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Exact solutions of nonlinear diffusion-convection equations

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Exact solutions of nonlinear diffusion-convection equations

A thesis submitted in fulfillment of the requirements for the award of the degree of

Doctor of Philosophy

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by

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Department of Mathematics

1997
This thesis is submitted to the University of Wollongong, and has not been submitted for a degree to any other University or Institution.

Maureen Edwards

March, 1997
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Abstract

We consider the class of nonlinear diffusion-convection equations which contain two arbitrary functions of the dependent variable. We perform a thorough symmetry analysis of the general equation in one, two and three spatial dimensions. We identify all special forms of the two arbitrary functions which admit special symmetry properties and for these cases, attempt to reduce the governing equation to an ordinary differential equation. We show that reduction of the governing equation to an ordinary differential equation is possible in many cases. We seek solutions to the reduced equations and hence are able to construct time-dependent similarity solutions to the governing equation.

We extend a previously derived method for reducing a power law case of our governing equation through our knowledge of the symmetries of the class of equations. As a result, we construct an infinite family of time-dependent solutions satisfying nonsingular initial conditions for special cases of the governing equation in both two and three dimensions.

Finally, we develop an inverse method by exploiting a linearisable form of our governing equation in one spatial dimension. The method is used to derive two solutions with distinct variable flux boundary conditions for an unsaturated soil.
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Introduction

1.1 Overview

In 1986, in a special volume of Water Resources Research, Garrison Sposito [81] closed his discussion on the physics of soil water physics with the following statement:

The present essay is by no means a comprehensive review, but it will have served a useful purpose if it helps dispel the notion that the physics of soil water physics was a settled scientific problem after 1931. Some merit might be accorded the proposition that is fact, little enough work on the physics has been done during the past 55 years. The present discussion can be considered in support of this proposition because it has raised, at least implicitly, three basic issues that demand resolution before our understanding of soil water physics can be regarded as satisfactory.

1. What are the possible groups of similarity transformations of the Richards Equation and how may they be used to classify the behavior of water in soils? The practical significance of this question can not be overestimated. In the example represented by (13), the implication is that if water movement is investigated experimentally in any one soil in

1
the class of Warrick-similar soils, water movement in all other soils in the class can be predicted [Warrick et al., 1985].

The remaining fundamental questions posed by Sposito are

2. *What is the most general form of the law of internal energy balance for soil water that is consistent with the Richards Equation?*

and

3. *What definition of the heating flux density vector will lead to a predictive model of coupled heat and water flow in soil that is both self-consistent and experimentally testable?*

In this thesis, we are ultimately interested in constructing exact time-dependent solutions to the class of nonlinear diffusion-convection equations in one, two and three spatial dimensions. In doing so, we provide a partial answer to the first fundamental question posed by Sposito [81]. In the past, researchers have often relied on trial and error in the search for specific types of solutions, for example, testing if an equation admits a travelling wave solution, or use of the Boltzmann similarity variable to reduce the number of independent variables in the general nonlinear diffusion equation. However, these seemingly *ad hoc* methods have a common underlying approach - the use of one-parameter groups of point symmetries.

In the late nineteenth century, the Norwegian mathematician Sophus Lie developed the theory of continuous transformation groups (Lie groups). This theory describes invertible point transformations of a differential equation (D.E.) which map every solution of the D.E. to another solution of the D.E. Of course, the assumption here is that a given D.E. has a known solution which will lead to the generation of a second, hopefully new, solution. In the event that we do not know the solution space of a given D.E. in advance, we can consider instead the equivalent
problem of finding the one-parameter groups of point symmetries which leave the
equation itself invariant.

Throughout this thesis we consider a specific class of partial differential equation
(P.D.E.) - the nonlinear diffusion-convection equation. Although only certain forms
of the diffusion-convection equation are significant physically, we consider the general
class of equations. Hence our governing equation involves two arbitrary functions of
the dependent variable. We rely on Lie point symmetry analysis of the governing
equation to flag cases which possess symmetry properties extra to the general case,
with the aim of forming similarity (self similar) solutions.

In the first part of this thesis, we outline the standard theory of Lie symmetry
methods for D.E.'s. The material contained in Chapter 2 is intended as an overview,
although some simple examples have been included for clarity. The ideas introduced
in this chapter form the framework for the analysis performed in the remainder of the
thesis. We close the chapter with a brief discussion of some generalised symmetry
techniques, however, these techniques are included for completeness and are not
exploited in this thesis.

In Chapter 3, we consider the class of nonlinear diffusion-convection equations
in one spatial dimension. The symmetry analysis for this class of equations is con­sidered
where the two free functions \( D(u) \) and \( K(u) \), occurring in the P.D.E. as
nonlinear coefficients, are arbitrary. Every form of the diffusion-convection equation
possesses symmetries which are translations in space and time - we list the special
cases of \( D \) and \( K \) which admit symmetries in addition to the trivial translational
symmetries. The optimal system of infinitesimal Lie symmetry operators for each
of the special cases is determined and hence a minimal complete set of reductions
is found. Although several authors have previously performed symmetry analyses
on this class of equations, our study surprisingly reveals that all of the previous	tabulations are incomplete.
Chapter 1. Introduction

Following the symmetry analysis of the one-dimensional diffusion-convection equation, the natural question to consider is which, if any, of the one-dimensional special cases will carry through to higher dimensions. In Chapter 4 we find that all but one of the one-dimensional special cases are also special in both the two and three-dimensional equation. For the two-dimensional equation, we need to reduce from three to one the number of independent variables to obtain an ordinary differential equation (O.D.E.). To achieve this, we determine the optimal system for each of the special cases and use the nontrivial symmetries from the optimal system to reduce by one the number of independent variables. The symmetry analysis of each of the reduced equations is performed, the new optimal system is determined and the equation is reduced to an O.D.E. We find that we are able to construct exact time-dependent similarity solutions of the two-dimensional diffusion-convection equation for two special combinations of $D$ and $K$. For two other special cases, we show that the form of the solution of the governing equation will be time-independent, while in the remaining cases, the O.D.E.'s have no known explicit solution.

We also construct an exact solution to a special form of the two-dimensional diffusion-convection equation in cylindrical polar coordinates by exploiting our knowledge of the symmetries of the equation to reduce by one the number of independent variables. However, a second reduction is then achieved using a physical constraint which does not arise as a result of point symmetry analysis. The solution obtained involves an arbitrary parameter and so we in fact contribute an infinite family of new solutions.

The second half of Chapter 4 involves the equivalent analysis of the three-dimensional diffusion-convection equation. To reduce the governing equation to an O.D.E. we need to reduce from four to one the number of independent variables. We demonstrate a reduction of the three-dimensional equation for one of the special combinations of $D$ and $K$ which admitted a time-dependent solution of the
two-dimensional equation. We show that a similar solution of the three-dimensional equation (involving the additional space variable) is possible. In addition we reduce a special form of the three-dimensional governing equation in spherical coordinates to an O.D.E. The first reduction is achieved using the symmetries of the governing equation. However, as in the two-dimensional case, the remaining reductions are not able to be explained by point symmetry analysis but are due to a pair of physical constraints. The solution obtained involves an arbitrary parameter and hence we again contribute an infinite family of new solutions.

From the symmetry analysis performed in Chapter 4, we showed that the Burgers' equation in two and three spatial dimensions admits symmetry properties additional to any other form of the diffusion-convection equation. In Chapter 5, we exploit these exceptionally large symmetry groups to obtain explicit nonsingular solutions to this nonlinear equation. In both the two and three-dimensional case, we determine the optimal system, reduce the number of independent variables by one and consider the symmetry analysis of the reduced equation. In this manner, we are able to reduce the two and three-dimensional Burgers' equation to various types of O.D.E.'s. In addition, we show that a number of the reduced equations are linear. We note that these linear P.D.E.'s could be solved using integral transform methods.

The linearisablity of a special form of the diffusion-convection equation in one dimension is exploited in Chapter 6 to obtain new meaningful solutions. The method we introduce allows us to construct exact flow solutions with continuously varying flux boundary conditions for an unsaturated soil. We demonstrate the sequence of transformations in our inverse approach for two examples; for the first solution, the varying water flux boundary condition resembles the passage of a peaking storm while for the second solution the varying water flux boundary condition suggests the continuous opening of a valve preceding a steady water supply.
In closing, we summarise our results in Chapter 7 and we outline some possible future directions for our research.

### 1.2 Physical motivation

Although more general conceptual schemes are being considered (Gray and Hassanzadeh [40]), the continuum approach to soil water modelling, based on extensions to Darcy’s law, continues to enjoy popular support many decades after its inception.

For one-dimensional vertical flow, a combination of the equation of continuity for conservation of water mass

\[
\frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial z},
\]

together with the Buckingham-Darcy law for unsaturated flow

\[
q = -D(\theta) \frac{\partial \theta}{\partial z} + K(\theta),
\]

leads to the one-dimensional diffusion-convection equation

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ D(\theta) \frac{\partial \theta}{\partial z} \right] - \frac{dK}{d\theta} \frac{\partial \theta}{\partial z}.
\]

(1.1) capillarity gravity

Equation (1.1) describes nonlinear processes arising in many physical situations, including porous media flow and diffusion in semiconductors. In the context of the flow of water through soil, the terminology used is

- \(\theta\) is volumetric water content,
- \(t\) is time,
- \(z\) is depth below the soil surface,
- \(D(\theta)\) is the concentration-dependent soil-water diffusivity and \(K(\theta)\) is the concentration-dependent hydraulic conductivity.
For soil moisture flow, $0 \leq \theta \leq 1$, where $\theta = 0$ represents a dry medium and $\theta = 1$ corresponds to a saturated medium. In typical repacked laboratory soils, $D(\theta)$, $D'(\theta)$, $K(\theta)$ and $K'(\theta)$ are strongly increasing concave functions. In dry soils, capillarity is much stronger than gravity. Hence at early wetting times we may neglect the nonlinear convection term to obtain the general nonlinear diffusion equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ D(\theta) \frac{\partial \theta}{\partial z} \right].$$

The general nonlinear diffusion-convection equation (1.1) will always admit a travelling wave solution

$$\theta = F(z - ct),$$

where $c$ is the speed of propagation of the wave. For many combinations of initial and boundary conditions, the solutions approach the travelling wave solution as $t \to \infty$. For higher-dimensional diffusion-convection equations

$$\frac{\partial \theta}{\partial t} = \nabla \cdot \left[ D(\theta) \nabla \theta \right] - \frac{dK}{d\theta} \frac{\partial \theta}{\partial z}, \quad (1.2)$$

equivalent travelling wave solutions may be found involving the additional space variables $x$ and $y$. The existence of these types of solutions is demonstrated in Chapters 3 and 4, where symmetry analysis of the diffusion-convection equation in one, two and three dimensions reveals that translation symmetries in time and in the corresponding space variables, exist for arbitrary $D$ and $K$.

Although we are interested in the general class of nonlinear diffusion-convection equations in one, two and three spatial dimensions, we will restrict the forms of $D$ and $K$ so that (1.2) does remain nonlinear. Hence we will not consider the case that $D$ and $K$ are both constant functions. In addition, we will also ignore the case $D$ general and $K$ a linear function, as in this instance we may transform (1.2) to a purely diffusive case by a change of reference frame. The general class of nonlinear diffusion equations has been thoroughly analysed by Galaktionov et al. [38].
Because of the nonlinearity of (1.2) together with the complicated boundary conditions occurring during field measurements or laboratory experiments, numerical approximation schemes have been required to solve many unsaturated flow problems. However, exact and approximate analytical methods have also played an important role in elucidating basic quantitative relationships among measurable quantities, in providing very efficient procedures for evaluating solutions to fundamental boundary value problems and in testing more generally applicable numerical solution schemes. We refer to the review by Philip [63] and to some additional work on similarity solutions [31] and on perturbation expansions [87]. The aim of this thesis is to extend the class of known exact time-dependent solutions to the diffusion-convection equation in one, two and three dimensions, by a thorough systematic analysis.
Chapter 2

One-parameter transformation groups

In this chapter we outline some ideas of Lie symmetry methods for D.E.'s. The theory of continuous transformation groups (Lie groups) was developed by Sophus Lie in the late nineteenth century and deals with point symmetries of a D.E. - that is, invertible point transformations which map every solution of the D.E. to another solution of the D.E. Lie showed that the symmetries of a D.E. form a group. In Chapters 3, 4 and 5, we are interested in the application of symmetry groups to the class of nonlinear diffusion-convection equations in one, two and three spatial dimensions to construct invariant (or similarity) solutions. Symmetry groups may also be applied to D.E.'s in the search for linearizing transformations. In Chapter 6, a transformation which linearises a special form of the one-dimensional diffusion-convection equation is exploited to derive new solutions. The material contained in this chapter is intended as an overview only. The reader is referred to Bluman and Kumei [7], Olver [57] and Ovsiannikov [59] for greater detail of the theory.
2.1 One-parameter transformation groups

Let \( x = (x_1, x_2, \ldots, x_n) \) lie in \( L \subseteq \mathbb{R}^n \). For each \( x \in L \), we consider a set of transformations

\[
x^* = X(x; \epsilon),
\]

depending on the parameter \( \epsilon \in J \subseteq \mathbb{R} \). Let \( \ast \) be a binary operation on \( J \). Then (2.1) forms a **group of transformations** on \( L \) if:

(i) \((J, \ast)\) forms a group (that is, the closure, identity, associativity and inversion properties of a group are satisfied),

(ii) for each \( \epsilon \in J \), the transformations are one-to-one onto \( L \),

(iii) when \( \epsilon \) is the group identity \( e \), \( x^* = x \); that is \( X(x; e) = x \), and

(iv) for \( \epsilon, \delta \in J \), \( X(X(x; \epsilon); \delta) = X(x; \epsilon \ast \delta) \).

A one-parameter group of transformations, in addition to satisfying the above four properties of a group, satisfies the following:

(v) \( \epsilon \) is a continuous parameter, that is, for any \( \epsilon, \delta \in J \), if \( \epsilon \leq \gamma \leq \delta \), then \( \gamma \in J \),

(vi) \( X \) is infinitely differentiable with respect to \( x \) in \( L \) and an analytic function of \( \epsilon \) in \( J \),

(vii) \( \epsilon \ast \delta \) is an analytic function on \( J \times J \).

There exists a variety of classes of one-parameter Lie groups of transformations, including scalings and translations. An example of a one-parameter Lie group of transformations in \( \mathbb{R}^2 \) is the rotation transformation

\[
x_1^* = x_1 \cos \epsilon - x_2 \sin \epsilon, \quad x_2^* = x_1 \sin \epsilon + x_2 \cos \epsilon.
\]

We now consider a one-parameter group of transformations which act on \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \). Expanding in the neighbourhood of \( \epsilon = 0 \), the transformation (2.1) may be written as

\[
x^* = x + \epsilon \left( \frac{\partial X}{\partial \epsilon} (x; \epsilon) \right)_{\epsilon=0} + \frac{\epsilon^2}{2} \left( \frac{\partial^2 X}{\partial \epsilon^2} (x; \epsilon) \right)_{\epsilon=0} + \ldots
\]

\[
= x + \epsilon X(x) + O(\epsilon^2),
\]

(2.2)
where $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n)$ is given by

$$\mathcal{X}_i = \left. \frac{\partial X_i}{\partial \epsilon} \right|_{\epsilon=0}, \quad 1 \leq i \leq n.$$

The transformation (2.3) is called the **infinitesimal transformation** of the Lie group of transformations (2.1), while the quantities $\mathcal{X}_i$ are called the **infinitesimals** of the one-parameter group.

Given an infinitesimal transformation (2.3), we can construct the corresponding one-parameter group by solving the O.D.E. initial value problem

$$\frac{dx^*}{d\epsilon} = \mathcal{X}(x^*), \quad x^*(0) = x.$$ 

For example, from the rotation transformation (2.2), the infinitesimal transformations are

$$x_1^* = x_1 + \epsilon x_2 + O(\epsilon^2), \quad x_2^* = x_2 - \epsilon x_1 + O(\epsilon^2). \quad (2.4)$$

Hence solving the initial value problem

$$\frac{dx_1^*}{d\epsilon} = x_2, \quad x_1^*(0) = x_1,$$

$$\frac{dx_2^*}{d\epsilon} = -x_1, \quad x_2^*(0) = x_2,$$

recovers the original one-parameter group. This is the first fundamental theorem of Lie. For a formal statement of the theorem and proof see [7], [57], [59].

We define the **group operator** $\Gamma$ of a one-parameter group with infinitesimals $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n)$ as the first order differential operator

$$\Gamma = \mathcal{X}_i(x) \frac{\partial}{\partial x_i}. \quad (2.5)$$

As an example, the group operator which corresponds to (2.2) is

$$\Gamma = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}.$$
The terms infinitesimal operator, Lie operator and group generator may also be used to describe the vector field (2.5). We will generally denote group operators as $\Gamma_1$, $\Gamma_2$, etc. An $n$-parameter Lie transformation group has $n$ associated group operators $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ which are linearly independent.

If we let $\Gamma_1$ and $\Gamma_2$ be two group operators, their commutator $[\Gamma_1, \Gamma_2]$ is the first order operator

$$[\Gamma_1, \Gamma_2] = \Gamma_1 \Gamma_2 - \Gamma_2 \Gamma_1.$$ 

An immediate consequence of this definition is the anticommutative property, that is

$$[\Gamma_1, \Gamma_2] = -[\Gamma_2, \Gamma_1].$$

A function $F(x)$ is an invariant function of the Lie group of transformations (2.1) if and only if for any $\epsilon \in J$,

$$F(x^*) \equiv F(x),$$

(2.6)

when $X^* = X(x; \epsilon)$. If $F(x)$ satisfies (2.6), then $F(x)$ is called an invariant of (2.1) and we say that $F(x)$ is invariant under (2.1).

The function $F(x)$ is invariant under (2.1) if and only if

$$\Gamma F(x) \equiv \chi_i \frac{\partial F}{\partial x_i} = 0.$$ 

In the two-dimensional case, a function $F$ is invariant if

$$F(x_1^*, x_2^*) \equiv F(x_1, x_2).$$

Hence $F$ will satisfy

$$\Gamma F(x_1, x_2) = \chi_1 \frac{\partial F}{\partial x_1} + \chi_2 \frac{\partial F}{\partial x_2} = 0.$$ 

In this case, the one-parameter group of point transformations has one independent invariant. We can find this invariant by solving the characteristic equation

$$\frac{dx_1}{\chi_1(x_1, x_2)} = \frac{dx_2}{\chi_2(x_1, x_2)}.$$
For example, consider the one-parameter group of transformations (2.4). The corresponding characteristic equation is
\[ \frac{dx_1}{x_2} = \frac{dx_2}{-x_1}, \]
with solution \( x_1^2 + x_2^2 = C^2 \), where \( C^2 \in \mathbb{R}^+ \) is the arbitrary constant of integration. Hence our invariant could be
\[ r = \sqrt{x_1^2 + x_2^2}. \]
Any other invariant is a function of \( r \), for example
\[ r^2 = x_1^2 + x_2^2 \]
is also an invariant.

In the multidimensional case, each one-parameter group of transformations (2.1) has exactly \((n - 1)\) functionally independent invariants, called a basis of invariants. We construct a basis of invariants for a group with operator (2.5) by solving the characteristic equation
\[ \frac{dx_1}{X_1(x)} = \ldots = \frac{dx_n}{X_n(x)}. \]

### 2.2 Prolongation formulas

In Chapters 3, 4 and 5, we are interested in determining the one-parameter groups of point transformations admitted by the class of nonlinear diffusion-convection equations in one, two and three spatial dimensions. Our groups of transformations will be of the form
\[ x^* = X(x, u; \epsilon) \]
\[ u^* = U(x, u; \epsilon) \]
acting on the space of variables
\[ x = (x_1, x_2, x_3, x_4), \]
\[ u = u(x), \]
including four independent variables $x$ and one dependent variable $u$. The transformations (2.7) are considered point transformations as they map the points $(x, u)$ into the points $(x^*, u^*)$. In contrast, for contact transformations, the transformed variables also depend on first order derivatives of $u$ with respect to the independent variables. For the class of diffusion-convection equations, our independent variables correspond to variables in space and time. Hence in later chapters we will generally use the notation

$$(x_1, x_2, x_3, x_4) \equiv (x, y, z, t).$$

For one dependent and four independent variables, we write the one-parameter group of transformations as

$$
\begin{align*}
x^*_i &= X_i(x, u; \epsilon) = x_i + \epsilon X_i(x, u) + O(\epsilon^2), \\
u^* &= U(x, u; \epsilon) = u + \epsilon U(x, u) + O(\epsilon^2),
\end{align*}
\tag{2.8}
$$

for $i = 1, 2, 3, 4$, with infinitesimal generator

$$
\Gamma = X_i(x, u) \frac{\partial}{\partial x_i} + U(x, u) \frac{\partial}{\partial u}.
$$

The $k$th extension of (2.8) is given by

$$
\begin{align*}
x^*_i &= X_i(x, u; \epsilon) = x_i + \epsilon X_i(x, u) + O(\epsilon^2), \\
u^* &= U(x, u; \epsilon) = u + \epsilon U(x, u) + O(\epsilon^2), \\
u^*_i &= U_i(x, u, \frac{1}{i}; \epsilon) = u_i + \epsilon U_i^{(1)}(x, u, \frac{1}{i}) + O(\epsilon^2), \\
&\vdots \\
u^*_{i_1 i_2 \ldots i_k} &= U_{i_1 i_2 \ldots i_k}(x, u, \frac{1}{i_1}, \ldots, \frac{1}{i_k}; \epsilon) \\
&= u_{i_1 i_2 \ldots i_k} + \epsilon U_{i_1 i_2 \ldots i_k}^{(k)}(x, u, \frac{1}{i_1}, \ldots, \frac{1}{i_k}) + O(\epsilon^2),
\end{align*}
$$

where $i = 1, 2, 3, 4$ and $i_l = 1, 2, 3, 4$ for $l = 1, 2, \ldots, k$ with $k = 2, 3, \ldots$. We are using the notation $\frac{k}{\mu}$ to denote all the $k$th order partial derivatives of $u$ with respect to $x = (x_1, x_2, x_3, x_4)$. For example,

$$
\begin{align*}
\frac{1}{u} &= (u_1, u_2, u_3, u_4), \\
\frac{2}{u} &= (u_{11}, u_{12}, u_{13}, u_{14}, u_{22}, u_{23}, u_{24}, u_{33}, u_{34}, u_{44}).
\end{align*}
$$
Hence we are extending from our base space \((x, u)\) to the \(k\)th jet space with coordinates \((x, u, \frac{u}{1}, \frac{u}{2}, \ldots, \frac{u}{k})\). Since the diffusion-convection equation is a second order P.D.E., in later chapters we will only be interested in the case \(k = 2\).

Only a restricted class of transformations on the jet space can be viewed as lifted point transformations, for which \(\frac{u}{k}\) makes sense as a collection of derivatives of \(u\). The extended one-parameter group of transformations has the \((k\)th extended) infinitesimal

\[
(\mathcal{X}(x, u), \mathcal{U}(x, u), \mathcal{U}^{(1)}(x, u, \frac{u}{1}), \ldots, \mathcal{U}^{(k)}(x, u, \frac{u}{1}, \ldots, \frac{u}{k}))
\]

with corresponding \((k\)th extended) infinitesimal generator

\[
\Gamma^{(k)} = \mathcal{X}_i(x, u) \frac{\partial}{\partial x_i} + \mathcal{U}(x, u) \frac{\partial}{\partial u} + \mathcal{U}^{(1)}(x, u, \frac{u}{1}) \frac{\partial}{\partial u_i} + \ldots + \mathcal{U}^{(k)}_{i_1 i_2 \ldots i_k}(x, u, \frac{u}{1}, \ldots, \frac{u}{k}) \frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}},
\]

\(k = 1, 2, \ldots\). The extended infinitesimals \(\{\mathcal{U}^{(k)}\}\) are given by (Olver [57], Bluman and Kumei [7])

\[
\mathcal{U}^{(1)} = D_i \mathcal{U} - (D_i \mathcal{X}_j) u_j, \quad i, j = 1, 2, 3, 4,
\]

and

\[
\mathcal{U}^{(k)}_{i_1 i_2 \ldots i_k} = D_{i_k} \mathcal{U}^{(k-1)}_{i_1 i_2 \ldots i_{k-1}} - (D_{i_k} \mathcal{X}_j) u_{i_1 i_2 \ldots i_{k-1} j},
\]

for \(i_1, 1, 2, 3, 4\) with \(l = 1, 2, \ldots, k\) with \(k = 2, 3, \ldots\), where \(D_i\) represents the total derivative operator

\[
D_i = \frac{D}{Dx_i} = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \ldots + u_{i_1 i_2 \ldots i_n} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_n}}, \quad i = 1, 2, 3, 4.
\]

For example,

\[
\mathcal{U}^{(1)}_1 = \frac{\partial \mathcal{U}}{\partial x_1} + \left[ \frac{\partial \mathcal{U}}{\partial u} - \frac{\partial \mathcal{X}_1}{\partial x_1} \right] u_1 - \frac{\partial \mathcal{X}_2}{\partial x_1} u_2 - \frac{\partial \mathcal{X}_3}{\partial x_1} u_3 - \frac{\partial \mathcal{X}_4}{\partial x_1} u_4
\]

\[
- \frac{\partial \mathcal{X}_1}{\partial u} (u_1)^2 - \frac{\partial \mathcal{X}_2}{\partial u} u_1 u_2 - \frac{\partial \mathcal{X}_3}{\partial u} u_1 u_3 - \frac{\partial \mathcal{X}_4}{\partial u} u_1 u_4.
\]
and
\[ U_{11}^{(2)} = \frac{\partial^2 U}{\partial x_1^2} + \left[ 2 \frac{\partial^2 U}{\partial x_1 \partial u} - \frac{\partial^2 \chi_1}{\partial x_1^2} \right] u_1 - \frac{\partial^2 \chi_2}{\partial x_1^2} u_2 - \frac{\partial^2 \chi_3}{\partial x_1^2} u_3 - \frac{\partial^2 \chi_4}{\partial x_1^2} u_4 \]
\[ + \left[ \frac{\partial U}{\partial u} - 2 \frac{\partial \chi_1}{\partial x_1} \right] u_{11} - 2 \frac{\partial \chi_2}{\partial x_1} u_{12} - 2 \frac{\partial \chi_3}{\partial x_1} u_{13} - 2 \frac{\partial \chi_4}{\partial x_1} u_{14} \]
\[ + \left[ \frac{\partial^2 U}{\partial u^2} - 2 \frac{\partial^2 \chi_1}{\partial x_1 \partial u} \right] (u_1)^2 - 2 \frac{\partial^2 \chi_2}{\partial x_1 \partial u} u_1 u_2 - 2 \frac{\partial^2 \chi_3}{\partial x_1 \partial u} u_1 u_3 - 2 \frac{\partial^2 \chi_4}{\partial x_1 \partial u} u_1 u_4 \]
\[ - \frac{\partial^2 \chi_1}{\partial u^2} (u_1)^3 - \frac{\partial^2 \chi_2}{\partial u^2} (u_1)^2 u_2 - \frac{\partial^2 \chi_3}{\partial u^2} (u_1)^2 u_3 - \frac{\partial^2 \chi_4}{\partial u^2} (u_1)^2 u_4 \]
\[ - 3 \frac{\partial \chi_1}{\partial u} u_{11} u_1 - \frac{\partial \chi_2}{\partial u} u_{21} u_1 - \frac{\partial \chi_3}{\partial u} u_{31} u_1 - \frac{\partial \chi_4}{\partial u} u_{41} u_1 \]
\[ - 2 \frac{\partial \chi_2}{\partial u} u_{11} u_{12} - 2 \frac{\partial \chi_3}{\partial u} u_{11} u_{13} - 2 \frac{\partial \chi_4}{\partial u} u_{11} u_{14}. \]

In a similar fashion, we may construct expressions for \( \chi_2^{(1)} \), \( \chi_3^{(1)} \), \( \chi_4^{(1)} \), \( \chi_{12}^{(2)} \), \( \chi_{22}^{(2)} \), etc. The ideas we have just outlined may be easily extended to obtain prolongations of a one-parameter group of transformations for any system of equations of order \( k \) with \( m \) dependent and \( n \) independent variables.

We now apply the \( k \)th extended infinitesimal generator (2.9) to the function \( F \) which defines our governing equation (or system of equations) under the restriction \( F(x,u,u,u,...,u) = 0 \).

We have condition(s) involving a known function \( F(x,u,\frac{1}{u},\frac{2}{u},...,\frac{k}{u}) \) and the unknown functions \( \chi_i(x,u) \) and \( U(x,u) \). Our expression(s) may be split into a number of independent equations as we consider all derivatives of \( u \) (that is, any \( \frac{1}{u},... ,\frac{k}{u} \)) to be independent variables. As a result, we are able to obtain a finite number of determining equations for \( \chi_i \) and \( U \) - these determining equations are generally overdetermined and are linear P.D.E.'s for \( \chi_i \) and \( U \). The solutions qualify as
components of symmetry vector fields.

We consider a simple example to demonstrate the ideas that we have outlined. Consider the diffusion equation in two spatial dimensions,

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.
\]  \hspace{1cm} (2.10)

We have one dependent and three independent variables. We write the one-parameter group of transformations as

\[
x^* = x + \epsilon \mathcal{X}(x, y, t, u) + O(\epsilon^2),
\]

\[
y^* = y + \epsilon \mathcal{Y}(x, y, t, u) + O(\epsilon^2),
\]

\[
t^* = t + \epsilon \mathcal{T}(x, y, t, u) + O(\epsilon^2),
\]

\[
u^* = u + \epsilon \mathcal{U}(x, y, t, u) + O(\epsilon^2),
\]

with infinitesimal generator

\[
\Gamma = \mathcal{X} \frac{\partial}{\partial x} + \mathcal{Y} \frac{\partial}{\partial y} + \mathcal{T} \frac{\partial}{\partial t} + \mathcal{U} \frac{\partial}{\partial u}.
\]

Since (2.10) is a second order P.D.E., the twice extended space has coordinates

\[
(x, y, t, u, u_x, u_y, u_t, u_{xx}, u_{xy}, u_{xt}, u_{yy}, u_{yt}, u_{tt}).
\]

Hence the twice-extended infinitesimal generator is

\[
\Gamma^{(2)} = \mathcal{X} \frac{\partial}{\partial x} + \mathcal{Y} \frac{\partial}{\partial y} + \mathcal{T} \frac{\partial}{\partial t} + \mathcal{U} \frac{\partial}{\partial u} + \mathcal{U}_x^{(1)} \frac{\partial}{\partial u_x} + \mathcal{U}_y^{(1)} \frac{\partial}{\partial u_y} + \mathcal{U}_t^{(1)} \frac{\partial}{\partial u_t} + \mathcal{U}_{xx}^{(2)} \frac{\partial}{\partial u_{xx}} + \mathcal{U}_{xy}^{(2)} \frac{\partial}{\partial u_{xy}} + \mathcal{U}_{xt}^{(2)} \frac{\partial}{\partial u_{xt}} + \mathcal{U}_{yy}^{(2)} \frac{\partial}{\partial u_{yy}} + \mathcal{U}_{yt}^{(2)} \frac{\partial}{\partial u_{yt}} + \mathcal{U}_{tt}^{(2)} \frac{\partial}{\partial u_{tt}}.
\]

We let

\[
F = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2},
\]

and we apply the twice-extended infinitesimal generator to \( F \) under the restriction

\[
\Gamma^{(2)} F \bigg|_{F=0} = 0.
\]
As a result, we obtain the set of determining equations

\[
\begin{align*}
\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial u} = \frac{\partial X}{\partial u} = \frac{\partial Y}{\partial u} = 0 \\
2 \frac{\partial^2 U}{\partial x \partial u} + \frac{\partial X}{\partial t} - \frac{\partial^2 X}{\partial x^2} - \frac{\partial^2 X}{\partial y^2} = 0 \\
\frac{\partial^2 U}{\partial y \partial u} + \frac{\partial Y}{\partial t} - \frac{\partial^2 Y}{\partial x^2} - \frac{\partial^2 Y}{\partial y^2} = 0 \\
\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} = 0 \\
2 \frac{\partial Y}{\partial y} - \frac{\partial T}{\partial t} = 0 \\
\frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x} = 0 \\
\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} = 0
\end{align*}
\]

(2.12)

A step by step description of the method used to obtain this set of equations is contained in Appendix A.

One solution of (2.12) is

\[ X = y, \ Y = -x, \ T = 0, \ U = 0, \]

with the corresponding characteristic equation

\[
\frac{dx}{y} = \frac{dy}{-x} = \frac{dt}{0} = \frac{du}{0}.
\]

One invariant is

\[ r = \sqrt{x^2 + y^2}, \]

while the remaining invariants are \( t \) and \( u \). Hence we can write the independent variable \( u \) as

\[ u = G(r, t), \quad \text{where} \ r = \sqrt{x^2 + y^2}. \]
Substitution of this functional form of \( u \) into (2.10) leads to

\[
\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r},
\]

(2.13)
a P.D.E. with one dependent and two independent variables. In later chapters we are interested in the symmetry analysis of classes of diffusion-convection equations with the goal of eventually reducing our governing P.D.E. to an O.D.E.

The Lie point symmetry analysis process may be performed algorithmically - there exist a number of software packages which generate and then attempt to solve the determining equations for any given system of D.E.'s (for example, Sherring's package Dimsym [78] and Head's program LIE [41]), while other packages can be used to determine the size and structure of the symmetry group from a standard form of the determining equations, even in the event that the determining equations cannot be solved (for example, Reid [66],[67]).

In the previous example, we demonstrated a reduction of the two-dimensional diffusion equation to a second P.D.E. with one dependent and two independent variables - that is, we reduced by one the number of independent variables. Often we may be interested in two (or possibly more) reductions of our governing equation. For instance, in the example of the diffusion equation (2.10), we would ultimately be aiming to reduce to an O.D.E., hence we would again wish to reduce by one the number of independent variables of (2.13). We are able to predict the possibility of a multiple reduction of a P.D.E if any two of the group generators of our equation obey the property

\[
[\Gamma_i, \Gamma_j] = \lambda \Gamma_i, \quad \lambda \in \mathbb{R}.
\]

(2.14)

That is, if our governing equation admits a \( n \)-parameter Lie transformation group with \( n \) associated group operators \( \Gamma_1, \Gamma_2, \ldots, \Gamma_n \), where two of the generators satisfy (2.14), then after a reduction of the original equation using \( \Gamma_i, \Gamma_j \) will be a generator (in the new variables) of the reduced equation and so a second reduction will be
possible. We call $\Gamma_j$ an **inherited symmetry**. It is also possible that the reduced equation may have symmetries which do not exist in symmetries of the original equation. For this reason, in later chapters we generally consider separately the symmetry analysis of each reduced equation and do not necessarily rely on the existence of inherited symmetries when seeking to reduce our governing equation to an O.D.E.

