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Keywords

spectral, collocation, method, for, pricing, perpetual, American, puts, stochastic, volatility

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A spectral-collocation method for pricing perpetual American puts with stochastic volatility

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Based on the Legendre pseudospectral method, we propose a numerical treatment for pricing perpetual American put option with stochastic volatility. In this simple approach, a nonlinear algebraic equation system is first derived, and then solved by the Gauss-Newton algorithm. The convergence of the current scheme is ensured by constructing a test example similar to the original problem, and comparing the numerical option prices with those produced by the classical Projected SOR (PSOR) method. The results of our numerical experiments suggest that the proposed scheme is both accurate and efficient, since the spectral accuracy can be easily achieved within a small number of iterations. Moreover, based on the numerical results, we also discuss the impact of stochastic volatility term on the prices of perpetual American puts.

AMS(MOS) subject classification.

Keywords. Spectral-collocation method, Perpetual American put options, Heston model.

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1 Introduction

Perpetual American options are options that can be exercised at any time without a definitive expiry date. This simple contract can be viewed as an approximation for long-dated American options. Also, analyzing such kind of options may in principle be used as a building block in an approximation procedure for American options with finite maturities [4].

The availability of a closed-form solution of perpetual American option under the BS (Black-Scholes) framework has already been achieved [13]. Empirical evidence, however, suggests that the BS model is inadequate to describe asset returns and the behavior of the option markets [1]. The literature advocates the introduction of stochastic volatility to reproduce the implied volatility smile observed in markets. Among those SV (stochastic volatility) models (e.g. [2, 9–11]), the one proposed by Heston [10] has received the most attention, primarily due to its great analytical traceability for European options, and we shall thus focus on this model throughout the paper. Since the proposed scheme is quite general, as far as pricing perpetual American options is concerned, the extension of the current approach to other SV models should be straightforward.

For perpetual American options under the Heston model, there is no analytical solution, primarily due to the fact that the optimal exercise price now remains unknown as a function of volatility, rather than a constant as in the BS case. In other words, the introduction of a second stochastic process has considerably complicated the solution process in pricing perpetual American options.

In the literature, several numerical approaches were introduced to solve the free boundary problem associated with the valuation of American options under the Heston model (e.g. [5, 18]), but they all concentrated on options with finite maturities. It should be pointed out that all these approaches are not suitable for pricing perpetual American options, since large time evolution is required to approximate the infinite maturity, resulting in both low accuracy and computational inefficiency. On the other hand, although the Projected SOR (PSOR) method can be adopted to solve directly the LCP (linear complementary problem) associated with the current valuation problem, the accuracy and the efficiency of this classical approach are still not satisfying. To date, no documented numerical schemes have been proposed to address this issue; an efficient numerical approach for the valuation of perpetual American options under the Heston model is the aim of this paper.

In this paper, a numerical approach based on the spectral-collocation (SC) method is proposed for the valuation of perpetual American puts under the Heston model. This approach consists of two steps. The first step is to derive a system of nonlinear algebraic equations by using the SC method. The second step is to transform the nonlinear system obtained in the first step into a nonlinear least-square problem (NLSP), and solve it with the Gauss-Newton algorithm. To make sure that our numerical solution converges to the correct one, a test example similar to the original problem is constructed first, and then, the numerical results of the option price are compared with those produced by the PSOR method. Our numerical experiments show that this approach is both accurate and efficient, since a desired spectral accuracy can be easily achieved with a small number of iterations.

The paper is organized as follows. In Section 2, we introduce the partial differential equation (PDE) system that the price of a perpetual American put option must satisfy under the Heston model. In Section 3, we introduce our numerical approach in details. In Section 4, numerical results and some useful discussions are presented. Concluding remarks are given in the last section.

