1-1-2005

On an infinite integral arising in the numerical integration of stochastic differential equations

David M. Stump

Australian National Bank, david_stump@national.com.au

James M. Hill

University of Wollongong, jhill@uow.edu.au

Follow this and additional works at: <https://ro.uow.edu.au/infopapers>

Part of the [Physical Sciences and Mathematics Commons](#)

Recommended Citation

Stump, David M. and Hill, James M.: On an infinite integral arising in the numerical integration of stochastic differential equations 2005, 397-413.
<https://ro.uow.edu.au/infopapers/1628>

On an infinite integral arising in the numerical integration of stochastic differential equations

Abstract

We study a stochastic integral that arises during the implementation of the Milstein method for the numerical integration of systems of stochastic differential equations. The distribution of the integral can be written as the inverse Fourier transform of a characteristic function with essential singularities. This leads to a generalized integral that can be expressed as an infinite series involving the derivatives of Meixner polynomials. The generating function of the polynomials in combination with the Mittag–Leffler expansion theorem is used to obtain a novel series representation for the integral and the motivating problem in particular. This new form is rapidly convergent and, therefore, well suited to numerical work.

Keywords

infinite, integral, arising, numerical, integration, stochastic, differential, equations

Disciplines

Physical Sciences and Mathematics

Publication Details

Hill, J. M. & Stump, D. (2005). On an infinite integral arising in the numerical integration of stochastic differential equations. *Proceedings of the Royal Society of London. Mathematical, Physical and Engineering Sciences*, 461 (2054), 397-413.

On an infinite integral arising in the numerical integration of stochastic differential equations

BY DAVID M. STUMP¹ AND JAMES M. HILL²

¹*Quantitative Support Group, National Australia Bank, 32/500 Bourke Street, Melbourne, Victoria 3000, Australia (david.stump@national.com.au)*

²*School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, Australia*

We study a stochastic integral that arises during the implementation of the Milstein method for the numerical integration of systems of stochastic differential equations. The distribution of the integral can be written as the inverse Fourier transform of a characteristic function with essential singularities. This leads to a generalized integral that can be expressed as an infinite series involving the derivatives of Meixner polynomials. The generating function of the polynomials in combination with the Mittag–Leffler expansion theorem is used to obtain a novel series representation for the integral and the motivating problem in particular. This new form is rapidly convergent and, therefore, well suited to numerical work.

Keywords: stochastic differential equations; Milstein integration scheme; Sheffer system; Meixner polynomials; Mittag–Leffler expansion

1. Introduction

Systems of stochastic differential equations arise in a variety of contexts in applied mathematics. One area of note is the finance industry, where stochastic volatility models are used to describe the long-time behaviour of equity prices. The integral that is the subject of this paper arises within the industrial context of the numerical integration of the Heston (1993) mean-reverting stochastic volatility model. Since the Heston model only serves to motivate this study, we include no additional details and only briefly mention it again. For a generalization of our integral, we derive a novel analytical expansion involving a system of orthogonal polynomials. This expression is subsequently exploited to deduce a representation that is particularly suited to numerical evaluation.

In order to explain the origin of our integral, we begin with some background on the numerical integration of systems of stochastic differential equations, which is a complicated subject and requires the use of specialized methods. We present only a brief discussion of some of the details of this topic. The interested reader should consult the treatise by Kloeden & Platen (1999) for a complete discussion (particularly in relation to the important topics of strong- and weak-order convergence).

Let t denote time, let $y_i(t)$ denote a set of unknown stochastic variables ($i = 1, \dots, N$), and let dW_j ($j = 1, \dots, M$) denote a set of independent Wiener processes.

A system of stochastic differential equations in these variables has the form

$$dy_i(t) = a_i(t, y_1, y_2, \dots, y_N) dt + \sum_{j=1}^M b_{ij}(t, y_1, y_2, \dots, y_N) dW_j, \quad (1.1)$$

where a_i and b_{ij} are, respectively, the vector of drifts and the matrix of volatilities. The dependence of the drifts and volatilities on the unknowns y_i is explicitly indicated in (1.1).

The simplest approach to the numerical integration of (1.1) is the Euler method, which for time-step Δt has the update equations

$$\Delta y_i = a_i \Delta t + \sum_{j=1}^M b_{ij} \Delta W_j, \quad (1.2)$$

where a_i and b_{ij} are evaluated at the start of the time interval. The Wiener process increments are given by $\Delta W_j = \phi_j \sqrt{\Delta t}$, where the ϕ_j are independent standard normal random variables. The Euler formulation is complete in only the $O(\sqrt{\Delta t})$ terms of the equation.

The next improvement in accuracy is provided by the Milstein method, which is obtained by appending to (1.2) the terms

$$\sum_{j=1}^N \sum_{l=1}^M \sum_{k=1}^M b_{jl} \frac{\partial b_{ik}}{\partial y_j} \mathcal{I}_{lk} \Delta t,$$

which also makes the system complete in $O(\Delta t)$ terms. Here, all quantities are evaluated at the start of the time interval, and the term $\mathcal{I}_{lk}$ is an $M \times M$ matrix of stochastic integrals of the form

$$\mathcal{I}_{lk} = \int_0^{\Delta t} W_l(t) dW_k(t). \quad (1.3)$$

Itô calculus is used to evaluate the diagonal terms, while Stratonovich calculus is used for the off-diagonal terms.

In terms of strong-order convergence, which relates to the pointwise approximation of the transitional probability density function, the Euler and Milstein methods are accurate, respectively, to $O(\sqrt{\Delta t})$ and $O(\Delta t)$, provided that the functions a_i and b_{ij} obey certain continuity and boundedness conditions. In terms of weak-order convergence, which applies when the resulting transitional probability density is used to take an expectation (the typical finance-industry use), both methods are accurate to $O(\Delta t)$, again provided that a_i and b_{ij} obey certain continuity and boundedness restrictions. It should also be noted that there are certain classes of functions a_i and b_{ij} which are exceptions to these rules when the Euler equations have the same accuracy as the Milstein formulation. However, these are special cases rather than the general situation. A complete discussion of the various requirements can be found in Kloeden & Platen (1999).