If our governing equation admits a $n$-parameter Lie transformation group with $n$ associated group operators, a reduction is possible using any arbitrary linear combination of the the operators

$$a_1 \Gamma_1 + a_2 \Gamma_2 + \ldots + a_n \Gamma_n, \quad a_1, a_2, \ldots a_n \in \mathbb{R}. \quad (2.15)$$

To ensure that a minimal complete set of reductions is found from the symmetries of the governing equation, the **optimal system** (Olver [57]; Ovsiannikov [59]) is found. An optimal system is a list of one-parameter subalgebras such that every one-parameter subalgebra is equivalent to a unique member of the list under some element of the adjoint representation of the group. In the adjoint representation, group representatives act on the Lie algebra viewed as a vector space according to

$$\text{Ad}(\exp(e^{i\Gamma_i})) \Gamma_j = e^{i\Gamma_i} \Gamma_j e^{-i\Gamma_i},$$

$$= \Gamma_j - i[\Gamma_i, \Gamma_j] + \frac{1}{2} e^{2i\Gamma_i} [\Gamma_i, [\Gamma_i, \Gamma_j]] - \ldots,$$  \quad (2.16)

where $[\Gamma_i, \Gamma_j]$ is the previously defined commutator of $\Gamma_i$ and $\Gamma_j$. The optimal system can be determined by a comparison of the Lie algebra with standard classifications previously evaluated (Patera and Winternitz [61]). Alternatively, the optimal system may be determined by taking a general element $\Gamma = a_1 \Gamma_1 + a_2 \Gamma_2 + \ldots + a_n \Gamma_n$ in the Lie algebra and simplifying it as much as possible by subjecting it to judiciously chosen adjoint transformations. We use the latter method for the determination of any optimal systems constructed in later chapters.
As an example, we consider the Lie algebra spanned by
\[ \Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial y}, \quad \Gamma_3 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}. \]

The commutator table is

\[
\begin{array}{ccc}
\Gamma_1 & \Gamma_2 & \Gamma_3 \\
\Gamma_1 & 0 & 0 & -\Gamma_2 \\
\Gamma_2 & 0 & 0 & \Gamma_1 \\
\Gamma_3 & \Gamma_2 & -\Gamma_1 & 0 \\
\end{array}
\]

To compute the adjoint representation, we use (2.16) in conjunction with the commutator table. For example,

\[
\text{Ad}(\exp(\varepsilon \Gamma_1))\Gamma_3 = \Gamma_3 - \varepsilon [\Gamma_1, \Gamma_3] + \frac{1}{2} \varepsilon^2 [\Gamma_1, [\Gamma_1, \Gamma_3]] - \ldots
\]

\[= \Gamma_3 + \varepsilon \Gamma_2,\]

\[
\text{Ad}(\exp(\varepsilon \Gamma_3))\Gamma_1 = \Gamma_1 - \varepsilon [\Gamma_3, \Gamma_1] + \frac{1}{2} \varepsilon^2 [\Gamma_3, [\Gamma_3, \Gamma_1]] - \ldots
\]

\[= \Gamma_1 \cos \varepsilon - \Gamma_2 \sin \varepsilon,\]

etc., and hence the table

\[
\begin{array}{ccc|ccc}
\text{Ad} & \Gamma_1 & \Gamma_2 & \Gamma_3 \\
\Gamma_1 & \Gamma_1 & \Gamma_2 & \Gamma_3 + \varepsilon \Gamma_2 \\
\Gamma_2 & \Gamma_1 & \Gamma_2 & \Gamma_3 - \varepsilon \Gamma_1 \\
\Gamma_3 & \Gamma_1 \cos \varepsilon - \Gamma_2 \sin \varepsilon & \Gamma_2 \cos \varepsilon + \Gamma_1 \sin \varepsilon & \Gamma_3 \\
\end{array}
\]

is constructed, where the entry \((i, j)\) indicates \(\text{Ad}(\exp(\varepsilon \Gamma_i))\Gamma_j\).

Given a nonzero vector
\[ \Gamma = a_1 \Gamma_1 + a_1 \Gamma_2 + a_3 \Gamma_3, \quad (2.17) \]

we want to simplify the maximum number of coefficients \(a_1, a_2, a_3\) possible by careful application of adjoint maps to \(\Gamma\). We assume that the constant \(a_3 \neq 0\) and
by rescaling $\Gamma$ we can assume that $a_3 = 1$. Using the adjoint table, we act on $\Gamma$ by $\text{Ad}(\exp(c_1 \Gamma_1))$ to obtain

$$\Gamma^I = \text{Ad}(\exp(c_1 \Gamma_1)) \Gamma = a_1 \Gamma_1 + a_2 \Gamma_2 + \Gamma_3 + c_1 \Gamma_2$$

$$= a_1 \Gamma_1 + \Gamma_3,$$

if we choose $c_1 = -a_2$. Hence we have eliminated the coefficient of $\Gamma_2$. We now act on the new vector $\Gamma^I$ by $\text{Ad}(\exp(c_2 \Gamma_2))$ to obtain

$$\Gamma^{II} = \text{Ad}(\exp(c_2 \Gamma_2)) \Gamma^I = a_1 \Gamma_1 + \Gamma_3 - c_2 \Gamma_1$$

$$= \Gamma_3,$$

when $c_2 = a_1$. Hence every one-dimensional subalgebra spanned by a vector $\Gamma$ with $a_3 \neq 0$ is equivalent to the subalgebra spanned by $\Gamma_3$. Any remaining subalgebras are spanned by vectors of the form (2.17) with $a_3 = 0$. We now assume that $a_1 \neq 0$ and scale so that $a_1 = 1$. Acting on $\Gamma$ by $\text{Ad}(\exp(c_3 \Gamma_3))$ gives

$$\Gamma^I = \text{Ad}(\exp(c_3 \Gamma_3)) \Gamma = (\Gamma_1 \cos c_3 - \Gamma_2 \sin c_3) + a_2(\Gamma_2 \cos c_3 + \Gamma_1 \sin c_3)$$

$$= (\cos c_3 + a_2 \sin c_3) \Gamma_1 + (a_2 \cos c_3 - \sin c_3) \Gamma_2$$

$$= \alpha \Gamma_1,$$

when we choose $c_3$ such that $\tan c_3 = a_2$, and $\Gamma^I$ may be scaled to give $\Gamma_1$. Finally, if $a_1 = 0$, we choose $a_2 = 1$ and act on $\Gamma$ by $\text{Ad}(\exp(c_4 \Gamma_3))$ to give

$$\Gamma^I = \text{Ad}(\exp(c_4 \Gamma_3)) \Gamma = \Gamma_2 \cos c_4 + \Gamma_1 \sin c_4$$

$$= \Gamma_1,$$

when $c_4 = \pi/2$ is chosen. Hence a minimal set of generators for the symmetry algebra under the action of the symmetry group in the adjoint representation is $\{\Gamma_1, \Gamma_3\}$. In a similar fashion, we obtain the optimal systems for the subalgebras of various forms of the diffusion-convection equation in later chapters.

An **equivalence transformation** for a class of D.E.’s maps every solution of an equation in the general class of D.E.’s to a solution of some other equation in the
Chapter 2. One-parameter transformation groups

general class. For example, the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ D(u) \frac{\partial u}{\partial x} \right]$$

is mapped to the diffusion equation

$$\frac{\partial u'}{\partial t'} = \frac{\partial}{\partial x'} \left[ D'(u') \frac{\partial u'}{\partial x'} \right],$$

where $D'(u') = -D(u')$, by the transformation

$$x' = x, \quad t' = -t, \quad u' = u.$$

This transformation is an equivalence transformation since (2.18) and (2.19) are both elements of the class of nonlinear diffusion equations. When determining the optimal systems for diffusion-convection equations in later chapters, we will often use this equivalence property to reduce the number of inequivalent subalgebras in the optimal system.

A nonlinear P.D.E. may usually be transformed to a linear P.D.E. if the nonlinear P.D.E. admits an infinite-parameter Lie group of point transformations. For example, the nonlinear diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \left( \frac{\partial u}{\partial x} \right)^2 \frac{\partial u}{\partial t}$$

admits the generator

$$\Gamma = c \frac{\partial}{\partial x},$$

where $c(u,t)$ satisfies

$$\frac{\partial^2 c}{\partial u^2} = \frac{\partial c}{\partial t}.$$ 

Hence (2.20) may be transformed to

$$\frac{\partial^2 \phi}{\partial \rho^2} = \frac{\partial \phi}{\partial \eta}$$

by the mapping

$$\rho = u, \quad \eta = t, \quad \phi = x.$$
We say that (2.20) is **linearisable**. However, the symmetry analysis performed in later chapters shows that no form of the diffusion-convection equation in one, two or three spatial dimensions admits an infinite-parameter Lie group of point transformations.

### 2.3 Generalised symmetry ideas

In addition to the ideas of one-parameter Lie groups of point transformations which we use extensively throughout the remainder of this thesis, there are many extended ideas which are currently being investigated. For example, we could look for the Lie-Backlund symmetries of a D.E. by considering transformations of the form

\[ x^* = x + \epsilon X(x, u, \frac{1}{x}, \frac{2}{x}, \ldots, \frac{p}{x}) + O(\epsilon^2), \]
\[ u^* = u + \epsilon U(x, u, \frac{1}{x}, \frac{2}{x}, \ldots, \frac{p}{x}) + O(\epsilon^2), \]

which are extensions of the transformations (2.8) considered earlier. The transformations (2.21) are more general as the infinitesimals \( X \) and \( U \) depend on derivatives \( \frac{1}{x}, \frac{2}{x}, \ldots, \frac{p}{x} \). As mentioned previously, the case \( p = 1 \) gives the contact transformations of the D.E. Lie-Backlund and point symmetries are local symmetries, as their infinitesimal generators only depend on local properties of solutions.

The class of known symmetries of a D.E. could also be extended by considering nonlocal symmetries, for example, when the infinitesimal generator depends on integrals of dependent variables. Nonlocal symmetries admitted by a D.E. are often found by considering a corresponding system of D.E.'s. For example, the nonlocal symmetries of the linear diffusion equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]

with generator

\[ \Gamma = X(x, u, u_{-1}) \frac{\partial}{\partial x} + U(x, u, u_{-1}) \frac{\partial}{\partial u}, \]
where
\[ u_{-1} = \int u \, dx, \]
are equivalent to the point symmetries of the system of equations
\[ \frac{\partial v}{\partial x} = u, \]
\[ \frac{\partial v}{\partial t} = \frac{\partial u}{\partial x}. \]
The variable \( v \) is a potential which arises from the conserved form of the linear diffusion equation. The reader is referred to [7] for further information.

A further extension to the symmetry methods previously discussed is the nonclassical method, introduced by Bluman and Cole [6]. The classical method only used the given equation \( F = 0 \) to obtain the set of linear determining equations. Hence the technique of finding the Lie point symmetries of a given D.E. is a classical method. In the nonclassical method, the equation \( F = 0 \) and the invariant surface condition, given by
\[ X_i(x,u) \frac{\partial u}{\partial x_i} = U(x,u) \]
are both used to find the set of determining equations. For the linear diffusion equation, the corresponding invariant surface condition is given by
\[ X(x,t,u) \frac{\partial u}{\partial x} + T(x,t,u) \frac{\partial u}{\partial t} = U(x,t,u). \]
It is generally assumed that \( T \neq 0 \) and hence the invariant surface condition is written as
\[ X(x,t,u) \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = U(x,t,u). \]
The case \( T = 0 \) should also be considered. The set of determining equations obtained using the nonclassical method are nonlinear for \( X \), \( T \) and \( U \). In some instances, the determining equations may be solved in general but lead only to the classical symmetries of the system. In most cases, however, the determining equations can
not be solved in general and some assumptions on the forms of the infinitesimals must be made. Hence it is not possible to find the most general solution to the determining equations. Recent developments to the nonclassical method include the “direct method” for finding explicit solutions to P.D.E.’s, introduced by Clarkson and Kruskal [24]. This method has been shown to be connected with the nonclassical method [50], [55]. Nucci [54] iterates the nonclassical method to obtain new nonlinear equations to find new solutions of the governing equation from invariant solutions of the heir-equations.

A recently introduced method extends the concept of a Lie point symmetry by combining the idea of a generalised (or Lie-Backlund) symmetry with the notion of a conditional symmetry, which was introduced as a nonclassical symmetry [6]. Fokas and Liu [34] introduce the concept of a generalised conditional symmetry, or generalisations of conditional symmetries. A function $\sigma(u)$ is defined as a generalised conditional symmetry of an equation

$$\frac{\partial u}{\partial t} = F(u)$$

if and only if

$$K'\sigma - \sigma'K = F(u, \sigma), \quad F(u, 0) = 0,$$

where $K(u)$ and $\sigma(u)$ are differentiable functions of $u, u_x, u_{xx}, \ldots$, $F(u, \sigma)$ is a differentiable function of $u, u_x, u_{xx}, \ldots$ and $\sigma, \sigma_x, \sigma_{xx}, \ldots$ and the prime denotes the Fréchet derivative. The definition implies that the equations

$$\frac{\partial u}{\partial t} = F(u) \quad \text{and} \quad \sigma = 0 \quad (2.22)$$

are compatible and hence in general they share a common manifold of solutions. The form of $\sigma$ may include arbitrary functions of the dependent variable $u$. Demanding that the equations (2.22) are compatible leads to a series of nonlinear P.D.E.’s in the arbitrary functions. However, the choice of $\sigma$ is somewhat ad hoc as the order of
the equation can be continually increased. In addition, it is often the case that the solutions obtained using this method could also have been recovered using either classical or nonclassical techniques.
Chapter 3

Classical symmetry reductions of the one-dimensional nonlinear diffusion-convection equation

The importance of the one-dimensional nonlinear diffusion-convection equation

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial z} \left( D(u) \frac{\partial u}{\partial z} \right) - \frac{dK}{du} \frac{\partial u}{\partial z} \]  

(3.1)

is well known, and there is a continuing high level of interest in the construction of exact solutions to this equation (e.g. [14], [64]). It is widely believed that the Lie symmetry analysis for this class of equations has been fully investigated. Surprisingly, this is not the case. The purpose of this chapter is to present a more comprehensive analysis.

The most extensive table of symmetries for the one-dimensional nonlinear diffusion-convection equation (3.1) is that given by Oron and Rosenau [58], extended to include the special case of Burgers' equation \((D(u) = \text{const}, K(u) = u^2)\) by Katoanga [47]. Recently, Yung et al. [96] have presented a more complete symmetry analysis. The work presented in this chapter was carried out independently of Yung et al., and
it was first published simultaneously (Edwards, [30]). Although most of the special cases of Eq. (3.1) which possess symmetries beyond the expected translations in space and time (which apply for general $D$ and $K$) have been listed, not all have appeared in a single list. This chapter collects together all of the special cases and modifies some cases previously listed incorrectly.

The symmetry analysis for the class of equations (3.1) is considered where $D(u)$ and $K(u)$ are arbitrary and the special cases of these functions which admit additional symmetries are listed. The corresponding optimal system of Lie symmetry subalgebras is found for each special case with extra symmetries. Once the optimal system has been determined, a reduction of the governing P.D.E. to an O.D.E. is made using the nontrivial generators of each optimal system. As a result, a number of new symmetries and new reductions are found.

### 3.1 Symmetry analysis

Classical Lie group theory is used to determine the classical symmetries of the general class of nonlinear diffusion-convection equation (3.1). These symmetries are obtained by considering the infinitesimal transformations

\[
\begin{align*}
    u_* &= e^{\Gamma} u = u + \epsilon U(z, t, u) + O(\epsilon^2), \\
    t_* &= e^{\Gamma} t = t + \epsilon T(z, t, u) + O(\epsilon^2), \\
    z_* &= e^{\Gamma} z = z + \epsilon Z(z, t, u) + O(\epsilon^2),
\end{align*}
\]

(3.2)

where

\[
\Gamma = Z \frac{\partial}{\partial z} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u}
\]

is the infinitesimal generator (e.g. [57], [7]). Equation (3.2) is then extended to first and second order by the prolongation formulae, where, for example,

\[
\frac{\partial u_*}{\partial z_*} = \frac{\partial u}{\partial z} + \epsilon U_1 + O(\epsilon^2),
\]
Chapter 3. One-dimensional diffusion-convection equation

with

\[ \frac{D}{Dz} \mathcal{U} - \left( \frac{D}{Dz} \mathcal{Z} \right) u_z - \left( \frac{D}{Dz} \mathcal{T} \right) \mathcal{U}_t \]

and \( D/Dz \) is the total derivative operator with respect to \( z \), that is,

\[ \frac{D}{Dz} F(z, t, u) = \frac{\partial F}{\partial z} + u_z \frac{\partial F}{\partial u}. \]

The invariance of (3.1) under the infinitesimal transformation (3.2) and the fact that the derivatives of \( u \) are independent leads to a set of determining equations for the symmetry group of the governing equation (3.1). These determining equations are linear P.D.E.'s in \( Z, T \) and \( U \), and are:

\[ \frac{\partial Z}{\partial u} = 0, \quad \frac{\partial T}{\partial u} = 0, \quad \frac{\partial T}{\partial z} = 0. \]

The most general infinitesimal symmetry of (3.1) is found by solving this system of linear P.D.E.'s. A number of algorithms and packages (e.g. [66],[75]) have been developed to automatically determine the symmetries of systems of P.D.E.'s, and the size of these symmetry groups. The current analysis was performed using the symmetry-finding package Dimsym [77],[78] under REDUCE. This package was chosen because it allows for the determination of symmetries for general classes of equations. In Dimsym, use of the Riquier-Janet theory (e.g. [66]) to place the
governing equations in standard form, for example, substitution of the equations into each other and the inclusion of any non-trivial integrability conditions, ensures that the local solvability property [57] holds and the hyperspace $F^i = 0$ is uniquely described. Erroneous results are avoided by Dimsym’s implementation of these two conditions.

In the attempt to calculate the symmetries of (3.1) for the general forms of $D(u)$ and $K(u)$, Dimsym reports in two ways any assumptions made on the form of these unknown functions. The first condition is when division by an expression which involves an unknown function occurs, and the assumption here is that the expression is not identically zero. The second condition arises when assumptions are made on the linear independence of various expressions when splitting the determining equations. This operation excludes specific cases if the expressions contain unknown functions. Solving the associated equation will recover these cases. In this way, by considering all assumptions made while solving the determining equations, a full classification of the symmetries of the governing equation (3.1) is possible. An example of the output generated by Dimsym is included in Appendix B, where the symmetry analysis of the general diffusion-convection equation with $D(u)$ and $K(u)$ arbitrary is performed.

Dimsym reports that the two symmetries

$$
\Gamma_1 = \frac{\partial}{\partial z} \quad \text{and} \quad \Gamma_2 = \frac{\partial}{\partial t}
$$

apply for all forms of $D(u)$ and $K(u)$. In the following discussion, $\Gamma_1$ and $\Gamma_2$ will not be mentioned explicitly, as this pair of symmetries exists for all possible combinations of the functions $D$ and $K$.

Holding $D(u)$ and $K(u)$ arbitrary, Dimsym reports that division has been made by the expressions

$$
\frac{dD}{du}
$$
and
\[ D(u)\frac{dD}{du} \frac{d^3D}{du^3} - 2D(u) \left( \frac{d^2D}{du^2} \right)^2 + \left( \frac{dD}{du} \right)^2 \frac{d^2D}{du^2}. \]

Equating the first expression to zero gives \( D(u) = \text{const} \), while equating the second expression to zero gives
\[ D(u) = \alpha(u + \beta)^m \quad \text{or} \quad D(u) = \alpha e^{mu} \]

and by rescaling, these may be simplified to
\[ D(u) = u^m \quad \text{or} \quad D(u) = e^{mu}. \]

Dimsym also performs an equation split while solving the determining equations, and the assumption made is that the expressions
\[ \frac{dK}{du} \quad \text{and} \quad 1 \]
are linearly independent. If the last expressions are taken to be linearly dependent, solving the O.D.E.
\[ \frac{dK}{du} = \alpha \]
leads to the form \( K(u) = \alpha u + \beta \). However, this is not considered a special case, since any form of the diffusion-convection equation with \( K(u) \) linear may be obtained by a change of reference frame from the case of pure diffusion, which has been thoroughly analysed by Galaktionov et al. [38].

From above, there are three forms of \( D(u) \) such that the nonlinear diffusion-convection equation may have extra symmetries, and these are
\[ D(u) = u^m, \quad D(u) = e^{mu} \quad \text{and} \quad D(u) = 1. \]

We consider each of these in turn.

When \( D(u) = u^m \), the assumptions made by Dimsym are that \( m, 3m + 4 \neq 0 \), and that the expressions
\[ u^m, u \frac{d^2K}{du^2}, \frac{dK}{du}, 1 \]
are linearly independent. This means that \( m = 0, m = -4/3 \) or any solution of the O.D.E.

\[
\alpha u \frac{d^2 K}{du^2} + \beta \frac{dK}{du} = \gamma u^m + \eta
\]  

may lead to a special case, that is, extra symmetry generators. The case \( m = 0 \) is considered separately later. Unlike the pure diffusion case [38], taking \( m = -4/3 \) does not lead to any new cases which are not contained in the case of general \( m \).

Solving (3.4) leads to a variety of specific forms for \( K(u) \) which may be tested to determine whether they admit extra symmetries. An alternative approach is to let

\[
\frac{d^2 K}{du^2} = \gamma u^m + \eta - \beta \frac{dK}{du \alpha u},
\]

where the assumption here is that \( \alpha \neq 0 \), and note the special cases reported by Dimsym. When this is done, Dimsym reports a number of cases that require further investigation, including the case that the expressions

\[
\frac{dK}{du}, u^m, 1
\]

are linearly independent. Assuming that these expressions are not independent leads to the case \( m = -2 \) being flagged by Dimsym. This is the only form of the one-dimensional diffusion-convection equation other than the Burgers equation that has more than three symmetry generators. Other forms of \( K(u) \) which satisfy (3.4) and admit extra symmetries are

\[
K(u) = u^n, K(u) = \ln(u), K(u) = u \ln(u) - u.
\]

We note that an arbitrary constant could be added to \( K'(u) \) by a change of reference frame. In the following analysis, \( D(u) \) and \( K'(u) \) are listed up to an arbitrary linear change of the variable \( u \).

When \( D(u) = e^{mu} \), Dimsym reports division by \( m \), and linear independence of the expressions

\[
e^{mu}, \frac{d^2 K}{du^2}, \frac{dK}{du}, 1.
\]
The case \( m = 0 \) will be considered separately. The only other condition is any solution of the O.D.E.

\[
\alpha \frac{d^2 K}{du^2} + \beta \frac{dK}{du} = \gamma e^{mu} + \eta \tag{3.5}
\]

may lead to special forms of the diffusion-convection equation which may possess more than the two trivial symmetries \( \Gamma_1 \) and \( \Gamma_2 \). Solving (3.5) leads to a variety of forms of \( K(u) \), however, extra symmetries are only found for the following forms,

\[
K(u) = e^{nu}, \quad K(u) = u^2.
\]

Finally, when \( D(u) = \text{const} \), i.e., let \( D(u) = 1 \), Dimsym reports the that expressions

\[
u, u \frac{dK}{du}, \frac{dK}{du}, 1
\]

have been assumed to be linearly independent, as have the expressions

\[
u \frac{d^2 K}{du^2}, \frac{d^2 K}{du^2}, \frac{dK}{du}, 1.
\]

Solving the O.D.E.s

\[
\alpha u \frac{dK}{du} + \beta \frac{dK}{du} = \gamma u + \eta
\]

and

\[
\alpha u \frac{d^2 K}{du^2} + \beta \frac{d^2 K}{du^2} + \gamma \frac{dK}{du} = \eta
\]

leads to three forms for \( K(u) \), which are

\[
K(u) = u^n, \\
K(u) = (\alpha u + \gamma) \ln(u) - \eta u, \\
K(u) = e^{nu}.
\]

The last two cases lead only to forms of \( K(u) \) which are subsets of the symmetries previously found when \( D(u) = u^n \) and \( D(u) = e^{nu} \). However, when \( K(u) = u^n \), the conditions arising from Dimsym are that \( n, n - 1 \neq 0 \) and the expressions

\[
u^2, u^{n+1}, u, u^n
\]
have been assumed to be linearly independent. The cases $n = 0$ and $n = 1$ mean that $K'(u) = \text{const}$, which we ignore for reasons mentioned earlier. Assuming that the expressions $u^2$, $u^{n+1}$, $u$, $u^n$ may be linearly dependent leads to the cases $n = 0$, 1 or 2. The cases $n = 0$ and $n = 1$ have already been considered and discarded. The case $n = 2$ is new and leads to Burgers’ equation, a special case of the nonlinear diffusion-convection equation which possesses five symmetries, compared with a maximum of four symmetries for any other form of $D(u)$ and $K(u)$.

We have found that extra symmetries exist for the following forms of $D(u)$ and $K(u)$:

\begin{align*}
D(u) &= u^m \quad K(u) = u^n, \\
D(u) &= u^m \quad K(u) = \ln u, \\
D(u) &= u^m \quad K(u) = u \ln(u) - u, \\
D(u) &= u^{-2} \quad K(u) = u^{-1}, \\
D(u) &= e^{mu} \quad K(u) = e^{nu}, \\
D(u) &= e^{mu} \quad K(u) = u^2, \\
\text{and} \quad D(u) &= 1 \quad K(u) = u^2.
\end{align*}

The symmetries (3.7a) and (3.7e) have been found previously by Sposito [82], who was primarily interested in those forms of Eq. (3.1) that apply to unsaturated flow in porous media. In fact, he notes that the exponential model (3.7e) is well suited to this purpose [53],[79]. Specific cases of (3.7a), (3.7b) and (3.7e) are listed by Oron and Rosenau [58]. A more complete analysis was performed by Yung et al. [96], however the symmetries of (3.7c) are listed incorrectly. Table 3.1 lists the known symmetries of the one-dimensional nonlinear diffusion-convection equation.

As previously noted, there are always at least the two symmetries

\begin{align*}
\Gamma_1 &= \frac{\partial}{\partial z} \quad \text{and} \quad \Gamma_2 = \frac{\partial}{\partial t},
\end{align*}

for arbitrary functions $D(u)$ and $K(u)$, which merely represent translations in space.
Table 3.1: Symmetries of the 1D diffusion-convection equation $m, n \in \mathbb{R}$.

<table>
<thead>
<tr>
<th>$D(u)$</th>
<th>$K(u)$</th>
<th>$\Gamma_i = T \frac{\partial}{\partial t} + Z \frac{\partial}{\partial z} + U \frac{\partial}{\partial u}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^m$</td>
<td>$u^n$</td>
<td>$\Gamma_3 = (m - 2n + 2)t \frac{\partial}{\partial t} + (m - n + 1)z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}$</td>
</tr>
</tbody>
</table>
| $u^{-2}$ | $u^{-1}$ | $\Gamma_3 = 2t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}$  
$\Gamma_4 = e^{-z} \left[ \frac{\partial}{\partial z} + u \frac{\partial}{\partial u} \right]$ |
| $u^m$  | $\ln(u)$ | $\Gamma_3 = (m + 2)t \frac{\partial}{\partial t} + (m + 1)z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}$ |
| $u^m$  | $uln(u) - u$ | $\Gamma_3 = mt \frac{\partial}{\partial t} + (mz + t) \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}$ |
| $e^{mu}$ | $e^{nu}$ | $\Gamma_3 = (m - 2n)t \frac{\partial}{\partial t} + (m - n)z \frac{\partial}{\partial z} + \frac{\partial}{\partial u}$ |
| $e^{mu}$ | $u^2$ | $\Gamma_3 = mt \frac{\partial}{\partial t} + (mz + 2t) \frac{\partial}{\partial z} + \frac{\partial}{\partial u}$ |
| $\text{const}$ | $u^2$ | $\Gamma_3 = 2t \frac{\partial}{\partial t} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u}$  
$\Gamma_4 = 2t^2 \frac{\partial}{\partial t} + 2tz \frac{\partial}{\partial z} + (z - 2ut) \frac{\partial}{\partial u}$  
$\Gamma_5 = 2t \frac{\partial}{\partial z} + \frac{\partial}{\partial u}$ |
and time. These symmetries have not been included in Table 3.1.

\section*{3.2 Optimal Systems}

The main use of symmetries is to obtain a reduction of variables (e.g. Sander \textit{et al.} \cite{73}). A reduction of variables of the one-dimensional diffusion-convection equation (3.1) will result in an O.D.E., which may or may not be solvable. The similarity variables used in the reduction of order follow from the solution of the characteristic equation

\begin{equation}
\frac{dz}{\mathcal{Z}} = \frac{dt}{\mathcal{T}} = \frac{du}{\mathcal{U}}.
\end{equation}

Reductions may be obtained from any symmetry which is an arbitrary linear combination

\begin{equation}
a_1 \Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3 + a_4 \Gamma_4 + a_5 \Gamma_5, \quad \text{for } D(u) = \text{const}, K(u) = u^2, \\
 a_1 \Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3 + a_4 \Gamma_4, \quad \text{for } D(u) = u^{-2}, K(u) = u^{-1}, \\
 a_1 \Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3, \quad \text{otherwise.}
\end{equation}

To ensure that a minimal complete set of reductions is found from the symmetries of the governing equation, the optimal system (Ovsiannikov \cite{59}; Olver \cite{57}) is found for each of the cases listed in Table 3.1. We determine the optimal system by taking a general element

\[ \Gamma = a_1 \Gamma_1 + a_2 \Gamma_2 + \ldots + a_n \Gamma_n \]

and simplifying it as much as possible by applying carefully chosen adjoint transformations. This method has been used for each of the cases in Table 3.1, although the actual working is shown only for three of the special functional forms of (3.1).

As an example, we consider the first case listed in Table 3.1. The symmetry algebra of the governing equation with \( D(u) = u^m \) and \( K(u) = u^n \) is spanned by the vector fields

\[ \Gamma_1 = \frac{\partial}{\partial z}, \quad \Gamma_2 = \frac{\partial}{\partial t}, \]
\[
\Gamma_3 = (m - 2n + 2)t \frac{\partial}{\partial t} + (m - n + 1)z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}.
\]

The commutator table is

<table>
<thead>
<tr>
<th></th>
<th>(\Gamma_1)</th>
<th>(\Gamma_2)</th>
<th>(\Gamma_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Gamma_1)</td>
<td>0</td>
<td>0</td>
<td>((m - n + 1)\Gamma_1)</td>
</tr>
<tr>
<td>(\Gamma_2)</td>
<td>0</td>
<td>0</td>
<td>((m - 2n + 2)\Gamma_2)</td>
</tr>
<tr>
<td>(\Gamma_3)</td>
<td>(- (m - n + 1)\Gamma_1)</td>
<td>(- (m - 2n + 2)\Gamma_2)</td>
<td>0</td>
</tr>
</tbody>
</table>

To compute the adjoint representation, we use (2.16) in conjunction with the commutator table. For example,

\[
\text{Ad}(\exp(\epsilon \Gamma_1)) \Gamma_3 = \Gamma_3 - \epsilon [\Gamma_1, \Gamma_3] + \frac{1}{2} \epsilon^2 [\Gamma_1, [\Gamma_1, \Gamma_3]] - \ldots
\]

\[
= \Gamma_3 - \epsilon (m - n + 1)\Gamma_1,
\]

\[
\text{Ad}(\exp(\epsilon \Gamma_2)) \Gamma_3 = \Gamma_3 - \epsilon [\Gamma_2, \Gamma_3] + \frac{1}{2} \epsilon^2 [\Gamma_2, [\Gamma_2, \Gamma_3]] - \ldots
\]

\[
= \Gamma_3 - \epsilon (m - 2n + 2)\Gamma_2,
\]

\[
\text{Ad}(\exp(\epsilon \Gamma_3)) \Gamma_1 = \Gamma_1 - \epsilon [\Gamma_3, \Gamma_1] + \frac{1}{2} \epsilon^2 [\Gamma_3, [\Gamma_3, \Gamma_1]] - \ldots
\]

\[
= \Gamma_1 + \epsilon (m - n + 1)\Gamma_1 + \frac{1}{2} \epsilon^2 (m - n + 1)^2 \Gamma_1 + \ldots
\]

\[
= e^{(m - n + 1)\epsilon} \Gamma_1,
\]

\[
\text{Ad}(\exp(\epsilon \Gamma_3)) \Gamma_2 = \Gamma_2 - \epsilon [\Gamma_3, \Gamma_2] + \frac{1}{2} \epsilon^2 [\Gamma_3, [\Gamma_3, \Gamma_2]] - \ldots
\]

\[
= \Gamma_2 + \epsilon (m - 2n + 2)\Gamma_2 + \frac{1}{2} \epsilon^2 (m - 2n + 2)^2 \Gamma_2 + \ldots
\]

\[
= e^{(m - 2n + 2)\epsilon} \Gamma_2,
\]

and hence the table

<table>
<thead>
<tr>
<th>(\text{Ad})</th>
<th>(\Gamma_1)</th>
<th>(\Gamma_2)</th>
<th>(\Gamma_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Gamma_1)</td>
<td>(\Gamma_1)</td>
<td>(\Gamma_2)</td>
<td>(\Gamma_3 - \epsilon (m - n + 1)\Gamma_1)</td>
</tr>
<tr>
<td>(\Gamma_2)</td>
<td>(\Gamma_1)</td>
<td>(\Gamma_2)</td>
<td>(\Gamma_3 - \epsilon (m - 2n + 2)\Gamma_2)</td>
</tr>
<tr>
<td>(\Gamma_3)</td>
<td>(e^{(m - n + 1)\epsilon} \Gamma_1)</td>
<td>(e^{(m - 2n + 2)\epsilon} \Gamma_2)</td>
<td>(\Gamma_3)</td>
</tr>
</tbody>
</table>

is constructed, where the entry \((i, j)\) indicates \(\text{Ad}(\exp(\epsilon \Gamma_i)) \Gamma_j\).