2 Perpetual American puts under the Heston model

In the Heston model [10], the underlying S_t, as a function of time, is assumed to follow the SDE (stochastic differential equation) of a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dw_1, \tag{2.1}$$

where μ is the drift rate, w_1 is a standard Brownian motion, and $\sqrt{v_t}$ is the standard deviation of the stock returns $\dfrac{dS_t}{S_t}$. Furthermore, the variance v_t (the square of the volatility) is assumed to be another stochastic process described by the following mean-reverting SDE:

$$dv_t = \kappa(\eta - v_t)dt + \sigma\sqrt{v_t}dw_2. \tag{2.2}$$

Here, η is the long-term mean of v_t, κ is the rate of relaxation to this mean, and σ is volatility of volatility. w_2 is also a standard Brownian motion, and is related to w_1 with a correlation factor $\rho \in [-1, 1]$. Eq. (2.2) is known in financial literature as the Cox-Ingersoll-Ross (CIR) process and in mathematical statistics as the Feller process [8, 9]. Various studies [3, 16] suggest that the volatility observed in the real market does indeed exhibit the mean-reverting characteristic described by Eq. (2.2).

As pointed out previously, perpetual American option is American option with infinite expiration. Let $U(v, S)$ denote the value of a perpetual American put option, with S being the underlying and v being the variance. Then, under the proposed processes (2.1)-(2.2), it can be easily shown that the valuation problem of a perpetual American put option can be formulated as a free boundary problem [15], in which the boundary location itself is part of the solution of the problem. In particular, $U(v, S)$ should satisfy

$$\begin{cases} \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma vS\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} + \kappa(\eta - v)\frac{\partial U}{\partial v} - rU = 0, \\ \lim_{S\to\infty} U(v, S) = 0, \\ U(v, S_f(v)) = K - S_f(v), \\ \frac{\partial U}{\partial S}\Big(v, S_f(v)\Big) = -1, \\ \lim_{v\to 0} U(v, S) = \max(K - S, 0), \\ \lim_{v\to\infty} U(v, S) = K. \end{cases} \tag{2.3}$$

One should notice that, once the stochastic volatility is taken into consideration, the valuation of the perpetual American puts is no longer as analytical achievable as the constant volatility case, because the optimal exercise price now remains unknown as a function of the volatility, while in the BS model, it is only an unknown constant. However, due to the time-independence, this pricing problem is somehow simplified, comparing with the valuation of American puts with finite maturities.

3 Numerical scheme based on the Legendre pseudospectral method

For a time-independent problem, such as the problem of pricing perpetual American options, a common approach is to start the problem as a time-dependent one and take the solution at the large time as the solution for the corresponding time-independent problem. However, a deficiency for this approach is that large time evolution is required to approximate the infinite maturity, resulting in low computational efficiency and accuracy. In this section, a new numerical scheme based on the Legendre pseudospectral method is introduced to solve the pricing of perpetual American puts efficiently and accurately.

In order to apply our new numerical scheme, we first adopt the Landau transform [12], i.e.,

$$x = \ln \frac{S}{S_f} \tag{3.4}$$

to convert the free boundary conditions to fixed boundary conditions. Furthermore, since the optimal exercise price is related to the option price by the conditions across the free boundary, we write it as

$$S_f(v) = K - U(v, S_f). \tag{3.5}$$

By substituting (3.4)-(3.5) into (2.3), we obtain

$$\begin{cases} \mathbb{L}U = 0, \\ \lim\limits_{x \to \infty} U(v, x) = 0, \\ U(v, 0) = K + \dfrac{\partial U}{\partial x}(v, 0), \\ \lim\limits_{v \to 0} U(v, x) = \max[K - e^x(K - U(v, 0)), 0], \\ \lim\limits_{v \to \infty} \dfrac{\partial U}{\partial v}(v, x) = 0, \end{cases} \tag{3.6}$$

where

$$\mathbb{L} = a(v)\frac{\partial^2}{\partial x^2} + b(v)\frac{\partial^2}{\partial v^2} + c(v)\frac{\partial^2}{\partial x \partial v} + d(v)\frac{\partial}{\partial x} + e(v)\frac{\partial}{\partial v} - r, \tag{3.7}$$

and

$$\begin{aligned} a(v) &= \frac{1}{2}v + \frac{1}{2}\sigma^2\xi^2 v - \rho\sigma v\xi, \qquad b(v) = \frac{1}{2}\sigma^2 v, \qquad c(v) = \rho\sigma v - \sigma^2 v\xi, \\ d(v) &= -\frac{1}{2}v + \frac{1}{2}\xi^2\sigma^2 v - \frac{1}{2}\sigma^2 v\beta + r - \kappa(\eta - v)\xi, \qquad e(v) = \kappa(\eta - v), \end{aligned}$$

$$\xi = \frac{1}{U(v, 0) - K}\frac{\partial U}{\partial v}(v, 0), \qquad \beta = \frac{1}{U(v, 0) - K}\frac{\partial^2 U}{\partial v^2}(v, 0).$$