From an option-pricing standpoint, the convergence results would imply that the Milstein method offers no advantage over the Euler method. However, it is an empirical fact that the Milstein method typically performs better than the Euler method

in an option-pricing context because of the higher accuracy in the drift terms. This is particularly true for longer-dated options, where the integrated effect of the drift can be substantial. However, the Milstein method comes at a substantial computational cost since it is time-consuming to evaluate the off-diagonal terms in equation (1.3). Because the Milstein method can be hundreds of times slower than the Euler method, it is standard industrial practice to use the Euler method. Nevertheless, interest remains in trying to fashion rapid implementations of the Milstein method on account of the potential for significant improvement in accuracy. We leave this issue and return to the subject of the paper.

We are specifically concerned with the distribution of a single integral of the form (1.3) for $l \neq k$, which is the context within which this problem arises since the Heston equations comprise a set of two stochastic differential equations in two Wiener processes. We now set $l = 1$ and $k = 2$ for the remainder of the discussion. This Stratonovich integral can be evaluated by applying the Karhunen–Loève expansion theorem for different systems of basis functions in t . The random Fourier series approach is most frequently used and yields the formula

$$\mathcal{I}_{12} = \frac{1}{2}\phi_1\phi_2\Delta t + \frac{\Delta t}{2\pi}L,$$

where L is given by the series

$$L = \sum_{r=1}^{\infty} \frac{1}{r} \{ \zeta_{2,r}(\phi_1\sqrt{2} + \eta_{1,r}) - \zeta_{1,r}(\phi_2\sqrt{2} + \eta_{2,r}) \}, \quad (1.4)$$

with the subscripted ζ and η quantities being standard normal random variables. (Consult Kloeden & Platen (1999, p. 198) for a more complete discussion of the origin of this formula.) In the implementation of the Milstein method, equation (1.4) is typically truncated at $K = O(1)/\Delta t$ terms with the addition of a remainder term to account for the tail of the series. This produces an approximation that achieves strong-order $O(\Delta t)$ accuracy, but at a substantial cost due to the number of random variables that must be computed.

An attractive alternative would be to draw the random variable L directly from the cumulative distribution $F(L)$. This would substantially decrease the amount of effort needed to construct the realizations (1.4). However, the success of such an approach depends on having an analytical representation of $F(L)$, which can be obtained from the method of characteristic functions. This analysis is summarized in the appendix and leads to the characteristic function

$$\Phi_L(\lambda) = \frac{\pi\lambda}{\sinh \pi\lambda} \exp[\frac{1}{2}\tilde{\phi}^2(1 - \pi\lambda \coth \pi\lambda)], \quad (1.5)$$

with $\tilde{\phi}^2 = \phi_1^2 + \phi_2^2$. In the appendix it is also shown that

$$F(L) = \frac{1}{2} + \int_0^\infty \frac{\sin L\lambda}{\sinh \pi\lambda} \exp[\frac{1}{2}\tilde{\phi}^2(1 - \pi\lambda \coth \pi\lambda)] d\lambda. \quad (1.6)$$

Despite the analytical appearance of (1.6), the presence of essential singularities at the values $\lambda = \pm ni$, where $n = 1, 2, 3, \dots$, significantly complicates the analytical inversion of this integral. The expression is easily evaluated with standard numerical integration packages, but this provides no insight into the structure of the distribution.

This paper considers the analytical evaluation of the generalized form of equation (1.6) given by

$$I(\alpha, \beta, \gamma) = \int_0^\infty \frac{\sin \alpha x}{\sinh \beta x} \exp[-\gamma x \coth \beta x] dx, \quad (1.7)$$

where α , β and γ are assumed to be positive real constants. In this context we show that $I(\alpha, \beta, \gamma)$ admits the infinite series expansion

$$I(\alpha, \beta, \gamma) = \frac{\pi}{2\beta} \sum_{n=0}^{\infty} \frac{(-\gamma/\beta)^n}{(n!)^2} [p_n(\xi) \tanh(\pi\xi/2)]^{(n)}, \quad (1.8)$$

where ‘ (n) ’ denotes the n th derivative with respect to $\xi = \alpha/\beta$, and the $p_n(\xi)$ are a family of Meixner polynomials, the first few of which are given by

$$\begin{aligned} p_0(\xi) &= 1, & p_1(\xi) &= \xi, & p_2(\xi) &= \xi^2 - 1, \\ p_3(\xi) &= \xi(\xi^2 - 5), & p_4(\xi) &= \xi^4 - 14\xi^2 + 9, \\ p_5(\xi) &= \xi(\xi^4 - 30\xi^2 + 89), & p_6(\xi) &= \xi^6 - 55\xi^4 + 439\xi^2 - 225. \end{aligned}$$

These polynomials are shown to be obtained from the Sheffer system generating function

$$q(\xi, s) = \sum_{n=0}^{\infty} \frac{p_n(\xi)}{n!} s^n = \frac{\exp[\xi \tan^{-1} s]}{\sqrt{1+s^2}}, \quad (1.9)$$

which is used to manipulate (1.8) into the novel series form

$$I(\alpha, \beta, \gamma) = 2 \sum_{k=0}^{\infty} \operatorname{Re} \left\{ \frac{[\alpha + i(2k+1)\beta - i\gamma]^k}{[\alpha + i(2k+1)\beta + i\gamma]^{k+1}} \right\}. \quad (1.10)$$

This result appears to be a new representation that shows a previously unknown type of structure.