Given a nonzero vector

\[
\Gamma = a_1 \Gamma_1 + a_1 \Gamma_2 + a_3 \Gamma_3,
\]

(3.12)
we want to simplify the maximum number of coefficients possible by careful application of adjoint maps to $\Gamma$. We first assume that $a_3 \neq 0$ and by rescaling $\Gamma$ we can assume that $a_3 = 1$. Using the adjoint table, we act on $\Gamma$ by $\text{Ad}(\exp(c_1 \Gamma_1))$ to obtain

$$
\Gamma^I = \text{Ad}(\exp(c_1 \Gamma_1))\Gamma = [a_1 - c_1(m - n + 1)]\Gamma_1 + a_2\Gamma_2 + \Gamma_3
$$

$$
= a_2\Gamma_2 + \Gamma_3,
$$

(3.13)

where the coefficient of $\Gamma_1$ is eliminated by choosing

$$
c_1 = \frac{a_1}{m - n + 1}, \quad \text{for } m - n + 1 \neq 0.
$$

The case $m - n + 1 = 0$ needs to be considered separately later and will have a different optimal system from the case with general $m$ and $n$. Acting on the new vector $\Gamma^I$ by $\text{Ad}(\exp(c_2 \Gamma_2))$ gives

$$
\Gamma^{II} = \text{Ad}(\exp(c_2 \Gamma_2))\Gamma^I = [a_2 - c_2(m - 2n + 2)]\Gamma_2 + \Gamma_3
$$

$$
= \Gamma_3,
$$

(3.14)

where the coefficient of $\Gamma_2$ is cancelled by choosing

$$
c_2 = \frac{a_2}{m - 2n + 2}, \quad \text{for } m - 2n + 2 \neq 0.
$$

Hence every one-dimensional subalgebra spanned by a vector $\Gamma$ with $a_3 \neq 0$ is equivalent to the subalgebra spanned by $\Gamma_3$, for $m - n + 1 \neq 0$ and $m - 2n + 2 \neq 0$.

The two cases $m - n + 1 = 0$ and $m - 2n + 2 = 0$ need to be considered separately.

Any remaining one-dimensional subalgebras are spanned by vectors of the form (3.12) with $a_3 = 0$. If $a_2 \neq 0$, we can scale so that $a_2 = 1$. Acting on $\Gamma$ by $\text{Ad}(\exp(c_3 \Gamma_3))\Gamma$ gives

$$
\Gamma^I = \text{Ad}(\exp(c_3 \Gamma_3))\Gamma = a_1 e^{(m-n+1)c_3}\Gamma_1 + e^{(m-2n+2)c_3}\Gamma_2,
$$

(3.15)

which is a scalar multiple of

$$
\Gamma^{II} = a_1 e^{(n-1)c_3}\Gamma_1 + \Gamma_2.
$$
Depending on the sign of \( a_1 \), the coefficient of \( \Gamma_1 \) can be made either \(+1\), \(-1\) or 0. Therefore, any one-dimensional subalgebra spanned by \( \Gamma \) with \( a_3 = 0, a_2 \neq 0 \) is equivalent to one spanned by \( \Gamma_1 + \Gamma_2, -\Gamma_1 + \Gamma_2 \) or \( \Gamma_2 \). Finally, the remaining case \( (a_2 = a_3 = 0) \) can be seen to be equivalent to \( \Gamma_1 \). Thus

\[
\{ \Gamma_3, \pm \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1 \}
\]

is a minimal set of generators for the symmetry algebra under the action of the symmetry group in the adjoint representation. In other words, these generators are representatives of the distinct conjugacy classes. Since each of these generators span the sub-algebras in the optimal system, henceforth such a list will also be referred to as the “optimal system”. If we admit the discrete symmetry \((z, t, u) \mapsto (-z, t, u)\) which maps \(-\Gamma_1 + \Gamma_2\) to \(\Gamma_1 + \Gamma_2\), the number of inequivalent subalgebras is reduced to four.

We now consider the first of the two special cases, that is, when \( n = m + 1 \). The commutator table from the more general example becomes

<table>
<thead>
<tr>
<th></th>
<th>( \Gamma_1 )</th>
<th>( \Gamma_2 )</th>
<th>( \Gamma_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>0</td>
<td>0</td>
<td>(-m\Gamma_2)</td>
</tr>
<tr>
<td>( \Gamma_3 )</td>
<td>0</td>
<td>(m\Gamma_2)</td>
<td>0</td>
</tr>
</tbody>
</table>

while the adjoint table is

<table>
<thead>
<tr>
<th>Ad</th>
<th>( \Gamma_1 )</th>
<th>( \Gamma_2 )</th>
<th>( \Gamma_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>( \Gamma_1 )</td>
<td>( \Gamma_2 )</td>
<td>( \Gamma_3 )</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>( \Gamma_1 )</td>
<td>( \Gamma_2 )</td>
<td>( \Gamma_3 + \epsilon m\Gamma_2 )</td>
</tr>
<tr>
<td>( \Gamma_3 )</td>
<td>( e^{-m\epsilon}\Gamma_2 )</td>
<td>( \Gamma_3 )</td>
<td></td>
</tr>
</tbody>
</table>
As in the general case, we start with the nonzero vector (3.12), with $a_3$ assumed to be nonzero, and rescale $\Gamma$ such that $a_3 = 1$. Using the adjoint table, we act on $\Gamma$ by $\text{Ad}(\exp(c_1 \Gamma_2))$ to obtain

$$\Gamma^I = \text{Ad}(\exp(c_1 \Gamma_2))\Gamma = a_1 \Gamma_1 + [a_2 + c_1 m] \Gamma_2 + \Gamma_3 = a_1 \Gamma_1 + \Gamma_3, \quad (3.16)$$

where the coefficient of $\Gamma_2$ is eliminated by choosing $c_1 = -a_2/m$, $m \neq 0$. When $m = 0$, $n = 1$, and we ignore this case as equation (3.1) with $K(u) = u$ can be transformed to the case of pure diffusion. We have not considered the symmetry analysis of any form of (3.1) when $K$ is a linear function. Examining the adjoint table again, we see that there are no entries which can act on $\Gamma^I$ to simplify the coefficient of $\Gamma_1$. Hence every one-dimensional subalgebra spanned by a vector $\Gamma$ with $a_3 \neq 0$ is equivalent to the subalgebra spanned by $c \Gamma_1 + \Gamma_3$ for some $c \in \mathcal{R}$.

Any remaining one-dimensional subalgebras are spanned by vectors of the form (3.12) with $a_3 = 0$. Assuming that $a_2 \neq 0$, we scale so that $a_2 = 1$. Acting on $\Gamma$ by $\text{Ad}(\exp(c_2 \Gamma_3))$ gives

$$\Gamma^I = \text{Ad}(\exp(c_2 \Gamma_3))\Gamma = a_1 \Gamma_1 + e^{-m c_2} \Gamma_2, \quad (3.17)$$

which is a scalar multiple of

$$\Gamma^H = a_1 e^{m c_2} \Gamma_1 + \Gamma_2.$$

Depending on the sign of $a_1$ the coefficient of $\Gamma_1$ can be made either $+1$, $-1$ or 0. Therefore, any one-dimensional subalgebra spanned by $\Gamma$ with $a_3 = 0$, $a_2 \neq 0$ is equivalent to one spanned by $\Gamma_1 + \Gamma_2$, $- \Gamma_1 + \Gamma_2$ or $\Gamma_2$. Finally, the remaining case ($a_2 = a_3 = 0$) can be seen to be equivalent to $\Gamma_1$. Thus the optimal system is

$$\{c \Gamma_1 + \Gamma_3, \, \pm \Gamma_1 + \Gamma_2, \, \Gamma_2, \, \Gamma_1\}.$$ 

If we admit the discrete symmetry $(z, t, u) \mapsto (-z, t, u)$ which maps $-\Gamma_1 + \Gamma_2$ to $\Gamma_1 + \Gamma_2$, the number of inequivalent subalgebras is reduced to four.
Chapter 3. One-dimensional diffusion-convection equation

For the second of the two special cases, that is, when \( n = (m + 2)/2 \), the commutator table from the more general example becomes

<table>
<thead>
<tr>
<th></th>
<th>( \Gamma_1 )</th>
<th>( \Gamma_2 )</th>
<th>( \Gamma_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>0</td>
<td>0</td>
<td>( \frac{m}{2} \Gamma_1 )</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \Gamma_3 )</td>
<td>( -\frac{m}{2} \Gamma_1 )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

while the adjoint table is

<table>
<thead>
<tr>
<th>Ad</th>
<th>( \Gamma_1 )</th>
<th>( \Gamma_2 )</th>
<th>( \Gamma_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>( \Gamma_1 )</td>
<td>( \Gamma_2 )</td>
<td>( \Gamma_3 + \frac{m}{2} \Gamma_1 )</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>( \Gamma_1 )</td>
<td>( \Gamma_2 )</td>
<td>( \Gamma_3 )</td>
</tr>
<tr>
<td>( \Gamma_3 )</td>
<td>( e^{-\frac{m}{2}} \Gamma_1 )</td>
<td>( \Gamma_2 )</td>
<td>( \Gamma_3 )</td>
</tr>
</tbody>
</table>

As in the general case, we start with the nonzero vector (3.12), with \( a_3 \) assumed to be nonzero, and rescale \( \Gamma \) such that \( a_3 = 1 \). Using the adjoint table, we act on \( \Gamma \) by \( \text{Ad}(\exp(c_1 \Gamma_1)) \) to obtain

\[
\Gamma^1 = \text{Ad}(\exp(c_1 \Gamma_1))\Gamma = [a_1 - c_1 \frac{m}{2}]\Gamma_1 + a_2 \Gamma_2 + \Gamma_3
\]

\[
= a_2 \Gamma_2 + \Gamma_3,
\]

(3.18)

where the coefficient of \( \Gamma_1 \) is eliminated by choosing \( c_1 = 2a_1/m \), \( m \neq 0 \). As discussed in the previous case, \( m = 0 \) does not arise in our analysis. Examining the adjoint table again, we see that there are no entries which can act on \( \Gamma^1 \) to simplify the coefficient of \( \Gamma_2 \). Hence every one-dimensional subalgebra spanned by a vector \( \Gamma \) with \( a_3 \neq 0 \) is equivalent to the subalgebra spanned by \( c\Gamma_2 + \Gamma_3 \) for \( c \in \mathbb{R} \).

Any remaining one-dimensional subalgebras are spanned by vectors of the form (3.12) with \( a_3 = 0 \). Assuming that \( a_2 \neq 0 \), we scale so that \( a_2 = 1 \). Acting on \( \Gamma \) by \( \text{Ad}(\exp(c_2 \Gamma_3)) \) gives

\[
\Gamma^1 = \text{Ad}(\exp(c_2 \Gamma_3))\Gamma = a_1 e^{\frac{m}{2} c_2} \Gamma_1 + \Gamma_2.
\]

(3.19)
Depending on the sign of \( a_1 \) the coefficient of \( \Gamma_1 \) can be made either +1, -1 or 0. Therefore, any one-dimensional subalgebra spanned by \( \Gamma \) with \( a_3 = 0, a_2 \neq 0 \) is equivalent to one spanned by \( \Gamma_1 + \Gamma_2, -\Gamma_1 + \Gamma_2 \) or \( \Gamma_2 \). Finally, the remaining case \( (a_2 = a_3 = 0) \) can be seen to be equivalent to \( \Gamma_1 \). Thus the optimal system is

\[
\{e\Gamma_2 + \Gamma_3, \pm \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1\}
\]

If we admit the discrete symmetry \((z,t,u) \mapsto (-z,t,u)\) which maps \(-\Gamma_1 + \Gamma_2\) to \(\Gamma_1 + \Gamma_2\), the number of inequivalent subalgebras is reduced to four.

The special form of (3.1) with \( D(u) = u^{-2} \) and \( K(u) = u^{-1} \) is spanned by the vector fields

\[
\Gamma_1 = \frac{\partial}{\partial z}, \quad \Gamma_2 = \frac{\partial}{\partial t},
\]

\[
\Gamma_3 = 2t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}, \quad \Gamma_4 = e^{-z} \left[ \frac{\partial}{\partial z} + u \frac{\partial}{\partial u} \right].
\]

The corresponding commutator table is

<table>
<thead>
<tr>
<th></th>
<th>( \Gamma_1 )</th>
<th>( \Gamma_2 )</th>
<th>( \Gamma_3 )</th>
<th>( \Gamma_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
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<td>0</td>
<td>0</td>
<td>(-\Gamma_4)</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
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<td>0</td>
<td>2( \Gamma_2 )</td>
<td>0</td>
</tr>
<tr>
<td>( \Gamma_3 )</td>
<td>0</td>
<td>(-2\Gamma_2)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \Gamma_4 )</td>
<td>( \Gamma_4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

while the adjoint table is

| \( \text{Ad} \) | \( \Gamma_1 \) | \( \Gamma_2 \) | \( \Gamma_3 \) | \( \Gamma_4 \) |
|----------------|----------------|----------------|----------------|
| \( \Gamma_1 \) | \( \Gamma_1 \) | \( \Gamma_2 \) | \( \Gamma_3 \) | \( e^c \Gamma_4 \) |
| \( \Gamma_2 \) | \( \Gamma_1 \) | \( \Gamma_2 \) | \( \Gamma_3 - 2e\Gamma_2 \) | \( \Gamma_4 \) |
| \( \Gamma_3 \) | \( e^{2c} \Gamma_2 \) | \( \Gamma_3 \) | \( \Gamma_4 \) |
| \( \Gamma_4 \) | \( \Gamma_1 - e\Gamma_4 \) | \( \Gamma_2 \) | \( \Gamma_3 \) | \( \Gamma_4 \) |
Given a nonzero vector

\[ \Gamma = a_1 \Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3 + a_4 \Gamma_4, \]

we again seek to simplify the maximum number of coefficients. In this case, we assume first that \( a_1 \neq 0 \), and hence assume that \( a_1 = 1 \). Using the adjoint table, we act on \( \Gamma \) by \( \text{Ad}(\exp(c_1 \Gamma_4)) \) to obtain

\[
\Gamma^I = \text{Ad}(\exp(c_1 \Gamma_4))\Gamma = \Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3 + [a_4 - c_1] \Gamma_4
\]

\[= \Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3, \quad (3.20)\]

where the coefficient of \( \Gamma_4 \) is eliminated by choosing \( c_1 = a_4 \). Acting on the new vector \( \Gamma^I \) by \( \text{Ad}(\exp(c_2 \Gamma_2)) \) gives

\[
\Gamma^{II} = \text{Ad}(\exp(c_2 \Gamma_2))\Gamma^I = \Gamma_1 + [a_2 - 2a_3 c_2] \Gamma_2 + a_3 \Gamma_3
\]

\[= \Gamma_1 + a_3 \Gamma_3, \quad (3.21)\]

where the coefficient of \( \Gamma_2 \) is cancelled by choosing \( c_2 = \frac{a_2}{2a_3} \), for \( a_3 \neq 0 \). Hence every one-dimensional subalgebra spanned by a vector \( \Gamma \) with \( a_1, a_3 \neq 0 \) is equivalent to the subalgebra spanned by \( \Gamma_1 + a_3 \Gamma_3 \). If \( a_3 = 0 \),

\[ \Gamma^I = \Gamma_1 + a_2 \Gamma_2 \]

and acting on \( \Gamma^I \) by \( \text{Ad}(\exp(c_3 \Gamma_3)) \) gives

\[
\Gamma^{II} = \text{Ad}(\exp(c_3 \Gamma_3))\Gamma^I = \Gamma_1 + a_2 e^{2c_3} \Gamma_2
\]

\[= \Gamma_1 + a_2 e^{2c_3} \Gamma_2, \quad (3.22)\]

and depending on the sign of \( a_2 \), the coefficient of \( \Gamma_2 \) can be made either \( +1, -1 \) or 0. Therefore, any one-dimensional subalgebra spanned by \( \Gamma \) with \( a_3 = 0, a_1 \neq 0 \) is equivalent to one spanned by \( \Gamma_1 + \Gamma_2, \Gamma_1 - \Gamma_2 \) or \( \Gamma_2 \).

Any remaining one-dimensional subalgebras are spanned by vectors of the form

\[ \Gamma = a_2 \Gamma_2 + a_3 \Gamma_3 + a_4 \Gamma_4. \]
Chapter 3. One-dimensional diffusion-convection equation

We now assume that \( a_3 \neq 0 \), and hence scale \( \Gamma \) so that \( a_3 = 1 \). Acting on \( \Gamma \) by \( \text{Ad}(\exp(c_1 \Gamma_2)) \) gives

\[
\Gamma^I = \text{Ad}(\exp(c_1 \Gamma_2))\Gamma = [a_2 - 2c_1] \Gamma_2 + \Gamma_3 + a_4 \Gamma_4 \\
= \Gamma_3 + a_4 \Gamma_4, \tag{3.23}
\]

when \( c_1 = a_2/2 \). Acting on \( \Gamma^I \) by \( \text{Ad}(\exp(c_2 \Gamma_1)) \) gives

\[
\Gamma^{II} = \text{Ad}(\exp(c_2 \Gamma_1))\Gamma^I = \Gamma_3 + a_4 e^{c_2} \Gamma_4 \tag{3.24}
\]

and depending on the sign of \( a_4 \), the coefficient of \( \Gamma_4 \) can be made either \( = 1, -1 \) or \( 0 \). Therefore, any one-dimensional subalgebra spanned by \( \Gamma \) with \( a_3 \neq 0, a_1 = 0 \) is equivalent to one spanned by \( \Gamma_3 + \Gamma_4, \Gamma_3 - \Gamma_4 \) or \( \Gamma_3 \).

When \( a_1 = a_3 = 0 \), we assume that \( a_4 \neq 0 \) and scale \( \Gamma \) so that \( a_4 = 1 \). Acting on \( \Gamma \) by \( \text{Ad}(\exp(c_1 \Gamma_3)) \) gives

\[
\Gamma^I = \text{Ad}(\exp(c_1 \Gamma_3))\Gamma = a_2 e^{2c_1} \Gamma_2 + \Gamma_4 \tag{3.25}
\]

and depending on the sign of \( a_2 \), the coefficient of \( \Gamma_2 \) can be made either \( +1, -1 \) or \( 0 \). Hence any one-dimensional subalgebra spanned by \( \Gamma \) with \( a_1 = a_3 = 0, a_4 \neq 0 \) is equivalent to one spanned by \( \Gamma_2 + \Gamma_4, -\Gamma_2 + \Gamma_4 \) or \( \Gamma_4 \).

Finally, the remaining case \( a_1 = a_3 = a_4 = 0 \) can be seen to be equivalent to \( \Gamma_2 \). Hence, for the case \( D(u) = u^{-2}, K(u) = u^{-1} \), the optimal system is

\[
\{\Gamma_1 + c\Gamma_3, \Gamma_1 \pm \Gamma_2, \Gamma_3 + \Gamma_4, \Gamma_3 - \Gamma_4, \Gamma_3, \pm \Gamma_2 + \Gamma_4, \Gamma_4, \Gamma_2\}.
\]

We admit the discrete symmetry \((z, t, u) \mapsto (z, -t, u)\) which maps \( \Gamma_1 + \Gamma_2 \) to \( \Gamma_1 - \Gamma_2 \) and \( \Gamma_2 + \Gamma_4 \) to \( -\Gamma_2 + \Gamma_4 \), reducing to eight the number of inequivalent subalgebras.

The special form of (3.1) with \( D(u) = 1 \) and \( K(u) = u^2 \) is spanned by the vector fields

\[
\Gamma_1 = \frac{\partial}{\partial z}, \quad \Gamma_2 = \frac{\partial}{\partial t}, \\
\Gamma_3 = 2t \frac{\partial}{\partial t} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u},
\]
\[ \Gamma_4 = 2t^2 \frac{\partial}{\partial t} + 2tz \frac{\partial}{\partial z} + (z - 2tu) \frac{\partial}{\partial u}, \]
\[ \Gamma_5 = 2t \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}. \]

The corresponding commutator table is

<table>
<thead>
<tr>
<th></th>
<th>$\Gamma_1$</th>
<th>$\Gamma_2$</th>
<th>$\Gamma_3$</th>
<th>$\Gamma_4$</th>
<th>$\Gamma_5$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
<td>$\Gamma_1$</td>
<td>$\Gamma_5$</td>
<td>0</td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>0</td>
<td>0</td>
<td>2$\Gamma_2$</td>
<td>2$\Gamma_3$</td>
<td>2$\Gamma_1$</td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>$-\Gamma_1$</td>
<td>$-2\Gamma_2$</td>
<td>0</td>
<td>2$\Gamma_4$</td>
<td>$\Gamma_5$</td>
</tr>
<tr>
<td>$\Gamma_4$</td>
<td>$-\Gamma_5$</td>
<td>$-2\Gamma_3$</td>
<td>$-2\Gamma_4$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\Gamma_5$</td>
<td>0</td>
<td>$-2\Gamma_1$</td>
<td>$-\Gamma_5$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

while the adjoint table is

<table>
<thead>
<tr>
<th>Ad</th>
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<th>$\Gamma_4$</th>
<th>$\Gamma_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1$</td>
<td>$\Gamma_1$</td>
<td>$\Gamma_2$</td>
<td>$\Gamma_3 - \epsilon \Gamma_1$</td>
<td>$\Gamma_4 - \epsilon \Gamma_5$</td>
<td>$\Gamma_5$</td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>$\Gamma_1$</td>
<td>$\Gamma_2$</td>
<td>$\Gamma_3 - 2\epsilon \Gamma_2$</td>
<td>$\Gamma_4 - 2\epsilon \Gamma_3 + 2\epsilon^2 \Gamma_2$</td>
<td>$\Gamma_2 - 2\epsilon \Gamma_1$</td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>$e^\epsilon \Gamma_1$</td>
<td>$e^{2\epsilon} \Gamma_2$</td>
<td>$\Gamma_3$</td>
<td>$e^{-2\epsilon} \Gamma_4$</td>
<td>$e^\epsilon \Gamma_5$</td>
</tr>
<tr>
<td>$\Gamma_4$</td>
<td>$\Gamma_1 + \epsilon \Gamma_5$</td>
<td>$\Gamma_2 + 2\epsilon \Gamma_3 + 2\epsilon^2 \Gamma_4$</td>
<td>$\Gamma_3 + 2\epsilon \Gamma_4$</td>
<td>$\Gamma_4$</td>
<td>$\Gamma_5$</td>
</tr>
<tr>
<td>$\Gamma_5$</td>
<td>$\Gamma_1$</td>
<td>$\Gamma_2 + 2\epsilon \Gamma_1$</td>
<td>$\Gamma_3 + \epsilon \Gamma_5$</td>
<td>$\Gamma_4$</td>
<td>$\Gamma_5$</td>
</tr>
</tbody>
</table>

Given a nonzero vector

\[ \Gamma = a_1 \Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3 + a_4 \Gamma_4 + a_5 \Gamma_5, \quad (3.26) \]

we again seek to simplify the maximum number of coefficients. Using the adjoint table, we act on $\Gamma$ by $\text{Ad}(\exp(\alpha \Gamma_2))$ to obtain

\[ \Gamma^1 = [a_1 - 2\alpha] \Gamma_1 + [a_2 - 2\alpha a_3 + 2\alpha^2 a_4] \Gamma_2 + [a_3 - 2\alpha a_4] \Gamma_3 + a_4 \Gamma_4 + a_5 \Gamma_5. \quad (3.27) \]
Chapter 3. One-dimensional diffusion-convection equation

Acting on $\Gamma^I$ by $\text{Ad}(\exp(\beta \Gamma_4))$ gives

$$\Gamma^{II} = [a_1 - 2\alpha] \Gamma_1 + [a_2 - 2\alpha a_3 + 2\alpha^2 a_4] \Gamma_2$$

$$+ [a_3 - 2\alpha a_4 + 2\beta (a_2 - 2\alpha a_3 + 2\alpha^2 a_4)] \Gamma_3$$

$$+ [2\beta (a_3 - 2\alpha a_4) + 2\beta^2 (a_2 - 2\alpha a_3 + 2\alpha^2 a_4) + a_4] \Gamma_4 + [\beta (a_1 - 2\alpha) + a_5] \Gamma_5.$$  

(3.28)

Let $\tilde{a}_i$ be the coefficient of $\Gamma_i$. We consider the coefficients of $\Gamma_2$, $\Gamma_3$ and $\Gamma_4$, that is

$$\tilde{a}_2 = 2a_4 \alpha^2 - 2a_3 \alpha + a_2,$$

$$\tilde{a}_3 = 2\beta a_2 + (1 - 4\alpha \beta) a_3 + 2\alpha a_4 (-1 + 2\alpha \beta)$$

$$\tilde{a}_4 = 2\beta^2 a_2 + 2\beta a_3 (1 - 2\alpha \beta) + a_4 (1 - 2\alpha \beta).$$

Let $\eta = a_3^2 - 2a_2 a_4$. There are three cases, depending on the sign of $\eta$.

Case 1: If $\eta > 0$, $\alpha$ is chosen as one of the real roots of $a_4 \alpha^2 - 2a_3 \alpha + a_2 = 0$ so that $\tilde{a}_2 = 0$. Setting

$$\beta = \frac{a_4}{2 \sqrt{a_3^2 - 2a_2 a_4}}$$

means that $\tilde{a}_4 = 0$ and $\tilde{a}_3 = -\sqrt{a_3^2 - 2a_2 a_4} \neq 0$ since $\eta > 0$. Thus $\Gamma^{II}$ can be written as

$$\Gamma^{II} = \tilde{a}_1 \Gamma_1 - \sqrt{a_3^2 - 2a_2 a_4} \Gamma_3 + \tilde{a}_5 \Gamma_5$$

$$= \tilde{a}_1 \Gamma_1 + \Gamma_3 + \tilde{a}_5 \Gamma_5,$$  

(3.29)
when $c_2 = -\tilde{a}_5$. Thus every choice of (3.26) with $\eta = a_3^2 - 2a_2a_4 > 0$ is equivalent to a multiple of $\Gamma_3$.

Case 2: If $\eta < 0$, let $\alpha = 0$ and choose $\beta = -a_3/(2a_2)$ so that

$$\tilde{a}_2 = a_2, \quad \tilde{a}_3 = 0, \quad \tilde{a}_4 = \frac{a_3^2}{2a_2} + a_4$$

and $\Gamma''_1$ can be written as

$$\Gamma''_1 = \tilde{a}_1 \Gamma_1 + \tilde{a}_2 \Gamma_2 + \tilde{a}_4 \Gamma_4 + \tilde{e}_5 \Gamma_5.$$  

Acting on $\Gamma''_1$ by $\text{Ad}(\exp(\epsilon \Gamma_3))$ gives

$$\Gamma''''_1 = \tilde{a}_1 \epsilon \Gamma_1 + \tilde{a}_2 \epsilon^2 \Gamma_2 + \tilde{a}_4 \epsilon^2 \Gamma_4 + \tilde{e}_5 \epsilon \Gamma_5$$

$$= \tilde{a}_2 \epsilon^2 [\Gamma_2 + \Gamma_4] + \tilde{a}_1 \epsilon \Gamma_1 + \tilde{e}_5 \epsilon \Gamma_5 \quad \text{if} \quad \tilde{a}_4 = a_2 \epsilon^4.$$  

Noting that since $\eta < 0$, $a_2 \neq 0$ we can rescale the coefficient of $[\Gamma_2 + \Gamma_4]$ to obtain

$$\Gamma''''_1 = \Gamma_2 + \Gamma_4 + \tilde{a}_1 \Gamma_1 + \tilde{e}_5 \Gamma_5.$$  

Acting on $\Gamma''''_1$ by $\text{Ad}(\exp(c_1 \Gamma_1))$ gives

$$\Gamma''' = \Gamma_2 + \Gamma_4 + \tilde{a}_1' \Gamma_1 + [\tilde{a}_5' - c_1] \Gamma_5$$

$$= \Gamma_2 + \Gamma_4 + \tilde{a}_1' \Gamma_1$$  

if $c_1 = \tilde{a}_5'$ and finally, by acting on $\Gamma'''$ by $\text{Ad}(\exp(c_2 \Gamma_5))$ gives

$$\Gamma''''_1 = \Gamma_2 + \Gamma_4 + [\tilde{a}_1' + 2c_2] \Gamma_1$$

$$= \Gamma_2 + \Gamma_4,$$  

when $c_2 = -\tilde{a}_1'/2$. Hence any choice of (3.26) with $\eta = a_3^2 - 2a_2a_4 < 0$ is equivalent to a multiple of $\Gamma_2 + \Gamma_4$.

Case 3: There are two subcases for $\eta a_3^2 - 2a_2a_4 = 0$, either not all the coefficients $a_2, a_3$ and $a_4$ disappear, or that $a_2 = a_3 = a_4 = 0$. Considering the first subcase, we let $\alpha = 0$ and choose $\beta = -a_4/(2a_2)$ so that

$$\tilde{a}_2 = a_2, \quad \tilde{a}_3 = 0, \quad \tilde{a}_4 = 0,$$
and \( \Gamma^\Pi \) can be written as

\[
\Gamma^\Pi = \tilde{a}_1 \Gamma_1 + \Gamma_2 + \tilde{c}_5 \Gamma_5,
\]

where the coefficient of \( \Gamma_2 \) can be rescaled since \( a_2 \neq 0 \). Acting on \( \Gamma^\Pi \) by \( \text{Ad}(\exp(c_1 \Gamma_1)) \) gives

\[
\Gamma^\Pi = (\tilde{a}_1 + 2c_1) \Gamma_1 + \Gamma_2 + \tilde{a}_5 e^\epsilon \Gamma_5
\]

\[= \Gamma_2 \tilde{a}_5 \Gamma_5,\tag{3.35}\]

when \( c_1 = -\tilde{a}_1/2 \). Acting on \( \Gamma^\Pi \) by \( \text{Ad}(\exp(\epsilon \Gamma_3)) \) gives

\[
\Gamma^\IV = e^{2\epsilon} \Gamma_2 + \tilde{a}_5 e^\epsilon \Gamma_5,
\]

which can be rescaled to become

\[
\Gamma^\IV = \Gamma_2 + \tilde{a}_5 e^{-\epsilon} \Gamma_5.
\]

Depending on the sign of \( \tilde{a}_5 \), the coefficient of \( \Gamma_5 \) can be made to be either \(+1, -1\) or 0. Hence any \( \Gamma \) with \( \eta = 0 \) but not all of \( a_2, a_3, a_4 = 0 \) is equivalent to either \( \Gamma_2 + \Gamma_5, \Gamma_2 - \Gamma_5 \) or \( \Gamma_2 \).

The second subcase is when \( a_2 = a_3 = a_4 = 0 \). In this case we can write \( \Gamma \) as

\[
\Gamma = a_1 \Gamma_1 + a_5 \Gamma_5.
\]

We assume that \( a_1 \neq 0 \), and hence assume that \( a_1 = 1 \). Acting on \( \Gamma \) by \( \text{Ad}(\exp(c_1 \Gamma_4)) \) gives

\[
\Gamma^\I = \Gamma_1 + [a_5 + c_1] \Gamma_5
\]

\[= \Gamma_1, \tag{3.37}\]

when \( c_1 = -a_5 \), otherwise if \( a_1 = 0, \Gamma = \Gamma_5 \). Hence, we have found an optimal system of one-dimensional subalgebras for the case \( D(u) = 1, K(u) = u^2 \) to be those spanned by

\[\{ \Gamma_3, \ Gamma_2 + \Gamma_4, \ Gamma_2 \pm \Gamma_5, \ Gamma_2, \ Gamma_1, \ Gamma_5 \} .\]
Table 3.2: Optimal system for diffusion-convection equations \( c \in \mathbb{R} \).

<table>
<thead>
<tr>
<th>( D(u) )</th>
<th>( K(u) )</th>
<th>Optimal System</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u^m )</td>
<td>( u^n )</td>
<td>( \Gamma_3, \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
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<td>( u^m )</td>
<td>( c\Gamma_1 + \Gamma_3, \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
<td>( u^m )</td>
<td>( u^{m+2} )</td>
<td>( c\Gamma_2 + \Gamma_3, \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
<td>( u^{-2} )</td>
<td>( u^{-1} )</td>
<td>( \Gamma_1 + c\Gamma_3, \Gamma_1 + \Gamma_2, \Gamma_3 + \Gamma_4, \Gamma_3 - \Gamma_4, \Gamma_2 + \Gamma_4, \Gamma_2, \Gamma_3, \Gamma_4 )</td>
</tr>
<tr>
<td>( u^m )</td>
<td>( \ln(u) )</td>
<td>( \Gamma_3, \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
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<td>( \ln(u) )</td>
<td>( c\Gamma_1 + \Gamma_3, \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
<td>( u^{-2} )</td>
<td>( \ln(u) )</td>
<td>( c\Gamma_2 + \Gamma_3, \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
<td>( u^m )</td>
<td>( u\ln(u) - u )</td>
<td>( \Gamma_3, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( u\ln(u) - u )</td>
<td>( c\Gamma_2 + \Gamma_3, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
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<td>( e^{nu} )</td>
<td>( \Gamma_3, \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
<td>( e^{mu} )</td>
<td>( e^{mu} )</td>
<td>( c\Gamma_1 + \Gamma_3, \Gamma_1 + \Gamma_2, \Gamma_2, \Gamma_1 )</td>
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<td>( e^{mu/2} )</td>
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<td>( e^{mu} )</td>
<td>( u^2 )</td>
<td>( \Gamma_3, \Gamma_2, \Gamma_1 )</td>
</tr>
<tr>
<td>( \text{const} )</td>
<td>( u^2 )</td>
<td>( \Gamma_3, \Gamma_2 + \Gamma_4, \Gamma_2 + \Gamma_5, \Gamma_5, \Gamma_1, \Gamma_2 )</td>
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</tbody>
</table>

We admit the discrete transformation \((z, t, u) \mapsto (-z, t, -u)\) which maps \( \Gamma_2 + \Gamma_5 \) to \( \Gamma_2 - \Gamma_5 \), reducing the number of inequivalent subalgebras to six.

The optimal system for each of the remaining entries in Table 3.1 have been determined, including all special cases which arise from particular values of \( m \) or \( n \) and are listed in Table 3.2.
3.3 Symmetry Reductions

For each special case of the governing equation (3.1) listed in Table 3.1, the corresponding optimal system given in Table 3.2 can be used to construct a minimal complete set of reductions. Reduction of the governing P.D.E. by two members of the same symmetry conjugacy classes is equivalent by a change of variable (Ovsiannikov [59]; Olver [57]). Therefore we need only consider reduction by members of the optimal system of symmetries, as these are representatives of the conjugacy classes.

