University of Wollongong

Research Online

Faculty of Engineering and Information
Sciences - Papers: Part B

Faculty of Engineering and Information
Sciences

2018

On a class of estimation and test for long memory

Hui Fu

Jiangnan University

Wenting Chen

Jiangnan University, wtchen@uow.edu.au

Xinjiang He

University of Wollongong, xinjiang@uow.edu.au

Follow this and additional works at: <https://ro.uow.edu.au/eispapers1>

Part of the [Engineering Commons](#), and the [Science and Technology Studies Commons](#)

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au

On a class of estimation and test for long memory

Abstract

This paper proposes a new analysis method of the estimation and test for long memory time series. We first introduce the definitions of the time scale series, strong variance scale exponent and weak variance scale exponent, and establish the mathematical equations for the variance scale exponents, with which the time series of the white noise, short memory and long memory can be accurately identified. Two statistics for the hypothesis tests of white noise, short memory and long memory time series are constructed, and the Monte Carlo performance for MSE of the weak variance scale exponent estimator and the empirical size and power of SLmemory statistic is subsequently demonstrated, giving practical recommendations of finite-sample. Finally, brief empirical examples are provided based on Sino-US stock index logarithmic return rate data.

Disciplines

Engineering | Science and Technology Studies

Publication Details

Fu, H., Chen, W. & He, X. (2018). On a class of estimation and test for long memory. *Physica A: Statistical Mechanics and its Applications*, 509 906-920.

On a Class of Estimation and Test for Long Memory

Hui Fu¹

Wenting Chen²

Xin-Jiang He^{3,*}

1. School of Business, Jiangnan University, Wuxi, China.

2. School of Business, Jiangnan University, Wuxi, China.

3. School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, Australia.

* Corresponding author.

Abstract: This paper proposes a new analysis method of the estimation and test for long memory time series. We first introduce the definitions of the time scale series, strong variance scale exponent and weak variance scale exponent, and establish the mathematical equations for the variance scale exponents, with which the time series of the white noise, short memory and long memory can be accurately identified. Two statistics for the hypothesis tests of white noise, short memory and long memory time series are constructed, and the Monte Carlo performance for MSE of the weak variance scale exponent estimator and the empirical size and power of *SLmemory* statistic is subsequently demonstrated, giving practical recommendations of finite-sample. Finally, brief empirical examples are provided based on Sino-US stock index logarithmic return rate data.

Keywords: Long Memory, Weak Variance Scale Exponent, *SLmemory* Statistic, Time Scale Series.

JEL Classification: C22, C13, C12

1. Introduction and setup

The research on the estimation of long memory time series dated back to 1951, when Hurst (1951) proposed the classical rescaled range (*R/S*) method to analyze the long-range dependence, and this approach was subsequently improved by excellent scholars (Mandelbrot and Wallis, 1969; Mandelbrot, 1972, 1975; Mandelbrot and Taqqu, 1979; Peters, 1999, 2002). In particular, fractional Gaussian noises process (Mandelbrot and Van Ness, 1968) was the first long memory model, with the long memory parameter H (Hurst exponent, also called self-similarity parameter) satisfying $0 < H < 1$, and it was further studied by Lo (1991), Kwiatkowski et al. (1992) and Giraitis et al. (2003), testing long memory properties following the *R/S*-type method. Subsequently, a large amount of research interest was devoted to studying the long memory issues in financial markets from the perspective of time domain analysis (Cheong et al., 2007; Chronopoulou and Viens, 2012; Niu and Wang, 2013; Lahmiri, 2015).

Fractional differencing noises process (Granger and Joyeux, 1980; Hosking, 1981) was the second long memory model, which formed a new technology path for estimation and test of long memory properties, with the long memory parameter (fractional differencing parameter d) satisfying $-0.5 < d < 0.5$. The related work include Geweke and Porter-Hudak (1983), Haslett and Raftery (1989), Beran (1994), Robinson (1995, 2005), among many others, and we refer interested readers to Lo (1991), Baillie (1996), Giraitis et al. (2003), Robinson (2003), (Palma, 2007) and Boutahar et al. (2007) for relevant and detailed literature review. There has also been a large amount of literature that investigates long memory properties from the perspective of frequency domain analysis (Floros et al., 2007; Christensen et al., 2010; Roueff and Von Sachs, 2011; Chkili et al., 2014).

It is widely acknowledged that the model parameter falling in the range of $0.5 < H < 1$ or $0 < d < 0.5$ implies long memory, long-range dependence, or persistence, while the other cases, i.e., $0.5 < H < 1$ or $-0.5 < d < 0$, are regarded as anti-persistence. However, there is a controversy on whether anti-persistence should be treated as long memory (Hosking, 1981; Lo, 1991; Palma, 2007). What is suggested by the present paper is that the situations of $0.5 < H < 1$ (or $0 < d < 0.5$) and $0 < H < 0.5$ (or $-0 < d < 0.5$) should be regarded as persistent and anti-persistent long memory, respectively, in the sense that persistent long memory indicates long-run positive correlation, while anti-persistent long memory implies long-run negative correlation.

The motivation of this paper can be illustrated from three different aspects. Firstly, most of the existing literature focuses on either time domain analysis or frequency domain analysis, such as time domain analysis represented by Hurst exponent and frequency domain analysis characterized

by fractional differencing parameter d . In other words, time domain analysis and frequency domain analysis of long memory are completely separated, and there lacks a unified time domain analysis and frequency domain analysis for long-term memory. Moreover, analysis of long-term memory in existing literature, focuses on either long memory parameter estimation, or the statistical test of long memory, and thus it is rather demanding to establish a unified analysis method of parameter estimation and long memory statistical test. Last but not the least, although the existing methods have already studied and analyzed the two most classic long-memory models (fractional Gaussian noises process and fractional differencing noises process), they are conducted in a relatively separated manner, and no existing methods have been proposed to study these two types of models uniformly. To address these, this paper proposes the concept of scaled time series. By constructing the relationship between the standard deviation and time scale among different time scale sequences, a unified analysis method for long memory parameter estimation and statistical test is obtained. This method can not only unify the time domain analysis and frequency domain analysis, but also simultaneously analyze the two most classic long memory models.

The main contributions of this paper can be arranged into three aspects. Firstly, the definitions of time scale series, strong variance scale exponent and weak variance scale exponent are firstly proposed, and the equations on the relationship between the short, long memory time series and variance scale exponents are established and proved. Two statistics for the hypothesis tests of white noise, short memory and long memory time series are constructed, forming a new system for theoretical analysis of stationary time series, especially for the estimation and test of long memory properties. Secondly, the asymptotic properties for estimation and test are shown to be unaffected by persistent or anti-persistent long memory, which is the main reason why we suggest the two types of (persistence and anti-persistence) long memory. Finally, time series properties are analyzed based on both perspectives of the time and frequency domain, which differ from the situations where the two technology paths and corresponding long memory models are confined to the time or frequency domain only.

The rest of the paper is organized as follow. In sections 2 and 3, we propose the definitions of time scale series, strong variance scale exponent and weak variance scale exponent, and establish the relationship between variance scale exponents and three typical time series, i.e., white noise, short memory and long memory time series, with the theoretical proofs left in the Appendix. Two statistics, $Wnoise$ and $SLmemory$, are constructed for the test of white noise, short memory and long memory time series. Section 4 investigates the finite-sample performance of the weak variance scale exponent estimator and the statistic, $SLmemory$, with Monte Carlo simulations, presenting practical recommendations for the choice of time scale n , followed by some conclusion remarks given in the last section.

2. Definitions and properties of strong and weak variance scale exponent.

We begin by specifying a few notations used in the following. Let $\{x(t), t=1, 2, \dots, N, N \rightarrow \infty\}$ be a stationary time series with the unknown mean μ and lag- i autocovariance γ_i . Let $f(\lambda)$ denote the spectral density of $\{x(t)\}$ defined over $|\lambda| \leq \pi$. d and H are used to represent the long memory parameter of the fractional differencing noises process (Granger and Joyeux, 1980; Hosking, 1981) and that of the fractional Gaussian noises process (Mandelbrot and Van Ness, 1968), respectively, and N is assumed to be the sample size. With any given stationary time series, we are always able to derive our newly proposed time scale series, whose definition is given as follows.