The paper is organized as follows. In § 2 we formulate the basic equations for the infinite series expansion of equation (1.7). In § 3 we show how the solution may be expressed in terms of the polynomials $p_n(\xi)$ and their derivatives. In § 4 we determine the generating function (1.9), and then in § 5 we use this and the Mittag-Leffler expansion theorem to transform equation (1.8) into the representation (1.10). In § 6 we develop some asymptotic representations of (1.7) for large ξ . Finally, in § 7 we compare our various analytical results with the numerical evaluation of several specific cases of (1.6). In § 8 we close with some remarks on the extension of this work.

2. Basic equations for the expansion

We consider the generalized form (1.7) of the integral expression in equation (1.6) and, since the exponential function can be expressed in a uniformly convergent series, we expand the integrand of (1.7) so that

$$I(\alpha, \beta, \gamma) = \sum_{n=1}^{\infty} \frac{(-\gamma)^n}{n!} L_n(\alpha, \beta), \quad (2.1)$$

where the $L_n(\alpha, \beta)$ are a sequence of infinite integrals defined by

$$L_n(\alpha, \beta) = \int_0^\infty \frac{\sin \alpha x}{\sinh \beta x} (x \coth \beta x)^n dx. \quad (2.2)$$

It is possible to use contour integration methods to evaluate integrals of the form (2.2). For the value $n = 0$, equation (2.2) becomes

$$L_0(\alpha, \beta) = \int_0^\infty \frac{\sin \alpha x}{\sinh \beta x} dx = \frac{\pi}{2\beta} \tanh\left(\frac{\pi \alpha}{2\beta}\right), \quad (2.3)$$

which is a well-known result (see Gradshteyn & Ryzhik 1965, eqn 3.981). However, as n increases, the formulae become complicated quickly due to the number of derivatives needed to compute residues, and contour integration ceases to be useful.

Instead of using contour integration methods, consider equation (2.2) and assume that we may differentiate under the integral sign. It is not difficult to show that for $n \geq 1$, we have the partial differential recurrence equation

$$n \frac{\partial^2 L_{n-1}}{\partial \alpha^2} + \frac{\partial L_n}{\partial \beta} + (n+1)L_{n+1} = 0. \quad (2.4)$$

On making the substitution $\beta x \rightarrow x$ in equation (2.2), we obtain

$$L_n(\alpha, \beta) = \frac{1}{\beta^{n+1}} \int_0^\infty \frac{\sin \xi x}{\sinh x} (x \coth x)^n dx = \frac{M_n(\xi)}{\beta^{n+1}}, \quad (2.5)$$

where $\xi = \alpha/\beta$ and $M_n(\xi) = L_n(\xi, 1)$. By expressing the partial derivatives with respect to α and β in equations (2.4) and (2.5) in terms of the variable ξ , we have, for $n \geq 1$,

$$\xi M'_n(\xi) = n M''_{n-1}(\xi) + (n+1)[M_{n+1}(\xi) - M_n(\xi)], \quad (2.6)$$

where a prime denotes differentiation with respect to ξ . With this notation, the known result equation (2.3) is put in the form

$$M_0(\xi) = \frac{1}{2} \pi \tanh\left(\frac{1}{2} \pi \xi\right) = \beta \int_0^\infty \frac{\sin \alpha x}{\sinh \beta x} dx, \quad (2.7)$$

which will prove to be a useful result.

3. Solution by orthogonal polynomials

We now consider the solution of equation (2.6) in terms of a family of orthogonal polynomials. From the differentiation of equation (2.7) with respect to β , we may deduce that

$$M_1(\xi) = [\xi M_0(\xi)]'. \quad (3.1)$$

Furthermore, using this equation and equation (2.6) with $n = 1$, we may simplify the result to obtain

$$M_2(\xi) = [(\xi^2 - 1)M_0(\xi)]''/2. \quad (3.2)$$

From a further application of equation (2.6) with $n = 2$ along with equations (3.1) and (3.2), we find, after several simplifications, the quite remarkable result

$$M_3(\xi) = [\xi(\xi^2 - 5)M_0(\xi)]'''/6. \quad (3.3)$$

After these preliminary calculations, it is quite reasonable to propose that generally for $n \geq 0$, we have

$$M_n(\xi) = [p_n(\xi)M_0(\xi)]^{(n)}/n!, \quad (3.4)$$

where $p_n(\xi)$ is a polynomial of degree n . Indeed, the substitution of equation (3.4) into equation (2.6) yields

$$\{[\xi p_n(\xi) - n^2 p_{n-1}(\xi) - p_{n+1}(\xi)]M_0(\xi)\}^{(n+1)} = 0, \quad (3.5)$$

and since $p_n(\xi)$ are polynomials of degree n , we may conclude that for $n \geq 1$

$$\xi p_n(\xi) = n^2 p_{n-1}(\xi) + p_{n+1}(\xi). \quad (3.6)$$

On taking $p_0(\xi) = 1$, we may readily verify that this recurrence relation yields precisely the polynomials occurring in equations (3.1), (3.2) and (3.3) that arise, respectively, from equation (3.6) with $n = 0$, $n = 1$ and $n = 2$. The additional polynomials $p_4(\xi)$, $p_5(\xi)$ and $p_6(\xi)$ listed in §1 are obtained from (3.6) for the values $n = 3, 4, 5$.

It is not difficult to show from equation (3.6) that these polynomials have a unitary leading coefficient with the general structure

$$p_{2n}(\xi) = \xi^{2n} - \left(\sum_{k=1}^{2n-1} k^2 \right) \xi^{2n-2} + \dots,$$

and

$$p_{2n+1}(\xi) = \xi \left\{ \xi^{2n} - \left(\sum_{k=1}^{2n} k^2 \right) \xi^{2n-2} + \dots \right\}.$$

In the following section we consider the determination of the generating function for $p_n(\xi)$.

