1-1-2014

Approximate ordinary differential equations for the optimal exercise boundaries of American put and call options

Marianito R. Rodrigo
University of Wollongong, marianit@uow.edu.au

Approximate ordinary differential equations for the optimal exercise boundaries of American put and call options

Abstract

We revisit the American put and call option valuation problems. We derive analytical formulas for the option prices and approximate ordinary differential equations for the optimal exercise boundaries. Numerical simulations yield accurate option prices and comparable computational speeds when benchmarked against the binomial method for calculating option prices. Our approach is based on the Mellin transform and an adaptation of the Kármán–Pohlausen technique for boundary layers in fluid mechanics.

Keywords

approximate, put, american, boundaries, differential, exercise, options, optimal, equations, call, ordinary

Disciplines

Engineering | Science and Technology Studies

Publication Details

Rodrigo, M. R. (2014). Approximate ordinary differential equations for the optimal exercise boundaries of American put and call options. European Journal of Applied Mathematics, 25 (1), 27-43.

Euro. Jnl of Applied Mathematics (2014), *vol.* 25, *pp.* 27–43.
doi:10.1017/S0956792513000260

Approximate ordinary differential equations for the optimal exercise boundaries of American put and call options

MARIANITO R. RODRIGO

School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, New South Wales, Australia
email: marianito_rodrigo@uow.edu.au

(*Received 29 January 2013; revised 19 July 2013; accepted 19 July 2013; first published online 15 August 2013*)

We revisit the American put and call option valuation problems. We derive analytical formulas for the option prices and approximate ordinary differential equations for the optimal exercise boundaries. Numerical simulations yield accurate option prices and comparable computational speeds when benchmarked against the binomial method for calculating option prices. Our approach is based on the Mellin transform and an adaptation of the Kármán–Pohlausen technique for boundary layers in fluid mechanics.

Key words: American put and call; Free boundary problem; Optimal exercise boundary; Black–Scholes kernel; Mellin transform

1 Introduction

An *option* is a contract between two parties (the *holder* and the *writer*) that gives the holder the right, but not the obligation, to buy/sell an underlying asset from/to the writer at a predetermined amount (the *exercise price* or *strike price*) on or before a predetermined date (the *expiry* or *maturity*) in the future. The option to buy is called a *call*, while the option to sell is called a *put*. A *European option* can be exercised only at expiry, while an *American option* can be exercised at any time prior to or at expiry. A fundamental problem in option valuation is to determine a fair price for the premium (or the time-zero option value) that the holder is willing to pay to enter into the contract with the writer.

In the case of an American option, it may be optimal for the holder to exercise early when the asset price reaches a critical asset value (the *optimal exercise price*). The collection of all of these optimal exercise prices is called the *optimal exercise boundary*. A distinguishing feature of an American option is that the optimal exercise boundary is unknown *a priori* but must be determined together with the option price.

There are several formulations of the American option valuation problem. McKean [22] first gave a formulation in the form of a free boundary problem for a partial differential equation (PDE). Alternatively, the valuation of American options can be formulated as a linear complementarity problem (see [11]) or as an optimal stopping problem (see [14, 17]).

There are currently many different approaches to the valuation of American options. Some of the more popular ones are as follows (this is by no means an exhaustive list):

(i) Binomial and Monte Carlo methods

(ii) Finite difference methods

(iii) Analytical approximations to the option price and the *early exercise premium* (i.e. the difference between the American and European option prices)

(iv) Numerical methods for solving integral equations for the optimal exercise boundary.

Under the risk-neutral assumption, the option price is expressed as a discounted expectation, with respect to the risk-neutral measure, of the payoff at expiry. The binomial model for option valuation was introduced by Cox *et al.* [9]. It traces the evolution of the asset price in discrete time by means of a binomial lattice. The numerical method is able to handle both European- and American-type options in a straightforward manner.

The Monte Carlo option valuation is also another numerical method that relies on the risk-neutrality assumption. Monte Carlo methods are particularly useful when valuing options with multiple sources of uncertainty or with complicated features. Tilley [29] was the first attempt in using Monte Carlo methods to solve the backward-in-time dynamic programming problem associated with the valuation of American-type contracts. Longstaff and Schwartz [20] developed a practical Monte Carlo method using the least squares technique to price American-style derivatives.

Finite difference methods are the most popular methods that have been utilized in valuing options based on a PDE approach. The basic idea is to replace the derivatives in the PDE by appropriate finite difference formulas and solve the resulting system by choosing either an explicit or implicit finite difference scheme. Some pioneering results in this area are due to Schwartz [27] and Brennan and Schwartz [2, 3].

Analytical approximations to the option price and the early exercise premium have also been obtained with the aim of simplifying the computational effort. Geske and Johnson [12] introduced the concept of a *compound option*, where the American option is decomposed into a finite number of European-style options, each of which when exercised leads to another option. MacMillan [21] and Barone-Adesi and Whaley [1] derived a PDE for the early exercise premium and disregarded one term to be able to solve the resulting PDE as well as to approximate the optimal exercise boundary.

Another class of approaches is based on solving an integral equation for the optimal exercise boundary. Kim [18] obtained such an integral equation (see [5, 6] as well for other formulations) and showed that the value of an American option is equal to that of the corresponding European option plus the early exercise premium. If the integral equation can be solved for the optimal exercise boundary, then the early exercise premium can be calculated, thus yielding the option value. Carr *et al.* [4] also derived another integral equation for the free boundary. Several authors dealt with numerically solving these integral equations by proposing different algorithms such as step function approximations [13] and quadratic approximations [25]. Sevcovic [28] derived a non-linear singular integral equation whose solution can be obtained by successive iterations. Ju [16] employed a multi-piece exponential function to approximate the optimal exercise boundary. As we can observe, valuing American options based on solving an integral equation for the optimal exercise boundary has led to the proposal of many different numerical algorithms to solve integral equations. In this paper, we take a different approach by deriving an approximate first-order ordinary differential equation (ODE) for the optimal exercise boundary. Like the integral equation, this ODE cannot be solved analytically but its numerical solution is essentially trivial. We will show later that the option values obtained using this new approach are very accurate when compared with those obtained via the binomial method (with comparable computational speeds).

Certainly, there are more approaches than those mentioned above that have been proposed in the literature. For a more recent detailed exposition, we refer the reader to a comprehensive review article by Zhu [30] and the exhaustive list of references in [10].

