Number sequences and phyllotaxis

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ABSTRACT

This thesis is largely concerned with the study of phyllotaxis (leaf arrangement) in plants. Most higher order plants exhibit a remarkable degree of regularity in the positioning of their leaves. In a sunflower, for instance, one can perceive two sets of opposed spirals which each partition the set of florets. Intriguingly, the numbers of spirals are almost certainly consecutive members of the Fibonacci sequence \((F_n = F_{n-1} + F_{n-2}, n \geq 2, F_0 = 0, F_1 = 1)\). This pattern manifests itself in 95% of those plants which produce their leaves sequentially.

We begin our study by considering the distribution of \(N\) points placed around the circle by a constant angle \(\alpha\). We might view this as a simplistic model of plant growth with each point representing a leaf appearing on a meristematic ring. The theory of continued fractions is used in offering a new proof of the Steinhaus Conjecture, a result which states that for any \(\alpha\) and any \(N\), the circle is divided into sub-intervals or gaps of at most three different lengths and at least two.

Various means of measuring the degree of uniformity of this sequence of points are examined, providing evidence of the supremacy of the golden number, \(\tau = (\sqrt{5} - 1)/2\). One such measure which we analyse is the “discrepancy” of sequences, a quantity studied in the theory of uniform distribution of sequences modulo one (see Kuipers and Niederreiter [68]). Two forms of this measure are actually considered, a relationship discovered between them, and some values of the two discrepancies are obtained using the theory of continued fractions (and the Steinhaus Conjecture).

We investigate further the manner in which the circle is divided by examining the transition from a sequence of gaps of two different lengths to another two-gap sequence. The problem is approached by labelling a gap according to its length (either large or small) - the sequence of strings thus obtained is analysed and, for the particular case where \(\alpha\)
has identical terms in its continued fraction expansion, a curious relationship is found between this sequence and the Bernoulli sequence. This latter sequence is \( d_1 d_2 d_3 \ldots \) where \( d_i = [(i + 1)\alpha] - [i\alpha] \), studied by Johann Bernoulli [6]. Particular attention is focused on the strings produced by \( \alpha = \tau \). A byproduct of the investigation provides the solution to the inductive equation \( G(n) = n - G(G(n - 1)) \), \( G(0) = 0 \), \( n \geq 1 \).

Higher dimensional models of phyllotaxis are considered. In particular, we study a phyllotaxis system arranged on the surface of a cylinder, with points lying at equal angular intervals on a regular helix. We determine the phyllotaxis system which ensures that the distance between neighbouring points is maximal, while the pitch of the spiral decreases during the process of growth. This is the contact pressure model considered by Adler [1], [3], a quantitative formulation of an hypothesis proposed by Schwenender [108] in 1878 to be the mechanism of Fibonacci phyllotaxis. We verify the consequences of his model by considering the analogous situation of contact pressure on the circle and extending the analysis to the cylinder.

In closing, we offer a model of phyllotaxis applicable to a general class of surfaces. We replace the condition that successive divergence angles be constant by the less restrictive assumption that the phyllotaxis system be "regular." The model simulates the emergence of Fibonacci phyllotaxis and has the advantage of being simple as well as biologically plausible.
CHAPTER ONE

Introduction

The problems considered in this thesis arise from the study of phyllotaxis which is concerned with the arrangement of leaves on the stem of plants. Most higher order plants exhibit a remarkable degree of regularity in the positioning of their leaves and it is this regularity that has attracted the attention of both botanists and mathematicians.

1.1 The phenomenon of phyllotaxis

In spiral phyllotaxis, where consecutive leaf primordia (the incipient sites of leaves) emerge singly and at a regular time interval (the plastochrone), one can trace a curve (the primary or genetic spiral) through these consecutive leaves. One can also observe two sets of "secondary" parallel spirals (the parastichies) of opposed orientation which each partition the set of leaves. What is interesting is that, if the intersection of two spirals always coincides with the position of a leaf, then the numbers of spirals in each set are, in 95% of cases, equal to consecutive members of the Fibonacci sequence. This sequence obeys the recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, $F_0 = 0$, $F_1 = 1$. In parallel to this observation the divergence angle subtended by consecutive leaves is quite close to the ratio of consecutive Fibonacci numbers. In the limit, $F_{n-1}/F_n$ is equal to the golden number or golden angle, $\tau = (\sqrt{5} - 1)/2$. (We measure angles in revolutions.)

If the parastichies are conspicuous so that they are determined by a leaf and its two nearest neighbours on either side, differing in age by $a$ and $b$ plastochrones, then the phyllotaxis is $(a, b)$. $a$ is then the number of parastichies which wind to the left and $b$ is the number of opposite orientation. The phyllotaxis is normal (or Fibonacci) if $a$ and $b$ are
consecutive Fibonacci numbers. *Anomalous* phyllotaxis, which occurs far less frequently, describes the case where $a$ and $b$ are consecutive elements of the sequence $1, 3, 4, 7, \ldots$

Figure 1.1 shows two sets of parastichies which partition the phyllotaxis system with 8 winding to the left and 5 to the right.

In many plants the phyllotaxis may change in time or space (this is often referred to as rising phyllotaxis). In a sunflower capitulum for instance, the phyllotaxis near the centre may be as low as $(3, 2)$, and perhaps as high as $(55, 89)$ as one proceeds outwards from the centre. Figure 1.2 represents a sunflower whose phyllotaxis rises as a function of the distance from the centre. In Figure 1.3 we illustrate the changes in direction between neighbouring florets by drawing lines between nearest neighbours. Figure 1.4 is produced by drawing a line from each floret to its second nearest (elder) neighbour. Figure 1.5 superimposes Figures 1.3 and 1.4, thus representing the changing phyllotaxis of our sunflower.

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*Figure 1.1 Parastichy system partitioned by 8 left-handed and 5 right-handed parastichies*
Figure 1.2 Representation of a sunflower

Figure 1.3 Neighbouring florets joined by straight lines
Figure 1.4 Second nearest neighbours joined by straight lines

Figure 1.5 The changing phyllotaxis of the sunflower - Figures 1.3 and 1.4 superimposed
The phyllotaxis may occur on a variety of surfaces. The simplest way to represent the pattern of leaves is to look down on the stem and assume that they are arranged on the circumference as points. An extension is to then allow the leaves to move out radially from the centre. The primordia are initiated at the meristem which is most usually dome-like: many researchers have chosen to model the structure as a cylinder and imagine that the primordia are arranged on helices. Only recently (see Ridley [98] and Chapter 6) have attempts been made to extend the modelling to a more general class of surfaces.

Experiments which investigate the primordia necessarily damage the plant tissue and interfere with the growth mechanism. Observations have therefore been of an indirect nature and this is one of the reasons why the mechanism of phyllotaxis is still not entirely understood. Various hypotheses have been entertained and these are reviewed in the next section.

1.2 Phyllotaxis review

The first to classify the arrangement of leaves into a spiral order was Bonnet in 1754. He proposed that the pattern emerged in such a way that leaves overlap as little as possible in order for air to circulate freely around them. Bonnet was the first to distinguish the phyllotactic pattern formed by a divergence angle of $2/5$, the quincuncial system. Other lower order systems were observed but not until 1831 did the first substantial geometrical treatment of the phenomenon appear. In the period 1831-36, Schimper [102], [103] and Braun [7] formulated their geometrical model of phyllotaxis, encompassing the earlier observations of botanists. Two new concepts governed their description; the divergence and the genetic spiral from which the parastichies appear as artifacts. Allowing only for rational divergences, Schimper recognised the prevalence of divergence angles equal to ratios of consecutive Fibonacci numbers. A similar contribution was made by the brothers Bravais and Bravais [8], [9] in 1837, 1839.
From these geometrical considerations theories arose attempting to explain the underlying mechanism which leads to the regular placement of leaves. One such explanation was proposed by Schwenender [108] in 1878. He postulated the existence of contact pressures, believing that leaf primordia in mutual contact would, due to the action of mechanical forces, form themselves into Fibonacci spirals. His proof involves a rather intriguing comparison of the forces acting on the girders of a gabled roof. Adler [1], [3] showed that Schwenender’s proof was incorrect and replaced it with his own more elaborate construction. Representing the phyllotaxis on the surface of a cylinder, Adler considered the leaves as points organised along a regular helix at equal intervals (so that the divergence angle is constant at each point in time). Contact pressure is expressed as that force which maximises the geodesic distance between neighbouring points. As the pitch of the spiral decreases due to the growth of the plant, the divergence angle changes, in time converging to the golden angle if certain initial conditions are met.

Adler extended his theory to other geometries by means of “appropriate geometrical transformations.” Ridley [97] questioned the validity of this approach and developed a computer simulation of contact pressure in a composite capitulum which succeeded in producing Fibonacci phyllotaxis, thus verifying Adler’s contact pressure theory.

Roberts [99] has proposed a model to account for the occurrence of semi-decussate phyllotaxis, where the divergence angle alternates between 1/2 and 1/4 of a revolution. He asserted that primordia move under the influence of contact pressure. In his model, primordia are initiated by a divergence angle close to the golden angle resulting in semi-decussate phyllotaxis if only the new primordium moves under contact pressure while the others remain fixed. Thus this type of phyllotaxis is shown to be a variant of normal Fibonacci phyllotaxis.

A major criticism of the contact pressure theory was raised by Church [13] who pointed out that the theory does not account for the occurrence of Fibonacci phyllotaxis.
in plants whose primordia do not arise in contact. For instance, the primordia of the fern *Apsidium filix-mas* appear on a rather broad apex, well separated from each other. Whether or not they touch at some stage in their development has so far not been ascertained experimentally. This difficulty was avoided by Roberts [101] who incorporated into his model the concept that chemical contact pressure results from competition between the primordia so that they grow preferentially into areas of high nutrient density. The model is a mathematical modification of Adler's contact pressure model. Williams [142] supports the contact pressure theory (without actually referring to it directly) stating that "physical constraint should play a part in the genesis of form" and that "phyllotaxis can scarcely be explained except in terms of mutual pressures."

Hofmeister [44] in 1868 suggested that each new primordium always arises in the widest gap between two older primordia. Richards [91], [92] believed that this proposal served as evidence in support of an hypothesis introduced by Schoute in 1913. Schoute proposed that the apical meristem and growing leaf primordia secrete a diffusible substance which inhibits the formation of new primordia in their immediate vicinity. The primordia may only emerge in areas where the concentration of the growth inhibitor is below some threshold value. Richards did not substantiate his claim and Adler [2], assuming that only the three youngest leaves produce inhibitory effects, has shown that any divergence may result.

Quantitative formulations of Schoute’s diffusion theory have appeared in the literature. Thomley [126] developed a one-dimensional model whereby primordia are represented as points on a meristematic ring, with a morphogen (growth inhibitor) diffusing along it being degraded at a rate proportional to its concentration. Primordia emerge singly at the point of lowest concentration of the morphogen field, which is assumed to be in a steady state. The predicted divergence angle depends on two parameters; the source strength and a quantity combining the rate of degradation and diffusion of the morphogen.
Simulations predict divergence angles in the bands 78-85, 97-101, 105-110, 132-134 and 146-180 (degrees). Richter and Schranner [95] have also offered an explanation in a one-dimensional framework. Assuming that the inhibitory strengths of the three youngest leaves is such that $i_3:i_2 = i_2:i_1$, they expect that $\Phi_{n,n-1}:\Phi_{n,n-2} = \Phi_{n,n-2}:\Phi_{n,n-3}$ where $\Phi_{n,m}$ is the angle between leaves $n$ and $m$. In the stationary case, $\Phi_{n,n-1}$ is constant and equal to the golden angle. The argument’s attraction derives from its simplicity and Richter and Schranner agree that “more rigour would be highly desirable.” More recently, Marzec and Kapraff [82] have considered a one-dimensional model, which is similar in spirit to that offered by Thomley. They define a class of concentration fields in which each leaf contributes its own field of morphogen. Briefly, they consider

$$C(\theta) = \sum_{i=1}^{N} \lambda^i f(\theta - \theta_i), \quad 0 < \lambda < 1,$$

and determine $\theta_1, \theta_2, \ldots, \theta_N$ such that $C(\theta)$ is minimised. $\theta_i$ denotes the position of the $i$th leaf on the circle where it is assumed that $\theta_{i+1} - \theta_i$ is constant. $f(\theta)$ is a monotonically decreasing even positive function, differentiable for positive $\theta$ and maximises when $\theta = 0$: the solution to Thomley’s diffusion system is such a candidate. Also considered are other “static” measures of concentration. It is shown numerically that the golden angle and other numbers related to it possess the desired minimisation properties for a wide range of concentration fields.

A computer simulation of the inhibitor-diffusion hypothesis on a cellular cylindrical surface has been constructed by Veen [135] and Veen and Lindenmayer [136]. They assume isotropic diffusion of the inhibitor which is produced by the leaf primordia and the apex or upper circle of the cylinder and degraded in a first-order reaction. Leaf primordia arise in cell sites where the concentration of the inhibitor is below some threshold value. Growth is simulated by the addition of a new row of cells with a specified concentration of inhibitor. The concentration profile is determined using finite difference approximation. The model succeeds in predicting many phyllotactic patterns observable
in nature depending of course upon parameters such as the rate of diffusion and decay of the inhibitor, the growth rates and the initial condition and configuration of cells. This model has been used by Hellendorn and Lindenmayer [41] in investigating the different types of phyllotactic patterns exhibited by the plant _Bryophyllum tubiflorum_. The model suggests that the changes in phyllotaxis are due either to a decrease in the production rate of the inhibitor at the apex or alternatively to an increase in the apical size. Young [146] has also formulated a model with diffusion occurring on the surface of a cylinder. The concentration of the inhibitor is assumed to have reached a steady-state before the initiation of new primordia which occurs when and where the concentration drops below a specified threshold value. Various phyllotactic patterns arise from the model. Schwabe and Clewer [107] have conducted a similar successful computer simulation, accounting for the most common types of phyllotaxis. They modelled the diffusion of an inhibitor subject to polar transport, which restricts its direction of movement. They considered conical as well as cylindrical surfaces and allowed the inhibitor to diffuse through the body and not just on the surface. Mitchison [88] has also investigated the inhibitor-diffusion hypothesis; he suggests that “an inhibitor mechanism can be formally equivalent to contact pressure” by considering the contours of inhibitor concentration as circles playing the role of “contact circles”. Turing [127] had earlier considered, as the basis for morphogenesis, a system of two interacting substances in a ring of cells. His attempts to extend the analysis (see [128]) to predict phyllotactic patterns were not completed before his death.

Models involving the short range activation and long range inhibition of morphogens forming a prepattern have been proposed to explore Schoute’s hypothesis. Meinhardt [85] has suggested that the two morphogens interacting with each other may be the mechanism which determines the precise positioning of leaves. Berding et. al [5] have investigated the reaction-diffusion system of Gierer and Meinhardt [34] in an attempt to produce a spiral prepattern similar in appearance to the sunflower head. They were able to simulate
the occurrence of Archimedian and logarithmic spirals but could not account for the prevalence of Fibonacci phyllotaxis to the exclusion of all other systems.

It should be mentioned that the existence of growth inhibitors is purely hypothetical. Those experiments which do support this hypothesis usually either involve microsurgery aimed at thwarting the transport of morphogens (see for example Wardlaw [139]) or they involve the introduction of extraneous substances which change the phyllotaxis (see for instance Schwabe [106], Maksimowych and Erickson [80]).

The rule of Hofmeister (that each new primordium arises in the widest gap) also inspired an alternative explanation of the mechanism of phyllotaxis. Van Iterson [130] in 1907 applied the tenet of Hofmeister establishing a geometry of "regular point systems" on surfaces of revolution such as cylinders and cones. Erickson [25] has presented the results of van Iterson using a more modern notation. Van Iterson's discussion on regular systems of touching circles on the surface of cylinders was largely adopted by Snow and Snow [111]-[113] forming the "first available space" theory whereby each new primordium arises in the widest gap between existing primordia as soon as the gap is large enough to accommodate it. They performed surgical experiments on plant apices designed to substantiate their theory. Mitchison [88] constructed a space-filling model whereby primordia are represented as circles of equal size initiated on the surface of an expanding cylinder. Assuming that the lattice of circles remains approximately regular and that each circle touches two circles formed prior to it, Mitchison claims that Fibonacci phyllotaxis arises as a natural consequence. (We indicate, in Section 6.3.2, some errors in Mitchison's reasoning which render his conclusions untenable.) Adler [2] presented a model with leaves formed on the surface of a cylinder. He showed that if each new primordium arises in the largest gap between two predecessors, nearer to the older of the two and in a constant ratio, then the divergence angle lies between 1/3 and 2/5. Recently, Williams and Brittain [143] have conducted a simulation of Snow and Snow's hypothesis.
Primordia are represented on a disc by touching circles of varying radii, initially placed in the widest available space. Assuming values for the plastochrone ratio (the ratio of the distances from the centre to the centre of successively formed primordia), a computer simulation predicts the divergence angles which reproduce those phyllotaxis systems most often found in nature.

Teleological models of phyllotaxis have been proposed by some researchers viewing the problem as phylogenetic rather than ontogenetic. In 1873, Wright [145] suggested that leaves were formed on a stem with as little overlap as possible in order for them to be optimally exposed to light and air. He investigated the amount of leaf overlap in systems produced by rational divergence angles with denominators less than 14. The angles $1/2, 2/3, 3/5, 5/8$ and $8/13$ were found to possess special properties. In 1907 Weisner claimed to have experimentally proved that the golden angle provided the plant with less leaf overlap than any other divergence angle although his method does not substantiate the claim (see Adler [1]). Leigh [74] investigated the hypothesis of Wright by examining the continued fraction expansion of a number concluding that a non-golden angle "may yield a more even distribution of leaves at one stage of growth, but only at the expense of greater overlap later." He also discussed the evolutionary significance of this result observing that plants which exhibit Fibonacci phyllotaxis often occupy habitats such as roadsides, swamps, dunes, second-growth forests - places which are subject to broad environmental changes. The distribution of points successively placed at a constant angle about a circle has also been studied by Marzec and Kappraff [82]. They computed the sum of the squared lengths of the subintervals formed on the circle and used this value as a measure of uniformity. They determined the set of numbers which, in an asymptotic sense, minimises this measure. The numbers were found to have a close association with the golden number.
Coxeter [18] put forward a proposal which he claimed to be “biologically plausible and mathematically interesting.” He considered a lattice of points \((r, \theta) = (n/N, n\alpha)\), \(n = 1, 2, 3, \ldots\) (in cylindrical polar co-ordinates) where \(\alpha\) is an irrational number between 0 and 1, and \(N\) is positive. He showed that only the golden number satisfies the condition that, for any vertical line on the cylinder passing through a lattice point, the neighbouring points lie on alternate sides as one proceeds upwards along it. This observation had actually been made previously by de Candolle [20] in 1881.

Wright [145] also realised the importance of the golden angle in providing a “compact arrangement in the bud”. Richards [91] represented a phyllotaxis system by assigning to point \(n\) the polar co-ordinates \((r, \theta) = (\sqrt{n-1}, n\tau)\) where \(n = 1, 2, 3, \ldots\) and forming a mosaic of roughly congruent figures around these points. Richards claimed without proof that the golden angle provided the most uniform packing of equal members on a plane surface. Ridley [96] substantiated this claim by investigating the “packing efficiency” of the set of points \((r, \theta) = (\sqrt{n}, n\alpha), \ n = 1, 2, 3, \ldots, 0 < \alpha < 1\). It is shown that the golden number ensures the most efficient packing in that the distance between the closest pair of points is greatest if \(\alpha\) is the golden number. These points lie on a “cyclotron spiral,” the genetic spiral proposed by Vogel [138] to describe the location of florets on a sunflower head. The construction allows for each seed to occupy an equal area. van der Struijs [129] independently proposed this model and applied the construction to the packing of microfibrils in the collagen fibril.

Most models of phyllotaxis have assumed that the leaves lie on a circle, disc, or the surface of a cylinder. Recently, Ridley [98] has offered a thorough mathematical treatment to describe phyllotaxis systems on general surfaces of revolution, with the primordia growing at arbitrary rates. Primordia are arranged according to a constant divergence angle: Ridley was able to extend his result in [96] to show that the golden angle provides the most efficient packing on a surface which has a continuously varying
lattice. The lattice is defined at each point on the surface and is used in describing changing phyllotaxis. It is shown that the nature of the change at any point depends on two parameters; the divergence angle and a quantity proportional to the ratio of the cross-sectional area of the plant and the surface area of the primordium.

Jean [48]-[50], [53]-[59] has offered a model of phyllotaxis using a rather different approach than those we have thus described. He prefers to view the problem as phyletic and describes the phenomenon in terms of a hierarchy (basically a graph or tree) and analyses the allowable hierarchies by defining parameters such as the complexity, stability and rhythm. In terms of these parameters, the entropy of plants is defined - Fibonacci phyllotaxis is then found to be the pattern which minimises this entropy.

1.3 Plan of thesis

This thesis is concerned with aspects relating to phyllotaxis (leaf arrangement) in plants. We begin our study, in Chapter 2, by considering the distribution of $N$ points placed consecutively around the circle at a constant (irrational) angle. This might be viewed as a simplistic model of phyllotaxis where the points correspond to the placement of primordia on a meristematic ring. We offer a new proof of the Steinhaus Conjecture which states that, for all $\alpha$ and all values of $N$, the points partition the circle into subintervals or gaps of at least two, and at most three, different lengths. Some extensions are made: for instance, where the circle is divided into gaps of two different lengths we determine the number of points that precede a given point on the circle. We also determine values of $\alpha$ which produce identical orderings of points.

In Chapter 3, we examine further the nature of the gap division by studying the transition from a circle composed of gaps of two different lengths to a like sequence of gaps. By labelling gaps according to length (either large or small) a sequence of strings is defined, describing the transition. Chapter 4 describes the special case where $\alpha$ has
identical terms in its continued fraction expansion. An interesting relationship between
the sequence of strings formed from this value of $\alpha$ and the sequence $d_1 d_2 d_3 \ldots$ where
$d_i = [(i + 1)\alpha] - [i\alpha]$ (the Bernoulli sequence - see Bernoulli [6] or Venkov [137], pp.
65-68). The sequence of strings is represented graphically - the graph derived from the
golden number, $\tau = (\sqrt{5} - 1)/2$, is analysed yielding a solution to the inductive equation
$G(n) = n - G(G(n - 1)), G(0) = 0, n \geq 2.$

Chapter 5 utilises results from Chapters 2 and 3 to analyse the degree of uniformity
of the sequence of points. It is shown that the golden number in many ways provides a
more even spacing. Two such measures of uniformity which we look at are the standard
and extreme discrepancies of sequences studied in connection with the theory of uniform
distribution modulo one (see Kuipers and Niederreiter [68]). We show how these two
measures are related and obtain some values of them in terms of the continued fraction
expansion of $\alpha$. The final result of this chapter investigates contact pressure on the
circle by determining the manner in which points are placed so that the smallest gap is
consistently as large as possible.

In the final chapter, we look at phyllotaxis systems of higher dimension. We first
investigate properties associated with the phyllotaxis system defined on the surface of
a cylinder with points equally spaced along a regular helix. Adler [1], [3] considered
this system in formulating a quantitative description of Schwenender's contact pressure
hypothesis. We verify the consequences of his model. Then we formulate a model
of phyllotaxis which attempts to explain the phenomena for a general class of surfaces.
Assuming that the system is "regular" and that the phyllotaxis rises as a function of time
and distance from the growing point, a simple rule governing the movement of leaves is
proposed which guarantees the emergence of Fibonacci phyllotaxis.