Finally in this section we note that there exists a substantial body of literature on orthogonal polynomials and that equation (3.6) is a special case of a more general recurrence equation known to be a necessary and sufficient condition to form an orthogonal system (see either Andrews *et al.* (2000, p. 244) or Erdelyi *et al.* (1953, p. 158)). Schoutens (2000) discusses Sheffer systems of polynomials, which are generated by the equation

$$f(z) \exp(xu(z)) = \sum_{m=0}^{\infty} Q_m(x) \frac{z^m}{m!},$$

where the $u(z)$ and $f(z)$ can be expanded in power series about the origin and satisfy $u(0) = 0$, $u'(0) \neq 0$ and $f'(0) \neq 0$. These polynomials are described by the general recurrence relation

$$Q_{m+1}(x) = (x + l + m\lambda)Q_m(x) + m[k + (m-1)\kappa]Q_{m-1}(x),$$

involving the four arbitrary constants l , λ , k and κ . With the choices $l = \lambda = 0$ and $k = \kappa = -1$, this equation reduces to equation (3.6). Polynomials with these characteristics are known as Meixner polynomials, and it has been shown that the functions f and u are obtained from the differential equations

$$\frac{f'}{f} = \frac{l + kz}{1 - \lambda z - \kappa z^2}, \quad u' = \frac{1}{1 - \lambda z - \kappa z^2}.$$

4. Generating function

In this section, we develop the generating function for the polynomials $p_n(\xi)$. The general case for Sheffer polynomials is reviewed in Schoutens (2000). We use a new independent derivation using classical methods, which is included here for completeness and tractability.

In attempting to obtain expressions for general $p_n(\xi)$, we introduce the generating function

$$P(\xi, z) = \sum_{n=0}^{\infty} p_n(\xi) z^n, \quad (4.1)$$

and from the recurrence relation (3.6) we may deduce in the usual way

$$z^3 \frac{\partial^2 P}{\partial z^2} + 3z^2 \frac{\partial P}{\partial z} + \left(z + \frac{1}{z} - \xi\right) P = \frac{1}{z}.$$

On making the substitution $P(\xi, z) = Q(\xi, z)/z$, this equation becomes

$$z^2 \frac{\partial^2 Q}{\partial z^2} + z \frac{\partial Q}{\partial z} + \left(\frac{1}{z^2} - \frac{\xi}{z}\right) Q = \frac{1}{z}.$$

Furthermore, introducing $y = 1/z$ gives

$$y \frac{\partial^2 Q}{\partial y^2} + \frac{\partial Q}{\partial y} + (y - \xi) Q = 1 \quad (4.2)$$

as the determining equation for the generating function $Q(\xi, y)$ given by

$$Q(\xi, y) = \sum_{n=0}^{\infty} \frac{p_n(\xi)}{y^{n+1}}. \quad (4.3)$$

We observe that for $\xi = 0$, equation (4.2) is a non-homogeneous Bessel equation. For ξ non-zero, equation (4.2) has arisen both within the context of the asymptotic solutions of differential equations, and in the determination of separable solutions of Laplace's equation in parabolic coordinates (see Buchholz 1969, pp. 33, 52). The two linearly independent solutions to equation (4.2) formally involve Whittaker functions, and the details are not simple, specifically the determination of the particular integral.

Alternatively, we might approach the solution of equation (4.2) via Laplace transforms. If we introduce

$$\hat{Q}(\xi, s) = \int_0^{\infty} e^{-ys} Q(\xi, y) dy,$$

then in the usual way we may deduce from equation (4.2) the first-order differential equation

$$(1 + s^2) \frac{d\hat{Q}}{ds} + (s + \xi) \hat{Q} = -\frac{1}{s},$$

which, although it admits a simple integrating factor $\sqrt{1 + s^2} \exp[\xi \tan^{-1} s]$, is not trivial to solve. Bearing in mind that a transform inversion is still required, we do not pursue this approach further.

The key to obtaining a simple generating function is to note that $Q(\xi, y)$ as given by equation (4.3) is a Laurent expansion, and therefore the possibility arises that we may exploit the standard Laplace transform

$$\frac{1}{y^{n+1}} = \frac{1}{n!} \int_0^\infty e^{-sy} s^n \, ds, \quad (4.4)$$

as follows. From equations (4.3) and (4.4) we have

$$Q(\xi, y) = \int_0^\infty e^{-sy} q(\xi, s) \, ds, \quad (4.5)$$

where the function $q(\xi, s)$ is defined by

$$q(\xi, s) = \sum_{n=0}^\infty \frac{p_n(\xi)}{n!} s^n. \quad (4.6)$$

Thus, instead of using the traditional Laplace transform, we look for a solution of equation (4.2) in terms of the function $q(\xi, s)$.

Now by differentiating equation (4.5) with respect to y , and using integration by parts, we have

$$y \frac{\partial Q}{\partial y} = - \int_0^\infty e^{-sy} \left(s \frac{dq}{ds} + q \right) ds,$$

and therefore

$$\frac{\partial}{\partial y} \left(y \frac{\partial Q}{\partial y} \right) = \int_0^\infty e^{-sy} s \left(s \frac{dq}{ds} + q \right) ds. \quad (4.7)$$

Thus, from equations (4.2), (4.5) and (4.7), we may deduce

$$\int_0^\infty e^{-ss} \left\{ (1 + s^2) \frac{dq}{ds} + (s - \xi) q \right\} ds = 1 - q(\xi, 0),$$

and on using $q(\xi, 0) = p_0(\xi) = 1$, we may conclude that $q(\xi, s)$ satisfies the first-order differential equation

$$(1 + s^2) \frac{dq}{ds} + (s - \xi) q = 0,$$

which readily integrates to give the generating function (1.9) on using $q(\xi, 0) = 1$. In the next section, we use (1.9) to simplify our infinite series representation of $I(\alpha, \beta, \gamma)$. Finally, we note that a similar generating function arises in integral representations of the Hurwitz zeta function (Boros *et al.* 2002), and has been used to evaluate a variety of integrals. However, it does *not* describe integrals involving this family of polynomials.