A reduction of variables of the one-dimensional diffusion-convection equation (3.1) will result in an O.D.E., which may or may not be solvable. The similarity variables used in the reduction of order follow from the solution of the characteristic equation

\[ \frac{dz}{z} = \frac{dt}{T} = \frac{du}{U}. \]

A symmetry of the type \( c \Gamma_1 + \Gamma_2, c \in \mathbb{R} \), regardless of the form of \( D(u) \) and \( K(u) \), will always generate a travelling wave solution. We do not show the reduced equation for these cases. This means that for all the entries in Table 3.2, excluding Burgers' equation and the case when \( D(u) = u^{-2} \) and \( K(u) = u^{-1} \), the first symmetry listed is the only nontrivial symmetry remaining in each optimal system.

As an example, we consider the reductions for the case \( D(u) = u^m \) and \( K(u) = u^n \). From Table 3.2, we see that excluding the travelling wave solution which would be generated by the case \( \Gamma_1 + \Gamma_2 \), the only other nontrivial symmetry is \( \Gamma_3 \), that is

\[ \Gamma_3 = (m - n + 1)z \frac{\partial}{\partial z} + (m - 2n + 2)t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}. \]

The corresponding characteristic equation is

\[ \frac{dz}{(m - n + 1)z} = \frac{dt}{(m - 2n + 2)t} = \frac{du}{u}. \]

The solution from the first two expressions gives

\[ \frac{1}{m - n + 1} \ln z = \frac{1}{m - 2n + 2} \ln t + \text{const}, \quad \text{for } m - n + 1, m - 2n + 2 \neq 0. \]
Rearranging, we obtain the invariant

\[ \rho = z t^{-\frac{(m-n+1)}{(m-2n+2)}}. \]

The solution from the first and third expression is

\[ \frac{1}{m-2n+2} \ln t = \ln u + \text{const} \]

and rearranging, we obtain the functional form

\[ u(z, t) = F(\rho) t^{\frac{1}{(m-2n+2)}}, \text{ where } \rho = z t^{-\frac{(m-n+1)}{(m-2n+2)}}. \]

The functional form for \( u \) is used in the governing equation (3.1), where for example

\[ \frac{\partial u}{\partial z} = \frac{dF}{d\rho} t^{-\frac{(m-n)}{(m-2n+2)}}. \]

Substituting the expression for \( u_z \), \( u_{zz} \) and \( u_t \) into the governing equation and simplifying, we obtain the O.D.E.

\[ \frac{1}{m-2n+2} \left[ F - (m-n+1) \rho \frac{dF}{d\rho} \right] = \frac{d}{d\rho} \left[ F^m \frac{dF}{d\rho} - F^m \right], \]

where \( n \neq \frac{m+2}{2}, m + 1 \).

Table 3.3 shows the reduced O.D.E.s which relate symmetry invariants for the generators of the first symmetry algebra of each of the optimal systems in Table 3.2, excluding the case when \( D(u) = u^{-2}, K(u) = u^{-1} \) and Burgers’ equation.

From Table 3.2, we see that the case when \( D(u) = u^{-2} \) and \( K(u) = u^{-1} \) has six symmetries which are nontrivial and do not generate travelling wave solutions. The reductions from these symmetries are listed in Table 3.4.

From Table 3.4, we see that for the fifth and sixth entries, the invariant \( \rho \) is equivalent to \( z \) and \( t \) respectively. We also note that for the first entry, the expression \( \frac{1}{c} \) in the invariant \( \rho \) indicates that we must consider the case \( c = 0 \) separately. However, when \( c = 0 \), the symmetry \( \Gamma_1 + c \Gamma_3 \) becomes trivial and hence is neglected.
Chapter 3. One-dimensional diffusion-convection equation

Table 3.3: Reduced O.D.E.s of the 1-D diffusion-convection equation \( m, n \in \mathbb{R} \).

<table>
<thead>
<tr>
<th>( D(u) )</th>
<th>( K(u) )</th>
<th>Reduced O.D.E.</th>
</tr>
</thead>
</table>
| \( u^m \) | \( u^n \) | \[
\frac{1}{m - 2n + 2} \left[ F - (m - n + 1)\rho \frac{dF}{d\rho} \right] = \frac{d}{d\rho} \left[ F^m \frac{dF}{d\rho} - F^n \right],
\]
with \( u = F(\rho) t^{\frac{m-2n+2}{m-2n+2}}, \quad \rho = z t^{\frac{m-n+1}{m-2n+2}} \)

| \( u^m \) | \( u^{m+1}, m \neq 0 \) | \[
\frac{1}{m} \left[ -F + c \frac{dF}{d\rho} \right] = \frac{d}{d\rho} \left[ F^m \frac{dF}{d\rho} - F^{m+1} \right],
\]
with \( u = F(\rho) t^{-1/m}, \quad \rho = z + \frac{c}{m} \ln(t) \)

| \( u^m \) | \( u^{(m+2)/2}, m \neq 0 \) | \[
\frac{-1 - \frac{2c(3m + 4)}{m^2}}{m^2} F^m + \frac{c(m + 2)}{m} F^{m/2} \frac{dF}{d\rho} = \frac{d}{d\rho} \left[ F^m \frac{dF}{d\rho} - F^{m+1} \right],
\]
with \( u = F(\rho) z^{2/m}, \quad \rho = t - \frac{2c}{m} \ln(z) \)

| \( \ln(u) \) | \( \ln(u) \) | \[
\frac{1}{m + 2} \left[ F - (m + 1)\rho \frac{dF}{d\rho} \right] = \frac{d}{d\rho} \left[ F^m \frac{dF}{d\rho} - \ln(F) \right],
\]
with \( u = F(\rho) t^{\frac{m+2}{m+2}}, \quad \rho = z t^{\frac{m+1}{m+2}} \)

| \( u^{-1} \) | \( \ln(u) \) | \[
F - c \frac{dF}{d\rho} = \frac{d}{d\rho} \left[ F^{-1} \frac{dF}{d\rho} - \ln(F) \right], \text{with } u = t F(\rho), \quad \rho = z - c \ln(t) \)

| \( u^{-2} \) | \( \ln(u) \) | \[
-1 = \frac{d}{d\rho} \left[ c^2 F^{-2} \frac{dF}{d\rho} - c F^{-1} - c \ln(F) - F \right],
\]
with \( u = F(\rho) z^{-1}, \quad \rho = t + c \ln(z) \)

continued on next page
### Reduced O.D.E.

<table>
<thead>
<tr>
<th>$D(u)$</th>
<th>$K(u)$</th>
<th>Reduced O.D.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^m$</td>
<td>$u \ln(u) - u$</td>
<td>$\frac{1}{m} \left[ F - (1 + m \rho) \frac{dF}{d\rho} \right] = \frac{d}{d\rho} \left[ \frac{F^m}{m} \frac{dF}{d\rho} - F \ln(F) + F \right]$, with $u = F(\rho)t^\frac{1}{m}$, $\rho = zt^{-1} - \frac{1}{m} \ln(t)$</td>
</tr>
<tr>
<td>$1$</td>
<td>$u \ln(u) - u$</td>
<td>$\frac{1}{c} F = \frac{d}{d\rho} \left[ \frac{dF}{d\rho} - F \ln(F) + F \right]$, with $u = F(\rho)e^t/c$, $\rho = z - \frac{1}{2}t^2$</td>
</tr>
<tr>
<td>$e^{mu}$</td>
<td>$e^{nu}$, $n \neq m$</td>
<td>$\frac{1}{m - 2n} \left[ 1 - (m - n) \rho \frac{dF}{d\rho} \right] = \frac{d}{d\rho} \left[ e^{mF} \frac{dF}{d\rho} - e^{nF} \right]$, with $u = F(\rho) + \frac{1}{m - 2n} \ln(t)$, $\rho = zt^\left(\frac{m-n}{m-2n}\right)$</td>
</tr>
<tr>
<td>$e^{mu}$</td>
<td>$e^{nu}$, $m \neq 0$</td>
<td>$\frac{1}{m} \left[ \frac{dF}{d\rho} - 1 \right] = \frac{d}{d\rho} \left[ e^{mF} \frac{dF}{d\rho} - e^{mF} \right]$, with $u = F(\rho) - \frac{1}{m} \ln(t)$, $\rho = z + \frac{c}{m} \ln(t)$</td>
</tr>
<tr>
<td>$e^{mu/2}$</td>
<td>$e^{nu/2}$, $m \neq 0$</td>
<td>$\frac{e^{mF/2} - 2}{m} e^{mF} = \frac{d}{d\rho} \left[ \frac{4c^2}{m^2} e^{mF} \frac{dF}{d\rho} - \frac{6c}{m^2} e^{mF} + \frac{2c}{m} e^{mF/2} - F \right]$, with $u = F(\rho) + \frac{2}{m} \ln(z)$, $\rho = t - \frac{2c}{m} \ln(z)$</td>
</tr>
<tr>
<td>$e^{mu}$</td>
<td>$u^2$, $m \neq 0$</td>
<td>$\frac{1}{m} \left[ 1 - (2 + m \rho) \frac{dF}{d\rho} \right] = \frac{d}{d\rho} \left[ e^{mF} \frac{dF}{d\rho} - F^2 \right]$, with $u = F(\rho) + \frac{1}{m} \ln(t)$, $\rho = zt^{-1} - \frac{2}{m} \ln(t)$</td>
</tr>
</tbody>
</table>

Although Burgers' equation has been extensively studied, the reductions from the nontrivial symmetries in the optimal system have been listed in Table 3.5 for completeness.

The reduced O.D.E.s listed in Tables 3.3, 3.4 and 3.5 may or may not be solvable
Table 3.4: Reduced O.D.E.s for $D(u) = u^{-2}$, $K(u) = u^{-1}$.

<table>
<thead>
<tr>
<th>$\Gamma$</th>
<th>Reduced O.D.E.</th>
</tr>
</thead>
</table>
| $\Gamma_1 + c\Gamma_3$ | \[
\frac{1}{c} \left[ F - \frac{1}{c} \frac{dF}{d\rho} \right] = \frac{d}{d\rho} \left[ f^{-2} \frac{dF}{d\rho} - F^{-1} \right],
\text{with } u = F(\rho) e^{\frac{1}{2} \rho}, \quad \rho = z - \frac{1}{2c} t
\] |
| $\Gamma_3 + \Gamma_4$ | \[
-\frac{1}{2} \frac{d}{d\rho} e^{2\rho} = F^{-2} \frac{d^2 F}{d\rho^2} - 2f^{-3} \left( \frac{dF}{d\rho} \right)^2 - 2F^{-2} \frac{dF}{d\rho} - F^{-1},
\text{with } u = F(\rho) e^{e^z + \rho}, \quad \rho = e^z - \frac{1}{2} \ln t
\] |
| $\Gamma_3 - \Gamma_4$ | \[
\frac{1}{2} \frac{d}{d\rho} e^{-2\rho} = F^{-2} \frac{d^2 F}{d\rho^2} - 2f^{-3} \left( \frac{dF}{d\rho} \right)^2 + 2F^{-2} \frac{dF}{d\rho} + F^{-1},
\text{with } u = F(\rho) e^{-e^z + 2}, \quad \rho = e^z + \frac{1}{2} \ln t
\] |
| $\Gamma_2 + \Gamma_4$ | \[
\frac{d}{d\rho} \left[ F^{-2} \frac{dF}{d\rho} + F \right] = 0, \text{with } u = F(\rho) e^z, \quad \rho = e^z - t
\] |
| $\Gamma_3$ | \[
\frac{1}{2} \frac{dF}{dz} = \frac{d}{dz} \left[ F^{-2} \frac{dF}{dz} - F^{-1} \right], \text{with } u = t^{1/2} F(z),
\] |
| $\Gamma_4$ | \[
\frac{dF}{dt} = 0, \text{with } u = F(t) e^z
\] |
Table 3.5: Reduced O.D.E.s of the 1-D Burgers’ equation.

<table>
<thead>
<tr>
<th>Γ</th>
<th>Reduced O.D.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Γ₃</td>
<td>(\frac{d}{d\rho} \left[ \frac{dF}{d\rho} - F^2 + \frac{1}{2} \rho F \right] = 0, \text{ with } u = F(\rho)t^{-1/2}, \rho = zt^{-1/2})</td>
</tr>
<tr>
<td>Γ₂ + Γ₄</td>
<td>(\frac{d^2 F}{d\rho^2} - 2F \frac{dF}{d\rho} - \rho = 0,) \text{ with } u = F(\rho)(1 + 2t^2)^{-1/2} + zt(1 + 2t^2)^{-1}, \rho = z(1 + 2t^2)^{-1/2})</td>
</tr>
<tr>
<td>Γ₂ + Γ₅</td>
<td>(\frac{d^2 F}{d\rho^2} - 2F \frac{dF}{d\rho} - 1 = 0, \text{ with } u = F(\rho) + t, \rho = z - t^2)</td>
</tr>
<tr>
<td>Γ₅</td>
<td>(\frac{dF}{dt} + \frac{1}{t} F = 0, \text{ with } u = F(t) + \frac{z}{2t})</td>
</tr>
</tbody>
</table>

in closed form. A solvable example is the first entry in Table 3.3 which may be integrated once, when \(n = m + 2\), to give

\[
-\frac{1}{m + 2} \rho F = F^m \frac{dF}{d\rho} - F^{m+2} + \text{const.}
\]

If the constant of integration is taken to be zero, this first order O.D.E. may be solved exactly for the cases \(m = 0\) and \(m = 1\). This leads to solutions of the type noted by Philip [64].

### 3.4 Discussion

We have considered the symmetry analysis of the general class of one-dimensional diffusion-convection equations (3.1) and have found all the functional forms of \(D(u)\)
and $K(u)$ which admit symmetries other than translations in space and time. For each of the special cases, the determination of the corresponding optimal system has ensured that a minimal complete set of reductions was found from the symmetries of the governing equation.
Chapter 4

Classical symmetry reductions of higher-dimensional nonlinear diffusion-convection equations

The nonlinear diffusion-convection equation

\[ \frac{\partial u}{\partial t} = \nabla \cdot (D(u) \nabla u) - \frac{dK}{du} \frac{\partial u}{\partial z} \]  

(4.1)

with \( \nabla \) the Laplacian operator in N spatial dimensions (N = 1, 2 or 3) has a variety of applications to porous media, including displacement of one liquid by another (Fokas and Yortsos [35]), unsaturated flow (Klute [48]), transport of a solute with adsorption to pore surfaces (Rosen [70]), and saturated flow in a swelling medium (Smiles and Rosenthal [80]). \( D(u) \) is the concentration-dependent diffusivity and, based on Darcy's law (Klute [48]) for hydrological flows, \( K \) will be viewed as the concentration-dependent conductivity.

Because of its practical importance, much work has been devoted to constructing exact and approximate solutions to Eq. (4.1). Time-dependent solutions have been found mainly in one spatial dimension. These solutions are of two types. The first
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type relies on special integrable models which can be transformed to the linear diffusion equation. The integrable models are the Burgers’ equation (Clothier et al. [26]) (where \(D\) is constant and \(K\) is quadratic), and the Fokas-Yortsos-Rosen equation (Fokas and Yortsos [35]; Rosen [70]; extending the work of Knight and Philip [49]) (where \(D(u) = a/(b - u)^2\) and \(K'(u) = \lambda/2(b - u)^2\) with \(a, b, \lambda \in \mathbb{R}\)). In the second method of solution, similarity solutions follow from classical Lie symmetry group reductions when \(D(u)\) is a power law or an exponential, as discussed in the previous chapter.

For higher dimensions, much less is known. Unlike the one-dimensional case there are no linearisable models in two dimensions (Broadbridge [11]) and we doubt that integrable models exist in three dimensions. Next we turn to the other best-known approach for obtaining exact solutions. The classical Lie group symmetries are investigated for the general class of equations (4.1) in two and three dimensions. We find that the only forms which have additional symmetries, beyond translations and rotations, are with power-law, log and exponential functions for \(D(u)\) and \(K(u)\). Full reduction to an O.D.E. is possible in only a restricted number of cases, including the two dimensional Burgers’ equation, i.e. \(D(u)\) constant and \(K(u)\) quadratic and some special power-law, log and exponential functions for \(D(u)\) and \(K'(u)\).

Recently, Philip and Knight [65] have obtained a two-step reduction to an O.D.E. for power law forms of \(D(u)\) and \(K(u)\) where (4.1) is expressed in polar coordinates. Solutions obtained in this way follow neither from integrable models nor from two-stage classical symmetry reductions. We show that only the first stage of this technique is the result of a classical group reduction. In fact, the reduced equation obtained by Philip and Knight after the first step can be shown to have no classical symmetries. The reduction used by Philip and Knight has a power-law time dependence. However, their solution method does not apply to the case \(D = u^{-1}\) in two dimensions or to the case \(D = u^{-2/3}\) in three dimensions. We extend the
method of Philip and Knight to these singular cases. We find that for these special
cases, we obtain a much wider set of solutions than are available in any other single
model.

4.1 Symmetry analysis - 2-D nonlinear diffusion-convection equation

We consider the classical Lie group symmetry analysis of the class of equations (4.1)
in two dimensions, that is

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ D(u) \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial z} \left[ D(u) \frac{\partial u}{\partial z} \right] - \frac{\partial K}{\partial u} \frac{\partial u}{\partial z}.
\] (4.2)

We are not specifically concerned with unsaturated flow in a porous medium, but
in the general equation, so that \( D(u) \) and \( K(u) \) are completely arbitrary. We wish
to identify any special forms of \( D \) and \( K \) which possess additional symmetries.

We consider the infinitesimal transformation

\[
\begin{align*}
\tilde{u} &= e^{\epsilon \Gamma} u = u + \epsilon U(x, z, t, u) + O(\epsilon^2), \\
\tilde{t} &= e^{\epsilon \Gamma} t = t + \epsilon T(x, z, t, u) + O(\epsilon^2), \\
\tilde{x} &= e^{\epsilon \Gamma} x = x + \epsilon X(x, z, t, u) + O(\epsilon^2), \\
\tilde{z} &= e^{\epsilon \Gamma} z = z + \epsilon Z(x, z, t, u) + O(\epsilon^2),
\end{align*}
\] (4.3)

where

\[
\Gamma = X \frac{\partial}{\partial x} + Z \frac{\partial}{\partial z} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u}
\]

is the infinitesimal generator [57], [7]. We then extend Eq. (4.3) to first and second
order by the prolongation formulae, for example,

\[
\frac{\partial u_{\ast}}{\partial x_{\ast}} = \frac{\partial u}{\partial x} + \epsilon U_1 + O(\epsilon^2),
\]
Chapter 4. Higher-dimensional nonlinear diffusion-convection equations

where

\[ U_1 = \frac{D}{Dx} U - \left( \frac{D}{Dx} \mathcal{A} \right) u_x - \left( \frac{D}{Dx} \mathcal{Z} \right) u_z - \left( \frac{D}{Dx} \mathcal{T} \right) u_t \] (4.4)

and\( \frac{D}{Dx} \) is the total derivative operator with respect to\( x \)

\[ \frac{D}{Dx} F(x, z, t, u) = \frac{\partial F}{\partial x} + u_x \frac{\partial F}{\partial u}. \]

Invariance of the governing equation (4.2) under the infinitesimal transformation (4.3) and assuming that the derivatives of\( u \) are independent leads to a set of determining relations, which are linear P.D.E.'s in \( \mathcal{A}, \mathcal{Z}, \mathcal{T}, \mathcal{U} \). As for the symmetry analysis of the one-dimensional diffusion-convection equation in the previous chapter, the symmetry analysis was performed using the software package Dimsym (Sherring [78]) under Reduce (Hearn [42]).

For totally arbitrary\( D \) and\( K \), the only symmetries are the generators of space and time translations,

\[ \Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial z}, \quad \Gamma_3 = \frac{\partial}{\partial t}. \]

Dimsym flags that during the solution of the determining equations, division is made by the expressions

\[ \frac{dD}{du} \]

and

\[ D \frac{dD}{du} \frac{d^3D}{du^3} - 2D \left( \frac{d^2D}{du^2} \right)^2 + \left( \frac{dD}{du} \right)^2 \frac{d^2D}{du^2}. \]

Equating the first expression to zero gives that\( D(u) = \text{const} \) while equating the second expression to zero and solving gives

\[ D(u) = \alpha (u + \beta)^m \quad \text{or} \quad D(u) = \alpha e^{mu} \]

and by rescaling, these three expressions may be simplified to

\[ D(u) = 1, \quad D(u) = u^m \quad \text{or} \quad D(u) = e^{mu}. \]
We note that these are the same functional forms of $D$ as those flagged by Dimsym during the symmetry analysis of the one-dimensional case. As well, Dimsym flags that the assumption has been made that
\[
\frac{dK}{du} \quad \text{and} \quad 1
\]
are linearly independent. If this pair of expressions is taken to be linearly dependent, solving the O.D.E.
\[
\frac{dK}{du} = \alpha
\]
leads to the form $K(u) = \alpha u + \beta$. However, as in the one-dimensional case, we do not consider this a special case, since any form of the diffusion-convection equation with $K(u)$ linear may be obtained by a change of reference frame from the case of pure diffusion, which has been thoroughly analysed by Galaktionov et al. [38]. Up to a linear change of variables, there are three forms of $D(u)$ for which (4.2) may have extra symmetries,
\[
D(u) = u^m, \quad D(u) = e^{mu} \quad \text{and} \quad D(u) = 1.
\]

Considering each of these forms of $D(u)$ in turn, and performing the symmetry analysis while holding $K(u)$ arbitrary leads to all of the combinations of $D(u)$ and $K(u)$ which have extra symmetries. In Table 4.1, $\Gamma_1, \Gamma_2, \Gamma_3$ are not listed, as they are common to all cases. We do not list the case $K'(u)$ constant as this can be transformed to the case of pure diffusion, analysed fully by Galaktionov et al. [38].

Unlike the one-dimensional Burgers’ equation, the two-dimensional Burgers’ equation
\[
\frac{\partial t}{\partial u} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - u \frac{\partial u}{\partial z}, \quad (4.5)
\]
cannot, in general be transformed to a linear equation (Broadbridge [11]). However, the two-dimensional Burgers’ equation (4.5) does have special symmetry properties. From Table 4.1, we see that it is the only form of Eq. (4.2) that has five linearly
Table 4.1: Symmetries for the 2-D diffusion-convection equation $m, n \in \mathbb{R}$.

<table>
<thead>
<tr>
<th>$D(u)$</th>
<th>$K'(u)$</th>
<th>$\Gamma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^m$</td>
<td>$u^n$</td>
<td>$\Gamma_4 = (m - 2n) t \frac{\partial}{\partial t} + (m - n) x \frac{\partial}{\partial x} + (m - n) z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$u^m$</td>
<td>$\ln(u)$</td>
<td>$\Gamma_4 = m t \frac{\partial}{\partial t} + m x \frac{\partial}{\partial x} + (mz + t) \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$e^{mu}$</td>
<td>$e^{nu}$</td>
<td>$\Gamma_4 = (m - 2n) t \frac{\partial}{\partial t} + (m - n) x \frac{\partial}{\partial x} + (m - n) z \frac{\partial}{\partial z} + \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$e^{mu}$</td>
<td>$u$</td>
<td>$\Gamma_4 = m t \frac{\partial}{\partial t} + m x \frac{\partial}{\partial x} + (mz + t) \frac{\partial}{\partial z} + \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>const</td>
<td>$u$</td>
<td>$\Gamma_4 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u}$</td>
</tr>
</tbody>
</table>

independent symmetry generators. The two nontrivial symmetries $\Gamma_4$ and $\Gamma_5$ are compatible because they obey the simple commutation property $[\Gamma_5, \Gamma_4] = -\Gamma_5$. This means that after a classical Lie point symmetry reduction of (4.5) by $\Gamma_5$, $\Gamma_4$ will be an inherited symmetry of the reduced equation and a second reduction will be possible. Hence, (4.5) will be reduced from a P.D.E., with one dependent variable and three independent variables, to an O.D.E. The two-dimensional Burgers' equation is one of six cases of the two-dimensional nonlinear convection-diffusion equation that can be fully reduced to an O.D.E. by classical Lie symmetry reductions. The optimal system and subsequent reductions of (4.5) are considered in the following
The remaining forms of the two-dimensional diffusion-convection equation which can be fully reduced to an O.D.E. by classical Lie symmetry reductions are

\[
D(u) = u^m \quad K'(u) = u^n, \\
D(u) = u^m \quad K'(u) = u^{m/2}, \\
D(u) = e^{mu} \quad K'(u) = e^{mu}, \\
D(u) = e^{mu} \quad K'(u) = e^{mu/2}, \\
D(u) = 1 \quad K(u) = \ln u. 
\] (4.6)

We consider each of these cases below.

If we consider the commutator table of the symmetries for the general power law case, that is \( D(u) = u^m, K'(u) = u^n \), we obtain

<table>
<thead>
<tr>
<th></th>
<th>( \Gamma_1 )</th>
<th>( \Gamma_2 )</th>
<th>( \Gamma_3 )</th>
<th>( \Gamma_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>((m - n)\Gamma_1)</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>((m - n)\Gamma_2)</td>
</tr>
<tr>
<td>( \Gamma_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>((m - 2n)\Gamma_3)</td>
</tr>
<tr>
<td>( \Gamma_4 )</td>
<td>(-(m - n)\Gamma_1)</td>
<td>(-(m - n)\Gamma_2)</td>
<td>(-(m - 2n)\Gamma_3)</td>
<td>0</td>
</tr>
</tbody>
</table>

We find that for arbitrary \( m \) and \( n \), \( n \neq m \), \( n \neq m/2 \), the optimal system is

\[
\{ \Gamma_4, \pm \Gamma_1 + c \Gamma_2 + \Gamma_3, \pm \Gamma_2 + \Gamma_3, c \Gamma_1 + \Gamma_2, \Gamma_1 \}
\]

for some \( c \in \mathbb{R} \). The only nontrivial symmetry of the optimal system is \( \Gamma_4 \). Reduction of the governing equation by \( \Gamma_4 \) leads to the P.D.E.

\[
\frac{n - m}{m - 2n} \eta^3 \eta^{1/(n-m)} \frac{\partial}{\partial \eta} \left[ \eta^{1/(n-m)} F \right] = \rho^2 \frac{\partial}{\partial \rho} \left[ (1 + \rho^2) F \right] + \eta^2 \frac{\partial}{\partial \eta} \left[ F \frac{\partial F}{\partial \eta} \right] \\
+ 2 \rho \eta \frac{\partial}{\partial \eta} \left[ F \frac{\partial F}{\partial \rho} \right] + \rho^2 \eta F \frac{\partial F}{\partial \rho},
\] (4.7)
Chapter 4. Higher-dimensional nonlinear diffusion-convection equations

where

\[ u = F(\rho, \eta) t^{1/(m-2n)}, \quad \rho = \frac{x}{z}, \quad \eta = x t^{(n-m)/(m-2n)}. \]

The reduced equation (4.7) has no symmetries, so no further reductions are possible. However, there are two combinations of \( m \) and \( n \) where reduction of the governing equation to an O.D.E. is possible.

The first case where reduction of the nonlinear diffusion-convection equation to an O.D.E is possible is when \( n = m \), i.e. \( D(u) = u^m \) and \( K'(u) = u^m \). In this case, we find that the optimal system is

\[ \{ c \Gamma_1 + d \Gamma_2 + \Gamma_4, \pm \Gamma_1 + c \Gamma_2 + \Gamma_3, \pm \Gamma_2 + \Gamma_3, c \Gamma_1 + \Gamma_2, \Gamma_1 \} \]

for some \( c, d \in \mathbb{R} \). If we reduce the governing equation using the first element from the optimal system \( (c, d \neq 0) \) then the resulting P.D.E. is

\[
\frac{1}{m} \left[ \frac{\partial F}{\partial \eta} - F \right] = \left[ \frac{1}{c^2 + \frac{1}{d^2}} \right] \frac{\partial}{\partial \rho} \left[ F^m \frac{\partial F}{\partial \rho} \right] + \frac{1}{c^2} \frac{\partial}{\partial \eta} \left[ F^m \frac{\partial F}{\partial \eta} \right] + \frac{2}{c^2} \frac{\partial}{\partial \eta} \left[ F^m \frac{\partial F}{\partial \rho} \right] + \frac{1}{d} F^m \frac{\partial F}{\partial \rho},
\]

where

\[ u = F(\rho, \eta) t^{-1/m}, \quad \rho = \frac{x}{c} - \frac{z}{d}, \quad \eta = \frac{x}{c} + \frac{\ln t}{m}. \]

Equation (4.8) admits the two symmetries,

\[ \Gamma_1^1 = \frac{\partial}{\partial \rho}, \quad \Gamma_2^1 = \frac{\partial}{\partial \eta}, \]

with corresponding optimal system \( \{ c_1 \Gamma_1^1 + \Gamma_2^1, \Gamma_1^1 \} \), for some \( c_1 \in \mathbb{R} \), where the notation \( \Gamma_j^i \) is used to denote the \( j^{th} \) symmetry of the reduced equation obtained using the \( i^{th} \) symmetry of the optimal system of the original equation. Hence a second reduction is possible using \( c_1 \Gamma_1^1 + \Gamma_2^1 (c_1 \neq 0) \) and the subsequent O.D.E is

\[
\frac{d}{d\gamma} \left[ \exp \left( \frac{c^2 c_1 d \gamma}{d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1} \right) G^m \frac{dG}{d\gamma} \right] + \frac{c^2 c_1^2 d^2}{m(d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1)} \exp \left( \frac{c^2 c_1 d \gamma}{d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1} \right) \left[ \frac{dG}{d\gamma} - c_1 G \right] = 0,
\]

(4.9)
Chapter 4. Higher-dimensional nonlinear diffusion-convection equations

where

\[ F = G(\gamma), \quad \gamma = \frac{\rho}{c_1} - \eta. \]

Equation (4.9) may be written as

\[
\frac{d}{d\gamma} \left[ \exp \left( \frac{c^2 c_1 d \gamma}{d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1} \right) G^m \frac{dG}{d\gamma} \right] + \frac{c^2 c_1^2 d^2}{m(d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1)} \frac{d}{d\gamma} \left[ \exp \left( \frac{c^2 c_1 d \gamma}{d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1} \right) G \right] = 0
\]

(4.10)

when

\[
\frac{c^2 c_1 d}{d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1} = 1,
\]

that is,

\[
c_1 = \frac{2d + c^2 \pm \sqrt{4d + c^2 - 4}}{2d}.
\]

Equation (4.10) may be integrated once to give the first order O.D.E.

\[
\exp \left( \frac{c^2 c_1 d \gamma}{d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1} \right) \left[ G^m \frac{dG}{d\gamma} + \frac{c^2 c_1^2 d^2}{m(d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1)} G \right] = A_1,
\]

where \( A_1 \in \mathbb{R} \) is the constant of integration. If we let \( A_1 = 0 \), the first order O.D.E. may be solved to give

\[
G(\gamma) = \left[ -\frac{c^2 c_1^2 d^2}{d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1} \gamma + A_2 \right]^{1/m},
\]

where \( A_2 \in \mathbb{R} \) is the second constant of integration. In terms of the original variables the solution is

\[
u = \left[ -\frac{c^2 c_1^2 d^2}{d^2 + c_1^2 d^2 + c_1^2 - 2d^2 c_1} \left\{ \frac{x}{c} \left( \frac{1}{c_1} - \frac{z}{c_1 d} - \frac{\ln t}{m} \right) \right\} + A_2 \right]^{1/m} t^{-1/m},
\]

where \( c_1 \) satisfies (4.11). Hence we have determined an exact time-dependent similarity solution of the two-dimensional diffusion-convection equation for \( D(u) = u^m \), \( K'(u) = u^m \).
Chapter 4. Higher-dimensional nonlinear diffusion-convection equations

The second case where reduction of the nonlinear diffusion-convection equation to an O.D.E is possible is when \( n = m/2 \), i.e. \( D(u) = u^m \) and \( K'(u) = u^{m/2} \). In this case, we find that the optimal system is

\[
\{ c \Gamma_3 + \Gamma_4, \pm \Gamma_1 + c \Gamma_2 + \Gamma_3, \pm \Gamma_2 + \Gamma_3, \Gamma_3, c \Gamma_1 + \Gamma_2, \Gamma_1 \}
\]

for some \( c \in \mathbb{R} \). If we reduce the governing equation using the first element from the optimal system \((c \neq 0)\) then the resulting P.D.E. is

\[
\frac{1}{c} e^{mn} \left[ F - \frac{\partial F}{\partial \eta} \right] = \rho^2 \frac{\partial}{\partial \rho} \left[ (1 + \rho^2) F^m \frac{\partial F}{\partial \rho} \right] + \frac{4}{m^2} \frac{\partial}{\partial \eta} \left[ F^m \frac{\partial F}{\partial \eta} \right]
\]

\[
+ \frac{4}{m} \frac{\partial}{\partial \eta} \left[ F^m \frac{\partial F}{\partial \rho} \right] + \rho^2 F^{m/2} e^{mn/2} \frac{\partial F}{\partial \rho} - \frac{2}{m} F^m \frac{\partial F}{\partial \eta},
\]

(4.12)

where

\[
u = F(\rho, \eta)e^{t/c}, \quad \rho = \frac{x}{z}, \quad \eta = \frac{2}{m} \ln x - \frac{t}{c}.
\]

This P.D.E. admits the nontrivial symmetry

\[
\Gamma_1^1 = \frac{\partial}{\partial \eta} + F \frac{\partial}{\partial F},
\]

and hence a second reduction is possible. The subsequent O.D.E is

\[
\rho^2 \frac{d}{d\rho} \left[ (1 + \rho^2) G^m \frac{dG}{d\rho} \right] + \frac{4(m+1)}{m} \rho G^m \frac{dG}{d\rho} + \rho^2 G^{m/2} \frac{dG}{d\rho} + \frac{2(m+2)}{m^2} G^{m+1} = 0,
\]

(4.13)

where

\[
F = e^n G(\rho).
\]

Although this O.D.E. may be solved (for example, when \( m = -1 \)), the solution in terms of the original variables will always be time independent, since

\[
u = F(\rho, \eta)e^{t/c}
\]

\[
= e^n G(\rho)e^{t/c}
\]

\[
= x^{2/m} G \left( \frac{x}{z} \right).
\]
Reduction of the governing equation (4.2) to an O.D.E. is also possible for two special cases of the governing equation when \(D(u) = e^{mu}, \ K'(u) = e^{nu}\). If we consider the symmetries for the general exponential case, we find that commutator table obtained is identical to the commutator table for the general power law case considered previously. Hence the optimal system for \(D(u) = e^{mu}, \ K'(u) = e^{nu}\) is the same as the optimal system for \(D(u) = u^{m}, \ K'(u) = u^{n}\),

\[
\{\Gamma_4, \pm \Gamma_1 + c \Gamma_2 + \Gamma_3, \pm \Gamma_2 + \Gamma_3, c \Gamma_1 + \Gamma_2, \Gamma_1\}
\]

for some \(c \in \mathbb{R}, \ n \neq m, \ n \neq m/2\). Reduction of the governing equation by the only nontrivial symmetry in the optimal system leads to a P.D.E. with one dependent and two independent variables. As in the power law case, this reduced equation has no symmetries, so no further reductions are possible.