Definition 2.1. Time Scale Series

Given a time series $\{x(t)\}$, the new time series $\{x_2^1(t) = x(2t-1) + x(2t), t=1, 2, \dots, M, M = \lfloor N/2 \rfloor\}$ and $\{x_2^2(t) = x(2t) + x(2t+1), t=1, 2, \dots, M, M = \lfloor (N-1)/2 \rfloor\}$ are both defined as time scale 2 series. Similarly, the time scale n series can be constructed as $\{x_n^{i+1}(t), t=1, 2, 3, \dots, M, M = \lfloor (N-i)/n \rfloor\}$, where $x_n^{i+1}(t) = x(nt+i) + x(nt-1+i) + \dots + x(nt-(n-1)+i)$, $i=0, 1, \dots, n-1, n/N \rightarrow 0$. Here, $\lfloor \cdot \rfloor$ is used to round the number to the next smaller integer, if its result is not an integer.

From Definition 2.1, several nice and useful properties of the newly defined time scale series can be easily identified, which are summarized in the four propositions listed below. It should be pointed out that as the proofs for these propositions are quite straightforward, we omit the details

here.

Proposition 2.1. If we denote $D[x_n^i(t)]$ ($i=1, 2, \dots, n$) as the variance of i -th time series in the time scale n series, we have $D[x_n^1(t)] = \dots = D[x_n^n(t)]$. In this case, we can simplify our notations by assuming $D[x_n(t)]$ as the variance of each time scale n series.

Proposition 2.2. The variance of the time scale n series, $D[x_n(t)]$, can be expressed as

$$D[x_n(t)] = nD[x(t)] + 2 \sum_{i=1}^n (n-i) \gamma_i, \quad n=1, 2, \dots.$$

Proposition 2.3. Lag- n autocovariance of the time series $\{x(t)\}$, γ_n , can be expressed in terms of the variance of the time scale series, i.e.,

$$\gamma_n = \frac{1}{2} [D[x_{n+1}(t)] - 2D[x_n(t)] + D[x_{n-1}(t)]], n=1, 2, \dots.$$

Here, $D[x_0(t)] = 0$, $D[x_1(t)] = D[x(t)]$.

Proposition 2.4. The sum of lag- i autocovariance γ_i can be determined as

$$\sum_{i=-n}^n \gamma_i = n [D[x_{n+1}(t)] - D[x_n(t)]], n=1, 2, \dots.$$

With the knowledge of the newly proposed time scale series, we are now ready to give the definitions of the strong variance scale exponent and weak variance scale exponent.

Definition 2.2. Strong Variance Scale Exponent

For any given time series $\{x(t)\}$, if the variance of the time scale n series satisfies

$$D[x_n(t)] = n^{2F_n} D[x(t)]$$

with F_n being equal to a constant F , then we call F as the strong variance scale exponent.

Definition 2.3. Weak Variance Scale Exponent

For any given time series $\{x(t)\}$, if the variance of time scale n series satisfies

$$D[x_n(t)] = f n^{2F_n} D[x(t)]$$

with F_n converging to a constant F' , as n tends to infinity, then we call F' and f as the weak variance scale exponent and the adjusted proportion coefficient, respectively.

With all the new concepts being introduced, we can now proceed to analyze the properties of the time scale series in both of the time and frequency domain, the details of which are illustrated in the following two subsections.

2.1. Time Domain Analysis

By making use of the four propositions arising from Definition 2.1, we can establish the relationship between the time scale series and four typical types of time series.

Theorem 2.1.1. If the time series $\{x(t)\}$ is a white noise, then

$$D[x_n(t)]/D[x(t)] = n.$$

Theorem 2.1.1 is actually a direct consequence of Proposition 2.2 and the property of the white noise, and the following proposition can be straightforwardly derived with Definition 2.2 and 2.3.

Proposition 2.1.1. If the time series $\{x(t)\}$ is a white noise, then the strong variance scale exponent, weak variance scale exponent and adjusted proportion coefficient should respectively satisfy:

$$F=0.5, F'=0.5, f=1.$$

Theorem 2.1.2. If the time series $\{x(t)\}$ is a short memory time series, ARMA(p, q), then

$$D[x_n(t)]/D[x(t)] \sim cn, \text{ as } n \text{ tends to infinity.}$$

Throughout this paper, $x_n \sim y_n$ is used to represent the property that $x_n/y_n=1$, as n tends to infinity.

The proof of Theorem 2.1.2 is left in the Appendix. Unlike the first case where $\{x(t)\}$ is a white noise, the strong variance scale exponent no longer exists for the current case, while the weak variance scale exponent and adjusted proportion coefficient can be specified as follows.

Proposition 2.1.2. If the time series $\{x(t)\}$ is a short memory time series, ARMA(p, q), then the weak variance scale exponent and adjusted proportion coefficient should respectively satisfy:

$$F'=0.5, f \neq 1.$$

Theorem 2.1.3. If the time series $\{x(t)\}$ is a fractional Gaussian noises process (Mandelbrot and Van Ness, 1968), then

$$D[x_n(t)] \sim n^{2H} D[x(t)], \text{ as } n \text{ tends to infinity.}$$

The details for the derivation of Theorem 2.1.3 are presented in the Appendix. As this is again the convergence of the variance of the time scale series, we have the following proposition.

Proposition 2.1.3. If the time series $\{x(t)\}$ is a fractional Gaussian noises process, then the weak variance scale exponent and adjusted proportion coefficient should respectively satisfy:

$$F'=H, f=1.$$

Theorem 2.1.4. If the time series $\{x(t)\}$ is a fractional differencing noises process (Granger and Joyeux, 1980; Hosking, 1981, ARFIMA(0, d , 0)), then

$$D[x_n(t)]/D[x(t)] \sim \frac{\Gamma(1-d)}{(1+2d)\Gamma(1+d)} n^{1+2d}, \text{ as } n \text{ tends to infinity.}$$

Here, $\Gamma(\cdot)$ denotes the Gamma function.

The detailed proof is organized in the Appendix. Considering again the definition of the weak variance scale exponent, we can easily obtain the results illustrated below.

Proposition 2.1.4. If the time series $\{x(t)\}$ is a fractional differencing noises process, then the weak variance scale exponent and adjusted proportion coefficient should respectively satisfy:

$$F'=d+0.5, f = \frac{\Gamma(1-d)}{(1+2d)\Gamma(1+d)}.$$

2.2. Frequency Domain Analysis

If the autocovariance of the stationary time series $\{x(t)\}$, γ_i , is absolutely summable, then the spectral density $f(\lambda)$ can be expressed as

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-ik\lambda} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) \cos(k\lambda).$$

Setting $\lambda=0$ yields

$$f(0) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k).$$

With the utilization of the above results as well as Proposition 2.2, the following results can be easily derived.

Theorem 2.2.1. If the time series $\{x(t)\}$ is a white noise, then

$$D[x_n(t)] = 2\pi n f(0), n = 1, 2, \dots$$

Theorem 2.2.2. If the time series $\{x(t)\}$ is a short memory time series, ARMA(p, q), then

$$D[x_n(t)] = 2\pi n f(0), \text{ as } n \text{ tends to infinity.}$$

It should be remarked here that Proposition 2.1.1 and Proposition 2.1.2 can also be obtained from

Theorem 2.2.1 and Theorem 2.2.2, respectively. It also needs to be pointed out that the autocovariance of the long memory time series is not absolutely summable, which implies that we are unable to obtain similar results for this particular type of time series. However, we manage to derive a generalized relationship theorem between the spectral density $f(\lambda)$ of the time series $\{x(t)\}$ and the variance $D[x_n(t)]$ of the time scale n series of $\{x(t)\}$, which is presented in the following theorem.

Theorem 2.2.3. For any stationary time series $\{x(t)\}$, we always have

$$D[x_k(t)] = 2 \int_{-\pi}^{\pi} f(\lambda) \frac{1 - \cos(k\lambda)}{[2 \sin(\lambda/2)]^2} d\lambda.$$

The proof of Theorem 2.2.3 is again left in the Appendix. The results in Theorem 2.2.3 can be further generalized as follows.

Theorem 2.2.4. If the time series $\{x(t)\}$ is generalized to ARFIMA(p, d, q), then

$$D[x_n(t); p, d, q] \sim c(\phi, \theta) \frac{\Gamma(1-d)}{(1+2d)\Gamma(1+d)} D[x(t)] n^{2d+1}, \text{ as } n \text{ tends to infinity.}$$

Here, $c(\phi, \theta) = 1$ if and only if $p=q=0$.