In this paper, we revisit the valuation problems for the American put and the American call. We give an alternative derivation of valuation formulas for the American put and call option prices that depend on their respective optimal exercise boundaries. These formulas are obtained by rewriting the valuation problems and using the Mellin transform (cf. [26], where an exact valuation formula for a European option with a general payoff was obtained using the Mellin transform). Then we derive an approximate ODE for each of the optimal exercise boundaries. As mentioned above, this is different from deriving an exact integral equation for the optimal exercise boundary but having to numerically approximate this much more complicated equation. In our case, the numerical solution of the ODE is straightforward, fast and accurate.

The outline of this paper is as follows. In Section 2 we derive some properties of the Black–Scholes kernel (to be defined later) and recall some properties of the Mellin transform. For the convenience of the reader, we give the formal derivation of the valuation formula for a European option with a general payoff given in [26]. In Section 3 we derive a valuation formula for the American put option price and an approximate ODE for the American put optimal exercise boundary. We consider the American call option valuation problem in Section 4, utilizing some of the results of Section 3 for the American put. A valuation formula for the American call option price is also given as well as an approximate ODE for the American call optimal exercise boundary. In Section 5 we present the results of numerical simulations. We solve the ODEs for the respective optimal exercise boundaries and substitute into the option valuation formulas. The option values are benchmarked with the values obtained using the binomial method. Finally, in Section 6, we give some concluding remarks.

In the remainder of this section, we introduce our notation and recall the free boundary formulation of the American put and call option valuation problems. We assume that the value of an option depends on the asset price with risk-neutral dynamics given by the stochastic differential equation

$$\mathrm{d}S_t = (r-q)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t, \tag{1.1}$$

where $\{S_t : t \in [0,T]\}$ is the asset price process, $\{W_t : t \in [0,T]\}$ is a standard Brownian motion with respect to the risk-neutral measure, $T > 0$ is the expiry, $r > 0$ is the risk-free rate, $q \geqslant 0$ is the dividend yield and $\sigma > 0$ is the volatility.

The generic European option price at time t is denoted by V_t and the payoff function is denoted by $\phi : [0,\infty) \to [0,\infty)$. For example, the put and call payoffs are $\phi(x) = (E-x)^+$

and $\phi(x) = (x - E)^+$, respectively, where $E > 0$ is the exercise price and $(z)^+ = \max(z, 0)$ for $z \in \mathbb{R}$. At expiry, $V_T = \phi(S_T)$. It is well known that V_t is given as $V_t = v(S_t, t)$, where the function $v = v(x, t)$ satisfies the Black–Scholes PDE

$$\mathscr{L}v(x,t) := \frac{\partial v}{\partial t}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 v}{\partial x^2}(x,t) + (r-q)x\frac{\partial v}{\partial x}(x,t) - rv(x,t) = 0.$$

Hence, the generic European option price function v solves the terminal value problem

$$\mathscr{L}v(x,t) = 0, \quad x \geqslant 0, \quad 0 \leqslant t < T,$$

$$v(x,T) = \phi(x), \quad x \geqslant 0.$$

The American put and call option valuation problems are formulated as follows. The *American put option valuation problem* is to find two functions $p_a = p_a(x,t)$ and $S^* = S^*(t)$ such that

(i) $p_a(x,T) = (E - x)^+$ for $x \geqslant 0$;
(ii) $p_a(x,t) = E - x$ for $0 \leqslant t < T$ and $0 \leqslant x < S^*(t)$;
(iii) $\mathscr{L}p_a(x,t) = 0$ for $0 \leqslant t < T$ and $x > S^*(t)$;
(iv) At $x = S^*(t)$, where $0 \leqslant t < T$, the following smooth pasting conditions are satisfied:

$$p_a(S^*(t),t) = E - S^*(t), \quad \frac{\partial p_a}{\partial x}(S^*(t),t) = -1.$$

On the other hand, the *American call option valuation problem* is to find two functions $c_a = c_a(x,t)$ and $S^* = S^*(t)$ such that

(i) $c_a(x,T) = (x - E)^+$ for $x \geqslant 0$;
(ii) $c_a(x,t) = x - E$ for $0 \leqslant t < T$ and $x > S^*(t)$;
(iii) $\mathscr{L}c_a(x,t) = 0$ for $0 \leqslant t < T$ and $0 \leqslant x < S^*(t)$;
(iv) At $x = S^*(t)$, where $0 \leqslant t < T$, the following smooth pasting conditions are satisfied:

$$c_a(S^*(t),t) = S^*(t) - E, \quad \frac{\partial c_a}{\partial x}(S^*(t),t) = 1.$$

2 Some properties of the Black–Scholes kernel and the Mellin transform

We begin by defining some useful auxiliary functions. Let

$$z_1(x,t,u) = \frac{\log x + (r - q + \sigma^2/2)(u-t)}{\sigma\sqrt{u-t}},$$
$$z_2(x,t,u) = \frac{\log x + (r - q - \sigma^2/2)(u-t)}{\sigma\sqrt{u-t}}.$$

Then it is straightforward to show that

$$xe^{-q(u-t)}N'(z_1(x,t,u)) - e^{-r(u-t)}N'(z_2(x,t,u)) = 0, \tag{2.1}$$

where N is the cumulative distribution function of a standard normal random variable. The *Black–Scholes kernel* was defined in [26] as

$$\mathscr{K}(x,t,u) = \frac{e^{-r(u-t)}}{\sigma\sqrt{u-t}} N'(z_2(x,t,u)). \tag{2.2}$$

Using the identity (2.1), an alternative form for the kernel is therefore

$$\mathscr{K}(x,t,u) = \frac{xe^{-q(u-t)}}{\sigma\sqrt{u-t}} N'(z_1(x,t,u)). \tag{2.3}$$

From (2.2) and (2.3), it follows that

$$\begin{aligned} \mathscr{K}\left(\frac{x}{y},t,u\right) &= \frac{\partial}{\partial y}\left[-xe^{-q(u-t)}N\left(z_1\left(\frac{x}{y},t,u\right)\right)\right], \\ \frac{1}{y}\mathscr{K}\left(\frac{x}{y},t,u\right) &= \frac{\partial}{\partial y}\left[-e^{-r(u-t)}N\left(z_2\left(\frac{x}{y},t,u\right)\right)\right]. \end{aligned} \tag{2.4}$$