The first of the two special exponential cases is when \(n = m\). In this case, as in the power-law case when \(n = m\), the optimal system is

\[
\{c \Gamma_1 + d \Gamma_2 + \Gamma_4, \pm \Gamma_1 + c \Gamma_2 + \Gamma_3, \pm \Gamma_2 + \Gamma_3, c \Gamma_1 + \Gamma_2, \Gamma_1\}
\]

for some \(c, d \in \mathbb{R}\). If we reduce the governing equation using the first element from the optimal system \((c, d \neq 0)\) then the resulting P.D.E. is

\[
\frac{1}{m} \left[ \frac{\partial F}{\partial \eta} - 1 \right] = \left[ \frac{1}{c^2} + \frac{1}{d^2} \right] \frac{\partial}{\partial \rho} \left[ e^{m \rho} \frac{\partial F}{\partial \rho} \right] + \frac{1}{c^2} \frac{\partial}{\partial \eta} \left[ e^{m \rho} \frac{\partial F}{\partial \eta} \right] + \frac{2}{c^2} \frac{\partial}{\partial \eta} \left[ e^{m \rho} \frac{\partial F}{\partial \rho} \right] + \frac{1}{d} e^{m \rho} \frac{\partial F}{\partial \rho},
\]

where

\[
u = F(\rho, \eta) - \frac{\ln t}{m}, \ \ \ \rho = \frac{x}{c} - \frac{z}{d}, \ \ \ \eta = \frac{x}{c} + \frac{\ln t}{m}.
\]

Equation (4.14) admits the symmetries

\[
\Gamma^1_1 = \frac{\partial}{\partial \rho}, \ \ \ \Gamma^1_2 = \frac{\partial}{\partial \eta},
\]

with corresponding optimal system \(\{c_1 \Gamma^1_1 + \Gamma^1_2, \Gamma^1_1\}\), for some \(c_1 \in \mathbb{R}\). Hence a second reduction is possible using \(c_1 \Gamma^1_1 + \Gamma^1_2 \ (c_1 \neq 0)\), and the subsequent O.D.E is

\[
\alpha \left[ m e^{mc} \left( \frac{dG}{d\gamma} \right)^2 + e^{mc} \frac{d^2G}{d\gamma^2} \right] + \frac{1}{c_1 d} e^{mc} \frac{dG}{d\gamma} + \frac{1}{m} \left[ \frac{dG}{d\gamma} + 1 \right] = 0,
\]
where

\[ F = G(\gamma), \quad \gamma = \frac{\rho}{c_1} - \eta \quad \text{and} \quad \alpha = \frac{d^2 + d^2c_1^2 + c^2 - 2d^2c_1}{c^2d^2c_1^2}. \]

Equation (4.15) may be integrated once to give the first order O.D.E.

\[
\frac{d^2 + d^2c_1^2 + c^2 - 2d^2c_1}{c^2d^2c_1^2} e^{mG} \frac{dG}{d\gamma} \frac{1}{mc_1d} e^{mG} + \frac{1}{m} [G + \gamma] = A_1 \tag{4.16}
\]

where \( A_1 \in \mathbb{R} \) is the constant of integration, however, (4.16) has no known explicit solution. Considering the two cases \( c = 0, d \neq 0 \) or \( c \neq 0, d = 0 \) leads to an O.D.E. similar to (4.16).

The second special exponential case is when \( n = m/2 \), that is \( D(u) = e^{mu} \) and \( K'(u) = e^{mu/2} \). For this case, as for the power-law case when \( n = m/2 \), the optimal system is given as

\[
\{ c\Gamma_3 + \Gamma_4, \pm \Gamma_1 + c\Gamma_2 + \Gamma_3, \pm \Gamma_2 + \Gamma_3, \Gamma_3, c\Gamma_1 + \Gamma_2, \Gamma_1 \}
\]

for some \( c \in \mathbb{R} \). If we reduce the governing equation using the first element from the optimal system then the resulting P.D.E. is

\[
\frac{1}{c} \left[ -\frac{\partial F}{\partial \eta} + 1 \right] = \frac{1}{\rho^2} e^{-mn} \frac{\partial}{\partial \rho} \left[ (1 + \rho^2)e^{mF} \frac{\partial F}{\partial \rho} \right] + \frac{4}{m} \rho e^{-mn} \frac{\partial}{\partial \rho} \left[ e^{mF} \frac{\partial F}{\partial \eta} \right]
\]

\[
+ \frac{4}{m^2} e^{-mn} \frac{\partial}{\partial \eta} \left[ e^{mF} \frac{\partial F}{\partial \eta} \right] - \frac{2}{m} \frac{\partial F}{\partial \eta} + \rho^2 e^{m(F-\eta)/2} \frac{\partial F}{\partial \rho},
\]

\[
(4.17)
\]

where

\[ u = F(\rho, \eta) + \frac{t}{c}, \quad \rho = \frac{x}{z}, \quad \eta = \frac{2}{m} \ln x - \frac{t}{c}. \]

Equation (4.17) admits the symmetry \( \Gamma_1^1 = \frac{\partial}{\partial \eta} + \frac{\delta}{\delta F} \) and hence a second reduction is possible. The subsequent O.D.E. is

\[
\frac{d}{d\rho} \left[ (1 + \rho^2)e^{mG} \frac{dG}{d\rho} \right] + \left[ e^{mG/2} + \frac{4}{\rho} e^{mG} \right] \frac{dG}{d\rho} + \frac{2}{m\rho^2} e^{mG} = 0, \tag{4.18}
\]
where \( F = G(\rho) + \eta \). However, as in the case when \( D(u) = u^m \) and \( K'(u) = u^{m/2} \), any solutions to this O.D.E. will be time independent, as

\[
\begin{align*}
    u &= F(\rho, \eta) + \frac{t}{c} \\
    &= G(\rho) + \eta + e^t \\
    &= G\left(\frac{x}{z}\right) + \frac{2}{m} \ln x.
\end{align*}
\]

The last case of the two-dimensional diffusion-convection equation able to be reduced to an O.D.E. is when \( D(u) = 1 \) and \( K'(u) = \ln u \). For this case the optimal system is

\[
\{c\Gamma_1 + d\Gamma_3 + \Gamma_4, c\Gamma_1 + \Gamma_3, c\Gamma_1 + \Gamma_2, \Gamma_1\}
\]

for some \( c, d \in \mathbb{R} \). Reducing the governing equation using the first symmetry of the optimal system \((c, d \neq 0)\) results in the P.D.E.

\[
\frac{1}{d} \left[ -\frac{\partial F}{\partial \rho} + F \right] = \frac{1}{c^2} \frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \eta^2} - \ln F \frac{\partial F}{\partial \eta},
\]

where

\[
u = F(\rho, \eta)e^{t/d}, \quad \rho = \frac{x}{c} - \frac{t}{d}, \quad \eta = z - \frac{t^2}{2d}.
\]

Equation (4.19) admits the symmetries

\[
\Gamma_1 = \frac{\partial}{\partial \rho}, \quad \Gamma_2 = \frac{\partial}{\partial \eta},
\]

with corresponding optimal system \(\{c_1\Gamma_1 + \Gamma_2, \Gamma_1\}\) for some \( c_1 \in \mathbb{R} \). Hence a second reduction is possible using \( c_1\Gamma_1 + \Gamma_2 \) \((c_1 \neq 0)\), and the subsequent O.D.E. is

\[
\frac{1 + c^2 c_1^2}{c^2 c_1^2} \frac{d^2 G}{d \gamma^2} + \ln G \frac{dG}{d\gamma} + \frac{1}{c_1 d} \frac{dG}{d\gamma} - \frac{1}{d} G = 0,
\]

where

\[ F = G(\gamma), \quad \gamma = \frac{\rho}{c_1} - \eta. \]

Using the transformation \( \phi = \frac{dG}{d\gamma} \), (4.20) may be reduced to the first order O.D.E.

\[
\frac{1 + c^2 c_1^2}{c^2 c_1^2} \phi \frac{d\phi}{dG} + \left( \ln G + \frac{1}{c_1 d} \right) \phi - \frac{G}{d} = 0,
\]
which is an Abel equation of the second kind with no known explicit solution.

Using the first symmetry of the optimal system with $c = 0$ results in an O.D.E. similar to (4.20). However, using the first symmetry of the optimal system with $d = 0$ leads to a different O.D.E. with an exact solution. The first reduction of the governing equation using $c \Gamma_1 + \Gamma_4 \ (c \neq 0)$ leads to the P.D.E.

$$\frac{\partial F}{\partial t} = \left( \frac{1}{c^2} + \frac{1}{t^2} \right) \frac{\partial^2 F}{\partial \rho^2} + \left( \frac{2}{c^2} + \frac{\ln F}{t} + \frac{\rho}{t} \right) \frac{\partial F}{\partial \rho} + \frac{1}{c^2} F, \quad (4.21)$$

where

$$u = F(\rho, t)e^{\rho/c}, \quad \rho = \frac{x}{c} - \frac{z}{t}.$$

Equation (4.21) admits the symmetries

$$\Gamma_1^1 = -\frac{1}{t} \frac{\partial}{\partial \rho}, \quad \Gamma_2^1 = -\frac{\partial}{\partial \rho} + F \frac{\partial}{\partial F},$$

with corresponding optimal system $\{c_1 \Gamma_1^1 + \Gamma_2^1, \Gamma_3^1\}$ for some $c_1 \in \mathbb{R}$. Hence a second reduction is possible using $c_1 \Gamma_1^1 + \Gamma_2^1 \ (c_1 \neq 0)$, and the resulting O.D.E. is

$$\frac{dG}{dt} = \frac{c^2 + c_1^2}{c^2(c_1 + t)^2} G - \frac{G \ln G}{c_1 + t}, \quad (4.22)$$

where

$$F = G(t) e^{-\rho/(c_1 + t)}.$$

Equation (4.22) has the solution

$$G(t) = \exp \left[ \frac{(c^2 + c_1^2) \ln(c_1 + t) + c^2 A}{c^2(c_1 + t)} \right],$$

where $A \in \mathbb{R}$ is the constant of integration. In terms of the original variables, the solution is

$$u = \exp \left[ \frac{(c^2 + c_1^2) \ln(c_1 + t) + c_1 x + c^2 z + c^2 A}{c^2(c_1 + t)} \right].$$
4.2 Exact solutions for a 2-D nonlinear diffusion-convection equation

We transform (4.2) from Cartesian space coordinates \((x, z)\) into cylindrical polar coordinates \((r, \gamma)\) by

\[ x = r \sin \gamma, \quad z = r \cos \gamma \]

to obtain the nonlinear P.D.E.

\[
\frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ r D(u) \frac{\partial u}{\partial r} \right] + \frac{1}{r^2} \frac{\partial}{\partial \gamma} \left[ D(u) \frac{\partial u}{\partial \gamma} \right] - K'(u) \left[ \cos \gamma \frac{\partial u}{\partial r} - \frac{\sin \gamma \partial u}{r} \right].
\]  

(4.23)

If we assume power-law functions for \(D(u)\) and \(K'(u)\), (i.e. \(D(u) = u^m, K'(u) = u^n\)), then the appropriate symmetries are,

\[
\Gamma_1 = \sin \gamma \frac{\partial}{\partial r} + \frac{\cos \gamma}{r} \frac{\partial}{\partial \gamma},
\]

\[
\Gamma_2 = \cos \gamma \frac{\partial}{\partial r} - \frac{\sin \gamma}{r} \frac{\partial}{\partial \gamma},
\]

\[
\Gamma_3 = \frac{\partial}{\partial t},
\]

\[
\Gamma_4 = (m - 2n)t \frac{\partial}{\partial t} + (m - n)r \frac{\partial}{\partial r} + u \frac{\partial}{\partial u}.
\]

(4.24)

For total material conservation, Philip and Knight [65] assume the \textit{ad hoc} functional form

\[ u = F(\rho, \gamma)t^{-\alpha}, \quad \rho = rt^{-\alpha/2} \]

(4.25)

and show that the values of \(\alpha\) and \(n\) must be

\[ \alpha = \frac{1}{m + 1}, \quad n = m + \frac{1}{2}, \quad m \neq -1. \]

If we consider the symmetries (4.24), this gives

\[
\Gamma_4 = 2(m + 1)t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} - 2u \frac{\partial}{\partial u}.
\]

(4.26)
From this symmetry we obtain the characteristic equation
\[
\frac{dr}{r} = \frac{d\gamma}{0} = \frac{dt}{2(m+1)t} = \frac{du}{-2u},
\]
which may be solved to yield the functional form
\[
u = F(\rho, \gamma) t^{-1/(m+1)}, \quad \rho = rt^{-1/2(m+1)}, \quad m \neq -1, \tag{4.27}
\]
which is precisely the form used by Philip and Knight [65] to obtain a reduction of variables. The special case \(m = -1\) is excluded from this reduction. The ad hoc functional form assumed in [65] is incapable of treating the case \(m = -1\). Our systematic symmetry approach identifies the correct invariant variable substitutions not only for their cases but also for this exceptional case. This shows that a systematic symmetry analysis obviates the need for a case by case search for appropriate variable substitutions.

From the work of Philip and Knight [65] which makes no reference to symmetry analysis, it is not clear how to treat the special case \(m = -1\). This inverse linear case arises specifically in the analysis of a diffusing electron cloud in thermal equilibrium (Lonngren and Hirose [51]). In the hydrology context, it has two applications. Firstly, in rigid field soils containing biomacropores, the soil water diffusivity may be weakly increasing (Clothier and White [26]). Such a diffusivity may well be represented by \(D(\theta) = a(b - \theta)^{-1}\) with \(b\) greater than the porosity. Secondly, in a saturated swelling paste, the effective material diffusivity may be a weakly decreasing function which could be represented by \(D_m(\theta) = a(b + \theta)^{-1}\) (Broadbridge [13]).

For the case \(m = -1\), the symmetry (4.26) used above may be replaced by
\[
\Gamma_4 = r \frac{\partial}{\partial r} - 2u \frac{\partial}{\partial u}.
\]
If we now consider a linear combination of \(\Gamma_3\) and \(\Gamma_4\), then for \(m = -1\), the characteristic equations take the form
\[
\frac{dr}{r} = \frac{d\gamma}{0} = \frac{dt}{\beta} = \frac{du}{-2u},
\]
where $\beta \in \mathbb{R}, \beta \neq 0$. These may be solved to obtain

$$u = F(\rho, \gamma) e^{-2t/\beta}, \rho = re^{-t/\beta}.$$ 

If we let $c = 2/\beta$, then

$$u = F(\rho, \gamma) e^{-ct}, \rho = re^{-ct/\beta}. \tag{4.28}$$

This will lead to a reduction by one of the number of independent variables of the
P.D.E. (4.23). This form of $u$ also ensures total conservation of material, so for this
 case $m = -1$, we can also seek a solution of (4.23) by extending the method of
Philip and Knight [65].

In fact, for this case, the functional form (4.28) involves an arbitrary constant
$c$, which will mean greater variability in the solution. We also note that, unlike the
functional form (4.25) with power law dependence on $t$, the new solution (4.28) has
the additional advantage that it is finite for $t = 0$. It is worthwhile noting that
although a functional form

$$u = F(\rho, \gamma) e^{-t/\beta}, \rho = r e^{-t/\beta}$$

is valid in general when $m = 2n$, it is only when $n = -1/2$, i.e. when $m = -1$, that
the global material conservation condition is satisfied.

This condition of global conservation requires zero flux at the origin. That is

$$\text{for } t > 0, \ r = 0, \ \frac{\partial u}{\partial r} = 2 \cos \gamma \ u^{3/2}. \tag{4.29}$$

Substitution of (4.28) into (4.23) gives

$$\frac{-c}{2\rho} \frac{\partial}{\partial \rho} [\rho^2 F] = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \rho F^{-1} \frac{\partial F}{\partial \rho} \right] + \frac{1}{\rho^2} \frac{\partial}{\partial \gamma} \left[ F^{-1} \frac{\partial F}{\partial \gamma} \right] - F^{-1/2} \left[ \cos \gamma \frac{\partial F}{\partial \rho} - \frac{\sin \gamma}{\rho} \frac{\partial F}{\partial \gamma} \right], \tag{4.30}$$

while the zero flux condition (4.29) becomes

$$t > 0, \ \rho = 0, \ \frac{\partial F}{\partial \rho} = 2 \cos \gamma \ F^{3/2}. \tag{4.31}$$
As in the case of similarity variables with power law time dependence (Philip and Knight [65]), the similarity reduction (4.28) implies that the flux has no component normal to the radii. This provides a constraint

\[ \frac{-1}{\rho} F^{-1} \frac{\partial F}{\partial \gamma} - 2F^{1/2} \sin \gamma = 0, \]  

(4.32)

which allows us to make a further reduction of variables. For our purposes, the only significance of global mass conservation is that it provides an extra restriction (4.32) which allows us to make a further reduction of variables. Without this extra restriction of global mass conservation, a further reduction of variables of the reduced equation obtained by Philip and Knight [65] is not possible, since their reduced equation is a transformed version of (4.7) which admits no point symmetries.

Of course, each diffusion-convection equation (4.1), being a conservation equation, locally conserves material, even if it is unbalanced globally due to boundary conditions. However, it is only the global mass conserving solutions that allow this further reduction. A further symmetry reduction of (4.30) is in fact possible, since this equation is equivalent to the P.D.E. (4.12) which was obtained by reducing (4.2) in the special case \( D(u) = u^m, \ K'(u) = u^{m/2} \) discussed in the previous section.

However, we have shown that the similarity solution obtained in this instance is time independent.

Taking \( \frac{-1}{\rho} \frac{\partial}{\partial \gamma} \) of (4.32) and substitution of this expression into (4.30) leads to

\[ \frac{-c}{2\rho} \frac{\partial}{\partial \rho} \left[ \rho^2 F \right] = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left[ \rho F^{-1} \frac{\partial F}{\partial \rho} \right] - \frac{2 \cos \gamma \frac{\partial}{\partial \rho} [\rho F^{1/2}]}{\rho}, \]  

(4.33)

an O.D.E. for each value of \( \gamma \). This may be integrated with respect to \( \rho \), with the constant of integration eliminated through use of (4.31). Thus we wish to solve the first order O.D.E.

\[ F^{-2} \frac{\partial F}{\partial \rho} = 2 \cos \gamma F^{-1/2} - \frac{c}{2} \rho \]  

(4.34)

subject to the condition

\[ \rho = 0, \ F = F_0 > 0 . \]
If we let \( \eta = F^{-1} \) then (4.34) becomes
\[
\frac{\partial \eta}{\partial \rho} = -2 \cos \gamma \eta^{1/2} + \frac{c}{2} \rho \tag{4.35}
\]
subject to \( \rho = 0, \ \eta = \eta_0 = F^{-1}_o \).

The O.D.E. (4.35) may be solved exactly (Kamke [46]) to give the solution (for \( c > 0 \))
\[
\left( \frac{\eta + \rho \eta^{1/2} \cos \gamma - \frac{c}{2} \rho^2}{\eta_0} \right) = \left( \frac{2 \eta^{1/2} + \rho (\cos \gamma - \sqrt{\cos^2 \gamma + c})}{2 \eta^{1/2} + \rho (\cos \gamma + \sqrt{\cos^2 \gamma + c})} \right)^{2 \cos \gamma \over \sqrt{\cos^2 \gamma + c}} \tag{4.36}
\]
The case \( c < 0 \) implies a physically unappealing solution which increases exponentially in \( t \). For such a globally mass-conserving solution to a dissipative equation, this backward evolution cannot occur if the solution is smooth. For example, from the analogous solution to (4.36) but with \( c < 0 \), we find that along the ray \( \gamma = \pi/2 \), the solution is given by
\[
u = \frac{e^{-ct}}{[\eta_0 + c \rho^2/4]}, \tag{4.37}
\]
which is singular at \( \rho = 2 \sqrt{\eta_0/|c|} \).

Previously, the reduction method of Philip and Knight [65] has produced closed-form two-dimensional solutions only in the cases \( (m, n) = (0, 1/2) \) and \( (m, n) = (1/2, 1) \). Here, we have contributed an infinite family of new solutions whose character changes as the parameter \( c \) varies.

In Figures 4.1 and 4.2, the analytic solution is presented for a comparatively large value \( c = 5.0 \) and \( F_o = 1.0 \). Since time is incorporated in the spatial similarity variables, the figures display the solutions at all times. In this case, the initial condition is already displayed in Figures 4.1 and 4.2 when we assume \( t \) to be zero. Compared to the initially singular solutions of Philip and Knight [65], our solutions have the additional advantage of being finite at \( t = 0 \). As can be seen from Figures 4.1 and 4.2, the initial condition is an almost symmetric distribution of material which could have resulted from a local injection. The injected slug is allowed to spread, without extra material being supplied at the origin.
Chapter 4. Higher-dimensional nonlinear diffusion-convection equations

Figure 4.1: Plot of (4.36) for $c = 5.0$ and $F_0 = 1.0$.

Figure 4.2: Contour plot of (4.36) for $c = 5.0$ and $F_0 = 1.0$. 
Figure 4.3: Plot of (4.36) for $c = 1.0$ and $F_0 = 1.0$.

Figure 4.4: Contour plot of (4.36) for $c = 1.0$ and $F_0 = 1.0$. 
In Figures 4.3 and 4.4, the analytic solution is presented for a comparatively small value, \( c = 1.0 \) and \( F_0 = 1.0 \). Again, the initial condition is evident when \( t = 0 \). The concentration peaks in Figures 4.3 and 4.4 are steeper than in Figures 4.1 and 4.2. The initial condition could have originated from an injection from a horizontal line segment source.

We note that the solutions are symmetric about the plane \( x = 0 \). As \( c \) becomes larger, the solution reduces to a one-dimensional steady state. The one-dimensional steady state solution to Eq. (4.1), which has been used to model evaporation from a soil with a water table, can be obtained for arbitrary \( D(u) \) and \( K(u) \) (Gardner [39]).

### 4.3 The 3-D nonlinear diffusion-convection equation

For classical Lie group analysis of the class of equations

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ D(u) \frac{\partial u}{\partial x} \right] + \frac{\partial}{\partial y} \left[ D(u) \frac{\partial u}{\partial y} \right] + \frac{\partial}{\partial z} \left[ D(u) \frac{\partial u}{\partial z} \right] - \frac{dK}{du} \frac{\partial u}{\partial z}. \tag{4.38}
\]

in three dimensions, we are again interested in the general equation, so that as in the two-dimensional analysis, \( D(u) \) and \( K(u) \) are arbitrary. Through symmetry analysis, we identify the special forms of \( D \) and \( K \) which possess extra symmetries.
The infinitesimal transformations appropriate in this instance are

\[ u_\ast = e^{\epsilon t} u = u + \epsilon U(x, y, z, t, u) + O(\epsilon^2), \]

\[ t_\ast = e^{\epsilon t} t = t + \epsilon T(x, y, z, t, u) + O(\epsilon^2), \]

\[ x_\ast = e^{\epsilon t} x = x + \epsilon X(x, y, z, t, u) + O(\epsilon^2), \]

\[ y_\ast = e^{\epsilon t} y = y + \epsilon Y(x, y, z, t, u) + O(\epsilon^2), \]

\[ z_\ast = e^{\epsilon t} z = z + \epsilon Z(x, y, z, t, u) + O(\epsilon^2), \]

and

\[ (4.39) \]

where the infinitesimal generator \( \Gamma \) [57], [7], is

\[ \Gamma = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} + Z \frac{\partial}{\partial z} + T \frac{\partial}{\partial t} + U \frac{\partial}{\partial u}. \]

Eq. (4.39) is then extended to first and second order, so that, for example

\[ \frac{\partial u_\ast}{\partial x_\ast} = \frac{\partial u}{\partial x} + \epsilon U_1 + O(\epsilon^2), \]

where

\[ U_1 = \frac{D}{Dx} U - \left( \frac{D}{Dx} X \right) u_x - \left( \frac{D}{Dx} Y \right) u_y - \left( \frac{D}{Dx} Z \right) u_z - \left( \frac{D}{Dx} T \right) u_t, \]

(4.40)

with \( D/Dx \) the total derivative operator with respect to \( x \),

\[ \frac{D}{Dx} F(x, y, z, t, u) = \frac{\partial F}{\partial x} + u_x \frac{\partial F}{\partial u}. \]

For arbitrary \( D \) and \( K \), the symmetries are

\[ \Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial y}, \quad \Gamma_3 = \frac{\partial}{\partial z}, \quad \Gamma_4 = \frac{\partial}{\partial t}, \]

\[ \Gamma_5 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \]
Table 4.2: Symmetries for the 3-D diffusion-convection equation $m, n \in \mathbb{R}$.

<table>
<thead>
<tr>
<th>$D(u)$</th>
<th>$K'(u)$</th>
<th>$\Gamma_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^m$</td>
<td>$u^n$</td>
<td>$\Gamma_6 = (m - 2n)t \frac{\partial}{\partial t} + (m - n)x \frac{\partial}{\partial x} + (m - n)y \frac{\partial}{\partial y} + (m - n)z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$u^m$</td>
<td>$\ln(u)$</td>
<td>$\Gamma_6 = ml \frac{\partial}{\partial t} + mx \frac{\partial}{\partial x} + my \frac{\partial}{\partial y} + (mz + t) \frac{\partial}{\partial z} + u \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$e^{mu}$</td>
<td>$e^{nu}$</td>
<td>$\Gamma_6 = (m - 2n)t \frac{\partial}{\partial t} + (m - n)x \frac{\partial}{\partial x} + (m - n)y \frac{\partial}{\partial y} + (m - n)z \frac{\partial}{\partial z} + \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$e^{mu}$</td>
<td>$u$</td>
<td>$\Gamma_6 = ml \frac{\partial}{\partial t} + mx \frac{\partial}{\partial x} + my \frac{\partial}{\partial y} + (mz + t) \frac{\partial}{\partial z} + \frac{\partial}{\partial u}$</td>
</tr>
<tr>
<td>$\text{const}$</td>
<td>$u$</td>
<td>$\Gamma_6 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u}$</td>
</tr>
</tbody>
</table>

where $\Gamma_1$ to $\Gamma_4$ are the generators of space and time translations, and $\Gamma_5$ is the generator of rotations about the vertical axis. As in the two-dimensional case, the only forms of $D(u)$, up to a linear change of variables, which have additional symmetries are power law and exponential. In Table 4.2, $\Gamma_1$ to $\Gamma_5$ are omitted, as they are common to all cases. The case $K'(u)$ constant has not been included as this form of (4.38) can be transformed to the case of pure diffusion, fully analysed by Galaktionov et al. [38].

The three-dimensional Burgers' equation, like the two-dimensional Burgers' equa-
Chapter 4. Higher-dimensional nonlinear diffusion-convection equations

tion, is the case which possesses the most symmetries. The symmetries $\Gamma_6$ and $\Gamma_7$
are compatible, as they obey the commutation property $[\Gamma_7, \Gamma_6] = -\Gamma_7$. We show
in the following chapter that the three-dimensional Burgers’ equation may be re-
duced to an O.D.E. by successive symmetry reductions. However, as in the case of
the two-dimensional diffusion-convection equation, there are additional forms of the
three-dimensional diffusion-convection equation (4.38) which may also be reduced
to an O.D.E. We demonstrate one case of (4.38) which can be reduced to an O.D.E.

We have seen previously for the two-dimensional diffusion-convection equation
that when $D(u) = u^m$ and $K'(u) = u^m$, we are able to reduce the governing
equation to an O.D.E. It is also possible to reduce the three-dimensional diffusion-
convection equation to an O.D.E. and derive an exact similarity solution for this
combination of $D(u)$ and $K'(u)$. In this case, the optimal system is

$$\{c_1 \Gamma_3 + c_2 \Gamma_5 + \Gamma_6, c_1 \Gamma_1 + c_2 \Gamma_3 + \Gamma_6, c_1 \Gamma_3 \pm \Gamma_4 + \Gamma_5, c_1 \Gamma_3 + \Gamma_5, c_1 \Gamma_1 + c_2 \Gamma_3 + \Gamma_4, c_1 \Gamma_1 + \Gamma_3, \Gamma_1\}$$

for some $c_1, c_2 \in \mathbb{R}$. Reducing the governing equation by the second element from
the optimal system ($c_1, c_2 \neq 0$) results in the P.D.E.

$$\frac{1}{m} [-F + \frac{\partial F}{\partial \eta}] = \left( \frac{1}{c_1^2} + \frac{1}{c_2^2} \right) \frac{\partial}{\partial \rho} \left[ F^m \frac{\partial F}{\partial \rho} \right] + \frac{\partial}{\partial y} \left[ F^m \frac{\partial F}{\partial y} \right]$$

$$+ \frac{1}{c_2} \frac{\partial}{\partial \eta} \left[ F^m \frac{\partial F}{\partial \eta} + 2F^m \frac{\partial F}{\partial \rho} \right] + \frac{1}{c_2} F^m \frac{\partial F}{\partial \rho},$$

(4.41)

where

$$u = F(y, \rho, \eta) t^{-1/m}, \quad \rho = \frac{x}{c_1} - \frac{z}{c_2}, \quad \eta = \frac{x}{c_1} + \frac{\ln \tau}{m}.$$ 

Equation (4.41) admits the symmetries

$$\Gamma_1^2 = \frac{\partial}{\partial y}, \quad \Gamma_2^2 = \frac{\partial}{\partial \eta}, \quad \Gamma_3^2 = c_1^2 \frac{\partial}{\partial \rho} - c_2^2 \frac{\partial}{\partial \eta},$$

with corresponding optimal system

$$\{c_3 \Gamma_1^2 + c_4 \Gamma_2^2 + \Gamma_3^2, c_3 \Gamma_1^2 + \Gamma_2^2, \Gamma_1^2\}$$
for some \( c_3, c_4 \in \mathbb{R} \). Reducing (4.41) using the first element from the optimal system \( (c_3, c_4) \neq 0 \) results in the P.D.E.

\[
- \frac{1}{m} \left[ G + \frac{c_4^2}{c_1^2c_4 - c_2^2} \frac{\partial G}{\partial \gamma} \right] = \left( \frac{1}{c_1^2} + \frac{1}{c_2^2} + \frac{1}{c_3^2} \right) \frac{\partial}{\partial \mu} \left[ G^m \frac{\partial G}{\partial \mu} \right] - \frac{1}{c_2} G^m \frac{\partial G}{\partial \mu}
\]

\[
+ \left( \frac{2}{c_1^2c_4 - c_2^2} + \frac{2}{c_3^2} \right) \frac{\partial}{\partial \mu} \left[ G^m \frac{\partial G}{\partial \gamma} \right]
\]

\[
+ \left( \frac{c_1^2}{c_1^2c_4 - c_2^2} + \frac{1}{c_3^2} \right) \frac{\partial}{\partial \gamma} \left[ G^m \frac{\partial G}{\partial \gamma} \right],
\]

(4.42)

where

\[
F = G(\gamma, \mu), \quad \gamma = \frac{y}{c_3} - \frac{c_1^2}{c_1^2c_4 - c_2^2}, \quad \mu = \frac{y}{c_3} - \rho.
\]

We have reduced our original equation with one dependent and four independent variables to a P.D.E. with one dependent and two independent variables. Equation (4.42) admits the two symmetries

\[
\Gamma_1^{2,1} = \frac{\partial}{\partial \mu}, \quad \Gamma_2^{2,1} = \frac{\partial}{\partial \gamma},
\]

with optimal system \( \{ c_5 \Gamma_1^{2,1} + \Gamma_2^{2,1}, \Gamma_1^{2,1} \} \) for some \( c_5 \in \mathbb{R} \). Hence we are able to make a further reduction to (4.42) using the first element from the optimal system \( (c_5 \neq 0) \) to obtain the O.D.E.

\[
\left\{ \frac{1}{c_5^2} \left( \frac{1}{c_1^2} + \frac{1}{c_2^2} + \frac{1}{c_3^2} \right) - \frac{2}{c_5} \left( \frac{1}{c_1^2c_4 - c_2^2} + \frac{1}{c_3^2} \right) + \frac{c_1^2}{(c_1^2c_4 - c_2^2)^2} + \frac{1}{c_3^2} \right\} \frac{d}{d\omega} \left[ H^m \frac{dH}{d\omega} \right]
\]

\[
= -\frac{1}{m} \left[ H + \frac{c_1^2}{c_1^2c_4 - c_2^2} \frac{dH}{d\omega} \right] + \frac{1}{c_2c_5} H^m \frac{dH}{d\omega},
\]

(4.43)

where

\[
G = H(\omega), \quad \omega = \gamma - \frac{\mu}{c_5}.
\]
Equation (4.43) may be integrated once to obtain the first order O.D.E.

\[
\exp \left( \frac{\omega}{c_2 c_3 \phi} \right) \left\{ H' \frac{dH}{d\omega} + \frac{c_1^2}{m(c_1^2 c_4 - c_2^2) \phi} H \right\} = A_1, \tag{4.44}
\]

where \( A_1 \in \mathbb{R} \) is the constant of integration, subject to the condition

\[
\frac{1}{c_2 c_3 \phi} = \frac{c_2^2 c_4 - c_2^2}{c_1^2},
\]

where

\[
\phi = \frac{1}{c_2^2} \left( \frac{1}{c_1^2} + \frac{1}{c_2^2} + \frac{1}{c_3^2} \right) - \frac{2}{c_3} \left( \frac{1}{c_1^2 c_4 - c_2^2} + \frac{1}{c_3^2} \right) + \frac{c_1^2}{(c_1^2 c_4 - c_2^2)^2} + \frac{1}{c_3^2}.
\]

If we let \( A_1 = 0 \), (4.44) may be solved to give

\[
H = \left[ A_2 - \frac{c_1^2}{(c_1^2 c_4 - c_2^2) \phi} \omega \right]^{1/m},
\]

where \( A_2 \in \mathbb{R} \) is the second constant of integration. In terms of the original variables, the solution is

\[
u = t^{-1/m} \left[ A_2 - \frac{c_1^2}{(c_1^2 c_4 - c_2^2) \phi} \left\{ \left( \frac{1}{c_1 c_5} - \frac{c_1}{c_1^2 c_4 - c_2^2} \right) x \\
+ \frac{1}{c_3} \left( 1 - \frac{1}{c_5} \right) y - \frac{1}{c_2 c_5} z - \frac{c_1^2}{m(c_1^2 c_4 - c_2^2) \ln t} \right\} \right]^{1/m}.
\]

We may also utilise the symmetries in Table 4.2 to obtain an exact solution to a three-dimensional diffusion-convection equation using the method of Philip and Knight [65]. Equation (4.38) is transformed from Cartesian space coordinates \((x, y, z)\) into spherical coordinates \((r, \gamma, \psi)\) using

\[
x = r \sin \gamma \cos \psi, \ y = r \sin \gamma \sin \psi, \ z = r \cos \gamma,
\]

yielding the nonlinear P.D.E.

\[
\frac{\partial u}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 D(u) \frac{\partial u}{\partial r} \right] + \frac{1}{r^2 \sin \gamma} \frac{\partial}{\partial \gamma} \left[ \sin \gamma D(u) \frac{\partial u}{\partial \gamma} \right]
\]

\[
+ \frac{1}{r^2 \sin^2 \gamma} \frac{\partial}{\partial \psi} \left[ D(u) \frac{\partial u}{\partial \psi} \right] - K'(u) \left[ \cos \gamma \frac{\partial u}{\partial r} - \sin \gamma \frac{\partial u}{\partial \gamma} \right], \tag{4.45}
\]
Assuming power law-functions for $D$ and $K'$ (that is, $D(u) = u^m, K'(u) = u^n$), the symmetries from Table 4.2 become

\[
\begin{align*}
\Gamma_1 &= \sin \gamma \cos \psi \frac{\partial}{\partial r} + \frac{\cos \gamma \cos \psi}{r} \frac{\partial}{\partial \gamma} - \frac{\sin \psi}{r \sin \gamma} \frac{\partial}{\partial \psi}, \\
\Gamma_2 &= \sin \gamma \sin \psi \frac{\partial}{\partial r} + \frac{\cos \gamma \sin \psi}{r} \frac{\partial}{\partial \gamma} + \frac{\cos \psi}{r \sin \gamma} \frac{\partial}{\partial \psi}, \\
\Gamma_3 &= \cos \gamma \frac{\partial}{\partial r} - \frac{\sin \gamma}{r} \frac{\partial}{\partial \gamma}, \\
\Gamma_4 &= \frac{\partial}{\partial t}, \\
\Gamma_5 &= \frac{\partial}{\partial \psi}, \\
\Gamma_6 &= (m - 2n) \frac{\partial}{\partial t} + (m - n) r \frac{\partial}{\partial r} + u \frac{\partial}{\partial u}.
\end{align*}
\]

(4.46)

Philip and Knight [65] assume the functional form

\[
u = F(\rho, \gamma, \psi) t^{-\alpha}, \quad \rho = rt^{-\alpha/3},
\]

(4.47)

which ensures total material conservation. The values of $\alpha$ and $n$ are shown to be

\[
\alpha = \frac{3}{3m + 2}, \quad n = m + \frac{1}{3}, \quad m \neq \frac{2}{3}.
\]

However, from $\Gamma_6$ (when $n \neq m$), solving the characteristic equation

\[
\frac{dr}{(m - n)r} = \frac{d\gamma}{0} = \frac{d\psi}{0} = \frac{dt}{(m - 2n)t} = \frac{du}{u}
\]

leads to the functional form

\[
u = F(\rho, \gamma, \psi) t^{-1/(2n - m)}, \quad \rho = rt^{(m-n)/(2n-m)},
\]

(4.48)

which will lead to a reduction of order of the P.D.E. for general $m$ and $n$, $(n \neq m)$.