It can be clearly seen that after the Landau transform is applied, the nonlinear feature of the problem is explicitly exposed in the governing equation.

On the other hand, it can be observed that the option pricing problem is defined on an unbounded domain

$$\Big\{(v,x)|v \geq 0, x \geq 0\Big\}.$$

To implement a calculation in a computer, we truncate the semi-infinite domain into a finite domain:

$$\Big\{(v,x) \in [0, v_{\max}] \times [0, x_{\max}]\Big\}.$$

Theoretically, $x_{\max}$ and $v_{\max}$ should be sufficiently large to eliminate the boundary effect. However, based on Willmott *et al.*'s estimate [17] that the upper bound of the asset price $S_{\max}$ is typically three or four times of the strike price, it is reasonable for us to set $x_{\max} = \ln 5$. On the other hand, the volatility value is usually very small. The highest value of the volatility that has ever been recorded on Chicago Board Options Exchange (CBOE) is only about 0.85 [19]. Thus, it is quite reasonable to set $v_{\max} = 1$.

Now, with the truncated computational domain in hand, it is enough for us to introduce our new approach for solving (3.6). The most crucial step in the implementation of the current scheme is to derive the differential matrices (cf. [14]) for PDE system (3.6). Suppose the spectral approximation solution of (3.6) can be written as

$$U(v,x) = \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} U(v_i, x_j)F_i(v)F_j(x),$$

where $F_k(x)$ is the $k-$th Lagrange basis function, and (v_i, x_j) are $N+1$ collocation points. Based on the definition of $F_j(v)$, it is quite straightforward to show that

$$\begin{aligned}\frac{\partial U}{\partial x}(v_l, x_k) &= \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} U(v_i, x_j)F_i(v_l)F_j'(x_k)\\ &= \sum_{j=1}^{N+1} U(v_l, x_j)F_j'(x_k),\end{aligned}$$

which is equivalent to $\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} = \mathbf{U}\mathbf{D}_{\mathbf{X}}^{\mathbf{T}}$, where $\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}}$, $\mathbf{U}$ and $\mathbf{D}_{\mathbf{X}}^{\mathbf{T}}$ are defined as:

$$\frac{\partial \mathbf{U}}{\partial \mathbf{x}} = (\frac{\partial U}{\partial x}(v_i, x_j))_{N+1,N+1}, \quad \mathbf{U} = (U(v_i, x_j))_{N+1,N+1}, \quad \mathbf{D}_{\mathbf{X}} = (F_j'(x_i))_{N+1,N+1}.$$

Similarly, we obtain

$$\frac{\partial \mathbf{U}}{\partial \mathbf{v}} = \mathbf{D}_{\mathbf{v}}\mathbf{U}, \quad \frac{\partial^2 \mathbf{U}}{\partial \mathbf{x}^2} = \mathbf{U}(\mathbf{D}_{\mathbf{X}}^2)^{\mathbf{T}}, \quad \frac{\partial^2 \mathbf{U}}{\partial \mathbf{v}^2} = \mathbf{D}_{\mathbf{v}}^2\mathbf{U}, \quad \frac{\partial^2 \mathbf{U}}{\partial \mathbf{x}\partial \mathbf{v}} = \mathbf{D}_{\mathbf{v}}\mathbf{U}\mathbf{D}_{\mathbf{x}}^{\mathbf{T}}. \tag{3.8}$$

