theta = 0$ and elsewhere, whereas for $\gamma = 0.39$, the inequality (5.40) is satisfied at $\theta = 0$ and also throughout the entire inner dead region which means that the value $\gamma = 0.37$ is too small but $\gamma = 0.39$ is sufficiently large to ensure the inequality is satisfied throughout the entire dead region. Therefore, upon considering the inequality (5.40) at $\theta = 0$, where the Cartesian stresses (5.41) have been used with constants $A, B, C, D, E$. 
and $G$ defined by (5.43) and (5.48), then the inequality

$$\sin^2(2\psi_0 - 2\gamma) < [\sin 2\gamma + \beta \sin 2\psi_0]^2,$$

(5.49)

may be deduced and upon assuming that $\sin(2\psi_0 - 2\gamma) < 0$, the inequality below is obtained,

$$\gamma > \frac{1}{2} \left[ \psi_0 + \pi - \cos^{-1}(\beta \cos \psi_0) \right].$$

(5.50)

However, of course (5.50) does not give a unique value of $\gamma$, but only determines a range for $\gamma$ which ensures that the inequality (5.40) is satisfied. Here the smallest value of $\gamma$ which satisfies (5.50) is adopted. In the following Section, the stress distributions for the three proposed models for a two-dimensional sand-pile are displayed graphically.

### 5.6 Numerical results

The experimental results shown in Figure 5.1 given by Smid and Novosad [51], were obtained at various stages during the pouring of a three-dimensional heap. When the heap was at height $h$, the horizontal and vertical stresses were measured at the base of the heap. The material used was sand for which the angle of repose was $32.6^\circ$ and the average bulk density was determined as $\rho = 1567 \text{ kg/m}^3$.

In the previous Sections, three possible models have been proposed for determining the stress distribution throughout a two-dimensional sand-pile. In order to numerically solve the two-dimensional sand-pile equations given by (5.7) subject to the two-point boundary conditions (5.8), an iterative scheme needs to be used to determine successive numerical solutions that converge to the solution that satisfies
both (5.7) and (5.8). On the assumptions that \( \psi'(\alpha) \) is finite and both \( F \) and \( \nu \) are negative, then a numerical solution satisfying all three conditions (5.8), (5.19), and (5.22) is unable to be determined. This is because \( \psi(\alpha) \) is beyond the first singularity of the system (5.7), or in other words, the first zero of \( \beta + \cos 2\nu'(\alpha) = 0 \). Accordingly, assume \( \psi'(\alpha) \) is infinite and that this occurs precisely at the first zero of \( \beta + \cos 2\psi(\alpha) = 0 \). Following [13], both a Shooting Method, which involves a Runge-Kutta scheme, and a Nonlinear Finite-Difference Method have been used to determine a solution to (5.26) subject to (5.8), where as previously stated (5.26) is deduced from eliminating \( F \) from (5.7). Note that both the Shooting Method and Finite-Difference Method give identical results and for the case \( \beta = 1 \) coincide with the exact solution given by (5.29). Figure 5.4(a) shows the numerically determined solution for \( \psi(\theta) \), for three values of \( \beta \), namely \( \beta = \cos \alpha, \beta = 0.75, \) and \( \beta = 1 \), where \( \alpha = (287/900)\pi \) which gives an angle of repose of 32.6°. and \( \psi(\alpha) \) is given by the first negative root of \( \beta + \cos 2\psi(\alpha) = 0 \). \( F(\theta) \) is then determined from (5.7) which is solved as a first order differential equation subject to (5.8) and is shown in Figure 5.4(b). Notice from Figure 5.5, which shows the variation in \( \sigma_{x\theta} \) and \( \sigma_{y\theta} \), that the zero stress conditions on the surface are only satisfied for \( \beta = 1 \). In this special case the vertical and horizontal force distributions, as determined from (5.14), are shown in Figure 5.6. It is clear from Figure 5.6 that for \( \beta = 1 \) the vertical and horizontal forces do not predict the dip exhibited by the experimental results presented in Figure 5.1.

For the alternative formulation proposed in Section 5.4, a numerical solution of (5.37) subject to the three boundary conditions (5.35) is unable to be determined.
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Figure 5.4: Variation of $\psi(\theta)$ and $F(\theta)$ for two-dimensional sand piles using an angle of repose of $32.6^\circ$ ($\alpha = 1.0018$) with three angles of internal friction, namely $\phi = \cos \alpha$, $\phi = 0.75$, and $\phi = \pi/2$. ((a) $\psi(\theta)$ and (b) $F(\theta)$).

Figure 5.5: Variation of $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ for two-dimensional sand piles using an angle of repose of $32.6^\circ$ ($\alpha = 1.0018$) with three angles of internal friction, namely $\phi = \cos \alpha$, $\phi = 0.75$, and $\phi = \pi/2$. ((a) $\sigma_{r\theta}$ and (b) $\sigma_{\theta\theta}$).
which in general is to be expected. However, if only the two boundary conditions (5.35)\textsubscript{2} and (5.35)\textsubscript{3} are considered, which ensures zero stress on the boundary of the sand-pile, then a numerical solution can be determine only for the physically relevant case of $\beta = \cos \alpha$, and the results are shown in Figure 5.7. The graphs shown in Figure 5.7 clearly do not posses the necessary symmetry since $\sigma_{r\theta}$ is evidently non-zero at the origin.

Finally, as described in Section 5.5 a two-dimensional sand-pile with an inner dead region and an outer yield region is examined, where $\theta = \gamma$ is the boundary between the two regions. Note that remarkably the numerical solution for the outer yield region always gives identically the special solution of (5.15) where $\gamma$ is determined to be the smallest value that satisfies the inequality (5.50). As described in Section 5.5, the stresses in the outer and inner regions are able to be analytically
Figure 5.7: Variation of $\sigma_{rr}$, $\sigma_{r\theta}$, and $\sigma_{\theta\theta}$ as defined by (5.31), for two-dimensional sand-piles for $\beta = \cos \alpha$. ((a) $\sigma_{rr}$, (b) $\sigma_{r\theta}$, and (c) $\sigma_{\theta\theta}$).
determined. Figure 5.8 shows the variation of $\sigma_{rr}$, $\sigma_{r\theta}$, and $\sigma_{\theta\theta}$ with respect to $\theta$ for $\beta = 2/3$, $\gamma = 0.382$, and $h = 10$. Note from Figure 5.8 that both the derivatives of $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ with respect to $\theta$ remain continuous across the boundary between the inner dead region and the outer yield region at $\theta = \gamma$, whereas the derivative of $\sigma_{rr}$ is discontinuous. This means that the derivatives of horizontal and vertical forces will also be discontinuous as shown in Figure 5.9. However, Figure 5.9 shows that the horizontal and vertical forces do indeed possess the qualitative features of the experimental results presented in Figure 5.1. In fact, the location of the maximum
Figure 5.9: Variation of the horizontal and vertical force distributions for a two-dimensional sand-pile with respect to $\theta$, for stresses from Figure 5.8.

vertical stress may be identified to be the boundary between the inner dead region and the outer yield region at $\theta = \gamma$.

5.7 Conclusions

Three possible models have been proposed to solve the problem of determining the force distribution at the base of a two-dimensional sand-pile. Initially, the Jenike solutions for radial flow in a converging wedge were made use of, but with gravity acting in the opposite direction. For the special case of an angle of internal friction equal to ninety degrees an exact analytical solution has been determined which coincides with a full numerical solution of the problem. However, for more realistic angles of internal friction, numerical results indicate that it is not possible to determine a solution which satisfies all the required conditions. While many materials
such as Coal and Silica do exhibit large angles of internal friction such as 80 degrees and 78.34 degrees respectively, the exact analytical solution for an angle of internal friction equal to ninety degrees does not exhibit the experimentally determined profile obtained by Smid and Novosad [51]. For the second proposed model, a stress distribution which is slightly more general than that proposed by Jenike for radial flow in a converging wedge is assumed. However, a numerical solution only for the special case of an angle of internal friction equal to the angle of repose is obtained but this solution exhibits a non-zero stress $\sigma_{r\theta}$ beneath the sand-pile vertex and therefore does not possess the correct symmetry. Finally, a sand-pile that has an inner dead region and an outer yield region have been assumed. By numerically solving the outer yield region shows that the solution follows the special exact solution of (5.15), which means that the stress profile throughout the entire sand-pile may be determined analytically, with stresses which are linear in both $x$ and $y$. This model does exhibit the experimentally determined M-shaped profile as obtained by Smid and Novosad [51], where the location of the maximum vertical pressure is at the boundary between the two regions at $\theta = \gamma$, where $\gamma$ is determined as the smallest value satisfying the strict inequality (5.50).
Chapter 6

Force distributions at the base of three-dimensional sand-piles

6.1 Introduction

Throughout the world, granular materials are commonly used in many industries. These industries, such as the chemical industry that use fine powders, and the mining industry that deals with large irregularly shaped ores, frequently store granulated material in heaps. Knowledge of the stress distribution throughout the heap and particularly at the base, is therefore of importance, as this will enable us to be able to predict the amount of settlement, caking, comminution and overall deterioration of the stored material. Knowing the location of the maximum vertical pressure ensures that stock-piles can be arranged to ensure the best endurance of the stored material. Intuitively, it might be expected that the maximum vertical pressure lies directly beneath the vertex of the sand-pile. However, Smid and Novosad [51]
showed experimentally that this is not the case, but rather that it occurs at some intermediate point, so that there is a ring of maximum vertical pressure. This result has attracted much attention, both in the popular scientific literature (see for example Watson [54, 55]) and has produced numerous discrete and computational models which attempt to explain this curious phenomenon (see Savage [48] for an extensive and critical review of the literature). In this Chapter, three-dimensional sand-piles are examined using the proper continuum mechanical theory of granular materials and a model is proposed which is not entirely at yield, but contains an inner dead region. This model does give rise to the essential profile of the M-shaped curves from the experimental work of Smid and Novosad [51]. Such a model is by no means unique and similar models incorporating inner elastic regions have been proposed by Cantelaube and Goddard[15], Cantelaube, Didwania and Goddard [14] and Didwania, Cantelaube and Goddard [20]. The work in this Chapter differs in that the inner dead region is not prescribed to be elastic, but merely is a possible equilibrium state. The solutions first used by Jenike [33, 35, 36] and Johanson [39] are utilized which describe gravity flows of granular materials in converging cones. Note that these solutions have been re-examined more recently by Bradley [11] and Spencer and Bradley [53] and for convenience the notation of the latter authors is adopted, with the exception that gravity in this problem is acting in the opposite direction to their problem. Also note that these authors adopt the unusual convention of the x-axis being vertical.

