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Remarks on some fundamental results about higher-rank graphs and their C*-algebras

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REMARKS ON SOME FUNDAMENTAL RESULTS ABOUT HIGHER-RANK GRAPHS AND THEIR C^*-ALGEBRAS

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Abstract. Results of Fowler and Sims show that every k-graph is completely determined by its k-coloured skeleton and collection of commuting squares. Here we give an explicit description of the k-graph associated to a given skeleton and collection of squares and show that two k-graphs are isomorphic if and only if there is an isomorphism of their skeletons which preserves commuting squares. We use this to prove directly that each k-graph Λ is isomorphic to the quotient of the path category of its skeleton by the equivalence relation determined by the commuting squares, and show that this extends to a homeomorphism of infinite-path spaces when the k-graph is row finite with no sources. We conclude with a short direct proof of the characterisation, originally due to Robertson and Sims, of simplicity of the C^*-algebra of a row-finite k-graph with no sources.

1. Introduction

A k-graph is a combinatorial object akin to a directed graph, in which each path λ has a k-dimensional shape $d(\lambda) \in \mathbb{N}^k$, called its degree, instead of a 1-dimensional length. C^*-algebras associated to graphs and k-graphs have attracted significant attention recently because they at once encompass a great many interesting examples [2, 6, 11, 17], and are remarkably tractable [3, 4, 5, 7, 12, 13, 21]. Indeed, Spielberg [25] showed how to construct every Kirchberg algebra from combinations of graph C^*-algebras and 2-graph C^*-algebras. However, k-graphs themselves are, from a combinatorial point of view, substantially more complicated than their 1-dimensional counterparts, and one of the keys to using them effectively is a good visual description.

A crucial feature of k-graphs is the factorisation property, which says that, given any path λ and any decomposition $d(\lambda) = m + n$, there is a unique factorisation $\lambda = \mu\nu$ such that $d(\mu) = m$ and $d(\nu) = n$. In particular, writing $e_1, \dots, e_k$ for the generators of $\mathbb{N}^k$, if ef is a path with $d(e) = e_i$ and $d(f) = e_j$, then $d(ef) = e_j + e_i$ so there is a unique expression $ef = f'e'$ where $d(f') = e_j$ and $d(e') = e_i$. This is called a *square* of Λ. We can regard the list $\mathcal{C}_\Lambda$ of all such squares as data associated with the *skeleton* of Λ, which is the k-coloured directed graph E_Λ with the same vertices as Λ and with edges $\bigcup_{i=1}^k d^{-1}(e_k)$, where edges of different degrees are coloured with different colours.

Theorem 2.2 of [9] characterises exactly which coloured graphs E and collections $\mathcal{C}$ of squares arise from k-graphs; and [9, Theorem 2.1] implies that for each such pair $(E, \mathcal{C})$ there is a unique k-graph up to isomorphism whose skeleton is E and whose commuting

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squares are those in $\mathcal{C}$. The latter theorem is an existence result; it does not explicitly describe the k-graph $\Lambda_{E,\mathcal{C}}$. It is more or less folklore (and can be dug out of the proof of [9, Theorem 2.1]) that $\Lambda_{E,\mathcal{C}}$ can be described along the lines outlined for $k = 2$ in [14, Section 6]: paths in $\Lambda_{E,\mathcal{C}}$ are described as paths in E in which the colours occur in a fixed preferred order. But this is unsatisfactory because it is difficult to recognise a path when it is written as a concatenation of sub-paths, or to decide when one path is a sub-path of another; to do so requires tedious calculations using the collection $\mathcal{C}$ of squares.

In Section 4 we provide a concrete description of the k-graph $\Lambda_{E,\mathcal{C}}$. Inspired by the construction of 2-graphs from two-dimensional shift-spaces in [18], we show that the paths in Λ can be regarded as coloured-graph morphisms from a collection of model k-coloured graphs into E. An advantage of this construction is that under this presentation, each path explicitly encodes all of its subpaths. In Section 5 we use this to provide an explicit proof that Λ is the quotient of the path category E^*_Λ of E_Λ by the equivalence relation $\sim$ determined by $\mathcal{C}$. We then show that if Λ is row-finite and has no sources in the sense of [14] then the topology on the infinite-path space of Λ coincides with the quotient topology on $E^\infty/{\sim}$. We also present an example showing that the corresponding statement is false for boundary paths in non-row-finite k-graphs. Our final section gives a direct and elementary proof that if Λ is a row-finite k-graph with no sources, then $C^*(\Lambda)$ is simple if and only Λ is both aperiodic and cofinal (see Section 6 for details). This result first appeared in [21], but the proof there was indirect, proceeding via reference to the results of [14], which were proved using groupoid technology. Since aperiodicity and cofinality have been characterised in a number of different ways in the literature, we use the presentations which are best suited to the description of $\Lambda_{E,\mathcal{C}}$ from Section 4: specifically, the description of aperiodicity introduced in [21], and the cofinality condition of [15]. The key graph-theoretic component, Lemma 6.2, of our proof has already found applications elsewhere: it was precisely the statement needed to establish the Cuntz-Krieger uniqueness theorem [1, Theorem 4.7] for the Kumjian-Pask algebras introduced there.

2. Background

A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets E^0, E^1 and functions $r, s : E^1 \to E^0$. Since all the graphs in this paper are directed, we will drop the adjective. We call elements of E^0 *vertices*, and elements of E^1 *edges*. For an edge $e \in E^1$, we call $s(e)$ the *source* of e and $r(e)$ the *range* of e. A *path of length* n is a sequence $\mu = \mu_1\mu_2\dots\mu_n$ of edges such that $s(\mu_i) = r(\mu_{i+1})$ for $1 \le i \le n-1$. We write $|\mu|$ for the length m of μ; for $v \in E^0$, we define $|v| = 0$. We denote by E^n the set of all paths of length n, and define $E^* := \bigcup_{n\in\mathbb{N}} E^n$. We extend r and s to E^* by setting $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$. By an infinite path in E, we mean a sequence $x = \nu_1\nu_2\dots$ where $r(\nu_{i+1}) = s(\nu_i)$ for all i, and we write $r(x) = r(\nu_1)$. We write E^∞ for the set of all infinite paths, and call $W_E := E^* \cup E^\infty$ the *path space* of E.

For $k \in \mathbb{N}$, a *k-graph* is a pair (Λ, d) where Λ is a countable category and d is a functor from Λ to $\mathbb{N}^k$ which satisfies the *factorisation property*: for every $\lambda \in \mathrm{Mor}(\Lambda)$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \mathrm{Mor}(\Lambda)$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$ (see [14, Definition 1.1]). Elements $\lambda \in \mathrm{Mor}(\Lambda)$ are called *paths*, and by convention we write $\lambda \in \Lambda$ to mean $\lambda \in \mathrm{Mor}(\Lambda)$. The functor d is called the degree map. We write r, s for the usual maps from Λ to its identity morphisms: formally, $r(\lambda) = \mathrm{id}_{\mathrm{cod}(\lambda)}$ and $s(\lambda) = \mathrm{id}_{\mathrm{dom}(\lambda)}$.

For $m \in \mathbb{N}^k$ and $v \in \mathrm{Obj}(\Lambda)$, we define $\Lambda^m := \{\lambda \in \Lambda : d(\lambda) = m\}$ and $v\Lambda^m := \{\lambda \in \Lambda^m : r(\lambda) = v\}$. More generally, given $\lambda \in \Lambda$ and $F, G \subseteq \Lambda$, we define $\lambda G = \{\lambda\nu : \nu \in G, r(\nu) = s(\lambda)\}$ and $F\lambda = \{\mu\lambda : \mu \in F, s(\mu) = r(\lambda)\}$; and then $F\lambda G = \bigcup_{\mu\in F} \mu\lambda G = \bigcup_{\nu\in G} F\lambda\nu$.

A morphism between k-graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \to \Lambda_2$ which respects the degree maps. The factorisation property implies that $v \mapsto \mathrm{id}_v$ is a bijection between $\mathrm{Obj}(\Lambda)$ and Λ^0, allowing us to identify $\mathrm{Obj}(\Lambda)$ with Λ^0. In particular, we will henceforth regard r and s as maps from Λ to Λ^0.

3. Coloured graphs and coloured-graph morphisms

Consider the free semigroup $\mathbb{F}_k$ on k-generators $\{c_1, \dots c_k\}$. A *k-coloured graph* is a graph E together with a map $c : E^1 \to \{c_1, \dots, c_k\}$, which we extend to a functor $c : E^* \to \mathbb{F}_k^+$. We write q for the canonical quotient map $q : \mathbb{F}_k^+ \to \mathbb{N}^k$ determined by $q(c_i) = e_i$ for all i. So each path $x \in E^*$ has both a colouring $c(x) \in \mathbb{F}_k^+$ and a shape $q(c(x)) \in \mathbb{N}^k$. If there are multiple k-coloured graphs around, we write c_E for the colour map associated to the graph E. In this paper, we will draw edges of colour c_1 as solid lines, edges of colour c_2 as dashed lines, and edges of colour c_3 as dotted lines.

A *graph morphism* ψ from a graph E to a graph F consists of functions $\psi^0 : E^0 \to F^0$ and $\psi^1 : E^1 \to F^1$ such that $r_F(\psi^1(e)) = \psi^0(r_E(e))$ and $s_F(\psi^1(e)) = \psi^0(s_E(e))$ for all $e \in E^1$. Given graph morphisms $\psi : E \to F$ and $\phi : F \to G$, we write $\phi \circ \psi$ for the graph morphism from E to G given by $(\phi \circ \psi)^i = \phi^i \circ \psi^i$ for $i = 0, 1$. A *coloured-graph morphism* is a graph morphism ψ such that $c_E(e) = c_F(\psi(e))$ for every $e \in E^1$.

The following example describes the model k-coloured graphs which will underly the construction used in our main theorem in Section 4. In the example, $n + v_i$ is a formal symbol intended to suggest an edge of colour c_i pointing from the integer-grid point $n + e_i$ to the integer-grid point n.

Example 3.1. For $m \in (\mathbb{N} \cup \{\infty\})^k$, we define a coloured graph $E_{k,m}$ by

$$E^0_{k,m} = \{n \in \mathbb{N}^k : 0 \le n \le m\}, \qquad E^1_{k,m} = \{n + v_i : n, n + e_i \in E^0_{k,m}\},$$
$$r(n + v_i) = n, \qquad s(n + v_i) = n + e_i \qquad \text{and} \qquad c(n + v_i) = c_i.$$

Fix $n+v_i \in E^1_{k,m}$ and $m \in \mathbb{N}^k$. We define $(n+v_i)+m := (n+m)+v_i$; and as a notational convenience, if E is a coloured graph and $x \in E^1$ with $c(x) = c_i$, we sometimes write $n + v_{c(x)}$ for the edge $n + v_i$. Given a coloured-graph morphism $\lambda : E_{k,m} \to E$ we say λ has degree m and write $d(\lambda) = m$, and define $r(\lambda) := \lambda(0)$ and $s(\lambda) := \lambda(m)$.

Given a k-coloured graph E and distinct $i, j \in \{1, \dots, k\}$, an *$\{i, j\}$-square* (or just a *square*) in E is a coloured-graph morphism $\phi : E_{k,e_i+e_j} \to E$. If $\lambda : E_{k,m} \to E$ is a coloured-graph morphism and ϕ is a square in E, then ϕ *occurs in* λ if there exists $n \in \mathbb{N}^k$ such that $\phi(x) = \lambda(x + n)$ for all $x \in E_{k,e_i+e_j}$.