If we demand total conservation of material, we then obtain the form

\[
u = F(\rho, \gamma, \psi) t^{-3/(3m+2)}, \quad \rho = rt^{-1/(3m+2)}, \quad m \neq \frac{-2}{3},
\]

(4.49)
which is the same as that obtained by Philip and Knight [65]. We see that the case \( m = -2/3 \) is not included by this reduction. In this case, \( \Gamma_6 \) is replaced by

\[
\Gamma_6 = r \frac{\partial}{\partial r} - 3u \frac{\partial}{\partial u}
\]

and taking a linear combination of \( \Gamma_6 \) and \( \Gamma_7 \), the characteristic equation is

\[
\frac{dr}{r} = \frac{d\gamma}{0} = \frac{d\psi}{0} = \frac{dt}{\beta} = \frac{du}{-3u},
\]

with \( \beta \in \mathbb{R}, \beta \neq 0 \). Letting \( c = 3/\beta \) and solving leads to the functional form

\[
u = F(\rho, \gamma, \psi) e^{-ct}, \quad \rho = re^{-ct/3}.
\]  \( (4.50) \)

We note that \( (4.50) \) always ensures total material conservation, and will lead to a reduction by one in the number of independent variables. As in the two-dimensional special case, the arbitrary constant \( c \) leads to greater variability in the solution.

Total material conservation requires zero flow at the origin:

\[
\text{for } t > 0, \quad r = 0, \quad \frac{\partial u}{\partial r} = \frac{3}{2} \cos \gamma \ u^{4/3}.
\]  \( (4.51) \)

Substitution of \( (4.50) \) into \( (4.45) \) gives

\[
-\frac{c}{3\rho^2} \frac{\partial}{\partial \rho} \left[ \rho^3 F \right] = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left[ \rho^2 F^{-2/3} \frac{\partial F}{\partial \rho} \right] + \frac{1}{\rho^2 \sin \gamma} \frac{\partial}{\partial \gamma} \left[ \sin \gamma F^{-2/3} \frac{\partial F}{\partial \gamma} \right]
\]

\[
+ \frac{1}{\rho^2 \sin^2 \gamma} \frac{\partial}{\partial \gamma} \left[ F^{-2/3} \frac{\partial F}{\partial \psi} \right] - F^{1/3} \left[ \cos \gamma \frac{\partial F}{\partial \rho} - \frac{\sin \gamma}{\rho} \frac{\partial F}{\partial \gamma} \right],
\]

\( (4.52) \)

while the zero flux condition \( (4.51) \) becomes

\[
\text{for } t > 0, \quad \rho = 0, \quad \frac{\partial F}{\partial \rho} = \frac{3}{2} \cos \gamma \ F^{4/3}.
\]  \( (4.53) \)

The similarity reduction \( (4.50) \) implies that the flux has no component normal to the radii, as in the case of similarity variables with power law dependence (Philip
and Knight [65]). This means that we have the two additional constraints

\[ \frac{-1}{\rho} F^{-2/3} \frac{\partial F}{\partial \gamma} - \frac{3}{2} \sin \gamma F^{2/3} = 0 \]

and

\[ \frac{\partial F}{\partial \psi} = 0. \]

Utilising these constraints reduces (4.52) to the O.D.E.

\[ \frac{-c}{3} \rho F' = F^{-2/3} \frac{\partial F}{\partial \rho} - \frac{3}{2} \cos \gamma F^{2/3}. \]

This may be solved exactly (Kamke [46]) to give the solution \((c > 0)\)

\[ \left( \eta + 2 \rho \eta^{1/2} \cos \gamma - \frac{4c}{9} \rho^2 \right) = \left( \eta^{1/2} + \rho (\cos \gamma - \sqrt{\cos^2 \gamma + 4c/9}) \right)^{\frac{\cos \gamma}{2\sqrt{\cos^2 \gamma + 4c/9}}} \]

\[ \left( \eta^{1/2} + \rho (\cos \gamma + \sqrt{\cos^2 \gamma + 4c/9}) \right)^{\frac{\cos \gamma}{2\sqrt{\cos^2 \gamma + 4c/9}}} \]

where \(\eta = 1/F\). The case \(c < 0\) has been neglected to avoid singularities.

Since this is axially symmetric, solutions are similar in appearance to those obtained in two dimensions, except that Cartesian coordinate \(x\) is replaced by \((x^2 + y^2)^{1/2}\).

### 4.4 Discussion

There are few known exact solutions of higher-dimensional Richards' equation (4.1). Unlike the one-dimensional diffusion-convection equation (3.1), there are no linearisable models in two dimensions (Broadbridge [11]), and we doubt that integrable models exist in three dimensions. Instead we have relied on two separate techniques to obtain solutions to the two and three-dimensional nonlinear diffusion-convection equation. The first method we have relied on is the method of classical similarity reductions, which is the other best-known method for obtaining exact solutions of P.D.E.s. As a result, we have been able to reduce the governing equation (4.1) to
an O.D.E. for some special combinations of $D(u)$ and $K'(u)$, to obtain some exact similarity solutions.

In addition, Philip and Knight [65] have obtained a two step reduction of (4.1) to an O.D.E. for power-law forms of $D(u)$ and $K(u)$ when the two-dimensional diffusion-convection equation is expressed in polar coordinates, and a three step reduction of (4.1) to an O.D.E. when the three-dimensional diffusion-convection equation is expressed in spherical coordinates. However, their solutions, which are not obtained from integrable models nor from two stage classical symmetry reductions, do not apply to the cases $D(u) = u^{-2}$ in two dimensions, and $D(u) = u^{-2/3}$ in three dimensions. We have extended the method of Philip and Knight [65] to obtain solutions for these singular cases for which the previously unknown appropriate similarity variables are found to be invariants of classical symmetries. These singular cases admit a much wider class of implicit solutions than was previously available, and these solutions have the advantage of having finite initial conditions.
Chapter 5

Exceptional symmetry reductions of Burgers’ equation in higher dimensions

Burgers’ equation (Burgers [21])

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} - u \frac{\partial u}{\partial z}
\]  

(5.1)

is a one-dimensional version of the Navier Stokes equations and has applications in gas dynamics (e.g. Hopf [45]; Cole [28]), and in plasma dynamics (e.g. Brauner and Penel [10]). Clothier et al. [25] applied it to unsaturated flow in field soils. Unlike the linear model, Burgers’ equation has a stable large-time travelling wave solution and it leads to a good prediction of time to ponding during rainfall. Recent solutions to boundary value problems of the one-dimensional Burgers’ equation include unsaturated flow in layered media (Broadbridge et al. [17]), and unsaturated flow in a finite column with water table (Hills and Warrick [44]).

Burgers’ equation is the best known example of a nonlinear P.D.E. that can be directly transformed to a linear equation. For this reason, and because of its
wide range of applications, several studies have been made of generalizations of Burgers’ equation in two or three spatial dimensions. The generalizations (Webb and Zank [91, 92]; Tamizhmani and Punithavathi [86]) do not conform to the higher-dimensional Richards’ equation (4.1). Henceforth, we consider higher-dimensional versions of Burgers’ equation

\[ \frac{\partial u}{\partial t} = \nabla^2 u - u \frac{\partial u}{\partial z}, \]

which is a form of Richards’ equation (4.1) with \( D(u) = \text{const}, K(u) = \text{quadratic} \).

Because of the domination of biological macropores in field soils, compared to recompacted laboratory soils, a significant amount of water may be held at low potential energy, in which case the variability of the diffusion coefficient is relatively weak and may be neglected (Clothier et al. [25]). In the field, Eq. (5.2) applies directly to unsaturated flow, in which case \( u \) may be viewed as a linear function of the volumetric water concentration \( \theta \).

The linearisability of the one-dimensional Burgers’ equation can be discerned from its large symmetry group. In fact, it has an infinite dimensional group of potential symmetries (Bluman and Kumei [7]) and an infinite hierarchy of higher order Lie-Bäcklund symmetries (Olver [56]; Fokas [33]). Although the higher-dimensional Burgers’ equation (5.2) is not directly linearisable, it is not widely appreciated that it still has an exceptionally large symmetry group, with a fifth symmetry in two dimensions and a seventh symmetry in three dimensions (Edwards and Broadbridge [31]). The symmetry analysis of the nonlinear diffusion-convection equation (4.1) in the previous chapter indicates that some full classical reductions and exact solutions may be obtained for this special form of the nonlinear diffusion-convection equation. In fact, we are able to obtain explicit nonsingular solutions to the nonlinear Eq. (5.2), and these constitute a feasible time-dependent water concentration field in a class of field soils which may be parameterised by saturated conductivity and sorptivity (White and Broadbridge [93]).
5.1 Similarity reductions of the 2-D Burgers’ equation

From the symmetry analysis of the general class of two-dimensional diffusion-convection equation (4.2) in the previous chapter, we found that the symmetries obtained for the two-dimensional Burgers’ equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} - u \frac{\partial u}{\partial z} \]  

(5.3)

are

\[ \Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial z}, \quad \Gamma_3 = \frac{\partial}{\partial t}, \]

\[ \Gamma_4 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u}, \]  

(5.4)

\[ \Gamma_5 = t \frac{\partial}{\partial z} + \frac{\partial}{\partial u}. \]

The translation generators \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) as well as a form of \( \Gamma_4 \) are common to power law forms of Richards’ equation (4.2). The extra symmetry \( \Gamma_5 \) is peculiar to Burgers’ equation (Edwards and Broadbridge [31]). The two nontrivial symmetries \( \Gamma_4 \) and \( \Gamma_5 \) are compatible because they obey the commutation property

\[ [\Gamma_5, \Gamma_4] = -\Gamma_5. \]  

(5.5)

This means that after a classical Lie point symmetry reduction of the governing equation (4.2) using \( \Gamma_5, \Gamma_4 \) will be an inherited symmetry of the reduced equation, and a second reduction is possible. Two consecutive reductions of the two-dimensional Burgers’ equation will result in an O.D.E. We have shown in Chapter 4 that reduction of the two-dimensional nonlinear diffusion-convection equation to an O.D.E., excluding travelling wave solutions, is possible only for five special combination of \( D(u) \) and \( K'(u) \). Hence Burgers’ equation is the sixth case of (4.2) which can be reduced to an O.D.E.
As already noted, reduction with \( \Gamma_5 \) followed by a reduction with the inherited symmetry \( \Gamma_4 \) will result in an O.D.E. The reduced equation itself may have symmetries. These may include inherited symmetries due to commutation relations similar to Eq. (5.5) among the symmetry operators of the original P.D.E. However, as shown later in an example, the reduced equation may have additional symmetries which are not inherited from the original. We have taken full account of this possibility by performing a full symmetry analysis \textit{ab initio} on each reduced equation.

Reductions could be obtained from any symmetry which is an arbitrary linear combination

\[
a_1 \Gamma_1 + a_2 \Gamma_2 + a_3 \Gamma_3 + a_4 \Gamma_4 + a_5 \Gamma_5.
\]

To ensure that a minimal complete set of reductions is obtained from the symmetries of the governing equation, the optimal system is determined (Ovsiannikov [59]; Olver [57]). The optimal system for the two-dimensional Burgers' equation is

\[
\{ \Gamma_1, a \Gamma_1 + \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_1 + \Gamma_3, \Gamma_3 + \Gamma_5, \Gamma_1 + a \Gamma_3 + \Gamma_5 \},
\]

where in each case, \( a \in \mathbb{R} \) is arbitrary. This is a maximal set of unrelated symmetry operators. Any other symmetries may be obtained from these by a further invariance transformation. Table 5.1 lists the similarity variables and the reduced equation for each of the nontrivial symmetries from the optimal set.

Classical Lie group analysis has been performed on each of the reduced P.D.E.s in Table 5.1, and the cases \( \Gamma_4 \) and \( \Gamma_1 + \Gamma_5 \) lead to no further symmetries. Of the remaining entries, we note that the reduction from \( \Gamma_5 \) is a linear P.D.E. and therefore possesses an infinite dimensional group of symmetries, which have not been listed. Furthermore, the linear P.D.E.s may be solved by integral transform methods. The first entry in Table 5.1 is in fact a form of the one-dimensional Burgers' equation, and so has a general solution. Symmetry analysis of this reduced equation leads to five symmetries, four of which are inherited, and one which is an additional symmetry.
### Table 5.1: Reduced P.D.E.'s of the 2-D Burgers' equation $a \in \mathbb{R}$.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Reduced P.D.E.</th>
</tr>
</thead>
</table>
| $a\Gamma_1 + \Gamma_2, a \neq 0$ | \[
    \frac{\partial F}{\partial t} = \left(1 + \frac{a^2}{a^2}\right) \frac{\partial^2 F}{\partial \rho^2} - F \frac{\partial F}{\partial \rho}, \quad \text{with } u = F(t, \rho), \rho = z - \frac{x}{a}
\]
| $\Gamma_4$ | \[
    -\frac{1}{2} \left(F + \rho \frac{\partial F}{\partial \rho} + \gamma \frac{\partial F}{\partial \gamma}\right) = \frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \gamma^2} - F \frac{\partial F}{\partial \gamma},
\]
\[
    \text{with } u = \frac{F(\rho, \gamma)}{\sqrt{t}}, \rho = \frac{x}{\sqrt{t}}, \gamma = \frac{z}{\sqrt{t}}
\]
| $\Gamma_5$ | \[
    \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2} - \frac{1}{t} F, \quad \text{with } u = F(x, t) + \frac{z}{t}
\]
| $\Gamma_1 + \Gamma_3$ | \[
    \frac{\partial^2 F}{\partial \rho^2} = F \frac{\partial F}{\partial z} - \frac{\partial^2 F}{\partial x^2} - \frac{\partial F}{\partial \rho}, \quad \text{with } u = F(z, \rho), \rho = x - t
\]
| $\Gamma_3 + \Gamma_5$ | \[
    1 = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \rho^2} - F \frac{\partial F}{\partial \rho}, \quad \text{with } u = F(x, \rho) + t, \rho = z - \frac{t^2}{2}
\]
| $\Gamma_1 + a\Gamma_3 + \Gamma_5, a \neq 0$ | \[
    \frac{1}{a} \left(1 - \frac{\partial F}{\partial \rho}\right) = \frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \gamma^2} - F \frac{\partial F}{\partial \gamma},
\]
\[
    \text{with } u = F(\rho, \gamma) + \frac{t}{a}, \rho = x - \frac{t}{a}, \gamma = z - \frac{t^2}{2}
\]
| $\Gamma_1 + \Gamma_5$ | \[
    \frac{\partial F}{\partial t} = \left(\frac{1}{t^2} + 1\right) \frac{\partial^2 F}{\partial \rho^2} + \left(\frac{\partial F}{\partial \rho} - 1\right) \frac{F}{t},
\]
\[
    \text{with } u = F(t, \rho) + \frac{z}{t}, \rho = x - \frac{z}{t}
\]
Hence the reduced equations obtained using $\Gamma_5$ and $a\Gamma_1 + \Gamma_2$ admit symmetries additional to those symmetries inherited from the governing equation (4.2).

The reduced equation from $\Gamma_3 + \Gamma_5$ has the two trivial symmetries

$$\frac{\partial}{\partial x}$$ and $$\frac{\partial}{\partial p},$$

and in this case the only nontrivial symmetry is $\alpha \frac{\partial}{\partial x} + \frac{\partial}{\partial p}$, $(\alpha \neq 0)$. The resulting O.D.E. is

$$\left( \frac{\alpha^2 + 1}{\alpha^2} \right) \frac{d^2G}{d\gamma^2} - G \frac{dG}{d\gamma} = 1,$$

where

$$F = G(\gamma), \gamma = \rho - \frac{1}{\alpha} x.$$

The reduced equation from $\Gamma_1 + a\Gamma_3 + \Gamma_5$ has the two trivial symmetries

$$\frac{\partial}{\partial \rho}$$ and $$\frac{\partial}{\partial \gamma},$$

and the only nontrivial symmetry possible is $\alpha \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \gamma}$, $(\alpha \neq 0)$. The O.D.E. obtained is

$$\left( \frac{\alpha^2 + 1}{\alpha^2} \right) \frac{d^2G}{d\eta^2} - \left( \frac{1}{a\alpha} + G \right) \frac{dG}{d\eta} = \frac{1}{a},$$

with

$$F = G(\eta), \eta = \gamma - \frac{1}{\alpha} \rho.$$

Equations (5.6) and (5.7) can be integrated once to a Riccati equation of the form

$$\frac{dG}{d\eta} = a_1 G^2 + a_2 G + a_3 \eta + a_4,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{R}$, with general solution (Murphy [52])

$$G(\eta) = -\frac{\frac{d}{d\eta} \left\{ e^{\alpha_2 \eta/2} \left[ c_1 AI(\tau) + c_2 BI(\tau) \right] \right\}}{a_1 e^{\alpha_2 \eta/2} \left[ c_1 AI(\tau) + c_2 BI(\tau) \right]}, \quad \tau = \frac{a_1 a_4 + \frac{\alpha^2}{4} + a_1 a_3 \eta}{(a_1 a_3)^{2/3}},$$

with $c, c_1, c_2 \in \mathbb{R}$ the arbitrary constants of integration. From (5.7), for example, the constants $a_1, a_2, a_3$ and $a_4$ have the form

$$a_1 = \frac{\alpha^2}{2(\alpha^2 + 1)}, \quad a_2 = \frac{\alpha}{a(\alpha^2 + 1)}, \quad a_3 = \frac{\alpha^2}{a(\alpha^2 + 1)} \quad \text{and} \quad a_4 = \frac{c\alpha^2}{\alpha^2 + 1}. $$
Chapter 5. Higher-dimensional Burgers’ equation

Here, \( c \in \mathbb{R} \) is the constant of integration which arises when Eq. (5.6) is integrated once to obtain an O.D.E. of the form of (5.8).

The reduced equation from \( \Gamma_1 + \Gamma_3 \) has the two trivial symmetries

\[
\frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial \rho},
\]

and so the only symmetry possible is \( \alpha \frac{\partial}{\partial z} + \frac{\partial}{\partial \rho}, \) \((\alpha \neq 0)\), resulting in a reduction to the O.D.E.

\[
\left(\frac{\alpha^2 + 1}{\alpha^2}\right) \frac{d^2 G}{d \gamma^2} + \left(1 + \frac{1}{\alpha} G\right) \frac{dG}{d \gamma} = 0,
\]

where

\[
F = G(\gamma), \quad \gamma = \rho - \frac{1}{\alpha} z.
\]

By integrating Eq. (5.10) once, we obtain an O.D.E. of the form of Eq. (5.8) where \( a_3 = 0 \), that is

\[
\frac{dG}{d \gamma} = a_1 G^2 + a_2 G + a_4,
\]

and this may be solved to obtain the general solution

\[
G = \begin{cases} 
\frac{G_1 - AG_2 \exp\left[a_1 \gamma (G_1 - G_2)\right]}{1 - A \exp\left[a_1 \gamma (G_1 - G_2)\right]}, & G_1 \neq G_2, \\
\frac{1}{A - a_1 \gamma} + G_1, & G_1 = G_2,
\end{cases}
\]

where \( G_1, G_2 \) are the real roots of the quadratic \( a_1 G^2 + a_2 G + a_4 = 0 \), and \( A \) is the arbitrary constant of integration. From (5.10), the constants \( a_1, a_2 \) and \( a_4 \) have the form

\[
a_1 = \frac{-\alpha}{2(\alpha^2 + 1)}, \quad a_2 = \frac{-\alpha^2}{\alpha^2 + 1}, \quad a_4 = \frac{c \alpha^2}{\alpha^2 + 1}.
\]
5.2 Similarity reductions of the 3-D Burgers’ equation

From the symmetry analysis of the general class of three-dimensional diffusion-convection equation (4.38) in the previous chapter, we found that the symmetries obtained for the three-dimensional Burgers’ equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - u \frac{\partial u}{\partial z}
\]  

(5.13)

are

\[
\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial y}, \quad \Gamma_3 = \frac{\partial}{\partial z}, \quad \Gamma_4 = \frac{\partial}{\partial t},
\]

\[
\Gamma_5 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},
\]

\[
\Gamma_6 = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} - u \frac{\partial}{\partial u},
\]

\[
\Gamma_7 = t \frac{\partial}{\partial z} + \frac{\partial}{\partial u}.
\]

The first five are translation and rotation generators. These and a form of \( \Gamma_6 \) are common to all power-law forms of Eq. (4.38) (Edwards and Broadbridge [31]). \( \Gamma_7 \) is peculiar to Burgers’ equation. The two nontrivial symmetries \( \Gamma_6 \) and \( \Gamma_7 \) are compatible because they obey the commutation property

\[
[\Gamma_7, \Gamma_6] = -\Gamma_7.
\]  

(5.15)

This means that after a classical Lie point symmetry reduction of the governing equation (5.13) using \( \Gamma_7 \), \( \Gamma_6 \) will be an inherited symmetry of the reduced equation, and a second reduction will be possible. As in the case of the two-dimensional diffusion-convection equation, this is only possible for some very special forms of \( D(u) \) and \( K'(u) \).
Although reduction with $\Gamma_7$ followed by a reduction with $\Gamma_6$ is possible, we again seek the full set of symmetries, inherited or otherwise, of each of the reduced equations. The optimal system is determined to ensure that a minimal complete set of reductions is obtained from the symmetries of the governing equations. The optimal system for the three-dimensional Burgers' equation is

$$\{a\Gamma_5 + \Gamma_6, \Gamma_4 + \Gamma_5 + a\Gamma_7, \Gamma_5 + a\Gamma_7, \Gamma_3 + \Gamma_5, \Gamma_2 + \Gamma_4 + a\Gamma_7,$$

$$\Gamma_4 + \Gamma_7, a\Gamma_2 + \Gamma_3 + \Gamma_7, a\Gamma_2 + \Gamma_3, \Gamma_2 + a\Gamma_7, \Gamma_7, \Gamma_1 + \Gamma_7, \Gamma_1\}$$

for some $a \in \mathbb{R}$. Table 5.2 lists the reduced equation and the corresponding similarity variables for each of the nontrivial symmetries from the optimal set. We note that the reduced equations in Table 5.2 corresponding to the symmetry generators $\Xi_{13}$ and $\Xi_{14}$ are linear, and so no further symmetry analysis has been performed. The entries for $\Xi_{13}$ and $\Xi_{14}$ are examples of reduced equations which have symmetries extra to those symmetries inherited from the governing equation (4.38). We consider each of the remaining cases to determine the largest possible number of reductions.

Table 5.2: Reduced P.D.E.'s of the 3-D Burgers' equation.

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>Reduced P.D.E.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Xi_1 = a\Gamma_5 + \Gamma_6$</td>
<td>[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - u \frac{\partial u}{\partial z} = -\frac{1}{2t} \left[ u + (x - ay)^2 \frac{\partial u}{\partial x} + (ax + y) \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} \right]]</td>
</tr>
<tr>
<td>$\Xi_2 = \Gamma_6$</td>
<td>[-\frac{1}{2} \left[ F + \rho \frac{\partial F}{\partial \rho} + \gamma \frac{\partial F}{\partial \gamma} + \eta \frac{\partial F}{\partial \eta} \right] = \frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \gamma^2} + \frac{\partial^2 F}{\partial \eta^2} - F \frac{\partial F}{\partial \eta}] with $u = \frac{F(\rho, \gamma, \eta)}{\sqrt{t}}, \rho = \frac{x}{\sqrt{t}}, \gamma = \frac{y}{\sqrt{t}}, \eta = \frac{z}{\sqrt{t}}$</td>
</tr>
<tr>
<td>Symmetry</td>
<td>Reduced P.D.E.</td>
</tr>
<tr>
<td>------------------</td>
<td>-------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>$\Xi_3 = \Gamma_4 + \Gamma_5 + a\Gamma_7$</td>
<td>$a - \frac{\partial F}{\partial \eta} = 4 \frac{\partial F}{\partial \rho} + 4 \rho \frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial^2 F}{\partial \eta^2} + \frac{\partial^2 F}{\partial \gamma^2} - F \frac{\partial F}{\partial \gamma}$</td>
</tr>
<tr>
<td></td>
<td>with $u = F(\rho, \gamma, \eta) + at$, $\rho = x^2 + y^2$, $\gamma = z - \frac{at^2}{2}$, $\eta = \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}} - t$</td>
</tr>
<tr>
<td>$\Xi_4 = \Gamma_5 + a\Gamma_7$</td>
<td>$\frac{\partial F}{\partial t} = 4 \frac{\partial F}{\partial \rho} + 4 \rho \frac{\partial^2 F}{\partial \rho^2} + \left( \frac{1}{\rho} + \frac{1}{a^2 t^2} \right) \frac{\partial^2 F}{\partial \gamma^2} + \frac{1}{at} F \frac{\partial F}{\partial \gamma}$</td>
</tr>
<tr>
<td></td>
<td>with $u = F(t, \rho, \gamma) + \frac{z}{t}$, $\rho = x^2 + y^2$, $\gamma = \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}} - \frac{z}{at}$</td>
</tr>
<tr>
<td>$\Xi_5 = \Gamma_5$</td>
<td>$\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} - F \frac{\partial F}{\partial z}$</td>
</tr>
<tr>
<td></td>
<td>with $u = F(z, r, t)$, $r = \sqrt{x^2 + y^2}$</td>
</tr>
<tr>
<td>$\Xi_6 = \Gamma_3 + \Gamma_5$</td>
<td>$\frac{\partial F}{\partial t} = 4 \frac{\partial F}{\partial \rho} + 4 \rho \frac{\partial^2 F}{\partial \rho^2} + \left( \frac{1}{\rho} + 1 \right) \frac{\partial^2 F}{\partial \gamma^2} + F \frac{\partial F}{\partial \gamma}$</td>
</tr>
<tr>
<td></td>
<td>with $u = F(t, \rho, \gamma)$, $\rho = x^2 + y^2$, $\gamma = \sin^{-1} \frac{y}{\sqrt{x^2 + y^2}} - z$</td>
</tr>
<tr>
<td>$\Xi_7 = \Gamma_2 + \Gamma_4 + a\Gamma_7$</td>
<td>$a - \frac{\partial F}{\partial \rho} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \gamma^2} - F \frac{\partial F}{\partial \gamma}$</td>
</tr>
<tr>
<td></td>
<td>with $u = F(x, \rho, \gamma) + at$, $\rho = y - t$, $\gamma = z - \frac{at^2}{2}$</td>
</tr>
<tr>
<td>$\Xi_8 = \Gamma_4 + \Gamma_7$</td>
<td>$1 = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial \rho^2} - F \frac{\partial F}{\partial \rho}$</td>
</tr>
<tr>
<td></td>
<td>with $u = F(x, y, \rho) + t$, $\rho = z - \frac{t^2}{2}$</td>
</tr>
<tr>
<td>Symmetry</td>
<td>Reduced P.D.E.</td>
</tr>
<tr>
<td>---------------------</td>
<td>--------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>( E_9 = a \gamma_2 + \gamma_3 + \gamma_7 ) [( a \neq 0 )]</td>
<td>[ \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2} + \left( \frac{1}{a^2} + \frac{1}{(t+1)^2} \right) \frac{\partial^2 F}{\partial \rho^2} + \frac{1}{t+1} (F + \rho) \frac{\partial F}{\partial \rho} ] with ( u = F(x,t,\rho) + \frac{y}{a}, \rho = \frac{y}{a} - \frac{z}{t+1} )</td>
</tr>
<tr>
<td>( E_{10} = a \gamma_2 + \gamma_3 ) [( a \neq 0 )]</td>
<td>[ \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2} + \left( \frac{1 + a^2 t^2}{a^2} \right) \frac{\partial^2 F}{\partial \rho^2} + F \frac{\partial F}{\partial \rho} ] with ( u = F(x,t,\rho), \rho = \frac{y}{a} - z )</td>
</tr>
<tr>
<td>( E_{11} = \gamma_1 + a \gamma_7 ) [( a \neq 0 )]</td>
<td>[ \frac{\partial F}{\partial t} = (t^2 + 1) \left( \frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial y^2} + \frac{1}{(t+1)^2} (F + \rho) \frac{\partial F}{\partial \rho} \right) ] with ( u = F(y,t,\rho) + x, \rho = x - \frac{z}{t+1} )</td>
</tr>
<tr>
<td>( E_{12} = \gamma_3 + \gamma_7 )</td>
<td>[ \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - \frac{1}{t+1} F, ] with ( u = F(x,y,t) + \frac{z}{t+1} )</td>
</tr>
<tr>
<td>( E_{14} = \gamma_7 )</td>
<td>[ \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - \frac{1}{t} F, ] with ( u = F(x,y,t) + \frac{z}{t} )</td>
</tr>
</tbody>
</table>

We introduce the notation \( \Xi_i^j, i, j = 1, 2, 3... \) to denote the \( j^{th} \) nontrivial optimal symmetry of the \( i^{th} \) reduced equation from Table 5.2 corresponding to \( \Xi_i \), and \( \Xi_i^{j,k} \) to denote the \( k^{th} \) nontrivial symmetry of the reduced equation obtained by the symmetry operator \( \Xi_i^j \).

**Case 1:** The reduced equation obtained from the symmetry \( \Xi_1 \) is the only case
where the characteristic equations appear to be unsolvable, and so we are unable
to determine an explicit set of invariants. To reduce the number of independent
variables, we use the direct substitution method (Bluman and Kumei [7]) and use
the invariant surface condition

\[ X \frac{\partial u}{\partial x} + Y \frac{\partial u}{\partial y} + Z \frac{\partial u}{\partial z} + T \frac{\partial u}{\partial t} = U, \tag{5.16} \]

where for this case

\[ X = x - ay, \quad Y = y + ax, \quad Z = z, \quad T = 2t, \quad U = -u. \]

Rearrangement of (5.16), and substitution into the governing equation leads to the
entry for \( \Xi_1 \) in Table 5.2. In this reduced equation \( t \) appears as a parameter, not as
an independent variable of differentiation.

A second reduction is possible due to the symmetry generator

\[ \Xi_1 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \]

and the subsequent reduced equation is

\[ \frac{\partial F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} - F \frac{\partial F}{\partial z} = - \frac{1}{2t} \left[ F + r \frac{\partial F}{\partial r} + z \frac{\partial F}{\partial z} \right], \tag{5.17} \]

where

\[ u = F(r, z), \quad r = \sqrt{x^2 + y^2}, \]

and (5.17) has no symmetries.