We refer interested readers to the Appendix for the complete proof of Theorem 2.2.4. Clearly, the weak variance scale exponent of the generalized time series can be easily determined.

Proposition 2.2.1. If the time series $\{x(t)\}$ is generalized to ARFIMA(p, d, q), then the weak variance scale exponent and adjusted proportion coefficient should respectively satisfy:

$$F' = d + 0.5, f = \frac{\Gamma(1-d)}{(1+2d)\Gamma(1+d)} c(\phi, \theta).$$

Again, $c(\phi, \theta) = 1$ if and only if $p=q=0$.

3. Wnoise and SLmemory Statistics

In order to analyze the memory properties of white noise, short memory and long memory time series, two hypothesis test statistics are constructed in the section.

If the stationary time series $\{x(t)\}$ is independent of a white noise, when n approaches infinity, applying Lindbergh Central Limit Theorem leads to

$$\sqrt{n} \left(\sum_{t=1}^n x(t) / n - \mu \right) \xrightarrow{L} N(0, D[x(t)])$$

and

$$\frac{x_n(t) - n\mu}{\sqrt{nD[x(t)]}} \xrightarrow{L} N(0, 1)$$

With $\hat{D}[x(t)]$ denoted as the sample variance, it is not difficult to obtain from Slutsky Theorem that

$$\frac{x_n(t) - n \sum_{t=1}^N x(t) / N}{\sqrt{n\hat{D}[x(t)]}} \xrightarrow{L} N(0, 1)$$

and

$$(N(n)-1) \frac{\hat{D}[x_n(t)]}{n\hat{D}[x(t)]} \xrightarrow{L} \chi^2(N(n)-1)$$

Here, $N(n)-1$ represents the freedom of the sample variance of $\{x_n(t)\}$.

Therefore, we can define

$$Wnoise(n) = (N(n)-1) \frac{\hat{D}[x_n(t)]}{n\hat{D}[x(t)]}$$

for $Wnoise$ statistic tests under independent white noise hypotheses and non-independent stochastic process alternatives.

As a generalization, if the stationary time series $\{x(t)\}$ is a short memory series, applying

central limit theorem (Anderson, 1971, Theorem 7.7.8; Brockwell and Davis, 1991, Theorem 7.1.2) yields

$$\sqrt{n} \left(\sum_{t=1}^n x(t) / n - \mu \right) \xrightarrow{L} N(0, \sum_{i=-\infty}^{\infty} \gamma_i)$$

Based on Slutsky Theorem and Proposition 2.4, we can obtain

$$\frac{x_n(t) - n \sum_{t=1}^N x(t) / N}{\sqrt{n(\hat{D}[x_{k+1}(t)] - \hat{D}[x_k(t)])}} \xrightarrow{L} N(0, 1)$$

Thus, we have

$$\frac{(N(n)-1) \hat{D}[x_n(t)]}{n(\hat{D}[x_{k+1}(t)] - \hat{D}[x_k(t)])} \xrightarrow{L} \chi^2(N(n)-1)$$

Obviously, we can use

$$SLmemory(n) = \frac{(N(n)-1) \hat{D}[x_n(t)]}{n(\hat{D}[x_{k+1}(t)] - \hat{D}[x_k(t)])}$$

for *SLmemory* statistic tests under white noise, short memory null hypotheses and long memory alternatives.

Furthermore, if the alternatives are limited in the situation of anti-persistent long memory, where the two classical long memory parameters satisfy $-0.5 < d < 0$, $0 < H < 0.5$ respectively, the reject field of the hypothesis test of *SLmemory* statistic can be specified as

$$\{SLmemory(k) \geq \chi_{1-\alpha}^2(N(k)-1)\}$$

with α being the significance level. On the other hand, if the alternatives are limited in the situation of persistent long memory, where two classical long memory parameters satisfy $d > 0$, $H > 0.5$ respectively, the reject field of the hypothesis test of *SLmemory* statistic should be

$$\{SLmemory(k) \leq \chi_{\alpha}^2(N(k)-1)\}$$

4. Monte Carlo Performance

In this section, we illustrate the asymptotic properties for the weak variance scale exponent estimator and *SLmemory* statistic, in order to examine their finite-sample performance, and to give advice on how to choose the time scale n in practice. We focus on the MSE for the weak variance scale exponent estimator, and the empirical size and power for *SLmemory* statistic. Throughout the simulation exercise, the number of replications is 1000.

4.1. Monte Carlo Study for Weak Variance Scale Exponent Estimator

The definition of the weak variance scale exponent can be rearranged as

$$y_i = \hat{f}x_i^{F'}$$

where $y_i = D[x_i(t)]/D[x(t)]$, $x_i = i^2$, $i = 2, L, n$, with n being the maximum time scale of finite-sample series. We can firstly estimate the parameter F' through the non-linear regression with the sample series $\{y_i\}$ and $\{x_i\}$, and then obtain the estimator of the weak variance scale exponent.

Let the stationary time series $\{x(t)\}$ be a linear generalized Gaussian process, ARFIMA(1, d , 1), in the form of $(1 - \phi B)(1 - B)^d X_t = (1 - \theta B)\varepsilon_t$, with unit standard deviation, for different values of ϕ , θ and d . Four sample sizes, $N=250, 500, 1000, 2000$, are considered, and the parameter d is estimated by using the weak variance scale exponent with the maximum time scale setting to be $n=[N^m]$, where m are the 13 numbers ranging from 0.2 to 0.8 with the step size chosen to be 0.05.

Table 1 displays the MSE of the weak variance scale exponent estimator calculated for different values of the maximum time scale $n=[N^m]$. It can be easily noticed that no matter d takes the value of $-0.1, 0$ or 0.1 , the MSE always shows an approximate U-shape curve with respect to the value of m . Therefore, $m=0.5$ and $n=[N^{0.5}]$ could be the optimal choice of the weak variance scale exponent estimator in practice.

4.2. Monte Carlo Study for *SLmemory* Statistic

The Monte Carlo study for *SLmemory* statistic investigates the percentage of the replications where the rejection of a short memory (or white noise) null hypothesis was observed. Thus, if a data generating process belongs to the null hypothesis, the empirical test sizes will be calculated, whereas the empirical power of the tests will be provided if it is a long memory process.

Let stationary time series $\{x(t)\}$ be a linear model, ARMA(1, 1), expressed in the form of $(1 - \phi B)X_t = (1 - \theta B)\varepsilon_t$, with unit standard deviation, for different values of ϕ and θ . We consider two sample sizes $N=500, 1000$, and estimate the parameter d using the weak variance scale exponent with optimal time scale $n=[N^m]$, where m again can take the value from 0.2 to 0.8 with the step size being 0.05. Table 2 exhibits the empirical size of the ARMA(1, 1) model. The significant characteristic that can be observed in Table 2 is that as m increases, the empirical size shows an approximate U-shape curve with the left values higher than the right ones.

What is presented in Table 3 is the power of the tests under long memory alternatives. We consider the long memory model, ARFIMA(1, d , 1), formulated as $(1 - \phi B)(1 - B)^d X_t = (1 - \theta B)\varepsilon_t$, with unit standard deviation, for different values of ϕ , θ and d . With two sample sizes chosen as $N=500, 1000$, an opposite phenomenon can be witnessed that the empirical power displays an approximate inverse U-shape curve with the left values higher than the right ones.

Combining the significant characteristics of the U-shape and inverse U-shape curve shown in Table 2 and 3, we advice that $m=0.4$ and $n=[N^{0.4}]$ can be optimal choice of *SLmemory* statistic in practice. The results of the Monte Carlo study on the empirical size and power for *SLmemory* statistic demonstrates significant advantages, compared with long memory tests in the literature (Lo, 1991; Giraitis et al., 2003, 2005), which investigate bigger values of the long memory parameter, $d=1/3, -1/3$ and $d=0.2, 0.3, 0.4$, respectively.

4.3. Application to Sino-US Stock Index Return rate Data

We illustrate the theory and simulations by applying the weak variance scale exponent estimation and *SLmemory* statistic to the logarithmic return rate data of Sino-US stock index. To make the results comparable with the simulations, we divide each logarithmic return rate series into three blocks with the length of each block being approximately 1000, as shown in Table 4.