Consider an ‘infinite’ three-dimensional sand-pile and set up the coordinate axis at the vertex of the sand-pile as indicated in Figure 6.1, with gravity shown acting
in the vertically upwards direction. The basic idea is to attempt to determine the stress distribution in a sand-pile of infinite height and then evaluate the horizontal and vertical forces acting along a horizontal plane at a finite height. Clearly the problem for a finite sand-pile, resting on a rigid horizontal plane is different to that examined here, but nevertheless the two force distributions would be expected to be similar. Here a sand-pile is proposed that is made up of two regions, an inner dead region and an outer yield region. For the two-dimensional sand-piles considered in the previous Chapter, a full numerical solution for the outer region shows that the solution actually follows a known simple exact solution of the governing equations for which the stresses are linear in both $x$ and $y$. From this a possible stress distribution is deduced in the inner dead region that is also linear in both $x$ and $y$ and which satisfies the equilibrium equations and the strict inequality of the Coulomb-Mohr yield condition which means that the material in the inner region is not at yield
and also that the stresses are not uniquely determined. However, the resulting force distribution that are found do have the essential profile of the M-shaped curves of the experimental curves of Smid and Novosad [51]. For three-dimensional sand-piles a similar situation applies except that in the outer plastic region the numerical solution determined does not correspond to any well known exact solution of the governing equations. Instead, in the inner dead region the stresses are assumed to involve quintic polynomial expressions in $\sin \Theta$ and $\cos \Theta$ which are subsequently shown to determine a stress distribution that has the essential M-shaped profile.

In the following Section the basic equations of the continuum theory of granular materials are briefly stated and expressions for the horizontal and vertical force resultants are given. As previously mentioned the notation of Spencer and Bradley [53] are followed, noting again that in this problem gravity acts in the opposite direction and the $x$-axis denotes the vertical direction. In the final Section of this Chapter, the numerically determined stress solutions for the model are shown graphically and are compared with the experimental force distribution curves.

### 6.2 Basic equations of continuum theory

For quasi-static axially symmetric flow, the stress components in a spherical polar coordinate system $(R, \Theta, \Phi)$ satisfy the equilibrium equations

\[
\frac{\partial \sigma_{RR}}{\partial R} + \frac{1}{R} \frac{\partial \sigma_{R\Theta}}{\partial \Theta} + \frac{1}{R} (2\sigma_{RR} - \sigma_{\Theta\Theta} - \sigma_{\Phi\Phi} + \sigma_{R\Theta} \cot \Theta) = -\rho g \cos \Theta,
\]

\[
\frac{\partial \sigma_{R\Theta}}{\partial R} + \frac{1}{R} \frac{\partial \sigma_{\Theta\Theta}}{\partial \Theta} + \frac{1}{R} (\sigma_{\Theta\Theta} - \sigma_{\Phi\Phi}) \cot \Theta + \frac{3}{R} \sigma_{R\Theta} = \rho g \sin \Theta,
\]
where \( \rho \) is the density, \( g \) is the acceleration due to gravity and \( \sigma_{RR}, \sigma_{\Theta \Theta}, \sigma_{\Phi \Phi} \) and \( \sigma_{R \Theta} \) denote the physical components of stress. Assuming the 'Haar-von Karman' region for which two principal stresses coincide, then

\[
\sigma_{RR} = -p + q \cos 2\Psi, \quad \sigma_{\Theta \Theta} = -p - q \cos 2\Psi.
\]

\[
(6.2)
\]

\[
\sigma_{R \Theta} = q \sin 2\Psi, \quad \sigma_{\Phi \Phi} = -p - q,
\]

may be deduced, where

\[
p = -\frac{1}{2} (\sigma_{RR} + \sigma_{\Theta \Theta}), \quad q = \left\{ \frac{1}{4} (\sigma_{RR} - \sigma_{\Theta \Theta})^2 + \sigma_{R \Theta}^2 \right\}^{1/2},
\]

\[
(6.3)
\]

while \( \Psi \) is defined by

\[
\tan 2\Psi = \frac{2\sigma_{R \Theta}}{(\sigma_{RR} - \sigma_{\Theta \Theta})},
\]

\[
(6.4)
\]

and physically \( \Psi \) is the angle between the direction of the maximum principal stress and the radial direction, measured in the direction of \( \Theta \) increasing.

Following Jenike [35] and Spencer and Bradley [53] solutions of the form

\[
\Psi = \Psi(\Theta), \quad q = -pgRG(\Theta),
\]

\[
(6.5)
\]

are examined and from the above equations the following governing equations may be deduced,

\[
\frac{dG}{d\Theta} = \frac{2G \sin \Psi \{ \cos \Psi - \beta \cosec \Theta \sin (\Theta + \Psi) \} + \beta \sin (\Theta + 2\Psi)}{\beta + \cos 2\Psi},
\]

\[
(6.6)
\]

\[
\frac{d\Psi}{d\Theta} + 1 = \frac{G\{ \beta^{-1} - 2\beta - 1 - (1 + \beta) \cosec \Theta \sin (\Theta + 2\Psi) \} + \cos \Theta + \beta \cos (\Theta + 2\Psi)}{2G(\beta + \cos 2\Psi)}.
\]

Now for a symmetrical stress distribution and for zero stress along the sand-pile slope, gives rise to the following conditions

\[
\Psi(0) = 0, \quad G(\alpha) = 0,
\]

\[
(6.7)
\]
where $\alpha$ denotes the semi-vertex angle. Observe from the equilibrium equations (6.1) that if $\sigma_{RR}, \sigma_{\Theta},$ and $\sigma_{\Phi\Phi}$ are assumed to be even functions of $\Theta$, then $\sigma_{R\Theta}$ is necessarily an odd function or skew-symmetric and therefore $\sigma_{R\Theta}$ vanishes at the origin, and hence the condition (6.7). Thus (6.6) must be solved subject to subject to (6.7). In general this can only be attempted numerically.

Similarly from the previous Section, expressions for the vertical and horizontal force distributions along the plane $Z = \text{constant} = h$ where $R = h \sec \Theta$ are determined to be

$$
\sigma_Z = \rho gh \frac{G'(\Theta)}{\cos \Theta} \left\{ \frac{1}{\beta} - \cos 2[\psi(\Theta) + \Theta] \right\}, \quad \sigma_r = -\rho gh \frac{G'(\Theta)}{\cos \Theta} \sin 2[\psi(\Theta) + \Theta].
$$

(6.8)

where in this context $r$ and $Z$ denote the cylindrical polar radius and height as indicated in Figure 6.1.

### 6.3 Solution structure incorporating an inner dead region and an outer yield region

In this Section, a three-dimensional sand-pile is assume which has as inner dead region for $0 \leq \Theta \leq \gamma$, and an outer yield region for $\gamma \leq \Theta \leq \alpha$, where $\Theta = \gamma$ is the boundary between the two regions as shown in Figure 6.1. In the following two Subsections the details for the two regions are provided.

#### 6.3.1 Outer yield region

In the outer yield region of the three-dimensional sand-pile, the stresses are at equilibrium and also at yield. This means that the stresses satisfy the equilibrium
equations (6.1) and the Coulomb-Mohr yield condition

\[ q = \beta p, \]  

(6.9)

where \( p \) and \( q \) are defined by (6.3), and therefore the governing equations (6.6).

Again, zero stress on the slope of the sand-pile is assumed which provides the boundary condition for \( G(\alpha) \) of (6.7)_2, and upon assuming that \( \Psi'(\alpha) \) is finite then \( \Psi(\alpha) \) is seen to be given by

\[ \Psi(\alpha) = -\frac{1}{2} \left\{ \pi + \alpha - \cos^{-1} \left( \frac{\cos \alpha}{\beta} \right) \right\}, \]  

(6.10)

which gives us two boundary conditions at \( \Theta = \alpha \) from which a numerical solution of (6.6) can be determined and solved backwards towards \( \Theta = \gamma \).

In the following Section the numerical solution of (6.6) subject to (6.7)_2 is determined and it is seen that (6.10) does not remain continuous throughout the entire region \( 0 \leq \Theta \leq \alpha \). Instead, the solution encounters a singularity, in the sense that the denominators of (6.6) become zero. This means there is a value of \( \Theta \) which \( \Psi \) satisfies the equation

\[ \cos 2\Psi + \beta = 0, \]  

(6.11)

and this value is denoted by \( \Theta = \gamma \). In other words, the location of the boundary between the inner dead region and the outer yield region is where the numerical solution first encounters the singularity defined by (6.11). When (6.6) is solved backwards from \( \Theta = \alpha \), and this defines the angle \( \gamma \).

In order to obtain a continuous solution throughout the entire sand-pile, the value of the outer yield stresses at the boundary \( \Theta = \gamma \) needs to be known. The
value \( \gamma \) is known and is determined numerically, and \( \Psi(\gamma) \) is defined by (6.11). so upon assuming that \( \Psi(\gamma) \) is the first negative root of (6.11), then

\[
\Psi(\gamma) = -\frac{\phi}{2} - \frac{\pi}{4},
\]

(6.12)

recalling that \( \beta = \sin \phi \). Now, assuming that \( G'(\gamma) \) and \( \Psi'(\gamma) \) are finite, then (6.6) at \( \Theta = \gamma \), gives

\[
G(\gamma) \sin 2\Psi(\gamma) - 2\sin \phi G(\gamma) \csc \gamma \sin \Psi(\gamma) \sin[\gamma + \Psi(\gamma)] \\
+ \sin \phi \sin[\gamma + 2\Psi(\gamma)] = 0,
\]

(6.13)

\[
(1 + \sin \phi)G(\gamma) \{2 - \csc \phi - \csc \gamma \sin[\gamma + 2\Psi(\gamma)]\} \\
+ \cos \gamma + \sin \phi \cos[\gamma + 2\Psi(\gamma)] = 0,
\]

and from (6.12), both (6.13)_1 and (6.13)_2 are found to give

\[
G(\gamma) = \frac{\sin \phi \sin \gamma \cos(\gamma - \phi)}{\sin \phi \sin(\gamma - \phi) - \sin(\gamma + \phi)}.
\]

(6.14)

Accordingly, \( \gamma \) is analytically indeterminant from this approach. Now, from (6.9), (6.2), (6.5), (6.12), and (6.14) expressions for the stresses \( \sigma_{RR}, \sigma_{R\Theta}, \sigma_{\Theta\Theta}, \) and \( \sigma_{\phi\phi} \) at \( \Theta = \gamma \) are determined to be

\[
\sigma_{RR}(R, \gamma) = \rho g R \frac{(1 + \sin^2 \phi) \sin \gamma \cos(\gamma - \phi)}{\sin \phi \sin(\gamma - \phi) - \sin(\gamma + \phi)},
\]

\[
\sigma_{\Theta\Theta}(R, \gamma) = \rho g R \frac{\cos^2 \phi \sin \gamma \cos(\gamma - \phi)}{\sin \phi \sin(\gamma - \phi) - \sin(\gamma + \phi)},
\]