**Case 2:** The reduced equation obtained using \( \Xi_2 \) has the symmetry

\[ \Xi_2 = -\rho \frac{\partial}{\partial \gamma} + \gamma \frac{\partial}{\partial \rho}, \]

leading to the reduced equation

\[ \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial \eta^2} - G \frac{\partial G}{\partial \eta} = - \frac{1}{2} \left[ G + r \frac{\partial G}{\partial r} + \eta \frac{\partial G}{\partial \eta} \right], \tag{5.18} \]
with

\[ F = G(r, \eta), \quad r = \sqrt{\rho^2 + \eta^2}, \]

and (5.18) has no symmetries.

**Case 3:** The reduced equation obtained using \( \Xi_3 \) has the symmetry

\[ \Xi^1_3 = \alpha \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \eta}, \]

which leads to the reduced equation, (\( \alpha \neq 0 \)),

\[
\alpha - \frac{\partial G}{\partial \xi} = 4 \frac{\partial G}{\partial \rho} + 4 \rho \frac{\partial^2 G}{\partial \rho^2} + \left( \frac{1}{\rho} + \frac{1}{\alpha^2} \right) \frac{\partial^2 G}{\partial \xi^2} + \frac{G}{\alpha} \frac{\partial G}{\partial \xi},
\]

(5.19)

with

\[ F = G(\rho, \xi), \quad \xi = \eta - \frac{\gamma}{\alpha}, \]

and (5.19) possesses only the trivial symmetry \( \Xi^1_{3,1} = \frac{\partial}{\partial \xi} \), which would lead to inadmissible solutions with no \( z \)-dependence.

**Case 4:** The reduced equation obtained using \( \Xi_4 \) has the symmetry

\[ \Xi^1_4 = (\alpha + \frac{1}{t}) \frac{\partial}{\partial \gamma} + \frac{a}{t} \frac{\partial}{\partial F}, \]

where \( \alpha \in \mathbb{R} \), which leads to the reduced equation

\[
\frac{\partial G}{\partial t} = 4 \frac{\partial G}{\partial \rho} + 4 \rho \frac{\partial^2 G}{\partial \rho^2} - \frac{\alpha G}{\alpha t + 1},
\]

(5.20)

where

\[ F = G(\rho, t) + \frac{a \gamma}{\alpha t + 1}. \]

Since (5.20) is linear, no further symmetry analysis has been performed.

**Case 5:** There are four nontrivial symmetries in the optimal system of the reduced equation obtained using \( \Xi_5 \) and they are
(i) $E_1 = 2t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} + z \frac{\partial}{\partial z} - F \frac{\partial}{\partial F}$, so that

$$F = \frac{G(\rho, \gamma)}{\sqrt{t}}, \rho = \frac{r}{\sqrt{t}}, \gamma = \frac{z}{\sqrt{t}}$$

to give

$$\frac{\partial^2 G}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial G}{\partial \rho} - \frac{\partial^2 G}{\partial \gamma^2} - G \frac{\partial G}{\partial \gamma} = -\frac{1}{2} \left[ G + \rho \frac{\partial G}{\partial \rho} + \gamma \frac{\partial G}{\partial \gamma} \right], \quad (5.21)$$

which has no symmetries.

(ii) $E_2 = t \frac{\partial}{\partial z} + \frac{\partial}{\partial F}$, so that

$$F = G(r, t) + \frac{z}{t}$$

to give

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} - \frac{G}{t}, \quad (5.22)$$

which is linear.

(iii) $E_3 = \frac{\partial}{\partial t} + t \frac{\partial}{\partial z} + \frac{\partial}{\partial F}$, so that

$$F = G(r, \rho) + t, \quad \rho = z - \frac{1}{2} t^2$$

and

$$1 = \frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial \rho^2} - G \frac{\partial G}{\partial \rho}. \quad (5.23)$$

Equation (5.23) has only the trivial symmetry $E_3^{1.1} = \frac{\partial}{\partial \rho}$.

(iv) $E_4 = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}$, leading to

$$F = G(r, \rho), \quad \rho = z + t,$$

to give

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r} \frac{\partial G}{\partial r} + \frac{\partial^2 G}{\partial \rho^2} - (G + 1) \frac{\partial G}{\partial \rho} = 0, \quad (5.24)$$

which has the symmetry

$$E_4^{1.1} = (G + 1) \frac{\partial}{\partial G} - \rho \frac{\partial}{\partial \rho} - r \frac{\partial}{\partial r}$$
so that
\[ G = \frac{1}{\rho} H(\eta) - 1, \quad \eta = \frac{\rho}{r}, \]
and (5.24) reduces to the nonlinear O.D.E.
\[ \left( \eta + \frac{1}{\eta} \right) \frac{d^2 H}{d\eta^2} + \left( 1 - \frac{H + 2}{\eta^2} \right) \frac{dH}{d\eta} + \frac{1}{\eta^3} H^2 + \frac{2}{\eta^3} H = 0. \quad (5.25) \]
In this case, the surfaces of constant concentration are vertical cones \( \eta = \text{constant} \) with a common vertex that moves in the OZ direction with constant speed.

The substitution \( H(\eta) = \eta \Phi(\eta) \) transforms Eq. (5.25) into
\[ (\eta^2 + 1) \frac{d^2 \Phi}{d\eta^2} + 3\eta \frac{d\Phi}{d\eta} - \Phi \frac{d\Phi}{d\eta} + \Phi = 0, \quad (5.26) \]
which may be integrated once to give the first order nonlinear O.D.E.
\[ (\eta^2 + 1) \frac{d\Phi}{d\eta} + \eta \Phi - \frac{1}{2} \Phi^2 = c_1, \quad (5.27) \]
where \( c_1 \in \mathbb{R} \) is the arbitrary constant of integration. If \( c_1 = 0 \), then Eq. (5.27) is a form of Bernoulli equation with solution
\[ \Phi(\eta) = \frac{1}{c_2 \sqrt{\eta^2 + 1} - \frac{1}{2} \eta}, \quad (5.28) \]
where \( c_2 \in \mathbb{R} \) is the second constant of integration.

In terms of the original dependent variable \( u(x, y, z, t) \), the solution to the three-dimensional Burgers' equation (5.13) is
\[ u = \frac{1}{c_2 \sqrt{(z + t)^2 + r^2 - \frac{1}{2} (z + t)}} - 1, \quad (5.29) \]
where \( r = \sqrt{x^2 + y^2} \). Taking \( u \) as a linear function of the volumetric water content, that is \( u = \theta + c \), Eq. (5.2) becomes
\[ \frac{\partial \theta}{\partial t} = \nabla^2 \theta - (\theta + c) \frac{\partial \theta}{\partial z}, \quad (5.30) \]
with solution
\[ \theta = \frac{1}{c_2 \sqrt{(z + t)^2 + r^2 - \frac{1}{2} (z + t)}} - (c + 1). \quad (5.31) \]
The condition that the water content $\theta = 0$ as $z \to \infty$ is satisfied when $c = -1$, and $(5.31)$ becomes

$$\theta = \frac{1}{c_2\sqrt{(z + t)^2 + r^2 - \frac{1}{2}(z + t)}}.$$  \hfill (5.32)

This form of $\theta$ also satisfies the additional conditions

$$\theta \to 0 \text{ as } x \to \infty \quad \text{and} \quad \theta \to 0 \text{ as } y \to \infty.$$  \hfill (5.33)

We also require that at some initial time $t = t_0$, with $x, y, z = 0$, that $\theta = 1$, which is satisfied when

$$c_2 = \frac{1}{2} + \frac{2}{t_0}.$$  

In Figure 5.1, the analytic solution is plotted against $r$ and $z$ for $t = 2.0$ and $t_0 = 0.01$, where $r = \sqrt{x^2 + y^2}$.

When $c_1 \neq 0$, it is possible to construct a class of solutions of Eq. (5.27) which
take the form
\[
\Phi = \frac{-\left[\alpha_1 F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \lambda; 1; \frac{1}{2} + \mu\right) - \alpha_2 F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \lambda; 1; \frac{1}{2} - \mu\right)\right]}{\sqrt{\eta^2 + 1} \left[\alpha_1 \left(\frac{1}{2} + \mu\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \lambda; 2; \frac{1}{2} + \mu\right) + \alpha_2 \left(\frac{1}{2} - \mu\right) F\left(\frac{1}{2} - \lambda, \frac{1}{2} + \lambda; 2; \frac{1}{2} - \mu\right)\right]},
\]
(5.34)
where \( \lambda = \sqrt{\frac{\alpha_1}{4} + 1}, \mu = \frac{\eta}{2\sqrt{\eta^2 + 1}} \), and \( F(a, b; c; \tau) \) is the hypergeometric function (Erdélyi et al. [32]). This class of solutions may be expressed in terms of \( \theta \) by using the transformation
\[
\theta = \frac{1}{r} \Phi(\eta) - (c + 1), \quad \text{with} \quad \eta = \frac{z + t}{r}.
\]
However, these solutions do not satisfy the conditions specified earlier.

**Case 6**: The optimal system of the reduced equation obtained using \( \Xi_6 \) has two nontrivial symmetries

(i) \( \Xi_6^1 = \alpha \frac{\partial}{\partial t} + t \frac{\partial}{\partial \gamma} - \frac{\partial}{\partial F} \), \( (\alpha \neq 0) \), so that
\[
F = G(\rho, \gamma) - \frac{t}{\alpha}, \quad \eta = \alpha \gamma - \frac{1}{2} t^2,
\]
to give
\[
-\frac{1}{\alpha} = 4 \frac{\partial G}{\partial \rho} + 4 \rho \frac{\partial^2 G}{\partial \rho^2} + \alpha^2 \left[\frac{1}{\rho} + 1\right] \frac{\partial^2 G}{\partial \eta^2} + \alpha \frac{\partial G}{\partial \eta},
\]
(5.35)
which has only one trivial symmetry \( \Xi_6^{1,1} = \frac{\partial}{\partial \eta} \).

(ii) \( \Xi_6^2 = t \frac{\partial}{\partial \gamma} - \frac{\partial}{\partial F} \), leading to
\[
F = G(\rho, t) - \frac{\gamma}{t}
\]
to give
\[
\frac{\partial G}{\partial t} = 4 \frac{\partial G}{\partial \rho} + 4 \rho \frac{\partial^2 G}{\partial \rho^2} - \frac{G}{t},
\]
(5.36)
which is linear.

**Case 7**: The optimal system of the reduced equation obtained using \( \Xi_7 \) has four nontrivial symmetries
(i) $\Xi_1^1 = \alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \rho}$, $(\alpha, \beta \neq 0)$, so that

$$F = G(\eta, \xi), \eta = \frac{x}{\alpha} - \frac{\gamma}{\beta}, \xi = \frac{x}{\alpha} - \rho,$$

to give

$$a + \frac{\partial G}{\partial \xi} = \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) \frac{\partial^2 G}{\partial \eta^2} + \left( 1 + \frac{1}{\alpha^2} \right) \frac{\partial^2 G}{\partial \xi^2} + G \frac{\partial G}{\beta \partial \eta},$$

which has the nontrivial symmetry

$$\Xi_1^{1,1} = c \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi}, (c \neq 0).$$

Equation (5.37) reduces to the O.D.E.

$$a + \frac{dH}{d\zeta} = \left[ \frac{1}{c^2} \left( \frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) + 1 + \frac{1}{\alpha^2} \right] \frac{d^2 H}{d\xi^2} - \frac{H}{c\beta} \frac{dH}{d\zeta},$$

where $G = H(\zeta), \zeta = \xi - \frac{x}{c}$.

(ii) $\Xi_1^2 = \beta \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \rho}$, $(\beta \neq 0)$, so that

$$F = G(x, \eta), \eta = \rho - \frac{\gamma}{\beta},$$

leading to

$$a - \frac{\partial G}{\partial \eta} = \frac{\partial^2 G}{\partial x^2} + \left( 1 + \frac{1}{\beta^2} \right) \frac{\partial^2 G}{\partial \eta^2} + G \frac{\partial G}{\beta \partial \eta},$$

which in turn has the symmetry

$$\Xi_1^{2,1} = c \frac{\partial}{\partial x} + \frac{\partial}{\partial \eta}, (c \neq 0),$$

and Eq. (5.39) is reduced to the O.D.E.

$$a - \frac{dH}{d\xi} = \left( 1 + \frac{1}{c^2} + \frac{1}{\beta^2} \right) \frac{d^2 H}{d\xi^2} + \frac{H}{\beta} \frac{dH}{d\xi},$$

where

$$G = H(\xi), \xi = \eta - \frac{x}{c}.$$
If however $a = 0$ in $\Xi_7$, then (5.39) possesses the additional symmetry

$$\Xi_7^{2,2} = (G + \beta) \frac{\partial}{\partial G} - x \frac{\partial}{\partial x} - \gamma \frac{\partial}{\partial \gamma},$$

and is reduced to the nonlinear O.D.E.

$$\left(1 + \eta^2 + \frac{\eta^2}{\beta^2}\right) \frac{d^2 H}{d\eta^2} - \frac{\eta H}{\beta} \frac{dH}{d\eta} + 5\eta \left(1 + \frac{1}{\beta^2}\right) \frac{dH}{d\eta} - \frac{H^2}{\beta} + 2 \left(1 + \frac{1}{\beta^2}\right) H = 0.$$

(iii) $\Xi_7^3 = \alpha \frac{\partial}{\partial x} + \frac{\partial}{\partial \rho}$, $(\alpha \neq 0)$, so that

$$F = G(\gamma, \eta), \quad \eta = \rho - \frac{x}{\alpha}$$

and

$$a - \frac{\partial G}{\partial \eta} = \left(1 + \frac{1}{\alpha^2}\right) \frac{\partial^2 G}{\partial \eta^2} + \frac{\partial^2 G}{\partial \gamma^2} - G \frac{\partial G}{\partial \gamma},$$

(5.41)

which has the symmetry

$$\Xi_7^{3,1} = c \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \eta}, \quad (c \neq 0),$$

and reduces Eq. (5.41) to the second order O.D.E.

$$a - \frac{dH}{d\xi} = \left(1 + \frac{1}{\alpha^2} + \frac{1}{c^2}\right) \frac{d^2 H}{d\xi^2} + \frac{H}{c} \frac{dH}{d\xi},$$

(5.42)

with

$$G = H(\xi), \quad \xi = \eta - \frac{\gamma}{c}.$$

(iv) $\Xi_7^4 = \alpha \frac{\partial}{\partial x} + \frac{\partial}{\partial \gamma}$, $(\alpha \neq 0)$, so that

$$F = G(\rho, \eta), \quad \eta = \gamma - \frac{x}{\alpha}$$

to get

$$a - \frac{\partial G}{\partial \rho} = \left(1 + \frac{1}{\alpha^2}\right) \frac{\partial^2 G}{\partial \eta^2} + \frac{\partial^2 G}{\partial \rho^2} - G \frac{\partial G}{\partial \eta}.$$

(5.43)
Equation (5.43) has the symmetry

\[ \Xi^4_1 = c \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \eta}, \quad (c \neq 0), \]

and reduces to

\[ a + \frac{1}{c} \frac{dH}{d\xi} = \left( 1 + \frac{1}{\alpha^2} + \frac{1}{c^2} \right) \frac{d^2H}{d\xi^2} - H \frac{dH}{d\xi}, \quad (5.44) \]

with

\[ G = H(\xi), \quad \xi = \eta - \frac{\rho}{c}. \]

The equations (5.38), (5.40), (5.42) and (5.44) can all be integrated once to obtain an O.D.E. either of the form of Eq. (5.8) for \( a \neq 0 \), and possess a solution of the form of Eq. (5.9), or of the form (5.11) when \( a = 0 \), and have a solution of the form of (5.12).

**Case 8:** There are three nontrivial symmetries in the optimal system of the reduced equation obtained using \( \Xi_8 \)

(i) \( \Xi^4_1 = \alpha \frac{\partial}{\partial \rho} + y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad (\alpha \neq 0), \) so

\[ F = G(r, \eta), \quad r = x^2 + y^2, \quad \eta = \sin^{-1} \frac{x}{\sqrt{x^2 + y^2}} - \frac{\rho}{\alpha}, \]

to give

\[ 1 = 4 \frac{\partial G}{\partial r} + 4r \frac{\partial^2 G}{\partial r^2} + \left( \frac{1}{r} + \frac{1}{\alpha^2} \right) \frac{\partial^2 G}{\partial \eta^2} + \frac{1}{\alpha} G \frac{\partial G}{\partial \eta}, \quad (5.45) \]

which has only one trivial symmetry, so no further reductions are possible.

(ii) \( \Xi^2_8 = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \), and the functional form of \( F \) is

\[ F = G(\rho, r), \quad r = x^2 + y^2 \]

to give

\[ 1 = 4 \frac{\partial G}{\partial \rho} + 4r \frac{\partial^2 G}{\partial r^2} + \frac{\partial^2 G}{\partial \rho^2} - G \frac{\partial G}{\partial \rho}, \quad (5.46) \]
which has only one trivial symmetry, so no further reductions are possible.

(iii) \( \Xi^3_8 = \alpha \frac{\partial}{\partial x} + \frac{\partial}{\partial \rho}, (\alpha \neq 0) \), so that

\[
F = G(y, \eta), \quad \eta = \rho - \frac{x}{\alpha}
\]
to give

\[
1 = \left( 1 + \frac{1}{\alpha^2} \right) \frac{\partial^2 G}{\partial \eta^2} + \frac{\partial^2 G}{\partial y^2} - G \frac{\partial G}{\partial \eta}, \quad (5.47)
\]
which has the nontrivial symmetry

\[
\Xi^3_{1,1} = c \frac{\partial}{\partial y} + \frac{\partial}{\partial \eta}, (c \neq 0),
\]
so that

\[
G = H(\xi), \quad \xi = \eta - \frac{y}{c}
\]
and (5.47) reduces to the O.D.E.

\[
1 = \left( 1 + \frac{1}{\alpha^2} + \frac{1}{c^2} \right) \frac{d^2 H}{d\xi^2} - H \frac{dH}{d\xi}. \quad (5.48)
\]

Equation (5.48) may be integrated once to obtain a first order O.D.E. which is of the form of Eq. (5.8) (with \( a_2 = 0 \)), and therefore has a solution of the form of (5.9).

**Case 9:** There are four nontrivial symmetries in the optimal system of the reduced equation obtained using \( \Xi_9 \)

(i) \( \Xi^1_9 = \alpha \frac{\partial}{\partial x} + \frac{1+t-\beta t}{1+t} \frac{\partial}{\partial \rho} + (\beta - 1) \frac{\partial}{\partial \rho}, \) so that

\[
F = G(t, \eta) + \frac{(\beta - 1)x}{\alpha}, \quad \eta = (1 + t - \beta t)x - \alpha(1 + t)\rho
\]
to give

\[
\frac{\partial G}{\partial t} = \left[ (1 + t - \beta t)^2 + \frac{\alpha^2}{a^2} \left( (1 + t)^2 + a^2 \right) \right] \frac{\partial^2 G}{\partial \eta^2} - \alpha G \frac{\partial G}{\partial \eta}, \quad (5.49)
\]
which has the nontrivial symmetry

\[
\Xi^1_{9,1} = (c + \alpha t) \frac{\partial}{\partial \eta} + \frac{\partial}{\partial G}
\]
and (5.49) may be reduced to

$$\frac{dH}{dt} = -\frac{\alpha H}{c + \alpha t},$$

leading to the solution

$$H = \frac{A}{c + \alpha t},$$

where

$$G = H(t) + \frac{\eta}{c + \alpha t},$$

and A is the arbitrary constant of integration.

(ii) \( \Xi^2_g = \frac{1+t-\delta t}{1+t} \frac{\partial}{\partial \rho} + (\beta - 1) \frac{\partial}{\partial \rho}, \) so that

$$F = G(x, t) + (\beta - 1)\left(\frac{1 + t}{1 + t - \beta t}\right)^\rho$$

and

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2} + (\beta - 1)\left(\frac{1 + t}{1 + t - \beta t}\right)G,$$

which is linear, so no further symmetry analysis has been performed.

(iii) \( \Xi^3_g = \alpha \frac{\partial}{\partial x} - \frac{t}{1+t} \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \rho}, \) \((\alpha \neq 0), \) to get

$$F = G(t, \gamma) + \frac{x}{\alpha}, \quad \gamma = \frac{x t}{\alpha} + (1 + t)\rho$$

and

$$\frac{\partial G}{\partial t} = \left[\frac{t^2}{\alpha^2} + \frac{(1 + t)^2 + a^2}{a^2}\right] \frac{\partial^2 G}{\partial \gamma^2} + G \frac{\partial G}{\partial \gamma},$$

which has the nontrivial symmetry

$$\Xi^{3,1}_g = (c - t) \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial G}$$

and (5.51) reduces to the O.D.E.

$$\frac{dH}{dt} = -\frac{H}{t - c}.$$
Chapter 5. Higher-dimensional Burgers' equation

with solution

\[ H = \frac{A}{t - c}, \]

where

\[ G = H(t) + \frac{\gamma}{c - t}. \]

(iv) \[ \Xi^4 = \frac{\partial}{\partial F} - \frac{t}{1 + t} \frac{\partial}{\partial \rho}, \] so that

\[ F = G(x, t) - \frac{1 + t}{t} \rho \]

and

\[ \frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2} - G, \] (5.52)

which is linear.

**Case 10:** There are six nontrivial symmetries in the optimal system of the reduced equation obtained using \( \Xi_{10} \), which are

(i) \[ \Xi^1_{10} = 2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \rho \frac{\partial}{\partial \rho} - F \frac{\partial}{\partial F}, \] so that

\[ F = \frac{G(\gamma, \eta)}{\sqrt{t}}, \quad \gamma = \frac{x}{\sqrt{t}}, \quad \eta = \frac{\rho}{\sqrt{t}}, \]

leading to

\[ \frac{\partial^2 G}{\partial \gamma^2} + \left( \frac{1 + a^2}{a^2} \right) \frac{\partial^2 G}{\partial \eta^2} + G \frac{\partial G}{\partial \eta} = -\frac{1}{2} \left[ G + \frac{\gamma}{\partial \gamma} + \frac{\eta}{\partial \eta} \right], \] (5.53)

which has no symmetries.

(ii) \[ \Xi^2_{10} = \alpha \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + t \frac{\partial}{\partial \rho} - \frac{\partial}{\partial F}, \quad (\alpha \neq 0), \] so that

\[ F = G(\gamma, \eta) - t, \quad \gamma = t - \frac{x}{\alpha}, \quad \eta = \rho - \frac{1}{2} t^2, \]

and leads to

\[ \frac{\partial G}{\partial \gamma} - 1 = \frac{1}{\alpha^2} \frac{\partial^2 G}{\partial \gamma^2} + \left( \frac{1 + a^2}{a^2} \right) \frac{\partial^2 G}{\partial \eta^2} + G \frac{\partial G}{\partial \eta}, \] (5.54)
and has the nontrivial symmetry

\[ \Xi_{10}^{2,1} = \beta \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial \eta}. \]

Equation (5.54) reduces to the O.D.E.

\[ -\frac{1}{\beta} \frac{dH}{d\xi} - 1 = \left( \frac{1}{\alpha^2 \beta^2} + \frac{1 + a^2}{a^2} \right) \frac{d^2H}{d\xi^2} + H \frac{dH}{d\xi}, \quad (5.55) \]

which may be integrated once to obtain an equation of the form generalised by (5.8), and therefore has a solution of the type given by (5.9), where

\[ G = H(\xi), \quad \xi = \eta - \frac{\gamma}{\beta}. \]

(iii) \[ \Xi_{10}^{3} = \frac{\partial}{\partial t} + t \frac{\partial}{\partial \rho} - \frac{\partial}{\partial \rho}, \]

so that

\[ F = G(x, \gamma) - t, \quad \gamma = \rho - \frac{1}{2} t^2 \]

to give

\[ -1 = \frac{\partial^2 G}{\partial x^2} + \left( \frac{1 + a^2}{a^2} \right) \frac{\partial^2 G}{\partial \gamma^2} + G \frac{\partial G}{\partial \gamma}, \quad (5.56) \]

which has the nontrivial symmetry

\[ \Xi_{10}^{3,1} = \beta \frac{\partial}{\partial x} + \frac{\partial}{\partial \gamma}, \]

and (5.56) reduces to the O.D.E.

\[ -1 = \left( \frac{1}{\beta^2} + \frac{1 + a^2}{a^2} \right) \frac{d^2H}{d\xi^2} + H \frac{dH}{d\xi}, \quad (5.57) \]

where

\[ G = H(\xi), \quad \xi = \gamma - \frac{x}{\beta}. \]

After one integration, the resultant first order O.D.E. is a special case of (5.8) where \( a_2 = 0 \), and has an exact solution of the form given by (5.9).
(iv) $\Xi_{10}^4 = \alpha \frac{\partial}{\partial x} + t \frac{\partial}{\partial \rho} - \frac{\partial}{\partial F}$, so that

$$F = G(t, \gamma) - \frac{\rho}{t}, \quad \gamma = \frac{x}{\alpha} - \frac{\rho}{t}$$

leading to

$$\frac{\partial G}{\partial t} = \left[ \frac{1}{\alpha^2} + \frac{1}{t^2} \left( \frac{1 + a^2}{a^2} \right) \right] \frac{\partial^2 G}{\partial \gamma^2} - \frac{1}{t} G \left( \frac{\partial G}{\partial \gamma} + 1 \right), \quad (5.58)$$

which has the nontrivial symmetry

$$\Xi_{10}^{4,1} = \left( \beta + \frac{1}{t} \right) \frac{\partial}{\partial \gamma} - \frac{1}{t} \frac{\partial}{\partial G}$$

and leads to

$$\frac{dH}{dt} = - \frac{\beta H}{\beta t + 1},$$

and has the solution

$$H = \frac{A}{\beta t + 1},$$

where $A$ is the constant of integration, and

$$G = H(t) - \frac{\gamma}{\beta t + 1}.$$

(v) $\Xi_{10}^5 = t \frac{\partial}{\partial t} - \frac{\partial}{\partial F}$, so that

$$F = G(x, t) - \frac{\rho}{t},$$

and leads to

$$\frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2} - \frac{G}{t}, \quad (5.59)$$

which is linear.

(vi) $\Xi_{10}^6 = \frac{\partial}{\partial x} + \frac{\partial}{\partial \gamma}$, so that

$$F = G(\rho, \gamma), \quad \gamma = x - t,$$
leading to

\[- \frac{\partial G}{\partial \gamma} = \frac{\partial^2 G}{\partial \gamma^2} + \left( \frac{1 + a^2}{a^2} \right) \frac{\partial^2 G}{\partial \rho^2} + G \frac{\partial G}{\partial \rho}, \tag{5.60}\]

which has the nontrivial symmetry

\[\Xi_{10}^{6,1} = c \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \gamma}, \; c \neq 0.\]

Equation (5.60) reduces to

\[- \frac{dH}{d\xi} = \left( 1 + \frac{1 + a^2}{a^2 c^2} \right) \frac{d^2 H}{d\xi^2} - \frac{1}{c} H \frac{dH}{d\xi}, \tag{5.61}\]

where

\[G = H(\xi), \; \xi = \gamma - \frac{\rho}{c}.\]

Equation (5.61) may be integrated once to obtain a first order O.D.E. of the form given by (5.11), and has an exact solution of the form (5.12).

**Case 11:** There are four nontrivial symmetries in the optimal system of the reduced equation obtained using \(\Xi_{11}\)

(i) \(\Xi_{11}^{1} = \alpha \frac{\partial}{\partial x} + \left( \frac{t}{t + 1} \right) \frac{\partial}{\partial \rho} - \frac{\partial}{\partial F}, \; (\alpha, \beta \neq 0), \) so that

\[F = G(t, \gamma) - \frac{\rho t}{t + \beta}, \; \gamma = \frac{x}{\alpha} - \frac{\rho t}{t + \beta},\]

to give

\[\frac{\partial G}{\partial t} = \left[ \frac{1}{\alpha^2} + \frac{a^2 + t^2}{(t + \beta)^2} \right] \frac{\partial^2 G}{\partial \gamma^2} - \frac{G}{t + \beta} \left( 1 + \frac{\partial G}{\partial \gamma} \right), \tag{5.62}\]

which has the nontrivial symmetry

\[\Xi_{11}^{1,1} = \left( c + \frac{t}{\beta(t + \beta)} \right) \frac{\partial}{\partial \gamma} + \frac{1}{t + \beta} \frac{\partial}{\partial G}, \; (c \in \mathbb{R}),\]

and so

\[G = H(t) + \frac{\beta \gamma}{(c \beta + 1)t + c \beta^2}.\]
In this case, (5.62) reduces to the O.D.E.

\[
\frac{dH}{dt} = -\frac{(c\beta + 1)H}{(c\beta + 1)t + c\beta^2},
\]

and has the solution

\[
H = \frac{A}{(c\beta + 1)t + c\beta^2},
\]

with \( A \in \mathbb{R} \) the constant of integration.

(ii) \( \Xi_{11}^2 = \alpha_\beta \frac{\partial}{\partial x} + \frac{\partial}{\partial \rho} - \frac{\partial}{\partial F}, \) so that

\[
F = G(t, \gamma) - \rho, \quad \gamma = \rho - \frac{x}{\alpha},
\]

and leads to

\[
\frac{\partial G}{\partial t} = \left( \frac{1}{\alpha^2} + \frac{a^2 + t^2}{t^2} \right) \frac{\partial^2 G}{\partial \gamma^2} + \frac{G}{t} \left( \frac{\partial G}{\partial \gamma} - 1 \right),
\]

which has the nontrivial symmetry

\[
\Xi_{11}^{2,1} = \left( c + \frac{1}{t} \right) \frac{\partial}{\partial \gamma} + \frac{1}{t} \frac{\partial}{\partial G}, \quad (c \in \mathbb{R}),
\]

and (5.63) reduces to

\[
\frac{dH}{dt} = -\frac{cH}{ct + 1},
\]

where

\[
G = H(t) + \frac{\gamma}{ct + 1},
\]

with solution

\[
H = \frac{A}{ct + 1},
\]

where \( A \in \mathbb{R} \) is the constant of integration.

(iii) \( \Xi_{11}^3 = \left( \frac{\beta t}{t + 1} \right) \frac{\partial}{\partial \rho} - \frac{\partial}{\partial F}, \quad (\beta \in \mathbb{R}), \) so that

\[
F = G(x, t) - \frac{\rho t}{t + \beta},
\]
leading to
\[ \frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2} - \frac{G}{t + \beta}, \] (5.64)
which is linear.

(iv) \( \Xi_{11}^4 = \alpha \frac{\partial}{\partial x} + \frac{1}{t} \frac{\partial}{\partial \rho}, \) \((\alpha \neq 0),\) so that
\[ F = G(t, \gamma), \quad \gamma = \rho t - \frac{x}{\alpha} \]
and
\[ \frac{\partial G}{\partial t} = \left( \frac{1}{\alpha^2} + \alpha^2 + t^2 \right) \frac{\partial^2 G}{\partial \gamma^2} + G \frac{\partial G}{\partial \gamma}, \] (5.65)
which has the nontrivial symmetry
\[ \Xi_{11}^{41} = (c + t) \frac{\partial}{\partial \gamma} - \frac{\partial}{\partial G}. \]
Equation (5.65) reduces to the O.D.E.
\[ \frac{dH}{dt} = -\frac{H}{t + c}, \]
where
\[ G = H(t) - \frac{\gamma}{t + c}, \]
and has the solution
\[ H = \frac{A}{t + c}, \]
where \( A \in \mathbb{R} \) is the constant of integration.

**Case 12:** There are four nontrivial symmetries in the optimal system of the reduced equation obtained using \( \Xi_{12} \)

(i) \( \Xi_{12}^1 = \alpha \frac{\partial}{\partial y} + \left( \frac{\beta}{t} + 1 \right) \frac{\partial}{\partial \rho} - \frac{\partial}{\partial F}, \) \((\alpha, \beta \neq 0),\) so that
\[ F = G(t, \gamma) - \frac{t \rho}{t + \beta}, \quad \gamma = \frac{y}{\alpha} - \frac{t \rho}{t + \beta}, \]
and leads to
\[
\frac{\partial G}{\partial t} = \left[ \frac{1}{\alpha^2} + \frac{1 + t^2}{(t + \beta)^2} \right] \frac{\partial^2 G}{\partial \gamma^2} - \frac{G}{t + \beta} \left( \frac{\partial G}{\partial \gamma} + 1 \right),
\]  
which has the nontrivial symmetry
\[
\Xi_{12}^{1,1} = \left( c + \frac{t}{\beta(t + \beta)} \right) \frac{\partial}{\partial \gamma} + \frac{1}{t + \beta} \frac{\partial}{\partial G}, \quad (c \in \mathbb{R}),
\]
and so Eq. (5.66) reduces to
\[
\frac{dH}{dt} = -\frac{H(c\beta + 1)}{(c\beta + 1)t + c\beta^2},
\]
with solution
\[
H = \frac{A}{(c\beta + 1)t + c\beta^2},
\]
where
\[
G = H(t) + \frac{\beta\gamma}{(c\beta + 1)t + c\beta^2},
\]
and \( A \in \mathbb{R} \) is the constant of integration.

(ii) \( \Xi_{12}^2 = \alpha \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \rho} - \frac{\partial}{\partial F}, \quad (\alpha \neq 0), \) so that
\[
F = G(t, \gamma) - \rho, \quad \gamma = \rho - \frac{y}{\alpha},
\]
and leads to
\[
\frac{\partial G}{\partial t} = \left( \frac{1}{\alpha^2} + \frac{1 + t^2}{t^2} \right) \frac{\partial^2 G}{\partial \gamma^2} + \frac{G}{t} \left( \frac{\partial G}{\partial \gamma} - 1 \right),
\]  
which has the nontrivial symmetry
\[
\Xi_{12}^{2,1} = \left( c + \frac{1}{t} \right) \frac{\partial}{\partial \gamma} + \frac{1}{t} \frac{\partial}{\partial G}, \quad c \in \mathbb{R},
\]
and (5.66) is reduced to the O.D.E.
\[
\frac{dH}{dt} = -\frac{cH}{ct + 1},
\]
with solution
\[
H = \frac{A}{ct + 1},
\]
Chapter 5. Higher-dimensional Burgers' equation

where

\[ G = H(t) + \frac{\gamma}{ct + 1} \]

and \( A \in \mathbb{R} \) the constant of integration.