Let E be a k-coloured graph. A *complete collection of squares* is a collection $\mathcal{C}$ of squares in E such that for each $x \in E^*$ with $c(x) = c_ic_j$ and $i \neq j$, there exists a unique $\phi \in \mathcal{C}$ such that $x = \phi(v_i)\phi(e_i + v_j)$. We write $\phi(v_i)\phi(e_i + v_j) \sim_{\mathcal{C}} \phi(v_j)\phi(e_j + v_i)$, so for each c_ic_j-coloured path $x \in E^*$, there is a unique c_jc_i-coloured path y such that $x \sim_{\mathcal{C}} y$. If $\mathcal{C}$ is clear from context, we just write $x \sim y$. A coloured-graph morphism $\lambda : E_{k,m} \to E$ is *$\mathcal{C}$-compatible* if every square occurring in λ belongs to $\mathcal{C}$.

For $p, q \in \mathbb{N}^k$ with $p \leq q$, define $E_{k,[p,q]}$ to be the subgraph of $E_{k,q}$ such that

$$E^0_{k,[p,q]} = \{n \in \mathbb{N}^k : p \leq n \leq q\},$$
$$E^1_{k,[p,q]} = \{x \in E^1_{k,q} : s(x), r(x) \in E^0_{k,[p,q]}\}.$$

Given a coloured-graph morphism $\lambda : E_{k,m} \to E$ and $p, q \in \mathbb{N}^k$ such that $p \leq q \leq m$, define $\lambda|^*_{E_{k,[p,q]}} : E_{k,q-p} \to E$ by

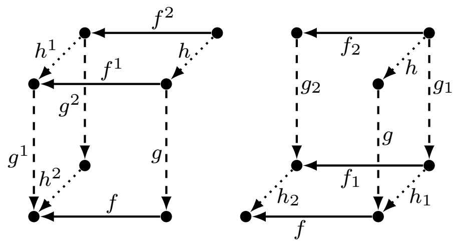
$$\lambda|^*_{E_{k,[p,q]}}(a) = \lambda(p + a). \tag{3.1}$$

The star is to remind us that this non-standard restriction involves a translation.

We say a complete collection of squares $\mathcal{C}$ in a k-coloured graph E is *associative* if for every path fgh in E such that f, g, h are edges of distinct colour, the edges $f_1, f_2, g_1, g_2, h_1, h_2$ and $f^1, f^2, g^1, g^2, h^1, h^2$ determined by

(3.2) $fg \sim g^1f^1$, $f^1h \sim h^1f^2$, and $g^1h^1 \sim h^2g^2$

$gh \sim h_1g_1$, $fh_1 \sim h_2f_1$, and $f_1g_1 \sim g_2f_2$



satisfy $f^2 = f_2, g^2 = g_2$ and $h^2 = h_2$.

Let E be a k-coloured graph, and $m \in \mathbb{N}^k \setminus \{0\}$. Let $|m| := \sum_{i=1}^k m_i$. Fix $x \in E^*$ and a coloured-graph morphism $\lambda : E_{k,m} \to E$. Recall that q is the quotient map from $\mathbb{F}_k^+$ to $\mathbb{N}^k$. We say x *traverses* λ if $q(c(x)) = d(\lambda)$ and $\lambda(q(c(x_1 \dots x_{l-1})) + v_{c(x_l)}) = x_l$ for all $0 < l \leq |m|$. By definition, $d(\lambda) = m$, and $|x| = |d(\lambda)|$. If $m = 0$, then $x \in E^0$ and $\mathrm{dom}(\lambda) = \{0\}$, and we say x traverses λ if $x = \lambda(0)$. Observe that for any coloured-graph morphism λ, and any decomposition $d(\lambda) = e_{i_1} + e_{i_2} + \cdots + e_{i_m}$ there is a corresponding path $x := \lambda(0 + v_{i_1})\lambda(e_{i_1} + v_{i_2}) \dots \lambda\big((d(\lambda) - e_{i_m}) + v_{i_m}\big)$ which traverses λ; in particular, for every finite coloured-graph morphism λ there is a path which traverses λ.

We can also make sense of infinite coloured paths which traverse infinite coloured-graph morphisms. If $x \in E^\infty$ and $\lambda : E_{k,p} \to E$ is a coloured-graph morphism of non-finite degree (so $p \in (\mathbb{N} \cup \{\infty\})^k \setminus \mathbb{N}^k$), then we say that x traverses λ if $x_1 \dots x_n$ traverses $\lambda|_{E_{k,d(x_1 \dots x_n)}}$ for every $n \in \mathbb{N}$.

Remark 3.2. Let E be a k-coloured graph and let $\lambda : E_{k,m} \to E$ be a coloured-graph morphism where $m \in \mathbb{N}^k$. Fix $p \leq m$. If $x \in E^*$ traverses $\lambda|_{[0,p]}$ and $y \in E^*$ traverses of $\lambda|^*_{[p,m]}$, then $d(\lambda) = m = p + (m - p) = q(c(x)) + q(c(y))$, and for $l \leq |xy| = |x| + |y|$, we have

$$\lambda(d((xy)_1 \dots (xy)_{l-1}) + v_{c((xy)_l)}) = \begin{cases} \lambda|_{[0,p]}(q(c(x_1 \dots x_{l-1})) + v_{c(x)_l}) & \text{if } l \leq |x| \\ \lambda|^*_{[p,m]}(q(c(y_1 \dots y_{l-p-1})) + v_{c(y_{l-p})}) & \text{otherwise} \end{cases}$$
$$= \begin{cases} x_l & \text{if } l \leq |x| \\ y_{l-p} & \text{otherwise,} \end{cases}$$

so xy traverses λ.

4. From k-coloured graphs to k-graphs.

In this section we present an explicit description of the unique k-graph associated to a k-coloured graph E and complete collection $\mathcal{C}$ of squares in E which is associative (see Theorem 4.4).

We begin by showing how a k-graph defines a skeleton and a collection of squares. If Λ is a k-graph, $\lambda \in \Lambda$, and $m \leq n \leq d(\lambda)$, then we write $\lambda(m,n)$ for the unique element of Λ^{m-n} such that $\lambda = \lambda'\lambda(m,n)\lambda''$ with $d(\lambda') = m$ and $d(\lambda'') = d(\lambda) - n$. We write $\lambda(n)$ for $s(\lambda(0,n)) \in \Lambda^0$.

Definition 4.1. Let Λ be a k-graph. We define a coloured graph E_Λ and a collection $\mathcal{C}_\Lambda$ of squares associated to Λ as follows. Let E_Λ be the k-coloured graph with $E^0_\Lambda = \{\overline{v} : v \in \Lambda^0\}$, $E^1_\Lambda = \bigcup_{i=1}^k \{\overline{f} : f \in \Lambda^{e_i}\}$, and $c(\overline{f}) = c_i \iff d(f) = e_i$. Define $\pi : E^0_\Lambda \to \Lambda$ by $\pi(\overline{v}) = v$ and $\pi : E^1_\Lambda \to \Lambda$ by $\pi(\overline{f}) = f$, and extend this to a map $\pi : E^*_\Lambda \to \Lambda$ by $\pi(\overline{f_1} \dots \overline{f_n}) = f_1 \cdots f_n$. For distinct $i, j \leq k$ and $\lambda \in \Lambda^{e_i+e_j}$ define a coloured-graph morphism $\phi_\lambda : E_{k,e_i+e_j} \to E_\Lambda$ by

$$\phi^0_\lambda(n) = \overline{\lambda(n)} \quad \text{and} \quad \phi^1_\lambda(n + v_i) := \overline{\lambda(n, n + e_i)}. \tag{4.1}$$

Let $\mathcal{C}_\Lambda := \bigcup_{i<j\leq k}\{\phi_\lambda : \lambda \in \Lambda^{e_i+e_j}\}$. We call E_Λ the *skeleton* of Λ.

Lemma 4.2. *Let Λ be a k-graph. Fix distinct $i, j \leq k$ and $\lambda \in \Lambda^{e_i+e_j}$. Then ϕ_λ is the unique coloured-graph morphism from $E_{k,e_i+e_j} \to E_\Lambda$ such that*

$$\pi(\phi_\lambda(0 + v_i)\phi_\lambda(e_i + v_j)) = \lambda = \pi(\phi_\lambda(0 + v_j)\phi_\lambda(e_j + v_i)). \tag{4.2}$$

Moreover $\mathcal{C}_\Lambda$ is a complete collection of squares in E_Λ which is associative.

Proof. Fix distinct $i, j \leq k$, and $\lambda \in \Lambda^{e_i+e_j}$. Then

$$\pi(\phi_\lambda(0 + v_i)\phi_\lambda(e_i + v_j)) = \pi\big(\overline{\lambda(0, e_i)}\,\overline{\lambda(e_i, e_i + e_j)}\big) = \lambda(0, e_i)\lambda(e_i, e_i + e_j) = \lambda.$$

The symmetric calculation shows that $\pi(\phi_\lambda(0 + v_j)\phi_\lambda(e_j + v_i)) = \lambda$ also. Hence ϕ_λ satisfies (4.2). To see that it is the unique such coloured-graph morphism, suppose that $f \in c^{-1}(i)$ and $g \in c^{-1}(j)$ and $\pi(fg) = \lambda$. Then the factorisation property forces $\pi(f) = \lambda(0, e_i)$ and $\pi(g) = \lambda(e_i, e_i + e_j)$. Since π is injective on E^1_Λ, it follows that $f = \overline{\lambda(0, e_i)} = \phi_\lambda(0 + v_i)$ and $g = \overline{\lambda(e_i, e_i + e_j)} = \phi_\lambda(e_i + v_j)$. A symmetric argument applies with i and j interchanged, and this proves the first statement of the lemma.

To see that the collection $\mathcal{C}_\Lambda$ is complete, fix $f, g \in E^1_\Lambda$ with $s(f) = r(g)$ and $c(f) \neq c(g)$, say $c(f) = c_i$ and $c(g) = c_j$. Then $\pi(f) \in \Lambda^{e_i}$ and $\pi(g) \in \Lambda^{e_j}$, so $\pi(fg) \in \Lambda^{e_i+e_j}$, and the factorisation property ensures that fg traverses $\phi_{\pi(fg)}$. Moreover, if $\lambda \in \Lambda^{e_i+e_j}$ is another path such that fg traverses ϕ_λ, then

$$\lambda = \lambda(0, e_i)\lambda(e_i, e_i + e_j) = \pi(\phi_\lambda(0 + v_i)\phi_\lambda(e_i + v_j)) = \pi(fg),$$

so $\phi_{\pi(fg)}$ is the unique element of $\mathcal{C}_\Lambda$ such that fg traverses $\phi_{\pi(fg)}$. For the associativity condition, suppose we have f, g, h, f^i, g^i, h^i, and f_i, g_i, h_i as in (3.2). By associativity of composition in Λ, we have

$$\pi(h_2 g_2 f_2) = \pi(fgh) = \pi(h^2 g^2 f^2),$$

so the factorisation property in Λ forces $\pi(h_2) = \pi(h^2)$, $\pi(g_2) = \pi(g^2)$ and $\pi(f_2) = \pi(f^2)$. Since π is injective on E^1_Λ, it follows that $h_2 = h^2$, $g_2 = g^2$ and $f_2 = f^2$ as required. □

Notation 4.3. Let E be a k-coloured graph, and let $\mathcal{C}$ be a complete collection of squares in E which is associative. For each $m \in \mathbb{N}^k$, we write $\Lambda^m_{(E,\mathcal{C})}$ for the set of all $\mathcal{C}$-compatible coloured-graph morphisms $E_{k,m} \to E$. Let $\Lambda_{(E,\mathcal{C})} := \bigcup_{m\in\mathbb{N}^k} \Lambda^m_{(E,\mathcal{C})}$. Let $d : \Lambda_{(E,\mathcal{C})} \to \mathbb{N}^k$ and $r, s : \Lambda_{(E,\mathcal{C})} \to \Lambda^0_E$ be as defined in Example 3.1. For $v \in E^0$ we define $\lambda_v : E_{k,0} \to E$ by $\lambda_v(0) = v$, and for $1 \le i \le k$ and $f \in E^1$ with $c(f) = c_i$ we define $\lambda_f : E_{k,e_i} \to E$ by $\lambda_f(0) = r(f)$, $\lambda_f(e_i) = s(f)$ and $\lambda_f(0 + v_i) = f$.