(6.15)

\[
\sigma_{R\Theta}(R, \gamma) = \rho g R \frac{\sin \phi \cos \phi \sin \gamma \cos(\gamma - \phi)}{\sin \phi \sin(\gamma - \phi) - \sin(\gamma + \phi)},
\]

\[
\sigma_{\phi\phi}(R, \gamma) = \rho g R \frac{(1 + \sin \phi) \sin \gamma \cos(\gamma - \phi)}{\sin \phi \sin(\gamma - \phi) - \sin(\gamma + \phi)}.
\]
and (6.15) enables us to ensure that the solution is continuous across the boundary at $\Theta = \gamma$. Also upon using (6.1) and (6.15) it is found that the derivatives of $\sigma_{R\Theta}$ and $\sigma_{\Theta\Theta}$ with respect to $\Theta$, at $\Theta = \gamma$, will be continuous across the boundary between the inner dead region and the outer yield region. To ensure that all stresses have continuous derivatives across the boundary, the value of the derivatives of the stresses in the outer yield region need to be determined at the boundary. First, the finite values of $\Psi'(\gamma)$ and $G'(\gamma)$ need to be determined in order to determine the derivatives of the stresses defined by (6.2). At $\Theta = \gamma$, $\Psi(\gamma)$ has been previously determined and is given by (6.12) and $G(\gamma)$ is given by (6.14), which means that the numerators and the denominators of both equations (6.6) vanish and therefore from $\ell'$Hopital's rule, (6.6) gives

$$G' [2\Psi' \cos \phi + \csc \gamma \sin (\gamma + \phi) - \sin \phi \csc \gamma \sin (\gamma - \phi)]$$

$$= \sin \phi \sin (\gamma - \phi) (1 + 2\Psi') - 2G' \sin \phi \csc \gamma \sin (\gamma - \cos (\gamma - \phi))$$

$$+ (1 + \sin \phi)G \sin \phi \csc^2 \gamma,$$

$$G' (1 + \sin \phi) [2 - \csc \phi - \csc \gamma \cos (\gamma - \phi)]$$

$$= -4G' \cos \phi \cos \phi - 2\Psi' [2G \cos \phi + (1 + \sin \phi)G \csc \gamma \sin (\gamma - \phi)$$

$$- \sin \phi \cos (\gamma - \phi)] - (1 + \sin \phi)G \cos \phi \csc^2 \gamma - \cos \phi \sin (\gamma - \phi),$$

where $G$ is defined by (6.14). Upon eliminating $G'$ from (6.16), then $\Psi'$ must satisfy the cubic equation

$$-8G \cos^2 \phi \Psi'^3 - 4 \cos \phi \Psi'^2 [4G \cos \phi - \sin \phi \cos (\gamma - \phi)]$$

$$+ 2\Psi' [(1 + \sin \phi)G \csc \gamma \{2 \cos^2 \gamma - 4 \cos \phi \sin \gamma \cos \gamma - 5 \cos^2 \phi$$

$$+ (1 - \sin \phi)(2 \cos^2 \gamma + 3 \sin \phi + 5 \sin \gamma \cos (\gamma - \phi))]$$

$$+ (1 + \sin \phi) \cos \gamma - (1 - \sin \phi) \cos \phi \cos (\gamma - \phi)] = 0,$$
and obviously, one solution is $\Psi'(\gamma) = 0$. The non-trivial solutions are given as the roots of a quadratic, thus

$$
\Psi'(\gamma) = \frac{-1}{4G \cos \phi} \left\{ 4G \cos \phi - \sin \phi \cos(\gamma - \phi) 
\pm \left[ 2(1 + \sin \phi)G^2 \csc^2 \gamma \left[ 6 \sin \phi \sin^2 \gamma + 2 \cos \phi \sin \gamma \cos \gamma \\
-2(1 - \sin \phi + 3 \sin^2 \phi) - 10 \sin \phi \cos \gamma \sin(\gamma - \phi) \right] \\
-4(1 + \sin \phi)G \sin \phi \sin(\gamma - \phi) + \sin^2 \phi \cos^2(\gamma - \phi) \right]^{1/2} \right\},
$$

(6.18)

where the numerical results presented in the following Section indicate that $\Psi'(\gamma)$ takes the $+$ sign in (6.18). Now, upon substituting (6.18) into (6.16) and using MAPLE, then $G'(\gamma)$ is found to be given by

$$
G'(\gamma) = - \left\{ 2(1 + \sin \phi)G \csc^2 \gamma \left[ \cos \phi \sin \gamma \cos \gamma - 2 \sin(\gamma - \phi) \sin(\gamma + \phi) \\
+ 4 \sin \phi(1 - \sin \phi) - 3 \sin \phi \cos \gamma \cos(\gamma + \phi) \right] \\
+(1 - 2 \sin \phi - 2 \sin^2 \phi) \sin(\gamma - \phi) + \sin(\gamma + \phi) \\
+(1 + \sin \phi) \sin \phi \csc \gamma \sin(\gamma - \phi) \cos(\gamma - \phi) \right\} \\
/ \left\{ 2 \cos \phi(1 + \sin \phi)[2 - \csc \phi - \csc \gamma \cos(\gamma - \phi)] \right\} \right\}
$$

(6.19)

$$
\pm \frac{2 \cos \phi - (1 + \sin \phi) \csc \gamma \sin(\gamma - \phi)}{2 \cos \phi(1 + \sin \phi)[2 - \csc \phi - \csc \gamma \cos(\gamma - \phi)]} \\
\times \left\{ \sin^2 \phi \cos^2(\gamma - \phi) + 2(1 + \sin \phi)G^2 \csc^2 \gamma \left[ 6 \sin \phi \sin^2 \gamma \\
+ 2 \cos \phi \sin \gamma \cos \gamma - 2(1 - \sin \phi + 3 \sin^2 \phi) \\
-10 \sin \phi \cos \gamma \sin(\gamma - \phi) \right] - 4(1 + \sin \phi)G \sin \phi \sin(\gamma - \phi) \right\}^{1/2},
$$

where the $\pm$ sign in (6.19) corresponds to the $\pm$ sign in (6.18), respectively. Now, from (6.9), (6.2), and (6.3) the derivatives of the stresses in the outer yield region
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at the boundary $\Theta = \gamma$ are given by

\[
\begin{align*}
\frac{\partial \sigma_{RR}}{\partial \Theta} &= \rho g R \left\{ G'(\gamma) [\cosec \phi + \sin \phi] - 2 \cos \phi G(\gamma) \Psi'(\gamma) \right\}, \\
\frac{\partial \sigma_{\Theta \Theta}}{\partial \Theta} &= \rho g R \left\{ G'(\gamma) [\cosec \phi - \sin \phi] + 2 \cos \phi G(\gamma) \Psi'(\gamma) \right\}, \\
\frac{\partial \sigma_{\Theta \Phi}}{\partial \Theta} &= \rho g R \left\{ \cos \phi G'(\gamma) + 2 \sin \phi G(\gamma) \Psi'(\gamma) \right\}, \\
\frac{\partial \sigma_{\Phi \Phi}}{\partial \Theta} &= \rho g R G'(\gamma) (1 + \cosec \phi),
\end{align*}
\]

where $G(\gamma)$ is given by (6.14), $\Psi'(\gamma)$ is given by (6.18), and $G'(\gamma)$ is given by (6.19).

In the following Subsection the stresses are assumed not to be at yield in an inner dead region, but satisfy only the equilibrium equations and remain continuous across the boundary at $\Theta = \gamma$.

6.3.2 Inner dead region

In the inner dead region of the sand-pile, the stresses are assumed to satisfy only the equilibrium equations (6.1), but not the equality of the Coulomb-Mohr yield condition (6.9). Accordingly, the stresses are not uniquely determined but must satisfy the strict inequality

\[
q < \beta p,
\]

where $p$ and $q$ are defined by (6.3). Now, an assumption needs to be made about the form of the stresses. Here, the non-zero spherical polar stresses are assumed to
be quintic expressions involving $\sin \Theta$ and $\cos \Theta$ of the form

$$
\begin{align*}
\sigma_{RR} &= pgR \left\{ c_1 \sin^5 \Theta + c_2 \sin^4 \Theta \cos \Theta + c_3 \sin^3 \Theta \cos^2 \Theta \\
&\quad + c_4 \sin^2 \Theta \cos^3 \Theta + c_5 \sin \Theta \cos^4 \Theta + c_6 \cos^5 \Theta \right\}, \\
\sigma_{\Theta \Theta} &= pgR \left\{ c_7 \sin^5 \Theta + c_8 \sin^4 \Theta \cos \Theta + c_9 \sin^3 \Theta \cos^2 \Theta \\
&\quad + c_{10} \sin^2 \Theta \cos^3 \Theta + c_{11} \sin \Theta \cos^4 \Theta + c_{12} \cos^5 \Theta \right\}, \\
\sigma_{R\Phi} &= pgR \left\{ c_{13} \sin^5 \Theta + c_{14} \sin^4 \Theta \cos \Theta + c_{15} \sin^3 \Theta \cos^2 \Theta \\
&\quad + c_{16} \sin^2 \Theta \cos^3 \Theta + c_{17} \sin \Theta \cos^4 \Theta + c_{18} \cos^5 \Theta \right\}, \\
\sigma_{\Phi \Phi} &= pgR \left\{ c_{19} \sin^5 \Theta + c_{20} \sin^4 \Theta \cos \Theta + c_{21} \sin^3 \Theta \cos^2 \Theta \\
&\quad + c_{22} \sin^2 \Theta \cos^3 \Theta + c_{23} \sin \Theta \cos^4 \Theta + c_{24} \cos^5 \Theta \right\},
\end{align*}
$$

(6.22)

where $c_i$ is a constant, for $i = 1, \ldots, 24$. Now, upon substituting (6.22) into the equilibrium equations (6.1), gives

$$
\begin{align*}
(3c_1 - c_7 - c_{14} - c_{19}) \sin^6 \Theta \\
+ (3c_2 - c_8 + 6c_{13} - 2c_{20} + 1) \sin^5 \Theta \cos \Theta \\
+ (3c_3 - c_9 + 5c_{14} - 3c_{16} - c_{21}) \sin^4 \Theta \cos^2 \Theta \\
+ (3c_4 - c_{10} + 4c_{15} - 4c_{17} - c_{22} + 2) \sin^3 \Theta \cos^3 \Theta \\
+ (3c_5 - c_{11} + 3c_{16} - 5c_{18} - c_{23}) \sin^2 \Theta \cos^4 \Theta \\
+ (3c_6 - c_{12} + 2c_{17} - c_{24} + 1) \sin \Theta \cos^5 \Theta + c_{18} \cos^6 \Theta = 0, \\
\end{align*}
$$