Table 4. Descriptive Statistics

CN. and US. Block denotes Shanghai Composite and Standard & Poor 500 index data respectively. Block 1 and 2 are both for a period of four years, ranging from January 1, 2000 to December 31, 2003 and from January 1, 2004 to December 31, 2007, respectively, while Block 3 represents the period from January 1, 2008 to May 25, 2012.

Table 1

MSE (in %) of weak variance scale exponent estimator

d	$N=250$				$N=500$				$N=1000$				$N=2000$			
	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$
	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$
-0.1	$n=[N^{0.2}]=3$				$n=[N^{0.2}]=3$				$n=[N^{0.2}]=3$				$n=[N^{0.2}]=4$			
	9.645	4.319	15.201	0.287	9.511	4.296	15.222	0.167	9.524	4.232	15.162	0.117	7.740	2.564	11.493	0.082
	6.817	2.648	10.685	0.216	6.832	2.621	10.693	0.106	6.833	2.598	10.690	0.048	5.488	1.490	8.050	0.027
	4.484	1.431	6.847	0.274	4.496	1.460	6.936	0.164	4.555	1.447	6.957	0.096	3.612	0.784	5.200	0.056
0	$n=[N^{0.25}]=3$				$n=[N^{0.25}]=4$				$n=[N^{0.25}]=5$				$n=[N^{0.25}]=6$			
	9.645	4.319	15.201	0.287	7.772	2.655	11.556	0.165	6.354	1.769	9.010	0.101	5.247	1.298	7.260	0.068
	6.817	2.648	10.685	0.216	5.475	1.532	8.038	0.123	4.410	0.990	6.230	0.063	3.604	0.691	4.962	0.036
	4.484	1.431	6.847	0.274	3.538	0.809	5.140	0.171	2.832	0.487	3.942	0.092	2.288	0.332	3.115	0.055
0.1	$n=[N^{0.3}]=5$				$n=[N^{0.3}]=6$				$n=[N^{0.3}]=7$				$n=[N^{0.3}]=9$			
	6.536	1.936	9.112	0.317	5.337	1.414	7.357	0.184	4.415	1.036	5.972	0.100	3.215	0.688	4.241	0.065
	4.443	1.139	6.244	0.277	3.635	0.784	4.982	0.164	2.973	0.552	4.015	0.086	2.107	0.344	2.776	0.052
	2.792	0.576	3.836	0.357	2.245	0.401	3.071	0.212	1.828	0.265	2.455	0.110	1.265	0.161	1.657	0.065
-0.1	$n=[N^{0.35}]=6$				$n=[N^{0.35}]=8$				$n=[N^{0.35}]=11$				$n=[N^{0.35}]=14$			
	5.473	1.525	7.395	0.343	3.873	0.974	5.141	0.212	2.490	0.565	3.213	0.126	1.740	0.381	2.214	0.080
	3.656	0.911	4.984	0.323	2.553	0.544	3.366	0.207	1.583	0.306	2.036	0.128	1.066	0.193	1.353	0.077
	2.243	0.481	2.992	0.407	1.515	0.309	2.004	0.258	0.906	0.174	1.165	0.155	0.593	0.112	0.748	0.093
0	$n=[N^{0.4}]=9$				$n=[N^{0.4}]=12$				$n=[N^{0.4}]=15$				$n=[N^{0.4}]=20$			
	3.520	1.010	4.461	0.431	2.390	0.679	3.015	0.280	1.643	0.419	2.064	0.153	1.048	0.269	1.295	0.101
	2.266	0.689	2.903	0.453	1.502	0.429	1.877	0.298	1.001	0.256	1.246	0.167	0.611	0.157	0.751	0.106
	1.329	0.462	1.631	0.557	0.857	0.320	1.060	0.356	0.555	0.180	0.686	0.194	0.333	0.121	0.397	0.125

Table 1 (continued)

MSE (in %) of weak variance scale exponent estimator

d	$N=250$				$N=500$				$N=1000$				$N=2000$			
	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$
	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$
-0.1	$n=[N^{0.45}]=11$				$n=[N^{0.45}]=16$				$n=[N^{0.45}]=22$				$n=[N^{0.45}]=30$			
	2.823	0.883	3.447	0.488	1.690	0.592	2.072	0.343	1.016	0.348	1.242	0.198	0.645	0.228	0.758	0.138
	1.823	0.679	2.251	0.536	1.074	0.451	1.282	0.391	0.625	0.270	0.736	0.234	0.382	0.180	0.442	0.160
	1.067	0.519	1.228	0.652	0.627	0.392	0.723	0.457	0.372	0.241	0.420	0.271	0.235	0.169	0.255	0.178
0	$n=[N^{0.5}]=15$				$n=[N^{0.5}]=22$				$n=[N^{0.5}]=31$				$n=[N^{0.5}]=44$			
	2.098	0.826	2.406	0.619	1.223	0.570	1.439	0.426	0.756	0.367	0.882	0.268	0.478	0.243	0.528	0.192
	1.438	0.772	1.660	0.714	0.851	0.551	0.958	0.528	0.509	0.336	0.562	0.321	0.327	0.241	0.352	0.237
	0.924	0.716	0.948	0.875	0.582	0.530	0.606	0.614	0.385	0.348	0.388	0.389	0.254	0.249	0.260	0.258
0.1	$n=[N^{0.55}]=20$				$n=[N^{0.55}]=30$				$n=[N^{0.55}]=44$				$n=[N^{0.55}]=65$			
	1.760	0.926	1.918	0.805	1.040	0.643	1.170	0.560	0.672	0.448	0.749	0.382	0.435	0.311	0.463	0.278
	1.367	0.978	1.485	0.959	0.839	0.724	0.894	0.714	0.538	0.472	0.562	0.468	0.377	0.350	0.387	0.353
	1.084	1.041	1.018	1.232	0.744	0.760	0.719	0.865	0.493	0.513	0.492	0.547	0.370	0.386	0.366	0.400
-0.1	$n=[N^{0.6}]=27$				$n=[N^{0.6}]=41$				$n=[N^{0.6}]=63$				$n=[N^{0.6}]=95$			
	1.739	1.179	1.787	1.130	1.115	0.836	1.155	0.800	0.703	0.599	0.778	0.535	0.520	0.429	0.517	0.423
	1.540	1.335	1.599	1.331	1.023	0.982	1.051	0.979	0.696	0.701	0.706	0.703	0.514	0.508	0.521	0.514
	1.503	1.541	1.379	1.758	1.091	1.140	1.043	1.239	0.723	0.777	0.731	0.795	0.594	0.592	0.568	0.624
0	$n=[N^{0.65}]=36$				$n=[N^{0.65}]=56$				$n=[N^{0.65}]=89$				$n=[N^{0.65}]=139$			
	1.969	1.566	1.959	1.544	1.343	1.162	1.361	1.121	0.899	0.829	0.941	0.794	0.695	0.603	0.654	0.629
	1.888	1.818	1.917	1.817	1.378	1.402	1.393	1.399	0.986	1.035	0.991	1.037	0.757	0.778	0.757	0.783
	2.145	2.207	1.996	2.415	1.580	1.697	1.583	1.724	1.164	1.198	1.146	1.236	0.922	0.909	0.893	0.952

Table 1 (*continued*)

MSE (in %) of weak variance scale exponent estimator

d	$N=250$				$N=500$				$N=1000$				$N=2000$			
	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$	$\varphi=0.5$	$\varphi=0$
	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$	$\theta=0$	$\theta=-0.5$	$\theta=-0.5$	$\theta=0$
-0.1	$n=[N^{0.7}]=47$				$n=[N^{0.7}]=77$				$n=[N^{0.7}]=125$				$n=[N^{0.7}]=204$			
	2.394	2.091	2.383	2.062	1.781	1.639	1.791	1.604	1.348	1.219	1.300	1.260	0.995	0.962	1.000	0.948
	2.480	2.425	2.498	2.426	1.967	2.021	1.974	2.019	1.476	1.586	1.477	1.589	1.130	1.166	1.129	1.171
	2.971	3.116	2.920	3.226	2.335	2.376	2.274	2.464	1.909	1.853	1.805	1.971	1.452	1.491	1.476	1.479
0	$n=[N^{0.75}]=62$				$n=[N^{0.75}]=105$				$n=[N^{0.75}]=177$				$n=[N^{0.75}]=299$			
	3.158	2.923	3.193	2.852	2.489	2.321	2.449	2.324	1.952	1.885	1.959	1.870	1.545	1.551	1.571	1.501
	3.535	3.427	3.547	3.419	2.786	2.872	2.787	2.871	2.305	2.411	2.300	2.416	1.774	1.753	1.770	1.764
	4.257	4.406	4.251	4.457	3.471	3.327	3.237	3.583	2.806	2.866	2.835	2.862	2.375	2.389	2.358	2.365
0.1	$n=[N^{0.8}]=82$				$n=[N^{0.8}]=144$				$n=[N^{0.8}]=251$				$n=[N^{0.8}]=437$			
	4.722	4.286	4.560	4.338	3.600	3.398	3.523	3.434	2.985	2.841	2.917	2.891	2.491	2.502	2.511	2.454
	5.501	4.992	5.539	4.971	4.227	4.438	4.225	4.428	3.453	3.592	3.449	3.599	2.723	2.716	2.755	2.775
	6.447	6.159	6.025	6.567	5.274	4.931	4.851	5.373	4.344	4.182	4.130	4.367	3.625	3.776	3.822	3.699

Table 2

Empirical test sizes (in %) of *SLmemory* statistic of time series under the null hypothesis of ARMA(1, 1) model, $(1-\phi B)X_t=(1-\theta B)\varepsilon_t$, with standard normal innovations.