On the other hand, in order to obtain the numerical values of the differential matrices without much effort, we should choose some proper collocation points. It can be shown that if we use the so-called Legendre-Gauss-Lobatto points $\{s_i\}_{i=1}^{N+1}$, the corresponding differential matrix $\mathbf{D_s}$ has the following structure [14]:

$$\begin{aligned}
&(\mathbf{D_s})_{i,j} = \frac{L_N(s_i)}{L_N(s_j)}\frac{1}{s_i - s_j}, \quad i \neq j, \quad i, j = 2 \cdots N, \\
&(\mathbf{D_s})_{i,i} = 0, \quad i \neq 1, i \neq N+1, \\
&(\mathbf{D_s})_{1,1} = -(\mathbf{D_s})_{N+1,N+1} = \frac{N(N+1)}{4}.
\end{aligned}$$

Since $\{s_i\}_{i=1}^{N+1} \subset [-1, 1]$, the following coordinate transformation should be applied, i.e.,

$$v_i = \frac{(s_i + 1)v_{max}}{2}, \quad x_j = \frac{(s_j + 1)x_{max}}{2},$$

which yields

$$\mathbf{D_v} = \frac{2}{v_{max}}\mathbf{D_s}, \quad \mathbf{D_x} = \frac{2}{x_{max}}\mathbf{D_s}.$$

By substituting (3.8) into (3.6), we obtain the nonlinear algebraic equations for the unknown matrix $\mathbf{U}$ as

$$\begin{cases} a_i(\mathbf{U}\mathbf{D}_{\mathbf{x}}^{2,\mathbf{T}})_{i,j} + b_i(\mathbf{D}_{\mathbf{v}}^2\mathbf{U})_{i,j} + c_i(\mathbf{D}_{\mathbf{v}}\mathbf{U}\mathbf{D}_{\mathbf{x}}^T)_{i,j} + d_i(\mathbf{U}\mathbf{D}_{\mathbf{x}}^T)_{i,j} + e_i(\mathbf{D}_{\mathbf{v}}\mathbf{U})_{i,j} - r(\mathbf{U})_{i,j} = 0, \\ \qquad\qquad i,j = 2\cdots N, \\ (\mathbf{U})_{1,j} = 0, \quad j = 2\cdots N, \\ (\mathbf{D}_{\mathbf{v}}\mathbf{U})_{N+1,j} = 0, \quad j = 2\cdots N, \\ (\mathbf{U})_{i,1} = K + (\mathbf{U}\mathbf{D}_{\mathbf{x}}^T)_{i,1}, \quad i = 2\cdots N, \\ (\mathbf{U})_{i,N+1} = 0, \quad i = 2\cdots N. \end{cases}$$

The above system can be solved efficiently with the utilization of the Gauss-Newton algorithm. The iterative process is as follows:
(i) If $\mathbf{U}^{(k)}$ is obtained after the k-th iterative step, we can compute $\mathbf{f}(\mathbf{U}^{(k)})$, and the corresponding Jacobian $\mathbf{J}_{\mathbf{f}}(\mathbf{U}^k)$, where $\mathbf{f}(\mathbf{U})$ is a vector of "residuals". Specifically,

$$\begin{aligned} \mathbf{f}_{(i-1)(N+1)+j} &= a_i(\mathbf{U}\mathbf{D}_{\mathbf{x}}^{2,\mathbf{T}})_{i,j} + b_i(\mathbf{D}_{\mathbf{v}}^2\mathbf{U})_{i,j} + c_i(\mathbf{D}_{\mathbf{v}}\mathbf{U}\mathbf{D}_{\mathbf{x}}^T)_{i,j} + d_i(\mathbf{U}\mathbf{D}_{\mathbf{x}}^T)_{i,j} \\ &\quad + e_i(\mathbf{D}_{\mathbf{v}}\mathbf{U})_{i,j} - r(\mathbf{U})_{i,j}, \quad i,j = 2\cdots N, \\ \mathbf{f}_j &= (\mathbf{U})_{1,j}, \quad j = 1\cdots N+1, \\ \mathbf{f}_j &= (\mathbf{D}_{\mathbf{v}}\mathbf{U})_{N+1,k}, \quad j = N(N+1)\cdots(N+1)^2, k = 1\cdots(N+1), \\ \mathbf{f}_{(i-1)(N+1)+1} &= (\mathbf{U})_{i,1} - K - (\mathbf{U}\mathbf{D}_{\mathbf{x}}^T)_{i,1}, \quad i = 2\cdots N, \\ \mathbf{f}_{(i-1)(N+1)+N+1} &= (\mathbf{U})_{i,N+1}, \quad i = 2\cdots N. \end{aligned}$$