The *Mellin transform* $\hat{f}$ at $\xi \in \mathbb{R}$ of a function $f : [0,\infty) \to \mathbb{R}$ is defined to be

$$\hat{f}(\xi) = \int_0^\infty x^{\xi-1} f(x)\,\mathrm{d}x,$$

provided the improper integral converges at ξ. For sufficiently differentiable and well-behaved real-valued functions f and g defined on $[0,\infty)$, it is known (see [23, pp. 362–363] for instance) that

$$\widehat{xf'}(\xi) = -\xi\hat{f}(\xi), \quad \widehat{x^2f''}(\xi) = (\xi+\xi^2)\hat{f}(\xi), \tag{2.5}$$

where we write xf' and x^2f'' to denote the functions whose values are $xf'(x)$ and $x^2f''(x)$, respectively. Moreover, if we define the *convolution* $f * g$ of f and g by

$$(f*g)(x) = \int_0^\infty \frac{1}{y} f\left(\frac{x}{y}\right) g(y)\,\mathrm{d}y,$$

then it can be shown that

$$(\widehat{f*g})(\xi) = \hat{f}(\xi)\hat{g}(\xi), \tag{2.6}$$

also known as the *convolution property*. In particular, it was shown in [26] that the Mellin transform of the Black–Scholes kernel $\mathscr{K}(\cdot,t,u)$ with respect to x is

$$\hat{\mathscr{K}}(\xi,t,u) = \int_0^\infty x^{\xi-1}\mathscr{K}(x,t,u)\,\mathrm{d}x = e^{-p(\xi)(u-t)}, \tag{2.7}$$

where p is the polynomial

$$p(\xi) = r + \left(r - q - \frac{\sigma^2}{2}\right)\xi - \frac{\sigma^2}{2}\xi^2. \tag{2.8}$$

The above properties of the Mellin transform make the Black–Scholes PDE amenable to the treatment based on the Mellin transform. Indeed, suppose that $v = v(x,t)$ satisfies the following terminal value problem:

$$\mathscr{L}v(x,t) = 0, \quad x \geqslant 0, \quad 0 \leqslant t < T, \tag{2.9a}$$

$$v(x,T) = \phi(x), \quad x \geqslant 0. \tag{2.9b}$$

Define

$$\hat{v}(\xi,t) = \int_0^\infty x^{\xi-1} v(x,t)\,\mathrm{d}x.$$

Formally applying the Mellin transform to (2.9) and using the properties in (2.5), we obtain

$$\frac{\partial \hat{v}}{\partial t} - p(\xi)\hat{v} = 0, \quad \hat{v}(\xi,T) = \hat{\phi}(\xi).$$

The solution of this linear problem is

$$\hat{v}(\xi,t) = e^{-p(\xi)(T-t)}\hat{\phi}(\xi) = \hat{\mathscr{K}}(\xi,t,T)\phi(\xi),$$

where we used (2.7) in the last term. Invoking the convolution property (2.6), the formal solution of (2.9) is given by

$$v(x,t) = \int_0^\infty \frac{1}{y}\mathscr{K}\left(\frac{x}{y},t,T\right)\phi(y)\,\mathrm{d}y. \tag{2.10}$$

We say that (2.10) is the formal solution since we have not specified the properties of ϕ. It was rigorously shown in [26] that if ϕ is continuous and bounded, then (2.10) is the unique classical solution of (2.9). An example is the put payoff function $\phi(x) = (E-x)^+$. A similar result follows if ϕ is continuous but unbounded, provided that it satisfies a growth condition at infinity. More precisely, suppose that ϕ is continuous and $\phi(x) = O(x)$ as $x \to \infty$. Following [26], it can be rigorously shown that (2.10) is the unique classical solution of (2.9), noting that the improper integral converges by the first equation in (2.4). An example is the call payoff function $\phi(x) = (x-E)^+$.

We remark that (2.10) could also have been obtained by applying a series of variable transformations to the terminal value problem (2.9) to eventually arrive at an initial value problem for the heat equation, whose formal solution can be derived using the Fourier transform. However, the Mellin transform has the advantage of being applicable directly to (2.9) without introducing variable transformations. We also point out that using the convolution property avoids the introduction of complex-valued integrals when using the definition of the inverse Mellin transform (cf. [15, 24]), thus foregoing the use of residue theory. Other recent references where the Mellin transform was used in the pricing of options are [7, 8].

3 American put option valuation problem

The American put option valuation problem can be reformulated as

$$\mathscr{L}p_a(x,t) = (-rE + qx)H(S^*(t) - x), \quad x \geqslant 0, \quad x \neq S^*(t), \quad 0 \leqslant t < T, \tag{3.1a}$$

$$p_a(x,T) = (E-x)^+, \quad x \geqslant 0, \tag{3.1b}$$

$$p_a(S^*(t),t) = E - S^*(t), \quad \frac{\partial p_a}{\partial x}(S^*(t),t) = -1, \quad 0 \leqslant t < T, \tag{3.1c}$$

where H is the usual Heaviside function,

$$H(z) = \begin{cases} 1 & \text{if } z > 0, \\ 0 & \text{if } z < 0. \end{cases}$$

Following [26], we will solve (3.1) using the Mellin transform. Let $\hat{p}_a(\cdot, t)$ denote the Mellin transform of $p_a(\cdot, t)$ with respect to x, i.e. let

$$\hat{p}_a(\xi, t) = \int_0^\infty x^{\xi-1} p_a(x, t)\, \mathrm{d}x.$$

Note that

$$\int_0^\infty x^{\xi-1}(-rE + qx)H(S^*(t) - x)\, \mathrm{d}x = -\frac{rE}{\xi} S^*(t)^\xi + \frac{q}{\xi+1} S^*(t)^{\xi+1}. \tag{3.2}$$

Taking the Mellin transform of (3.1a) and using the previous equation, we obtain

$$\frac{\partial \hat{p}_a}{\partial t} - p(\xi)\hat{p}_a = -\frac{rE}{\xi} S^*(t)^\xi + \frac{q}{\xi+1} S^*(t)^{\xi+1}, \tag{3.3}$$

where p is as defined in (2.8). On the other hand, the Mellin transform of (3.1b) is

$$\hat{p}_a(\xi, T) = \int_0^\infty x^{\xi-1}(E - x)^+\, \mathrm{d}x = \frac{E^{\xi+1}}{\xi(\xi+1)}. \tag{3.4}$$

Then the solution of the linear problem (3.3), (3.4) is

$$\begin{aligned} \hat{p}_a(\xi, t) &= e^{-p(\xi)(T-t)} \frac{E^{\xi+1}}{\xi(\xi+1)} \\ &\quad - \int_t^T e^{-p(\xi)(u-t)} \left[-\frac{rE}{\xi} S^*(u)^\xi + \frac{q}{\xi+1} S^*(u)^{\xi+1} \right] \mathrm{d}u. \end{aligned}$$