The first two appendices of the thesis are concerned with the theory of continued
fractions. Appendix A introduces notation and applies some of the results from Chapter 2
to Diophantine approximation theory. Appendix B offers a geometrical interpretation of
the continued fraction expansion of a number in terms of the partitioning of a rectangle.
Appendix C contains some results needed for Chapter 5 while Appendix D considers the
diffusion of morphogen on a ring.
CHAPTER TWO

The Three Gap Theorem

2.1 Introduction

Consider the distribution of a sequence of points arranged on a circle of unit circumference in the following manner: take any point on the circle as the origin and call this the point 0. Generate the next point at a clockwise distance of \( \alpha \) (or clockwise angle of \( \alpha \) revolutions) where \( \alpha \) is irrational. Call this point 1. In this way generate point \( q + 1 \) from point \( q \) at the same angle \( \alpha \) until a total of \( N \) points 0, 1, 2, \ldots, \( N - 1 \) have been generated. Note that point \( q \) is at a total distance (or angle) of \( q\alpha \) from the origin.

We will be concerned with determining the order in which these points appear on the circle and with the nature of the arcs or gaps between these ordered points. To determine this order we permute the integers 0, 1, 2, \ldots, \( N - 1 \) into the sequence

\[
U_N(\alpha) = (u_1(N), u_2(N), \ldots, u_N(N)),
\]

where

\[
\{u_j(N)\alpha\} < \{u_{j+1}(N)\alpha\}, \quad j = 1, 2, \ldots, N - 1.
\]

Thus the sequence \( \{u_j(N)\alpha\}, j = 1, 2, \ldots, N \) is simply the sequence \( \{q\alpha\}, q = 0, 1, 2, \ldots, N - 1 \) arranged in ascending order. Hence \( U_N(\alpha) \) is the ordered sequence of points as they appear on a circle of \( N \) points. Usually, \( N \) is specified and \( u_j(N) \) is then denoted by \( u_j \). Also note that \( u_1(N) = 0 \). For convenience we define \( u_{N+1}(N) \) (point \( N \)) to be zero.

We define \( d_{i,j}(N) = d_{i,j} \) to be the shortest clockwise distance between point \( i \) and point \( j \) and in future refer to this as the distance between \( i \) and \( j \). It is clear that

\[
d_{i,j} = \{j\alpha\} - \{i\alpha\} = (j - i)\alpha, \quad 0 \leq i, j < N, \quad (2.1.1)
\]
where \( \{ \} \) is the fractional operator. Hence, for real \( y \), \( \{y\} = y - [y] \), where \([\ ]\) is the truncation (or integer part) operator and \([y]\) is the largest integer not greater than \( y \). We remark that \( \{y\} = y \mod 1 \), where \( y \mod x = x(y/x) = y - x[y/x] \), for real \( x \) and \( y \). Thus \( y \mod x \) is the remainder upon division of \( y \) by \( x \) and \( 0 \leq y \mod x < x \). Note that
\[
\{y\} = \{y - \lfloor y \rfloor + x\} = \{y - x + [x] - [y]\} = \{y - x\}.
\]

From (2.1.1),
\[
d_{0,q} = d_{k,k+q} = 1 - d_{k+q,k} = \{q\alpha\}; \quad k = 0, 1, 2, \ldots, N - q - 1, \quad 0 < q < N \tag{2.1.2}
\]
\[
d_{q,0} = d_{k,k-q} = 1 - d_{k-q,k} = 1 - \{q\alpha\}; \quad k = q, q + 1, \ldots, N - 1, \quad 0 < q < N \tag{2.1.3}
\]

We define the length of the \( j \)th gap as follows.
\[
g_j(N) = d_{u_j, u_{j+1}}, \quad j = 1, 2, \ldots, N. \tag{2.1.4}
\]

There are \( N \) gaps and, in particular
\[
g_1(N) = \min_{0 \leq q < N} d_{0,q} = d_{0, u_2}. \tag{2.1.5}
\]
\[
g_N(N) = \min_{0 \leq q < N} d_{q,0} = d_{u_N, 0}. \tag{2.1.6}
\]

Also,
\[
d_{0,u_{j+1}} = \sum_{k=1}^{j} g_k(N).
\]

Often we will simply denote \( g_j(N) \) by \( g_j \).

The 'age' of gap \( g_j(N) \) is \( a_j(N) \) where
\[
a_j(N) = N - 1 - \max(u_j, u_{j+1}). \tag{2.1.7}
\]

Consider three consecutive points \( a, b \) and \( c \) and call \( a \) the predecessor to \( b \), denoted by \( a = \text{Pre}(b) \). We call \( c \) the successor to \( b \), denoted by \( c = \text{Suc}(b) \). It
is seen that if $2 \leq j \leq N$, then $u_j = \text{Pre}(u_{j+1}) = \text{Suc}(u_{j-1})$. Also, $\text{Suc}(0) = u_2$ and $\text{Suc}(u_N) = u_{N+1} = 0$.

We state the Steinhaus Conjecture for which we offer a new proof. For most values of $N$ there are three different gap lengths, either $g_1$, $g_N$ or $g_1 + g_N$; for particular values of $N$ (related to the convergents of $\alpha$) there are only two different gap lengths, $g_1$ and $g_N$. Furthermore, in the three gap case, the sequence of points is such that $q - \text{Pre}(q)$ is either $u_2$, $-u_N$ or $u_2 - u_N$. In the two gap case, $q - \text{Pre}(q)$ is either $u_2$ or $-u_N$.

We extend the above, allowing us to generate the sequence $U_N(\alpha)$ as well as $g_j$ for $j = 1, 2, \ldots, N$. Our proof uses the theory of continued fractions, the elements of which are outlined in Appendix A.1. All notation, not explicitly defined in the main sections, is introduced in Appendix A, Section A.1.

2.2 The predecessor and successor of the origin

In this section we determine the value of the points $u_2(N) = \text{Suc}(0)$ and $u_N(N) = \text{Pre}(0)$ in terms of the denominators of the convergents to $\alpha$ (see Appendix A, equations (A.1.8) and (A.1.13)).

Firstly consider the trivial case where $N \leq q_1$. Then, $u_j = j - 1, 1 \leq j \leq N$. This follows from (A.1.17), since $p_0/q_0 < \alpha < p_1/q_1$ or $a_0 < \alpha < a_0 + 1/q_1$ from (A.1.8). Thus $\{q\alpha\} = q\{\alpha\}$ for $q < N \leq q_1$. In the following we will only consider the case where $N > q_1$.

We note that every integer, greater than $q_1$, may be located in exactly one of the intervals:

$$(q_{n-1}, q_n], \quad (q_{n,i-1}, q_{n,i}], \quad i = 2, 3, \ldots, a_n, \quad (n \geq 2).$$

The following results presuppose that $N$ is located in one such interval.
The following lemma states in effect that every best approximation (of the second kind) to \( \alpha \) is a total convergent. (See Appendix A, Section A.2.) This is well known and for the proof we refer the reader to Khintchine [63] or Hardy and Wright [37]. We define \( \|q \alpha\| \) as the distance from \( q \alpha \) to its nearest integer (see (A.1.22)).

**Lemma 2.1**

\[
\min_{0 < q < q_n,i} \|q \alpha\| = \|q_{n-1} \alpha\|, \quad i = 1, 2, \ldots, a_n, \quad (n \geq 2).
\]

**Lemma 2.2**

\[
q_{n-1} = \begin{cases} 
  u_2, & \text{n odd,} \\
  u_N, & \text{n even,}
\end{cases}
\]

where \( q_{n-1} < N < q_n,i, \ 1 \leq i \leq a_n, \ (n \geq 2) \).

**Proof**  
From (A.1.22b) we may restate Lemma 2.1 as follows:

\[
\min_{0 < q < q_n,i} (\{q \alpha\}, 1 - \{q \alpha\}) = \min (\{q_{n-1} \alpha\}, 1 - \{q_{n-1} \alpha\}),
\]

\[
= \begin{cases} 
  \{q_{n-1} \alpha\}, & n \text{ odd,} \\
  1 - \{q_{n-1} \alpha\}, & n \text{ even.}
\end{cases} \quad (2.2.1)
\]

Equation (2.2.1) follows from (A.1.23). From (2.2.1) and equations (2.1.2) and (2.1.3) which define \( d_{0,q} \) and \( d_{q,0} \) we find that

\[
\min_{0 < q < N} d_{0,q} = d_{0,q_{n-1}}, \quad n \text{ odd,}
\]

\[
\min_{0 < q < N} d_{q,0} = d_{q_{n-1},0}, \quad n \text{ even,}
\]

where \( q_{n-1} < N \leq q_n,i, \ 1 \leq i \leq a_n, \ (n \geq 2) \).

Thus we have shown that \( q_{n-1} \) is the successor to the origin when \( n \) is odd and the predecessor to the origin when \( n \) is even. It now remains to evaluate \( u_2 \) when \( n \) is even and \( u_N \) when \( n \) is odd.
In Appendix A, Section A.3, the concept of a second best approximation of the second kind to $\alpha$ is introduced. There we show that the following lemma implies that these second best approximations are partial convergents to $\alpha$.

**Lemma 2.3**

\[ \min_{0 < q < q_{n,i}} \| q \alpha \| = \| q_{n,i-1} \alpha \|. \quad i = 1, 2, \ldots, a_n, \ (n \geq 2), \]

where $k$ is integer so that $0 < k q_{n-1} < q_{n,i}$ or $k < q_{n-2}/q_{n-1} + i$ or $k \leq i \leq a_n$.

**Proof** We emphasise that $q$ is integer, lies between 0 and $q_{n,i}$, and may not be a multiple of $q_{n-1}$.

We write

\[ q = \mu q_{n-1} + v q_{n,i}, \]
\[ p = \mu p_{n-1} + v p_{n,i}. \]

Solving for $\mu$ and $v$ using (A.1.15) yields the integer solution:

\[ \mu = (-1)^n (p q_{n,i} - q p_{n,i}), \]
\[ v = (-1)^{n-1} (p q_{n-1} - q p_{n-1}). \]

Neither $\mu$ nor $v$ may equal 0 (if $\mu = 0$, then $q = v q_{n,i}$, while if $v = 0$, $q = \mu q_{n-1}$ - two obvious contradictions). Note that

\[ q \alpha - p = \mu (q_{n-1} \alpha - p_{n-1}) + v (q_{n,i} \alpha - p_{n,i}). \]

Now, $0 < q = \mu q_{n-1} + v q_{n,i} < q_{n,i}$. This shows that $v$ and $\mu$ are of opposite sign. From Appendix A (see (A.1.24) and (A.1.25)), $(q_{n-1} \alpha - p_{n-1})$ and $(q_{n,i} \alpha - p_{n,i})$ are also of opposite sign. Thus,

\[ |q \alpha - p| = |\mu| \|q_{n-1} \alpha\| + |v| \|q_{n,i} \alpha\|, \]

\[ \geq \|q_{n-1} \alpha\| + \|q_{n,i} \alpha\|. \]
That is, \((A.1.27)\),
\[\|q\alpha\| \geq \|q_{n,i-1}\alpha\|\]
Equality occurs when \(|\mu| = |v| = 1\). If \(\mu = -v = 1\), then \(q = q_{n-1} - q_{n,i} < 0\) (a contradiction), while \(-\mu = v = 1\) implies that \(q = -q_{n-1} + q_{n,i} = q_{n,i-1}\). Thus we conclude that
\[\|q\alpha\| > \|q_{n,i-1}\alpha\|, \quad 0 < q < q_{n,i}, \quad q \neq kq_{n-1}, \quad k = 1, 2, \ldots, i.\]

\textbf{Lemma 2.4}
\[q_{n,i-1} = \begin{cases} u_N, & n \text{ odd}, \\ u_2, & n \text{ even}, \end{cases}\]
where \(q_{n,i-1} < N \leq q_{n,i}, \quad 2 \leq i \leq a_n, \quad (n \geq 2)\).

\textbf{Proof} From (A.1.22b) an equivalent formulation of Lemma 2.3 is the following:
\[\min_{0 < q < q_{n,i}, \quad q \neq kq_{n-1}} (\{q\alpha\}, 1 - \{q\alpha\}) = \min (\{q_{n,i-1}\alpha\}, 1 - \{q_{n,i-1}\alpha\})\]
\[= \begin{cases} 1 - \{q_{n,i-1}\alpha\}, & n \text{ odd}, \\ \{q_{n,i-1}\alpha\}, & n \text{ even}. \end{cases} \tag{2.2.2}\]
Equation (2.2.2) follows from (A.1.26).

The condition \(q \neq kq_{n-1}, \quad k = 1, 2, \ldots, i\) need not appear in the lemma since (for odd \(n\)),
\[d_{kq_{n-1,0}} = 1 - k\|q_{n-1}\alpha\|,\]
\[> 1 - \frac{k}{q_{n-1}} > \frac{1}{2}.\]
The inequality derives from (A.1.20). The first step follows from (A.1.28). Thus,
\[d_{q_{n,i-1,0}} < 1/2 < d_{kq_{n-1,0}} \leq d_{iq_{n-1,0}}, \quad (\text{where } n \text{ is odd}).\]
That is, the distance from point \(kq_{n-1}\) to the origin is greater than the corresponding distance from point \(q_{n,i-1}\). The case is similar for even \(n\). Hence,
\[\min_{0 < q < N} d_{q,0} = d_{q_{n,i-1,0}}, \quad n \text{ odd},\]
where \( q_{n,i-1}, 2 < i < a_n \), (\( n \geq 2 \)). The lemma then follows.

\[ \begin{align*}
\min_{0 < q < N} d_0, q &= d_0, q_{n,i-1}, \\
&= n \text{ even},
\end{align*} \]

where \( q_{n,i-1} < N \leq q_{n,i}, 2 \leq i \leq a_n \), (\( n \geq 2 \)). The lemma then follows.

**Lemma 2.5**

\[
q_{n-2} = \begin{cases} 
q_{n-1}, & n \text{ odd,} \\
q_{n,i-1}, & n \text{ even,}
\end{cases}
\]

where \( q_{n-1} < N \leq q_{n,1} \), (\( n \geq 2 \)).

**Proof** This is a particular case of Lemma 2.4 with \( i = 1 \) and the lower bound of \( N \) replaced by \( q_{n-1} \).

Amalgamating lemmas 2.2, 2.4 and 2.5 allows one to find the predecessor and successor to the origin for any value of \( N \). This is expressed in the following theorem.

**Theorem 2.6**

\[
\begin{align*}
q_{n-1}, & \quad n \text{ odd,} \\
q_{n,i-1}, & \quad n \text{ even,}
\end{align*}
\]

\[ u_N = \begin{cases} 
q_{n,i-1}, & n \text{ odd,} \\
q_{n-1}, & n \text{ even,}
\end{cases} \quad (2.2.4) \]

where \( q_{n,i-1} < N \leq q_{n,i}, 2 \leq i \leq a_n \), (\( n \geq 2 \)).

Also, for \( q_{n-1} < N \leq q_{n,1} \), (\( n \geq 2 \)),

\[
\begin{align*}
q_{n-1}, & \quad n \text{ odd,} \\
q_{n-2}, & \quad n \text{ even,}
\end{align*}
\]

\[ u_N = \begin{cases} 
q_{n-2}, & n \text{ odd,} \\
q_{n-1}, & n \text{ even.}
\end{cases} \quad (2.2.6) \]

Recall that the phyllotaxis of a system is determined by a leaf and its nearest neighbours on alternate sides. For the circle, \((u_2(N), u_N(N))\) is the phyllotaxis corresponding to \( N \) points.
Let $V(\alpha)$ denote the ordered sequence of distinct pairs $(u_2, u_N)$ where $N = q_1, q_1 + 1, \ldots$. We call $V(\alpha)$ the phyllotaxis path of $\alpha$. From Theorem 2.6,

$$V(\alpha) = ((q_0, q_1), (q_2, q_1), (q_2, q_1), \ldots, (q_2, q_2, q_1)) = (q_2, q_1),$$

$$(q_4, q_3, 1), (q_2, q_3, 2), \ldots, (q_2, q_3, a_3) = (q_2, q_3),$$

$$(q_4, q_3), (q_4, q_3), \ldots, (q_4, a_4, q_3) = (q_4, q_3),$$

$\ldots$$

which we write as

$$V(\alpha) = ((q_0, q_1), (q_n, i, q_{n-1}'); \ i = 2, 3, \ldots, a_n, n = 1, 2, \ldots),$$

(2.2.7)

where

$$(q_n, i, q_{n-1}) = \begin{cases} (q_{n-1}, q_n, i), & n \text{ odd,} \\ (q_n, i, q_{n-1}), & n \text{ even.} \end{cases}$$

(2.2.8)

Note that $V(\alpha)$ characterises $\alpha$. That is, $V(\alpha) = V(\beta)$ only if $\alpha = \beta$.

**Corollary 2.7**

The first point to replace either $\text{Suc}(0)$ or $\text{Pre}(0)$ is the point $\text{Pre}(0) + \text{Suc}(0)$.

**Proof** This follows from Theorem 2.6. If $u_2(N) \neq u_2(N - 1)$ then $u_2(N) = u_2(N - 1) + u_{N-1}(N - 1)$ and $u_N(N) = u_{N-1}(N - 1)$. If $u_N(N) \neq u_{N-1}(N - 1)$ then $u_N(N) = u_2(N - 1) + u_{N-1}(N - 1)$ and $u_2(N) = u_2(N - 1)$. $\blacksquare$

**Corollary 2.8**

$$\|u_2 + u_N\alpha\| = \|u_2\alpha\| - \|u_N\alpha\|.$$

**Proof** We use equation (A.1.27). For $q_{n-1} < N \leq q_n, 1$,

$$\|u_2 + u_N\| = \|q_{n, 1}\alpha\|,$$

$$= \|q_{n-2}\alpha\| - \|q_{n-1}\alpha\|,$$

$$= \begin{cases} \|u_N\alpha\| - \|u_2\alpha\|, & n \text{ odd,} \\ \|u_2\alpha\| - \|u_N\alpha\|, & n \text{ even.} \end{cases}$$
For $q_{n,i-1} < N \leq q_{n,i}$, $2 \leq i \leq a_n$,

$$
\|u_2 + u_N\| = \|q_{n,i} \alpha\| = \|q_{n,i-1} \alpha\| - \|q_{n-1} \alpha\|,
$$

$$
= \begin{cases} 
\|u_N \alpha\| - \|u_2 \alpha\|, & n \text{ odd}, \\
\|u_2 \alpha\| - \|u_N \alpha\|, & n \text{ even}.
\end{cases}
$$

But $\|u_2 + u_N\alpha\| > 0$. This completes the proof. ■

**Corollary 2.9**

$N \leq u_2 + u_N$.

**Proof** If $q_{n-1} < N \leq q_{n,1}$, then $u_2 + u_N = q_{n,1} \geq N$, while if $q_{n,i-1} < N \leq q_{n,i}$, $2 \leq i \leq a_n$, then $u_2 + u_N = q_{n,i} \geq N$. ■

**Proposition 2.10**

The upper bound of $k$ (that is $i$) in Lemma 2.3 may be replaced by the following.

$$
m_n = \begin{cases} 
i, & i \leq (a_n + 1)/2, \\
a_n - i + 1, & i > (a_n + 1)/2.
\end{cases}
$$

**Proof** For $n \geq 2$, we require $k$ to not exceed $i$ and to satisfy the following inequality,

$$
\|k q_{n-1} \alpha\| < \|q_{n,i-1} \alpha\|.
$$

From (A.1.27) and (A.1.28), it follows that

$$
k < \frac{\|q_{n-2} \alpha\|}{\|q_{n-1} \alpha\|} - i + 1.
$$

From (A.1.1) and (A.1.21) it is seen that

$$
k \leq a_n - i + 1.
$$

Hence,

$$
k \leq \min(i, a_n - i + 1), \quad i = 1, 2, \ldots, a_n, \quad (n \geq 2).
$$
We have thus determined the successors and predecessors to the origin and we extend this to find the successor of any point on the circle. That is, we determine the complete sequence $U_N(\alpha)$.

2.3 The two gap case

We examine the case where the circle is divided into gaps of two distinct lengths before considering the three gap case. We emphasise that this case occurs when there are, in total, $u_2 + u_N$ points distributed on the circle. That is,

$$N = u_2 + u_N = q_n, \quad i = 1, 2, \ldots, a_n, \quad (n \geq 2),$$

(2.3.1)

so that the total number of points on the circle is equal to the denominator of a total or partial convergent to $\alpha$. In this section and its sub-sections it will be assumed that $N = u_2 + u_N$.

Firstly consider those points located at a distance $\{q\alpha\}$ from a point $j$ where $0 < q < N$ (and thus, $0 < j < N - q$). Of these points, $j + u_2$ is the successor of point $j$ since $\{u_2\alpha\}$ is the smallest of all possible distances, $\{q\alpha\}$.

That is, for $0 < j < N - q$,

$$\min_{0 < q < N} d_{j, j+q} = \min_{0 < q < N} d_{0, q} = d_{0, u_2} = d_{k, k+u_2}, \quad 0 \leq k < N - u_2. \quad (2.3.2)$$

Thus, $k + u_2 = \text{Suc}(k)$ where $k = 0, 1, 2, \ldots, N - 1 - u_2$.

Similarly, for $q \leq j < N$,

$$\min_{0 < q < N} d_{j, j-q} = \min_{0 < q < N} d_{q, 0} = d_{u_N, 0} = d_{k, k-u_N}, \quad u_N \leq k < N. \quad (2.3.3)$$

Thus, $k - u_N = \text{Suc}(k)$ where $k = u_N, u_N + 1, \ldots, N - 1$. (Note that $N = u_2 + u_N$).

We now evaluate the sequence of points, $U_N(\alpha)$, and from this result show how to determine the number of points that precede any given point on the circle. Later we present results concerning the distribution of gaps.
2.3.1 The sequence $U_N(\alpha)$

**Theorem 2.11**

\[
\text{Suc}(m) - m = \begin{cases} 
  u_2, & 0 \leq m < u_N, \\
  -u_N, & u_N \leq m < N.
\end{cases} \tag{2.3.4}
\]

**Proof** From (2.3.2), $k = \text{Pre}(k + u_2)$ $(k = 0, 1, 2, \ldots, u_N - 1)$ while it is seen from (2.3.3) that each remaining point $r$ $(r = u_N, u_N + 1, \ldots, u_N + 1)$ has, as its successor, point $r - u_N$. That is, $r - u_N = \text{Suc}(r)$.

In terms of the sequence $U_N(\alpha)$ where $N = u_2 + u_N$, $k$ and $k + u_2$ are consecutive elements of $U_N(\alpha)$ if $k$ is equal to one of the integers $0, 1, 2, \ldots, u_N - 1$. If not, $k$ and $k - u_N$ are consecutive elements.

**Corollary 2.12**

\[u_j = ((j - 1)u_2) \mod N, \quad j = 1, 2, \ldots, N. \tag{2.3.5}\]

**Proof** From Theorem 2.11, for $u_N \leq u_j < N$,

\[u_j - u_N = u_j + u_2 - N,\]
\[= (u_j + u_2) \mod N.\]

The second step is valid since $N \leq u_j + u_2 < 2N$, $u_N \leq u_j < N$.

We may now rewrite (2.3.4) to state

\[u_{j+1} = (u_j + u_2) \mod N, \quad j = 0, 1, 2, \ldots, N,
\]
\[u_1 = 0.\]

The corollary thus follows.

**Corollary 2.13**

\[u_j = ((-1)^{n-1}(j - 1)q_{n-1}) \mod q_{n,i}, \quad j = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, a_n, \quad (n \geq 2). \tag{2.3.6}\]
Proof: Substituting expressions for \( u_2 \) from Theorem 2.6 into (2.3.5) yields the following statements;

For odd \( n \),

\[
\begin{align*}
    u_j &= (j - \gamma_{n-1}) \mod q_{n,i}.
\end{align*}
\]

For even \( n \) (using (A.1.13)),

\[
\begin{align*}
    u_j &= (j - 1)q_{n,i} - q_{n-1} \mod q_{n,i}, \\
    &= (j - 1)(q_{n,i} - q_{n-1}) \mod q_{n,i}, \\
    &= (-1)(j - 1)q_{n-1} \mod q_{n,i},
\end{align*}
\]

where, in both cases, \( j = 1, 2, \ldots, N \) and \( i = 1, 2, \ldots, a_n \). The corollary thus follows.