(ii) Linearize $\mathbf{f}$ with the current value $\mathbf{U}^{(k)}$, i.e.,

$$\begin{aligned} \mathbf{f}(\mathbf{U}) &\approx \mathbf{f}(\mathbf{U}^{(k)}) + \mathbf{J}_{\mathbf{f}}(\mathbf{U}^{(k)})(\mathbf{U} - \mathbf{U}^{(k)}), \\ &= \mathbf{A}^{(k)}\mathbf{U} - \mathbf{b}^{(k)}, \end{aligned}$$

where $\mathbf{A}^{(k)} = \mathbf{J_f}(\mathbf{U}^{(k)})$ and $\mathbf{b}^{(k)} = \mathbf{J_f}(\mathbf{U}^{(k)})\mathbf{U}^{(k)} - \mathbf{f}(\mathbf{U}^{(k)})$.

(iii) Solve the following linear-square problem:

$$\| \mathbf{f}(\mathbf{U}) \| = \| \mathbf{A}^{(k)}\mathbf{U} - \mathbf{b}^{(k)} \|^2,$$

and obtain $\mathbf{U}^{(k+1)} = (\mathbf{A}^{(k),\mathbf{T}}\mathbf{A}^{(k)})^{-1}\mathbf{A}^{(k),\mathbf{T}}\mathbf{b}^{(k)}$.

(iv) Repeat (i)-(iii) until $\| U^{(k+1)} - U^{(k)} \| < \epsilon$ is satisfied. The tolerance ϵ is set to 10^{-6} for all the results presented in this paper.

One should notice that in the above Gauss-Newton method, it is quite important to choose a proper initial guess of $\mathbf{U}$, i.e., $\mathbf{U}^{(0)}$, since the algorithm may converge slowly or not at all if the initial guess is too far away from the final solution. For most of the practical parameter-settings, we recommend to use the analytical formula for perpetual American puts under the BS model, with the corresponding variance setting to v, as a good initial guess. This is because, firstly, the formula satisfies all the boundary conditions automatically, and secondly, it should be close to the final solution of (3.6), since with the same parameter-settings, the option prices under the two different models should not differ too much.

4 Numerical results and discussions

In this section, we shall present the numerical results as well as some useful discussions. The section is organized into three subsections, according to three important issues that should be addressed.

4.1 A test example

As demonstrated earlier, no analytical solution for the case of perpetual vanilla American option under the Heston model has yet been derived. Thus, in order to illustrate the reliability of the current scheme, we should conduct a test example, for which, the analytical solution can be artificially constructed. The test example is as follows:

$$\begin{cases} \mathbb{L}U + rK = 0, \\ \lim\limits_{x \to x_{max}} U(v,x) = K + ve^{x_{max}}, \\ U(v,0) = K + \dfrac{\partial U}{\partial x}(v,0), \\ \lim\limits_{v \to 0} U(v,x) = K, \\ \lim\limits_{v \to v_{max}} U(v,x) = K + v_{max}e^{x}, \end{cases} \tag{4.9}$$

where $\mathbb{L}$ is the same operator as shown in (3.7). This test example can be viewed as the pricing of some kind of perpetual exotic American option under the Heston model, and therefore, it keeps the essential nonlinear feature of the original problem. On the other hand, the price of this option is equal to $U_{exact} = K + ve^{x}$.

Since this test example has almost the same structure as our original problem, so if the error of the numerical results of this example is reasonably small, we should have confidence that the proposed scheme is quite accurate in solving (3.6) as well.