Note that (2.7) allows us to rewrite this as

$$\begin{aligned} \hat{p}_a(\xi, t) &= \hat{\mathscr{K}}(\xi, t, T) \frac{E^{\xi+1}}{\xi(\xi+1)} \\ &\quad - \int_t^T \hat{\mathscr{K}}(\xi, t, u) \left[-\frac{rE}{\xi} S^*(u)^\xi + \frac{q}{\xi+1} S^*(u)^{\xi+1} \right] \mathrm{d}u. \end{aligned}$$

Invoking the convolution property (2.6), while recalling (3.2) and (3.4), we get

$$\begin{aligned} p_a(x, t) &= \int_0^\infty \frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, T\right) (E - y)^+\, \mathrm{d}y \\ &\quad - \int_t^T \int_0^\infty \frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, u\right) (-rE + qy) H(S^*(u) - y)\, \mathrm{d}y\, \mathrm{d}u \\ &= \int_0^E \frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, T\right) (E - y)\, \mathrm{d}y \\ &\quad - \int_t^T \int_0^{S^*(u)} \frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, u\right) (-rE + qy)\, \mathrm{d}y\, \mathrm{d}u. \end{aligned}$$

Using the properties in (2.4) and recalling that $N(z) + N(-z) = 1$ for all $z \in \mathbb{R}$, we therefore obtain the following explicit formula for the American put,

$$\begin{aligned} p_a(x,t) = p_e(x,t) + rE \int_t^T e^{-r(u-t)} N\left(-z_2\left(\frac{x}{S^*(u)}, t, u\right)\right) \mathrm{d}u \\ - qx \int_t^T e^{-q(u-t)} N\left(-z_1\left(\frac{x}{S^*(u)}, t, u\right)\right) \mathrm{d}u, \end{aligned} \tag{3.5}$$

where

$$p_e(x,t) = Ee^{-r(T-t)} N\left(-z_2\left(\frac{x}{E}, t, T\right)\right) - xe^{-q(T-t)} N\left(-z_1\left(\frac{x}{E}, t, T\right)\right) \tag{3.6}$$

is the well-known Black–Scholes formula for the European put. Note that S^* is still unspecified since we have not yet used the smooth pasting conditions (3.1c). However, (3.5) and (3.6) solve (3.1a) and (3.1b) *exactly* for an arbitrary S^*. We need to find a single function S^* that simultaneously satisfies both equations in (3.1c). We also observe that (3.6) follows from the more general formula (2.10).

Before we proceed to find S^*, let us go back to the Black–Scholes PDE (2.9a). Formally integrating with respect to x from $x = a(t)$ to $x = b(t)$, it is straightforward to show that

$$\begin{aligned} \int_{a(t)}^{b(t)} x \frac{\partial v}{\partial x}(x,t)\,\mathrm{d}x &= [xv(x,t)]_{a(t)}^{b(t)} - \int_{a(t)}^{b(t)} v(x,t)\,\mathrm{d}x, \\ \int_{a(t)}^{b(t)} x^2 \frac{\partial^2 v}{\partial x^2}(x,t)\,\mathrm{d}x &= \left[x^2 \frac{\partial v}{\partial x}(x,t) - 2xv(x,t)\right]_{a(t)}^{b(t)} + 2\int_{a(t)}^{b(t)} v(x,t)\,\mathrm{d}x, \\ \int_{a(t)}^{b(t)} \frac{\partial v}{\partial t}(x,t)\,\mathrm{d}x &= -v(b(t),t)\dot{b}(t) + v(a(t),t)\dot{a}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} v(x,t)\,\mathrm{d}x, \end{aligned}$$

where we used the Leibniz Rule in the last equation. Then (2.9a) becomes

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} v(x,t)\,\mathrm{d}x + (\sigma^2 - 2r + q)\int_{a(t)}^{b(t)} v(x,t)\,\mathrm{d}x \\ - [v(b(t),t)\dot{b}(t) - v(a(t),t)\dot{a}(t)] \\ + \left[\frac{\sigma^2}{2}x^2 \frac{\partial v}{\partial x}(x,t) - (\sigma^2 - r + q)xv(x,t)\right]_{a(t)}^{b(t)}. \end{aligned} \tag{3.7}$$

Recall from (3.1a) that p_a satisfies

$$\frac{\partial p_a}{\partial t} + \frac{\sigma^2}{2}x^2 \frac{\partial^2 p_a}{\partial x^2} + (r-q)x\frac{\partial p_a}{\partial x} - rp_a = 0$$

for all $x > S^*(t)$. Formally taking $a(t) = S^*(t)$, $b(t) = \infty$ and $v(x,t) = p_a(x,t)$, (3.7) simplifies to

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{S^*(t)}^{\infty} p_a(x,t)\,\mathrm{d}x + (\sigma^2 - 2r + q)\int_{S^*(t)}^{\infty} p_a(x,t)\,\mathrm{d}x + p_a(S^*(t),t)\dot{S}^*(t) \\ - \left[\frac{\sigma^2}{2}S^*(t)^2\frac{\partial p_a}{\partial x}(S^*(t),t) - (\sigma^2 - r + q)S^*(t)p_a(S^*(t),t)\right]. \tag{3.8}$$

Similarly, we know from (2.10) that p_e satisfies

$$\frac{\partial p_e}{\partial t} + \frac{\sigma^2}{2}x^2\frac{\partial^2 p_e}{\partial x^2} + (r-q)x\frac{\partial p_e}{\partial x} - rp_e = 0$$

for all $x \geqslant 0$ (and therefore for all $x > S^*(t)$). With $a(t) = S^*(t)$, $b(t) = \infty$ and $v(x,t) = p_e(x,t)$, (3.7) reduces to

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{S^*(t)}^{\infty} p_e(x,t)\,\mathrm{d}x + (\sigma^2 - 2r + q)\int_{S^*(t)}^{\infty} p_e(x,t)\,\mathrm{d}x + p_e(S^*(t),t)\dot{S}^*(t) \\ - \left[\frac{\sigma^2}{2}S^*(t)^2\frac{\partial p_e}{\partial x}(S^*(t),t) - (\sigma^2 - r + q)S^*(t)p_e(S^*(t),t)\right]. \tag{3.9}$$