5. Alternative series expansion

In this section we develop an alternative series expansion to equation (2.1). We begin by combining equations (2.1), (2.5) and (3.4) to obtain the representation (1.8) in terms of our orthogonal polynomials $p_n(\xi)$. We now define the function $f_n(\xi)$ by

$$f_n(\xi) = p_n(\xi) \tanh(\pi\xi/2), \quad (5.1)$$

and use the Mittag-Leffler expansion theorem to develop an expression for $f_n^{(n)}(\xi)/n!$. In order to apply the theorem we note the following. First, as $\xi \rightarrow \infty$, $|f_n(\xi)| \sim |\xi|^n$ in accordance with the properties of the polynomials $p_n(\xi)$ and the tanh function. Second, the function $f_n(\xi)$ has simple poles at the values $\xi = a_j$, where $j = (\dots, -3, -1, 1, 3, \dots)$. Third, at the poles a_j , the function $f_n(\xi)$ has the residues

$$b_{nj} = \lim_{\xi \rightarrow a_j} = \frac{(\xi - a_j)p_n(\xi) \sinh(\pi\xi/2)}{(\pi/2)(\xi - a_j) \sinh(\pi a_j/2)} = \frac{2}{\pi} p_n(a_j).$$

We next consider the ratio $f_n(\xi)/(\xi - z)^{n+1}$ and examine the various singularities. At the poles a_j of $f_n(\xi)$ we have the residues $b_{nj}/(a_j - z)^{n+1}$, while at $\xi = z$ we have the residue $f_n^{(n)}(z)/n!$. Now consider the integral of $f_n(\xi)/(\xi - z)^{n+1}$ around a square contour C in the ξ -plane which surrounds the poles a_j in the range $-J \leq a_j \leq J$ as well as the pole at $\xi = z$. We may then use Cauchy's theorem to obtain

$$\frac{1}{2\pi i} \int_C \frac{f_n(\xi) d\xi}{(\xi - z)^{n+1}} = \frac{f_n^{(n)}(z)}{n!} + \sum_{j=-J}^J \frac{b_{nj}}{(a_j - z)^{n+1}}, \quad (5.2)$$

and on setting $z = 0$ we have

$$\frac{1}{2\pi i} \int_C \frac{f_n(\xi) d\xi}{\xi^{n+1}} = \frac{f_n^{(n)}(0)}{n!} + \sum_{j=-J}^J \frac{b_{nj}}{a_j^{n+1}}. \quad (5.3)$$

From the definition (5.1) for $f_n(\xi)$ and the properties of the $p_n(\xi)$, it is clear that $f_n(\xi)$ is even when n is odd, and odd when n is even. This implies that $f_n^{(n)}(\xi)$ is always odd, and therefore $f_n^{(n)}(0) = 0$. Thus, by subtracting equation (5.3) from equation (5.2) and using the definition b_{nj} , we deduce

$$\begin{aligned} \frac{1}{2\pi i} \int_C f_n(\xi) \left\{ \frac{1}{(\xi - z)^{n+1}} - \frac{1}{\xi^{n+1}} \right\} d\xi \\ = \frac{f_n^{(n)}(z)}{n!} + \frac{2}{\pi} \sum_{j=-J}^J p_n(a_j) \left\{ \frac{1}{(a_j - z)^{n+1}} - \frac{1}{a_j^{n+1}} \right\}. \end{aligned} \quad (5.4)$$

Now the integral in this equation has the magnitude

$$|\xi|^n \left\{ \frac{1}{|\xi|^{n+1}} - \frac{1}{|\xi - z|^{n+1}} \right\} |\xi| = 1 - \frac{1}{|1 - z/\xi|^{n+1}},$$

which tends to zero as we let the contour $C \rightarrow \infty$, and thus by Jordan's lemma the integral vanishes and equation (5.4) becomes

$$\frac{f_n^{(n)}(z)}{n!} = \frac{2}{\pi} \sum_{j=-\infty}^{\infty} p_n(a_j) \left\{ \frac{1}{a_j^{n+1}} - \frac{1}{(a_j - z)^{n+1}} \right\}. \quad (5.5)$$

But from the properties of the polynomials $p_n(\xi)$ it is clear that for n both odd and even, the function $p_n(\xi)/\xi^{n+1}$ is always odd, and therefore, because $a_j = -a_{-j}$, we may deduce that

$$\sum_{j=-\infty}^{\infty} \frac{p_n(a_j)}{a_j^{n+1}} = 0.$$

With this expression and on replacing z with ξ , the expansion (5.5) becomes

$$\frac{f_n^{(n)}(z)}{n!} = -\frac{2}{\pi} \sum_{j=-\infty}^{\infty} \frac{p_n(a_j)}{(a_j - z)^{n+1}}. \quad (5.6)$$

As a simple check, we set $n = 0$ and $p_0(\xi) = 1$ to obtain the meromorphic series representation of $\tanh(\pi\xi/2)$ (see Gradshteyn & Ryzhik 1965, eqn 1.421).

We now exploit equation (5.6) to deduce an expression for equation (1.7). From equations (1.8), (5.1) and (5.6), we find

$$I(\alpha, \beta, \gamma) = -\frac{1}{\beta} \sum_{n=0}^{\infty} \frac{(-\gamma/\beta)^n}{n!} \sum_{j=-\infty}^{\infty} \frac{p_n(a_j)}{(a_j - \xi)^{n+1}},$$

and assuming that we may invert the order of summation, we obtain

$$I(\alpha, \beta, \gamma) = \frac{1}{\beta} \sum_{j=-\infty}^{\infty} \frac{1}{(\xi - a_j)} \sum_{n=0}^{\infty} \frac{p_n(a_j)}{n!} \left(\frac{\gamma/\beta}{\xi - a_j} \right)^n. \quad (5.7)$$

On introducing the generating function (1.9) into equation (5.7) we find that

$$I(\alpha, \beta, \gamma) = \frac{1}{\beta} \sum_{j=-\infty}^{\infty} \frac{\exp\{a_j \tan^{-1}[(\gamma/\beta)/(\xi - a_j)]\}}{\sqrt{(\xi - a_j)^2 + (\gamma/\beta)^2}}. \quad (5.8)$$