2.3.2 Number of points preceding a given point

Corollaries 2.12 and 2.13 allow us to evaluate \( u_j \) from \( j \). We now consider the inverse problem of evaluating \( j \) given \( m \) (in the two gap case) where \( u_j = m \). \( j \) is of course the number of points that precede point \( m \).

Theorem 2.14

\[
\begin{align*}
    j &= 1 + (mp_{n,i}) \mod q_{n,i}, \quad i = 1, 2, \ldots, a_n, \ (n \geq 2), \\
\end{align*}
\]

where \( m = u_j \).

Proof: For (2.3.6), there exists a non-negative integer \( r \) so that

\[
\begin{align*}
    u_j &= (-1)^n r q_{n,i} - (-1)^n (j - 1)q_{n-1},
\end{align*}
\]

which is a linear Diophantine equation with unknowns \( j \) and \( r \).

From (A.1.15),

\[
\begin{align*}
    u_j \left(( -1)^n p_{n-1}q_{n,i} - (-1)^n q_{n-1}p_{n,i}\right) = u_j.
\end{align*}
\]
Thus, a particular solution to (2.3.8) is

\[ j - 1 = j_0 = u_j p_{n,i}, \]
\[ r = r_0 = u_j p_{n-1}. \]

The general solution is

\[ j - 1 = u_j p_{n,i} + q_{n,i} t, \]
\[ r = u_j p_n + q_{n-1} t, \]

where \( t \) is integer. But \( 0 \leq j - 1 < q_{n,i} \). Thus, the theorem follows.

### 2.3.3 Gap sizes

**Theorem 2.15**

For \( N = u_2 + u_N = q_{n,i} \), the circle is divided into gaps of only two different lengths:

\[ g_1 = \|u_2\alpha\|, \quad g_N = \|u_N\alpha\|. \]

Further,

\[ g_j = \begin{cases} 
  g_1, & 0 \leq u_j < u_N, \\
  g_N, & u_N \leq u_j < N.
\end{cases} \]

**Proof** Using (2.1.1),

\[ g_j = d_{u_j,u_{j+1}}, \]
\[ = d_{0,u_{j+1}} - d_{0,u_j}, \]
\[ = \{u_{j+1}\alpha\} - \{u_j\alpha\}, \]
\[ = \{(u_{j+1} - u_j)\alpha\}. \]

From (2.3.4),

\[ g_j = \begin{cases} 
  \{u_2\alpha\}, & 0 \leq u_j < u_N, \\
  1 - \{u_N\alpha\}, & u_N \leq u_j < N.
\end{cases} \]

The theorem now follows from definition (A.1.22c). \( \blacksquare \)

We note from (2.3.4) that there are exactly \( u_N \) gaps equal in length to \( g_1 \) and \( N - u_N = u_2 \) gaps equal to \( g_N \). Thus,

\[ u_N g_1 + u_2 g_N = 1. \]  

(2.3.9)
2.4 The three gap case

In this section, we derive results concerning the distribution of points and gaps about a circle for general $N$. In particular, we show that if $N$ is not equal to $u_2 + u_N$, the circle is divided into gaps of exactly three different lengths. First we determine the sequence $U_N(\alpha)$ and the set of numbers which produce the same ordering of points.

2.4.1 The sequence $U_N(\alpha)$.

The following theorem presents the rule for generating the sequence $U_N(\alpha)$. Note that $m \in U_N(\alpha)$.

**Theorem 2.16**

\[
\text{Suc}(m) - m = \begin{cases} 
  u_2, & 0 \leq m < N - u_2, \\
  u_2 - u_N, & N - u_2 \leq m < u_N, \\
  -u_N, & u_N \leq m < N.
\end{cases} \tag{2.4.1}
\]

**Proof** From (2.3.2), (2.3.3),

\[
\text{Suc}(k) - k = \begin{cases} 
  u_2, & k = 0, 1, 2, \ldots, N - 1 - u_2, \\
  -u_N, & k = u_N, u_N + 1, \ldots, N - 1.
\end{cases} \tag{2.4.2}
\]

In particular, when $N = u_2 + u_N$,

\[
\text{Suc}(k) - k = \begin{cases} 
  u_2, & k = 0, 1, 2, \ldots, u_N - 1, \\
  -u_N, & k = u_N, u_N + 1, \ldots, u_2 + u_N - 1.
\end{cases} \tag{2.4.3}
\]

(This is merely Theorem 2.11).

From the sequence of $u_2 + u_N$ points, remove the $i$ points $u_2 + u_N - 1, u_2 + u_N - 2, \ldots, u_2 + u_N - i$ where $i < \min(u_2, u_N)$. Thus remove the successors of points $r$, $r = u_N - i, u_N - i + 1, \ldots, u_N - 1$. Then $N$ points are left where $\max(u_2, u_N) < N \leq u_2 + u_N$.

From (2.4.3),

\[
\begin{align*}
\text{Suc}(r) &= r + u_2, \\
\text{Suc}(\text{Suc}(r)) &= r + u_2 - u_N,
\end{align*}
\]
where
\[ r = u_N - i, u_N - i + 1, \ldots, u_N - 1, \]
\[ = N - u_2 + 1, N - u_2 + 2, \ldots, u_N - 1. \]

Thus, in the circle of \( N \) points \((\max(u_2, u_N) < N \leq u_2 + u_N)\), the successor of point \( r \) is \( r + u_2 - u_N \) where \( r = N - u_2, N - u_2 + 1, \ldots, u_N - 1 \). Combining this with (2.4.2) yields
\[
\text{Suc}(k) - k = \begin{cases} 
  u_2, & k = 0, 1, 2, \ldots, N - 1 - u_2, \\
  u_2 - u_N, & k = N - u_2, N - u_2 + 1, \ldots, u_N - 1, \\
  -u_N, & k = u_N, u_N + 1, \ldots, N - 1.
\end{cases}
\]

where \( \max(u_2, u_N) < N \leq u_2 + u_N \).

The theorem follows from Corollary 2.9 as \( N \) never exceeds \( u_2 + u_N \).

It is noted that (2.4.1) reduces (as it should) to the two gap case (2.3.4) when \( N = u_2 + u_N \). In this case \( q - \text{Pre}(q) \) is equal to either \( u_2 \) or \(-u_N\). For any other value of \( N \), \( q - \text{Pre}(q) \) equals \( u_2 \), \(-u_N \) or \( u_2 - u_N \).

**Corollary 2.17**
\[
\text{Pre}(m) - m = \begin{cases} 
  u_N, & 0 \leq m < N - u_N, \\
  u_N - u_2, & N - u_N \leq m < u_2, \\
  -u_2, & u_2 \leq m < N.
\end{cases}
\]  

**Proof** This is an immediate consequence of Theorem 2.16.

**2.4.2 Invariance in \( U_N(\alpha) \)**

We now determine the limits for \( \alpha \) such that \( U_N(\alpha) \) remains constant.

Let \( E(a_0, a_1, \ldots, a_n) \) denote the set of all numbers for which the first \( n + 1 \) terms in their continued fraction expansions have the given values \( a_0, a_1, \ldots, a_n \).
E \left( a_0 \right) \text{ lies in the interval} \\
I \left( a_0 \right) = [a_0, a_0 + 1].

E \left( a_0, a_1 \right) \text{ lies in} \\
I \left( a_0, a_1 \right) = \left( a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, a_0 + \frac{1}{a_1} \right],

\text{while } E \left( a_0, a_1, a_2 \right) \text{ lies in} \\
I \left( a_0, a_1, a_2 \right) = \left[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + 1}}, a_0 + \frac{1}{a_1 + \frac{1}{a_2}} \right).

\text{In general, } E \left( a_0, a_1, \ldots, a_n \right) \text{ is contained in} \\
I \left( a_0, a_1, \ldots, a_n \right) = \left\{ \left\{ a_0; a_1, \ldots, a_{n-1}, a_n + 1 \right\}, \left\{ a_0; a_1, \ldots, a_n \right\}, \left\{ a_0; a_1, \ldots, a_{n-1}, a_n \right\}, \left\{ a_0; a_1, \ldots, a_{n+1} \right\} \right\}, \\
\text{ where } n \text{ odd, } \\
\left\{ \left\{ a_0; a_1, \ldots, a_{n+1} \right\}, \left\{ a_0; a_1, \ldots, a_{n-1}, a_n \right\}, \left\{ a_0; a_1, \ldots, a_{n+1} \right\}, \left\{ a_0; a_1, \ldots, a_{n+1} \right\} \right\}, \\
\text{ where } n \text{ even.} \tag{2.4.5}

\text{That is,} \\
I \left( a_0, a_1, \ldots, a_n \right) = \left\{ \left\{ \frac{P_n + P_{n-1}}{q_n + q_{n-1}}, \frac{P_n}{q_n} \right\}, \right\{ \frac{P_n}{q_n}, \frac{P_n + P_{n-1}}{q_n + q_{n-1}} \right\}, \\
\text{ where } n \text{ odd, } \\
\left\{ \left\{ \frac{P_n}{q_n}, \frac{P_n + P_{n-1}}{q_n + q_{n-1}} \right\}, \right\{ \frac{P_n + P_{n-1}}{q_n + q_{n-1}}, \frac{P_n}{q_n} \right\}, \\
\text{ where } n \text{ even.} \tag{2.4.6}

\text{Also, if } \alpha \in I \left( a_0, a_1, \ldots, a_n \right) \text{ then} \\
\alpha = \left\{ a_0; a_1, \ldots, a_n, t_{n+1} \right\}, \tag{2.4.7}

\text{where } 1 < t_{n+1} < \infty.

\textbf{Theorem 2.18}

\text{If } \alpha, \beta \in I \left( a_0, a_1, \ldots, a_n \right) \text{ then } U_N \left( \alpha \right) = U_N \left( \beta \right) \text{ for } 0 < N \leq 2q_n + q_{n-1}.

\textbf{Proof} \quad \text{The first } n + 1 \text{ terms in the continued fraction expansions of } \alpha \text{ and } \beta \text{ are the same only if the first } n + 1 \text{ total convergents are the same.}

\text{Inspection of Theorem 2.6 shows that } \alpha \text{ and } \beta \text{ generate the same number pair } u_2, u_N \text{ for } 0 < N \leq q_n + q_{n-1}. \text{ If } q_{n+1,1} < N \leq q_{n+1,2} \text{ (so that } a_{n+1} > 1 \text{) or } q_{n+1} < N \leq q_{n+2,1}
(\(a_{n+1} = 1\)) the pair \(u_2, u_N\) for \(\alpha\) and \(\beta\) is still the same, thus increasing the range for \(N\) to \(0 < N \leq 2q_n + q_{n-1}\).

The sequences \(U_N(\alpha)\) and \(U_N(\beta)\) depend solely on the values for \(u_2\) and \(u_N\) as indicated, for example, by Theorem 2.16. Thus the theorem follows. ■

**Theorem 2.19**

If \(\alpha \in I(a_0, a_1, \ldots, a_n)\) and \(\beta \in I(a_0, a_1, \ldots, a_{n-1}, b), b > a_n\) then \(U_N(\alpha) = U_N(\beta)\) only if \(0 < N \leq q_n + q_{n-1}, (n \geq 2)\).

**Proof** The proof is similar to that of Theorem 2.18. In this case the number pair \(u_2, u_N\) differs for \(\alpha\) and \(\beta\) only if \(N > q_n + q_{n-1}\). ■

**Theorem 2.20**

If the phyllotaxis is \((a, b)\) then

\[ B a - A b = 1, \]

\[ A < \alpha < \frac{B}{b}, \]

where \(A = [a \alpha], B = [b \alpha] + 1\).

**Proof** \((a, b) = (u_2(N), u_N(N))\) for some value of \(N\). Hence, from (2.3.9), \(a(1 - \{b \alpha\} + b \{a \alpha\} = 1\) or \(a(1 + [b \alpha]) - b[a \alpha] = 1\). (2.4.9) follows easily since \(A/a = [a \alpha]/a = \alpha - \{a \alpha\}/a < \alpha\) and \(B/b = ([b \alpha] + 1)/b = \alpha + 1 - \{b \alpha\}/b > \alpha\). ■

**Corollary 2.21**

If the phyllotaxis is \((q_{n,i}, q_{n-1})\), \((n \geq 2)\), then

\[ \frac{p_{n-1}}{q_{n-1}} < \alpha < \frac{p_{n,i}}{q_{n,i}}, \quad n \text{ odd}, \]
\[
\frac{p_{n,i}}{q_{n,i}} < \alpha < \frac{p_{n-1}}{q_{n-1}}, \quad n \text{ even.}
\]

**Proof**  This follows directly from Theorem 2.20 with \((a, b) = (q_{n,i}, q_{n-1})\). 

### 2.4.3 Gap sizes

**Theorem 2.22**

For \(N \neq u_2 + u_N\), \(g_j\) is equal to either \(g_1, g_N\) or \(g_1 + g_N\). Further, \(g_j = \begin{cases} g_1, & 0 \leq u_j < N - u_2, \\ g_1 + g_N, & N - u_2 \leq u_j < u_N, \\ g_N, & u_N \leq u_j < N. \end{cases}\) \((2.4.10)\)

**Proof**  From Theorem 2.16 and (2.1.1), \(g_j = d_{u_j, u_{j+1}} = \begin{cases} \{u_2 \alpha\}, & 0 \leq u_j < N - u_2, \\ \{(u_2 - u_N) \alpha\}, & N - u_2 \leq u_j < u_N, \\ 1 - \{u_N \alpha\}, & u_N \leq u_j < N. \end{cases}\)

Since both \(\{u_2 \alpha\}\) and \(1 - \{u_N \alpha\}\) are less than 1/2, it follows from (A.1.22c) that \(g_1 = \{u_2 \alpha\} = \|u_2 \alpha\|\) and \(g_N = 1 - \{u_N \alpha\} = \|u_N \alpha\|\). Also, \(\{(u_2 - u_N) \alpha\}\) \(\leq 1/2\) and thus \(\{(u_2 - u_N) \alpha\} = \|(u_2 - u_N) \alpha\|\), \(\|u_2 \alpha\| + \|u_N \alpha\|\), \(g_1 + g_N\), which follows from the fact that \(u_2 \alpha\) less its nearest integer is positive, while \(u_N \alpha\) less its nearest integer is negative. Thus, the largest gap is equal in length to the sum of the other two distinct gap lengths.  \[\]
Corollary 2.23

For $q_{n-1} < N \leq q_n$, $(n \geq 2)$,

$$\min (g_1, g_N) = \begin{cases} \|u_2 \alpha\| = g_1, & n \text{ odd}, \\ \|u_N \alpha\| = g_N, & n \text{ even}, \\ = \|q_{n-1} \alpha\|. \end{cases}$$

Proof This follows from Theorems 2.6 and 2.22.

2.5 Examples

(i) $\alpha = \tau = \frac{\sqrt{5} - 1}{2} = \{0; 1, 1, \ldots\}$.

In this case, $a_n = 1$, $n = 1, 2, \ldots$ and

$$\begin{align*}
p_n &= F_n = F_{n-1} + F_{n-2}, & n \geq 1, \\
q_n &= F_{n+1} = F_n + F_{n-1}, & n \geq 1,
\end{align*}$$

where $F_{-1} = 1$, $F_0 = 0$.

Sequence $U_N(\alpha)$: From Theorems 2.6 and 2.16, for $F_n < N \leq F_{n+1}$,

$$u_{j+1} - u_j = \begin{cases} F_n, & 0 \leq u_j \leq N - F_n, \\
F_{n-2}, & N - F_n < u_j < F_{n-1}, & n \text{ odd}, \\
-F_{n-1}, & F_{n-1} \leq u_j < N, \\
F_{n-1}, & 0 \leq u_j \leq N - F_{n-1}, \\
-F_{n-2}, & N - F_{n-1} < u_j < F_n, & n \text{ even}, \\
-F_n, & F_n \leq u_j < N, \end{cases}$$

Thus, for $\alpha = \tau$, points Pre(0) and Suc(0) always equal the denominators of total convergents.
From (2.3.6), for \( N = F_{n+1} \),

\[ u_j = (-1)^{n+1}(j - 1)F_n \mod F_{n+1}. \]

From (2.3.7), for \( N = F_{n+1} \),

\[ j = 1 + (mF_n) \mod F_{n+1}, \]

where \( m = u_j \).

Gap sizes: from Theorem 2.22 and (A.1.32b), for \( F_n < N \leq F_{n+1} \),

\[
g_j = \begin{cases} 
\tau^n, & 0 \leq u_j \leq N - F_n, \\
\tau^{n-2}, & N - F_n < u_j < F_{n-1}, n \text{ odd}, \\
\tau^{n-1}, & F_{n-1} \leq u_j < N, \\
\tau^{n-1}, & 0 \leq u_j \leq N - F_{n-1}, \\
\tau^{n-2}, & N - F_{n-1} < u_j < F_n, n \text{ even}, \\
\tau^n, & F_n \leq u_j < N, 
\end{cases}
\]

(ii) \( \alpha = \sqrt{2} = \{ 1; 2, 2, \ldots \} \).

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<th>( i )</th>
<th>( p_{n,i} )</th>
<th>( q_{n,i} )</th>
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</table>
A Two Gap Sequence: \( N = q_{4,1} = 17 \) \((n \) is even).

From (2.3.6),
\[
u_j = -12(j - 1) \mod 17, \quad j = 1, 2, \ldots, 17.
\]
Thus,
\[
U_{17}(\sqrt{2}) = (0, 5, 10, 15, 3, 8, 13, 1, 6, 11, 4, 9, 14, 2, 7, 12).
\]

The number of points that precede point \( m \) is equal in length to, from (2.3.7),
\[
j - 1 = 24m \mod 17. \quad \text{Thus for instance } 24 \mod 17 = 7 \text{ points precede point 1.}
\]

The difference between successive elements of \( U_{17}(\sqrt{2}) \) is either 5 or -12. Thus, the two gap sequence is made up of gaps equal to either \( g_1 = |12\sqrt{2} - 17| \) or \( g_{17} = |5\sqrt{2} - 7| \).

A Three Gap Sequence: \( N = 25 \).

Note that \( q_{4,1} < N \leq q_{4,2} = q_4 \).

We find that
\[
U_{25}(\sqrt{2}) = (0, 17, 5, 22, 10, 15, 3, 20, 8, 13, 1, 18, 6, 23, 11, 16, 4, 21, 9, 14, 2, 19, 17, 24, 12).
\]
Gaps are equal to either \( g_1 = |17\sqrt{2} - 24| \), \( g_{25} = |12\sqrt{2} - 17| \) or \( g_1 + g_{25} = |5\sqrt{2} - 7| \).

2.6 Discussion

Various proofs of the Steinhaus Conjecture have been offered in the literature (Świerczkowski [121], Sós [114], [115], Surányi [120], Halton [36], Slater [110], Knuth [67]), p. 511). Our purpose here is to survey the different approaches that have been adopted (using the notation we have used) and to compare them with our own.

Our approach was to first determine \( u_2 \) and \( u_N \). This was achieved via an investigation into Diophantine Approximation Theory. We show in Appendix A (Sections
A.2. and A.3) that Lemmas 2.1 and 2.3 are concerned with best and second best approximations of the second kind to $\alpha$. It is proved that best approximations are total convergents and second best partial convergents. This fact was used to determine $u_2$ and $u_N$ in terms involving the continued fraction expansion of $\alpha$. Then, largely through the application of the identities (2.3.2) and (2.3.3), the recurrence relation (Theorem 2.16) describing the sequence $U_N(\alpha)$ was found.

Most authors have chosen a different path. Świerczkowski [121] for instance determined the recurrence relation after showing that the first point to replace either Suc (0) or Pre (0) is the point Suc (0) + Pre (0). This result appears in our work as Corollary 2.7. Świerczkowski did not relate any of his results to continued fractions. As a corollary he proved a conjecture of J. Oderfeld, determining the gap sizes for the case $\alpha = (\sqrt{5} - 1)/2$ by using Corollary 2.7 to show inductively that Suc (0) and Pre (0) are consecutive Fibonacci numbers.

The proof offered by Sós ([114], [115]) is similar in approach to that of Świerczkowski. Sós [114] determined the recurrence relation, Theorem 2.16, by first proving Corollary 2.7. In [115] Sós identified the points Suc (0) and Pre (0) with the denominators of convergents to $\alpha$.

Sós [115] also showed that, if the terms in the continued fraction expansion of $\alpha$ are unbounded, then

$$
\liminf_{N \to \infty} Nh_N = 0, \quad \limsup_{N \to \infty} Nh_N = 1,
$$

$$
\liminf_{N \to \infty} NH_N = 1, \quad \limsup_{N \to \infty} NH_N = \infty,
$$

where $H_N$ and $h_N$ denote, respectively, the lengths of the largest and smallest gap belonging to a circle of $N + 1$ points. These results were also conjectured by H. Steinhaus and first proved by Hartman [39] using the theory of continued fractions. Sós asserts that her proof is simpler.
Surányi [120] enclosed $\alpha$ in an interval whose endpoints are consecutive members of the Farey sequence of certain order, such that $A/a < \alpha < B/b$, where $aB - bA = 1$ and $A/a, B/b \in H_N$. $H_N$ denotes the Farey sequence of order $N$ where $\max(a, b) \leq N \leq a + b - 1$. He then deduced that $a = u_2$ and $b = u_N$ and that $U_N(\alpha) = U_N(A/a)$. From this latter result Theorem 2.16 is then verified.

The proof offered by Halton [36] involves a description of the dynamics of the gap division with reference to quantities derived from the continued fraction expansion of $\alpha$. However he did not concern himself with ordering the points.

Slater [110] approached the problem by first deducing Theorem 2.16. From this relation it is easily shown that $u_N g_1 + u_2 g_N = 1$ which is our equation (2.3.9). To find $u_2$ and $u_N$ this equation is solved subject to the constraint $A/u_2 < \alpha < B/u_N$ where $A$ and $B$ are the nearest integers to $u_2 \alpha$ and $u_N \alpha$ respectively. Thus Slater was able to express $u_2$ and $u_N$ in terms of the continued fraction expansion of $\alpha$.

Slater also considered the problem of determining the relationship between the successive integer values of $j$ for which $\{j \alpha\} < \Phi$ where $0 < \Phi < 1$. He showed that the 'gaps' between the successive $j$ may take on at most three different values, one being the sum of the other two. This problem is originally discussed in Slater [110].

A more recent proof is outlined by Knuth [67], Theorem S, p. 511. The problem is approached by describing the change in structure induced by the addition of points to the circle. For example, it is noted that each additional point falls in one of the largest existing gaps.
CHAPTER THREE

Gap Sequences

3.1 Introduction

In the preceding chapter we showed that the sequence \( \{q\alpha\} = q\alpha \mod 1, \ q = 0, 1, 2, \ldots, N - 1 \) when represented on the circle of unit circumference, so that \( \{q\alpha\} \) designates the point lying a clockwise circumferential distance of \( \{q\alpha\} \) from the origin, partitions the circumference into arcs or gaps of three, and sometimes two, different lengths for any irrational \( \alpha \).

We proved that if \( N = q_{n,i}, \ i = 1, 2, \ldots, a_n, \ (n \geq 2) \), then the circle is composed of gaps of only two different lengths. In this chapter we will primarily be concerned with describing the change in gap structure induced by the transition from a circle of \( q_{n-1} \) or \( q_{n,i} \) gaps to one of \( q_{n,1} \) or \( q_{n,i+1} \) gaps respectively (\( i = 1, 2, \ldots, a_n - 1 \)).

In the next chapter, we apply some of the results found here to the case \( \alpha = \{ 0; a, a, \ldots \} \). Recursive relationships describing this number's gap structure are found and related to its "characteristic."