N	n	$\phi=0$ $\theta=0$			$\phi=0.5$ $\theta=0$			$\phi=-0.5$ $\theta=0$			$\phi=0.8$ $\theta=0$		
		$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$
500	$n=[N^{0.2}]=3$	0.223	0.123	0.040	0.999	0.995	0.978	0.994	0.986	0.962	1.000	1.000	1.000
	$n=[N^{0.25}]=4$	0.204	0.108	0.037	0.973	0.922	0.564	0.520	0.414	0.285	1.000	1.000	1.000
	$n=[N^{0.3}]=6$	0.187	0.101	0.037	0.578	0.336	0.028	0.501	0.405	0.257	1.000	1.000	0.981
	$n=[N^{0.35}]=8$	0.167	0.106	0.041	0.261	0.076	0.003	0.431	0.338	0.226	0.989	0.947	0.457
	$n=[N^{0.4}]=12$	0.169	0.115	0.048	0.100	0.028	0.009	0.357	0.272	0.164	0.655	0.275	0.000
	$n=[N^{0.45}]=16$	0.180	0.120	0.062	0.103	0.053	0.027	0.292	0.220	0.122	0.224	0.024	0.001
	$n=[N^{0.5}]=22$	0.176	0.122	0.067	0.106	0.071	0.037	0.248	0.199	0.137	0.063	0.023	0.006
	$n=[N^{0.55}]=30$	0.179	0.143	0.093	0.144	0.103	0.074	0.239	0.187	0.124	0.082	0.056	0.026
	$n=[N^{0.6}]=41$	0.199	0.168	0.115	0.169	0.139	0.091	0.239	0.184	0.138	0.119	0.089	0.059
	$n=[N^{0.65}]=56$	0.231	0.186	0.150	0.248	0.197	0.148	0.264	0.218	0.179	0.194	0.170	0.132
	$n=[N^{0.7}]=77$	0.251	0.217	0.178	0.265	0.225	0.189	0.286	0.254	0.205	0.248	0.213	0.176
	$n=[N^{0.75}]=105$	0.329	0.300	0.277	0.317	0.288	0.266	0.336	0.309	0.279	0.322	0.292	0.254
	$n=[N^{0.8}]=144$	0.363	0.345	0.318	0.363	0.338	0.312	0.363	0.348	0.321	0.373	0.356	0.339
1000													
	$n=[N^{0.2}]=3$	0.210	0.102	0.022	1.000	1.000	1.000	1.000	1.000	0.997	1.000	1.000	1.000
	$n=[N^{0.25}]=5$	0.179	0.094	0.028	0.982	0.941	0.667	0.862	0.816	0.700	1.000	1.000	1.000
	$n=[N^{0.3}]=7$	0.150	0.082	0.024	0.679	0.476	0.094	0.613	0.516	0.358	1.000	1.000	1.000
	$n=[N^{0.35}]=11$	0.144	0.075	0.028	0.221	0.068	0.003	0.395	0.298	0.172	0.989	0.944	0.505

Table 2 (*continued*)

Empirical test sizes (in %) of *SLmemory* statistic of time series under the null hypothesis of ARMA(1, 1) model, $(1 - \phi B) X_t = (1 - \theta B) \varepsilon_t$, with standard normal innovations.

N	n	$\phi=0$ $\theta=0$			$\phi=0.5$ $\theta=0$			$\phi=-0.5$ $\theta=0$			$\phi=0.8$ $\theta=0$		
		$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$
1000	$n = [N^{0.4}] = 15$	0.140	0.076	0.026	0.104	0.026	0.004	0.324	0.253	0.147	0.750	0.440	0.013
	$n = [N^{0.45}] = 22$	0.124	0.069	0.031	0.087	0.038	0.014	0.256	0.183	0.101	0.228	0.042	0.000
	$n = [N^{0.5}] = 31$	0.157	0.091	0.046	0.106	0.064	0.026	0.240	0.166	0.090	0.082	0.025	0.012
	$n = [N^{0.55}] = 44$	0.172	0.128	0.060	0.142	0.098	0.044	0.213	0.153	0.096	0.085	0.059	0.032
	$n = [N^{0.6}] = 63$	0.196	0.143	0.089	0.156	0.119	0.070	0.204	0.154	0.093	0.138	0.100	0.063
	$n = [N^{0.65}] = 89$	0.234	0.193	0.132	0.217	0.169	0.129	0.251	0.206	0.143	0.170	0.143	0.110
	$n = [N^{0.7}] = 125$	0.267	0.239	0.191	0.285	0.244	0.188	0.296	0.248	0.196	0.231	0.193	0.150
	$n = [N^{0.75}] = 177$	0.322	0.285	0.243	0.304	0.275	0.240	0.317	0.288	0.251	0.284	0.255	0.217
	$n = [N^{0.8}] = 251$	0.383	0.355	0.322	0.376	0.349	0.329	0.390	0.362	0.332	0.373	0.346	0.322

Table 3

Empirical power of the tests (in %) based on $SLmemory$ statistic, the alternatives considered are ARFIMA(1, d , 1) model, $(1 - \varphi B)(1 - B)^d X_t = (1 - \theta B)\varepsilon_t$, with standard normal innovations.

N	n	$d=0.1$						$d=-0.1$					
		$\varphi=0.5$ $\theta=0$			$\varphi=0$ $\theta=0$			$\varphi=0.5$ $\theta=0$			$\varphi=0$ $\theta=0$		
		$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$
500	$n=[N^{0.2}]=3$	0	0	0	0.362	0.53	0.827	0.038	0.092	0.381	0.337	0.417	0.607
	$n=[N^{0.25}]=4$	0.001	0.003	0.044	0.465	0.652	0.932	0.349	0.536	0.908	0.37	0.487	0.675
	$n=[N^{0.3}]=6$	0.072	0.206	0.702	0.645	0.836	0.991	0.825	0.943	0.994	0.437	0.542	0.708
	$n=[N^{0.35}]=8$	0.331	0.598	0.977	0.732	0.894	0.996	0.86	0.939	0.981	0.523	0.637	0.769
	$n=[N^{0.4}]=12$	0.73	0.936	0.997	0.874	0.965	0.991	0.804	0.87	0.942	0.58	0.679	0.789
	$n=[N^{0.45}]=16$	0.88	0.986	0.999	0.895	0.968	0.992	0.76	0.819	0.891	0.616	0.698	0.805
	$n=[N^{0.5}]=22$	0.927	0.976	0.987	0.921	0.963	0.983	0.719	0.79	0.87	0.617	0.686	0.799
	$n=[N^{0.55}]=30$	0.925	0.949	0.972	0.906	0.942	0.961	0.685	0.749	0.837	0.642	0.704	0.8
	$n=[N^{0.6}]=41$	0.884	0.907	0.933	0.879	0.901	0.931	0.7	0.758	0.813	0.675	0.709	0.784
	$n=[N^{0.65}]=56$	0.844	0.869	0.903	0.837	0.87	0.9	0.67	0.74	0.804	0.658	0.714	0.783
	$n=[N^{0.7}]=77$	0.78	0.808	0.84	0.773	0.803	0.841	0.648	0.685	0.728	0.628	0.672	0.73
	$n=[N^{0.75}]=105$	0.689	0.711	0.746	0.69	0.715	0.748	0.62	0.655	0.677	0.593	0.633	0.676
	$n=[N^{0.8}]=144$	0.633	0.658	0.669	0.627	0.647	0.671	0.572	0.59	0.613	0.558	0.584	0.608
1000													
	$n=[N^{0.2}]=3$	0	0	0	0.102	0.204	0.504	0	0.001	0.016	0.131	0.196	0.378
	$n=[N^{0.25}]=5$	0	0	0.005	0.29	0.468	0.828	0.458	0.663	0.942	0.23	0.335	0.523

Table 3 (continued)

Empirical power of the tests (in %) based on $SLmemory$ statistic, the alternatives considered are ARFIMA(1, d , 1) model, $(1 - \varphi B)(1 - B)^d X_t = (1 - \theta B)\varepsilon_t$, with standard normal innovations.