Our first main theorem shows that the notation above describes a k-graph whose skeleton is isomorphic to E under an isomorphism which carries the commuting squares of Λ to the elements of $\mathcal{C}$.

Theorem 4.4. *Fix a k-coloured graph E and a complete collection of squares $\mathcal{C}$ in E which is associative. If $\mu : E_{k,m} \to E$ and $\nu : E_{k,n} \to E$ are $\mathcal{C}$-compatible coloured-graph morphisms such that $s(\mu) = r(\nu)$, then there exists a unique $\mathcal{C}$-compatible coloured-graph morphism $\mu\nu : E_{k,m+n} \to E$ such that $(\mu\nu)|_{E_{k,m}} = \mu$ and $(\mu\nu)|^*_{E_{k,[m,m+n]}} = \nu$. Under this composition map, the set $\Lambda = \Lambda_{(E,\mathcal{C})}$ of Notation 4.3, endowed with the structure maps defined there, is a k-graph. There is an isomorphism $\rho : E \to E_\Lambda$ such that $\rho^0(v) = \overline{\lambda_v}$ for all $v \in E^0$ and $\rho^1(f) = \overline{\lambda_f}$ for all $f \in E^1$; and this ρ satisfies $\rho \circ \phi \in \mathcal{C}_\Lambda$ for all $\phi \in \mathcal{C}$.*

Our second main theorem says that the k-graph $\Lambda_{(E,\mathcal{C})}$ is uniquely determined, up to isomorphism, by the isomorphism class of $(E, \mathcal{C})$.

Theorem 4.5. *Fix a k-graph Γ, a k-coloured graph E and a complete collection $\mathcal{C}$ of squares in E which is associative. Suppose that $\psi : E_\Gamma \to E$ is a coloured-graph isomorphism such that $\psi \circ \phi \in \mathcal{C}$ for all $\phi \in \mathcal{C}_\Gamma$. Then for each $\gamma \in \Gamma$ there is a $\mathcal{C}$-compatible coloured-graph morphism $\theta_\gamma : E_{k,d(\gamma)} \to E$ such that*

$$\theta^0_\gamma(m) = \psi^0(\overline{\gamma(m)}) \quad \text{for } m \in E^0_{k,d(\gamma)}, \text{ and} \tag{4.3}$$

$$\theta^1_\gamma(m + v_i) = \psi^1(\overline{\gamma(m, m + e_i)}) \quad \text{for } m + v_i \in E^1_{k,d(\gamma)}. \tag{4.4}$$

Moreover, the map $\theta : \gamma \mapsto \theta_\gamma$ is an isomorphism $\Gamma \cong \Lambda_{(E,\mathcal{C})}$.

The key technical result which we need to prove Theorems 4.4 and 4.5 says that every path in the coloured graph E determines a unique element of Λ. We first use the associativity condition to prove this in the special case of a tri-coloured path of length three, and then deal with arbitrary paths using an inductive argument.

Lemma 4.6. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E which is associative. If $f, g, h \in E^1$ are of distinct colour and fgh is a path in E, then there is a unique $\mathcal{C}$-compatible coloured-graph morphism $\lambda : E_{k,d(fgh)} \to E$ such that fgh traverses λ.*

Proof. The completeness of $\mathcal{C}$ implies that there exist paths f^i, g^i, h^i and f_i, g_i, h_i satisfying the equations (3.2). Let λ be the coloured-graph morphism such that each of fgh, fh_1g_1, $h_2f_1g_1$, $h_2g_2f_2$, g^1f^1h, and $g^1h^1f^2$ traverses λ. Associativity of $\mathcal{C}$ ensures that λ is $\mathcal{C}$-compatible. Since the values of the f^i, g^i, h^i and f_i, g_i, h_i are determined by f, g, h and $\mathcal{C}$, if fgh traverses μ also, then $\mu = \lambda$. □

Proposition 4.7. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E which is associative. For every $x \in E^*$ there is a unique $\mathcal{C}$-compatible coloured-graph morphism $\lambda_x : E_{k,d(x)} \to E$ such that x traverses λ_x.*

Remark 4.8. The notation of Proposition 4.7 is consistent with that of Notation 4.3 since λ_v and λ_f (see Notation 4.3) are the unique morphisms such that v traverses λ_v and f traverses λ_f.

Proof of Proposition 4.7. We prove this by induction on $|x|$. If $|x| = 0$, then the result is trivial.

Now suppose as an inductive hypothesis that for every $y \in E^*$ with $|y| \leq n$, the path y traverses a unique coloured-graph morphism $\lambda_y : E_{k,d(y)} \to E$. Fix a path $x \in E^*$ with $|x| = n+1$, and express $x = yf$ where $f \in E^1$, with $c(f) = c_i$, say.

Let $m := q(c(y))$. By the inductive hypothesis, y traverses a unique $\mathcal{C}$-compatible coloured-graph morphism λ_y. We complete the proof by consideration of three cases: $|\{j \neq i : m_j > 0\}| = 0$, $|\{j \neq i : m_j > 0\}| = 1$, and then $|\{j \neq i : m_j > 0\}| \geq 2$.

Suppose first that $|\{j \neq i : m_j > 0\}| = 0$. Then $E_{k,d(x)} = E_{k,m} \cup E_{k,[m,m+e_i]}$, so the formulae

$$\lambda_x|_{E_{k,m}} = \lambda_y, \quad \lambda_x(m+v_i) = f \quad \text{and} \quad \lambda_x(m+e_i) = s(f) \tag{4.5}$$

completely specify a coloured-graph morphism λ_x such that x traverses λ_x. Furthermore, λ_x is the unique such coloured-graph morphism: if x also traverses μ, then μ satisfies the formulae (4.5). This completes the proof when $|\{j \neq i : m_j > 0\}| = 0$.

Suppose for the rest of the proof that $|\{j \neq i : m_j > 0\}| \geq 1$ (we will consider separately later the cases $|\{j \neq i : m_j > 0\}| = 1$ and $|\{j \neq i : m_j > 0\}| \geq 2$). Then

$$E_{k,m+e_i} = E_{k,m} \cup \Big(\bigcup_{j \neq i, m_j > 0} E_{k,m+e_i-e_j} \Big) \cup \Big(\bigcup_{j \neq i, m_j > 0} E_{k,[m-e_j,m+e_i]} \Big); \tag{4.6}$$

(the union here is taken inside the enveloping graph $E_{k,m+e_i}$, and is not a disjoint union; for example, $E_{k,m+e_i-e_j} \cap E_{k,m+e_i-e_l} = E_{k,m+e_i-e_j-e_l}$). For each $j \neq i$ such that $m_j > 0$, fix, for the remainder of the proof, a path z^j which traverses $\lambda_y|_{E_{k,m-e_j}}$.

Claim 1. *Suppose that $j \neq i$ satisfies $m_j > 0$. Let ϕ^j be the unique square in $\mathcal{C}$ traversed by $\lambda_y((m-e_j)+v_j)f$. Let $g^j = \phi^j(0+v_i)$ and $h^j = \phi^j(e_i+v_j)$, so $g^j h^j \sim \lambda_y((m-e_j)+v_j)f$. Then there is a unique coloured-graph morphism $\lambda^j : E_{k,m-e_j+e_i} \to E$ such that $\lambda^j|_{E_{k,m-e_j}} = \lambda_y|_{E_{k,m-e_j}}$ and $\lambda^j\big((m-e_j)+v_i\big) = g^j$.*

To prove Claim 1, observe that $|z^j g^j| = n$, so the inductive hypothesis implies that $z^j g^j$ traverses a unique $\mathcal{C}$-compatible coloured-graph morphism λ^j. Since z^j traverses both $\lambda^j|_{E_{k,m-e_j}}$ and $\lambda_y|_{E_{k,m-e_j}}$, the inductive hypothesis implies that the two are equal. This proves Claim 1.

Suppose now that $|\{j \neq i : m_j > 0\}| = 1$; let j be the unique element of this set. Then Claim 1 and (4.6) imply that there is a well-defined function $\lambda_x : E_{k,m+e_i} \to E$ such that

$$\lambda_x|_{E_{k,m}} = \lambda_y, \quad \lambda_x|_{E_{k,m+e_i-e_j}} = \lambda^j \quad \text{and} \quad \lambda_x|^*_{E_{k,[m-e_j,m+e_i]}} = \phi^j. \tag{4.7}$$

This λ_x is $\mathcal{C}$-compatible by construction, and x traverses λ_x. For uniqueness, fix a $\mathcal{C}$-compatible coloured-graph morphism μ traversed by x. Then $z^j g^j h^j$ traverses μ.

Hence y traverses $\mu|_{E_{k,m}}$ and $z^j g^j$ traverses $\mu|_{E_{k,m-e_j+e_i}}$. The inductive hypothesis forces $\mu|_{E_{k,m}} = \lambda_y$ and $\mu|_{E_{k,m-e_j+e_i}} = \lambda^j$. That μ is $\mathcal{C}$-compatible implies that $\mu|^*_{E_{k,[m-e_j,m+e_i]}} = \phi^j$. So $\mu = \lambda_x$. This proves the lemma when there is a unique $j \neq i$ such that $m_j \neq 0$ as claimed.

We now consider the last remaining case: suppose that there are at least two distinct $j, l \neq i$ such that $m_j, m_l > 0$.

Claim 2. *For distinct $j, l \neq i$ with $m_j, m_l \neq 0$, we have $\lambda^j|_{E_{k,m+e_i-e_j-e_l}} = \lambda^l|_{E_{k,m+e_i-e_j-e_l}}$.*

To establish Claim 2, observe that since i, j, l are all different, Lemma 4.6 implies that $\lambda_y\big((m-e_j-e_l)+v_l\big)\lambda_y\big((m-e_j)+v_j\big)f$ traverses a unique $\mathcal{C}$-compatible graph morphism $\psi^{j,l}$. We show that

$$\lambda^j|^*_{E_{k,[m-e_l-e_j,m+e_i-e_j]}} = \psi^{j,l}|_{E_{k,e_i+e_l}} \quad \text{and} \quad \lambda^l|^*_{E_{k,[m-e_l-e_j,m+e_i-e_l]}} = \psi^{j,l}|_{E_{k,e_i+e_j}}. \tag{4.8}$$

By symmetry, it suffices to establish that $\lambda^j|^*_{E_{k,[m-e_l-e_j,m+e_i-e_j]}} = \psi^{j,l}|_{E_{k,e_i+e_l}}$. Since $\lambda_y\big((m-e_j)+v_j\big)f \sim g^j h^j$, and since $\psi^{j,l}$ is a $\mathcal{C}$-compatible coloured-graph morphism,

$$\lambda_y\big((m-e_j-e_l)+v_l\big)g^j h^j = \lambda^j\big((m-e_j-e_l)+v_l\big)\lambda^j\big((m-e_j)+v_i\big)h^j \qquad \text{traverses } \psi^{j,l}.$$

Since $\mathcal{C}$ is a complete collection of squares, $\lambda^j|^*_{E_{k,[m-e_l-e_j,m+e_i-e_j]}} = \psi^{j,l}|_{E_{k,e_i+e_l}}$. This proves (4.8).

To complete the proof of Claim 2, note that

$$\lambda^j|_{E_{k,m-e_j-e_l}} = \lambda_y|_{E_{k,m-e_j-e_l}} = \lambda^l|_{E_{k,m-e_j-e_l}}.$$

Suppose that z traverses this morphism. Equation (4.8) implies that $z\psi^{j,l}(0+v_i)$ traverses each of $\lambda^j|_{E_{k,m+e_i-e_j-e_l}}$ and $\lambda^l|_{E_{k,m+e_i-e_j-e_l}}$. The inductive hypothesis now establishes Claim 2.