Finally, if we make use of the relationship

$$\tan^{-1} x = \frac{1}{2i} \log \left(\frac{1 + ix}{1 - ix} \right)$$

and make the index change $k = 2j + 1$, where $k = (\dots, -2, -1, 0, 1, \dots)$, so that $a_j = i(2k + 1)$, then we may express equation (5.8) in the elegant form

$$I(\alpha, \beta, \gamma) = \frac{1}{\beta} \sum_{k=-\infty}^{\infty} \frac{[\xi + i(2k + 1) - i\gamma/\beta]^k}{[\xi + i(2k + 1) + i\gamma/\beta]^{k+1}}. \quad (5.9)$$

If we note that $\xi = \alpha/\beta$ is purely real, then it is not hard to show that this expression reduces to equation (1.10).

A form that is particularly useful for numerical evaluation can be obtained by introducing

$$r_1 e^{i\theta_1} = \alpha + i(2k + 1)\beta - i\gamma, \quad r_2 e^{i\theta_2} = \alpha + i(2k + 1)\beta + i\gamma,$$

where

$$r_1 = \{\alpha^2 + [\beta(2k + 1) - \gamma]^2\}^{1/2}, \quad \theta_1 = \tan^{-1} \left(\frac{\beta(2k + 1) - \gamma}{\alpha} \right),$$

$$r_2 = \{\alpha^2 + [\beta(2k + 1) + \gamma]^2\}^{1/2}, \quad \theta_2 = \tan^{-1} \left(\frac{\beta(2k + 1) + \gamma}{\alpha} \right),$$

so that equation (1.10) can be expressed as

$$I(\alpha, \beta, \gamma) = 2 \sum_{k=0}^{\infty} \frac{r_1^k}{r_2^{k+1}} \cos[k(\theta_2 - \theta_1) + \theta_2]. \quad (5.10)$$

From the definition of θ_2 we also have

$$\cos \theta_2 = \frac{\alpha}{r_2}, \quad \sin \theta_2 = \frac{\beta(2k+1) + \gamma}{r_2},$$

and from the relationship

$$\tan^{-1}(A) - \tan^{-1}(B) = \tan^{-1}\left(\frac{A-B}{1+AB}\right),$$

we may deduce

$$\theta_2 - \theta_1 = \tan^{-1}\left[\frac{2\alpha\gamma}{\alpha^2 + \beta^2(2k+1)^2 - \gamma^2}\right]. \quad (5.11)$$

We can therefore expand the cosine term in equation (5.10) to obtain

$$I(\alpha, \beta, \gamma) = 2 \sum_{k=0}^{\infty} \frac{r_1^k}{r_2^{k+2}} \{\alpha \cos k(\theta_2 - \theta_1) - [\beta(2k+1) + \gamma] \sin k(\theta_2 - \theta_1)\}, \quad (5.12)$$

which makes use of equation (5.11).

We note the following features of the solution. First, $I(\alpha, \beta, \gamma)$ tends to zero as $\alpha \rightarrow 0$, which is readily apparent from the definition (1.7). Second, for $\gamma \rightarrow 0$, I becomes the known result (2.3). Third, an analysis for large k of the numerator and denominator in equation (5.12) shows that the series converges on $O(1/k^2)$ despite the appearance of the index k as an exponent, which implies digamma function $\psi(z)$ behaviour as $\alpha \rightarrow \infty$.

Finally, as an aside, we note that equation (5.9) is a meromorphic function with poles lying along the imaginary axis of the ξ -plane and with the order of the poles increasing linearly with denumeration from the origin. It is possible to map a function with this pole structure into the reciprocal of the product of two Barnes G -functions (see Gradshteyn & Ryzhik 1965, eqn 8.333). However, there remains an undetermined entire function that makes it difficult to represent this series as an infinite product.

6. Asymptotic analysis

We now consider the asymptotic behaviour of equation (1.7) for large ξ . This case is of particular interest because it describes the tail of the distribution $F(L)$. To gain some insight into the structure of the solution, we return to equation (1.7) and express this as the Fourier transform inversion integral

$$I(\alpha, \beta, \gamma) = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{\sinh \beta x} \exp[-\gamma x \coth \beta x] dx. \quad (6.1)$$

For large values of α , the contributions to the integral are concentrated around $x = 0$. If we expand the sinh and coth functions in locally convergent power series and retain terms linear in x , then we obtain the approximation

$$I(\alpha, \beta, \gamma) \sim \frac{e^{-\gamma/\beta}}{2\beta i} \int_{-\infty}^{\infty} \left(\frac{1}{x} - \frac{\beta^2 x}{6}\right) e^{i\alpha x - \gamma\beta x^2/3} dx. \quad (6.2)$$

Figure 1. A plot of the numerical (solid lines) and analytical (points) results for the cumulative distribution function $F(L)$ for $\gamma/\beta = \tilde{\phi}^2/2$ equal to (i) 0, (ii) 1 and (iii) 4.

This form is particularly suitable for values of the parameter ratio $\gamma/\beta > 1$ where the decay from the coth term dominates that from the sinh function. Equation (6.2) can be evaluated with standard methods and leads to

$$I \sim \frac{\pi e^{-\gamma/\beta}}{\beta} \left\{ \Phi\left(\delta\sqrt{\frac{3}{2}}\right) - \frac{1}{2} \right\} - \frac{\sqrt{3\pi}}{8\gamma} \delta e^{-3\delta^2/4 - \gamma/\beta}, \quad (6.3)$$

where $\delta = \alpha/\sqrt{\beta\gamma}$ and Φ is the cumulative standard normal distribution. This expression becomes asymptotically accurate as $\gamma/\beta \rightarrow \infty$.