Table 1: **The test example. Parameters are $\kappa = 2$, $\eta = 0.2$, $\sigma = 0.04$, $r = 0.5$, $\rho = 0.1$, $K = 10.0$.**

N	Error	Residual	Number of iterations
2	0.2236	3.7178e-8	3
3	0.0248	1.2669e-7	3
4	1.5e-3	3.2130e-8	4
5	1.8029e-4	6.2445e-14	5
6	1.5336e-5	9.7619e-14	5
7	1.0661e-6	5.0749e-13	5
8	6.2249e-8	9.4133e-13	5
9	2.9092e-9	1.2353e-12	5
10	9.1443e-10	7.4091e-12	5
11	2.3013e-9	8.5294e-12	5
12	9.1677e-10	8.8039e-12	6

Table 1 shows the numerical results of the test example with the initial guess $U_0 = 3 + K + v^2 x$. Here, N stands for the number of collocation points along each direction, and the error is defined as the maximum point-wise error. Furthermore, since the Gauss-Newton algorithm is adopted to solve the nonlinear algebraic equation system, we have also displayed the number of iterations and the final residuals measured in the L_2-norm.

From this table, it can be observed that a rapid convergence of our numerical solution to the exact solution can be easily achieved with a small number of iterations. Most remarkably, a desired spectral accuracy can be obtained even when very coarse grids are adopted. Therefore, the proposed scheme is both accurate and efficient in solving nonlinear problems, especially those with a structure similar to the test example, such as our problem for perpetual American puts with stochastic volatility.

4.2 SC scheme VS PSOR method

As mentioned earlier, the pricing of perpetual American puts under the Heston model can be also solved by the well-known PSOR method introduced in [7]. It is quite interesting to make a comparison of the two different approaches. In Table 2, we compared the option prices calculated by the PSOR method and the SC scheme. Furthermore, the corresponding number of iterations and the total CPU-time cost are also displayed. All the experiments here were performed within Matlab7.5 on an Intel Pentium 4, 3GHZ machine.

Table 2: **Comparison of the option prices at $v = 0.4$, calculated by the PSOR method and the SC method. Model parameters are $\kappa = 0.8$, $\eta = 0.45$, $\sigma = 0.4$, $r = 0.4$, $\rho = -0.1$, $K = \$10.0$. The iteration convergence tolerance for PSOR is 10^{-10}.**

	Grid number (N_v, N_x)	Asset values 8	9	10	11	No. of iterations	CPU -time(s)
PSOR	(25,50)	2.2982	1.8159	1.4764	1.2281	458	6.4
	(50,100)	2.3156	1.8368	1.4983	1.2497	1725	16.5
	(100,150)	2.3244	1.8476	1.5095	1.2604	2484	246.2
	(100,200)	2.3234	1.8470	1.5091	1.2602	6396	1127.8
SC	(15,15)	2.3299	1.8531	1.5128	1.2596	6	2.1
	(20,20)	2.3273	1.8500	1.5093	1.2560	7	4.1
	(25,25)	2.3280	1.8514	1.5111	1.2581	7	9.3
	(30,30)	2.3259	1.8485	1.5078	1.2546	8	20.9

It can be clearly seen from this table that for the finest grid, the option prices produced by the two numerical approaches agree well with each other, but with substantially different CPU-time consumption. Obviously, the CPU-times required by the PSOR method are significantly more than the SC approach. Furthermore, it can also be observed that the option prices produced by the SC method with the most coarse grid (the grid number

$(15, 15)$) are already very close to those computed by the PSOR method with the finest grid (the grid number $(100, 200)$), and yet the CPU time needed for the former is remarkably less than 3 seconds, a fraction of roughly 600th of the latter!