(6.23)

$$
\begin{align*}
(-c_8 + 4c_{13} - 1) \sin^6 \Theta + (6c_7 - 2c_9 + 4c_{14} - c_{19}) \sin^5 \Theta \cos \Theta \\
+ (5c_8 - 3c_{10} + 4c_{15} - c_{20} - 2) \sin^4 \Theta \cos^2 \Theta \\
+ (4c_9 - 4c_{11} + 4c_{16} - c_{21}) \sin^3 \Theta \cos^3 \Theta \\
+ (3c_{10} - 5c_{12} + 4c_{17} - c_{22} - 1) \sin \Theta^2 \cos^4 \Theta \\
+ (2c_{11} + 4c_{18} - c_{23}) \sin \Theta \cos^5 \Theta + (c_{12} - c_{24}) \cos^6 \Theta = 0,
\end{align*}
$$
which is certainly satisfied provided each of the coefficients vanish, which yields

\[ c_6 = -\frac{1}{12}(2c_2 + 3c_4 - 3c_8 - 2c_{10} - 3c_{12} + 10), \]

\[ c_9 = \frac{1}{3}(3c_1 + 3c_3 + 7c_5 - 7c_7 - 3c_{11}), \]

\[ c_{13} = \frac{1}{4}(c_8 + 1), \quad c_{14} = \frac{1}{15}(15c_1 + 10c_3 + 14c_5 - 35c_7 - 6c_{11}). \]

\[ c_{15} = \frac{1}{4}(2c_2 - 3c_8 + 2c_{10} + 3), \quad c_{16} = c_{11} - c_5. \]

\[ c_{17} = \frac{1}{8}(2c_2 + 3c_4 - 3c_8 - 2c_{10} + 5c_{12} + 6), \quad c_{18} = 0, \]

\[ c_{19} = \frac{1}{15}(30c_1 - 6c_3 - 14c_5 + 20c_7 + 6c_{11}), \]

\[ c_{20} = 2c_2 + 2c_8 - c_{10} + 1, \]

\[ c_{21} = \frac{1}{3}(12c_1 + 12c_3 + 16c_5 - 28c_7 - 12c_{11}), \]

\[ c_{22} = \frac{1}{2}(2c_2 + 3c_4 - 3c_8 + 4c_{10} - 5c_{12} + 4), \]

\[ c_{23} = 2c_{11}, \quad c_{24} = c_{12}. \]

Now, in order to determine the remaining unknown constants, assume firstly that the stresses are continuous throughout the entire sand-pile. This means that at the boundary between the inner dead region and the outer yield region at \( \Theta = \gamma \) that
the stress solution must be forced to remain continuous. Therefore, equating (6.15) with (6.22) at \( \Theta = \gamma \), and using MAPLE gives

\[
c_1 = -\frac{H}{6 \sin^5 \gamma \cos \gamma} \left\{ \begin{array}{l}
50 \cos \phi \cos^2 \gamma \cos(\gamma - \phi) + 23 \cos \gamma (\sin^2 \phi - \cos^2 \gamma) \\
-7 \sin \phi \cos \phi \sin \gamma - 7 \cos \gamma (\cos^2 \gamma + \sin \phi) + \cos^2 \phi \cos \gamma \\
+ \frac{5c_2 \cos \gamma (3 \cos^2 \gamma - 2)}{24 \sin^3 \gamma} + \frac{5c_4 \cos^3 \gamma \gamma}{16 \sin^3 \gamma} - \frac{c_8 (3 \cos^4 \gamma - 2 \cos^2 \gamma + 14)}{48 \sin^3 \gamma \cos \gamma} \\
+ \frac{c_{10} \cos \gamma (9 \cos^2 \gamma + 14)}{24 \sin^3 \gamma} + \frac{c_{11} \cos^2 \gamma (3 \cos^2 \gamma + 4)}{3 \sin^4 \gamma} \\
+ \frac{7c_{12} \cos^3 \gamma (3 \cos^2 \gamma + 5)}{16 \sin^5 \gamma} + \frac{15 \cos^4 \gamma + 15 \cos^2 \gamma - 7}{24 \sin^3 \gamma \cos \gamma},
\end{array} \right.
\]

\[
c_3 = \frac{H}{6 \sin^4 \gamma \cos^3 \gamma} \left\{ \begin{array}{l}
28 \sin \phi \cos \gamma \cos(\gamma - \phi) - 7 \sin \phi \sin \gamma \cos \gamma \\
+ 32 \cos \phi \sin \gamma \cos^2 \gamma \cos(\gamma - \phi) + 7 \cos(\gamma + \phi) \sin(\gamma - \phi) \\
-6 \cos^2 \gamma \cos(\gamma - \phi) \sin(\gamma + \phi) - 3 \left(1 + \sin \phi + 2 \sin^2 \phi \right) \\
- \frac{c_2 (8 \cos^4 \gamma - 17 \cos^2 \gamma + 14)}{24 \sin^3 \gamma \cos \gamma} + \frac{c_4 \cos \gamma (16 \cos^2 \gamma - 21)}{16 \sin^3 \gamma} \\
+ \frac{c_8 (11 \cos^4 \gamma - 10 \cos^2 \gamma + 14)}{48 \sin^3 \gamma \cos^3 \gamma} + \frac{c_{10} (20 \cos^4 \gamma - \cos^2 \gamma - 14)}{24 \sin^3 \gamma \cos \gamma} \\
- \frac{2c_{11} (3 \cos^2 \gamma + 2)}{3 \sin^2 \gamma} - \frac{5c_{12} \cos \gamma (8 \cos^2 \gamma + 7)}{16 \sin^3 \gamma} \\
+ \frac{20 \cos^6 \gamma - 3 \cos^4 \gamma - 19 \cos^2 \gamma + 7}{24 \sin^3 \gamma \cos^3 \gamma},
\end{array} \right. \tag{6.25}
\]
and also

\[
c_5 = -\frac{H}{2\sin^2 \gamma \cos^3 \gamma} \left\{ 4 \sin \phi \cos \gamma \cos(\gamma - \phi) - \sin \phi \sin \gamma \cos \gamma + 2 \sin \phi \cos \phi \cos^2 \gamma + \cos(\gamma + \phi) \sin(\gamma - \phi) \right\}
\]

\[
\quad \quad + \frac{c_2 (\cos^2 \gamma + 2)}{8 \sin \gamma \cos \gamma} + \frac{9c_4 \cos \gamma}{16 \sin \gamma} - \frac{c_8 (5 \cos^4 \gamma + 2 \cos^2 \gamma + 2)}{16 \sin \gamma \cos^3 \gamma}
\]

\[
\quad \quad - \frac{c_{10} (5 \cos^2 \gamma - 2)}{8 \sin \gamma \cos \gamma} + c_{11} + \frac{15c_{12} \cos \gamma}{16 \sin \gamma} + \frac{5 \cos^4 \gamma + \cos^2 \gamma - 1}{8 \sin \gamma \cos^3 \gamma}.
\]  

(6.26)

\[
c_7 = -\frac{H}{2\sin^5 \gamma} \left\{ 4 \cos \phi \cos \gamma \cos(\gamma - \phi) \right. \\
\quad \quad \left. - 2 \cos^2 \phi \sin^2 \gamma - (1 + \sin \phi) \cos^2 \gamma \right\}
\]

\[
\quad \quad - \frac{c_8 \cos \gamma}{2 \sin \gamma} + \frac{c_{10} \cos^3 \gamma}{2 \sin^3 \gamma} + \frac{c_{11} \cos^4 \gamma}{\sin^4 \gamma} + \frac{3c_{12} \cos^5 \gamma}{2 \sin^5 \gamma} + \frac{\cos \gamma}{2 \sin^3 \gamma},
\]

where \( H = G \csc \phi \). Note that (6.25) and (6.26) ensures that the stresses are continuous at the boundary between the inner dead region and the outer yield region at \( \Theta = \gamma \). Secondly, assume that the derivatives of the stresses at the boundary are continuous. As previously stated, if the stresses are continuous and satisfy the equilibrium equations (6.1) throughout the entire sand-pile, then (6.1) shows that the derivatives of \( \sigma_{R\Theta} \) and \( \sigma_{\Theta\Theta} \) must also be continuous throughout the entire sand-pile, and hence, only the continuity of the derivatives of \( \sigma_{RR} \) and \( \sigma_{\Phi\Phi} \) needs to be
examined. Therefore, from (6.20) and (6.22) and using MAPLE, gives

\[ c_4 = \frac{H}{9 \sin^3 \gamma \cos^2 \gamma} \left\{ -8 \cos^2 \phi \sin \gamma \cos \gamma (82 \cos^4 \gamma - 74 \cos^2 \gamma + 19) \\
-4 \sin \phi \cos \phi \sin^2 \gamma (146 \cos^4 \gamma - 37 \cos^2 \gamma - 28) \\
+48\Psi'(\gamma) \sin \phi \cos \phi \sin^6 \gamma + \sin \gamma \cos \gamma (629 \cos^4 \gamma - 862 \cos^2 \gamma + 314) \\
- \sin \phi \sin \gamma \cos \gamma (19 \cos^4 \gamma - 218 \cos^2 \gamma + 118) \right\} \\
- \frac{2c_2(5 \cos^2 \gamma - 2)}{9 \cos^2 \gamma} + \frac{c_{10}(25 \cos^4 \gamma + 16 \cos^2 \gamma - 14)}{9 \sin^2 \gamma \cos^2 \gamma} \]

\[ + \frac{8c_{11}(3 \cos^4 \gamma + 2 \cos^2 \gamma - 2)}{3 \sin^3 \gamma \cos \gamma} + \frac{5c_{12}(8 \cos^4 \gamma + 8 \cos^2 \gamma - 7)}{3 \sin^4 \gamma} \]

\[ + \frac{(2 \cos^2 \gamma + 7)(5 \cos^2 \gamma - 2)}{9 \sin^2 \gamma \cos^2 \gamma} + \frac{G''(\gamma)}{24 \sin \phi \sin^3 \gamma \cos^2 \gamma} \left[ 24 \cos^2 \phi \sin^4 \gamma \right] \]

\[ +(1 - \sin \phi)(\cos^4 \gamma + 22 \cos^2 \gamma - 14) - 50 \cos^4 + 52 \cos^2 \gamma - 20 \right], \]

\[ c_8 = -\frac{H}{\sin \phi \sin^4 \gamma} \left\{ 4 \sin^2 \phi \cos \phi \sin \gamma (10 \cos^2 \gamma - 1) \\
+8 \sin \phi \cos^2 \phi \cos \gamma (5 \cos^2 \gamma - 2) - \sin^2 \phi \cos \gamma (5 \cos^2 \gamma + 4) \\
- \sin \phi \cos \gamma (29 \cos^2 \gamma - 20) \right\} \\
+ \frac{3c_{10} \cos^2 \gamma}{\sin^2 \gamma} + \frac{8c_{11} \cos^3 \gamma}{\sin^3 \gamma} + \frac{15c_{12} \cos^4 \gamma}{\sin^4 \gamma} - \frac{(1 - 4 \cos^2 \gamma)}{\sin^2 \gamma} \]

\[ - \frac{\cos^2 \gamma (1 + \sin \phi)G'(\gamma)}{\sin \phi \sin^3 \gamma}, \]

where \( G'(\gamma) \) is given by (6.19), \( \Psi'(\gamma) \) is given by (6.18), and \( H = G \sec \phi \). Therefore, this ensures that the derivatives of the stresses throughout the entire sand-pile will remain continuous across the boundary at \( \Theta = \gamma \). Now, recall that the stress
distribution must be symmetrical where \( \sigma_{RR}, \sigma_{\Theta \Theta}, \) and \( \sigma_{\Phi \Phi} \) are even functions and \( \sigma_{R \Theta} \) is an odd function. This means that the derivatives of \( \sigma_{RR}, \sigma_{\Theta \Theta}, \) and \( \sigma_{\Phi \Phi} \) directly beneath the vertex of the sand-pile must be zero, and the value of \( \sigma_{R \Theta} \) directly beneath the vertex must also be zero. From (6.22)\(_3\), and noting from (6.24) that \( c_{18} = 0 \), then \( \sigma_{R \Theta} \) must be zero directly beneath the vertex at \( \Theta = 0 \). From (6.1)\(_2\) and (6.22), at \( \Theta = 0 \) shows that for \( \partial \sigma_{\Theta \Theta} / \partial \Theta \) to be zero, then \( c_{23} = 2c_{11} \) is required and which is also required directly from satisfying the equilibrium equations (6.1) as given in (6.24). However, upon differentiating (6.22)\(_2\) with respect to \( \Theta \), for \( \partial \sigma_{\Theta \Theta} / \partial \Theta (R, 0) = 0 \), \( c_{11} = 0 \), and hence \( c_{23} = 0 \). For \( \partial \sigma_{\Phi \Phi} / \partial \Theta \), upon differentiating (6.22)\(_4\) with respect to \( \Theta \) and as \( c_{23} = 0 \), then \( \partial \sigma_{\Phi \Phi} / \partial \Theta (R, 0) = 0 \). Finally, for \( \partial \sigma_{RR} / \partial \Theta \), upon differentiating (6.22)\(_1\) with respect to \( \Theta \) then \( c_2 \) is given by

\[
c_2 = -\frac{H}{2\sin^4 \gamma} \left\{ 12\Psi'(\gamma) \sin \phi \cos \phi \sin^3 \gamma + 32 \sin \phi \cos \phi \sin \gamma (3 \cos^2 \gamma + 1) + \sin \phi \cos \gamma (11 \cos^2 \gamma - 38) + 6 \cos^2 \phi \cos \gamma (19 \cos^2 \gamma - 7) - \cos \gamma (121 \cos^2 \gamma - 94) \right\}
\]