N	n	$d=0.1$						$d=-0.1$					
		$\varphi=0.5$ $\theta=0$			$\varphi=0$ $\theta=0$			$\varphi=0.5$ $\theta=0$			$\varphi=0$ $\theta=0$		
		$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.05$	$\alpha=0.01$
1000	$n = [N^{0.3}] = 7$	0.012	0.052	0.305	0.434	0.643	0.943	0.848	0.943	0.996	0.342	0.432	0.623
	$n = [N^{0.35}] = 11$	0.25	0.51	0.949	0.625	0.855	0.992	0.838	0.919	0.973	0.43	0.56	0.75
	$n = [N^{0.4}] = 15$	0.549	0.807	0.998	0.749	0.937	0.998	0.753	0.844	0.93	0.491	0.612	0.757
	$n = [N^{0.45}] = 22$	0.787	0.951	0.997	0.859	0.965	0.995	0.699	0.796	0.899	0.56	0.657	0.8
	$n = [N^{0.5}] = 31$	0.912	0.991	0.996	0.926	0.985	0.995	0.717	0.787	0.872	0.621	0.705	0.811
	$n = [N^{0.55}] = 44$	0.94	0.972	0.988	0.932	0.967	0.988	0.697	0.759	0.856	0.634	0.709	0.818
	$n = [N^{0.6}] = 63$	0.907	0.928	0.961	0.896	0.926	0.96	0.674	0.746	0.821	0.647	0.714	0.8
	$n = [N^{0.65}] = 89$	0.848	0.878	0.916	0.843	0.878	0.918	0.671	0.719	0.784	0.641	0.703	0.753
	$n = [N^{0.7}] = 125$	0.782	0.814	0.853	0.786	0.817	0.848	0.644	0.694	0.747	0.634	0.684	0.739
	$n = [N^{0.75}] = 177$	0.748	0.768	0.795	0.748	0.765	0.795	0.634	0.667	0.716	0.631	0.663	0.693
	$n = [N^{0.8}] = 251$	0.654	0.684	0.71	0.658	0.68	0.712	0.595	0.621	0.648	0.6	0.625	0.65

	Sample Sizes	Mean	Std. Dev	Minimum	Maximum	Skewness	Kurtosis
CN. Block 1	957	0.0095	1.3666	-6.5437	9.4014	0.7929	7.5504
CN. Block 2	968	0.1298	1.6209	-9.2562	7.8903	-0.4333	3.1152
CN. Block 3	1069	-0.0761	1.9062	-8.0437	9.0348	-0.1541	2.8143
US. Block 1	1004	-0.0278	1.3814	-6.0045	5.5744	0.1447	1.2812
US. Block 2	1006	0.0276	0.7612	-3.5343	2.8790	-0.3088	1.7889
US. Block 3	1110	-0.0097	1.7393	-9.4695	10.9572	-0.2300	6.3233

The weak variance exponent estimators, the *SLmemory* statistic values and corresponding *P*-values are displayed in Table 5. The evidence against the null hypothesis, being in favor of the long memory alternative, is strong for the second blocks of Sino-US stock index series, where the weak variance exponent estimators are 0.6379 and 0.4239 respectively. The first and third blocks of Sino-US stock index series show that their weak variance exponent estimators are all close to 0.5 and the corresponding *P*-values are all beyond 0.3, which implies that the data exhibit some forms of accepting the null hypothesis discussed in the present paper. In a word, the weak variance exponent estimators are consistent with statistical tests for Sino-US stock index series, and the results demonstrate that there is long memory in some periods and no long memory in some other periods.

Table 5. Estimation and Test Results for Sino-US Stock Index Data

	CN. Block 1	CN. Block 2	CN. Block 3	US. Block 1	US. Block 2	US. Block 3
$\hat{F}'$	0.5093	0.6379	0.5306	0.4596	0.4239	0.5045
$\hat{f}$	0.9752	0.5831	0.9059	1.0177	1.0046	0.6811
<i>SLmemory</i>	57.5542	41.5572	59.4155	66.4455	86.64281	63.36272
Freedom	62	63	65	65	66	68
<i>P</i> -value	0.3636	0.0169	0.3279	0.4269	0.0451	0.3633

It can be found again that applying $n=[N^{0.5}]$ for the weak variance scale exponent estimator and $n=[N^{0.4}]$ for *SLmemory* statistic test should be recommended in practice. For the application to Sino-US stock data, it is not difficult to find that there exist different long memory properties between Sino-US stock data series in the period of Block 2, since the long memory properties of Shanghai Composite and Standard & Poor 500 index data series are persistent and anti-persistent, respectively, with persistence and anti-persistence representing positive and negative correlated long-range effect, respectively. The possible economics interpretation for the phenomenon observed in the periods of Block 2 can be that Shanghai Composite index market was more vulnerable to the impact of external events, which implies that the influence can last for a relatively long time period, while Standard & Poor 500 index market can quickly return to stability under the influence of external shocks.

5. Conclusion

This paper introduces a new system for theoretical analysis of stationary time series, especially for the estimation and test of long memory properties. Combining the analysis in both of the time and frequency domain, the properties of the strong and weak variance scale exponent of the white noise, short memory ARMA(p, q) and long memory ARFIMA(p, d, q) are presented, and in particular, the equations between weak variance scale exponent and long memory parameter d ($-0.5 < d < 0.5$), H ($0 < H < 1$) are established. Furthermore, we construct *Wnoise* and *SLmemory* statistic tests under the case of independent white noise null hypotheses and non-independent stochastic process alternatives, and the case of white noise, short memory null hypothesis and long

memory alternatives, respectively. The estimation and tests embrace the possibility of memory properties of persistence or anti-persistence in theory, being different from the V/S statistic tests (Giraitis et al., 2003, 2005), where only persistence situations are allowed.

A Monte Carlo study for the weak variance scale exponent and *SLmemory* statistic is carried out for the selection of the time scale n of finite-sample. By minimizing the MSE for estimation and optimizing empirical size and power for tests, we find that $n=[N^{0.5}]$ and $n=[N^{0.4}]$ can be recommended in practice for the weak variance scale exponent estimator and *SLmemory* statistic test, respectively. In fact, the time scale n can also be analogous to the bandwidth (Abadir et al., 2009).

Further study can be led into analyzing the properties of the strong and weak variance scale exponent for non-stationary or non-invertible time series, more complicated long memory models and corresponding statistical tests. Surely, on the estimation of autocovariance, Proposition 2.3 can be regarded as a replacement of the Durbin-Levinson algorithm.

Acknowledgments

The second author would like to gratefully acknowledge the financial support from the Fundamental Research Funds for the Central Universities NO. JUSRP11611, National Natural Science Foundation of China NO. 11601189 and NO. 71701081, and Natural Science Foundation of Jiangsu Province NO. BK20160156.