All results in this chapter apply to any irrational \( \alpha \).

3.2 The string of gap types

Suppose that the circle is partitioned into gaps of only two different lengths which we describe as large and small. We label a large gap \( l \) and call a small gap \( s \). Let

\[
\Phi_{n,i} = \phi_{n,i}^1 \phi_{n,i}^2 \ldots \phi_{n,i}^{q_n,i},
\]  

(3.2.1)
\( \Phi_{n,a_n} = \Phi_n \) \hfill (3.2.2)

where \( i = 1, 2, \ldots, a_n \), denote the string of gap types for \( N = q_{n,i}, i = 1, 2, \ldots, a_n, (n \geq 1) \), ordered clockwise around the circle so that \( \phi_j \) denotes the gap type (either \( s \) or \( l \)) formed by the points \( u_j(q_{n,i}) \) and \( u_{j+1}(q_{n,i}) \). Assume that \( \Phi_0 = s \).

Let \( P_{n,i+1} \) be the function which transforms \( \Phi_{n,i} \) into \( \Phi_{n,i+1} \) such that

\[
\Phi_{n,1} = P_{n,1}(\Phi_{n-1}), \quad \Phi_{n,i+1} = P_{n,i+1}(\Phi_{n,i}), \quad i = 1, 2, \ldots, a_n - 1. \hfill (3.2.3) \]

(3.2.4)

Also assume that \( P_{n,i} \) is a homomorphism. That is, \( P_{n,i}(sl) = P_{n,i}(s)P_{n,i}(l) \).

Let \( P_n \) be the function defined over \( \Phi_{n-1} \) which gives the string \( \Phi_n \). \( P_n \) is then defined such that

\[
\Phi_n = P_n(\Phi_{n-1}) = P_{n,a_n}(P_{n,a_n-1}(\ldots P_{n,2}(P_{n,1}(\Phi_{n-1}))\ldots))). \hfill (3.2.5) \]

The following theorem describes the manner by which the string of gap types develops as more points are included on the circle. In particular, it is shown that each point divides only those gaps which are large.

**Theorem 3.1**

\[ P_{n,i}(l) = \begin{cases} sl, & n \text{ odd}, \\ ls, & n \text{ even}, \end{cases} \]

where \( i = 1, 2, \ldots, a_n \), \( P_{n,1}(s) = l \), \( P_{n,i}(s) = s, \quad i = 2, 3, \ldots, a_n \).

**Proof** As \( N \) increases from \( q_{n-1} \) to \( q_{n,1} \) each successive point divides exactly one of the large gaps present when \( N = q_{n-1} \). Each of the large gaps present when \( N = q_{n,i} \) \((i = 1, 2, \ldots, a_n - 1)\) is divided as \( N \) is increased to \( q_{n,i+1} \). To see this, first note from Theorem 2.16 and Corollary 2.17 that

\[
N - 1 - \text{Pre}
\]

(3.2.6)
This implies that each additional point \( N - 1 \) enters a large gap of length \( \{(u_2 - u_N)\alpha\} = \|u_2\alpha\| + \|u_N\alpha\| \), dividing it into two gaps of length, in clockwise order, \( \|u_2\alpha\| \) and \( \|u_N\alpha\| \).

The large gap labelled \( l \) now changes, from Corollary 2.23, to \( sI \) if \( n \) is odd and \( l s \) if \( n \) is even.

To see that each of the large gaps (and only these gaps) are divided as \( N \) ranges from \( q_{n-1} \) to \( q_{n,1} \), we note from (3.2.6) and (3.2.7) that \( 0 \leq \text{Suc}(N-1), \text{Pre}(N-1) < q_{n-1} \), which means that no two points enter the same gap present when \( N = q_{n-1} \). Also, the number of points added is equal to the number of large gaps present when \( N = q_{n-1} \). The same type of argument applies for \( q_{n,i} < N \leq q_{n,i+1} \) (\( i = 1, 2, \ldots a_n - 1 \)).

The theorem now follows by noting that each small gap present when \( N = q_{n-1} \) (length \( \|q_{n-2}\alpha\| \)) is labelled as large when \( N = q_{n,1} \) while each small gap remains undivided for \( q_{n,1} < N \leq q_n \). This may be deduced from Corollary 2.23 with reference to Theorem 2.6. |

**Corollary 3.2**

\[
P_n(l) = \begin{cases} s^{a_n} l, & n \text{ odd,} \\ l s^{a_n}, & n \text{ even,} \end{cases}
\]

\[
P_n(s) = \begin{cases} s^{a_n-1} l, & n \text{ odd,} \\ l s^{a_n-1}, & n \text{ even.} \end{cases}
\]

**Proof** The result follows directly from (3.2.5) and Theorem 3.1. |

In the proof of Theorem 3.1, it was stated that each point divides a large gap of length \( g_1 + g_N \), forming two new gaps, one equal in length to the smallest gap present.
Thus it is natural to define the ratio of gap division as

$$r_N(\alpha) = \frac{\min(g_1, g_N)}{g_1 + g_N}. \quad (3.2.9)$$

**Proposition 3.3**

$$r_N(\alpha) = \begin{cases} \frac{1}{1+i_n}, & q_{n-1} < N \leq q_{n,1}, \\ \frac{1}{2-i+n}, & q_{n,i-1} < N \leq q_{n,i}, \end{cases}$$

where $i = 2, 3, \ldots, a_n, (n \geq 2)$.

**Proof** From Corollary 2.23,

$$r_N(\alpha) = \begin{cases} \frac{\|q_{n-1}\alpha\|}{\|q_{n-1}\alpha\|+\|q_{n-2}\alpha\|}, & q_{n-1} < N \leq q_{n,1}, \\ \frac{\|q_{n-1}\alpha\|}{\|q_{n-1}\alpha\|+\|q_{n,i-1}\alpha\|}, & q_{n,i-1} < N \leq q_{n,i}, \end{cases} \quad (3.2.10)$$

where $i = 2, 3, \ldots, a_n, (n \geq 2)$. The proposition now follows from (A.1.21) and (A.1.27).

The following theorem shows that a component of $\Phi_{n,i}$ is symmetric.

**Theorem 3.4**

Let $A_{n,i} = \phi_n^{1}, B_{n,i} = \phi_n^{2} \phi_n^{3} \ldots \phi_n^{q_{n,i}-1}, C_{n,i} = \phi_n^{q_{n,i}}$. Then,

$$B_{n,i}^* = B_{n,i},$$

where $B_{n,i}^*$ is the string $B_{n,i}$ in reverse order.

**Proof** It is required to show that

$$\phi_{n,i}^j = \phi_{n,i}^{q_{n,i}-j+1}, \quad j = 2, 3, \ldots, q_{n,i} - 1.$$  

From Corollary 2.13,

$$u_j = (-1)^{n-1}(j - 1)q_{n-1} \mod q_{n,i}, \quad j = 1, 2, \ldots, q_{n,i}, \quad i = 1, 2, \ldots, a_n.$$
Thus,

\[ u_{q_n,i-j+2} = ((-1)^{n-1}(q_{n,i} - (j - 1))q_{n-1}) \mod q_{n,i}, \]

Hence,

\[ u_j + u_{q_n,i-j+2} = q_{n,i}. \]

Thus,

\[ u_{j+1} - u_j = q_{n,i} - u_{q_n,i-j+1} - (q_{n,i} - u_{q_n,i-j+2}), \]

Hence, \( g_j = g_{q_n,i-j+1} \) for \( j = 2, 3, \ldots, q_n,i - 1 \).

Observation: Corollary 3.2 may be interpreted as follows. Each large gap present at \( q_{n-1} \) points \((n \geq 2)\) is partitioned into \( a_n \) small gaps of length \( \|q_{n-1}\alpha\| \) and a new large gap of length \( \|q_{n-1}\alpha\| + \|q_n\alpha\| \) as we go to \( q_n - 1 \) points. If we pretend that each of the \( \text{large gaps are circles of unit circumference then (after point } q_{n-1} + q_{n-2} \text{ is added) they appear as if they are being divided by an angle of } 1/t_n \) for odd \( n \) and \( 1 - 1/t_n \) for even \( n \).

3.3 A related string

In this section we apply the same sort of analysis to a string closely related to \( \Phi_{n,i} \) through the operator \( R \) defined below. We find an application for these results in the next chapter.

Define \( R(\Theta) \) such that

\[ R(\Theta) = R(\theta_1 \theta_2 \ldots \theta_k) = \theta_2 \theta_3 \ldots \theta_{k-1} \theta_1 \theta_k, \]

where \( \Theta = \theta_1 \theta_2 \ldots \theta_k \) denotes a string of \( k \) letters where \( \theta_i = s \) or \( l \). Then let

\[ \Omega_{n,i} = R(\Phi_{n,i}), \]

\[ = \phi_{n,i}^{q_{n,i}-1} \phi_{n,i} 1 \phi_{n,i}^{q_{n,i}}, \]

\[ = \omega_{n,i}^{q_{n,i}-2} \omega_{n,i} q_{n,i}, \]

(3.3.1)
where $\Omega_{n,a_n} = \Omega_n$.

Here we determine the rules relating $\Omega_{n,1}$ to $\Omega_{n-1}$ and $\Omega_{n,i+1}$ to $\Omega_{n,i}$ ($i = 1, 2, \ldots, a_n - 1$). Thus let

$$
\begin{align*}
\Omega_{n,1} &= Q_{n,1}(\Omega_{n-1}), \\
\Omega_{n,i+1} &= Q_{n,i+1}(\Omega_{n,i}), \quad i = 1, 2, \ldots, a_n - 1.
\end{align*}
$$

We assume that $Q_{n,i}$ ($i = 1, 2, \ldots, a_n$) are homomorphic.

**Theorem 3.5**

$$
Q_{n,1}(s) = l, \quad Q_{n,1}(l) = l s,
$$
$$
Q_{n,i}(s) = s, \quad Q_{n,i}(l) = s l, \quad i = 2, 3, \ldots, a_n.
$$

**Proof** We are required to show that the given functions $Q_{n,i}$ ($i = 1, 2, \ldots, a_n$) are such that

$$
Q_{n,1}(R(\Phi_{n-1})) = R(P_{n,1}(\Phi_{n-1})), \quad (3.3.2)
$$
$$
Q_{n,i+1}(R(\Phi_{n,i})) = R(P_{n,i+1}(\Phi_{n,i})), \quad i = 2, 3, \ldots, a_n. \quad (3.3.3)
$$

We prove (3.3.2) only since the proof is isomorphic to that of (3.3.3). For notational convenience let $\Phi_{n-1} = \lambda_1 \lambda_2 \ldots \lambda_{q_{n-1}}$.

A simple consequence of Corollary 3.2 is that

$$
\lambda_1 = \begin{cases} 
\mathcal{G}, & \text{n odd}, \\
\mathcal{S}, & \text{n even}. 
\end{cases} \quad (3.3.4)
$$

Consider the two cases:

(i) $n$ odd

In this case it follows that

$$
Q_{n,1}(\Theta) = l P_{n,1}(\Theta) l^{-1}, \quad (3.3.5)
$$
since \( Q_{n,1}(s) = P_{n,1}(s) \) and \( Q_{n,1}(l) = l P_{n,1}(l) l^{-1} \), where \( l l^{-1} \) is the empty string and \( s l^{-1} \) is undefined. \( Q_{n,1}(\emptyset) \) is then the string \( P_{n,1}(\emptyset) \) with its last element, \( l \), now the leading element.

Via (3.3.4) and (3.3.5),

\[
Q_{n,1}(R(\Phi_{n-1})) = Q_{n,1}(\lambda_2 \lambda_3 \ldots \lambda_{q_n-1} \lambda_1 \lambda_{q_n-1}),
\]

\[
= l P_{n,1}(\lambda_2 \lambda_3 \ldots \lambda_{q_n-1} \lambda_1 \lambda_{q_n-1}) l^{-1},
\]

\[
= l P_{n,1}(\lambda_2 \lambda_3 \ldots \lambda_{q_n-1}) s l,
\]

\[
= R(s l P_{n,1}(\lambda_2 \lambda_3 \ldots \lambda_{q_n-1}) l),
\]

\[
= R(P_{n,1}(l \lambda_2 \lambda_3 \ldots \lambda_{q_n-1} s)),
\]

\[
= R(P_{n,1}(\lambda_1 \lambda_2 \ldots \lambda_{q_n-1})),
\]

\[
= R(P_{n,1}(\Phi_{n-1})).
\]

(ii) \( n \) even

In this case

\[
P_{n,1}(\emptyset) = Q_{n,1}(\emptyset).
\]  \hfill (3.3.6)

From (3.3.4) and (3.3.6),

\[
Q_{n,1}(R(\Phi_{n-1})) = P_{n,1}(\lambda_2 \lambda_3 \ldots \lambda_{q_n-1} \lambda_1 \lambda_{q_n-1}),
\]

\[
= P_{n,1}(\lambda_2 \lambda_3 \ldots \lambda_{q_n-1}) l l s,
\]

\[
= R(l P_{n,1}(\lambda_2 \lambda_3 \ldots \lambda_{q_n-1})),
\]

\[
= R(P_{n,1}(\lambda_1 \lambda_2 \ldots \lambda_{q_n-1})),
\]

\[
= R(P_{n,1}(\Phi_{n-1})).
\]

Let

\[
\Omega_n = Q_n(\Omega_{n-1}) = Q_{n,a_n}(Q_{n,a_n-1}(\ldots Q_{n,2}(Q_{n,1}(\Omega_{n-1}))(\ldots)).
\]  \hfill (3.3.7)
Corollary 3.6

\[ Q_n(t) = s^{a_n - 1} t \],

\[ Q_n(s) = s^{a_n - 1} l. \]

**Proof**  The result follows immediately from Theorem 3.5 and (3.3.7).

3.4 Age structure

Corollary 3.7

Each point divides the oldest of the large gaps which has attained the age \( \min (u_2, u_N) - 1. \)

**Proof**  It has already been shown (Theorem 3.1) that each additional point \( N - 1 \) divides a large gap. It remains to be shown that this gap is oldest.

From (3.2.6) and (3.2.7) the maximum age that each gap attains before it is divided is equal to

\[ N - 2 - \max (N - 1 - u_2, N - 1 - u_N) = N - 2 - (N - 1 + \max (-u_2, -u_N)), \]

\[ = \min (u_2, u_N) - 1. \]

If this is not the oldest of the large gaps then \( d_{a,b} \), say, is older. Since \( d_{a,b} \) is a large gap it follows that \( b = a + u_2 - u_N \). Without loss of generality assume that \( u_2 > u_N \). Then \( d_{a,b} \) is of age

\[ N - 2 - \max (a, b) = N - 2 - b, \]

\[ = N - 2 - a - u_2 + u_N. \]

It then follows that \( N - 2 - a - u_2 + u_N > u_N - 1 \) or

\[ N - 1 - a > u_2. \] (3.4.1)

From Theorem 2.16, if \( b - a = u_2 - u_N \), then \( N - u_2 \leq a < u_N \). Thus,

\[ N - 1 - u_N < N - 1 - a \leq u_2 - 1, \]

which contradicts (3.4.1) and hence completes the proof.
CHAPTER FOUR

Gap Sequences for $\alpha = \{0; a, a, \ldots\}$

4.1 Introduction

In this chapter we apply some of the results developed in Chapter 3 to the number $\alpha = \{0; a, a, \ldots\}$. It is found that the sequence of strings $\Phi_0, \Phi_1, \Phi_2, \ldots$ may be generated recursively. The strings $\Omega_0, \Omega_1, \Omega_2, \ldots$ are also found to obey a recursive relationship and their relationship to the characteristic of $\alpha$ is discussed. In Section 4.2, the characteristic of $\alpha$ is defined and the connection between it and the distribution of gaps is demonstrated in Section 4.4 after recurrence relations for $\Phi_n$ and $\Omega_n$ are presented in Section 4.3. In Section 4.5 the recursive relationship for $\Omega_n$ is displayed graphically and the resulting diagram for $\alpha = \{0; 1, 1, \ldots\} = (\sqrt{5} - 1)/2$ is analysed yielding a solution to the inductive equation

$$G(n) = n - G(G(n - 1)), \quad n \geq 1 \text{ where } G(0) = 0.$$  

4.2 The characteristic of $\alpha$

To determine the sequence $[m\alpha], \ m = 0, 1, 2, \ldots$, for irrational $\alpha$ Johann Bernoulli [6] considered its sequence of differences $d_1, d_2, d_3, \ldots$ where

$$d_m = [(m + 1)\alpha] - [m\alpha], \quad m = 1, 2, 3, \ldots \quad (4.2.1)$$

It may be readily shown that $d_m$ may only equal $[\alpha]$ or $[\alpha] + 1$. If we replace $[\alpha]$ by $s$ (for small) and $[\alpha] + 1$ by $l$ (for large), then we obtain a string of such characters. This we will refer to as the characteristic of $\alpha$.

String operations may be used to generate the characteristic from its first few terms. Bernoulli was the first to discover the rules which were the basis of these string operations.
These were reformulated in a more attractive form by Christoffel [11]. However, it had to wait until Markoff [81] before the first proofs were offered. The following method of constructing the characteristic is described in Venkov [137], p. 65-68.

Let $s_1^a$ denote the a-fold concatenation of the string $s_1$ with itself. Thus for example $(l s)^2 = l s l s$. $s_1^0$ is the empty string. Markoff showed that the characteristic of $\alpha$ is equal to $\beta_1 \beta_2 \beta_3 \ldots$ where

$$\beta_n = \beta_{n-1}a_n^{-1} \beta_{n-2} \beta_{n-1}, \quad \beta_0 = s, \quad \beta_1 = s a_1^{-1} l. \quad (4.2.2)$$

We mention that if $\alpha$ is rational, say $\alpha = \{ a_0; a_1, a_2, \ldots, a_N \}$, then $\beta_1 \beta_2 \ldots \beta_{N-1} (\beta_N)^{\infty}$ is the characteristic where $N$ is even (so that the number of terms is odd). If $N$ is odd, the number of terms can be made odd as $a_N$ can be replaced by $a_N - 1$, 1 if $a_N > 1$. If $a_N = 1$ (and $\alpha \neq 1$) then $a_{N-1}$, $a_N$ can be replaced by $a_{N-1} + 1$.

Example Let $\alpha = \{0; 1, 2, 3\} = \{0; 1, 2, 2, 1\} = 7/10$. Then,

$$\beta_0 = s,$$
$$\beta_1 = l,$$
$$\beta_2 = \beta_1 \beta_0 \beta_1 = l s l,$$
$$\beta_3 = \beta_2 \beta_1 \beta_2 = l s l l l s l,$$
$$\beta_4 = \beta_2 \beta_3 = l s l l l s l.$$

The characteristic is then given by $\beta_1 \beta_2 \beta_3 (\beta_4)^{\infty}$ where $s$ represents zero and $l$ unity.

Fraenkel et. al [28] offer an alternative method of construction: they show that the characteristic is equal to $\lim_{n \to \infty} \delta_n$ where

$$\delta_n = \delta_{n-1} a_n \delta_{n-2}, \quad \delta_0 = s, \quad \delta_1 = s a_1^{-1} l. \quad (4.2.3)$$

They actually form the characteristic by means of ‘shift operators’. It may be shown however that the above recurrence relation is an equivalent formulation.

Note that if $\alpha = \{ a_0; a_1, a_2, \ldots, a_N \}$ then $\delta_N^{\infty}$ is the characteristic (if $N$ is even).
Example Let \( \alpha = \{0; 1, 2, 3\} = \{0; 1, 2, 2, 1\} = \frac{7}{10} \). Then,
\[
\begin{align*}
\delta_0 &= s, \\
\delta_1 &= l, \\
\delta_2 &= ll s, \\
\delta_3 &= l l l s l s, \\
\delta_4 &= l l l s l l l s.
\end{align*}
\]
Thus, \( \delta_4^{\infty} \) is the characteristic.

This method generalises the work done by Stolarsky [119] who shows how to generate the characteristic for the particular case where \( \alpha = 1 + \{0; a, a, \ldots\} \), the positive root of \( \alpha^2 + (a-2)\alpha - a = 0 \).

In this chapter, we present a new proof of [28] and [119] for the case \( \alpha = \{0; a, a, \ldots\} \), the positive root of \( \alpha^2 + a\alpha - 1 = 0 \). The proof demonstrates an interesting connection between the distribution of the sequence \( n\alpha \mod 1 \) (for \( n = 0, 1, 2, \ldots \)) and the characteristic of \( \alpha \).

4.3 The case \( \alpha = \{0; a, a, \ldots\} \)

For the remainder of this chapter assume that \( \alpha = \{0; a, a, \ldots\} \). For this case we show how the strings \( \Phi_n \) and \( \Omega_n \) are generated recursively and how \( \Omega_n \) is related to the characteristic of \( \alpha \).

The following propositions relate to the transition rules formulated in Theorem 3.1 for this special case. As before, \( \Theta = \theta_1 \theta_2 \ldots \theta_k \) denotes a string of \( k \) letters, where \( \theta_i = s \) or \( l \).

**Proposition 4.1**

\[
P_n(\Theta) = \begin{cases} 
 l^{-1} P_{n-1}(\Theta) l, & \text{if } n \text{ odd}, \\
 l P_{n-1}(\Theta) l^{-1}, & \text{if } n \text{ even},
\end{cases}
\]

where \( s_1^{-1} s_1 = s_1 s_1^{-1} \) is the empty string (see the proof of Theorem 3.5 for the definition of \( s_1^{-1} \)).
The result follows since

\[
\begin{align*}
P_n(l) &= \begin{cases} 
  l \cdot P_{n-1}(l), & n \text{ odd}, \\
  l \cdot P_{n-1}(l)^{-1}, & n \text{ even}, 
\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
P_n(s) &= \begin{cases} 
  l \cdot P_{n-1}(s), & n \text{ odd}, \\
  l \cdot P_{n-1}(s)^{-1}, & n \text{ even}. 
\end{cases} \\
\end{align*}
\]

We introduce the following proposition prior to establishing the recurrence relation for \( \Phi_n \).

**Proposition 4.2**

\[ P_n(\Theta)^* = P_{n-1}(\Theta^*), \]

where \( \Theta^* \) is the string \( \Theta \) in reverse order.

**Proof**

\[
\begin{align*}
P_n(\Theta)^* &= (P_n(\theta_1) P_n(\theta_2) \ldots P_n(\theta_k))^*, \\
&= P_n(\theta_k)^* P_n(\theta_{k-1})^* \ldots P_n(\theta_1)^*, \\
&= P_{n-1}(\theta_k) P_{n-1}(\theta_{k-1}) \ldots P_{n-1}(\theta_1), \\
&= P_{n-1}(\theta_k \theta_{k-1} \ldots \theta_1), \\
&= P_{n-1}(\Theta^*), 
\end{align*}
\]

where we have used the fact that \( P_n(s)^* = P_{n-1}(s) \) and \( P_n(l)^* = P_{n-1}(l) \).

**4.4 Recurrence relation for \( \Phi_n \)**

**Theorem 4.3**

\[ \Phi_n = \Phi_{n-1}^a \Phi_{n-2}, \quad \Phi_0 = s, \quad \Phi_1 = s^{a-1} l. \]
Proof The result is true for \( n = 2, 3 \). Using the induction hypothesis we show that the result holds in general by verifying it for \( n = k + 1 \), assuming that it holds for \( n = k \) (and \( n = k - 1 \)):

\[
\Phi_{k+1} = P_{k+1}(\Phi_k),
\]

\[
= P_{k+1}(\Phi_{k-1}^a \Phi_{k-1}),
\]

\[
= P_{k+1}(\Phi_{k-1})^a P_{k+1}(\Phi_{k-2}),
\]

\[
= P_{k-1}(\Phi_{k-1})^a P_{k-1}(\Phi_{k-2}).
\]

Thus,

\[
\Phi_{k+1} = \Phi_k^a \Phi_{k-1},
\]

which follows from Proposition 4.2 and the fact that \( P_{k+1} = P_{k-1} \).