(6.28)

\[
+ \frac{c_{10}(5 \cos^2 \gamma + 4)}{2\sin^2 \gamma} + \frac{15c_{12} \cos^2 \gamma (\cos^2 \gamma + 2)}{2\sin^4 \gamma} - \frac{5 \cos^2 \gamma - 4}{2\sin^2 \gamma} \]

\[
+ \frac{G'(\gamma) [\sin^2 \gamma (6 \sin^2 \phi - \sin \phi + 5) - 3(1 + \sin \phi)]}{2 \sin \phi \sin^3 \gamma}.
\]

Therefore, it remains only to specify constants \( c_{10} \) and \( c_{12} \). The former is arbitrarily chosen so as to satisfy the Coulomb-Mohr inequality while the latter is completely arbitrary and the value \( c_{12} = -0.08 \) is adopted.
6.4 Numerical results

The experimental results of Smid and Novosad [51], were obtained at various stages during the pouring of a three-dimensional heap. When the heap was at height $h$, the horizontal and vertical stresses were measured at the base of the heap. The material used was sand for which the angle of repose was $32.6^\circ$ and the average bulk density was determined to be $\rho = 1567 \text{kg/m}^3$. From Burden and Faries [13], a Shooting Method incorporating a Runge-Kutta scheme of order 4 has been used to do all the numerical solutions presented in this Chapter.

In Section 6.3, a three-dimensional sand-pile with an inner dead region and an outer yield region is examined, where $\Theta = \gamma$ is the boundary between the two regions. Unlike the two-dimensional sand-pile presented in the previous Chapter, the numerical solution in the outer yield region does not seem to follow a known exact special solution. This inevitably means that $\gamma$ must be determined numerically and for $\beta = 0.69$ the numerically determined value is $\gamma = 0.3$. As detailed in Section 6.3, the stresses in the outer yield region are given by (6.22), and the stress distribution throughout the entire sand-pile is shown in Figure 6.2. Note that all the stresses and their derivatives remain continuous across the boundary at $\Theta = \gamma$, and $\sigma_{RR}, \sigma_{\Theta \Theta}$, and $\sigma_{\Phi \Phi}$ are even functions of $\Theta$, while $\sigma_{R\Theta}$ is an odd function. This means that unlike the two-dimensional sand-pile, the horizontal and vertical stresses and their derivatives also remain continuous as shown in Figure 6.3. Also note that the horizontal and vertical stresses in Figure 6.3 also have the M-shaped profile.
Figure 6.2: Variation of $\sigma_{RR}, \sigma_{\Theta\Theta}, \sigma_{R\Theta},$ and $\sigma_{\Phi\Phi}$ for three-dimensional sand-piles with inner dead region and outer yield region using an angle of repose of 32.6° ($\alpha = 1.0018$) for $\beta = 0.69$ ((a) $\sigma_{RR}$, (b) $\sigma_{\Theta\Theta}$, (c) $\sigma_{R\Theta}$, and (d) $\sigma_{\Phi\Phi}$).

6.5 Conclusions

A possible model has been proposed to solve the problem of determining the force distribution at the base of a three-dimensional sand-pile. The Jenike solutions for radial flow in converging cone shaped hoppers have been used, but with gravity acting in the opposite direction. For three-dimensional piles, a sand-pile which has
Figure 6.3: Variation of the horizontal and vertical stresses in a three-dimensional sand-pile. (a) horizontal and (b) vertical).

an inner dead region and an outer yield region has been assumed. Unlike the case for a two-dimensional sand-pile as presented in the previous Chapter, the numerical solution in the outer yield region appears not to coincide with a simple well known analytical solution. Despite this, one possible solution for the polar stresses in the dead region involves quintic expressions of $\sin \Theta$ and $\cos \Theta$ has been examined and which gives rise to the observed M-shaped profile.
Chapter 7

An exact parametric solution for flow in a converging wedge

7.1 Introduction

The problem of a granular material falling under gravity but constrained to flow through either a converging wedge or a cone arises in many industrial processes and was first studied by Jenike [33, 35, 36] and Johanson [39]. These authors examine radial flow solutions for which momentum equations and the Coulomb-Mohr yield condition reduce to two highly nonlinear coupled ordinary differential equations for the determination of the stress field. One particularly simple solution of these equations is noted but it is not sufficiently general to satisfy the necessary boundary conditions, and in general the coupled ordinary differential equations must be solved numerically as demonstrated by Spencer and Bradley [53] and Bradley [11]. The purpose of this Chapter is to show that these equations admit an exact parametric
Table 7.1: Measured values of $\beta = \sin \phi$ for certain granular materials, where $\phi$ is the angle of internal friction.

<table>
<thead>
<tr>
<th>Granular material</th>
<th>Measured values of $\beta = \sin \phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coal</td>
<td>0.939 0.958 0.973 0.985</td>
</tr>
<tr>
<td>Alumina cake</td>
<td>0.941</td>
</tr>
<tr>
<td>Waste rock</td>
<td>0.974</td>
</tr>
<tr>
<td>Silica</td>
<td>0.979</td>
</tr>
</tbody>
</table>

solution in the special case when the angle of internal friction is equal to $\pi/2$. This mathematically meaningful angle of internal friction physically corresponds to a granular material capable of sustaining a vertical slope. Of course such materials do not exist, but there do exist materials that have large values of internal friction close to $\pi/2$ as shown in Table 7.1. For such materials, it is expected that the exact parametric solution for $\phi = \pi/2$ to give a reasonable estimate of their exact solution. Spencer and Bradley [53] and Bradley [11] have re-examined the Jenike radial flow solutions with a view to the determination of the associated double-shearing flow field (see Spencer [52]). Here for convenience, the notation adopted by Spencer and Bradley [53] is followed.

In the following Section the basic equations of continuum theory for plane flow of an ideal cohesionless material which satisfies the Coulomb-Mohr yield condition are briefly stated. The coupled ordinary differential equations are stated for the determination of the Jenike stress field and a particularly simple exact solution is noted. In Section 7.3, certain additional relations involving boundary values from the coupled equations are deduced and the boundary conditions are stated. A single
second order ordinary differential equation is given for the stress angle $\psi$ which is defined by (7.4). In Appendix C the exact parametric solution of the second order ordinary differential equation in the special case when the angle of internal friction of the material $\phi$ is equal to $\pi/2$ is derived and the solution itself in terms of two arbitrary constants is given in Section 7.4. In Section 7.5, typical numerical results are presented which in particular confirm that the full numerical solution coincides with the exact solution.

7.2 Basic equations of continuum theory

For quasi-static plane flow under gravity through a wedge hopper as indicated in Figure 7.1, the stress components in a cylindrical polar coordinate system $(r, \theta, z)$
satisfy the equilibrium equations

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho g \cos \theta, \tag{7.1}
\]

\[
\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = -\rho g \sin \theta,
\]

where \(\rho\) is the density, \(g\) is the acceleration due to gravity and \(\sigma_{rr}, \sigma_{\theta\theta}\) and \(\sigma_{r\theta}\) denote the in-plane physical stress components. Following Spencer and Bradley [53] these components can be expressed in the standard form

\[
\sigma_{rr} = -p + q \cos 2\psi, \quad \sigma_{\theta\theta} = -p - q \cos 2\psi, \quad \sigma_{r\theta} = q \sin 2\psi, \tag{7.2}
\]

where \(p\) and \(q\) are defined as

\[
p = -\frac{1}{2} (\sigma_{rr} + \sigma_{\theta\theta}), \quad q = \left\{ \frac{1}{4} (\sigma_{rr} - \sigma_{\theta\theta})^2 + \sigma_{r\theta}^2 \right\}^{1/2}, \tag{7.3}
\]

while \(\psi\) is given by

\[
\tan 2\psi = \frac{2\sigma_{r\theta}}{\sigma_{rr} - \sigma_{\theta\theta}}, \tag{7.4}
\]

and physically \(\psi\) is the angle between the maximum principal stress axis and the radial direction, in the direction of increasing \(\theta\). For a cohesionless material, the Coulomb-Mohr yield condition takes the form

\[
q = p \sin \phi, \tag{7.5}
\]

where \(\phi\) is assumed to be a material constant and referred to as the angle of internal friction.

Following Jenike [35] and Spencer and Bradley [53], solutions of the form

\[
\psi = \psi(\theta), \quad q = \rho gr F(\theta), \tag{7.6}
\]
are examined and from the above equations the following governing equations may be deduced,

\[
\frac{dF}{d\theta} = \frac{F \sin 2\psi + \sin \phi \sin(2\nu + \theta)}{\sin \phi + \cos 2\nu}.
\]

\[
\frac{d\psi}{d\theta} + 1 = \frac{F \cot \phi \cos \phi + \cos \theta + \sin \phi \cos(2\nu + \theta)}{2F(\sin \phi + \cos 2\nu)}.
\]

Now, looking for a symmetrical stress distribution gives the condition

\[
\psi(0) = 0,
\]

and at the wall \( \theta = \alpha \), a Coulomb friction condition is assumed, so that

\[
\sigma_{r\theta} = -\sigma_{\theta\theta} \tan \mu, \quad \text{at} \quad \theta = \alpha,
\]

where \( \mu \) is the angle of wall friction and \( \alpha \) denotes the semi-vertex angle. Thus, (7.2) and (7.5) gives

\[
\sin[2\psi(\alpha) - \mu] = \frac{\sin \mu}{\sin \phi},
\]

which is meaningful provided \( \mu \leq \phi \). If \( \mu \geq \phi \) then the wall is 'perfectly rough' and the material slips on itself at the wall. In this case

\[
\psi(\alpha) = \frac{\phi}{2} + \frac{\pi}{4},
\]

and observe that for \( \psi \) positive, this value of \( \psi \) provides the first singularity of both equations of (7.7) in the sense that this value of \( \psi(\alpha) \) satisfies \( \cos 2\psi = -\sin \phi \). Thus (7.7) needs to be solved subject to (7.8) and either (7.9) or (7.10), depending on the value of \( \mu \). In general this must be achieved numerically (Spencer and Bradley [53]), and some results are given in the final Section of the paper. Note here that a
special exact solution of (7.7) is

\[ \psi(\theta) = -\theta + \psi^*, \quad F(\theta) = -\frac{\sin \phi}{\cos^2 \phi} \left[ \cos \theta + \sin \phi \cos(\theta - 2\psi^*) \right]. \] (7.11)

for some constant \( \psi^* \).