Appendix. Proofs of the theorems and an auxiliary lemma

Proof of Theorem 2.1.2. Proposition 2.2 can be rewritten as

$$D[x_n(t)] = \sum_{i=-n}^n (n-i)\gamma_i = n\gamma_0 \sum_{i=-n}^n \rho_i - \sum_{i=-n}^n i\gamma_i.$$

The accumulation of autocorrelation ρ_i for any short memory time series converges to a constant, i.e.,

$$\lim_{n \rightarrow \infty} \sum_{i=-n}^n \rho_i \rightarrow c, \quad c \text{ is a constant.}$$

Moreover, applying Kronecker lemma on short memory time series yields

$$\frac{1}{n} \sum_{i=1}^n i\gamma_i \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Therefore, we finally arrive at

$$D[x_n(t)]/D[x(t)] \sim cn, \quad \text{as } n \rightarrow \infty.$$

Proof of Theorem 2.1.3. The Lag- n autocovariance γ_n of a fractional Gaussian noises process $\{x(t)\}$ satisfies

$$\gamma_n = \frac{\gamma_0}{2} \left[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H} \right], \quad \text{as } n \rightarrow \infty.$$

Thus, with the utilization of Proposition 2.3,

$$\gamma_n = \frac{1}{2} [D[x_{n+1}(t)] - 2D[x_n(t)] + D[x_{n-1}(t)]], \quad n = 1, 2, \dots$$

We can easily obtain

$$D[x_n(t)] : D[x(t)]n^{2H}, \quad \text{as } n \rightarrow \infty.$$

Proof of Theorem 2.1.4. Proposition 2.2 can be alternatively expressed as

$$D[x_n(t)] = n\gamma_0 + 2\gamma_0 \left[(n+d) \sum_{k=1}^n \rho_k - \sum_{k=1}^n (k+d) \rho_k \right]$$

If we denote $S_1(n) = \sum_{k=1}^n \rho_k$, $S_2(n) = \sum_{k=1}^n (k+d) \rho_k$, a further calculation leads to

$$S_1(n) = \sum_{k=1}^n \rho_k = \frac{d}{1-d} + \frac{d(1+d)}{(1-d)(2-d)} + \dots + \frac{d(1+d) \cdots (n-1+d)}{(1-d)(2-d) \cdots (n-d)}$$

$$\begin{aligned}
&= \frac{1}{2} \left(\frac{1+d}{1-d} - 1 \right) + \frac{1}{2} \frac{1+d}{1-d} \left(\frac{2+d}{2-d} - 1 \right) + \dots + \frac{1}{2} \frac{(1+d)(2+d) \dots (n-1+d)}{(1-d)(2-d) \dots (n-1-d)} \left[\frac{(n+d)}{(n-d)} - 1 \right] \\
&= -\frac{1}{2} + \frac{1}{2} \frac{\Gamma(1-d)\Gamma(n+1+d)}{\Gamma(1+d)\Gamma(n+1-d)}
\end{aligned}$$

and

$$\begin{aligned}
S_2(n) &= \sum_{k=1}^n (k+d) \rho_k = d \left[\frac{1+d}{1-d} + \frac{(1+d)(2+d)}{(1-d)(2-d)} + \dots + \frac{(1+d)(2+d) \dots (n-1+d)(n+d)}{(1-d)(2-d) \dots (n-1-d)(n-d)} \right] \\
&= d \left\{ \frac{1+d}{1+2d} \left(\frac{2+d}{1-d} - 1 \right) + \frac{1+d}{1+2d} \left[\frac{(2+d)(3+d)}{(1-d)(2-d)} - \frac{(2+d)}{(1-d)} \right] + L \right. \\
&\quad \left. + \frac{1+d}{1+2d} \left[\frac{(2+d)(3+d) \dots (n+d)(n+1+d)}{(1-d)(2-d) \dots (n-1-d)(n-d)} - \frac{(2+d)(3+d) \dots (n+d)(n-d)}{(1-d)(2-d) \dots (n-1-d)(n-d)} \right] \right\} \\
&= d \left[-\frac{1+d}{1+2d} + \frac{1}{1+2d} \frac{\Gamma(1-d)\Gamma(n+2+d)}{\Gamma(1+d)\Gamma(n+1-d)} \right]
\end{aligned}$$

the substitution of which into the expression of $D[x_n(t)]$ yields

$$D[x_n(t)] = \gamma_0 \left[\frac{d}{1+2d} + \frac{1}{1+2d} \frac{\Gamma(1-d)\Gamma(n+1+d)}{\Gamma(1+d)\Gamma(n-d)} \right]$$

Using the Stirling formula, $\Gamma(x) \sim \sqrt{2\pi} e^{1-x} (x-1)^{x-1/2}$, as $x \rightarrow \infty$, we can obtain

$$D[x_n(t)]/D[x(t)] \sim \frac{1}{1+2d} \frac{\Gamma(1-d)}{\Gamma(1+d)} n^{1+2d}, \quad -1/2 < d < 1/2, \text{ as } n \rightarrow \infty.$$

Proof of Theorem 2.2.3. If Proposition 2.2 is rewritten as

$$D[x_k(t)] = k\gamma_0 + 2 \sum_{i=1}^k (n-i)\gamma_i = k \int_{-\pi}^{\pi} f(\lambda) d\lambda + 2 \int_{-\pi}^{\pi} f(\lambda) \sum_{i=1}^{k-1} (k-i) \cos(i\lambda) d\lambda$$

applying the summation formula for $\sum_{i=1}^k \cos(i\lambda)$ and $\sum_{i=1}^k k \cos(i\lambda)$ (Gradshteyn and Ryzhik, 2007, 37-38) can directly lead to Theorem 2.2.3.

Proof of Theorem 2.2.4. For a generalized time series ARFIMA(p, d, q), its spectral density $f(\lambda)$ can be specified as

$$f(\lambda; p, d, q) = \frac{\sigma^2}{2\pi} \left[2 \sin \frac{\lambda}{2} \right]^{-2d} \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2, \quad -\frac{1}{2} < d < \frac{1}{2} \text{ and } d \neq 0.$$

(1) if $p=q=0$, the spectral density $f(\lambda)$ can be simplified as

$$f(\lambda; 0, d, 0) = \frac{\sigma^2}{2\pi} \left[2 \sin \frac{\lambda}{2} \right]^{-2d}$$

From Theorem 2.2.3, we can easily obtain

$$\begin{aligned}
D[x_n(t); 0, d, 0] &= 2 \int_{-\pi}^{\pi} f(\lambda; 0, d, 0) \frac{1 - \cos(n\lambda)}{[2 \sin(\lambda/2)]^2} d\lambda \\
&= 2 \int_{-\pi}^{\pi} \frac{\sigma^2}{2\pi} \left[2 \sin \frac{\lambda}{2} \right]^{-2d-2} (1 - \cos n\lambda) d\lambda
\end{aligned}$$

Thus, applying the integral formulae, $\int_0^{\pi} \cos^{\nu-1} x \cos ax dx = \frac{\pi}{2^{\nu} \nu} B\left(\frac{\nu+a+1}{2}, \frac{\nu-a+1}{2}\right)$,

$\int_0^{\frac{\pi}{2}} \sin^{\mu-1} x dx = 2^{\mu-2} B(\frac{\mu}{2}, \frac{\mu}{2})$ (Gradshteyn and Ryzhik, 2007, 37-38), yields the desired result.

(2) if $p \neq 0$, or $q \neq 0$, Theorem 2.2.3 gives

$$D[x_n(t); p, d, q] = 2 \int_{-\pi}^{\pi} \frac{\sigma^2}{2\pi} \left[2 \sin \frac{\lambda}{2} \right]^{-2d-2} \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2 (1 - \cos n\lambda) d\lambda$$

According to Lemma A, it is not difficult to obtain

$$\frac{\min \{|\theta(1)|^2, |\theta(-1)|^2\}}{\max \{|\phi(1)|^2, |\phi(-1)|^2\}} D[x_n(t); 0, d, 0] \leq D[x_n(t); p, d, q] \leq \frac{\max \{|\theta(1)|^2, |\theta(-1)|^2\}}{\min \{|\phi(1)|^2, |\phi(-1)|^2\}} D[x_n(t); 0, d, 0]$$

Therefore

$$D[x_n(t); p, d, q] = c(\phi, \theta) D[x_n(t); 0, d, 0]$$

with $c(\phi, \theta)$ being a constant, satisfying
$$\frac{\min \{|\theta(1)|^2, |\theta(-1)|^2\}}{\max \{|\phi(1)|^2, |\phi(-1)|^2\}} \leq c(\phi, \theta) \leq \frac{\max \{|\theta(1)|^2, |\theta(-1)|^2\}}{\min \{|\phi(1)|^2, |\phi(-1)|^2\}}.$$

This has completed the proof.