An alternative expansion suitable for smaller values of the parameter γ/β can be obtained from equation (1.8). Using the properties of the orthogonal polynomials $p_n(\xi)$ and the tanh function, we can deduce that for large values of ξ

$$M_n = [p_n(\xi) \tanh(\pi\xi/2)]^{(n)} \sim n! \tanh(\pi\xi/2) + O(\xi^n / \cosh^2 \pi\xi/2).$$

For the values $n = 1, 2, 3$ we use equations (2.7), (3.1) and (3.2) to obtain

$$M_1(\xi) = M_0(\xi) + \xi M'_0(\xi), \quad M_2(\xi) = 2M_0(\xi) + 2\xi M'_0(\xi) + (\xi^2 - 1)M''_0(\xi).$$

If we use these expressions in full and only the leading-order terms of M_n for all values of $n \geq 4$ in equation (1.8), then we obtain the approximation

$$I \sim \frac{\pi}{2\beta} \left\{ \sum_{n=0}^{\infty} \frac{(-\gamma/\beta)^n}{n!} M_0 - \frac{\gamma}{\beta} \xi M'_0 + \frac{1}{4} \left(\frac{\gamma}{\beta} \right)^2 [2\xi M'_0 + (\xi^2 - 1)M''_0] \right\},$$

which becomes

$$I \sim \frac{\pi}{2\beta} e^{-\gamma/\beta} M_0 - \frac{\pi}{2\beta} \left\{ \frac{\gamma}{\beta} \xi M'_0 - \frac{1}{4} \left(\frac{\gamma}{\beta} \right)^2 [2\xi M'_0 + (\xi^2 - 1)M''_0] \right\}. \quad (6.4)$$

This formula is most useful for small values of the parameter γ/β . It is straightforward to work out the derivatives of the tanh function in this expression. We now

Figure 2. A plot of the analytical results (solid lines), the normal approximation (stars), and the tanh approximation (crosses) for the value $\gamma/\beta = \tilde{\phi}^2/2 = 1$.

have two asymptotic expansions, equations (6.3) and (6.4), to compare against the numerical and analytical evaluations of (1.7).

7. Numerical example

We now compare the results of the analytic expressions for the cumulative distribution of the random variable L with the numerical evaluation of the definition (1.6). In order to do this, we manipulate (1.6) into the form

$$F(L) = \frac{1}{2} + e^{\tilde{\phi}^2/2} I(L, \pi, \tilde{\phi}^2 \pi/2),$$

Figure 3. A plot of the analytical results (solid lines), the normal approximation (stars), and the tanh approximation (crosses) for the value $\gamma/\beta = \tilde{\phi}^2/2 = 4$.

where the arguments of I are α , β and γ , respectively. Note the additional multiplication by the factor $\exp(\gamma/\beta)$.

First we compare the results of the analytical formula equation (5.12) with the numerical evaluation of equation (1.6) using a standard integration package as shown in figure 1. The results are very close and confirm the accuracy of the analytical approach. Because the series converges on $O(1/k^2)$, accurate estimates of $F(L)$ in the extreme tail require several thousand series terms for, say, six-digit accuracy. For all values of $\gamma/\beta = \tilde{\phi}^2/2$, $F(L) \rightarrow 1$ as $\xi = L/\pi \rightarrow \infty$, which can be seen from equations (6.3) and (6.4).

We now compare the behaviour of equation (5.12) with our asymptotic estimates for the values of $\gamma/\beta = 1$ and 4. Before we consider these specific cases, we note

some practical aspects of our integral within the context of the Milstein method. Through the use of moment-generating functions, it can be shown that the random variable $\tilde{\phi}^2/2$ has the cumulative exponential distribution $1 - \exp(-\tilde{\phi}^2/2)$. Thus, during the implementation of the Milstein method, the value of $\gamma/\beta = \tilde{\phi}^2/2$ exceeds, on average, the values 1 and 4 in 36% and 1.8% of the integration steps. These two values therefore represent moderate and high levels of the parameter from a practical standpoint.

First, we consider the value $\gamma/\beta = 1$. Figure 2a shows a comparison of the results of the two approximations (6.3) and (6.4) with the series (5.12) for $\gamma/\beta = 1$ over a wide range of L , while figure 2b concentrates on the tail of the distribution where $F(L) \geq 0.99$. The normal approximation (6.3) is surprisingly good outside the tail, but breaks down as $F(L) \rightarrow 1$. The tanh approximation is not good for small values of $\xi = L/\pi$ but becomes quite good in the tail.

Next, we consider the value $\gamma/\beta = 4$. Figure 3a shows results over a wide range of L , while figure 3b shows the tail region $F(L) \geq 0.99$. The normal approximation is quite good throughout the distribution, while the tanh approximation is not particularly good anywhere except the very remote tail. This is not surprising since as $\gamma/\beta \rightarrow \infty$, $F(L)$ approaches a normal distribution.

The key feature to note about the tail of the distribution $F(L)$ is that for finite γ/β , even large values, the very remote regions show exponential rather than normal behaviour. As γ/β gets large, the region where the normal distribution is more accurate expands to cover almost all values of L and pushes the exponential region further into the remote tail. However, for large enough L , the behaviour of $F(L)$ will still be exponential. In accordance with the Fisher–Tippett theorem, this means that the extreme value distribution of $F(L)$ is always attracted to the Frechet class of distributions.

8. Closing remarks

The contribution of this paper is the development of a novel method for the evaluation of integrals of the form (1.7) using Sheffer systems of polynomials. The key features are the identification of the integral with a specific family of Meixner polynomials and then the use of the Mittag–Leffler expansion theorem in combination with the polynomial-generating function, which allows equation (1.8) to be manipulated into the form (1.10). The method adopted here highlights the ubiquitous relationship between stochastic processes and orthogonal polynomials, and the major ingredients of the procedure may well be applicable to the evaluation of other integrals.

Appendix A.