### 7.3 Mathematical analysis

From the coupled ordinary differential equations and the boundary conditions at \( \theta = 0 \) and \( \theta = \alpha \), certain additional relations can be determined which apply on these boundaries. First at \( \theta = 0 \), (7.7) and (7.8) gives

\[ F'(0) = 0, \quad \psi'(0) = \frac{1}{2} \left\{ \csc \phi - 3 + \frac{1}{F(0)} \right\}. \] (7.12)

where primes throughout this Chapter denotes differentiation with respect to \( \theta \).

Now at \( \theta = \alpha \), there are two possible boundary conditions, depending on the value of \( \mu \), namely (7.9) and (7.10). For \( \mu \leq \phi \), (7.7) and (7.9) gives

\[ F'(\alpha) = \frac{F(\alpha) \sin \mu + \sin \phi \sin(\alpha + \mu)}{\sqrt{\sin^2 \phi - \sin^2 \mu}} - \frac{\sin \phi \sin \alpha \cos \mu}{\cos^2 \phi \sqrt{\sin^2 \phi - \sin^2 \mu}} + \frac{\sin \phi \sin \alpha}{\cos^2 \phi}, \] (7.13)

\[ \psi'(\alpha) = -\frac{3}{2} + \frac{\cos \mu}{2F(\alpha) \sqrt{\sin^2 \phi - \sin^2 \mu}} + \frac{\sin \phi \cos(\alpha + \mu)}{2F(\alpha) \sqrt{\sin^2 \phi - \sin^2 \mu}}, \]

where the positive square root has been taken, due to the fact that the negative square root is non-physical. Note from (7.13) that for \( \phi = \mu \), both \( F'(\alpha) \) and \( \psi'(\alpha) \) become infinite and that the numerical results indicate that these remain infinite for \( \mu > \phi \).
Chapter 7: Exact parametric solution for flow in a converging wedge

Note that, upon eliminating $F(\theta)$ from (7.7), then the following second order differential equation for $\psi(\theta)$ may be deduced,

$$
\begin{align*}
&\left(\sin \phi + \cos 2\psi\right)\left[\cos \theta + \sin \phi \cos(2\psi + \theta)\right] \psi'' \\
= &2(\psi' + 1) \left\{ \sin 2\psi \left[\cos \theta + \sin \phi \cos(2\psi + \theta)\right] \psi' \\
&- 2\sin \phi \left(\sin \phi + \cos 2\psi\right) \sin(2\psi + \theta)\psi' \\
&- (3\sin^2 \phi + 2\sin \phi \cos 2\psi - 1) \sin(2\psi + \theta) \right\}.
\end{align*}
$$

(7.14)

The formal exact parametric solution of this equation for $\phi = \pi/2$ is derived in Appendix C and which applies for $\mu < \phi$. In the following Section, this solution is stated and the constants of integration which satisfy the two boundary conditions given by (7.8) and (7.9) are determined. In Section 7.5, the numerical solution of the system of coupled ordinary differential equations (7.7) subject to the boundary conditions (7.8) and (7.9) is briefly presented.

### 7.4 Exact solution for the special case $\phi = \pi/2$

As shown in Appendix C, equation (7.14) for the special case of $\phi = \pi/2$ admits the following exact parametric solution for $\psi(\theta)$

$$
\tan \psi = \frac{I(\omega)}{C_2} \left\{ 1 - \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \right\} - \frac{C_2}{2} \omega^{1/2} e^{-\omega/2},
$$

(7.15)

$$
\tan \theta = C_2 \left\{ 2e^{\omega/2} \omega^{-1/2} - I(\omega) \right\}^{-1},
$$
Chapter 7: Exact parametric solution for flow in a converging wedge

where the integral $I(\omega)$ is defined by

$$I(\omega) = \int_0^\omega t^{-1/2} e^{t/2} dt + C_1.$$  \hspace{1cm} (7.16)

where $C_1$ and $C_2$ denote two arbitrary constants of integration.

In the special case of $\phi = \pi/2$, (7.8), (7.9) and (7.13)\textsubscript{2} gives

$$\psi(0) = 0, \quad \psi(\alpha) = \mu, \quad \psi'(\alpha) = -1 + \frac{\cos(\alpha + \mu)}{2F(\alpha) \cos \mu},$$  \hspace{1cm} (7.17)

and obviously in this case $\mu \leq \phi$. Now the constants $C_1$ and $C_2$ are required to be determined such that the boundary conditions (7.17)\textsubscript{1} and (7.17)\textsubscript{2} are satisfied.

From (7.15) and (7.16) it is clear that $\psi(0) = 0$ provided the parameter value $\omega \equiv 0$ is identified to correspond to $\theta = 0$, hence, $C_1$ is chosen such that

$$I(\omega) = \int_0^\omega t^{-1/2} e^{t/2} dt,$$  \hspace{1cm} (7.18)

and in the following this is assumed to be the case. Now to determine $C_2$, the parameter value $\omega \equiv \omega_0$ is identified to correspond with $\theta = \alpha$ for some $\omega_0$. Then from (7.17)\textsubscript{2} it is found that (7.15) becomes

$$\tan \mu = \frac{I(\omega_0)}{C_2} \left\{ 1 - \frac{\omega_0^{1/2} e^{-\omega_0/2} I(\omega_0)}{2} \right\} - \frac{C_2}{2} \omega_0^{1/2} e^{-\omega_0/2},$$  \hspace{1cm} (7.19)

$$\tan \alpha = C_2 \left\{ 2 e^{\omega_0/2} \omega_0^{-1/2} - I(\omega_0) \right\}^{-1},$$

and (7.19)\textsubscript{2} gives

$$C_2 = \tan \alpha \left\{ \frac{2 e^{\omega_0/2}}{\omega_0^{1/2}} - I(\omega_0) \right\},$$  \hspace{1cm} (7.20)

which upon substituting into (7.19)\textsubscript{1} gives

$$\frac{\sin \alpha}{\cos \mu} \sin(\alpha + \mu) = \frac{\omega_0^{1/2}}{2} e^{-\omega_0/2} I(\omega_0).$$  \hspace{1cm} (7.21)
which is a transcendental equation for $\omega_0$. Thus, $C_2$ is determined from (7.20) where $\omega_0$ is a solution of (7.21).

Now, a relation needs to be determined for $F(\theta)$ in terms of the parameter $\omega$. Upon differentiating (C.15) and (7.15)$_2$ with respect to $\theta$, gives

$$\psi'(\theta) + 1 = \frac{\omega \cos^2(\psi + \theta)}{\sin^2 \theta} = \frac{\cos(\psi + \theta)}{2F \cos \psi}.$$  \hspace{1cm} (7.22)

where the latter equality follows from the differential equation (7.7)$_2$ with $\phi = \pi/2$. Then rearranging (7.22) for $F(\theta)$, gives

$$F(\theta) = \frac{\sec \psi \sec(\psi + \theta)}{2\omega \csc^2 \theta} = \frac{\left\{1 + \left(\frac{y - x}{1 + xy}\right)^2\right\}^{1/2}(1 + y^2)^{1/2}}{2\omega(1 + x^{-2})} = \frac{x^2(1 + y^2)}{2\omega(1 + xy)(1 + x^2)^{1/2}}.$$  \hspace{1cm} (7.23)

where $x$ and $y$ are defined by (C.3). On simplifying (7.23), the following expression for $F(\theta)$ in terms of the parameter $\omega$ may be deduced,

$$F(\theta) = \frac{e^{-\omega/2}[C_2^2 + I(\omega)^2]}{4\omega^{1/2} \left\{C_2^2 + [2e^{\omega/2} - I(\omega)]^2\right\}^{1/2}},$$  \hspace{1cm} (7.24)

where $C_2$ is given by (7.20). Also note from (7.22) that $\psi'(0) + 1 = (2F(0))^{-1}$ may be concluded which is entirely consistent with (7.12)$_2$ in the special case $\phi = \pi/2$.

### 7.5 Numerical results

The numerical results shown in Figures 7.2 and 7.3 were obtained using an iterative scheme to determine successive numerical solutions which converge to the solution which satisfies the appropriate boundary conditions. For $\mu \leq \phi$, both a Shooting Method, employing a Runge-Kutta scheme for the system of coupled first order ordinary differential equations (7.7), and a Finite-Difference Method for the second
Figure 7.2: Variation of $\psi(\theta)$ for three values of $\phi$ for which $\mu \leq \phi$, where the angle of wall friction $\mu$ is $\pi/12$.

Figure 7.3: Variation of $F(\theta)$ for three values of $\phi$ for which $\mu \leq \phi$, where the angle of wall friction $\mu$ is $\pi/12$. 
Figure 7.4: Absolute error between the exact parametric solution (7.15) and the numerical solution for $\psi(\theta)$ for $\phi = \pi/2$.

order ordinary differential equation (7.14) subject to (7.8) and (7.9) have been used and both methods give same results. Figures 7.2 and 7.3 show the variation of $\psi(\theta)$ and $F(\theta)$ respectively for three values of $\phi > \mu$, assuming an average bulk density $\rho = 1567$ kg/m$^3$ and an angle of wall friction of $\mu = \pi/12$. Note that the solutions shown in Figures 7.2 and 7.3 satisfy the relations (7.12) and (7.13) on the boundaries $\theta = 0$ and $\theta = \alpha$. For $\phi = \pi/2$, the general numerical solution and the exact parametric solution give the same curve with absolute errors as shown in Figures 7.4 and 7.5.