Subtracting (3.9) from (3.8), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^*(t)}^{\infty} [p_a(x,t) - p_e(x,t)]\,\mathrm{d}x &+ (\sigma^2 - 2r + q)\int_{S^*(t)}^{\infty} [p_a(x,t) - p_e(x,t)]\,\mathrm{d}x \\ &= -[p_a(S^*(t),t) - p_e(S^*(t),t)]\dot{S}^*(t) \\ &\quad + \frac{\sigma^2}{2}S^*(t)^2\left[\frac{\partial p_a}{\partial x}(S^*(t),t) - \frac{\partial p_e}{\partial x}(S^*(t),t)\right] \\ &\quad - (\sigma^2 - r + q)S^*(t)[p_a(S^*(t),t) - p_e(S^*(t),t)].\end{aligned}$$

Note that this equation depends on S^*, p_a and p_e. What we want is an expression that depends only on S^* and p_e, and that utilizes the boundary conditions for p_a in (3.1c). We now make the simplifying assumption that

$$\int_{S^*(t)}^{\infty} p_a(x,t)\,\mathrm{d}x \approx \int_{S^*(t)}^{\infty} p_e(x,t)\,\mathrm{d}x, \tag{3.10}$$

i.e. the 'average' values of p_a and p_e over $[S^*(t),\infty)$ are approximately equal for all $0 \leqslant t < T$. This assumption is certainly valid when $t \approx T$ since the integral terms in (3.5) are negligible. Moreover, it can be verified numerically (e.g. by the binomial method) that

$$\lim_{x\to\infty}\frac{p_a(x,t)}{p_e(x,t)} = 1$$

for all $0 \leqslant t < T$. The idea behind (3.10) is borrowed from the Kármán–Pohlausen technique in fluid mechanics, where the free boundary represents the thickness of a boundary layer [31, pp. 421–423]. Using the boundary conditions in (3.1c), an approximate

ODE for the American put optimal exercise boundary is therefore

$$\dot{S}^*(t) = \frac{\sigma^2}{2} S^*(t)^2 \frac{(\partial p_e/\partial x)(S^*(t), t) + 1}{p_e(S^*(t), t) + S^*(t) - E} - (\sigma^2 - r + q)S^*(t). \tag{3.11}$$

In summary, our valuation formulas are given by the expressions (3.5) and (3.6) for the American and European put option prices, respectively, together with the approximate ODE (3.11) for the American put optimal exercise boundary.

4 American call option valuation problem

The American call option valuation problem can be reformulated as

$$\mathscr{L} c_a(x,t) = (rE - qx)H(x - S^*(t)), \quad x \geqslant 0, \quad x \neq S^*(t), \quad 0 \leqslant t < T, \tag{4.1a}$$

$$c_a(x, T) = (x - E)^+, \quad x \geqslant 0, \tag{4.1b}$$

$$c_a(S^*(t), t) = S^*(t) - E, \quad \frac{\partial c_a}{\partial x}(S^*(t), t) = 1, \quad 0 \leqslant t < T. \tag{4.1c}$$

Unlike the American put, here we cannot apply the Mellin transform directly to (4.1) since the Mellin transform of (4.1b) does not exist. We therefore follow a different route. Consider the terminal value problem

$$\mathscr{L} w(x,t) = (-rE + qx)H(S^*(t) - x), \quad x \geqslant 0, \quad x \neq S^*(t), \quad 0 \leqslant t < T, \tag{4.2a}$$

$$w(x, T) = (E - x)^+, \quad x \geqslant 0. \tag{4.2b}$$

Note that this is very similar to the American put valuation problem (3.1) except that there are no smooth pasting conditions. But in the previous section, we obtained (3.5) by considering only (3.1a) and (3.1b), keeping S^* arbitrary. Thus, using (3.5), the solution of (4.2) is

$$\begin{aligned} w(x,t) = p_e(x,t) + rE \int_t^T e^{-r(u-t)} N\left(-z_2\left(\frac{x}{S^*(u)}, t, u\right)\right) \mathrm{d}u \\ - qx \int_t^T e^{-q(u-t)} N\left(-z_1\left(\frac{x}{S^*(u)}, t, u\right)\right) \mathrm{d}u, \end{aligned} \tag{4.3}$$

where p_e is given by (3.6) but S^* is still unspecified here.

Now let us define the function

$$c_a(x,t) = x - E + w(x,t). \tag{4.4}$$

We claim that (4.4) solves (4.1a) and (4.1b). From the relation $H(z) + H(-z) = 1$ for all $z \neq 0$, we see that

$$\begin{aligned}\mathscr{L}c_a(x,t) &= \mathscr{L}(x-E) + \mathscr{L}w(x,t)\\ &= (r-q)x - r(x-E) + (-rE+qx)H(S^*(t)-x)\\ &= -qx + rE + (-rE+qx)H(S^*(t)-x)\\ &= (qx-rE)[H(S^*(t)-x)-1]\\ &= (rE-qx)H(x-S^*(t)),\end{aligned}$$

verifying (4.1a). In addition,

$$c_a(x,T) = x - E + w(x,T) = x - E + (E-x)^+ = (x-E)^+,$$

which satisfies (4.1b).

We can rewrite (4.4) as follows. Recall the *put-call parity* that relates the European put and call option functions:

$$c_e(x,t) - p_e(x,t) = xe^{-q(T-t)} - Ee^{-r(T-t)}. \tag{4.5}$$

Substituting (4.3) into (4.4), we get

$$\begin{aligned}c_a(x,t) = x - E &+ p_e(x,t)\\ &+ rE\int_t^T e^{-r(u-t)}N\left(-z_2\left(\frac{x}{S^*(u)},t,u\right)\right)\,\mathrm{d}u\\ &- qx\int_t^T e^{-q(u-t)}N\left(-z_1\left(\frac{x}{S^*(u)},t,u\right)\right)\,\mathrm{d}u.\end{aligned}$$

Using (4.5), this is equivalent to

$$\begin{aligned}c_a(x,t) = c_e(x,t) &+ x\left[1-e^{-q(T-t)}\right] - E\left[1-e^{-r(T-t)}\right]\\ &+ rE\int_t^T e^{-r(u-t)}\left[1-N\left(z_2\left(\frac{x}{S^*(u)},t,u\right)\right)\right]\,\mathrm{d}u\\ &- qx\int_t^T e^{-q(u-t)}\left[1-N\left(z_1\left(\frac{x}{S^*(u)},t,u\right)\right)\right]\,\mathrm{d}u.\end{aligned}$$

Evaluating the integrals gives

$$\begin{aligned}c_a(x,t) = c_e(x,t) &+ qx\int_t^T e^{-q(u-t)}N\left(z_1\left(\frac{x}{S^*(u)},t,u\right)\right)\,\mathrm{d}u\\ &- rE\int_t^T e^{-r(u-t)}N\left(z_2\left(\frac{x}{S^*(u)},t,u\right)\right)\,\mathrm{d}u,\end{aligned} \tag{4.6}$$

where

$$c_e(x,t) = xe^{-q(T-t)}N\left(z_1\left(\frac{x}{E},t,T\right)\right) - Ee^{-r(T-t)}N\left(z_2\left(\frac{x}{E},t,T\right)\right) \tag{4.7}$$

from (4.5) and (3.6). We observe that (4.7) follows from the more general formula (2.10). What remains is to determine S^* such that (4.1c) holds.