4.5 Recurrence relation for \( \Omega_n \)

The transition rule \( Q_n \) is somewhat simpler than its counterpart \( P_n \) as the comparison between Corollaries 3.2 and 3.6 demonstrates. The recurrence relation for \( \Omega_n \) is correspondingly as simple.

Theorem 4.4

\[
\Omega_n = \Omega_{n-1}^a \Omega_{n-2}, \quad \Omega_0 = s, \quad \Omega_1 = s^{a-1} l.
\]

Proof A simple induction suffices. From Corollary 3.6,

\[
Q_n(l) = Q(l) = s^{a-1} l s,
\]

\[
Q_n(s) = Q(s) = s^{a-1} l.
\]

The result holds for \( n = 2, 3 \). Assuming that the result holds for \( n = k \) (and \( n = k - 1 \)),

\[
\Omega_{k+1} = Q_{k+1}(\Omega_k),
\]

\[
= Q(\Omega_k),
\]

\[
= Q(\Omega_{k-1})^a Q(\Omega_{k-2}),
\]

\[
= Q(\Omega_{k-1})^a \Omega_{k-2},
\]

\[
= \Omega_k^a \Omega_{k-1},
\]

and thus the result holds for \( n = k + 1 \).
4.6 The connection between $\Omega_n$ and the characteristic of $\alpha$

Comparing Theorem 4.4 with Fraenkel et. al's result (that is, equation (4.2.3)) demonstrates that $\Omega_n$ is simply the first $q_n$ elements of the characteristic. We proceed to prove that $\Omega_n$ does indeed form the characteristic for the special case $\alpha = \{0; a, a, \ldots\}$ by exploiting the relationship between $\Phi_n$ and $\Omega_n$. The relationship seems a curious one, made more so by the fact that it is not true for all $\alpha$. Our proof sheds light on the set of numbers for which the relationship does hold. We begin by proving the following.

Lemma 4.5

$$[k \alpha] = \left[ k \frac{p_{n,i}}{q_{n,i}} \right], \quad k = 1, 2, \ldots, q_{n,i} - 1, \; (n \geq 2),$$

where $\alpha$ is any irrational number.

**Proof** From equation (2.3.6) there exists a non-negative integer $m$ so that

$$u_j = (-1)^n m q_{n,i} - (-1)^n (j - 1) q_{n-1}, \quad (4.6.1)$$

where $j = 1, 2, \ldots, q_{n,i}$ or $u_j = 0, 1, 2, \ldots, q_{n,i} - 1$. Solving this linear Diophantine equation by use of (A.1.15) yields

$$j - 1 = q_{n,i} \left\{ u_j \frac{p_{n,i}}{q_{n,i}} \right\}, \quad (4.6.2)$$

$$m = u_j p_{n-1} - \left[ u_j \frac{p_{n,i}}{q_{n,i}} \right] q_{n-1}. \quad (4.6.3)$$

From (4.6.1) it may be shown that

$$\{u_j \alpha\} = (j - 1)\|q_{n-1} \alpha\| + m\|q_{n,i} \alpha\|. \quad (4.6.4)$$

Substituting expressions for $j$ and $m$ from (4.6.2), (4.6.3) and using (A.1.29) and (A.1.30) yields

$$\{u_j \alpha\} = u_j \alpha - \left[ u_j \frac{p_{n,i}}{q_{n,i}} \right].$$

That is,

$$\left[ u_j \alpha \right] = \left[ u_j \frac{p_{n,i}}{q_{n,i}} \right].$$

where $u_j = 0, 1, 2, \ldots, q_{n,i} - 1$. Thus, the lemma is proven. \( \blacksquare \)
We now prove the connection between $\Omega_n$ and the characteristic of $\alpha$. For notational convenience, let

$$\Phi_n = \phi_{n,1} \phi_{n,2} \ldots \phi_{n,qn}, \quad (4.6.5)$$

$$\Omega_n = \omega_{n,1} \omega_{n,2} \ldots \omega_{n,qn}. \quad (4.6.6)$$

Theorem 4.6

For $\alpha = \{0; a, a, \ldots\}$, 

$$\omega_{n,j} = \begin{cases} s, & d_j = 0, \\ l, & d_j = 1. \end{cases}$$

Proof. From the definition of $\Omega_n$ (equation (3.3.1), the theorem is equivalent to the statement:

$$\phi_{n,1} = \begin{cases} s, & d_{qn-1} = 0, \\ l, & d_{qn-1} = 1, \end{cases}$$

$$\phi_{n,j} = \begin{cases} s, & d_{j-1} = 0, \\ l, & d_{j-1} = 1, \end{cases}$$

$$\phi_{n,qn} = \begin{cases} s, & d_{qn} = 0, \\ l, & d_{qn} = 1, \end{cases}$$

where $j = 2, 3, \ldots, q_n - 1$. We proceed to prove this.

Equation (2.3.6) with $i = a_n$ is equivalent to

$$u_j = q_n \left\{ (-1)^{n-1}(j-1) \frac{q_{n-1}}{q_n} \right\},$$

$$= (-1)^{n-1}(j-1)q_{n-1} - q_n \left[ (-1)^{n-1}(j-1) \frac{q_{n-1}}{q_n} \right].$$
since \( y \mod x = x\{y/x\} = x(y/x - \lfloor y/x \rfloor) \) for real \( x, y \). Thus, for \( j = 2, 3, \ldots, q_n - 1 \),

\[
u_{j+1} - u_j = (-1)^{n-1} q_{n-1} - (-1)^{n-1} q_n \left( \left\lfloor \frac{q_{n-1}}{q_n} \right\rfloor - \left\lfloor j - 1 \frac{q_{n-1}}{q_n} \right\rfloor \right). \tag{4.6.7}
\]

From Lemma 4.5 and equation (A.1.31a),

\[
u_{j+1} - u_j = (-1)^{n-1} q_{n-1} - (-1)^{n-1} q_n ([j \alpha] - [(j - 1)\alpha]). \tag{4.6.8}
\]

Hence, for \( j = 2, 3, \ldots, q_n - 1 \),

\[
u_{j+1} - u_j = \begin{cases} (-1)^{n-1} q_{n-1}, & d_{j-1} = 0, \\ (-1)^{n-1}(q_{n-1} - q_n), & d_{j-1} = 1. \end{cases}
\]

Therefore,

\[
g_j = \begin{cases} (-1)^{n-1} q_{n-1} \alpha, & d_{j-1} = 0, \\ (-1)^{n-1}(q_n - q_{n-1}) \alpha, & d_{j-1} = 1. \end{cases}
\]

That is,

\[
g_j = \begin{cases} \|q_{n-1} \alpha\|, & d_{j-1} = 0, \\ \|q_{n-1} \alpha\| + \|q_n \alpha\|, & d_{j-1} = 1. \end{cases}
\]

Hence,

\[
\phi_{n,j} = \begin{cases} s, & d_{j-1} = 0, \\ l, & d_{j-1} = 1, \end{cases}
\]

where \( j = 2, 3, \ldots, q_n - 1 \).

To complete the proof, first note from (A.1.16) that

\[
[q_n \alpha] = \begin{cases} p_n, & n \text{ even}, \\ p_n - 1, & n \text{ odd}. \end{cases}
\]

From Lemma 4.5, \([(q_n - 1)\alpha] = [(q_n - 1)p_n/q_n] = p_n - 1 \). Therefore,

\[
d_{q_n-1} = [q_n \alpha] - [(q_n - 1)\alpha],
\]

\[
= \begin{cases} 0, & n \text{ odd}, \\ 1, & n \text{ even}. \end{cases}
\]

From Theorem 2.6,

\[
g_1 = \begin{cases} \|q_{n-1} \alpha\|, & n \text{ odd}, \\ \|q_{n-1} \alpha\| + \|q_n \alpha\|, & n \text{ even}. \end{cases}
\]
which verifies the result concerning \( \phi_n,1 \).

The result for \( \phi_n,q_n \) follows similarly. From Lemma C1 (Appendix C), \( \left[(q_n + 1)\alpha\right] = p_n \). Hence,

\[
\delta_{q_n} = \left[(q_n + 1)\alpha\right] - [q_n\alpha],
\]

\[
= \begin{cases} 
0, & n \text{ even}, \\
1, & n \text{ odd}, 
\end{cases}
\]

and, from Theorem 2.6,

\[
\gamma_{q_n} = \begin{cases} 
\|q_{n-1}\alpha\|, & n \text{ even}, \\
\|q_{n-1}\alpha\| + |q_n\alpha|, & n \text{ odd}, 
\end{cases}
\]

This completes the proof. \( \Box \)

Note that the step from (4.6.7) to (4.6.8) cannot be made for all \( \alpha \), thus proving that the correspondence between the string of gap types and the characteristic is not true for all \( \alpha \).

**Corollary 4.7**

Suppose that \( \alpha = \{0; a_1, a_2, \ldots\} \) where \( a_j = a_{n-j+1} \) for \( j = 1, 2, \ldots, n \). Then,

\[
\omega_{n,j} = \begin{cases} 
s, & d_j = 0, \\
l, & d_j = 1. 
\end{cases}
\]

**Proof** For this value of \( \alpha \), \( \frac{p_n}{q_n} = \frac{q_{n-1}}{q_n} = \{0; a_n, a_{n-1}, \ldots, a_2, a_1\} \) and thus the step from (4.6.7) to (4.6.8) can be made. \( \Box \)

**4.7 Graphical description**

In this section we develop a graphical representation of the sequence of strings

\[
\Omega_{1,1}, \Omega_{1,2}, \ldots, \Omega_{1,a} = \Omega_1, \Omega_{2,1}, \Omega_{2,2}, \ldots, \Omega_{2,a} = \Omega_2, \Omega_{3,1}, \ldots
\]

(4.7.1)

where, for the purpose of illustration, we have returned to the special case \( \alpha = \{0; a, a, \ldots\} \).

Then, \( Q_n \) and \( Q_{n,i} \) are independent of \( n \) and thus we write \( Q \) for \( Q_n \) and \( Q_i \) for \( Q_{n,i} \).
The graphs obtained illustrate the recursive nature of the sequence (4.7.1), also indicated by Theorem 4.4 and the fact that $Q_n$ and $Q_{n,i}$ are independent of $n$.

Let $D$ be the ordered pair $(V, A)$ where $V$ denotes the union of sets,

$$\left\{ \omega_{n,i}^1, \omega_{n,i}^2, \ldots, \omega_{n,i}^{q_{n,i}} \right\},$$

for $n = 1, 2, \ldots, i = 1, 2, \ldots, a$.

$A$ is the set of pairs of elements of $V$ such that $\left\{ \omega_{n,i}^j, Q_{i+1}(\omega_{n,i}^j) \right\}$ belongs to $A$ where it is understood that if $Q_{i+1}(\omega_{n,i}^j)$ is a string of two letters, say $\theta_1 \theta_2$, then $\left\{ \omega_{n,i}^j, \theta_1 \right\}, \left\{ \omega_{n,i}^j, \theta_2 \right\}$ belong to $A$.

We represent $D$ by a diagram (or tree) where the elements of $V$ are represented by points or nodes such that if $\theta_1$ and $\theta_2$ are nodes, they are linked by a line or edge if $\left\{ \theta_1, \theta_2 \right\}$ is in $A$. Assuming that no edges cross, we suppose that the nodes belonging to level $n a + i$ when read from left to right correspond to the string $\Omega_{n,i}$. Also included in $D$ is the pair $\left\{ \Omega_0, \Omega_{1,1} \right\}$ but not the element $\Omega_0$.

Example \[ \alpha = \{ 0; 2, 2, \ldots \} = \sqrt{2} - 1. \]

For this case,

$$\Omega_{n,1} = Q_1(\Omega_{n-1}),$$

$$\Omega_{n,2} = \Omega_n = Q_2(\Omega_{n,1}),$$

where

$$Q_1(s) = l, \quad Q_1(l) = ls,$$

$$Q_2(s) = s, \quad Q_2(l) = sl.$$

Then,

$$\Omega_0 = s,$$

$$\Omega_{1,1} = Q_1(s) = l,$$

$$\Omega_{1,2} = \Omega_1 = Q_2(l) = sl,$$

$$\Omega_{2,1} = Q_1(sl) = lls,$$

$$\Omega_{2,2} = \Omega_2 = Q_2(lls) = slls, \ldots$$
The first few levels of the corresponding tree for this number are shown in Figure 4.1. In Figure 4.2, the complete tree is portrayed implicitly. Its recursive nature is explicitly revealed - the whole tree is labelled $H$: wherever a $H$ appears it may be replaced by any tree $H$ which has been constructed in the same way. Figure 4.3 is an expanded version of Figure 4.2. Figure 4.4 also portrays tree $H$ where only the $H$ appearing in the bottom left hand corner of Figure 4.2 is replaced by the tree $H$ portrayed in Figure 4.1.

4.8 Special case $\alpha = \tau = \{0; 1, 1, \ldots\} = \frac{\sqrt{5} - 1}{2}$

For this case,

$$\Omega_n = Q(\Omega_{n-1}),$$

where

$$Q(s) = l, \quad Q(l) = ls.$$  

Then,

$$\Omega_0 = s,$$

$$\Omega_{1,1} = \Omega_1 = Q(s) = l,$$

$$\Omega_{2,1} = \Omega_2 = Q(l) = ls,$$

$$\Omega_{3,1} = \Omega_3 = Q(ls) = lsl,$$

$$\Omega_{4,1} = \Omega_4 = Q(lsl) = lsls, \ldots$$

Figure 4.5 illustrates the first few levels while Figure 4.6 portrays the implicit representation of the tree $G$ obtained for this number describing the sequence of strings (4.7.1).

We mention here that Jean [48]-[50], [53]-[59] has undertaken an intensive study of this structure, though without realising its recursive representation. He models phyllotaxis systems as hierarchies and assigns to each a measure, the entropy, and shows that the Fibonacci tree (our tree $G$) minimises this measure. Jean's definition of entropy depends on parameters like the stability, complexity and rhythm of the hierarchy - tree $G$ has,
Figure 4.1 Graphical representation of (4.7.1) for $\alpha = \sqrt{2} - 1$: the first few levels

Figure 4.2 Tree H: implicit graphical representation of (4.7.1) for $\alpha = \sqrt{2} - 1$
Figure 4.3 Tree H: expanded version of Figure 4.2

Figure 4.4 Tree H
Figure 4.5 Graphical representation of (4.7.1) for $\alpha = \tau$: the first few levels

Figure 4.6 Tree $G$: implicit representation of (4.7.1) for $\alpha = \tau$
intuitively, unique and special associations with these concepts. Jean's analysis is quite complex and we venture that the underlying philosophy which motivates his study may more easily find its expression in tree G's recursive nature and the fact that it is the simplest representation of any phyllotaxis system.

Figure 4.6 (tree G) appears in Hofstadter [45], pp. 135-137, as an example of an infinite geometric structure defined recursively. Hofstadter numbers the nodes of the tree such that the node $\omega_{n,j}$ has number $F_{n+2} + j$. He states (without proof) that the tree may be constructed through the following recursive equation,

$$G(k) = \begin{cases} 
0, & k = 0, \\
k - G(G(k-1)), & k \geq 1,
\end{cases}$$

(4.8.1)

by placing a node numbered $G(k)$ below node $k$ for all values of $k$. Here, we find an explicit solution to this equation by two methods.

(A) We assume that (4.8.1) does indeed describe the tree and then, by using our previous means of generating the tree, we determine in closed form the number of the node lying below any given node.

(B) This approach is more direct: we merely show that the solution found by (A) does satisfy (4.8.1).

We state the result in the following theorem and proceed to prove it by methods (A) and (B).

**Theorem 4.8**

$$G(k) = [(k + 1)\tau].$$

**Proof (A)** Firstly suppose that we number each level $\Omega_n$ of the tree such that node $\omega_{n,j}$ has number $j$. All the nodes from the preceding level each contribute, through the transition rule $Q$, exactly one node of type $l$ to level $\Omega_n$. The number of nodes $\omega_{n,m}$,
m \leq j \text{ of type } l \text{ is } \sum_{i=1}^{j} [(i + 1)\tau] - [i\tau] = [(j + 1)\tau] \text{ and hence this is the number of the node lying below } \omega_{n,j}.

However, the actual numbering of the node \(\omega_{n,j}\) is \(k = F_{n+2} + j\) and thus the node below is \(F_{n+1} + [(j + 1)\tau]\). But from Lemma C1 (Appendix C), \([(k + 1)\tau] = [(F_{n+2} + j + 1)\tau] = F_{n+1} + [(j + 1)\tau]\) and hence the theorem follows. \(\blacksquare\)

**Proof (B)** Suppose that we decompose \(k\) by the method in Appendix C (Lemma C2) such that

\[
k = \sum_{i=j}^{m} b_i F_{i+1}, \quad b_j = 1,
\]

which we mention is the Zeckendorf expansion of \(k\) (see Zeckendorf [147]). Note that if \(b_i = 1\), then \(b_{i-1} = 0\).

From **Theorem C2**

\[
[k\tau] = \begin{cases}
\sum_{i=j}^{m} b_i F_i, & j \text{ even}, \\
-1 + \sum_{i=j}^{m} b_i F_i, & j \text{ odd},
\end{cases}
\]

The following result may be proved by induction (we omit the proof):

\[
F_{j+1} - 1 = \begin{cases}
\frac{j}{2} + \sum_{i=1}^{(j-1)/2} F_{2i}, & j \text{ even}, \\
\sum_{i=1}^{(j-1)/2} F_{2i+1}, & j \text{ odd},
\end{cases}
\]

To prove the theorem we show that

\[
[k\tau] = k - 1 - \lfloor((k - 1)\tau) + 1)\tau\rfloor.
\]
First note from (4.8.2) that

\[ k - 1 = \sum_{i=j+2}^{m} b_i F_{i+1} + F_{j+1} - 1. \]  \hspace{1cm} (4.8.6)

Note that \( b_{j+1} = 0 \) since \( b_j = 1 \). (This is always true only with the Zeckendorf expansion.)

There are two cases to investigate:

(i) even \( j \)

From (4.8.4) we may rewrite (4.8.6) so that

\[ k - 1 = \sum_{i=j+2}^{m} b_i F_{i+1} + \sum_{i=1}^{j/2} F_{2i}, \]

\[ = \sum_{i=1}^{m} b_i F_{i+1}, \]

where

\[ b_{2i+1} = 1, \quad i = 0, 1, 2, \ldots, \frac{j}{2} - 1. \]  \hspace{1cm} (4.8.7)

Hence, from (4.8.3),

\[ [(k - 1)\tau] + 1 = \sum_{i=1}^{m} b_i F_i, \]

\[ = \sum_{i=0}^{m-1} a_i F_{i+1}, \quad a_i = b_{i+1}. \]

This latter step is made so that (4.8.3) may be applied. From (4.8.3),

\[ [((k - 1)\tau] + 1)\tau] = \sum_{i=0}^{m-1} a_i F_i, \]

\[ = \sum_{i=1}^{m} b_i F_{i-1}. \]

But, from (4.8.7),

\[ \sum_{i=1}^{j} b_i F_{i+1} = \sum_{i=1}^{j/2-1} F_{2i}. \]
Thus, from (4.8.4),

\[ \left[ \left[ (k - 1)\tau \right] + 1 \right] \tau = \sum_{i=j+2}^{m} b_i F_{i-1} + F_{j-1} - 1. \]

Thus, from (4.8.3),

\[ \[k\tau\] + \left[ \left[ (k - 1)\tau \right] + 1 \right] \tau = \sum_{i=j}^{m} b_i F_i + \sum_{i=j+2}^{m} b_i F_{i-1} + F_{j-1} - 1, \]

\[ = \sum_{i=j+2}^{m} b_i F_{i+1} + F_{j+1} - 1, \]

\[ = k - 1, \]

which verifies (4.8.5) for this case.

(ii) odd \( j \)

From (4.8.4), (4.8.6) is rewritten such that

\[ k - 1 = \sum_{i=j+2}^{m} b_i F_{i+1} + \sum_{i=1}^{(j-1)/2} F_{2i+1}, \]

\[ = \sum_{i=2}^{m} b_i F_{i+1}, \]

where

\[ b_{2i} = 1, \quad i = 1, 2, \ldots, (j - 1)/2. \] (4.8.8)

From (4.8.3),

\[ \left[ (k - 1)\tau \right] + 1 = 1 + \sum_{i=2}^{m} b_i F_i, \]

\[ = 1 + \sum_{i=2}^{j-1} b_i F_i + \sum_{i=j+2}^{m} b_i F_i, \]

\[ = F_j + \sum_{i=j+2}^{m} b_i F_i, \]
which follows from (4.8.8) and (4.8.4). Thus,

\[
[\{(k - 1)\tau + 1\}\tau] = \left[ \sum_{i=j}^{m} b_i F_i \tau \right],
\]

\[
= \left[ \sum_{i=j-1}^{m-1} a_i F_{i+1} \tau \right], \quad a_i = b_{i+1},
\]

\[
= \sum_{i=j}^{m} b_i F_{i-1},
\]

which follows from (4.8.3). Hence, from (4.8.3),

\[
[k \tau + \{(k - 1)\tau + 1\}\tau] = -1 + \sum_{i=j}^{m} b_i F_i + \sum_{i=j}^{m} b_i F_{i-1},
\]

\[
= -1 + \sum_{i=j}^{m} b_i F_{i+1},
\]

\[
= k - 1,
\]

which completes the proof. \(\blacksquare\)

**Corollary 4.9**

\[ G(F_i) = F_{i-1}, \quad i \geq 2. \]

**Proof** From Theorem 4.8 we are required to show that

\[ [(F_i + 1)\tau] = F_{i-1}. \]  \hspace{1cm} \text{(4.8.9)}

From (A.1.32b),

\[ F_i \tau - F_{i-1} = (-\tau)^i, \]

Hence,

\[ (F_i + 1)\tau = F_{i-1} + (-\tau)^i + \tau. \]

Thus (4.8.9), together with the corollary, follow since \(0 < (-\tau)^i + \tau < 1.\) \(\blacksquare\)
The following proof of Theorem 4.8 was suggested by one of the referees.

**Proof (C)** We verify equation (4.8.5) by letting \( p = [(k - 1)\tau] \) and \( \beta = \{ (k - 1)\tau \} \).

Then,
\[
p + 1 \in \begin{cases} 
(k\tau, (k + 1)\tau), & 0 < \beta < 1 - \tau, \\
((k - 1)\tau, k\tau), & 1 - \tau < \beta < 1.
\end{cases}
\]

Thus,
\[
k = \begin{cases} 
[(p + 1)/\tau], & 0 < \beta < 1 - \tau, \\
[(p + 1)/\tau] + 1, & 1 - \tau < \beta < 1,
\end{cases}
\]
\[
= \begin{cases} 
p + 1 + [(p + 1)\tau], & 0 < \beta < 1 - \tau, \\
p + 2 + [(p + 1)\tau], & 1 - \tau < \beta < 1,
\end{cases}
\]

which is equivalent to (4.8.5) since
\[
[k\tau] = \begin{cases} 
p, & 0 < \beta < 1 - \tau, \\
p + 1, & 1 - \tau < \beta < 1.
\end{cases}
\]

(Note that \( \beta \) cannot equal 0 or \( 1 - \tau \) since \( \tau \) is irrational.)
CHAPTER FIVE

Measures of Uniformity

5.1 Introduction

In this chapter we examine ways of measuring the degree of uniformity of sequences distributed in the unit interval. Our main concern is with the sequence formed from the fractional parts of consecutive non-negative multiples of an irrational number denoted by \( \omega(\alpha) \);

\[
\omega(\alpha) = \omega = (\omega_1, \omega_2, \omega_3 \ldots),
\]

where \( \omega_i = (i - 1)\alpha \mod 1, \ i = 1, 2, \ldots \)

The aim is to demonstrate that when \( \omega(\alpha) \) is represented on the circle of unit circumference, \( \omega(\tau) \) determines points which in some ways are most evenly spaced.