We consider the development of the characteristic function $\Phi_L(\lambda)$ for the distribution $f(L)$ of the random variable L defined in equation (1.4). We introduce the characteristic function $\Phi_L(\lambda)$ defined by the Fourier transform pair

$$\Phi_L(\lambda) = \int_{-\infty}^{\infty} f(L) e^{i\lambda L} dL \quad \text{and} \quad f(L) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_L(\lambda) e^{-i\lambda L} d\lambda. \quad (\text{A } 1)$$

Next, we split the random variable $L = L_1 - L_2$, where the variables L_1 and L_2 depend individually on ϕ_1 and ϕ_2 , according to

$$L_1 = \sum_{r=1}^{\infty} \frac{1}{r} [\zeta_{2,r}(\phi_1 \sqrt{2} + \eta_{1,r})] \quad \text{and} \quad L_2 = \sum_{r=1}^{\infty} \frac{1}{r} [\zeta_{1,r}(\phi_2 \sqrt{2} + \eta_{2,r})]. \quad (\text{A } 2)$$

The properties of characteristic functions are well known and include

$$\Phi_L(\lambda) = \Phi_{L_1}(\lambda) \Phi_{L_2}(-\lambda). \quad (\text{A } 3)$$

We develop an expression for $\Phi_L(\lambda)$ by substituting the definition (A 2)₁ into (A 1)₁ and using the independence of the ζ and η normal random variables to obtain

$$\begin{aligned} \Phi_{L_1}(\lambda) = \prod_{r=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2}(\zeta_{2,r})^2 + \frac{i\phi_1 \sqrt{2}}{r} \zeta_{2,r} \right\} d\zeta_{2,r} \\ \times \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2}(\eta_{1,r})^2 + \frac{i\zeta_{2,r}\lambda}{r} \eta_{1,r} \right\} d\eta_{1,r}. \end{aligned} \quad (\text{A } 4)$$

The integrals in this product are evaluated to obtain the expression

$$\Phi_{L_1}(\lambda) = \exp G(\lambda) = \prod_{r=1}^{\infty} \frac{r}{\sqrt{r^2 + \lambda^2}} \exp \left\{ -\frac{\lambda^2 \phi_1^2}{r^2 + \lambda^2} \right\},$$

where $G(\lambda)$ is the cumulant function. Next, we use the decomposition $G(\lambda) = -\phi_1^2 A(\lambda) - B(\lambda)/2$, where

$$A(\lambda) = \sum_{r=1}^{\infty} \frac{\lambda^2}{\lambda^2 + r^2} \quad \text{and} \quad B(\lambda) = \sum_{r=1}^{\infty} \log \left(1 + \frac{\lambda^2}{r^2} \right).$$

The summations can be found in standard handbooks (e.g. Gradshteyn & Ryzhik 1965) or evaluated with contour integration techniques. The results are

$$A(\lambda) = \frac{1}{2} \pi \lambda \coth \pi \lambda - \frac{1}{2} \quad \text{and} \quad B(\lambda) = \log \left[\frac{\sinh \pi \lambda}{\lambda} \right]. \quad (\text{A } 5)$$

We now substitute (A 5) into $G(\lambda)$ to obtain the characteristic function

$$\Phi_{L_1}(\lambda) = \sqrt{\frac{\pi \lambda}{\sinh \pi \lambda}} \exp \left[\frac{1}{2} \phi_1^2 (1 - \pi \lambda \coth \pi \lambda) \right]. \quad (\text{A } 6)$$

A similar expression holds for $\Phi_{L_2}(\lambda)$, where ϕ_1 is replaced with ϕ_2 . Finally, we employ (A 6) and the expression for $\Phi_{L_2}(\lambda)$ in (A 3) to obtain the characteristic function

$$\Phi_L(\lambda) = \frac{\pi \lambda}{\sinh \pi \lambda} \exp \left[\frac{1}{2} \tilde{\phi}^2 (1 - \pi \lambda \coth \pi \lambda) \right], \quad (\text{A } 7)$$

where $\tilde{\phi}^2 = \phi_1^2 + \phi_2^2$. Not surprisingly, the characteristic function depends upon the sum of the squares of the individual values ϕ_1 and ϕ_2 .

To finish up the process, we apply the inverse Fourier transform (A 1)₂ to (A 7). Despite the analytical appearance of (A 7), the presence of essential singularities at $\lambda = \pm ni$, where $n = 1, 2, \dots$, makes this expression unsuitable for contour integration

inversion techniques. Instead, the expression is most simply cast as the numerical inversion integral

$$F(L) = \frac{1}{2} + \int_0^\infty \frac{\sin L\lambda}{\sinh \pi\lambda} \exp\left[\frac{1}{2}\tilde{\phi}^2(1 - \pi\lambda \coth \pi\lambda)\right] d\lambda, \quad (\text{A } 8)$$

for the cumulative distribution function $F(L)$.

References

- Andrews, G. E., Askey, R. & Roy, R. 2000 *Special functions, encyclopedia of mathematics and its applications*, no. 71, p. 244. Cambridge University Press.
- Boros, G., Espinosa, O. & Moll, V. H. 2002 On some families of integrals solvable in terms of polygamma and negapolygamma functions. Preprint, Tulane University.
- Buchholz, H. 1969 *The confluent hypergeometric function*. Springer Tracts in Natural Philosophy, vol. 15, pp. 33, 52. Springer.
- Erdelyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. 1953 *Higher transcendental functions*, vol. 2, p. 158. New York: McGraw-Hill.
- Gradsteyn, I. S. & Ryzhik, I. M. 1965 *Tables of integrals, series and products*, 4th edn. Academic.
- Heston, S. L. 1993 A closed form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.* **6**, 327–343.
- Kloeden, P. E. & Platen, E. 1999 *Numerical solution of stochastic differential equations*, 3rd edn. Springer.
- Schoutens, W. 2000 *Stochastic processes and orthogonal polynomials*. Lecture Notes in Statistics, vol. 146, pp. 45–62. Springer.