From (4.1a) we see that c_a solves the Black–Scholes PDE

$$\frac{\partial c_a}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 c_a}{\partial x^2} + (r-q)x\frac{\partial c_a}{\partial x} - rc_a = 0$$

for $0 \leqslant x < S^*(t)$. Choosing $a(t) = 0$, $b(t) = S^*(t)$ and $v(x,t) = c_a(x,t)$, (3.7) becomes

$$\begin{aligned} 0 = {} & \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{S^*(t)} c_a(x,t)\,\mathrm{d}x + (\sigma^2 - 2r + q)\int_0^{S^*(t)} c_a(x,t)\,\mathrm{d}x - c_a(S^*(t),t)\dot{S}^*(t) \\ & + \left[\frac{\sigma^2}{2} S^*(t)^2 \frac{\partial c_a}{\partial x}(S^*(t),t) - (\sigma^2 - r + q)S^*(t)c_a(S^*(t),t)\right]. \end{aligned} \tag{4.8}$$

Similarly, from (2.10) we know that c_e satisfies the Black–Scholes PDE

$$\frac{\partial c_e}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 c_e}{\partial x^2} + (r-q)x\frac{\partial c_e}{\partial x} - rc_e = 0$$

for all $x \geqslant 0$ (and therefore for all $0 \leqslant x < S^*(t)$). Take $a(t) = 0$, $b(t) = S^*(t)$ and $v(x,t) = c_e(x,t)$ in (3.7), yielding

$$\begin{aligned} 0 = {} & \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{S^*(t)} c_e(x,t)\,\mathrm{d}x + (\sigma^2 - 2r + q)\int_0^{S^*(t)} c_e(x,t)\,\mathrm{d}x - c_e(S^*(t),t)\dot{S}^*(t) \\ & + \left[\frac{\sigma^2}{2} S^*(t)^2 \frac{\partial c_e}{\partial x}(S^*(t),t) - (\sigma^2 - r + q)S^*(t)c_e(S^*(t),t)\right]. \end{aligned} \tag{4.9}$$

Subtracting (4.9) from (4.8), we have

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{S^*(t)} [c_a(x,t) - c_e(x,t)]\,\mathrm{d}x + (\sigma^2 - 2r + q)\int_0^{S^*(t)} [c_a(x,t) - c_e(x,t)]\,\mathrm{d}x \\ & \quad = [c_a(S^*(t),t) - c_e(S^*(t),t)]\dot{S}^*(t) \\ & \qquad - \frac{\sigma^2}{2} S^*(t)^2 \left[\frac{\partial c_a}{\partial x}(S^*(t),t) - \frac{\partial c_e}{\partial x}(S^*(t),t)\right] \\ & \qquad + (\sigma^2 - r + q)S^*(t)[c_a(S^*(t),t) - c_e(S^*(t),t)]. \end{aligned}$$

Analogous to the put case, we make the simplifying assumption that

$$\int_0^{S^*(t)} c_a(x,t)\,\mathrm{d}x \approx \int_0^{S^*(t)} c_e(x,t)\,\mathrm{d}x$$

for all $0 \leqslant t < T$ (i.e. the ‘average’ values of c_a and c_e over $[0, S^*(t)]$ are approximately equal for all $0 \leqslant t < T$). In addition, similar to the put case, it can be verified numerically that

$$\lim_{x \to 0^+} \frac{c_a(x,t)}{c_e(x,t)} = 1$$

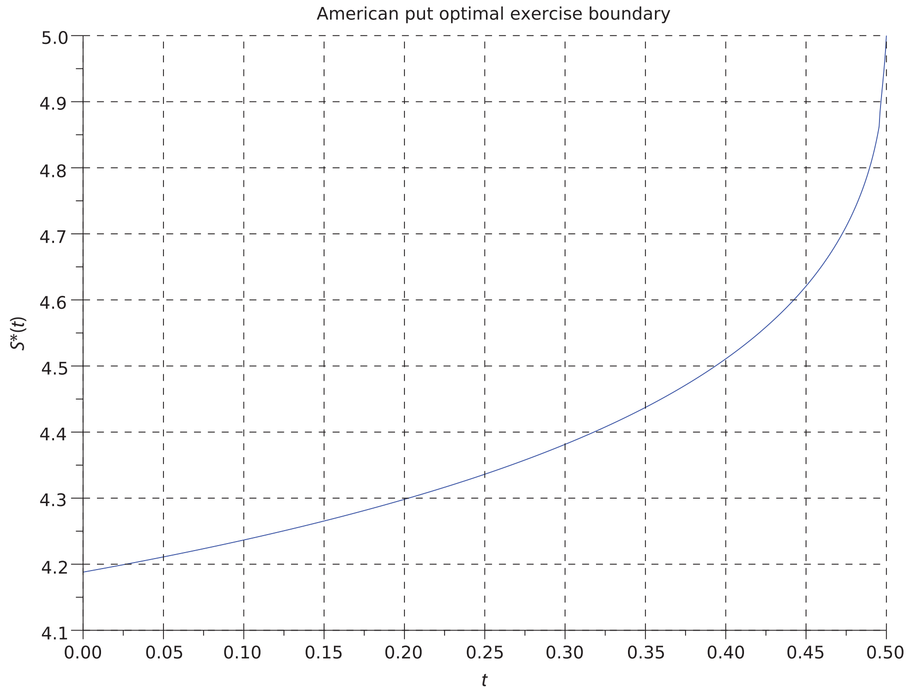

FIGURE 1. (Colour online) American put optimal exercise boundary, where $r = 0.03$, $q = 0.02$, $\sigma = 0.2$, $E = 5$ and $T = 0.5$.

for all $0 \leqslant t < T$. Using the boundary conditions in (4.1c), an approximate ODE for the American call optimal exercise boundary is therefore

$$\dot{S}^*(t) = \frac{\sigma^2}{2} S^*(t)^2 \frac{(\partial c_e/\partial x)(S^*(t), t) - 1}{c_e(S^*(t), t) - S^*(t) + E} - (\sigma^2 - r + q)S^*(t). \tag{4.10}$$

Summarizing, our valuation formulas are given by the expressions (4.6) and (4.7) for the American and European call option prices, respectively, together with the approximate ODE (4.10) for the American call optimal exercise boundary.