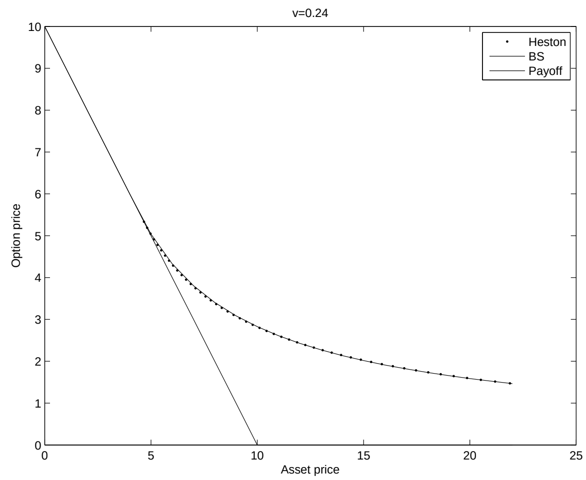
On the other hand, by calculating the option prices with different parameter settings, it is found that the convergence rate of the PSOR method deteriorates when the parameter κ increases, while for the SC scheme, it seems to be less parameter-dependent. Therefore, due to the high efficiency and accuracy, it is suggested that the SC scheme, superior to the PSOR method, be adopted to value perpetual American puts under the Heston model.

Before leaving this subsection, it should be remarked that the numerical solution produced by the SC method may be adopted as a benchmark for future studies. This is because the high order convergence of the current scheme is ensured by testing the constructed perpetual American exotic option, as shown in section 4.1. Furthermore, the convergence of our solution to the exact one has also been numerically guaranteed by the comparison with the PSOR method.

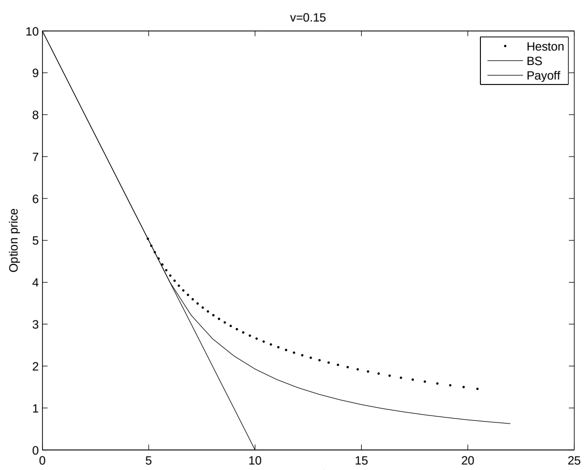
4.3 The impact of stochastic volatility

As mentioned earlier, the option prices of perpetual American puts under the BS model can be calculated with a simple and elegant formula [13]. With the current numerical scheme, it is enough for us to make a comparison of the pricing difference for two perpetual American put option contracts being otherwise identical except the volatility terms. Such a comparison is quite interesting, since it can give us a quantitative sense on the largest effect of the stochastic volatility on the price of American puts. This is because the impact on the pricing of an option from the stochastic volatility usually becomes progressively larger as the life of the option increases [11], and for perpetual case, the impact should undoubtedly be the most significant.

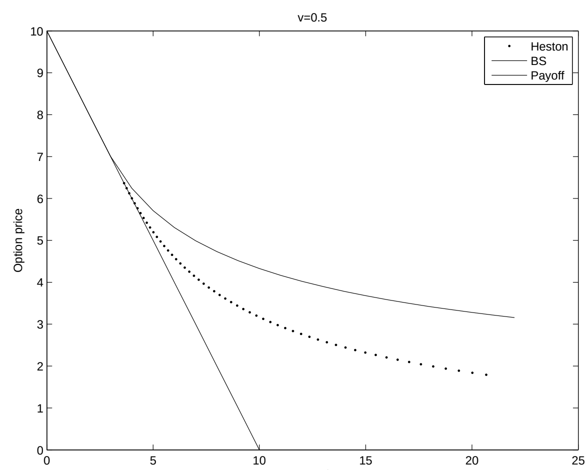
Plotted in Fig 2 are the comparison of the option prices under the two different models, with the variance rate under the BS model being v. It is interesting to notice that there is a special value of v, at which, the option prices under the two different models agree well with each other, as shown in Fig 1-a. Moreover, for v less than that of this "special" value, the prices calculated with the BS model are all lower than the corresponding ones calculated with the Heston model (Fig 1-b), while for v values larger than that point, the two sets of prices are reversed (Fig 1-c).