For $j \neq i$ such that $m_j > 0$, let ϕ^j and λ^j be as in Claim 1. Then Claim 2 implies that the formulae

$$\lambda_x|_{E_{k,m}} = \lambda_y|_{E_{k,m}}, \quad \lambda_x|_{E_{k,m+e_i-e_j}} = \lambda^j, \quad \text{and} \quad \lambda_x|^*_{E_{k,[m-e_j,m+e_i]}} = \phi^j$$

determine a well-defined coloured-graph morphism $\lambda_x : E_{k,m+e_i} \to E$. Moreover λ_x is $\mathcal{C}$-compatible because each square occurring in λ_x occurs in λ_y, in one of the λ^j or in one of the ϕ^j.

To see that λ_x is the unique $\mathcal{C}$-compatible coloured-graph morphism which x traverses, fix a $\mathcal{C}$-compatible coloured-graph morphism μ traversed by x. Then y traverses $\mu|_{E_{k,m}}$, so the inductive hypothesis implies that $\mu|_{E_{k,m}} = \lambda_y$. Fix $j \neq i$ such that $m_j > 0$. That $\mathcal{C}$ is a complete collection of squares and that $\lambda_y\big((m-e_j)+v_j\big)f$ traverses $\mu|^*_{E_{k,[m-e_j,m+e_i]}}$ implies that $\mu|^*_{E_{k,[m-e_j,m+e_i]}} = \phi^j$. In particular, $\mu\big((m-e_j)+v_i\big) = g^j$, and hence $z^j g^j$ traverses $\mu|_{E_{k,m-e_j+e_i}}$. The the inductive hypothesis forces $\mu|_{E_{k,m+e_i-e_j}} = \lambda^j$. It now follows from (4.6) that $\mu = \lambda_x$. □

Corollary 4.9. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E which is associative. If $\mu : E_{k,m} \to E$ and $\nu : E_{k,n} \to E$ are $\mathcal{C}$-compatible coloured-graph morphisms such that $s(\mu) = r(\nu)$, then there exists a unique $\mathcal{C}$-compatible* *coloured-graph morphism* $\mu\nu : E_{k,m+n} \to E$*, called the* composition *of* μ *and* ν *such that* $(\mu\nu)|_{E_{k,m}} = \mu$ *and* $(\mu\nu)|^*_{E_{k,[m,m+n]}} = \nu$.

Proof. Fix $x, y \in E^*$ such that x traverses μ and y traverses ν. Proposition 4.7 implies that xy traverses a unique $\mathcal{C}$-compatible coloured-graph morphism $\mu\nu$. Then x traverses $(\mu\nu)|_{E_{k,m}}$, and y traverses $(\mu\nu)|^*_{E_{k,[m,m+n]}}$, so Proposition 4.7 implies that $(\mu\nu)|_{E_{k,m}} = \mu$ and $(\mu\nu)|^*_{E_{k,[m,m+n]}} = \nu$.

Moreover, if λ is any other coloured-graph morphism such that $\lambda|_{E_{k,m}} = \mu$ and $\lambda|^*_{E_{k,[m,m+n]}} = \nu$ then Remark 3.2 shows that xy traverses λ so uniqueness in Proposition 4.7 forces $\lambda = \mu\nu$. □

Remark 4.10. Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E which is associative. Fix $m \le n$ in $\mathbb{N}^k$ and suppose that $\lambda : E_{k,n} \to E$ is a $\mathcal{C}$-compatible coloured-graph morphism. Corollary 4.9 implies that $\mu := \lambda|_{E_{k,m}}$ and $\nu := \lambda|^*_{E_{k,[m,n]}}$ satisfy $\mu\nu = \lambda$. Suppose that $\mu' : E_{k,m} \to E$ and $\nu' : E_{k,n-m} \to E$ are another two $\mathcal{C}$-compatible coloured-graph morphisms such that $\mu'\nu' = \lambda$. Then $\mu' = \lambda|_{E_{k,m}} = \mu$ and $\nu' = \lambda|^*_{E_{k,[m,n]}} = \nu$. So μ and ν are the unique coloured-graph morphisms with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.

Corollary 4.11. *Let* E *be a* k*-coloured graph and let* $\mathcal{C}$ *be a complete collection of squares in* E *which is associative. If* $\lambda : E_{k,l} \to E$, $\mu : E_{k,m} \to E$ *and* $\nu : E_{k,n} \to E$ *are* $\mathcal{C}$*-compatible coloured-graph morphisms such that* $s(\lambda) = r(\mu)$ *and* $s(\mu) = r(\nu)$*, then* $\lambda(\mu\nu) = (\lambda\mu)\nu$.

Proof. Fix $x_\lambda, x_\mu, x_\nu \in E^*$ such that x_λ traverses λ, x_μ traverses μ and x_ν traverses ν. Repeated applications of Remark 3.2 show that $x_\lambda x_\mu$ traverses $\lambda\mu$. Hence $x_\lambda x_\mu x_\nu = (x_\lambda x_\mu)x_\nu$ traverses $(\lambda\mu)\nu$. Similarly, $x_\lambda x_\mu x_\nu = x_\lambda(x_\mu x_\nu)$ traverses $\lambda(\mu\nu)$. So Proposition 4.7 implies that $(\lambda\mu)\nu = \lambda(\mu\nu)$. □

Proof of Theorem 4.4. The first statement of the theorem is precisely Corollary 4.9. We must check that Λ is a category. For composable μ, ν we have

$$s(\mu\nu) = (\mu\nu)(d(\mu\nu)) = (\mu\nu)|^*_{E_{k,[d(\mu),d(\mu)+d(\nu)]}}(d(\nu)) = \nu(d(\nu)) = s(\nu),$$

and similarly $r(\mu\nu) = r(\mu)$. Associativity of composition follows from Corollary 4.11. For $v \in E^0$, we have $r(\lambda_v) = \lambda_v(0) = v$ and $s(\lambda_v) = \lambda_v(d(\lambda_v)) = \lambda_v(0) = v$. Moreover, if $r(\mu) = \lambda_v$ and $s(\nu) = \lambda_v$, then Remark 4.10 implies that $\mu = \lambda_v\mu$ and $\nu = \nu\lambda_v$. Hence Λ is a category.

Since $\mathbb{N}^k$ as a category has only one object, d trivially respects r and s. It follows immediately from the definition of composition (see Corollary 4.9) that d respects composition. So d is a functor. Remark 4.10 shows that d satisfies the factorisation property. So (Λ, d) is a k-graph.

It remains to show that ρ defines an isomorphism of E with E_Λ and that $\rho \circ \phi \in \mathcal{C}_\Lambda$ for each $\phi \in \mathcal{C}$. The map $v \mapsto \overline{\lambda_v}$ is a bijection. We established above that $f \mapsto \lambda_f$ is a range- and source-preserving bijection between $c^{-1}(c_i) \subset E^1$ and Λ^{e_i}. We defined $E^1_\Lambda = \{\overline{f} : f \in \bigcup_{i=1}^k \Lambda^{e_i}\}$ (see Definition 4.1). For each $f \in E^1$, λ_f is the unique coloured-graph morphism traversed by f, and $\overline{\lambda_f} \in E^1_\Lambda$ satisfies $c_{E_\Lambda}(\overline{\lambda_f}) = c_i = c(f)$, $r(\overline{\lambda_f}) = \overline{\lambda_f(0)} = \overline{\lambda_{r(f)}}$, and $s(\overline{\lambda_f}) = \overline{\lambda_f(e_i)} = \overline{\lambda_{s(f)}}$. Since ρ^1 is bijective, the pair

$(\rho^0, \rho^1) : E \to E_\Lambda$ is an isomorphism of coloured graphs. To see that it preserves squares, fix $\psi \in \mathcal{C}$. Then $\rho \circ \psi$ is the square ϕ_ψ of (4.1) and hence belongs to $\mathcal{C}_{\Lambda_{(E,\mathcal{C})}}$ as required. □

Proof of Theorem 4.5. For $\gamma \in \Gamma$ define $\theta_\gamma : E_{k,m} \to E$ as in (4.3) and (4.4). Then $r(\theta^1_\gamma(m + v_i)) = r(\psi^1(\overline{\gamma(m, m + e_i)})) = \psi^0(\gamma(m)) = \theta^0_\gamma(m)$ and similarly at the source, so θ_γ is a graph morphism. Since ψ^1 preserves colour, we have

$$c_E(\theta^1_\gamma(m + v_i)) = c_E(\psi^1(\overline{\gamma(m, m + e_i)})) = c_{E_\Gamma}(\overline{\gamma(m, m + e_i)}) = c_i = c_{E_{k,d(\gamma)}}(m + v_i),$$

so θ_γ is a coloured-graph morphism.

To see that θ_γ is $\mathcal{C}$-compatible, fix a square α occurring in θ_γ. Then there exist $m \in \mathbb{N}^k$ and $i, j \le k$ such that $\alpha(x) = \theta_\gamma(x + m)$ for all $x \in E_{k,e_i+e_j}$. Let $\lambda := \gamma(m, m + e_i + e_j)$. Then $\alpha^0(n) = \theta^0_\gamma(m+n) = \psi^0(\lambda(n))$ for $0 \le n \le e_i+e_j$, and $\alpha^1(n+v_l) = \theta^1_\gamma(m+n+v_l) = \psi^1(\overline{\lambda(n, n + e_l)})$ whenever $n, n + e_l \le e_i + e_j$. That is, $\alpha = \psi \circ \phi_\lambda$ where $\phi_\lambda \in \mathcal{C}_\Gamma$ is as in Definition 4.1. By hypothesis, that $\phi_\lambda \in \mathcal{C}_\Gamma$ implies that $\alpha \in \mathcal{C}$, and hence θ_γ is $\mathcal{C}$-compatible. Hence $\theta_\gamma \in \Lambda^{d(\gamma)}_{(E,\mathcal{C})}$.

The assignment $\gamma \mapsto \theta_\gamma$ is a degree, range and source preserving map $\theta : \Gamma \to \Lambda_{(E,\mathcal{C})}$. To see that θ is injective, fix $\gamma, \gamma' \in \Gamma$ and suppose that $\theta_\gamma = \theta_{\gamma'}$. Write $\gamma = \gamma_1 \dots \gamma_n$ where each $d(\gamma_i) \in \{e_1, \dots, e_k\}$, and $\gamma' = \gamma'_1 \dots \gamma'_n$ where each $d(\gamma'_i) = d(\gamma_i)$. For $i \le n$ define $p_i := \sum_{j=1}^{i} d(\gamma_j)$. Then for each $i \le n$,

$$\psi^1(\overline{\gamma_i}) = \theta^1_\gamma(p_{i-1}, p_i) = \theta^1_{\gamma'}(p_{i-1}, p_i\Big) = \psi^1(\overline{\gamma'_i}).$$

Since ψ^1 is injective, it follows that $\overline{\gamma_i} = \overline{\gamma'_i}$ and hence $\gamma_i = \gamma'_i$. So θ is injective.

To see that θ preserves composition, fix $\gamma, \gamma' \in \Gamma$ with $s(\gamma) = r(\gamma')$, and fix paths $\psi^1(\overline{\gamma_1}) \dots \psi^1(\overline{\gamma_m})$ and $\psi^1(\overline{\gamma'_1}) \dots \psi^1(\overline{\gamma'_n})$ which traverse θ_γ and $\theta_{\gamma'}$. Then

$$\psi^1(\overline{\gamma_1}) \dots \psi^1(\overline{\gamma_m})\psi^1(\overline{\gamma'_1}) \dots \psi^1(\overline{\gamma'_n})$$

traverses both $\theta_{\gamma\gamma'}$ and $\theta_\gamma\theta_{\gamma'}$. So $\theta_{\gamma\gamma'} = \theta_\gamma\theta_{\gamma'}$ by Proposition 4.7. So θ is a functor.