The following section introduces two measures of discrepancy of sequences which arise in the theory of uniform distribution modulo one. This theory was first developed by Hermann Weyl [140] in 1916. (See Kuipers and Niederreiter [68].) In Section 5.3 we consider the way in which gaps are divided. In particular, we look at the ratio of gap division and the age of the divided gap in order to show that \( \tau \) provides a more even spacing of points. Section 5.4 establishes asymptotic results connected with the lengths of the smallest and largest gaps indicating the spacing properties of \( \omega(\tau) \). In Section 5.5, we determine the phyllotaxis path which ensures that the smallest gap is consistently as large as possible.
5.2 Uniform distribution modulo one and discrepancy

In this section we define the term uniform distribution modulo one and introduce two functions to measure the deviation of a given sequence from an ideal, uniform distribution. In Section 5.2.2, after representing (in Section 5.2.1) the sequence on the circle, an interesting connection is demonstrated between these two measures. In Section 5.2.3 these results are used in conjunction with results from Chapter 2 to determine some values of the two discrepancies for the sequence \( \omega(\alpha) \). An asymptotic result relating to one of the measures is offered as a corollary.

**Definition 5.1**  Let \( v = (x_1, x_2, x_3, \ldots) \) be an infinite sequence of real numbers located in the unit interval. \( v \) is said to be uniformly distributed modulo one if

\[
\lim_{N \to \infty} \frac{A([\beta, \gamma]; v; N)}{N} = (\gamma - \beta),
\]

for all possible choices of \( \beta \) and \( \gamma \), where \( A([\beta, \gamma]; v; N) \) counts the number of the first \( N \) elements of \( v \) which belong to \([\beta, \gamma]\).

Thus a sequence could be said to be uniformly distributed modulo one (in an asymptotic sense) if each subinterval of the unit interval receives its "fair share" of elements. For instance \( \omega(\alpha) \) is uniformly distributed modulo one for all irrational \( \alpha \).

**Definition 5.2**  The *standard* discrepancy of the first \( N \) elements of \( v \) is

\[
D_N(v) = \sup_{0 \leq \beta \leq \gamma < 1} \left| \frac{A([\beta, \gamma]; v; N)}{N} - (\gamma - \beta) \right|.
\]  (5.2.1)

The discrepancy is a measure of how closely the first \( N \) elements of \( v \) approximate a uniform distribution. A related measure is now defined.

**Definition 5.3**  The *extreme* discrepancy of the first \( N \) elements of \( v \) is

\[
D_N^*(v) = \sup_{0 \leq \gamma < 1} \left| \frac{A([0, \gamma]; v; N)}{N} - \gamma \right|.
\]  (5.2.2)
We mention that \( v \) is uniformly distributed if and only if \( \lim_{N \to \infty} D_N(v) = \lim_{N \to \infty} D^*_N(v) = 0 \). (See Kuipers and Neiderreiter [68], Theorem 1.1 and Corollary 1.1.)

5.2.1 Representation of \( v \) on the circle

Suppose that the sequence \( v \) is represented on the circle \( C \) of unit circumference, rather than on the unit interval. Let \([\beta : \gamma]\) denote the arc from \( \beta \) to \( \gamma \) (\( 0 \leq \beta, \gamma < 1 \)) on the circle in the direction of increasing co-ordinate. That is,

\[
[\beta : \gamma] = \begin{cases} 
[\beta, \gamma], & 0 \leq \beta \leq \gamma < 1, \\
C - (\gamma, \beta), & 0 \leq \gamma < \beta < 1.
\end{cases} 
\]  

(5.2.3)

Note that the length of such an arc is equal to \( \{ \gamma - \beta \} \), the fractional part of \( \gamma - \beta \).

With respect to this representation, the discrepancy of \( v \) may be given by

\[
\hat{D}_N(v) = \sup_{0 \leq \beta, \gamma < 1} \left| \frac{A([\beta : \gamma]; v; N)}{N} - \{ \gamma - \beta \} \right|. 
\]  

(5.2.4)

Lemma 5.4

\[
D_N(v) = \hat{D}_N(v).
\]

Proof

Let

\[
R(\beta, \gamma) = A([\beta : \gamma]; v; N) / N - \{ \gamma - \beta \}. 
\]  

(5.2.5)

Clearly,

\[
\hat{D}_N(v) = \sup \left\{ \sup_{0 \leq \beta, \gamma < 1} R(\beta, \gamma), \sup_{0 \leq \beta, \gamma < 1} -R(\beta, \gamma) \right\}. 
\]

But \(-R(\beta, \gamma) \leq R(\gamma, \beta)\) since \(A([\beta : \gamma]; v; N) + A([\gamma : \beta]; v; N) \geq N\) and \( \{-x\} = 1 - \{x\} \) for real \( x \). Hence,

\[
\hat{D}_N(v) = \sup_{0 \leq \beta, \gamma < 1} \left( \frac{A([\beta : \gamma]; v; N)}{N} - \{ \gamma - \beta \} \right), 
\]  

(5.2.6)

or

\[
\hat{D}_N(v) = \sup \left\{ \sup_{0 \leq \beta \leq \gamma < 1} R(\beta, \gamma), \sup_{0 \leq \gamma < \beta < 1} R(\beta, \gamma) \right\}. 
\]  

(5.2.7)
From (5.2.3),

\[ R(\beta, \gamma) = \begin{cases} \frac{A([\beta,\gamma];v;N)}{N} - (\gamma - \beta), & 0 \leq \beta \leq \gamma \leq 1, \\ \beta - \gamma - \frac{A([\gamma,\beta];v;N)}{N}, & 0 \leq \gamma < \beta \leq 1. \end{cases} \]

Replacing \( R(\beta, \gamma) \) in (5.2.7) by this expression completes the proof.

Thus if the sequence is represented on the unit interval or the circle of unit circumference, the measure of discrepancy is the same.

### 5.2.2 A relation between \( D_N^*(v) \) and \( D_N(v) \)

The following proposition relates the two functions, \( D_N^*(v) \) and \( D_N(v) \).

**Proposition 5.5**

\[ D_N(v) = D_N^*(v) + \inf\left( D_N^+(v), D_N^-(v) \right), \]

where

\[ D_N^+(v) = \sup_{0 \leq \gamma < 1} \left( \frac{A([0,\gamma];v;N)}{N} - \gamma \right), \]
\[ D_N^-(v) = \sup_{0 \leq \gamma < 1} \left( \gamma - \frac{A([\gamma,0];v;N)}{N} \right), \]
\[ D_N^*(v) = \sup \left( D_N^+(v), D_N^-(v) \right). \]

**Proof** We need only show that \( D_N(v) = D_N^+(v) + D_N^-(v) \). Without loss of generality assume that the elements \( x_j, 1 \leq j \leq N \) are arranged in ascending order of magnitude. For notational convenience, let \( x_0 = 0 \) and \( x_{N+1} = 1 \). Then the numbers \( x_0, x_1, \ldots, x_{N+1} \) partition the unit interval so that

\[ D_N^+(v) = \sup_{\substack{x_j \leq \gamma < x_{j+1} \leq 1 \leq j \leq \gamma \leq x_{j+1}}} \frac{A([0,\gamma];v;N)}{N} - \gamma. \]
Evaluating the supremum over each subinterval \([x_j, x_{j+1})\) gives

\[
D^+_N(v) = \sup_{j=1,2,...,N} \left( \frac{j}{N} - x_j \right). \tag{5.2.8}
\]

Similarly,

\[
D^-_N(v) = \sup_{j=1,2,...,N} \left( x_j - \frac{j-1}{N} \right). \tag{5.2.9}
\]

(5.2.6) and the fact that \(R(\beta, \gamma) \leq R(x_i, x_j)\) where \(x_i \leq \beta < x_{i+1}, x_j \leq \gamma < x_{j+1}\) (for suitable \(i\) and \(j\)) yield

\[
D_N(v) = \sup_{0 \leq i, j \leq N} \left\{ \frac{A([x_i:x_j]; v; N)}{N} - \{x_j - x_i\} \right\}
\]

Alternatively,

\[
D_N(v) = \sup_{0 \leq i, j \leq N} \left( \frac{j-i+1}{N} - x_j + x_i \right),
\]

\[
= \sup_{0 \leq j \leq N} \left( \frac{j}{N} - x_j \right) + \sup_{0 \leq i \leq N} \left( x_i - \frac{i-1}{N} \right),
\]

\[
= D^+_N(v) + D^-_N(v).
\]

This follows from (5.2.8) and (5.2.9). \(\blacksquare\)

5.2.3 The discrepancy of \(\omega(\alpha)\)

In this section we evaluate the discrepancies \(D^+_N(\omega)\) and \(D^-_N(\omega)\) for some particular values of \(N\) related to the simple continued fraction expansion of \(\alpha\). Those values of \(N\) are in fact the denominators of partial and total convergents. Recall (from Theorem 2.15) that the circle, in this case, is divided into gaps of only two different lengths.

**Theorem 5.6**

\[
D^+_n(\omega) = \begin{cases} 
\frac{1}{q_{n,i}} + (q_{n,i} - 1)(\frac{p_{n,i}}{q_{n,i}} - \alpha), & \text{n odd,} \\
\sup \left( \frac{1}{q_{n,i}}, (q_{n,i} - 1)(\alpha - \frac{p_{n,i}}{q_{n,i}}) \right), & \text{n even,}
\end{cases}
\]

\[
D^-_n(\omega) = \frac{1}{q_{n,i}} + (q_{n,i} - 1)\left| \frac{p_{n,i}}{q_{n,i}} - \alpha \right|.
\]
Proof Suppose that the first $N$ elements of $\omega$ are arranged in ascending order of magnitude. Let $\{u_j(\alpha)\}, j = 1, 2, \ldots, N$ be that ordering. Corollary 2.13 states that
\[
u_j = \left((-1)^{n-1}(j - 1)q_{n-1}\right) \mod q_{n,i}, \quad j = 1, 2, \ldots, q_{n,i}, \quad i = 1, 2, \ldots, a_n.
\]
Theorem 2.14 states that
\[
j - 1 = (u_j p_{n,i}) \mod q_{n,i},
\]
or
\[
j - 1 = q_{n,i} \left\{u_j \frac{p_{n,i}}{q_{n,i}}\right\}. \quad (5.2.10)
\]
From Lemma 4.5,
\[
\{u_j \alpha\} = u_j \alpha - \left\{u_j \frac{p_{n,i}}{q_{n,i}}\right\}. \quad (5.2.11)
\]
Substitution of (5.2.10) into (5.2.9) with the inclusion of (5.2.11) yields
\[
D_{q_{n,i}}^-(\omega) = \sup_{j=1,2,\ldots,q_{n,i}} u_j \left(\alpha - \frac{p_{n,i}}{q_{n,i}}\right).
\]
\[
= \sup_{u_j=0,1,\ldots,q_{n,i}-1} u_j \left(\alpha - \frac{p_{n,i}}{q_{n,i}}\right),
\]
\[
= \begin{cases} 
0, & n \text{ odd,} \\
(q_{n,i} - 1)\left(\alpha - \frac{p_{n,i}}{q_{n,i}}\right), & n \text{ even.}
\end{cases}
\]
This follows from the fact that $\alpha - p_{n,i}/q_{n,i}$ is negative for odd $n$ and positive otherwise.

To determine $D_{q_{n,i}}^+(\omega)$ note that
\[
D_{q_{n,i}}^+(\omega) = \frac{1}{q_{n,i}} - \inf_{j=1,2,\ldots,q_{n,i}} \left\{u_j \alpha\right\} - \frac{j - 1}{q_{n,i}}.
\]
Following the same procedure as above, it is found that
\[
D_{q_{n,i}}^+(\omega) = \begin{cases} 
\frac{1}{q_{n,i}} + (q_{n,i} - 1)(\frac{p_{n,i}}{q_{n,i}} - \alpha), & n \text{ odd,} \\
\frac{1}{q_{n,i}}, & n \text{ even.}
\end{cases}
\]
The theorem now follows from Proposition 5.5. \qed
Corollary 5.7

\[ 1 \leq \liminf_{N \to \infty} ND_N(\omega) \leq 1 + \frac{1}{\sqrt{5}}. \]

**Proof** The lower bound is easily found. (See Kuipers and Neiderreiter [68], pp. 90).

For the upper bound, first note that

\[ \liminf_{N \to \infty} ND_N(\omega) \leq \liminf_{n \to \infty} q_n D_{q_n}(\omega). \]

From Theorem 5.6,

\[ \liminf_{n \to \infty} q_n D_{q_n}(\omega) = \liminf_{n \to \infty} 1 + (q_n - 1)\|q_n\alpha\|, \]

\[ = 1 + \liminf_{n \to \infty} q_n\|q_n\alpha\|. \]

From a theorem of Hurwitz (see Theorem 5.10 or Hardy and Wright [37], Theorem 194),

\[ \sup_{\alpha} \liminf_{q \to \infty} q\|q\alpha\| = \sup_{\alpha} \liminf_{n \to \infty} q_n\|q_n\alpha\|, \]

\[ = \frac{1}{\sqrt{5}}. \]

The supremum occurs at all values of \( \alpha \) which have \( t_j = \frac{(1 + \sqrt{5})/2}{2} \) for some non-negative integer \( j \). That is, all the values of \( \alpha \) which are **equivalent** to \( \tau \). Thus the corollary follows. \( \blacksquare \)

Numerical experiments support the following.

**Conjecture**

\[ \sup_{\alpha} \liminf_{N \to \infty} ND_N(\omega) = 1 + \frac{1}{\sqrt{5}}, \]

uniquely attained by \( \alpha \) equivalent to \( \tau \).

### 5.3 Division of gaps

**Theorem 5.8**

\[ \max_{\alpha \in \mathbb{Z}_N^+} \min_{\alpha} r_N(\alpha) = \tau^2 = \frac{3 - \sqrt{5}}{2}, \]
uniquely attained by $\alpha = \tau$.

**Proof** From Proposition 3.3, for $n = 2, 3, \ldots$,

$$
\min_{q_{n-1} < N \leq q_n} r_N(\alpha) = \min_{n} \frac{1}{1 + t_n} = \frac{1}{1 + \max t_n}.
$$

Suppose that the maximum does not occur at $\alpha = \tau$. Consider then $\alpha = \alpha' \neq \tau$ which has $a_k > 1$ for some non-negative integer $k$. Then $t_k > 2$ and hence $r_N(\alpha') < 1/3$. But $1/3 < r_N(\tau) = \tau^2$ which completes the proof.  

Note that the ratio of gap division is constant only for $\alpha = \tau$.

The following theorem combined with Corollary 3.7 shows that if $\alpha = \tau$ the oldest gap is always large.

**Theorem 5.9**

$\alpha = \tau$ is unique in that each point divides the oldest gap. ($\alpha > \sqrt{2}$).

**Proof** From Corollary 3.2, for $\alpha = \tau$, $\Phi_n = P_n(\Phi_{n-1})$, where

$$
P_n(l) = \begin{cases} 
sl, & n \text{ odd}, \\
ls, & n \text{ even},
\end{cases}
$$

Thus the formation of a small gap coincides with that of a large gap. Hence no small gap is older than a large gap. From Corollary 3.7 each point divides the oldest of the large gaps; that is the oldest gap.

To show that this property is unique to $\alpha = \tau$, suppose that $\alpha \neq \tau$ and that $a_k > 1$ for some non-negative integer $k$. Then, from Theorem 3.1, $P_{n,i}(s) = s$, $i = 2, 3, \ldots, a_k$. Thus small gaps, formed from the division of large gaps, age as $N$ is increased from $q_{k,1}$ to $q_{k+1}$ while the new large gaps are divided.  

5.4 Asymptotic results

Let $S_N(\alpha)$ and $L_N(\alpha)$ denote respectively the lengths of the smallest and largest gap present for $N$ points on the circle. From Theorems 2.15 and 2.22,

\begin{align*}
S_N(\alpha) &= \min(g_1, g_N), \quad (5.4.1) \\
L_N(\alpha) &= \begin{cases} 
\max(g_1, g_N), & N = u_2 + u_N, \\
g_1 + g_N, & N \neq u_2 + u_N.
\end{cases} \quad (5.4.2)
\end{align*}

Corollary 2.23 explicitly determines $S_N(\alpha)$.

Theorem 5.10

\[ \sup \liminf_{N \to \infty} N S_N(\alpha) = \frac{1}{\sqrt{5}}, \]

uniquely attained by $\alpha$ equivalent to $\tau$.

Proof The theorem derives from a theorem of Hurwitz which states that any irrational number $\alpha$ has an infinity of approximations which satisfy

\[ q |q \alpha - p| < \frac{1}{\sqrt{5}}, \]

where $1/\sqrt{5}$ is best possible in the sense that

\[ \sup \liminf_{q \to \infty} q |q \alpha - p| = \frac{1}{\sqrt{5}}, \]

which is uniquely attained by $\alpha$ equivalent to $\tau$.

From Lemma 2.1,

\[ \sup \liminf_{q \to \infty} q |q \alpha - p| = \sup \liminf_{n \to \infty} q_{n-1} \|q_{n-1} \alpha\|. \]

The theorem now follows from Corollary 2.23 since $S_N(\alpha) = \|q_{n-1} \alpha\|$ for $q_{n-1} < N \leq q_n$.

The following theorem is proved by Graham and van Lint [35].
Theorem 5.11

\[ \inf_{\alpha} \lim_{N \to \infty} \sup N L_N(\alpha) = 1 + \frac{2\sqrt{5}}{5}, \]

attained by \( \alpha \) equivalent to \( \tau \).

\[ \sup_{\alpha} \lim_{N \to \infty} \inf N L_N(\alpha) = \frac{1 + \sqrt{2}}{2}, \]

attained by \( \alpha \) equivalent to \( \sqrt{2} - 1 \).

5.5 Contact pressure on the circle

In this section we determine the phyllotaxis path which consistently maximises the length of the smallest gap. We show that \( V(\tau) \) is the path which guarantees that the smallest gap, for all numbers of points, is as large as possible. That is, if the phyllotaxis does not rise through consecutive elements of \( V(\tau) \), then the smallest gap thus formed is not as large. At each stage, the value of \( \alpha \) is constant but with the addition of points may change though only in such a way that the phyllotaxis path is retained.

The force which maximises the length of the smallest gap is Adler's interpretation ([1], [3]) of contact pressure first postulated by Schwenender [108] to be the cause of Fibonacci phyllotaxis. Adler was concerned with points equally spaced along a regular helix on the surface of a cylinder. The phyllotaxis rises as the pitch of the spiral decreases. Our geometry is on the circle, though we extend the result in Section 6.2 to the cylindrical representation of Adler.

Theorem 5.12

Suppose that the phyllotaxis is initially \((2, 1)\) and \(1/2 < \alpha < 2/3\). Under the conditions of contact pressure, the phyllotaxis path is \( V(\tau) \).

Proof We prove the result by induction. Initially, \(1/2 < \alpha < 2/3\) or \(\alpha = \{0; 1, 1, \sqrt{3}\}, 1 < \sqrt{3} < \infty\) such that point 2 is closest to the origin. The phyllotaxis must, from Theorem
2.6, rise to (2, 3). From Proposition 3.3, point 2 is further from the origin if \(a_3 = 1\) than if \(a_3 > 1\) since then it divides the gap \(d_{0,1}\) in a larger ratio. Thus, \(\alpha = \{0; 1, 1, 1, t_4\}\). This ensures (from Theorem 2.6) that the phyllotaxis will rise to (5, 3). Thus the first three terms in the phyllotaxis path belong to \(V(\tau)\).

Now assume that the phyllotaxis rises along Fibonacci pairs and reaches a phyllotaxis \((F_{n-1}, F_n)\), where \(n\) is even. Then \(\alpha = \{0; 1, 1, 1, \ldots, 1, t_n\}\) \((n - 1)\) ones, \(1 < t_n < \infty\). The phyllotaxis will rise to \((F_{n+1}, F_n)\), and then to \((F_{n+1}, F_{n+2})\) if \(a_n = 1\). From Proposition 3.3, the gap \(d_{F_n,0}\) is larger if \(a_n = 1\) than if \(a_n > 1\). The case is similar for odd \(n\). Thus the phyllotaxis rises along the path \(V(\tau)\) and, also, \(\alpha\) converges to the golden angle.
CHAPTER SIX

A Model of Phyllotaxis

6.1 Introduction

In this chapter we present a simple model to account for the phenomenon of Fibonacci phyllotaxis. We first consider Adler's [1], [3] contact pressure model which applies to phyllotaxis systems defined on the surface of cylinders (and extended to other geometries). In this model, the system of leaves depends on two parameters which vary in time; the divergence angle and the (vertical) internode distance between consecutively formed leaves. The assumptions inherent in our model (presented in Section 6.3) are less restrictive: we do not assume that consecutive leaves differ by a constant angle but rather that the phyllotaxis is approximately “regular.” The model also allows for rising phyllotaxis and applies to phyllotaxis systems on a general class of surfaces. Adler also applies his contact pressure argument to different surfaces by employing “appropriate geometric transformations” from the cylinder to the particular surfaces. Ridley [97] has criticized this method since, though the transformations are conformal (angle preserving), they invariably distort distances rendering the results difficult to justify.

Whereas the model of Adler involves the maximisation of the distance between neighbouring leaves, our model does not involve the consideration of metric properties of the phyllotaxis system. We feel that the principal of growth advocated here is more natural: its attraction lies in its simplicity and its biological plausibility.

We first consider the phyllotaxis system considered by Adler: we apply this analysis to verify the consequences of his model, also drawing upon results from Chapter 5 (in particular Theorem 5.12). It should be mentioned that Ridley [98] has recently modelled
phyllotaxis systems on general surfaces of revolution, obtaining results concerning changes in phyllotaxis by defining a lattice at each point on the surface and showing that the golden number provides the most efficient packing.

6.2 Cylindrical representation

In Section 5.5 we determined the phyllotaxis path which ensures that the smallest gap on the circle for any number of points is as large as possible. Here we consider a phyllotaxis system composed of points $P_0, P_1, P_2, \ldots$ on the surface of a cylinder of girth one lying on a helix at equal intervals. The geodesic vertical and horizontal components of the distance between $P_k$ and $P_{k+1}$ are respectively $r(t)$ and $\alpha(t)$, where $t$ denotes time.

Rather than have $P_k$ as the point with cylindrical co-ordinates $\{k\alpha(t)\},kr(t))$ we choose the following, more convenient, co-ordinate system.

$$P_k = \begin{cases} 
\{k\alpha(t)\}, kr(t), & \{k\alpha(t)\} < 1/2, \\
\{k\alpha(t)\} - 1, kr(t), & \{k\alpha(t)\} > 1/2, \\
\{||k\alpha(t)||, kr(t)\}, & \{k\alpha(t)\} < 1/2, \\
\{-||k\alpha(t)||, kr(t)\}, & \{k\alpha(t)\} > 1/2.
\end{cases}$$

For a given $t$, the distance from the origin to $P_k$ is denoted by $d_{0,k}(\alpha)$ where

$$d_{0,k}^2 = ||k\alpha(t)||^2 + k^2r^2(t).$$

We shall define the phyllotaxis path for this system and show that under the conditions of contact pressure (maximisation of the distance between neighbouring points) the phyllotaxis path is the same for both the circle and cylinder.
As defined by Adler [1], [3] \((a, b)\) is a visible opposed paristichy pair if the phyllotaxis system can be partitioned into \(a\) left parastichies and \(b\) right parastichies, such that the intersection of two parastichies always coincides with a point.

**Theorem 6.1**

\((a, b)\) is a visible opposed parastichy pair only if \((a, b) = (u_2(N), u_N(N))\) for some value of \(N\), where \(u_2\) and \(u_N\) derive from \(\alpha = \alpha(t)\).