(a) $v = 0.24$



(b) $v = 0.15$



(c) $v = 0.5$

Figure 1: Comparison of the option prices under two different models. Model parameters are $r = 0.1$, $\sigma = 0.45$, $\rho = 0.1$, $\eta = 0.2$, $\kappa = 4$, $K = \$10.0$

In order to give a financially meaningful explanation for the pricing bias, causing by the stochastic volatility term, it is desirable to calculate a set of "special" v values first.

Displayed in Tables 3-5 are the values of "special" v with different parameter settings. Specifically, they are calculated by fixing the parameters (ρ, σ), (κ, σ) and (ρ, κ) respectively. Remarkably, it is quite reasonable for us to concentrate on the variation of the "special" v related to κ, σ, ρ and η only, because these four parameters are introduced once the stochastic volatility is taken into consideration, and it is believed that they can provide enough information on the stochastic volatility term.

Table 3: **"Special"** v **with** $\rho = 0.1$**,** $\sigma = 0.45$**,** $r = 0.1$**,** $K = \$10.0$

	$\eta = 0.2$	$\eta = 0.3$	$\eta = 0.4$
$\kappa = 1$	$v = 0.245$	$v = 0.335$	$v = 0.425$
$\kappa = 2$	$v = 0.245$	$v = 0.345$	$v = 0.425$
$\kappa = 3$	$v = 0.245$	$v = 0.320$	$v = 0.385$
$\kappa = 4$	$v = 0.245$	$v = 0.3$	$v = 0.385$

Table 4: **"Special"** v **with** $\kappa = 1$**,** $\sigma = 0.45$**,**$r = 0.1$**,** $K = \$10.0$

	$\eta = 0.2$	$\eta = 0.3$	$\eta = 0.4$
$\rho = \pm 0.1$	$v = 0.245$	$v = 0.335$	$v = 0.425$
$\rho = \pm 0.5$	$v = 0.245$	$v = 0.335$	$v = 0.425$

Table 5: **"Special"** v **with** $\rho = 0.1$**,** $\kappa = 1$**,** $r = 0.1$**,** $K = \$10.0$

	$\eta = 0.2$	$\eta = 0.3$	$\eta = 0.4$
$\sigma = 0.1$	$v = 0.230$	$v = 0.285$	$v = 0.395$
$\sigma = 0.2$	$v = 0.245$	$v = 0.315$	$v = 0.395$
$\sigma = 0.3$	$v = 0.245$	$v = 0.335$	$v = 0.395$
$\sigma = 0.4$	$v = 0.245$	$v = 0.345$	$v = 0.415$

From Tables 3-5, it is interesting to observe that the "special" v is approximately equal to the long-term mean (η) of the volatility process. Once η is fixed, the "special" v does not change significantly with respect to the changes of other parameters. This could be probably explained as follows.

For the mean-reverting process (2.2), it is reasonable to infer that as time goes to infinity, the variance v of the volatility should approach its long-term mean η asymptotically. Supposing that the spot variance $v_c < \eta$ ($v_c > \eta$), as time involves to infinity, it should overall increase (decrease) to η. For the BS model, it just ignored the growing (decreasing) tendency of the variance, resulting in option prices lower (higher) than the corresponding ones with stochastic volatility. Note that the option prices are monotonically increasing with respect to v. When $v_c \approx \eta$, the overall change of the variance in the long run is not significant, and thus the option prices under the two different models are almost the same. One should notice that, for the American puts with finite maturities, the above explanation is not true. This is because for finite maturity, the overall tendency of v depends on several factors, such as its correlation with the asset price, its long term mean and so on; one cannot simply determine which one is the dominant factor.

5 Conclusion

In this paper, we have considered a spectral-collocation method for the numerical pricing of perpetual American puts when Heston's stochastic volatility model is used. The option price can be obtained with two steps. The first step is to derive a system of nonlinear algebraic equations by means of the Legendre pseudospectral method, while the second step is to transform the above nonlinear system into a nonlinear least-square problem, which can be solved by the Gauss-Newton algorithm. Our numerical experiments suggest that the current approach is indeed fast and accurate in solving perpetual American puts with stochastic volatility. Moreover, based on the numerical results, a financially meaningful explanation for the effect of the stochastic volatility on the prices of perpetual American puts is also provided.

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