5 Numerical simulations

In this section we compare the option values obtained from our valuation formulas and those from the binomial method.

First we consider the American put. Suppose that

$$r = 0.03, \quad q = 0.02, \quad \sigma = 0.2, \quad E = 5, \quad T = 0.5.$$

It is known [19] that when $r > q$, then $S^*(T) = E$. We solve (3.11) with $S^*(T) = E$ backwards in time until $t = 0$, discretizing the ODE and applying Newton's method to solve for the optimal exercise prices at each discretized time value. The resulting American put optimal exercise boundary is shown in Figure 1.

Table 1. *Comparison of American put option prices using* (3.5), (3.6) *and* (3.11) *with the binomial method, where* $r = 0.03$, $q = 0.02$, $\sigma = 0.2$, $E = 5$ *and* $T = 0.5$

$x = S_0$	Binomial	(3.5), (3.6), (3.11)	Error
1.25	3.75	3.75	8.882×10^{-16}
2.5	2.5	2.5	4.085×10^{-8}
3.75	1.25	1.253731	0.003731
5	0.268481	0.269903	0.0014215
6.25	0.017602	0.017672	0.000071
7.5	0.000457	0.000460	0.000003
8.75	0.000007	0.000007	0.000000
10	7.055×10^{-8}	7.403×10^{-8}	3.489×10^{-9}

Using the optimal exercise boundary function just obtained, we look at the value of $S^*(0)$ and evaluate the formulas in (3.5) and (3.6) at $t = 0$ for different values of x (corresponding to different asset prices S_0 at $t = 0$). In particular, we choose $x = S_0$ equal to

$$\frac{E}{4}, \quad \frac{E}{2}, \quad \frac{3E}{4}, \quad E, \quad \frac{5E}{4}, \quad \frac{3E}{2}, \quad \frac{7E}{4}, \quad 2E.$$

For comparison, we also calculate the corresponding American put option prices using the binomial method as well as the absolute values of the differences between the computed prices. We summarize the results in Table 1.

Let us now look at the American call. Suppose that

$$r = 0.02, \quad q = 0.03, \quad \sigma = 0.2, \quad E = 5, \quad T = 0.5.$$

It is known [19] that when $r < q$, then $S^*(T) = E$. Numerically solving (4.10) with $S^*(T) = E$ using the same method for the American put, we obtain the American call optimal exercise boundary shown in Figure 2. Evaluating the formulas in (4.6) and (4.7) at $t = 0$ and at the same values of $x = S_0$ above, and also comparing the prices using the binomial method, we have the results in Table 2.

Thus, numerical simulations show that the option prices calculated using the analytical formulas and the binomial method yield very good agreement. We remark that the computational times for both methods are comparable.

6 Concluding remarks

In this paper we revisited the American put and call option valuation problems. Using a unified approach based on the Mellin transform for both types of options, we derived analytical formulas for the American put and call option prices as well as approximate first-order ODEs for the respective optimal exercise boundaries. Although these ODEs are analytically intractable, their numerical solution is straightforward unlike other approaches that rely on the numerical solution of integral equations for optimal exercise boundary. The numerical results indicate a very high accuracy and fast computational speed when benchmarked with the binomial method.

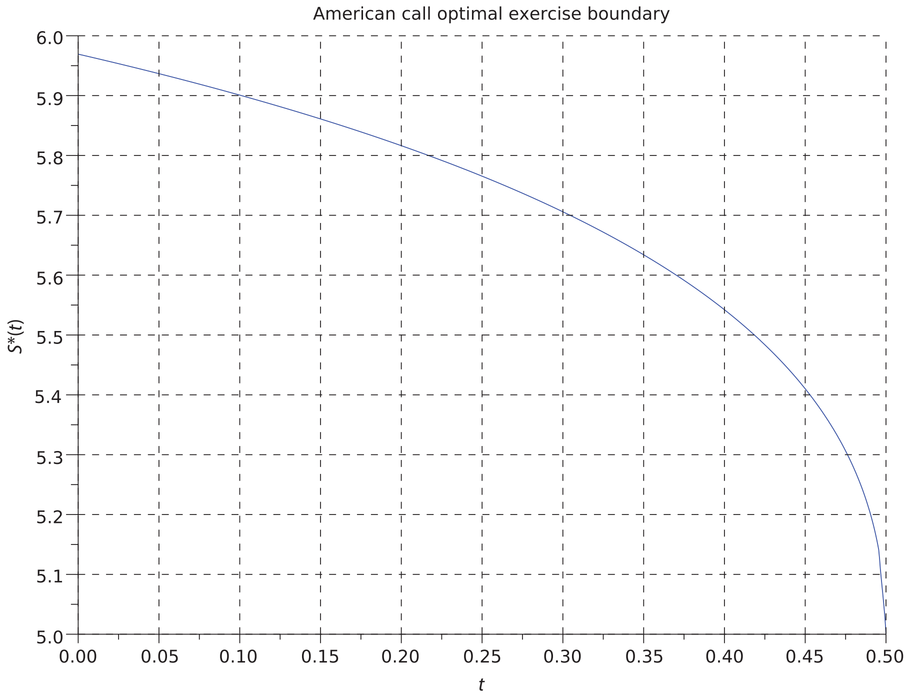

FIGURE 2. (Colour online) American call optimal exercise boundary, where $r = 0.02$, $q = 0.03$, $\sigma = 0.2$, $E = 5$ and $T = 0.5$.