**Proof** \((a, b)\) is a visible opposed parastichy pair only if there is no point inside the parallelogram defined by the origin \(P_0\) and the points \(P_a, P_b\) and \(P_{a+b}\). This condition is satisfied only when \((a, b)\) is equal to \((u_2(N), u_N(N))\) for some \(N\). To see this first note that the area of the parallelogram defined by \(P_0, P_{u_2}\) and \(P_{u_N}\) is equal to \(|u_2\|u_N\alpha\|r(t) + u_N\|u_2\alpha\|r(t)| = r(t)|\), which follows from \((2.3.9)\). This is the smallest possible area and hence no point lies inside this parallelogram.

Now consider the point \((a, b) \neq (u_2(N), u_N(N))\) for any \(N\), and the parallelogram \(J\) formed by \(P_0, P_a, P_b\). If \(a \neq u_2(N), b = u_N(N)\) then \(P_{u_2+b}\) lies inside \(J\). If \(a = u_2(N), b \neq u_N(N)\) then \(P_{a+u_N}\) lies inside \(J\). If \(a \neq u_2(N), b \neq u_N(N)\) for any \(N\), then \(P_{a+u_M}\) and \(P_{u_2+b}\) lie inside \(J\) where \(M = \max\{N:\ u_2(N) < a \text{ and } u_N(N) < b\}\). Thus, \((a, b) = (u_2(N), u_N(N))\) for some value of \(N\). Hence the theorem.

We define (as in Adler [1], [3]) the phyllotaxis to be \((a, b)\) where \((a, b)\) is a conspicuous opposed parastichy pair, determined by a point and its nearest neighbours to the left and to the right. We assume that the phyllotaxis is initially \((u_2(q_1), u_{q_1}(q_1))\) and determine the change in phyllotaxis as \(r(t)\) decreases with increasing \(r\).

The following theorem combined with Theorem 6.1 shows that a conspicuous opposed parastichy pair is also visible. Adler [3] proved this with the restriction that
the phyllotaxis changes under the influence of contact pressure. Our theorem shows that this restriction is not necessary.

**Theorem 6.2**

Suppose that the phyllotaxis is, at some time \(t\), equal to \((a, b)\). As \(r(t)\) decreases with increasing \(t\), the phyllotaxis rises from \((a, b)\) to \((c, d)\) where \((a, b)\) and \((c, d)\) are consecutive elements of \(V(\alpha)\). The value for \(\alpha = \alpha(t)\) may be determined from Theorem 2.20 (or Corollary 2.21).

**Proof** By induction. Initially the phyllotaxis is \((u_2(q_1), u_{q_1}(q_1))\) so that \(P_1\) and \(P_{q_1}\) are the closest points to the left and to the right of \(P_0\) - the point \(P_{q_1}\) is the closest point. \(P_1+u_{q_1}\) is closer in angle to the origin than \(P_1\) and, by Corollary 2.7, is the first point to replace \(P_1\) as \(r(t)\) decreases so that the phyllotaxis rises to \((1 + u_{q_1}(q_1), u_{q_1}(q_1)) = (u_2(q_{2,1}), u_{q_{2,1}}(q_{2,1}))\).

Now suppose that \(P_a\) and \(P_b\) lie closest, and on opposite sides, to \(P_0\) and that \((a, b)\) belongs to \(V(\alpha)\). By the same argument \(P_{a+b}\) is the first point to replace \(P_a\) or \(P_b\) as \(r(t)\) decreases. The phyllotaxis then rises to either \((a + b, b)\) (if \{\((a + b)\alpha\} < 1/2\) or \((a, a + b)\) (if \{\((a + b)\alpha\} > 1/2\), the consecutive element to \((a, b)\) in \(V(\alpha)\).

The following verifies the consequences of Adler’s model.

**Theorem 6.3**

Under the influence of contact pressure (maximisation of the distance between neighbouring points) the phyllotaxis path is \(V(\tau)\), where the phyllotaxis is initially \((2, 1)\) (with \(1/2 < \alpha < 2/3\)) and \(r(t)\) decreases with increasing \(t\).
Proof For a given $\alpha$, as $r(t)$ decreases with $t$, the phyllotaxis path on the cylinder is identical to the path on the circle. With $t$ fixed, $d_{0,q}^2(\alpha) \geq d_{0,q}^2(\beta)$ only if $\|q\alpha\| \geq \|q\beta\|$. Thus the distance between neighbouring points is maximal only if the angular component is maximal. Theorem 5.12 then verifies the result. 

Corollary 6.4

If $r(t)$ decreases with increasing $t$ then, under the conditions of contact pressure, for some $n$

$$\alpha(t) \in I(0; 1, 1, \ldots, 1), \quad (n \text{ ones}),$$

where the phyllotaxis is initially $(2, 1)$. (I is defined by equations (2.4.5) and (2.4.6).)

Proof From Theorem 6.3, the phyllotaxis rises through consecutive elements of $V(\tau)$. Corollary 2.21 (and (2.4.6)) then determine the changes in $\alpha(t)$. 

Note that Corollary 6.4 shows that the presence of contact pressure and decreasing $r(t)$ ensure convergence of the divergence angle to the golden angle, since the phyllotaxis rises through each element of $V(\tau)$. (That is, $n \to \infty$.)

6.3 A simple model

In this section we model the phenomenon of Fibonacci phyllotaxis on a general class of surfaces. We first present the rule which describes its development and then proceed to offer an explanation which guarantees its emergence. We assume that the phyllotaxis system is "regular" and that the phyllotaxis is constant with respect to the age of the leaf from which it is being determined. We also allow the phyllotaxis to rise as a function of time and distance from the growing point. The meaning of these conditions will be made clearer as we formulate the model.
6.3.1 Formulation

Let \((a, b) \rightarrow (c, d)\) denote the transition from a phyllotaxis of \((a, b)\) to one of \((c, d)\).

Lemma 6.5

Suppose that the phyllotaxis is initially \((2, 1)\). If \((a, b)\) is the phyllotaxis at some stage of growth then the transition

\[
(a, b) \rightarrow \begin{cases} 
(a, a + b), & a > b, \\
(a + b, b), & a < b,
\end{cases}
\]

(6.3.1)

describes the development of Fibonacci phyllotaxis.

Proof Let \(v_n\) denote the \(n\)th element of \(V(\tau)\). Then,

\[
v_n = \begin{cases} 
(F_n, F_{n+1}), & n \text{ odd}, \\
(F_{n+1}, F_n), & n \text{ even}.
\end{cases}
\]

It is a simple matter to show that \(v_n \rightarrow v_{n+1}\) obeys the rule (6.3.1). 

We offer an explanation of the emergence of Fibonacci phyllotaxis for a phyllotaxis system which is approximately regular at all stages of growth, such that for each leaf, if its nearest neighbours to the left and to the right differ in age by \(a\) and \(b\) plastochrones respectively, then the nearest leaf which lies angularly between these neighbours differs by \(a + b\) plastochrones in age. The position of a leaf is described by the triple \((\rho, \theta, z)\). The two co-ordinates \(\theta\) and \(z\) fix the position of the point on the surface (analogous to cylindrical co-ordinates) and \(\rho\) determines the shape of the surface for each \(z\). It is supposed that \(\rho\) is independent of \(\theta\) (so that we are considering a surface of revolution) and is a monotonically increasing function of \(z\). We refer to \(z\) as the surface distance since it is measured along the surface from the origin. Also suppose that the difference in surface distance between pairs of leaves differing by the same age decreases in time, thus guaranteeing that the phyllotaxis rises as a function of both time and distance from the growing point (the apical meristem). Without this assumption (since \(\rho\) increases with
z) the phyllotaxis would rise as a function of distance only. We assume that the surface distance is greater for older leaves than for younger leaves.

The cylindrical representation discussed in Section 6.2 (the lattice of leaves lying on a regular helix at constant angular intervals) satisfies this regularity condition. Note though that the regularity condition allows for more latitude in the relative positioning of leaves.

Suppose that the initial phyllotaxis is (2, 1). If leaves were generated by a constant angle $\alpha$ the assumption would be equivalent to having $\alpha$ lie between 1/2 and 3/4. We remark that an initial phyllotaxis equal to (1, 2) is an equivalent assumption since it leads to a symmetric situation, and is equivalent to having $\alpha$ lie between 1/4 and 1/2. Figure 6.1 illustrates the possible positioning of a leaf (labelled 0) and its neighbours 1, 2 and 3. The numbering refers to the differences in age between the leaves. The leaf labelled 0 lies further from the origin than the other three leaves since it is actually 3 plastochrones old when leaf 3 is newly formed. We trace the positioning of this leaf in relation to the other leaves and to the surface of the plant. The phyllotaxis of the plant is determined from this leaf and thus we are considering a phyllotaxis which changes in both time and space. For simplicity we suppose that the phyllotaxis determined from leaf 0 is typical of any other leaf. That is, if we trace the development of any leaf, the phyllotaxis determined by it would be the same as for our leaf 0 with respect to time (here time refers to the age of the leaves). That is, the phyllotaxis is constant with respect to the age of the leaf from which it is being determined.

Now as time increases (remembering that the regularity condition always applies) leaf 3 is positioned either to the right or left of leaf 0, taking the place of either leaf 1 or 2. If Fibonacci phyllotaxis develops so that the phyllotaxis becomes (2, 3), then the leaf is positioned to the right of leaf 0. For the moment let's suppose that this occurs and consider the case where leaf 0 has, as its nearest neighbours to its left and right, leaves $a$
Figure 6.1 Possible positioning of leaves 0, 1, 2 and 3
and \( b \). In time leaf \( a + b \) is the next candidate to change the phyllotaxis. We will refer to this leaf as the *candidate*. This leaf is positioned either to the left or to the right of leaf 0, taking the place of leaf \( a \) or leaf \( b \) respectively when the phyllotaxis changes. Call this “gap” between leaves 0 and \( a \) \( g_a \) and the alternative “gap” \( g_b \).

**Theorem 6.6**

Fibonacci phyllotaxis is assured if the candidate enters the older of the two eligible gaps.

**Proof** Assume that the phyllotaxis is \((a, b)\). Suppose first that \( a < b \). Then the candidate \((a + b)\) replaces leaf \( a \) since \( g_a \) is the older gap, thus changing the phyllotaxis to \((a + b, b)\). If \( a > b \) the candidate replaces leaf \( b \) since \( g_b \) is older than \( g_a \). The phyllotaxis then rises to \((a, a + b)\). That is,

\[
(a, b) \rightarrow \begin{cases} 
(a, a + b), & a > b, \\
(a + b, b), & a < b,
\end{cases}
\]

which is (6.3.1). Thus, if the rule is followed by the candidate leaf 3 and obeyed thereafter, the result is Fibonacci phyllotaxis.

We mention that anomalous phyllotaxis occurs if the phyllotaxis is initially \((3, 1)\) and the rule (Theorem 6.6) is followed: that is, each candidate enters the older of the two available gaps.

### 6.3.2 Discussion

Our model supports the first available space theory of Snow and Snow [111] - [113] who, exploiting van Iterson's [130] geometrical study of phyllotaxis systems composed of touching circles, proposed that each new primordium arises in the widest gap between existing primordia as soon as the gap is large enough to accommodate it. We let this principle hold for all primordia so that each primordium, as the phyllotaxis rises, tends to
widest gap between existing (older) primordia. It seems plausible that the oldest gap is also the widest and hence our model supports this modified version of Snow and Snow's hypothesis. The model also lends credence to Schoute's diffusion hypothesis wherein each new primordia (and the apical meristem) secrete a diffusible substance, a morphogen, which inhibits the formation of new primordia in their immediate vicinity. The primordia emerge in areas where the concentration of the morphogen is below some threshold value. Schwabe and Clewer [107] have recently modified this theory in constructing their computer simulation. (Their model successfully accounts for a wide range of natural phyllotaxis systems.) They subjected their morphogen to polar transport, largely restricting its direction of movement to the transverse and downward directions. The placement and movement of all primordia is affected (as the phyllotaxis rises) by the emanation of the morphogen from every primordia. It then seems reasonable to conclude that our model supports this modification of Schoute's hypothesis since it would be expected that the concentration of morphogen is lower in the area bounded by the older of the two gaps.

We note that Fibonacci phyllotaxis would also ensue if the candidates positioned themselves alternately to the right and to the left of leaf 0. Coxeter [18] considered something similar but in a static frame of reference. He considered the phyllotaxis system composed of the points (in cylindrical co-ordinates) \( (n/N, n\alpha) \), \( n = 1, 2, \ldots \) where \( N \) is a positive constant and \( \alpha \) is irrational and lies between 0 and 1. He showed that only \( \alpha = \tau \) has the property that, for any vertical line on the cylinder passing through a lattice point, the neighbouring points lie on alternate sides as one proceeds upwards along it.

We mention here that Mitchison [88] claimed to be considering a model composed of an approximately regular lattice of touching circles on the surface of an expanding cylinder: his arguments, however, implicitly assume that successive divergence angles are equal. He claimed that Fibonacci phyllotaxis arises if each circle touches two circles formed prior to it by showing that \( (a, b) \) phyllotaxis \( (a < b) \) gives way to \( (a + b, b) \) since \( \| (a + b)\alpha \| = \| a\alpha \| - \| b\alpha \| \) (assuming that \( \| b\alpha \| < \| a\alpha \| \)). However, this does not ensure Fibonacci phyllotaxis since it is also required that \( \| (a + b)\alpha \| \) be less then \( \| b\alpha \| \).
This appendix introduces notation and concepts involving the simple continued fraction expansion of $\alpha$.

### A.1 Formulae

We define the (simple) continued fraction expansion of the positive real number $\alpha$ by the following algorithm:

We write $t_0 = \alpha$ and define (for $n = 0, 1, 2, \ldots$),

$$a_n = \lfloor t_n \rfloor,$$  \hspace{1cm} (A.1.1)

$$t_{n+1} = \frac{1}{\{t_n\}}.$$  \hspace{1cm} (A.1.2)

(Recall that $[ \cdot ]$ is the truncation operator and that $\{x\} = x - [x]$.)

In this way the continued fraction expansion is expressed as

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}},}$$  \hspace{1cm} (A.1.3)

which, for notational convenience, we write as

$$\alpha = \{a_0; a_1, a_2, a_3, \ldots\}.$$  \hspace{1cm} (A.1.4)

Note that $a_0$ is the integer part of $\alpha$.

It is evident that

$$t_n = \{a_n; a_{n+1}, \ldots\},$$  \hspace{1cm} (A.1.5)
\[ \alpha = \{ a_0; a_1, a_2, \ldots, a_{n-1}, t_n \}. \]  

We note that the algorithm terminates (so that the number of terms in the continued fraction expansion is finite) if and only if \( \alpha \) is rational.

Let
\[ c_n = \{ a_0; a_1, a_2, \ldots, a_n \}. \]  

We call \( c_n \) the \( n \)th total convergent of \( \alpha \). Each total convergent is intimately related to preceding convergents through the following:
\[ c_n = \frac{p_n}{q_n}, \]  
where
\[ p_n = a_n p_{n-1} + q_{n-2}, \quad n = 0, 1, 2, \ldots, \]  
\[ q_n = a_n q_{n-1} + q_{n-2}, \quad n = 0, 1, 2, \ldots, \]
and
\[ p_{-2} = q_{-1} = 0, \quad q_{-2} = p_{-1} = 1. \]

Thus, from (A.1.6) we may write
\[ \alpha = \frac{t_n p_{n-1} + p_{n-2}}{t_n q_{n-1} + q_{n-2}}, \]  
from which,
\[ t_n = \frac{p_{n-2} - q_{n-2} \alpha}{q_{n-1} \alpha - p_{n-1}}. \]

But \( t_n \) is greater than unity. Hence,
\[ (q_{n-2} \alpha - p_{n-2})(q_{n-1} \alpha - p_{n-1}) < 0, \quad n = 0, 1, 2, \ldots, \]  
and
\[ |q_{n-2} \alpha - p_{n-2}| > a_n |q_{n-1} \alpha - p_{n-1}|, \quad n = 0, 1, 2, \ldots \]
Partial convergents, $c_{n,i}$, are defined by the (irreducible) fractions

$$c_{n,i} = \{a_0; a_1, a_2, \ldots, a_{n-1}, i\} = \frac{p_{n,i}}{q_{n,i}} = \frac{p_{n-2} + ip_{n-1}}{q_{n-2} + iq_{n-1}}, \quad i = 1, 2, \ldots, a_n. \quad (A.1.13)$$

We note the following results which may be easily proven. (See, for example, Khintchine [63].)

$$p_{n-1}q_{n-2} - q_{n-1}p_{n-2} = (-1)^n, \quad (A.1.14)$$

$$p_{n-1}q_n - q_{n-1}p_{n,i} = (-1)^n, \quad (A.1.15)$$

$$q_n \alpha - p_n = \frac{(-1)^n}{t_{n+1}q_n + q_{n-1}}, \quad (A.1.16)$$

$$c_0 < c_2 < c_4 < \ldots < \alpha < \ldots < c_5 < c_3 < c_1, \quad (A.1.17)$$

$$\lim_{n \to \infty} |c_n - c_{n-1}| = 0, \quad (A.1.18)$$

$$\lim_{n \to \infty} c_n = \alpha, \quad (A.1.19)$$

$$\frac{1}{q_{n-1} + q_n} < ||q_n \alpha|| < \frac{1}{q_{n+1}}, \quad (A.1.20)$$

$$t_{n+1} = \frac{||q_{n-1}\alpha||}{||q_n\alpha||}, \quad (A.1.21)$$

where

$$||q\alpha|| = |q\alpha - p|, \quad p = [q\alpha + 1/2], \quad (A.1.22a)$$

$$\|q\alpha\| = \min(\{q\alpha\}, 1 - \{q\alpha\}), \quad (A.1.22b)$$

$$\|q\alpha\| = \begin{cases} \{q\alpha\}, & \text{for } \{q\alpha\} \leq 1/2, \\ 1 - \{q\alpha\}, & \text{for } \{q\alpha\} > 1/2, \end{cases} \quad (A.1.22c)$$

$$\|q\alpha\| = \begin{cases} \{q\alpha\}, & \text{for } \{q\alpha\} > p, \\ 1 - \{q\alpha\}, & \text{for } \{q\alpha\} < p. \end{cases} \quad (A.1.22d)$$

Equations (A.1.22b), (A.1.22c), (A.1.22d) are simply alternative ways of expressing the absolute difference between $q\alpha$ and its nearest integer, that is $||q\alpha||$. We note that $||q_{n-1}\alpha|| = |q_{n-1}\alpha - p_{n-1}|$ and $||q_{n,i}\alpha|| = |q_{n,i}\alpha - p_{n,i}|, \quad i = 1, 2, \ldots, a_n$. Further, from (A.1.16) or (A.1.17) if $n$ is odd, $q_{n-1}\alpha > p_{n-1}$ while for $n$ even, $q_{n-1}\alpha < p_{n-1}$. Thus,
Further, from (A. 1.13),
\[ q_{n,i} \alpha - p_{n,i} = (q_{n-2} \alpha - p_{n-2}) + i (q_{n-1} \alpha - p_{n-1}). \]  
(A. 1.24)

But, from (A.1.12),
\[ i |q_{n-1} \alpha - p_{n-1}| < |q_{n-2} \alpha - p_{n-2}|, \quad i = 1, 2, \ldots, a_{n-1}. \]  
(A.1.25)

Hence, \( q_{n,i} \alpha - p_{n,i} \) is of the opposite sign to \( q_{n-1} \alpha - p_{n-1} \). Thus, \( q_{n,i} \alpha > p_{n,i} \) for even \( n \) and \( q_{n,i} \alpha < p_{n,i} \) for odd \( n \). From (A.1.22d) it follows that
\[ \|q_{n,i} \alpha\| = \begin{cases} 1 - \{q_{n,i} \alpha\}, & n \text{ odd,} \\ \{q_{n,i} \alpha\}, & n \text{ even.} \end{cases} \]  
(A.1.26)

The same argument implies that
\[ \|q_{n,i} \alpha\| = \|q_{n-2} \alpha\| - i \|q_{n-1} \alpha\|. \]  
(A.1.27)

Also,
\[ \|k q_{n-1} \alpha\| = k \|q_{n-1} \alpha\|, \quad k = 1, 2, \ldots, a_n, \quad (n \geq 1), \]  
(A.1.28)
\[ q_{n,i} \|q_{n-1} \alpha\| + q_{n-1} \|q_{n,i} \alpha\| = 1, \]  
(A.1.29)
\[ p_{n,i} \|q_{n-1} \alpha\| + p_{n-1} \|q_{n,i} \alpha\| = \alpha. \]  
(A.1.30)

If \( \alpha = \{0; a, a, \ldots\} = \left(\sqrt{a^2 + 4} - a\right)/2 \), then
\[ p_n = \frac{\alpha^n + (-1/\alpha)^n}{\alpha + 1/\alpha} = q_{n-1}, \]  
(A.1.31a)
\[ q_n \alpha - p_n = (-\alpha)^{n+1}. \]  
(A.1.31b)

In particular, if \( \alpha = \tau = \{0; 1, 1, \ldots\} = \frac{\sqrt{5} - 1}{2} \), then
\[ p_n = \frac{\tau^n + (-1/\tau)^n}{\sqrt{5}} = q_{n-1}, \]  
(A.1.32a)
Let (for this special case),

\[ q_n \alpha - p_n = (-\tau)^{n+1}. \]  \hspace{1cm} (A.1.32b)

where \( F_1 = 1, F_0 = 0 \). These numbers are the so-called Fibonacci numbers.

\[ p_n = F_n = F_{n-1} + F_{n-2}, \quad n \geq 1. \]  \hspace{1cm} (A.1.32c)

where \( F_{-1} = 1, F_0 = 0 \). These numbers are the so-called Fibonacci numbers.

**A.2 Best approximations to \( \alpha \)**

We are primarily concerned with best approximations of the second kind (BA2's) to \( \alpha \), as defined by Khintchine [63];

\( a/b \) is a BA2 to \( \alpha \) if

\[ \min_{0 < q \leq b} \| q\alpha \| = \| b \alpha \|, \]

where \( q \) is integer. (Note that \( a = [ b \alpha + 1/2 ] \) and the minimum, as usual, is unique).

From Lemma 2.1, it follows that

\[ \min_{0 < q \leq b} \| q\alpha \| = \| q_n \alpha \|, \]  \hspace{1cm} (A.2.1)

where \( n = \max \{ k : q_k \leq b \} \).

To conform with this definition of a BA2 we may replace \( b \) by \( q_n \). Lemma 2.1 thus expresses the fact that the total convergents to \( \alpha \) provide the unique sequence of BA2's to \( \alpha \).

We mention that a best approximation of the first kind (BA1) to \( \alpha \) is a fraction \( a/b \) which is closer to any other fraction with denominator not greater than \( b \) in the sense that

\[ \min_{0 < q \leq b} |\alpha - \frac{p}{q}| = |\alpha - \frac{a}{b}|. \]
It is known that all total, and only some partial, convergents are BA1's. In van Ravenstein et al. [133] we determine the set of fractions which qualify as BA1's in order to introduce the reader to the theory of continued fractions.

A.3 Second best approximations to $\alpha$

We define $c/d$ to be the second best approximation of the second kind $(BA22)$ to $\alpha$ if

$$|d\alpha - c| < |q\alpha - p|, \quad 0 < q \leq d, \quad \frac{p}{q} \neq \frac{a}{b} \quad c, \quad (A.3.1)$$

where $a/b$ is the $BA2$ to $\alpha$ where $b$ satisfies $0 < b \leq d$ and is maximal. That is, if we exclude from consideration the $BA2$ to $\alpha$ ($a/b$) then $c/d$ is the remaining best approximation which we call a $BA22$.

In terms of our usual notation, $c/d$ is a $BA22$ to $\alpha$ if

$$\min_{0 < q \leq d, \quad \frac{p}{q} \neq \frac{a}{b}} \|q\alpha\| = \|d\alpha\|, \quad (A.3.2)$$

where $p = \lfloor q\alpha + 1/2 \rfloor$.