7.6 Conclusions

Within this Chapter the coupled nonlinear ordinary differential equations governing a granular material falling under gravity, but constrained to flow through a converging wedge has been examined. These equations are shown to admit an exact
parametric solution in the special case when the angle of internal friction is equal to \( \pi/2 \). This is the only known exact solution of these equations involving two arbitrary constants.
Appendix A

Solution details for the first order differential equation \((2.29)_2\)

In this Appendix the details for the determination of an approximate solution of \((2.26)\) for small \(\beta\) are presented. Note that while \((2.29)_1\) can be solved for \(v_0\), the constant of integration \(C\) cannot be determined from the boundary condition for \(v_0(\xi)\), but as described in Section 2.3.1, this constant is determined from the condition on \(\sigma\) at \(r = r_0\). On solving \((2.29)_2\) as a standard first order linear differential equation for \(V_0\) and determining in the process several integrals that are integrated using MAPLE, a solution is obtained in terms of elliptic integrals. On making the substitution \(v_0 = \xi^2 w_0\) in \((2.29)_1\), a standard separable first order ordinary differential which readily integrates to yield the solution \((2.30)\) may be deduced. Observe from \((2.30)\) that the solution is valid only for \(\xi \leq (C/2)^{1/2}\) and for \(\xi > (C/2)^{1/2}\) the appropriate solution of \((2.29)_1\) is assumed to be \(v_0 \equiv 0\), which is indeed a solution.
Appendix A: Solution details for the differential equation (2.29)

Now (2.29) and (2.30), assuming the minus sign in (2.30), gives

\[ \frac{dV_0}{dk} = \left[ 1 + \left( 1 - k^2 \xi^2 \right)^{1/2} \right]^2 \frac{1}{\xi (1 - k^2 \xi^2)} V_0 = -\frac{4\xi \left[ 1 - \left( 1 - k^2 \xi^2 \right)^{1/2} \right]}{\xi' \left( 1 - k^2 \xi^2 \right)^{1/2}} + \frac{k^2 \xi^3}{(1 - k^2 \xi^2)}. \]  

(A.1)

where \( k^2 = 2/C \), which is a first order linear differential equation for \( V_0 \) in terms of \( \xi \) and \( \xi' \), for which the integrating factor \( R(\xi) \) is found to be

\[ R(\xi) = \frac{-k^2 (k^2 \xi^2 - 1)^{1/2}}{(1 - k^2 \xi^2)^{1/2} - 1}. \]  

(A.2)

and after integrating, the following result can be determined, namely

\[ \frac{-k^2 V_0 (k^2 \xi^2 - 1)^{1/2}}{(1 - k^2 \xi^2)^{1/2} - 1} = I_1 - I_2, \]  

(A.3)

where

\[ I_1 = \int \frac{4k^2 \xi (k^2 \xi^2 - 1)^{1/2}}{\xi' (1 - k^2 \xi^2)^{1/2} \left[ 1 - (1 - k^2 \xi^2)^{1/2} \right]} d\xi. \]  

(A.4)

and

\[ I_2 = \int \frac{k^4 \xi^3 (k^2 \xi^2 - 1)^{1/2}}{(1 - k^2 \xi^2) \left[ 1 - (1 - k^2 \xi^2)^{1/2} \right]^2} d\xi. \]  

(A.5)

Now, in order to integrate \( I_1 \), \( \xi' \) needs to be rewritten in terms of \( \xi \) and to do this recall that \( \xi = r - r_0^2/r \), which can be solved for \( r \) to give

\[ r = \frac{1}{2} \left[ \xi + \left( \xi^2 + 4r_0^2 \right)^{1/2} \right], \]  

(A.6)

and so from noting that \( \xi' = 1 + r_0^2/r^2 \), then

\[ \xi' = \frac{2 \left( \xi^2 + 4r_0^2 \right)^{1/2}}{\xi + \left( \xi^2 + 4r_0^2 \right)^{1/2}}. \]  

(A.7)

is obtained. Thus \( I_1 \) can be rewritten as

\[ I_1 = I_3 + I_4 = \int \frac{2k^2 \xi^2 (k^2 \xi^2 - 1)^{1/2}}{(1 - k^2 \xi^2)^{1/2} \left( \xi^2 + 4r_0^2 \right)^{1/2} \left[ 1 - (1 - k^2 \xi^2)^{1/2} \right]} d\xi + \int \frac{2k^2 \xi (k^2 \xi^2 - 1)^{1/2}}{(1 - k^2 \xi^2)^{1/2} \left[ 1 - (1 - k^2 \xi^2)^{1/2} \right]} d\xi. \]  

(A.8)
Appendix A: Solution details for the differential equation (2.29)

where $I_3$ can also be rewritten as

$$I_3 = I_5 + I_6 = \int \frac{2i}{(\xi^2 + 4r_0^2)^{1/2}} d\xi + \int \frac{2i}{(\xi^2 + 4r_0^2)^{1/2}} d\xi.$$  \hspace{1cm} (A.9)

Now upon using MAPLE, the integrals $I_2, I_4$ and $I_6$ become

$$I_2 = 2 \tan^{-1} \left\{ \frac{1}{k^2} \right\} - \frac{2 \log \xi \left[ \frac{(k^2 \xi^2 - 1)^{1/2}}{k^2 \xi^2 - 1} \right]}{(k^2 \xi^2 - 1)^{1/2}} + \frac{k^2 \xi^2 - 1}{2}. \hspace{1cm} (A.10)$$

$$I_4 = 2 \tan^{-1} \left\{ \frac{1}{k^2} \right\} + \frac{2 \log \xi \left[ \frac{(k^2 \xi^2 - 1)^{1/2}}{k^2 \xi^2 - 1} \right]}{(k^2 \xi^2 - 1)^{1/2}} + 2 \left( k^2 \xi^2 - 1 \right)^{1/2}. \hspace{1cm} (A.10)$$

$$I_5 = 2i \log \left[ \xi + (\xi^2 + 4r_0^2k^2)^{1/2} \right],$$

and Gradshteyn and Ryzhik [25] (page 326) shows that $I_6$ can be written as

$$I_6 = 2i \left( 1 + 4k^2r_0^2 \right)^{1/2} \left\{ F(\gamma, \nu) - E(\gamma, \nu) \right\} + 2i \xi \left( \frac{1 - k^2 \xi^2}{4r_0^2 + \xi^2} \right)^{1/2}, \hspace{1cm} (A.11)$$

where $\gamma$ and $\nu$ are defined as

$$\gamma = \sin^{-1} \left\{ \frac{1 + 4k^2r_0^2}{4r_0^2 + \xi^2} \right\}, \hspace{1cm} \nu = \frac{1}{(1 + 4k^2r_0^2)^{1/2}}, \hspace{1cm} (A.12)$$

and $F(\gamma, \nu)$ and $E(\gamma, \nu)$ are elliptic integrals of the first and second kind respectively.

From either Gradshteyn and Ryzhik [25] or Abramowitz and Stegun [1], an elliptic integral of the first kind $F(\gamma, \nu)$ is defined as

$$F(\gamma, \nu) = \int_0^{\sin \gamma} \frac{1}{\left[ (1 - t^2)(1 - \nu^2t^2) \right]^{1/2}} dt,$$ \hspace{1cm} (A.13)

and an elliptic integral of the second kind $E(\gamma, \nu)$ is defined as

$$E(\gamma, \nu) = \int_0^{\sin \gamma} \frac{[1 - \nu^2t^2]^{1/2}}{[1 - t^2]^{1/2}} dt.$$ \hspace{1cm} (A.14)

Thus from (A.10) and (A.3), equation (2.31) is obtained where $C = 2/k^2$ and $\gamma$ and $\nu$ are defined by (2.32). Note that from the boundary condition (2.19), the constant of integration for $V_0$ is zero.
Appendix B

The stress values where they fail to exist within a rat-hole

In this Appendix the point at which the stresses fail to exist within a rat-hole for a shear-index granular material is determined. The relations (4.31) are established which for general $n$ connect the stress values at the point of failure. As noted in Section 4.3, the relations (4.29) and (4.30) are immediately apparent for $n = 1$ and $n = 2$, because in these special cases further mathematical analysis is possible.

Upon comparing (4.29) with the Warren Spring equation (4.3), then it is found to be reasonable to speculate for general $n$ that

$$1 - \frac{\sigma_{rr}}{t} = \left( \frac{\sigma_{rz}}{c} \right)^n. \quad (B.1)$$

Moreover, on examination of (4.30) it would not be unreasonable to assume that for general $n$

$$\sigma_{zz} = \sigma_{rr} + \alpha \left( \frac{\sigma_{rz}}{c} \right)^m, \quad (B.2)$$
where $\alpha$ and $m$ are certain unknown constants, yet to be determined. Now on introducing $\omega = \sigma_{rz}/c$, then from (B.1) and (B.2) the quantities $A$ and $B$ which are defined by (4.34) become

$$A = \alpha^2 \omega^{2m} + 4c^2 \omega^2, \quad B = \omega^n - \frac{\alpha}{2t} \omega^m.$$  \hspace{1cm} (B.3)

and from these relations

$$B^2 - \frac{(n-1)}{n^2 t^2} A = \left[ \omega^n + \frac{(n-2)\alpha}{2nt} \omega^m \right]^2 - 2 \frac{(n-1)}{n} \left( \frac{\alpha}{t} \omega^{m+n} + \frac{2\lambda^2}{n} \omega^2 \right), \hspace{1cm} (B.4)$$

may be deduced. Now, the quantity on the left hand side is required to be a perfect square, which leads to the conclusion that $\alpha$ and $m$ needs to be chosen such that $m + n = 2$ and $\alpha/t = -2\lambda^2/n$, and hence (B.2) becomes

$$\sigma_{zz} = \sigma_{rr} - \frac{2\lambda c}{n} \left( \frac{\sigma_{rz}}{c} \right)^{2-n}.$$  \hspace{1cm} (B.5)

Observe that (B.5) is entirely consistent with the special cases $n = 1$ and $n = 2$ given by (4.30), and that (B.4) becomes

$$B - \left[ B^2 - \frac{(n-1)}{n^2 t^2} A \right]^{1/2} = \omega^n + \frac{\lambda^2}{n} \omega^{2-n} - \left[ \omega^n + \frac{\lambda^2}{n^2} (2 - n) \omega^{2-n} \right] = \frac{2\lambda^2}{n^2} (n-1) \omega^{2-n}, \hspace{1cm} \text{(B.6)}$$

from which it is a trivial matter to show that the yield condition (4.33) is satisfied identically, and thus confirms (B.5) as the correct expression.

From Figures 4.7, 4.8 and 4.9 it would seem that the stresses fail to exist where $\sigma_I = 0$ and $\sigma_{III} = -f_c$. From equation (4.14) these conditions give rise to

$$\sigma_{rr} + \sigma_{zz} + \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} = 0,$$

(B.7)

$$\sigma_{rr} + \sigma_{zz} - \left[ (\sigma_{rr} - \sigma_{zz})^2 + 4\sigma_{rz}^2 \right]^{1/2} = -2f_c.$$
Appendix B: The stress values where they fail to exist within a rat-hole

from which the following may be deduced,

$$\sigma_{rr}\sigma_{zz} = \sigma_{rz}^2.$$  \hfill (B.8)

$$\sigma_{rr}\sigma_{zz} + f_c(\sigma_{rr} + \sigma_{zz}) + f_c^2 = \sigma_{rz}^2.$$  

On using (B.1) in conjunction with (B.8) gives

$$\sigma_{rr}^2 + c^2\left(1 - \frac{\sigma_{rr}}{t}\right)^{2/n} + f_c\sigma_{rr} = 0.$$  \hfill (B.9)

as an exact equation for the determination of $\sigma_{rr}$ at the point at which the stresses no longer exist and both $\sigma_I = 0$ and $\sigma_{III} = -f_c$ are satisfied.