To see that θ is surjective, fix $\lambda \in \Lambda_{(E,\mathcal{C})}$ and a path $f_1 \dots f_m$ which traverses λ. Then each $f_i \in E^1$, and since ψ^1 is surjective, each $f_i = \psi^1(g_i)$ for some $g_i \in E^1_\Gamma$. Each $g_i = \overline{\gamma_i}$ for some $\gamma_i \in \Gamma$. Let $\gamma := \gamma_1 \dots \gamma_n$. Then $f_1 \dots f_m = \psi^1(\overline{\gamma_1}) \dots \psi^1(\overline{\gamma_m})$ traverses both λ and θ_γ. So Proposition 4.7 implies that $\theta_\gamma = \lambda$ and hence θ is surjective. Thus θ is an isomorphism $\Gamma \cong \Lambda_{(E,\mathcal{C})}$. □

5. Topology of path spaces

In [16, Proposition 4.3] the authors appeal to general category-theoretic results [23] to see that given a k-coloured graph E and a complete collection of squares $\mathcal{C}$ in E which is associative, the corresponding k-graph $\Lambda_{(E,\mathcal{C})}$ is isomorphic to the quotient of the category E^* under the equivalence relation $\sim$ generated by

$$\bigcup_{n\ge2}\{(x, y) \in E^n \times E^n : \text{there exists } i < n \text{ such that } x_j = y_j \text{ whenever } j \notin \{i, i + 1\} \text{ and } x_ix_{i+1} \sim_{\mathcal{C}} y_iy_{i+1}\}. \tag{5.1}$$

We start this section with a direct proof of this assertion by showing that each equivalence class for $\sim$ is the set of paths which traverse some $\lambda \in \Lambda_{(E,\mathcal{C})}$. We show that the quotient map extends to a surjection from the space of all paths in E to the space of all paths in Λ.

We then restrict attention to k-graphs which are row-finite with no sources in the sense that $0 < |v\Lambda^{e_i}| < \infty$ for all $v \in \Lambda^0$ and $i \leq k$ (see [14]). In this context, the space Λ^∞ of infinite paths in Λ (see Remark 5.3 for a precise definition) — under the topology with basic open sets $\mathcal{Z}(\mu) := \{x \in \Lambda^\infty : x(0, d(\mu)) = \mu\}$ indexed by $\mu \in \Lambda$ — is a locally compact Hausdorff space. Furthermore, it is the unit space of the groupoid $\mathcal{G}_\Lambda$ used to define $C^*(\Lambda)$ in [14].

We show that Λ^∞ is the topological quotient of the space

$$\partial^c E := \big\{x \in E^\infty : |\{i : c(x_i) = c_j\}| = \infty \text{ for each } j \leq k\big\}. \tag{5.2}$$

We also show that $\partial^c E$ is a closed subspace of E^∞. Lastly, we present an example which shows that these results do not necessarily hold if Λ is not row-finite.

The following elementary lemma can be deduced from more general results in the literature (for example [10, Theorem 3.9]), but we provide a straightforward proof for completeness. Recall that q denotes the quotient map from $\mathbb{F}_k^+$ to $\mathbb{N}^k$.

Lemma 5.1. *Fix $w, w' \in \mathbb{F}_k^+$ and suppose that $q(w) = q(w')$. Then there is a finite sequence $(w^i)_{i=1}^m$ in $\mathbb{F}_k^+$ such that $w^1 = w$, $w^m = w'$, and for each $i < m$ there exists $j_i < |w|$ such that $w^i_l = w^{i+1}_l$ for $l \notin \{j_i, j_i + 1\}$, $w^i_{j_i} = w^{i+1}_{j_i+1}$ and $w^i_{j_i+1} = w^{i+1}_{j_i}$.*

Proof. The result is trivial if $|w| = 0$. Suppose $|w| \geq 1$ and the result holds for words of length $|w| - 1$. Since $q(w) = q(w')$ there exists j such that $w_j = w'_1$. Let

$$\begin{aligned}
w^2 &= w_1 \dots w_{j-2} w_j w_{j-1} w_{j+1} \dots w_{|w|},\\
w^3 &= w_1 \dots w_j w_{j-2} w_{j-1} w_{j+1} \dots w_{|w|},\\
&\ \vdots\\
w^j &= w_j w_1 \dots w_{j-2} w_{j-1} w_{j+1} \dots w_{|w|}.
\end{aligned}$$

Let $x = w_1 \dots w_{j-2} w_{j-1} w_{j+1} \dots w_{|w|}$ and $x' = w'_2 \dots w'_{|w|}$. Then $w^j = w'_1 x$, $w' = w'_1 x'$, $q(x) = q(x')$ and $|x| = |w| - 1$. Apply the inductive hypothesis to x and x' to obtain $x^1, \dots, x^n$. The sequence $w^1, \dots w^j, w'_1 x^2, \dots, w'_1 x^n$ does the job. □

Proposition 5.2. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E which is associative. Let $\sim$ be the equivalence relation on E^* generated by (5.1). For $x, y \in E^*$, we have $x \sim y$ if and only if x and y traverse the same $\mathcal{C}$-compatible graph morphism λ. The structure maps $s([x]) := s(x)$, $r([x]) := r(x)$, $d([x]) := q(c(x))$ and $[x][y] := [xy]$ are well-defined on $E^*/{\sim}$, and under these operations $E^*/{\sim}$ is a k-graph which is isomorphic to $\Lambda_{(E,\mathcal{C})}$.*

Proof. For a pair (x, y) as in (5.1), we have $r(x) = r(y)$, $s(x) = s(y)$, and $q(c(x)) = q(c(y))$, so the formulas $s([x]) := s(x)$, $r([x]) := r(x)$ and $d([x]) := q(c(x))$ are well-defined.

If $x \sim y$, then there is a finite sequence of pairs (x^l, x^{l+1}), $1 \leq l \leq m - 1$, each of the form described in (5.1) such that $x^1 = x$ and $x^m = y$. So it suffices to fix (x, y) as in (5.1) and show that x and y traverse the same $\mathcal{C}$-compatible coloured-graph morphism. For this, let ϕ be the square in $\mathcal{C}$ traversed by $x_i x_{i+1}$ and hence also by

$y_i y_{i+1}$. By Proposition 4.7, $x_1 \dots x_{i-1} = y_1 \dots y_{i-1}$ traverses a unique $\mathcal{C}$-compatible morphism μ and $x_{i+2} \dots x_n = y_{i+2} \dots y_n$ traverses a unique $\mathcal{C}$-compatible morphism ν. By Corollary 4.9, there is a unique $\mathcal{C}$-compatible $\lambda = \mu\phi\nu$ which agrees, upon restriction, with each of μ, ϕ and ν. Each of x and y traverse this λ.

Now suppose that x and y traverse a common $\mathcal{C}$-compatible morphism λ. Then in particular $q(c(x)) = q(c(y))$. By Lemma 5.1 there is a finite sequence $(w^i)_{i=1}^m$ in $\mathbb{F}_k^+$ such that $w^1 = c(x)$, $w^m = c(y)$, and for each $i \leq m-1$ there exists $j_i < |x|$ such that $w^i_l = w^{i+1}_l$ for $l \notin \{j_i, j_i+1\}$, and $w^i_{j_i} = w^{i+1}_{j_i+1}$ and $w^i_{j_i+1} = w^{i+1}_{j_i}$. For each i, let z^i be the path which traverses λ such that $c(x^i) = w^i$, and for each $i \leq m$ and $l \leq |x|$, let $p^i_l := q(c(x^i_1 \cdots x^i_l))$. Then for each $i \leq m$, both $x^i_1 \dots x^i_{j_i-1}$ and $x^{i+1}_1 \dots x^{i+1}_{j_i-1}$ traverse $\lambda\big(0, p^i_{j_i-1})\big)$, so they are equal, and likewise $x^i_{j_i+2} \dots x^i_{|x|} = x^{i+1}_{j_i+2} \dots x^{i+1}_{|x|}$. Moreover each of $x^i_{j_i} x^i_{j_i+1}$ and $x^{i+1}_{j_i} x^{i+1}_{j_i+1}$ traverses $\lambda|^*_{[p^i_{j_i-1}, p^i_{j_i+1}]}$, which, since λ is $\mathcal{C}$-compatible, belongs to $\mathcal{C}$. Thus the pair (x^i, x^{i+1}) is a pair of paths as in (5.1), and it follows that $x \sim y$ as required.

By the preceding two paragraphs, the assignment $\rho : \lambda \mapsto [x]$ for any x which traverses λ is a well-defined bijection from $\Lambda_{(E,\mathcal{C})}$ to $E^*/{\sim}$ which preserves range, source and degree. By definition of composition in $\Lambda_{(E,\mathcal{C})}$, if x traverses μ and y traverses μ, then xy traverses $\lambda\mu$. So if $[x] = [x']$ and $[y] = [y']$, then x and x' both traverse μ, and y and y' both traverse ν, so xy and $x'y'$ both traverse $\mu\nu$. Thus

$$[xy] = \rho(\mu\nu) = [x'y'],$$

showing that the composition on $E^*/\sim$ is well-defined. So ρ is a degree-preserving bijective functor, and hence an isomorphism of k-graphs. □

We recall the k-graphs $\Omega_{k,m}$ described in [19, Examples 2.2]. For $m \in (\mathbb{N} \cup \{\infty\})^k$, define $\Omega_{k,m}$ be the category with $\mathrm{Obj}(\Omega_{k,m}) = \{n \in \mathbb{N}^k : n \leq m\}$, $\mathrm{Mor}(\Omega_{k,m}) = \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\}$, $s(p,q) = q$, $r(p,q) = p$ and $(p,q)(q,r) = (p,r)$. Then with $d(p,q) = q - p$, the pair $(\Omega_{k,m}, d)$ is a row-finite k-graph. By convention, $\Omega_k = \Omega_{k,(\infty,\dots,\infty)}$. Note that there is only one possible complete collection of squares $\mathcal{C}$ in the k-coloured graph $E_{k,m}$, this collection is also associative, and the k-graph $\Lambda_{E_{k,m},\mathcal{C}}$ of Theorem 4.4 is isomorphic to $\Omega_{k,m}$.

Remark 5.3. Let Λ be a k-graph. For $m \in \mathbb{N}^k$, the factorisation property gives a bijection $\lambda \mapsto x_\lambda$ between Λ^m and the set of graph morphisms from $\Omega_{k,m}$ to Λ: for $\lambda \in \Lambda$ and $p \leq q \leq d(\lambda)$, $x_\lambda(p,q)$ is the unique element of Λ^{q-p} such that $\lambda = \lambda' x_\lambda(p,q)\lambda''$ for some λ', λ''. By analogy, for $m \in (\mathbb{N} \cup \{\infty\})^k$, we call a k-graph morphism $x : \Omega_{k,m} \to \Lambda$ a *path of degree* m in Λ, and we write $d(x)$ for m and $r(x)$ for $x(0)$. We continue to denote the collection of all such paths by Λ^m. It is conventional to identify λ with x_λ, and in particular to denote $x_\lambda(p,q)$ by $\lambda(p,q)$; so $\lambda = \lambda(0,p)\lambda(p,q)\lambda(q,d(\lambda))$ whenever $0 \leq p \leq q \leq d(\lambda)$.

We shall write W_Λ for the *path space* $W_\Lambda := \bigcup_{m \in (\mathbb{N} \cup \{\infty\})^k} \Lambda^m$ of Λ.

Proposition 5.4 (cf. [14, Remarks 2.2]). *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E which is associative. The map $x \mapsto \lambda_x$ from E^* to $\Lambda = \Lambda_{(E,\mathcal{C})}$ of Proposition 4.7 extends uniquely to a degree-preserving map $\pi : W_E \to W_\Lambda$*

such that for $x \in W_E$ and $i \in \mathbb{N}$ with $i \leq |x|$, $\pi(x)(0, d(x_1 \dots x_i)) = \lambda_{x_1 \dots x_i}$. Moreover, π is surjective.