Table 2. *Comparison of American call option prices using* (4.6), (4.7) *and* (4.10) *with the binomial method, where* $r = 0.02$, $q = 0.03$, $\sigma = 0.2$, $E = 5$ *and* $T = 0.5$

$x = S_0$	Binomial	(4.6), (4.7), (4.10)	Error
1.25	5.742×10^{-25}	0	5.742×10^{-25}
2.5	3.527×10^{-8}	3.702×10^{-8}	1.744×10^{-9}
3.75	0.004251	0.004268	0.000018
5	0.26841	0.269903	0.001422
6.25	1.253438	1.264891	0.011453
7.5	2.5	2.500431	0.000431
8.75	3.75	3.750007	0.000007
10	5	5.000000	8.175×10^{-8}

We derived the American call option price by using the ansatz (4.4), which we can rewrite as

$$c_a(x,t) - w(x,t) = x - E, \tag{6.1}$$

where w satisfies a similar PDE and terminal condition as the American put, but not the boundary conditions. Analogously, the put-call parity is

$$c_e(x,t) - p_e(x,t) = xe^{-q(T-t)} - Ee^{-r(T-t)}.$$

Comparing these two relations, we see that (6.1) can be viewed as a *quasi put-call parity* for American options since w is 'almost' like the American put. Indeed, we see that

$$xe^{-q(T-t)} - E \leqslant c_a(x,t) - w(x,t) = x - E \leqslant x - Ee^{-r(T-t)}.$$

On the other hand, it is known that

$$xe^{-q(T-t)} - E \leqslant c_a(x,t) - p_a(x,t) \leqslant x - Ee^{-r(T-t)},$$

which is sometimes referred as the *put-call parity for American options*. Due to the above similarities, the term 'quasi put-call parity' seems appropriate.

Although we considered only American vanilla options here, the Mellin-transform-based technique that we used is potentially applicable to other American-style derivatives where the underlying asset is lognormally distributed. In such case, the resulting PDE is of Black–Scholes type and the Mellin transform can be applied directly without introducing a series of variable transformations. The properties of the Black–Scholes kernel that we gave here are particularly useful. Furthermore, the method of deriving an (approximate) ODE for the free boundary can be straightforwardly adapted to these more general valuation problems. Other applications of the Mellin transform to option pricing include options with underlying jump diffusion processes, barrier options and arithmetic Asian options (with or without American-style features). These works are currently in progress.

Acknowledgement

The author wishes to thank Dr. Rogemar Mamon for stimulating discussions.

References

[1] Barone-Adesi, G. & Whaley, R. (1987) Efficient analytic approximation of American option values. *J. Fin.* **17**, 91–111.

[2] Brennan, M. & Schwartz, E. (1977) The valuation of American put options. *J. Fin.* **32**, 449–462.

[3] Brennan, M. & Schwartz, E. (1978) Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. *J. Financ. Quant. Anal.* **13**, 461–474.

[4] Carr, P., Jarrow, R. & Myneni R. (1992) Alternative characterizations of American put options. *Math. Fin.* **2**, 87–106.

[5] Chiarella, C. & Ziogas, A. (2006) A Fourier transform analysis of the American call option on assets driven by jump-diffusion processes. URL: `http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1106029`. Accessed on 3 August 2013.

[6] Chiarella, C. & Ziogas, A. (2009) American call options under jump-diffusion processes – a Fourier transform approach. *Appl. Math. Fin.* **16**, 37–79.

[7] Company, R., González, A. L. & Jódar, L. (2006) Numerical solution of modified Black–Scholes equation pricing stock options with discrete dividend. *Math. Comput. Modelling* **44**, 1058–1068.

[8] Company, R., Jódar, L., Rubio, G. & Villanueva, R. J. (2007) Explicit solution of Black–Scholes option pricing mathematical models with an impulsive payoff function. *Math. Comput. Modelling* **45**, 80–92.

[9] Cox, J., Ross, S. & Rubinstein, M. (1979) Option pricing: A simplified approach. *J. Financ. Econ.* **7**, 229–263.

[10] Detemple, J. (2005) *American-Style Derivatives: Valuation and Computation*, CRC Financial Mathematical Series, Chapman and Hall, New York, NY.

[11] Dewynne, J., Howison, S., Rupf, I. & Wilmott, P. (1993) Some mathematical results in the pricing of American options. *Eur. J. Appl. Math.* **4**, 381–398.

[12] Geske, R. & Johnson, H. (1984) The American put options valued analytically. *J. Fin.* 39, 1511–1524.

[13] Huang, J., Subrahmanyam, M. & Yu, G. (1996) Pricing and hedging American options: A recursive integration method. *Rev. Financ. Stud.* **9**, 277–330.

[14] Jacka, S. (1991) Optimal stopping and the American put. *Math. Fin.* **1**, 1–14.

[15] Jódar, L., Sevilla-Perris, J., Cortés, J. C. & Sala, R. (2005) A new direct method for solving the Black–Scholes equation. *Appl. Math. Lett.* **18**, 29–32.

[16] Ju, N. (1998) Pricing an American option by approximating its early exercise boundary as a multipiece exponential function. *Rev. Financ. Stud.* **11**, 627–646.

[17] Karatzas, I. (1988) On the pricing of American options. *Appl. Math. Optim.* **17**, 37–60.

[18] Kim, I. (1990) The analytical valuation of American puts. *Rev. Financ. Stud.* **3**, 547–572.

[19] Kwok, Y. (1998) *Mathematical Models of Financial Derivatives*, Springer Finance, Singapore.

[20] Longstaff, F. & Schwartz, E. (2001) Valuing American options by simulation: A simple least-squares approach. *Rev. Financ. Stud.* **14**, 113–147.

[21] MacMillan, L. (1986) An analytical approximation for the American put price. *Adv. Futures Options Res.* **1**, 119–139.

[22] McKean, H. (1965) Appendix: A free boundary problem for the heat equation arising from a problem of mathematical economcs. *Ind. Manage. Rev.* **6**, 32–39.

[23] Myint-U, T. & Debnath, L. (1987) *Partial Differential Equations for Scientists and Engineers*, North Holland, Netherlands.

[24] Panini, R. & Srivastav, R. (2004) Option pricing with Mellin transforms. *Math. Comput. Modelling* **40**, 43–56.

[25] Press, W., Teukolsky, S., Vetterling, W. & Flannery, B. (1992) *Numerical Recipes in C: The Art of Scientific Computing*, Cambridge University Press, Cambridge, UK.

[26] Rodrigo, M. R. & Mamon, R. S. (2007) An application of Mellin transform techniques to a Black–Scholes equation problem. *Anal. Appl.* **5**, 51–66.

[27] Schwartz, E. (1977) The valuation of warrants: Implementing a new approach. *J. Financ. Econ.* **4**, 79–93.

[28] Sevcovic, D. (2001) Analysis of the free boundary for the pricing of an American call option. *Eur. J. Appl. Math.* **12**, 25–37.

[29] Tilley, J. (1993) Valuing American options in a path simulation model. *Transact. Soc. Actuaries* **45**, 83–104.

[30] Zhu, S.-P. (2011) On various quantitative approaches for pricing American options. *New Math. Nat. Comput.* **7**, 313–332.

[31] Zwillinger, D. (1989) *Handbook of Differential Equations*, Academic Press, Philadelphia PA.