Now based on an exact analysis of (B.9) for the two special cases of $n = 1$ and $n = 2$, leads to the proposal of the alternate equation for (B.9), namely

$$\left(1 + \frac{c^2}{t^2}\right)^{(2-n)/n}\sigma_{rr}^2 + \left(f_c - \frac{2c^2}{n}\right)\sigma_{rr} + c^2 = 0.$$  \hfill (B.10)

Further, on using (B.10) and again making a comparison with the exact analysis for $n = 1$ and $n = 2$, the following alternative approximate expression for $f_c$ for general $n$ is proposed, namely

$$f_c = \frac{2c^2}{n}\frac{1}{t} + 2c\left(1 + \frac{c^2}{t^2}\right)^{(2-n)/2n},$$  \hfill (B.11)

which is chosen so as to be consistent with the known exact values (4.35) for $n = 1$ and $n = 2$ and also that the quadratic (B.10) becomes a perfect square. Curiously, Figure 4.4 vindicates this judicious choice for $f_c$ while Figures 4.10 and 4.11 demonstrate the utility of approximating (B.9) by (B.10). It must be emphasized that equations (B.10) and (B.11) are speculative but give reasonably accurate results.
Appendix C

Derivation of the exact solution of (5.26) and (7.14) for $\beta = 1$

In this Appendix the exact parametric solution of the second order differential equation for $\psi(\theta)$, namely (5.26) or (7.14), is derived for the special case of $\beta = 1$. This corresponds to assuming an angle of internal friction of $\phi = \pi/2$. For $\beta = 1(\phi = \pi/2)$, both (5.26) and (7.14) simplifies to give

$$\cos \psi \cos(\psi + \theta) \psi'' = -2(\psi' + 1) \left\{ \sin(\psi + \theta) \cos \psi + \cos(\psi + \theta) \sin \psi \right\}, \quad (C.1)$$

which can be rearranged to yield

$$[\sec^2(\psi + \theta)(\psi' + 1)]' + 2 \tan \psi [\sec^2(\psi + \theta)(\psi' + 1)] = 0. \quad (C.2)$$

Thus, upon making the transformations

$$x = \tan \theta, \quad y = \tan(\psi + \theta). \quad (C.3)$$
so that

\[
\tan \psi = \frac{y - x}{1 + xy}, \quad (C.4)
\]

then equation (C.2) can be shown to simplify to yield

\[
d^2y \over dx^2 + \frac{2y}{1 + xy} \frac{dy}{dx} = 0. \quad (C.5)
\]

Now this equation remains invariant under the stretching group of transformations

\[
x_1 = \lambda x, \quad y_1 = \lambda^{-1} y. \quad (C.6)
\]

and therefore upon introducing the new variable \( z = xy \) and making the Euler transformation \( s = \log x \), equation (C.5) becomes

\[
\frac{d^2z}{ds^2} - 3 \frac{dz}{ds} + 2z + \frac{2z}{1 + z} \left( \frac{dz}{ds} - z \right) = 0. \quad (C.7)
\]

Now, equation (C.7) can be reduced to an Abel equation of the second kind by making the substitution \( u = dz/ds \) and taking \( z \) as the independent variable. However, although a first order differential equation, (C.7) appears not to be integrable by this procedure. Alternatively, upon introducing \( \omega \) such that

\[
\omega = \frac{dz}{ds} - z, \quad (C.8)
\]

then equation (C.7) is equivalent to

\[
\frac{dz}{ds} = z + \omega, \quad \frac{d\omega}{ds} = \frac{2\omega}{z + 1}. \quad (C.9)
\]

On eliminating \( z \) from these equations and introducing

\[
v = \frac{1}{\omega} \frac{d\omega}{ds}. \quad (C.10)
\]
Appendix C: Derivation of the exact parametric solution for $\delta = 1$

then the standard first order differential equation

$$\frac{dv}{d\omega} + \frac{1}{2} \left( 1 - \frac{1}{\omega} \right) v = -\frac{1}{\omega},$$

may be readily deduced. This equation readily integrates to give

$$v = 2 - \omega^{1/2} e^{-\omega/2} I(\omega),$$  \hspace{1cm} (C.12)

where $I(\omega)$ is the integral defined by (7.16). From (C.10) and (C.12) a second integration may be performed to obtain

$$2\omega^{-1/2} e^{\omega/2} - I(\omega) = \frac{C_2}{\chi},$$  \hspace{1cm} (C.13)

from which $(7.15)_2$ may be readily deduced. Equation $(7.15)_1$ follows from the fact that $v = 2/(z + 1)$ and therefore

$$z = \frac{2}{v} - 1 = \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \left\{ 1 - \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \right\}^{-1}.$$  \hspace{1cm} (C.14)

But $z = xy$ and hence

$$y = \tan(\psi + \theta) = \frac{I(\omega)}{C_2},$$  \hspace{1cm} (C.15)

and $(7.15)_1$ now follows from this equation and (C.4). Therefore, the formal parametric solution given by (7.15) and (7.24) has been established, namely

$$\tan \theta = C_2 \left\{ 2e^{\omega/2} \omega^{-1/2} - I(\omega) \right\}^{-1},$$

$$\tan \psi(\theta) = \frac{I(\omega)}{C_2} \left\{ 1 - \frac{\omega^{1/2}}{2} e^{-\omega/2} I(\omega) \right\} - \frac{C_2}{2} \omega^{1/2} e^{-\omega/2},$$  \hspace{1cm} (C.16)

$$F(\theta) = \frac{e^{-\omega/2} \left[ C_2^2 + I(\omega)^2 \right]}{4\omega^{1/2} \left\{ C_2^2 + [2e^{\omega/2} \omega^{-1/2} - I(\omega)]^2 \right\}^{1/2}}.$$
Appendix C: Derivation of the exact parametric solution for $\beta = 1$

where $I(\omega)$ is defined by

$$I(\omega) = \int_0^\omega t^{-1/2}e^{t/2}dt + C_1. \quad (C.17)$$

and $C_1$ and $C_2$ denote two arbitrary constants of integration.

In this Appendix, the two constants of integration are determined for a two-dimensional sand-pile with an angle of internal friction of $\varphi = \pi/2$. In this special case these arbitrary constants must be chosen so that $\psi(\theta)$ satisfies

$$\psi(0) = 0, \quad \psi(\alpha) = -\frac{\pi}{2}. \quad (C.18)$$

From (C.16) it can be seen that the first of these conditions is satisfied provided the parameter value $\omega \equiv 0$ is associated with $\theta = 0$ and the integral $I(\omega)$ is taken to be defined by

$$I(\omega) = \int_0^\omega t^{-1/2}e^{t/2}dt. \quad (C.19)$$

and from (C.15) or (C.16) it is clear that

$$I(\omega) = C_2 \tan(\psi + \theta). \quad (C.20)$$

Now, from (C.16) at $\theta = \alpha$ and (C.18) gives

$$\tan \alpha = \frac{C_2}{2e^{\omega_0^2/2}\omega_0^{-1/2} + C_2 \cot \alpha}. \quad (C.21)$$

where $\omega_0$ is used to denote the value of the parameter corresponding to $\theta = \alpha$ and $\tan(\alpha - \pi/2) = -\cot \alpha$ has been used. From (C.21) it is clear that $\omega_0 = -\infty$.

Accordingly, the parameter is changed from $\omega$ to $-\lambda$, and the substitution $t = -s$ is made in the integral (C.19) so that the exact parametric solution (C.16) now
Appendix C: Derivation of the exact parametric solution for \( \beta = 1 \)

becomes

\[
\tan \theta = -C_3 \left\{ 2e^{-\lambda/2} \lambda^{-1/2} + J(\lambda) \right\}^{-1},
\]

\[
\tan \psi(\theta) = \frac{J(\lambda)}{C_3} \left\{ 1 + \frac{\lambda^{1/2}}{2} e^{\lambda/2} J(\lambda) \right\} + \frac{C_3}{2} \lambda^{1/2} e^{\lambda/2}.
\] \tag{C.22}

\[
F(\theta) = -\frac{e^{\lambda/2} [C_3^2 + J(\lambda)^2]}{4 \lambda^{1/2} \left\{ C_3^2 + [2e^{-\lambda/2} \lambda^{-1/2} + J(\lambda)]^2 \right\}^{1/2}},
\]

where \( C_2 = iC_3 \) and the integral \( J(\lambda) \) is defined by

\[
J(\lambda) = \int_0^\lambda s^{-1/2} e^{-s/2} ds = 2^{3/2} \int_0^{(\lambda/2)^{1/2}} e^{-x^2} dx = (2\pi)^{1/2} \text{erf} (\lambda/2)^{1/2}.
\] \tag{C.23}

where \( \text{erf} \) denotes the usual error function. Now since \( \lambda = \infty \) is the parameter value corresponding to \( \theta = \alpha \), then an estimate of \( J(\lambda) \) is required for \( \lambda \) tending to infinity.

Using the standard asymptotic expansion for the complimentary error function (see for example Carslaw and Jaeger [16]), then

\[
J(\lambda) = (2\pi)^{1/2} \left\{ 1 - \text{erfc} \left( \frac{\lambda}{2} \right)^{1/2} \right\} = (2\pi)^{1/2} - \frac{2e^{-\lambda/2}}{\lambda^{1/2}} \left\{ 1 - \frac{1}{\lambda} + O \left( \frac{1}{\lambda^2} \right) \right\}, \tag{C.24}
\]

may be deduced and then this equation and (C.22) gives

\[
\tan \theta = -C_3 \left\{ (2\pi)^{1/2} + \frac{2e^{-\lambda/2}}{\lambda^{3/2}} \left[ 1 + O \left( \frac{1}{\lambda} \right) \right] \right\}^{-1}.
\]

\[
\tan \psi(\theta) = \left( \frac{\pi}{C_3} + \frac{C_3}{2} \right) \lambda^{1/2} e^{\lambda/2} + O(1), \tag{C.25}
\]

\[
F(\theta) = -\frac{e^{\lambda/2}}{4 \lambda^{1/2}} \left\{ (C_3^2 + 2\pi)^{1/2} - 4 \frac{e^{-\lambda/2}}{\lambda^{1/2}} \left( \frac{2\pi}{C_3^2 + 2\pi} \right)^{1/2} \right\} \left[ 1 + O \left( \frac{1}{\lambda} \right) \right].
\]

From (C.25) it may be deduced that

\[
C_3 = -(2\pi)^{1/2} \tan \alpha. \tag{C.26}
\]
and that

\[
\tan \psi(\theta) = -\left( \frac{\pi \lambda}{2} \right)^{1/2} \frac{e^{\lambda/2}}{\sin \alpha \cos \alpha} + O(1),
\]

which confirms \( \psi(\alpha) = -\pi/2 \) as \( \lambda \) tends to infinity, but note that \( F(\alpha) \) tends to infinity rather than zero. In the case with \( \beta = 1 \) the alternative conditions \( \psi(0) = 0 \) and \( \psi(\alpha) = -\pi/2 \) have been satisfied, which apply only for this special case. From the asymptotic expressions (C.27) it may be confirmed that \( \sigma_{r\theta} \) and \( \sigma_{\theta\theta} \) both tend to zero.

Thus for a two-dimensional sand-pile with angle of internal friction of \( \phi = \pi/2 \), an exact parametric solution is given by (C.22) for \( 0 \leq \lambda \leq \infty \) and \( J(\lambda) \) is defined by (C.23) and the constant \( C_3 \) is defined by equation (C.26).
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