Remark 5.5. We have used the same symbol π both for the map from W_E to W_Λ of Proposition 5.4, and for the map from E_Λ to Λ of Definition 4.1. This notation is consistent because Theorem 4.4 yields a coloured-graph isomorphism $E \cong E_\Lambda$ which carries elements of $\mathcal{C}$ to elements of $\mathcal{C}_\Lambda$.

Proof of Proposition 5.4. For $x \in W_E$ and $m \leq n \leq d(x)$, let j be the least element of $\mathbb{N}$ such that $d(x_1 \dots x_j) \geq n$, and define $\pi(x)(m, n) := \lambda_{x_1 \dots x_j}|^*_{E_{k,[m,n]}}$. Proposition 4.7 implies that for $j \leq l$, we have $\lambda_{x_1 \dots x_l}|_{E_{k,d(x_1 \dots x_j)}} = \lambda_{x_1 \dots x_j}$. Hence $\pi(x)(0, d(x_1 \dots x_j)) = \lambda_{x_1 \dots x_j}$ for all $j \leq |x|$. The factorisation property in Λ implies that $\pi(x)$ is a k-graph morphism from $\Omega_{k,d(x)}$ to Λ. For uniqueness of π, observe that by uniqueness of factorisations in Λ, any $y \in W_\Lambda$ such that $y(x_1 \dots x_i) = \lambda_{x_1 \dots x_i}$ for all $i \leq d(x)$ must satisfy $y(m, n) = \lambda_{x_1 \dots x_j}|^*_{E_{k,[m,n]}}$ whenever $d(x_1 \dots x_j) \geq n$.

To see that π is surjective first note that if $\lambda \in \Lambda$ then any path x which traverses λ satisfies $\pi(x) = \lambda$. So fix $y \in W_\Lambda \setminus \Lambda$. Fix a sequence $(m_j)_{j=0}^\infty$ such that $m_0 = 0$, $m_{j+1} - m_j \in \{e_1, \dots, e_k\}$ for all j and $\bigvee_{j \in \mathbb{N}} m_j = d(y)$. For each $j \in \mathbb{N}$ define $x_j := y(m_{j-1}, m_j) \in E^1$. Then $x = x_1 x_2 \cdots \in W_E$, and $\pi(x)(m, n) = y(m, n)$ for all m, n by uniqueness of factorisations in Λ, so $\pi(x) = y$. □

If $\pi : W_E \to W_\Lambda$ is the surjection of Proposition 5.4, then Proposition 4.9 implies that $\pi(x)\pi(y) = \pi(xy)$ when x and y are finite with $r(y) = s(x)$.

Now let Λ be a row-finite k-graph with no sources. Recall that Λ^∞ is the collection of k-graph morphisms from Ω_k to Λ, and $\partial^c E$ is the collection of infinite paths in E which contain infinitely many edges of each colour.

Remark 5.6. Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E which is associative. Let $\Lambda = \Lambda_{(E,\mathcal{C})}$ be the corresponding k-graph as in Theorem 4.4. Identify Λ^0 with E^0. Then for each $v \in \Lambda^0$ and $i \leq k$, we have $|v\Lambda^{e_i}| = |\{e \in E^1 : r(e) = v \text{ and } c(e) = c_i\}|$. Hence Λ is row-finite and has no sources if and only if $0 < |\{e \in E^1 : r(e) = v \text{ and } c(e) = c_i\}| < \infty$ for all $v \in E^0$ and $i \leq k$.

Recall that if Λ is a row-finite k-graph then the topology on Λ^∞ has basic open sets $\mathcal{Z}(\mu) = \{x \in \Lambda^\infty : x(0, d(\mu)) = \nu\}$ indexed by $\mu \in \Lambda$, and is a locally compact Hausdorff topology. If E is a k-coloured graph and $\mathcal{C}$ a complete collection of squares in E such that $\Lambda_{(E,\mathcal{C})}$ is row-finite with no sources, then E is row-finite and has no sources as well. So the sets $\mathcal{Z}(y)$ where $y \in E^*$ form a basis for a locally compact Hausdorff topology on E^∞, and we endow $\partial^c E$ with the subspace topology.

Proposition 5.7. *Let E be a k-coloured graph and let $\mathcal{C}$ be a complete collection of squares in E which is associative. Let $\Lambda = \Lambda_{(E,\mathcal{C})}$ as in Theorem 4.4. Suppose that Λ is row-finite and has no sources. Then the surjection of W_E onto W_Λ of Proposition 5.4 restricts to a surjection $\pi : \partial^c E \to \Lambda^\infty$. Moreover $U \subseteq \Lambda^\infty$ is open if and only if $\pi^{-1}(U) \subseteq \partial^c E$ is open.*

Proof. That $\partial^c E$ is precisely $\pi^{-1}(\Lambda^\infty)$ follows from the definitions of the two sets and of π. Since $\pi : W_E \to W_\Lambda$ is surjective, it follows that its restriction to $\partial^c E$ is surjective onto Λ^∞.

Suppose that U is open in Λ^∞, and fix $x \in \pi^{-1}(U)$. We seek a basic open set B_x in $\partial^c E$ such that $x \in B_x \subset \pi^{-1}(U)$. Since U is open, there exists $\mu \in \Lambda$ such that $\pi(x) \in \mathcal{Z}(\mu) \subset U$. Fix $n \in \mathbb{N}$ such that $q(c(x_1 \dots x_n)) > d(\mu)$. Then $\pi(x_1 \dots x_n) \in \mathcal{Z}(\mu)$. Let $y_x = x_1 \dots x_n$. Then $x \in \mathcal{Z}(y_x)$. To see $\mathcal{Z}(y_x) \subset \pi^{-1}(U)$, fix $y \in \mathcal{Z}(y_x)$; say $y = y_x y'$. Then $\pi(y) = \pi(x_1 \dots x_n y') = \pi(x_1 \dots x_n)\pi(y') \in \mathcal{Z}(\mu) \subset U$, so $y \in \pi^{-1}(U)$ as required.

For the reverse implication, suppose that $\pi^{-1}(U)$ is open in $\partial^c E$, and fix $\lambda \in U$. We seek a basic open set B_λ such that $\lambda \in B_\lambda \subset U$. Fix $x \in E^\infty$ which traverses λ. Then $x \in \partial^c E$, and $x \in \pi^{-1}(U)$ which is open. Hence there exists a basic open set $B_x \in \partial^c E$ such that $x \in B_x \subset \pi^{-1}(U)$. So $B_x = \mathcal{Z}(y_x)$ for some $y_x \in E^*$, and

$$\lambda = \pi(x) = \pi(y_x x') = \pi(y_x)\pi(x') \in \mathcal{Z}(\pi(y_x)).$$

To see that $\mathcal{Z}(\pi(y_x)) \subset U$, let $\mu \in \mathcal{Z}(\pi(y_x))$. Write $\mu = \pi(y_x)\mu'$, and let $x_{\mu'}$ be a path in E^* which traverses μ'. Then $y_x x_{\mu'} \in \mathcal{Z}(y_x) \subset \pi^{-1}(U)$, which implies that $\mu = \pi(y_x x_{\mu'}) \in U$. □

Proposition 5.7 implies that when $\Lambda_{E,\mathcal{C}}$ is row-finite with no sources, the topology on Λ^∞ is the quotient topology inherited from $\partial^c E$ under π. In particular, π is continuous.

The role in the proof of Proposition 5.7 of the hypothesis that Λ is row-finite and has no sources is not readily apparent. Indeed the proposition is valid for arbitrary k-graphs Λ when Λ^∞ is endowed with the topology with basis $\{\mathcal{Z}(\mu) : \mu \in \Lambda\}$. We have included the hypothesis because, from the point of view of C^*-algebras associated to higher-rank graphs, the result is only interesting for row-finite k-graphs with no sources. We close the section by discussing why this is, what would be the corresponding result for arbitrary k-graphs, and why it does not hold.

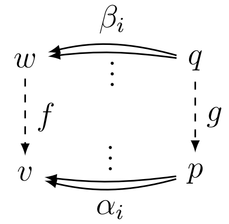
When Λ is row-finite with no sources, Λ^∞ is homeomorphic to the unit space of the groupoid $\mathcal{G}_\Lambda$ of [14]; it is also homeomorphic to the spectrum of the commutative subalgebra of $C^*(\Lambda)$ spanned by the projections $s_\lambda s_\lambda^*$ (see [26] and the opening of Section 6). If Λ is not row-finite, this is no longer the case: Λ^∞ need not even be locally compact. To see this, suppose that Λ is the 1-graph with one vertex and infinitely many edges $\{f_i : i \in \mathbb{N}\}$. Then given any $x \in \Lambda^\infty$, any neighbourhood of x contains $\mathcal{Z}(x_1 \dots x_n)$ for some n, and the cover $\mathcal{Z}(x_1 \dots x_n) = \bigcup_{i=1}^\infty \mathcal{Z}(x_1 \dots x_n f_i)$ has no finite subcover.

Instead, given a finitely aligned k-graph, let

$$\begin{aligned}\Lambda^{\leq\infty} := \{x \in W_\Lambda : &\text{ there exists } n \leq d(x) \text{ such that} \\ &(n \leq p \leq d(x) \text{ and } p_i = d(x)_i) \text{ implies } x(p)\Lambda^{e_i} = \emptyset\}\end{aligned}$$

as in [20]. Endow W_Λ with the topology with basic open sets $\mathcal{Z}(\mu \setminus G) := \mathcal{Z}(\mu) \setminus \Big(\bigcup_{\lambda\in G} \mathcal{Z}(\mu\lambda)\Big)$, where μ ranges over Λ and G ranges over all finite subsets of $s(\mu)\Lambda$. Then the unit space of $\partial\Lambda$ of the groupoid $\mathcal{G}_\Lambda$ constructed in [8] is the closure of $\Lambda^{\leq\infty}$ in W_Λ; this is also homeomorphic to the spectrum of $\overline{\operatorname{span}}\{s_\lambda s_\lambda^* : \lambda \in \Lambda\}$ [26]. So the natural question to ask for finitely aligned k-graphs is whether this topology on $\partial\Lambda$ coincides with the quotient topology determined by the surjection $\pi : \pi^{-1}(\partial\Lambda) \to \partial\Lambda$ where $\pi^{-1}(\partial\Lambda)$ is given the relative topology coming from W_E.

Example 5.8. Let E be the 2-coloured graph pictured below.



Let $\mathcal{C}$ be the collection of graph morphisms $\lambda_i : E_{2,(1,1)} \to E$ such that $\alpha_i g$ and $f\beta_i$ both traverse λ_i for each i. This is a complete collection of squares in E. Since E has only two colours $\mathcal{C}$ is associative. Let Λ be the 2-graph constructed from $(E, \mathcal{C})$ as in Theorem 4.4, and let $\pi : W_E \to W_\Lambda$ be the surjection of Proposition 5.4. Then $v\Lambda^{\leq\infty} = \Lambda^{(1,1)} = \{\lambda_i : i \in \mathbb{N}\}$ and $\pi(\alpha_i g) = \lambda_i = \pi(f\beta_i)$ for all i.

We claim that $\alpha_i g \to v$ in W_E but that $\lambda_i \to f$ in W_Λ. To see that $\alpha_i g \to v$ in W_E, fix a basic open set $\mathcal{Z}(y \setminus F) \subset E^*$ containing v. Then $y = v$. Since F is finite, there are only finitely many i such that either α_i or $\alpha_i g$ belongs to F. Let $N_0 := \max\{i : \alpha_i \in F \text{ or } \alpha_i g \in F\}$. Then $\alpha_n g \in \mathcal{Z}(v \setminus F)$ for all $n \geq N_0$, whence $\alpha_i g \to v$ as $i \to \infty$.