Now, from Lemma 2.1 it is clear that

$$\frac{a}{b} = \frac{P_n}{q_n},$$

where $n = \max \{ k : q_k \leq d \}$.

From Lemma 2.3,

$$\min_{0 < q \leq d, \quad \frac{p}{q} \neq \frac{a}{b}} \|q\alpha\| = \|q_{n+1,i} \alpha\|, \quad i = 0, 1, 2, \ldots, a_{n+1} - 1, \quad (A.3.3)$$

where $i = \max \{ k : q_{n+1,k} \leq d \}$. 
Thus, to conform with the definition of a BA22, we may replace $d$ by $q_{n+1,i}$ for $i = 1, 2, \ldots, a_{n+1} - 1$. (If $i = 0$, the definition of a BA22 is violated since

$$\min_{0 < q \leq d} \|q \alpha\| = \|q_{n-1} \alpha\| \neq \|d \alpha\|,$$

as $d \geq q_n$. We note that if $d = q_n$, we are considering BAl's to $\alpha$).

We conclude that a BA22 to $\alpha$ is necessarily a partial convergent to $\alpha$. 

\[ \]
APPENDIX B

Continued Fraction Geometry

We have shown in detail how the sequence \( \{n\alpha\} = n\alpha \mod 1, \ n = 0, 1, 2, \ldots, N \) is related geometrically to the simple continued fraction expansion of \( \alpha \). Here we examine the continued fraction by partitioning a rectangle of dimension \( \alpha \times 1 \) in the following manner:

Assume that \( 0 < \alpha < 1 \) and that the ratio of the lengths of the shortest to longest sides of the rectangle is equal to \( \alpha : 1 \). Now consider the square of dimension \( \alpha^2 \). Partition the rectangle into an integral number (say \( \alpha_1 \)) of these squares together with a rectangle whose shortest side is of length less than \( \alpha \). We will call this rectangle \( r_1 \). It is of dimension \( \alpha \times k_1 \) where \( k_1 < \alpha \). We let \( t_2 = \alpha/k_1 \) so that \( t_2 \) denotes the ratio of the longest to shortest side of this rectangle.

It is seen that

\[
1 = a_1\alpha + k_1, \\
= a_1\alpha + \frac{\alpha}{t_2}.
\]

Hence,

\[
\alpha = \frac{1}{a_1 + \frac{1}{t_2}}. \tag{B.1}
\]

Now partition the rectangle \( r_1 \) into a number \( (a_2) \) of squares of dimension \( k_1^2 \) along with a rectangle of dimension \( k_1 \times k_2 \) where \( k_2 < k_1 \). This rectangle we call \( r_2 \). Also let \( t_3 = k_1/k_2 \).

Thus,
\[ \alpha = a_2 k_1 + k_2, \]
\[ = a_2 k_1 + \frac{k_1}{t_3}, \]
\[ = a_2 \frac{\alpha}{t_2} + \frac{\alpha}{t_2 t_3}. \]

Hence,
\[ t_2 = a_2 + \frac{1}{t_3}. \]

Therefore, from (B.1),
\[ \alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{t_3}}}. \quad \text{(B.2)} \]

The procedure may be repeated to yield the continued fraction expansion of \( \alpha \). If \( \alpha \) is irrational, the procedure may be repeated infinitely often. If \( \alpha \) is rational, then there exists a finite \( n \) such that \( r_n \) is partitioned into an integral number of squares of dimension \( k_{n-1}^2 \) with no remaining rectangle.
APPENDIX C

The Evaluation of \([N\alpha], N=1,2,\ldots\)

We have shown how one may evaluate the integer parts of positive consecutive multiples of a number by forming its characteristic. Here we present an alternative method by which we decompose the number into terms related to its continued fraction expansion. The method appears in Fraenkel et. al [27] and is central to their paper. We offer a new and shorter proof. The following lemma is proved in Fraenkel et. al [28].

Lemma C1

Suppose that \(n > 0\) and \(0 < q < q_n\). Then, \([q + q_{n-1}\alpha] = p_{n-1} + [q\alpha]\).

The following result is well known (see, for example, Fraenkel [29].) We offer a proof since the result is often stated in the literature but demonstrated less often.

Lemma C2

There is a unique decomposition of any natural number \(N\) in the form

\[ N = \sum_{i=0}^{m} b_i q_i, \]

where the \(b_i\)'s are integers; \(0 \leq b_0 < q_1, 0 \leq b_i \leq a_{i+1}, i > 0\) and \(b_{i-1} = 0\) if \(b_i = a_{i+1}\). The expressions \(a_i\) and \(q_i\) come from the continued fraction expansion of some positive irrational \(\alpha\).

Proof

There exists a non-negative integer \(m\) such that \(q_m \leq N < q_{m+1}\). By the division algorithm, \(N = b_m q_m + N_{m-1}, 0 \leq N_{m-1} < q_m\), where \(b_m \leq a_{m+1}\) since \(a_{m+1} + \)

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where the \(b_i\)'s are integers; \(0 \leq b_0 < q_1, 0 \leq b_i \leq a_{i+1}, i > 0\) and \(b_{i-1} = 0\) if \(b_i = a_{i+1}\). The expressions \(a_i\) and \(q_i\) come from the continued fraction expansion of some positive irrational \(\alpha\).

Proof

There exists a non-negative integer \(m\) such that \(q_m \leq N < q_{m+1}\). By the division algorithm, \(N = b_m q_m + N_{m-1}, 0 \leq N_{m-1} < q_m\), where \(b_m \leq a_{m+1}\) since \(a_{m+1} + \)
1) \( q_m \geq q_{m+1} > N \). Similarly, \( N_{m-1} = b_{m-1}q_{m-1} + N_{m-2}, \ 0 \leq N_{m-2} < q_{m-1} \), where \( b_{m-1} \leq a_m \).

Repeating the process gives the unique decomposition

\[
N = \sum_{i=0}^{m} b_i q_i, \quad b_i \leq a_{i+1}.
\]

Now, if \( b_j = a_{j+1} \) then \( N_j = a_{j+1}q_j + N_{j-1}, \ 0 \leq N_{j-1} < q_j \), where \( 0 \leq N_j < q_{j+1} \). Thus, \( N_{j-1} < q_{j-1} \) since \( a_{j+1}q_j + q_{j-1} > N_j \). But \( N_{j-1} = b_{j-1}q_{j-1} + N_{j-1}, \ 0 \leq N_{j-2} < q_{j-1} \). Therefore, \( b_{j-1} = 0 \).

A result which follows from the fact that the expansion is unique is that

\[
\sum_{i=0}^{n} b_i q_i < q_{n+1}.
\]

This may also be verified by induction.

**Theorem C2**

Let \( N = \sum_{i=k}^{m} b_i q_i \). Then

\[
[N \alpha] = \begin{cases} 
\sum_{i=k}^{m} b_i p_i, & k \text{ even}, \\
-1 + \sum_{i=k}^{m} b_i p_i, & k \text{ odd},
\end{cases}
\]

where \( b_k \neq 0 \) (ie. \( k = \max \{ j : b_j > 0 \} \))

**Proof** If \( N = \sum_{i=k}^{m} b_i q_i, \ b_k \neq 0 \), then

\[
[N \alpha] = \begin{bmatrix} 
\left( \sum_{i=k}^{m-1} b_i q_i + b_m q_m \right) \alpha, \\
\left( \sum_{i=k}^{m-1} b_i q_i + (b_m - 1)q_m + q_m \right) \alpha
\end{bmatrix}.
\]
By equation (C.1),
\[
\sum_{i=k}^{m-1} b_i q_i + (b_m - 1)q_m < b_m q_m \leq a_{m+1}q_m < q_{m+1}.
\]
Hence, by Lemma C1,
\[
\lfloor N \alpha \rfloor = p_m + \left[ \sum_{i=k}^{m-1} b_i q_i + (b_m - 1)q_m \right] \alpha.
\]
Further application of (C.1) and Lemma C1 leads to
\[
\lfloor N \alpha \rfloor = b_m p_m + \left[ \sum_{i=k}^{m-1} b_i q_i \right] \alpha.
\]
Clearly we are led to
\[
\lfloor N \alpha \rfloor = \sum_{i=k+1}^{m} b_m p_m + [b_k q_k \alpha].
\]
From the theory of continued fractions (equation (A.1.16)),
\[
b_k q_k \alpha - b_k p_k = \frac{b_k (-1)^k}{t_{k+1}p_k + p_{k+1}},
\]
where \( t_{k+1} = \{ a_{k+1}; a_{k+2}, a_{k+3}, \ldots \} \). Thus, \(-1 < b_k (q_k \alpha - p_k) < 1\), since \( 0 \leq b_k \leq a_{k+1} \).
Hence,
\[
[b_k q_k \alpha] = \begin{cases} 
  b_k p_k, & \text{if } k \text{ even}, \\
  b_k p_k - 1, & \text{if } k \text{ odd}.
\end{cases}
\]
This completes the proof. 

APPENDIX D

Diffusion on a ring

In this appendix we determine the concentration of diffusant on a ring which is supplied at a known rate at some point. We first examine the steady state or time independent case considered by Thomley [125] in his formulation of a one-dimensional model of Scoute's hypothesis (see the introduction). We then consider the transient or time independent situation with the ring fixed and then allowed to vary in time.

D.1 Time independent case

Thomley [125] considered the following problem of steady state diffusion in a ring with a constant source of diffusant (in this case morphogen) supplied at some arbitrary origin:

\[ D \frac{d^2 M}{dx^2} - \hat{k}_2 M = 0, \quad 0 < \hat{x} < \hat{x}_m, \quad \text{(D.1.1)} \]

\[ M(0) = M(\hat{x}_m), \quad \text{(D.1.2)} \]

\[ \frac{S}{2} = -D \frac{dM}{d\hat{x}}, \quad \hat{x} = 0^+, \quad \text{(D.1.3)} \]

where \( M(\hat{x}) \) is the (steady state) concentration of morphogen at co-ordinate position \( \hat{x} \), \( D \) is the diffusion coefficient, \( \hat{k}_2 \) is the rate of degradation of morphogen, \( \hat{x}_m \) is the circumference of the ring, and \( S \) is the source strength.

These equations may be readily solved to yield the solution,

\[ M(\hat{x}) = \frac{S \cosh \left( \sqrt{\hat{k}_2/D} \left( \hat{x} - \hat{x}_m/2 \right) \right)}{2\sqrt{\hat{k}_2 D} \sinh \left( \hat{x}_m \sqrt{k/4D} \right)}. \quad \text{(D.1.4)} \]
D.2 Time dependent case

We now consider time-dependent diffusion in a ring described by the following equations:

\[
\frac{\partial \hat{c}}{\partial t} = D \frac{\partial^2 \hat{c}}{\partial x^2} - k_1 \hat{c} + k_2 e^{-\alpha t} \delta(\hat{x}), \quad 0 \leq \hat{x} \leq 2\pi x_m, \quad (D.2.1)
\]

\[
\hat{c}(\hat{x}, 0) = \hat{f}(\hat{x}), \quad (D.2.2)
\]

\[
\hat{c}(0, t) = \hat{c}(2\pi x_m, t), \quad (D.2.3)
\]

where \(\delta(\hat{x})\) is the usual Dirac delta function.

As in the steady-state formulation a first-order reaction term is present. Now however, we have let the point source strength decay exponentially with time.

The notation is similar to that used in the steady state problem, but now with the following changes and additions: \(\hat{c}(\hat{x}, \hat{t})\) is the concentration of morphogen at time \(\hat{t}\) and co-ordinate position \(\hat{x}\), \(k_1\) is the initial source strength, \(x_m\) is the radius of the ring, and \(\hat{f}(\hat{x})\) is the initial concentration of morphogen.

We work with the following dimensionless variables

\[
x = \frac{\hat{x}}{x_m}, \quad \hat{t} = \frac{D \hat{t}}{x_m^2}, \quad (D.2.4)
\]

and \(c(x, t) = \hat{c}(\hat{x}, \hat{t})\) so that problem (D.2.1) - (D.2.3) becomes

\[
\frac{\partial c}{\partial \hat{t}} = D \frac{\partial^2 c}{\partial x^2} - k_2 c + k_1 e^{-\alpha \hat{t}} \delta(x), \quad 0 \leq x \leq 2\pi, \quad (D.2.5)
\]

\[
c(x, 0) = f(x), \quad (D.2.6)
\]

\[
c(0, \hat{t}) = c(2\pi, \hat{t}), \quad (D.2.7)
\]

where

\[
(k_1, k_2, \alpha) = \frac{x_m^2}{D} \left(\hat{k}_1, \hat{k}_2, \hat{\alpha}\right). \quad (D.2.8)
\]
D.2.1 Method of solution

We simplify the problem by the use of Dankwert’s transformation,

\[ u(x, t) = c(x, t)e^{k_2t}. \] (D.2.9)

Problem (D.2.5) - (D.2.7) becomes

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + k_1e^{(k_2-\alpha)t} \delta(x), \quad 0 \leq x \leq 2\pi, \tag{D.2.10}
\]

\[ u(x, 0) = f(x), \] (D.2.11)

\[ u(0, t) = u(2\pi, t). \] (D.2.12)

Now let \( u(x, t) = v(x, t) + w(x, t) \) where \( v(x, t) \) satisfies the homogeneous problem,

\[
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \tag{D.2.13}
\]

\[ v(x, 0) = f(x), \] (D.2.14)

\[ v(0, t) = v(2\pi, t), \] (D.2.15)

and \( w(x, t) \) satisfies the following:

\[
\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + k_1e^{(k_2-\alpha)t} \delta(x), \tag{D.2.16}
\]

\[ w(x, 0) = 0, \] (D.2.19)

\[ w(0, t) = w(2\pi, t). \] (D.2.18)

Equations (D.2.13) - (D.2.15) describe Fourier’s problem of the ring (see Widder [141], p.100 or Carslaw and Jaeger [10], p.160). The solution for \( v(x, t) \) is

\[ v(x, t) = \int_0^{2\pi} \theta(x - r, t)f(r)dr, \] (D.2.19),

where

\[
\theta(x, t) = \sum_{n=-\infty}^{\infty} k(x + 2n\pi, t), \tag{D.2.20}
\]

\[ k(x, t) = \begin{cases} 
\frac{1}{2\sqrt{\pi t}}e^{-x^2/4t}, & t > 0, \\
0, & t \leq 0. \end{cases} \tag{D.2.21} \]
To solve for \( w(x, t) \) "roll out" the ring into a straight line and extend its domain to the entire real line. To satisfy the periodicity condition (D.2.18) the source located at \( x = 0 \) is duplicated at all integer multiples of \( 2\pi \). This approach is equivalent to the method of images (see Carslaw and Jaeger [10]).

We thus solve
\[
\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + F(x, t), \quad -\infty < x < \infty, \tag{D.2.22}
\]
\[
w(x, 0) = 0, \tag{D.2.23}
\]
where
\[
F(x, t) = k_1 e^{(k_2 - \alpha)t} \sum_{n=-\infty}^{\infty} \delta(x + 2n\pi). \tag{D.2.24}
\]
The production term \( F(x, t) \) ensures that \( w(x, t) \) is periodic in \( x \).

A solution to this problem in terms of the Green's function \( k(x, t) \) is
\[
w(x, t) = k_1 \int_0^t e^{(k_2 - \alpha)(t - r)} \theta(x, t - r) dr, \tag{D.2.25}
\]
\[
= k_1 \int_0^t \frac{e^{(k_2 - \alpha)(t - r)}}{2\pi \sqrt{\pi(t - r)}} \sum_{n=-\infty}^{\infty} e^{-(x + 2n\pi)^2/4(t - r)} dr, \tag{D.2.26}
\]
\[
= \frac{k_1}{2} \sum_{n=-\infty}^{\infty} \int_0^t e^{(k_2 - \alpha)(t - r)} \frac{e^{-(x + 2n\pi)^2/4r}}{\sqrt{\pi r}} dr. \tag{D.2.27}
\]

Let \( w^*(x, p) \) denote the Laplace transform of \( w(x, t) \) with respect to \( t \) so that
\[
w^*(x, p) = \int_0^{\infty} e^{-pt} w(x, t) dt. \tag{D.2.29}
\]
Applying Laplace transforms to (D.2.27) yields by the convolution theorem
\[
w^*(x, p) = \frac{k_1}{2} \sum_{n=-\infty}^{\infty} \frac{e^{-\sqrt{p}(x + 2\pi)}}{\sqrt{p(p + \alpha - k_2)}}. \tag{D.2.30}
\]
Inverting \( w^*(x, p) \) gives
\[
w(x, t) = \frac{k_1 e^{(k_2 - \alpha)t}}{4\sqrt{k_2 - \alpha}} \sum_{n=-\infty}^{\infty} \left( e^{-(x + 2n\pi)/(2\sqrt{t - \sqrt{k_2 - \alpha}})} \text{erfc} \left( \frac{x + 2n\pi}{2\sqrt{t - \sqrt{k_2 - \alpha}}} \right) - e^{-(x + 2n\pi)/(2\sqrt{t + \sqrt{k_2 - \alpha}})} \text{erfc} \left( \frac{x + 2n\pi}{2\sqrt{t + \sqrt{k_2 - \alpha}}} \right) \right). \tag{D.2.30}
\]
where \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-r^2} \, dr \). Thus,

\[
c(x, t) = e^{-k_2 t} \mu(x, t),
\]

\[
= e^{-k_2 t} \int_{0}^{2\pi} \theta(x - r, t) f(r) \, dr
\]

\[
+ \frac{k_1 e^{-\alpha t}}{4\sqrt{k_2 - \alpha}} \sum_{n=\infty}^{\infty} \left( e^{-\frac{(x+2n\pi)^2}{k_2 - \alpha}} \text{erfc} \left( \frac{x + 2n\pi}{2\sqrt{\alpha}} \right) \right),
\]

(D.2.31)

an expression which converges quickly for small \( t \).

An alternative expression for \( c(x, t) \), and one that converges well for large \( t \), may be found by expressing \( \theta(x, t) \) in terms of its Fourier series,

\[
\theta(x, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \cos nx.
\]

(D.2.32)

From (D.2.25) we find that

\[
w(x, t) = \frac{k_1 (e^{(k_2 - \alpha) t} - 1)}{2\pi k_2 - \alpha} + \frac{k_1}{\pi} \sum_{n=1}^{\infty} \frac{e^{(k_2 - \alpha) t} - e^{-n^2 t}}{n^2 + k_2 - \alpha} \cos nx.
\]

(D.2.33)

Thus,

\[
c(x, t) = e^{-k_2 t} \int_{0}^{2\pi} \theta(x - r, t) f(r) \, dr
\]

\[
+ \frac{k_1(e^{-\alpha t} - e^{-k_2 t})}{2\pi k_2 - \alpha} + \frac{k_1}{\pi} \sum_{n=1}^{\infty} \cos nx \left( e^{-\alpha t} - e^{-\left(n^2 + k_2\right) t} \right).
\]

(D.2.34)

In the limit as \( t \) tends to infinity, \( c(x, t) \) goes to zero for positive \( \alpha \). We suppose that \( \alpha = 0 \) so that the source strength of the morphogen is constant. Then,

\[
\lim_{t \to \infty} c(x, t) = \frac{k_1}{2\pi k_2} + \frac{k_1}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + k_2},
\]

(D.2.35)

\[
= \frac{k_1}{2\sqrt{k_2}} \frac{\cosh \sqrt{k_2 (\pi - x)}}{\sinh \sqrt{k_2 \pi}}.
\]

(D.2.36)
Note that (D.2.35) is simply the Fourier series expansion of (D.2.36), which is the steady state solution (for \( \alpha = 0 \)), and satisfies the following problem:

\[
k_2 \hat{u} = \frac{d^2 \hat{u}}{dx^2} + k_1 \delta(x), \quad 0 \leq x \leq 2\pi,
\]

\[
\hat{u}(0) = \hat{u}(2\pi), \quad (D.2.38)
\]

\[
\hat{u}'(0) = -\hat{u}'(2\pi), \quad (D.2.39)
\]

where \( \hat{u}(x) = \lim_{t \to \infty} c(x, t) \). This is perhaps a more attractive statement of the steady state diffusion problem than that of Thornley, and one that is readily amenable to solution via the use of Finite Fourier transforms. The prime denotes differentiation with respect to \( x \).

### D.3 Expanding Ring

We examine a similar problem defined over the region

\[
\{(x, t): 0 < x < 2R(t), \quad 0 < t < t_0\},
\]

where \( t_0 \) is a positive constant. Thus we consider diffusion in an expanding ring which has circumference \( 2R(t) \) at time \( t \). In particular we study the following set of equations,

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + k_1 e^{k_2 t} \delta(x), \quad 0 < x < 2R(t),
\]

\[
u(x, 0) = 0, \quad (D.3.2)
\]

\[
u(2R(t), t) = \nu(0, t). \quad (D.3.3)
\]

We extend the domain of \( u \) to the real line and, in order to satisfy the periodicity condition (D.3.3), the source term at \( x = 0 \) is duplicated at all integer multiples of \( 2R(t) \). We now solve

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(x, t), \quad -\infty < x < \infty, \quad (D.3.4)
\]

\[
u(x, 0) = 0, \quad (D.3.5)
\]

where

\[
F(X, t) = k_1 e^{k_2 t} \sum_{n=-\infty}^{\infty} \delta(x + 2nR(t)). \quad (D.3.6)
\]
We are left with the (equivalent) problem of ascertaining the concentration of morphogen in an infinite one dimensional medium, affected by the presence of moving sources of morphogen situated at all integer multiples of $2R(t)$. 

Note that the concentration of morphogen at time $t$ and co-ordinate position $x$ due to the production of a quantity of morphogen, $\Phi(r)\, dr$, emitted at time $r$ and position $2nR(r)$, is

$$
\frac{\Phi(r)\, dr}{2\sqrt{\pi(t-r)}} e^{-(x-2nR(t-r))^2/4(t-r)}.
$$

(D.3.7)

Summing from $r = 0$ to $r = t$ gives

$$
\int_0^t \frac{\Phi(t)}{2\sqrt{\pi(t-r)}} e^{-(x-2nR(t-r))^2/4(t-r)} \, dr,
$$

which is the contribution to the total concentration from the source at $x = 2nR(r)$ from time $r = 0$ to $r = t$. Thus the concentration of morphogen due to an infinite number of sources situated at $x = 2nR(r), n = 0, \pm 1, \pm 2, \ldots$, each producing a quantity of morphogen, $k_1 e^{k_2r}$, at time $r$ (from $r = 0$ to $r = t$) is

$$
\frac{k_1}{2\sqrt{\pi}} \int_0^t \frac{e^{k_2r}}{\sqrt{t-r}} \sum_{n=-\infty}^{\infty} e^{-(x+2nR(t-r))^2/4(t-r)} \, dr,
$$

$$
= \frac{k_1}{2\sqrt{\pi}} \int_0^t \frac{e^{k_2(t-r)}}{\sqrt{r}} \sum_{n=-\infty}^{\infty} e^{-(x+2nR(r))^2/4(r)} \, dr,
$$

$$
= \frac{k_1}{2\pi} \int_0^t \frac{e^{k_2(t-r)}}{R(r)} \left(1 + 2 \sum_{n=1}^{\infty} \cos \frac{nx}{R(r)} e^{-n^2r/R^2(r)}\right) \, dr.
$$

(D.3.9)

The last step follows from Poisson's summation formula. Progress from here is difficult without knowing the explicit form of $R(r)$. 

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PUBLICATIONS


Two papers are currently being prepared for publication. The first ("A Characteristic of The Three Gap Theorem," to be submitted to *Discr. Math*) deals largely with material from Chapters 3 and 4. The second paper ("A Model of Phyllotaxis," to be submitted to *Nature*) is similar in content to Section 6.3.