A. Auxiliary lemma

Lemma A. Let $\phi(e^{-i\lambda}) = 1 - \sum_{k=1}^p a_k e^{-i\lambda k}$ and $\theta(e^{-i\lambda}) = 1 - \sum_{k=1}^q b_k e^{-i\lambda k}$ be autoregressive and moving-average multinomial of a stationary and invertible time series ARMA(p, q), respectively, and have no common zeros. Then we have

$$\frac{\min \{|\theta(1)|^2, |\theta(-1)|^2\}}{\max \{|\phi(1)|^2, |\phi(-1)|^2\}} \leq \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2 \leq \frac{\max \{|\theta(1)|^2, |\theta(-1)|^2\}}{\min \{|\phi(1)|^2, |\phi(-1)|^2\}}, \text{ and } \left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2 \neq 1$$

Proof of Lemma A. As $\phi(e^{-i\lambda})$ and $\theta(e^{-i\lambda})$ have no common zeros, we can obtain

$$\left| \frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})} \right|^2 \neq 1$$

With $\phi(e^{-i\lambda})$ and $\theta(e^{-i\lambda})$ being autoregressive and moving-average multinomial respectively, it is not difficult to derive

$$\begin{aligned} |\phi(e^{-i\lambda})|^2 &= \left| 1 - \sum_{k=1}^p a_k e^{-i\lambda k} \right|^2 = \left| 1 + \left(\sum_{k=1}^p a_k \right)^2 - 2 \cos \lambda \sum_{k=1}^p a_k \right| \\ |\theta(e^{-i\lambda})|^2 &= \left| 1 - \sum_{k=1}^q b_k e^{-i\lambda k} \right|^2 = \left| 1 + \left(\sum_{k=1}^q b_k \right)^2 - 2 \cos \lambda \sum_{k=1}^q b_k \right| \end{aligned}$$

Considering the conditions that the time series is stationary and invertible, we can deduce that

$|\phi(e^{-i\lambda})|^2$ and $|\theta(e^{-i\lambda})|^2$ are both monotonic functions of λ , and satisfy

$$\min \{|\phi(1)|^2, |\phi(-1)|^2\} \leq |\phi(e^{-i\lambda})|^2 \leq \max \{|\phi(1)|^2, |\phi(-1)|^2\}, \text{ and } |\phi(e^{-i\lambda})|^2 \neq 1$$

$$\min \{|\theta(1)|^2, |\theta(-1)|^2\} \leq |\theta(e^{-i\lambda})|^2 \leq \max \{|\theta(1)|^2, |\theta(-1)|^2\}, \text{ and } |\theta(e^{-i\lambda})|^2 \neq 1$$

This has completed the proof.

References

- Abadir, K. M., Distaso, W., Giraitis, L., 2009. Two estimator of the long-run variance: Beyond short memory. *Journal of Econometrics* 150, 56-70.
- Anderson, T. W., 1971. *The Statistical Analysis of Time Series*, New York: Wiley.
- Baillie, R. T., 1996. Long memory processes and fractional integration in econometrics. *Journal of Econometrics* 73, 5-59.
- Beran, J., 1994. On a class of M-estimators for Gaussian long-memory models. *Biometrika* 81(4), 755-766.
- Boutahar, M., Marimoutou, V., Nouira, L., 2007. Estimation methods of the long memory parameter: Monte Carlo analysis and application. *Journal of Applied Statistics* 34, 261-301.
- Brockwell, P., Davis, R., 1991. *Time Series: Theory and Methods*(2nd ed.), Berlin: Springer-Verlag.
- Chronopoulou A, Viens F G, 2012. Estimation and pricing under long-memory stochastic volatility[J]. *Annals of Finance*, 8(2-3), 379-403.
- Cheong C W, Nor A H S M, Isa Z, 2007. Asymmetry and long-memory volatility: Some empirical evidence using GARCH[J]. *Physica A: Statistical Mechanics and its Applications*, 373: 651-664.
- Chkili, W., Hammoudeh, S., & Nguyen, D. K., 2014. Volatility forecasting and risk management for commodity markets in the presence of asymmetry and long memory. *Energy Economics*, 41(1), 1-18.
- Christensen, B. J., Nielsen, M. Ø., & Zhu, J. , 2010. Long memory in stock market volatility and the volatility-in-mean effect: the FIEGARCH-M model. *Journal of Empirical Finance*, 17(3), 460-470.
- Floros, C., Jaffry, S., & Valle Lima, G., 2007. Long memory in the Portuguese stock market. *Studies in Economics and Finance*, 24(3), 220-232.
- Fox, R., Taqqu, M. S., 1986. Large-Sample Properties of Parameter Estimates for Strongly Dependent Stationary Gaussian Time Series. *Annals of Statistics* 14(2), 517-532.
- Geweke, J., Porter-Hudak, S., 1983. The Estimation and Application of Long Memory Time Series Models. *Journal of Time Series Analysis* 4(4), 221-238.
- Giraitis, L., Kokoszka, P., Leipus, R., Teyssiere, G., 2003. Rescaled variance and related tests for long memory in volatility and levels. *Journal of Econometrics* 112(2), 265-294.
- Giraitis, L., Kokoszka, P., Leipus, R., Teyssiere, G., 2005. Corrigendum to ‘Rescaled variance and related tests for long memory in volatility and levels’. *Journal of Econometrics* 126(2), 571-572.
- Granger, Clive W. J., Joyeux, R., 1980. An Introduction to Long-memory Time Series Models and Fractional Differencing. *Journal of Time Series Analysis* 1(1), 15-29.
- Gradshteyn, I. S., Ryzhik, I. M., 2007. *Table of Integrals, Series, and Products*(Seventh Edition). Academic Press, Elsevier Inc.
- Haslett, J., Raftery, A. E., 1989. Space-Time Modelling with Long-Memory Dependence: Assessing Ireland’s Wind Power Resource. *Journal of the Royal Statistical Society. Series C (Applied Statistics)* 38(1), 1-50.
- Hosking, J. R. M., 1981. Fractional differencing. *Biometrika* 68(1), 165-176.
- Hurst, H. E., 1951. Long-term storage capacity of reservoirs. *Transactions of the American Society of Civil Engineers* 116, 770-779.
- Kwiatkowski, D., Phillips, P.C.B., Schmidt, P., Shin, Y., 1992. Testing the null hypothesis of stationarity against the alternative of a unit root: how sure are we that economic time series have a unit root? *Journal of Econometrics* 54, 159-178.
- Lahmiri, S., 2015. Long memory in international financial markets trends and short movements during 2008 financial crisis based on variational mode decomposition and detrended fluctuation analysis. *Physica A: Statistical Mechanics and its Applications*, 437(11), 130-138.
- Li, M., Cattani, C., & Chen, S. Y. (2011). Viewing sea level by a one-dimensional random function with long memory. *Mathematical Problems in Engineering*, 2011.
- Lo, A. W., 1991. Long-term memory in stock market prices. *Econometrica* 59 (5), 1279-1313.
- Mandelbrot, B. B., Wallis, J. R., 1968. Noah, Joseph, and Operational Hydrology. *Water Resources Research* 4(5), 909-918.

- Mandelbrot, B. B., Wallis, J. R., 1969. Robustness of the Rescaled Range R/S in the measurement of Noncyclic Long-run Statistical Dependence. *Water resources Research* 5(5), 967-988.
- Mandelbrot, B. B., 1975. Limit theorems of the self-normalized range for weakly and strongly dependent processes. *Probability Theory and Related Fields* 31(4), 271-285.
- Mandelbrot, B. B., Taqqu, M.S., 1979. Robust R/S analysis of long run serial correlation. 42nd Session of the International Statistical Institute, Manila, Book 2, 69-99.
- Niu, H., & Wang, J., 2013. Volatility clustering and long memory of financial time series and financial price model. *Digital Signal Processing*, 23(2), 489-498.
- Palma, W., 2007. *Long-Memory Time Series (Theory and Methods)*. New Jersey: John Wiley & Sons, Inc.
- Robinson, P. M., 1995. Gaussian Semiparametric Estimation of Long Range Dependence. *The Annals of Statistics* 23(5), 1630-1661.
- Robinson, P. M., 2003. *Time Series with Long Memory*. New York: Oxford University Press Inc.
- Robinson, P. M., 2005. Robust covariance matrix estimation: HAC estimates with long memory/anti-persistence correction. *Econometric Theory* 21, 171-180.
- Roueff, F., & Von Sachs, R., 2011. Locally stationary long memory estimation. *Stochastic Processes and their Applications*, 121(4), 813-844.