To see that $\lambda_i \to \pi(f)$ in W_Λ, fix a basic open set $\mathcal{Z}(\mu \setminus G) \subset \Lambda$ containing f. Then either $\mu = \pi(f)$ or $\mu = v$. We show that $\lambda_i \in \mathcal{Z}(\mu \setminus G)$ for large i. First suppose that $\mu = \pi(f)$. Then G is a finite collection of paths of the form $\pi(\beta_i\nu)$. Let $N_1 = \max\{i : \pi(\beta_i\nu) \in G \text{ for some } \nu\}$. Then $\lambda_n = \pi(f\beta_n) \in \mathcal{Z}(f \setminus G)$ for all $n \geq N_1$. Now suppose that $\mu = v$. Since G does not contain $\pi(f)$, it is a finite subset of $\{\pi(\alpha_i), \lambda_i : i \in \mathbb{N}\}$. Let $N_2 = \max\{i : \pi(\alpha_i) \in G \text{ or } \lambda_i \in G\}$. Then $\lambda_n \in \mathcal{Z}(v \setminus G)$ for all $n \geq N_2$. Hence $\lambda_i \to f$ as $i \to \infty$.

We now have $\pi(\lim \alpha_i g) = \pi(v) \neq \pi(f) = \lim \lambda_i = \lim \pi(\alpha_i g)$, so π is not continuous.

6. Simplicity of C^*-algebras of higher-rank graphs

Suppose that Λ is a k-graph which is row-finite and has no sources. For such Λ, a *Cuntz-Krieger Λ-family* in a C^*-algebra B consists of partial isometries $\{t_\lambda : \lambda \in \Lambda\}$ satisfying the *Cuntz-Krieger relations* [14]:

(CK1) $\{t_v : v \in \Lambda^0\}$ are mutually orthogonal projections;
(CK2) $t_{\lambda\mu} = t_\lambda t_\mu$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$;
(CK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(CK4) $t_v = \sum_{\lambda \in v\Lambda^m} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

The graph C^*-algebra $C^*(\Lambda)$ is the C^*-algebra generated by a universal Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$; it follows from [14, Proposition 2.11] that each vertex projection s_v is nonzero.

As in [21], we say that Λ is *aperiodic* if for every vertex $v \in \Lambda^0$ and each pair $m \neq n \in \mathbb{N}^k$ there is a path $\lambda \in v\Lambda$ such that $d(\lambda) \geqslant m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

Lemma 3.2 of [21] implies that this formulation of aperiodicity in terms of finite paths is equivalent to the aperiodicity condition used in [14]. So the next theorem follows from [14, Theorem 4.6].

Theorem 6.1 (The Cuntz-Krieger uniqueness theorem)**.** *Let Λ be a row-finite, aperiodic k-graph with no sources. Suppose that $\{t_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family, and let π be the homomorphism of $C^*(\Lambda)$ such that $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$. If each t_v is nonzero, then π is faithful.*

The proof of this theorem in [14] uses a groupoid model for $C^*(\Lambda)$. Here we outline a direct proof that flows from the finite-path formulation of aperiodicity via the following lemma.

Lemma 6.2. *Let (Λ, d) be an aperiodic k-graph with no sources. Suppose that $v \in \Lambda^0$ and $l \in \mathbb{N}^k$. Then there exists $\lambda \in \Lambda$ such that $r(\lambda) = v$, $d(\lambda) \geq l$ and*

$$\alpha, \beta \in \Lambda v,\ d(\alpha), d(\beta) \leq l \text{ and } \alpha \neq \beta \Longrightarrow (\alpha\lambda)(0, d(\lambda)) \neq (\beta\lambda)(0, d(\lambda)). \tag{6.1}$$

Proof. We list pairs (m, n) of distinct elements of $\mathbb{N}^k$ with $0 \leq m, n \leq l$ as $\{(m^{(i)}, n^{(i)}) : 1 \leq i \leq p\}$. Then an induction on i shows that there exist μ_i and $l^{(i)} \in \mathbb{N}^k$ such that $r(\mu_1) = v$, $r(\mu_i) = s(\mu_{i-1})$ for $i \geq 1$, $d(\mu_i) = (m^{(i)} \vee n^{(i)}) + l^{(i)}$, and $\mu_i(m^{(i)}, m^{(i)} + l^{(i)}) \neq \mu_i(n^{(i)}, n^{(i)} + l^{(i)})$. We now choose an arbitrary path λ' with $d(\lambda') \geq l$ and $r(\lambda') = s(\mu_p)$, and claim that $\lambda := \mu_1\mu_2 \cdots \mu_p\lambda'$ has the required properties. We trivially have $d(\lambda) \geq l$.

Suppose that α and β are distinct paths with source v and $d(\alpha) \vee d(\beta) \leq l$. If $d(\alpha) = d(\beta) = d$, say, then the initial segments $\alpha = (\alpha\lambda)(0, d)$ and $\beta = (\beta\lambda)(0, d)$ are not equal, and $\alpha\lambda \neq \beta\lambda$. So suppose that $d(\alpha) \neq d(\beta)$, say $(d(\alpha), d(\beta)) = (m^{(i)}, n^{(i)})$. Let $d := \sum_{j=1}^{i-1} d(\mu_j)$. Then

$$(\alpha\lambda)(d(\alpha) + d + n^{(i)}, d(\alpha) + d + n^{(i)} + l^{(i)}) = \mu_i(n^{(i)}, n^{(i)} + l^{(i)})$$

is not the same as $\mu_i(m^{(i)}, m^{(i)} + l^{(i)}) = (\beta\lambda)(d(\beta) + d + m^{(i)}, d(\beta)d + m^{(i)} + l^{(i)})$. Since $d(\beta) + d + m^{(i)} = d + m^{(i)} + n^{(i)} = d(\alpha) + d + n^{(i)}$, it follows that

$$(\alpha\lambda)(d + m^{(i)} + n^{(i)}, d + m^{(i)} + n^{(i)} + l^{(i)}) \neq (\beta\lambda)(d + m^{(i)} + n^{(i)}, d + m^{(i)} + n^{(i)} + l^{(i)}).$$

The presence of the factor λ' forces $d(\lambda) \geq d + m^{(i)} + n^{(i)} + l^{(i)}$, so $(\alpha\lambda)(0, d(\lambda)) \neq (\beta\lambda)(0, d(\lambda))$, as required. ☐

Remark 6.3. The technical condition established in Lemma 6.2 is actually equivalent to aperiodicity. There is no doubt a direct combinatorial proof of this, but to see it quickly, observe that our proof of Theorem 6.1 shows that the conclusion of Lemma 6.2 implies that every ideal of $C^*(\Lambda)$ contains a vertex projection, and paragraphs 2–4 of the proof of Theorem 6.6 below show that if every ideal of $C^*(\Lambda)$ contains a vertex projection, then Λ is aperiodic.

Proposition 6.4. *Suppose that Λ is a row-finite aperiodic k-graph with no sources, and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz-Krieger Λ-family in a C^*-algebra B such that $t_v \neq 0$ for all $v \in \Lambda^0$. Let F be a finite subset of Λ and let $a : (\mu, \nu) \mapsto a_{\mu,\nu}$ be a $\mathbb{C}$-valued function on $F \times F$ such that $s(\mu) = s(\nu)$ whenever $a_{\mu,\nu} \neq 0$. Then*

$$\Big\| \sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^* \Big\| \geq \Big\| \sum_{\mu,\nu \in F, d(\mu) = d(\nu)} a_{\mu,\nu} t_\mu t_\nu^* \Big\|.$$

Proof. Let $a := \sum_{\mu,\nu \in F} a_{\mu,\nu} t_\mu t_\nu^*$ and let $a_0 := \sum_{\mu,\nu \in F, d(\mu)=d(\nu)} a_{\mu,\nu} t_\mu t_\nu^*$. Define $n := \bigvee_{\mu \in F} d(\mu)$, and let $G := \bigcup_{\mu \in F} F s(\mu) \Lambda^{n - d(\mu)}$. So if $\mu, \nu \in F$ with $s(\mu) = s(\nu)$ and

$d(\mu\alpha) = n$, then $\mu\alpha, \nu\alpha \in G$. By applying (CK4) at $s(\mu)$ for each $\mu, \nu \in F$, we can express

$$a = \sum_{\mu,\nu\in G} b_{\mu,\nu} t_\mu t_\nu^* \quad \text{and} \quad a_0 = \sum_{\mu,\nu\in G, d(\mu)=d(\nu)} b_{\mu,\nu} t_\mu t_\nu^*,$$

where $b_{\mu,\nu} \neq 0$ implies $d(\mu) = n$ and $s(\mu) = s(\nu)$.

For each $v \in s(G)$, apply Lemma 6.2 with $l = \bigvee_{\nu\in G} d(\nu)$ to find $\lambda_v \in v\Lambda$ such that $d(\lambda) \geq l$ and

$$(\alpha\lambda_v)(0, l) \neq (\beta\lambda_v)(0, l) \text{ for distinct } \alpha, \beta \in Gv,$$

and let $Q_v := \sum_{\alpha\in Gv, d(\alpha)=n} t_{\alpha\lambda_v} t_{\alpha\lambda_v}^*$. Then (CK3) implies that the Q_v are mutually orthogonal projections. Hence

$$\Big\| \sum_{v\in s(G)} Q_v a Q_v \Big\| \leq \|a\|. \tag{6.2}$$

We show that

$$\sum_{v\in s(G)} Q_v a Q_v = \sum_{v\in s(G)} Q_v a_0 Q_v. \tag{6.3}$$

For $\mu, \nu \in G$ with $s(\mu) = s(\nu)$ and $d(\mu) = n$, a quick calculation using (CK4) gives

$$Q_v t_\mu t_\nu^* = \delta_{v,s(\mu)} t_{\mu\lambda_{s(\mu)}} t_{\nu\lambda_{s(\mu)}}^*. \tag{6.4}$$

Suppose $d(\mu) \neq n$ and fix $\alpha \in G \cap \Lambda^n$. Then $(\alpha\lambda_{s(\alpha)})(0, l) \neq (\nu\lambda_{s(\nu)})(0, l)$, and hence $t_{\nu\lambda_{s(\mu)}}^* t_{\alpha\lambda_{s(\alpha)}} = 0$. This and (6.4) give $Q_v t_\mu t_\nu^* Q_v = 0$ for all v, and (6.3) follows.

Finally we show that

$$\Big\| \sum_{v\in s(G)} Q_v a_0 Q_v \Big\| = \|a_0\|. \tag{6.5}$$

Routine calculations using the Cuntz-Krieger relations and that the t_v are all nonzero show that $\{t_\mu t_\nu^* : \mu, \nu \in G \cap \Lambda^n, s(\mu) = s(\nu)\}$ is a family of nonzero matrix units spanning an isomorphic copy of $\bigoplus_{v\in s(G)} M_{Gv\cap\Lambda^n}(\mathbb{C})$, and that $\{t_{\mu\lambda_{s(\mu)}} t_{\nu\lambda_{s(\nu)}}^* : \mu, \nu \in G \cap \Lambda^n, s(\mu) = s(\nu)\}$ is a family of nonzero matrix units for the same finite-dimensional C^*-algebra. Hence $t_\mu t_\nu^* \mapsto t_{\mu\lambda_{s(\mu)}} t_{\nu\lambda_{s(\nu)}}^*$ determines an isomorphism of finite-dimensional subalgebras of $C^*(\Lambda)$, so is isometric. Calculations like (6.4) show that $\sum_{v\in s(G)} Q_v t_\mu t_\nu^* Q_v = t_{\mu\lambda_{s(\mu)}} t_{\nu\lambda_{s(\mu)}}^*$ whenever $\mu, \nu \in G \cap \Lambda^n$ with $s(\mu) = s(\nu)$, and (6.5) follows.

Combining (6.3), (6.2) and (6.5) proves the Proposition. □

For the following proof, recall from the opening of [14, Section 3] that there is a strongly continuous action $\gamma : \mathbb{T}^k \to \operatorname{Aut}(C^*(\Lambda))$ characterised by

$$\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda = z_1^{d(\lambda)_1} z_2^{d(\lambda)_2} \cdots z_k^{d(\lambda)_k} s_\lambda$$

for all $\lambda \in \Lambda$. Averaging over this action yields a faithful conditional expectation $\Phi : C^*(\Lambda) \to \overline{\operatorname{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\}$ (see [14, Lemma 3.3]) such that $\Phi(s_\mu s_\nu^*) = \delta_{d(\mu),d(\nu)} s_\mu s_\nu^*$ for all $\mu, \nu \in \Lambda$.

Proof of Theorem 6.1. Follow the first paragraph of the proof of [14, Theorem 3.4] to see that π is injective on $C^*(\Lambda)^\gamma = \overline{\operatorname{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\}$.

Proposition 6.4 implies that the formula

$$\sum_{\mu,\nu\in F} a_{\mu,\nu} t_\mu t_\nu^* \mapsto \sum_{\mu,\nu\in F, d(\mu)=d(\nu)} a_{\mu,\nu} t_\mu t_\nu^*$$

is well defined on finite linear combinations (if two linear combinations are equal, Proposition 6.4 implies that the norm of the difference of their images is zero), and norm-decreasing, and hence extends by continuity to a linear map $\Psi : \pi_t(C^*(\Lambda)) \to \overline{\operatorname{span}}\{t_\mu t_\nu^* : d(\mu) = d(\nu)\}$ such that $\Psi(t_\mu t_\nu^*) = \delta_{d(\mu),d(\nu)} t_\mu t_\nu^*$.

To complete the proof, we argue as in the last two lines of the proof of [14, Theorem 3.4]: Let Φ be the faithful conditional expectation on $C^*(\Lambda)$ described above. By linearity and continuity, $\pi \circ \Phi = \Psi \circ \pi$. Suppose that $\pi(a) = 0$. Then $\Psi(\pi(a^*a)) = 0$ and hence $\pi(\Phi(a^*a)) = 0$. Since π is injective on $C^*(\Lambda)^\gamma$, it follows that $\Phi(a^*a) = 0$. Since Φ is a faithful expectation, we then have $a^*a = 0$ and hence $a = 0$. □

Let Λ be a row-finite graph without sources. As in [15], we say that Λ is *cofinal* if for every pair $v, w \in \Lambda^0$ there exists $n \in \mathbb{N}^k$ such that $v\Lambda s(\lambda) \neq \emptyset$ for all $\lambda \in w\Lambda^n$.

Remark 6.5. For row-finite graphs without sources, [15, Proposition A.2] implies that this notion of cofinality is equivalent to [15, Definition 3.3], and hence by [15, Theorem 5.1] to the usual one involving infinite paths.

Modulo the different formulation of cofinality, the following characterisation of simplicity appeared in [21], and was generalised to locally convex k-graphs in [22] and finitely aligned k-graphs in [15, 24].

Theorem 6.6. *Let Λ be a row-finite k-graph with no sources. Then $C^*(\Lambda)$ is simple if and only if Λ is both aperiodic and cofinal.*

In the proof we use the infinite-path representation. By [14, Proposition 2.3], for $x \in \Lambda^\infty$, $\lambda \in \Lambda r(x)$ and $n \in \mathbb{N}^k$, there are unique elements $\sigma^n(x)$ and λx of Λ^∞ such that

$$\sigma^n(x)(p, q) = x(n + p, n + q) \text{ and } (\lambda x)(p, q) = (\lambda x(0, q))(p, q)$$

for $p \leq q \in \mathbb{N}^k$. Let $\{\xi_x : x \in \Lambda^\infty\}$ be the usual orthonormal basis for $\ell^2(\Lambda^\infty)$. Then for each $\lambda \in \Lambda$ there is a partial isometry S_λ on $\ell^2(\Lambda^\infty)$ such that $S_\lambda \xi_x = \xi_{\lambda x}$, and $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family which gives a representation π_S of $C^*(\Lambda)$ on $\ell^2(\Lambda^\infty)$.

The following lemma is a special case of the implication (ii) $\Longrightarrow$ (i) in [15, Theorem 5.1], but the proof simplifies significantly in our setting.

Lemma 6.7. *Let Λ be a row-finite k-graph with no sources which is not cofinal. Then there exist a vertex $v \in \Lambda^0$ and an infinite path $x \in \Lambda^\infty$ such that $v\Lambda x(n) = \emptyset$ for all $n \in \mathbb{N}^k$.*

Proof. Since Λ is not cofinal, there exist $v, w \in \Lambda^0$ such that, for each $n \in \mathbb{N}^k$, there exists $\lambda \in w\Lambda^n$ with $v\Lambda s(\lambda) = \emptyset$. Choose $n^{(i)} \to \infty$ in $\mathbb{N}^k$, and $\lambda_i \in w\Lambda^{n^{(i)}}$ such that $v\Lambda s(\lambda_i) = \emptyset$. Let $1_k = (1, \ldots, 1) \in \mathbb{N}^k$. Since Λ is row-finite, there exists $\mu_1 \in v\Lambda^{1_k}$ such that $S_1 := \{j \in \mathbb{N} : \lambda_j(0, 1_k) = \mu_1\}$ is infinite. An induction argument now shows that there is a sequence $(\mu_i)_{i=1}^\infty$ in $v\Lambda$ such that for every $i \geq 2$, we have $\mu_i \in \mu_{i-1}\Lambda^{1_k}$, and $S_i := \{j \in S_{i-1} : \lambda_j(0, i \cdot 1_k)\}$ is infinite. In particular, for any $i \in \mathbb{N}$ and $j \in S_i$ that $v\Lambda s(\lambda_j) = \emptyset$ forces $v\Lambda s(\mu_i) = \emptyset$. Since $d(\mu_i) \to (\infty, \dots, \infty)$, [14, Remarks 2.2] imply that there is an infinite path x such that $x(0, d(\mu - i)) = \mu_i$ for all i. Now since $v\Lambda x(d(\mu_i)) = v\Lambda s(\mu_i) = \emptyset$ for all i, we have $v\Lambda x(n) = \emptyset$ for all n, so the infinite path x has the required properties. □

Proof of Theorem 6.6. First suppose that Λ is aperiodic and cofinal, and let I be a nonzero ideal in $C^*(\Lambda)$. To see that $I = C^*(\Lambda)$, we fix $\mu \in \Lambda$ and aim to show that s_μ belongs to I. Since Λ is aperiodic, the Cuntz-Krieger uniqueness theorem (Theorem 6.1) implies that I contains a vertex projection s_v. Applying cofinality with this v and $w = s(\mu)$ gives $n \in \mathbb{N}^k$ such that for each $\lambda \in s(\mu)\Lambda^n$ there exists $\nu_\lambda \in v\Lambda s(\lambda)$. Then

$$s_\mu = \sum_{\lambda \in s(\mu)\Lambda^n} s_{\mu\lambda} s_{s(\lambda)} s_\lambda^* = \sum_{\lambda \in s(\mu)\Lambda^n} s_{\mu\lambda}(s_{\nu_\lambda}^* s_v s_{\nu_\lambda}) s_\lambda^* \in I,$$

as required.

For the other direction, we first suppose that Λ is not aperiodic, so that there exist $v \in \Lambda^0$ and distinct $m, n \in \mathbb{N}^k$ such that, for every $\lambda \in v\Lambda$ with $d(\lambda) \geq m \vee n$,

$$\lambda(m, m + d(\lambda) - (m \vee n)) = \lambda(n, n + d(\lambda) - (m \vee n)). \tag{6.6}$$

Then for every $x \in v\Lambda^\infty$ and $l \in \mathbb{N}^k$, we can apply (6.6) to $\lambda = x(0, (m \vee n) + l)$ and deduce that $x(m, m + l) = x(n, n + l)$, whence $\sigma^m(x) = \sigma^n(x)$.

We now fix $\lambda \in v\Lambda^{m \vee n}$, let $\mu = \lambda(0, m)$ and $\nu = \lambda(0, n)$, and aim to prove that $a := s_\lambda s_\lambda^* - s_\mu s_\nu^* s_\lambda s_\lambda^*$ is nonzero and belongs to $\ker \pi_S$; since we know that $\ker \pi_S$ does not contain any vertex projections, this will prove that $\ker \pi_S$ is a nontrivial ideal. To see that $\pi_S(a) = 0$, fix $x \in \Lambda^\infty$ and compute

$$\pi_S(a)\xi_x = (S_\lambda S_\lambda^* - S_\mu S_\nu^* S_\lambda S_\lambda^*)\xi_x. \tag{6.7}$$

If $x(0, d(\lambda)) \neq \lambda$, then (6.7) vanishes. If $x(0, d(\lambda)) = \lambda$, then $x \in v\Lambda^\infty$, the argument in the previous paragraph gives $\sigma^m(x) = \sigma^n(x)$, and hence

$$\nu\sigma^n(x) = x = \mu\sigma^m(x) = \mu\sigma^n(x),$$

which implies $S_\mu S_\nu^* \xi_x = \xi_{\mu\sigma^n(x)} = \xi_x$ and $\pi_S(a)\xi_x = 0$. Thus $\pi_S(a) = 0$.

To see that $a \neq 0$, we choose $z \in \mathbb{T}^k$ such that $z^{m-n} = -1$. Then $\gamma_z(s_\mu s_\nu^* s_\lambda s_\lambda^*) = -s_\mu s_\nu^* s_\lambda s_\lambda^*$, and hence

$$\pi_S(a + \gamma_z(a)) = \pi_S(2s_\lambda s_\lambda^*) = 2S_\lambda S_\lambda^* \neq 0,$$

forcing $a \neq 0$. Thus $C^*(\Lambda)$ is not simple.

Now suppose that Λ is not cofinal. By Lemma 6.7, there exist $v \in \Lambda^0$ and $x \in \Lambda^\infty$ such that $v\Lambda x(n) = \emptyset$ for all $n \in \mathbb{N}^k$. Let

$$[x]_\sigma := \{y \in \Lambda^\infty : \text{ there exist } p, q \in \mathbb{N}^k \text{ such that } \sigma^p(x) = \sigma^q(y)\}.$$

We claim that

$$y \in [x]_\sigma \Longrightarrow r(y) \neq v. \tag{6.8}$$

To see this, fix $y \in [x]_\sigma$ and $p, q \in \mathbb{N}^k$ such that $\sigma^p(x) = \sigma^q(y)$. Then $x(p) = y(q)$ and hence $v\Lambda y(q) = \emptyset$ by choice of x. In particular, $y(0, q) \notin v\Lambda y(q)$, so $r(y) \neq v$, as claimed.

We now consider the subspace $\mathcal{H}_x := \overline{\operatorname{span}}\{\xi_y : y \in [x]_\sigma\}$ of $\ell^2(\Lambda^\infty)$. For $y \in [x]_\sigma$ and $s(\lambda) = r(y)$, we have $\lambda y \in [x]_\sigma$, and hence $\mathcal{H}_x$ is invariant for S_λ. On the other hand, $S_\lambda^*\xi_y$ vanishes unless $y(0, d(\lambda)) = \lambda$, and then $S_\lambda^*\xi_y = \xi_{\sigma^{d(\lambda)}(y)}$, which also belongs to $\mathcal{H}_x$. Thus $\mathcal{H}_x$ is reducing for π_S, and $\phi_x : a \mapsto \pi_S(a)|_{\mathcal{H}_x}$ is a homomorphism of $C^*(\Lambda)$ into $\mathcal{B}(\mathcal{H}_x)$. Since $\phi_x(s_{r(x)})\xi_x = \xi_x \neq 0$, $\ker \phi_x$ is not all of $C^*(\Lambda)$. Equation (6.8), on the other hand, implies that $\phi_x(s_v)\xi_y = 0$ for all $y \in [x]_\sigma$, so $s_v \in \ker \phi_x$. Thus $C^*(\Lambda)$ is not simple. □

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