(iii) \( \Xi_{12}^3 = (\beta \xi + 1) \frac{\partial}{\partial \rho} - \frac{\partial}{\partial \phi}, \quad (\beta \in \mathbb{R}) \), so that

\[ F = G(y, t) - \frac{t \rho}{t + \beta}, \]

and leads to

\[ \frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial y^2} - \frac{G}{t + \beta}, \quad (5.68) \]

which is linear.

(iv) \( \Xi_{12}^4 = \alpha \frac{\partial}{\partial y} + \frac{1}{t} \frac{\partial}{\partial \rho}, \quad (\alpha \neq 0) \), so that

\[ F = G(t, \gamma), \quad \gamma = \frac{y}{\alpha} - t \rho, \]

leading to

\[ \frac{\partial G}{\partial t} = \left( \frac{1}{\alpha^2} + 1 + t^2 \right) \frac{\partial^2 G}{\partial \gamma^2} - G \frac{\partial G}{\partial \gamma}, \quad (5.69) \]

and has the nontrivial symmetry

\[ \Xi_{12}^{4,1} = (c + t) \frac{\partial}{\partial \gamma} + \frac{\partial}{\partial G}, \quad c \in \mathbb{R}. \]

In this case, (5.69) reduces to the O.D.E.

\[ \frac{dH}{dt} = -\frac{H}{t + c}, \]

where

\[ G = H(t) + \frac{\gamma}{t + c}, \]

with solution

\[ H = \frac{A}{t + c}, \]

where \( A \in \mathbb{R} \) is the constant of integration.
5.3 Discussion

Unlike the one-dimensional diffusion-convection equation (3.1), there are no linearisable models in two dimensions (Broadbridge [11]), and we doubt that integrable models exist in three dimensions. Instead we have relied on classical similarity reductions, which is the other best-known method for obtaining exact solutions. For the higher-dimensional version of Burgers’ equation (5.2), we have been able to obtain a range of exact solutions through two-step classical symmetry reductions in the case of the two-dimensional Burgers’ equation, and through three-step classical symmetry reductions in the case of the three-dimensional Burgers’ equation.

For the three-dimensional Burgers’ equation, the range of results can be grouped as three distinct types. Equations (5.49), (5.51), (5.58), (5.62), (5.63), (5.65), (5.66), (5.67) and (5.69) all have solutions, which after substitution are cases of the general solution

$$u = \frac{a_3 + a_4x + a_5y + a_2z}{a_1 + a_2t},$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathbb{R}$.

Equations (5.38), (5.40), (5.42), (5.44), (5.48), (5.55), (5.57) and (5.61) all have solutions which are of the form of either (5.9) or (5.12), obtained by solving a Riccati equation. Equation (5.25), which initially appears to be a special case, also transforms to a Riccati equation (5.27).

Equations (5.20), (5.22), (5.36), (5.50), (5.52), (5.59), (5.64) and (5.68) are all linear, and so may be solved by suitable integral transform methods. Since a number of nontrivial linear equations have been obtained in this way, a full investigation of their solutions is beyond the scope of this chapter.

Exploitation of classical symmetry analysis and use of optimal systems has led to new exact solutions of Burgers’ equation in two and three dimensions. Nonclassical point symmetry analysis shows that no extra symmetries exist (Arrigo [2]), and the
set of exact solutions is unable to be extended via this approach.
Chapter 6

Closed-form solutions for unsaturated flow under variable flux boundary conditions

The classical Lie symmetry analysis has been performed for the general class of nonlinear diffusion-convection equations

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(u)\nabla u) - \frac{dK}{du} \frac{\partial u}{\partial z}$$

(6.1)

in one, two and three dimensions in the previous chapters. Equation (6.1) in two spatial dimensions has been shown to possess no linearisable models (Broadbridge [11]) and so we doubt that integrable models exist in three spatial dimensions. Hence we have relied on classical symmetry reductions to obtain reductions from the governing P.D.E. to an O.D.E. and as a result we have been able to construct solutions to some special forms of the nonlinear diffusion-convection equation. In addition, the method of Philip and Knight [65] has been extended to obtain closed-form solutions to the diffusion-convection equation when $D(u) = u^{-1}$ and $K'(u) = u^{-1/2}$ in two dimensions and $D(u) = u^{-2/3}$ and $K'(u) = u^{-1/3}$ in three dimensions.
Chapter 6. Closed-form solutions for unsaturated flow

Fortunately, for one-dimensional flows, there exist useful integrable nonlinear parabolic equations closely related to (6.1). These integrable equations may be characterised by the existence of Lie-Bäcklund symmetries [16], [35] or by certain infinite-dimensional Lie symmetry groups [8]. Up to a contact transformation, the integrable second order evolution equations belong to one of four classes [84]:

(i) the linear class, with canonical form

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x)u \quad (g \text{ arbitrary}),
\]

including the linear model for (6.1),

\[
\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial z^2} - \frac{K_s}{\theta - \theta_n} \frac{\partial \theta}{\partial z},
\]

(6.2)

with \(D, K_s, \theta_s\) and \(\theta_n\) constant,

(ii) the Burgers class [21], with canonical form

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + 2u \frac{\partial u}{\partial x} + g(x) \quad (g \text{ arbitrary}),
\]

including the weakly nonlinear Burgers model [25] for (6.1),

\[
\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial z^2} - 2K_s \frac{\theta - \theta_n}{(\theta - \theta_n)^2} \frac{\partial \theta}{\partial z},
\]

(6.3)

(iii) the Fujita class [37], with canonical form

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ u^{-2} \frac{\partial u}{\partial x} \right],
\]

including the realistic model Richards equations [19],[74], [93],

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ \frac{a}{(b - \theta)^2} \frac{\partial \theta}{\partial z} \right] - \left[ \frac{\lambda}{2(b - \theta)^2} - \gamma \right] \frac{\partial \theta}{\partial z},
\]

(6.4)

and (iv) the Freeman-Satsuma class [36], with canonical form

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ u^{-2} \frac{\partial u}{\partial x} \right] - 2,
\]
Chapter 6. Closed-form solutions for unsaturated flow

including the Richards equation with plant root absorption term [16],

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[ \frac{a}{(b-\theta)^2} \frac{\partial \theta}{\partial z} \right] - \nu \frac{a}{(b-\theta)^2} \frac{\partial \theta}{\partial z} - Re^{-\nu z} .
\] (6.5)

Each of the equations (6.2) - (6.5) has contributed to the theory of unsaturated flow. In particular, (6.4) corresponds to realistic choices for the soil hydraulic conductivity and diffusivity functions,

\[
K(\theta) = \frac{\lambda}{2(b-\theta)} + \beta + \gamma(b-\theta) ,
\] (6.6)

\[
D(\theta) = \frac{a}{(b-\theta)^2} ,
\] (6.7)

with \(\lambda, \beta, \gamma, a\) and \(b\) constant. Furthermore, the corresponding moisture characteristic function for the so-called soil-water pressure head

\[
\Psi = \int_{\theta_s}^{\theta} \frac{D(\theta)}{K(\theta)} d\theta
\] (6.8)

has a very realistic shape [19], unlike that of any other exactly solvable model. However, although this model allows \(\psi'(\theta_s)\) to be very large, it does not incorporate an air-exit pressure head.

The integrability of (6.4) is apparent from the work of Fokas and Yortsos [35]. Solutions of practical nonlinear boundary value problems, involving this equation, were first presented by Fokas and Yortsos [35], Rogers et al. [69] and Rosen [70]. Realistic solutions to vertical unsaturated flow, under boundary conditions of constant water supply rate, were given independently by Broadbridge and White [19] and by Sander et al. [74]. These works were closely related mathematically [71] but they had significant differences in the way in which the model parameters were related to physical properties of soils [20]. It was a natural step to subsequently derive exact solutions for deep drainage and for redistribution in a finite column under zero flux boundary conditions [15], [72], [90]. The problem of infiltration into a finite column with impervious base was mathematically more challenging but still
tractable by exact analysis [14]. These exact solutions have been incorporated in
more elaborate multi-component models of environmental processes [27]. At a more
fundamental level, they have provided quantitative relationships among measurable
hydrological quantities [18], [60], [68], [88], [94], [95] and they have been used to
validate numerical schemes for solving more general problems [76].

Warrick et al. considered not only drainage but also constant-rate evaporation
[90], as well as unsaturated flow under conditions of stepwise constant water flux at
the boundary [89]. This is the only known useful closed-form solution of (6.4) with
time-varying flux at the soil surface. There is now a widely held impression that
closed-form solutions to (6.4) can be constructed only for constant-flux boundary
conditions, or more generally for piecewise-constant-flux boundary conditions. Al­
though these boundary conditions are meaningful idealised representations of simple
irrigation, redistribution and evaporation processes, it would be preferable to have
the freedom to impose continuously variable flux boundary conditions [29]. A gen­
eral practical approach to this problem has been devised by Barry and Sander [4].
However, in the case of continuously variable flux boundary conditions, their solution
requires the numerical solution of an integral equation.

It is our objective to provide some closed-form solutions to the Broadbridge-
White version [19] of (6.4) with the surface flux varying in a continuous, nontrivial
and environmentally relevant manner. In order to achieve this, we consider an
alternative method of transforming (6.4) to the linear diffusion equation without
arriving at the Burgers’ equation as an intermediate step.
6.1 Alternative procedure for linearisation

In the formulation of Broadbridge and White [19] the equation for vertical unsaturated flow, based on Darcy's law, is

\[
\frac{\partial \Theta}{\partial t_\ast} = \frac{\partial}{\partial z_\ast} \left[ \frac{c(c-1)}{(c-\Theta)^2} \frac{\partial \Theta}{\partial z_\ast} \right] - \frac{c^2(c-1)}{(c-\Theta)^2} \frac{\partial \Theta}{\partial z_\ast} + (c-1) \frac{\partial \Theta}{\partial z_\ast}, \tag{6.9}
\]

where \( \Theta \) is normalised water content,

\[\Theta = \frac{\theta - \theta_n}{\theta_s - \theta_n} \text{ with } \theta \text{ volumetric water content ranging from minimum initial level } \theta_n \text{ to saturation level } \theta_s, \]

\( z_\ast \) is dimensionless depth \( z/l_s \) taken positive downwards,

\( t_\ast \) is dimensionless time \( t/t_s \),

and \( c \) is a nonlinearity parameter ranging from just over 1 for highly nonlinear soils to infinity for soils with constant diffusivity.

The macroscopic sorptive length scale and gravity time scale are respectively

\[l_s = \frac{1}{K_s - K_n} \int_{\theta_n}^{\theta_s} D(\theta) \, d\theta = \frac{h(c)}{c(c-1) (\theta_s - \theta_n)(K_s - K_n)},\]

and

\[t_s = \frac{l_s (\theta_s - \theta_n)}{K_s - K_n} = \frac{h(c)}{c(c-1)(K_s - K_n)^2},\]

where \( S \) is the sorptivity, \( K_s = K(\theta_s) \) and \( K_n = K(\theta_n) \), with \( K(\theta) \) the concentration-dependent hydraulic conductivity. Methods for measuring the model parameters, and the relationship between this and other unsaturated flow models, are given by White and Broadbridge [93]. For a soil water diffusivity

\[D(\theta) = \frac{a}{(b-\theta)^2},\]

“\( a \)” is related to squared sorptivity according to \( a/S^2 = h(c) \), where \( c = (b-\theta_n)/(\theta_s - \theta_n) \) and \( h(c) \) is closely approximated by a rational function [93]. In particular, the significance of the factor \( \frac{h(c)}{c(c-1)} \), whose value is between 0.5
and \( \frac{c}{4} \), is discussed further by White and Sully [95] and by Warrick and Broadbridge [88].

Most hydrological applications of the integrable equation (6.9) have made use of the material depth coordinate originally introduced by Storm [83] in the context of nonlinear heat conduction. This coordinate is

\[
Z = [c(c-1)]^{-1/2} \int_0^{z_*} [c - \Theta(z_1, t_*)] \, dz_1. \tag{6.10}
\]

In terms of \( Z \), \( T(= t_*) \) and the Kirchhoff variable

\[
\mu = \frac{c(c-1)}{c - \Theta},
\]

equation (6.9) is

\[
\frac{\partial \mu}{\partial T} = \frac{\partial^2 \mu}{\partial Z^2} - 2[c(c-1)]^{1/2} \left\{ \frac{\mu}{c-1} - (2P(T) + 1) \right\} \frac{\partial \mu}{\partial Z}, \tag{6.11}
\]

where \( P(T) \) is linearly related to the surface volumetric water flux \( v(0,t) \),

\[
P(T) = \frac{1}{4c(c-1)} \left[ (c-1)\Theta^2 - \frac{c(c-1)}{(c-\Theta)^2} \frac{\partial \Theta}{\partial z_*} \right] \bigg|_{z_*=0}
\]

\[
= \frac{v(0,t) - K_n}{4c(c-1)(K_* - K_n)}, \tag{6.12}
\]

In the special case that \( P(T) \) is constant, (6.11) is merely Burgers' equation, which can be transformed to the linear diffusion equation. This is the reason why this approach to linearisation is most suited to constant-flux, or more generally piecewise constant flux boundary conditions.

There are many possible routes connecting (6.9) to the linear diffusion equation. In our current strategy, we first transform the convective terms in (6.9) to zero. The linear convection term is subsumed in a change of Galilean reference frame

\[
x = z_* + (c - 1)t_*, \tag{6.13}
\]
so that
\[
\frac{\partial \Theta}{\partial t_\ast} = \frac{\partial}{\partial x} \left[ \frac{c(c - 1)}{(c - \Theta)^2} \frac{\partial \Theta}{\partial x} \right] - \frac{c^2(c - 1)}{(c - \Theta)^2} \frac{\partial \Theta}{\partial x}.
\] (6.14)

Less well known is the fact that the remaining nonlinear convection term may be transformed to zero [12] by a simple change of variable

\[
w = \frac{c - \Theta}{[c(c - 1)]^{1/2}} e^{cx},
\]
and
\[
y = \frac{1}{c} \left( 1 - e^{-cx} \right),
\] (6.15)
implying
\[
\frac{\partial w}{\partial t_\ast} = \frac{\partial}{\partial y} \left[ w^{-2} \frac{\partial w}{\partial y} \right].
\] (6.16)

If we introduce a potential \( \phi \) such that
\[
\frac{\partial \phi}{\partial y} = w
\]
then it is sufficient for (6.16) that \( \phi \) satisfies
\[
\frac{\partial \phi}{\partial t_\ast} = \left( \frac{\partial \phi}{\partial y} \right)^{-2} \frac{\partial^2 \phi}{\partial y^2}.
\] (6.17)

Now (6.17) is known to transform directly to the linear diffusion equation [23] by the hodograph transformation
\[
Q = y, \quad \chi = \phi
\]
so that
\[
\frac{\partial Q}{\partial t_\ast} = \frac{\partial^2 Q}{\partial \chi^2}.
\] (6.18)

To the best of our knowledge, this fact was first used by Vein (cited by Ames [1]), who linearised the related equation
\[
\frac{\partial W}{\partial t_\ast} = W^2 \frac{\partial^2 W}{\partial y^2},
\]
which is (6.16) with \( W = w^{-1} \).
Chapter 6. Closed-form solutions for unsaturated flow

The most significant fact to emerge from the above analysis is that given any solution to the linear diffusion equation (6.18), we may construct an exact solution to the realistic model unsaturated flow equation (6.9) by applying the above changes of variable in reverse order. Given a solution $Q(x, t_*)$ to the linear diffusion equation (6.18),

$$ (w, y) = \left( \left( \frac{\partial Q(x, t_*)}{\partial x} \right)^{-1}, Q(x, t_*) \right) $$

(6.19)

is a parametric solution to (6.16). Consequently, a parametric solution to (6.9) is given by

$$ \Theta = c - \left[ \frac{\partial Q(x, t_*)}{\partial x} \right]^{-1} [c(c - 1)]^{1/2} (1 - cQ(x, t_*)) $$

(6.20)

and

$$ z_* = \frac{1}{c} \ln \frac{1}{1 - cQ(x, t_*)} - (c - 1)t_* $$

(6.21)

Given a solution $Q(x, t_*)$ to the linear diffusion equation, the required changes of variable may be carried out explicitly and easily with the aid of a symbolic manipulation package such as MAPLE.

6.2 Some new exact solutions

The general parametric representation (6.20) - (6.21) allows us to construct a wide variety of exact flow solutions with variable water flux at the surface of an unsaturated soil. In this formulation, we have control over the initial conditions. We have some freedom in the choice of initial conditions but for simplicity, as in Broadbridge and White [19], we are choosing uniform concentration as the initial condition. That is,

$$ \Theta(z_*, 0) = 0. $$

By (6.20), this is guaranteed by

$$ Q = \frac{1}{c} + k \exp \left( - [c(c - 1)]^{1/2} x \right) \quad \text{at } t_* = 0 $$
with \( k \) constant. Solutions of the linear diffusion equation, with this initial condition, are of the general form

\[
Q = \frac{1}{c} + k \exp \left\{ \left[ c(c - 1) t* - [c(c - 1)]^{1/2} \chi \right] \right\} + \mathcal{L}^{-1}_{p \to t*} \left\{ A(p)e^{-p^{1/2} \chi} + B(p)e^{p^{1/2} \chi} \right\},
\]

where \( \mathcal{L}^{-1} \) is the inverse Laplace transform, with \( A(p) \) and \( B(p) \) functions of the Laplace variable \( p \).

Our physical variables are restricted to

\[
0 \leq z_* < \infty
\]

and

\[
\Theta_{\text{min}} \leq \Theta \leq 1,
\]

where \( \Theta_{\text{min}} = -\theta_n/\left(\theta_s - \theta_n\right) \). This implies from (6.13), (6.15) and (6.19) that the solution \( \frac{1}{c} - Q(\chi, t*) \) to the linear diffusion equation must be bounded by

\[
0 < \frac{1}{c} - Q \leq \frac{1}{c} e^{-c(c-1)t*},
\]

and that it is a decreasing function with

\[
-[c(c - 1)]^{1/2} \frac{c}{c - 1} \leq \frac{\partial}{\partial \chi} \ln \left( \frac{1}{c} - Q \right) \leq -[c(c - 1)]^{1/2} \frac{c}{c - \Theta_{\text{min}}}.
\]

Given a standard solution to the classical diffusion equation, we may restrict the variable \( \chi \) so that (6.23) is satisfied, and then attempt to adjust the free functions \( A(p) \) and \( B(p) \) until (6.24) is valid.

Given an admissible solution \( Q(\chi, t*) \) to the classical linear diffusion equation, the value \( \chi_0(t*) \) of the \( \chi \)-coordinate at the soil surface is given implicitly by (6.21) as

\[
Q(\chi_0, t*) = \frac{1}{c} \left( 1 - e^{-c(c-1)t*} \right).
\]

Subsequently, the time-dependent water content at the surface is given directly by (6.20). The dimensionless water flux \( V \), defined by \( V = v/(K_s - K_n) \), where \( v \) is
volumetric water flux (with dimensions of velocity), is evaluated from

\[ V - K_n = \frac{(c-1)\Theta^2}{c-\Theta} \frac{c(c-1)}{(c-\Theta)^2} \frac{\partial \Theta}{\partial z_*} \]

\[ = (c-1)[c(c-1)]^{1/2}w(1 - cy) - 2c(c-1) + [c(c-1)]^{1/2}w^{-2} \frac{\partial w}{\partial y} \quad (6.25) \]

\[ = (c-1)[c(c-1)]^{1/2}(1 - cQ) \left[ \frac{\partial Q}{\partial \chi} - 2c(c-1) - [c(c-1)]^{1/2} \frac{\partial Q}{\partial t_*} \right] \right|_{\chi = \chi_*} - 2c(c-1) + K_n, \]

where \( K_n = K_n/(K_s - K_n) \). Hence the value of dimensionless water flux at the soil surface is

\[ R(t_*) = [c(c-1)]^{1/2} \left\{ (c-1)e^{-c(c-1)t_*} \frac{\partial Q}{\partial \chi} \right\}_{\chi = \chi_*} - 2c(c-1) + K_n. \]

In the illustrative examples presented here, we assume \( c = 1.15 \), representing a moderately nonlinear soil, and \( K_n = 0.1725 \). In the first example, using Carslaw and Jaeger [22], we take

\[ Q = c_1 \frac{\chi}{2\sqrt{\pi t^*_2}} e^{-\chi^2/4t_*} + k \exp \left( c(c-1)t_* - [c(c-1)]^{1/2} \chi \right) + \frac{1}{c}. \]

In Figure 6.1, we plot the surface flux in the case \( k = -2 \) and \( c_1 = -1, -4 \) and \(-10\). The water supply pattern is reminiscent of the broad structure of a rain storm, after high frequency fluctuations have been filtered out. In Figure 6.1, the water supply rate rises to a maximum soon after the arrival of the front of the storm, and thereafter it gradually diminishes. The intensity of the model storm may be boosted by increasing the magnitude of the parameter \( c_1 \).

In Figure 6.2, we plot the soil water content profiles in the case \( c_1 = -1 \). During the intense early stages of the model storm, water content rises rapidly near the soil surface. Later, as the rainfall rate weakens, water content decreases near the surface and the water is redistributed, with the peak concentration progressing deeper into the soil. We admit that hysteresis effects, which are significant in the redistribution process, have not been accounted for here.
Figure 6.1: Peaked water supply profile from analytic solution; $c = 1.15$, $k = -2$. $c_1 = -1$ (---), $-4$ (----), $-10$ (-----).

Figure 6.2: Water content profile for water supply of Figure 6.1 ($c_1 = -1$); $T = 0.1, 0.2, 0.3, 0.4, 0.7, 1.0, 1.4$, with curves taken in the order of increasing depth, near $\Theta = 0$. 
Figure 6.3: Water supply rate for opening valve; $c = 1.15, k = -2, c_1 = -1, h = -5$.

Figure 6.4: Water content profile for water supply of Figure 6.3; $T = 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$, with curves taken in the order of increasing depth.
In the second example, we take
\[
Q = c_1 \left\{ \frac{1}{\sqrt{\pi t_*}} e^{-x^2/4t_*} - he^{hx + h^2t_*} \text{erfc} \left( \frac{x}{2\sqrt{t_*}} + h\sqrt{t_*} \right) \right\} + k \exp \left( c(c-1)t_* - [c(c-1)]^{1/2} x \right) + \frac{1}{c}.
\]
(6.26)

In Figure 6.3, we illustrate the surface flux when $c_1 = -1, h = -5$ and $k = -2$. The flux rises continuously and after some early adjustment, it is approximately constant. This water supply pattern is reminiscent of an artificial reticulation system which is turned on by slowly opening a valve, before it operates at a steady supply rate. The water concentration profile is shown in Figure 6.4. Except for a small deficit in cumulative infiltration due to the valve opening time, the profiles are generally in agreement with those due to a constant supply rate. We note that in the case depicted in Figure 6.1, the large-time steady supply rate exceeds the hydraulic conductivity at saturation, implying that incipient ponding must eventually occur. Because of an approximate general relationship between cumulative infiltration, and infiltration rate at incipient ponding [18], the delay in ponding is easily estimated from the deficit in cumulative infiltration.

6.3 Discussion

Contrary to popular belief, it is possible to construct meaningful closed-form solutions to realistic versions of the Richards' unsaturated flow equation with smoothly varying flux boundary conditions, as demonstrated here. These solutions may be used to test general approximate analytic relationships, such as that between cumulative infiltration at ponding, and infiltration rate at ponding. Also, the solutions may be used to test the more general procedure of Barry and Sander [4] which relies on the numerical solution of an integral equation. However, for any reader contemplating using our inverse method for constructing solutions to Richards' equation,
we offer a word of warning. We have found that an arbitrary choice of solution (6.22) of the linear diffusion equation will not necessarily lead to a physically valid solution of Richards' equation via (6.20) - (6.21). The constraints (6.23) - (6.24) are important.
Chapter 7

Outlook

In this thesis we have demonstrated the benefits of the systematic symmetry analysis of a class of physically interesting nonlinear P.D.E.'s. The analysis of the diffusion-convection equation in one, two and three spatial dimensions, with the diffusive and convective terms unknown functions of the dependent variable, has revealed the cases for which the governing equation admits symmetries beyond translations in time and space (and in the three-dimensional case, the rotation generator about the vertical axis). In all of these cases, the diffusivity is either a power-law or exponential function of concentration. The determination of the optimal system and subsequent decrease by one in the number of independent variables has provided a minimal complete set of reductions for each of the special symmetry cases, that is, for each of the special combinations of $D$ and $K$ for which the governing equation admits extra symmetries. In the one-dimensional case, one reduction reduces the governing equation to an O.D.E., and we have provided a listing of all reduced equations using nontrivial elements of the optimal systems.

In the two and three-dimensional cases, additional reductions are needed to obtain an O.D.E. In both two and three dimensions, the symmetry analysis of the reduced equation was considered to ensure that all symmetries are found, including
symmetries of the reduced equation which may not have been inherited from the original equation. At each step in the process, we determined the corresponding optimal system and as a result, we were able to obtain a minimal complete set of reductions to O.D.E.'s for the higher-dimensional diffusion-convection equation. We found that for special combinations of \( D \) and \( K \) in the two-dimensional case, solutions to the O.D.E. which are time dependent may be constructed. We were also able to show that of the remaining cases for which we were able to reduce to an O.D.E., the solutions would either not be time dependent or the O.D.E. has no known explicit solution. Although we demonstrated only one exact time-dependent solution to the three-dimensional equation using purely symmetry methods, we believe that the cases that are special for the two-dimensional equation will be mirrored by the three-dimensional equation.

The systematic symmetry analysis of the diffusion-convection equation in two and three dimensions was also invaluable in understanding the first reduction used by Philip and Knight [65]. As a result of the symmetry classification performed, we were able to overcome the singular cases encountered in [65], where an ad hoc functional form was assumed. Hence in the case \( D(u) = u^{-1} \) and \( K'(u) = u^{-1/2} \) in two dimensions and the case \( D(u) = u^{-2/3} \) and \( K'(u) = u^{-1/3} \) in three dimensions, we were able to completely reduce the governing equation to an O.D.E. following the Philip and Knight technique. In both two and three dimensions, the solutions that we found involved an arbitrary parameter \( c \). This contributes an infinite family of solutions whose character changes as the parameter \( c \) varies and extends the small set of closed-form solutions previously known.

The special symmetry properties of the one-dimensional Burgers' equation are well documented. However, it was not recognised that Burgers' equation also has symmetry properties beyond any other form of the diffusion-convection equation in higher dimensions. We have shown that the two-dimensional Burgers' equation may
be reduced to O.D.E.'s which admit explicit solutions. In addition, reducing the two-dimensional Burgers' equation using the symmetry which is special compared to the symmetries of other forms of the diffusion-convection equation in two dimensions leads to a linear P.D.E. That is, although the two-dimensional Burgers' equation is not linearisable, we may transform it to a linear P.D.E. with one less independent variable. The three-dimensional Burgers' equation may also be reduced once to obtain P.D.E.'s which are linear. In addition, we have found that in the cases where we are able to reduce to an O.D.E., equations are of two distinct types. The first type has a general solution of the form

\[ u = \frac{a_3 + a_4x + a_5y + a_2z}{a_1 + a_2t}, \]

where \( a_1, a_2, a_3, a_4, a_5 \in \mathbb{R} \), while the second type have a general solution obtained by solving a Riccati equation.

Finally, we have demonstrated an inverse method for constructing solutions to a linearisable form of the diffusion-convection equation in one dimension. The strategy used did not rely on the transformation of the governing equation to the Burgers' equation as an intermediate step and hence solutions with variable water flux boundary conditions for an unsaturated soil are found and two examples were illustrated.

We note that there are some natural extensions following from the work completed in this thesis. Firstly, an exploration of solutions of the linear reduced equations obtained for Burgers' equation in two and three dimensions could be performed to obtain a number of new solutions. The functional forms of the original dependent variable are generally of the type

\[ u = F(\rho, \gamma, t) + \frac{z}{t + a_1}, \]

where \( a_1 \in \mathbb{R} \) and \( \rho \) and \( \gamma \) are invariants which do not depend on \( z \). Hence the solutions would not be defined as \( z \to \infty \). However, meaningful solutions may be possible on a restricted domain for \( z \).
Secondly, the inverse method introduced in Chapter 6 has great potential for generating solutions with variable water flux at the surface of an unsaturated soil. We have demonstrated the technique twice using two solutions of the linear diffusion equation and applying the inverse method to obtain solutions satisfying two distinct sets of flux boundary conditions. By constructing other solutions to the linear diffusion equation, it may be possible to achieve new closed-form solutions to the nonlinear diffusion-convection equation with different smoothly varying flux boundary conditions.

Another possible extension of the material in this thesis is the symmetry classification of the general nonlinear diffusion-convection equation with a source. A full analysis has been performed for two spatial dimensions when the source term is a function of the dependent variable [3]. In the soil water context, the additional source term represents a sink or source term. Broadbridge and Rogers [16] have considered the Lie-Bäcklund symmetries of a one-dimensional diffusion-convection equation with a sink. In this case the sink term represents the extraction by plant roots and depends on the depth \( z \) only. This is a realistic assumption, as Talsma and Gardner [85] have observed that some eucalypt species extract water at a constant rate, provided the water content remains above a critical value, until wilting point is reached. A classification of the general equation involving arbitrary functions for the diffusion, convection and source terms may reveal cases that admit similarity solutions. The use of symmetry determination packages such as Dimsym makes the symmetry analysis of general classes of equations involving unknown functions feasible.

The question of the existence of nonclassical symmetries and nonlocal symmetries of a P.D.E. has been raised separately by many researchers, however, the issue of whether a P.D.E. may admit nonlocal, nonclassical symmetries seems not to have been addressed. Using the nonclassical method results in an overdetermined
nonlinear system of determining equations which in most cases can not be solved in
general. We consider the nonlinear diffusion equation
\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left[ D(u) \frac{\partial u}{\partial x} \right]
\]  
(7.1)
as an example. To determine the nonlocal (or potential) symmetries of this equation,
we consider the point symmetry analysis of the system of equations
\[
\frac{\partial v}{\partial x} = u,
\]
\[
\frac{\partial v}{\partial t} = D(u) \frac{\partial u}{\partial x},
\]
with the one-parameter group of transformations
\[
x^* = x + \epsilon \mathcal{X}(x, t, u, v) + O(\epsilon^2),
\]
\[
t^* = t + \epsilon \mathcal{T}(x, t, u, v) + O(\epsilon^2),
\]
\[
u^* = u + \epsilon \mathcal{U}(x, t, u, v) + O(\epsilon^2),
\]
\[
v^* = v + \epsilon \mathcal{V}(x, t, u, v) + O(\epsilon^2).
\]
At least one of the infinitesimals \( \mathcal{X}, \mathcal{T} \) or \( \mathcal{U} \) must depend on the potential variable \( v \), otherwise we have merely recovered point symmetries of the original equation.
The nonlocal symmetry analysis of this system of equations has been performed by
Bluman et al. [9], where they show that there are two forms of \( \text{D}(u) \) which admit
potential symmetries. One of the forms is
\[
\text{D}(u) = \lambda (u + \kappa)^{-2},
\]
where \( \lambda, \kappa \in \mathbb{R} \). If we consider the case \( \lambda = 1 \) and \( \kappa = 0 \), the potential symmetries are
\[
\Gamma_1 = -xv \frac{\partial}{\partial x} + u(v + xu) \frac{\partial}{\partial u} + 2t \frac{\partial}{\partial v},
\]
\[
\Gamma_2 = -x(v^2 + 2t) \frac{\partial}{\partial x} + 4t^2 \frac{\partial}{\partial t} + u(6t + v^2 + 2xuv) \frac{\partial}{\partial u} + 4tv \frac{\partial}{\partial v},
\]
\[
\Gamma_3 = \phi(v, t) \frac{\partial}{\partial x} - u^2 \frac{\partial \phi(v, t)}{\partial v} \frac{\partial}{\partial u},
\]
where $\phi(v, t)$ satisfies the linear heat equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial v^2}.$$ 

If we now consider the nonlocal, nonclassical symmetries of the system of equations, we require only invariant solutions, that is

$$v = G(x, t) \rightarrow v_* = G(x_*, t_*),$$

$$u = F(x, t) \rightarrow u_* = F(x_*, t_*),$$

with invariant surface conditions

$$x \frac{\partial v}{\partial x} + T \frac{\partial v}{\partial t} = \mathcal{V},$$

$$x \frac{\partial u}{\partial x} + T \frac{\partial u}{\partial t} = \mathcal{U},$$

where we assume that $T = 1$. We find two nonlinear determining equations which cannot be solved in general. If we make the assumptions

$$\frac{\partial X}{\partial u} = \frac{\partial Y}{\partial u} = 0,$$

we may solve the determining equations to obtain the following symmetries:

1. $X = 0$, $T = 1$, $U = \frac{u}{v^2}$ and $Y = -\frac{1}{v}$, which leads to the solutions

$$v^2 = 2 \left( \frac{x - B}{A} - t \right) \rightarrow v = \sqrt{2 \left( \frac{x - B}{A} - t \right)}$$

$$u = \frac{1}{A \sqrt{2 \left( \frac{x - B}{A} - t \right)}},$$

where $A, B \in \mathbb{R}$ are arbitrary constants.

2. $X = -\frac{3x}{v^2}$, $T = 1$, $U = \frac{6u}{v^2} - \frac{6xu^2}{v^3}$ and $Y = -\frac{3}{v}$ which leads to the implicit relationship between the dependent and independent variables

$$x = v \left[ A \left( \frac{v^2}{2} + 3t \right) + B \right]$$
and \( u = v_x \), where \( A, B \in \mathbb{R} \) are arbitrary constants.

We knew in advance that these symmetries would exist, as the linear heat equation may be transformed to the nonlinear diffusion equation (7.1) via a hodograph transformation. Hence nonclassical symmetries of the heat equation are nonclassical, nonlocal symmetries of (7.1). The examples presented here correspond to nonclassical symmetries of the linear heat equation found in [43]. But the question of whether a P.D.E. will admit nonlocal, nonclassical symmetries at this stage seems not to have been addressed and is worth posing. The main problem is that since assumptions need to be made in order to solve the determining equations, it is not possible to know when a complete set of nonlocal nonclassical symmetries has been found.

In this thesis we have contributed to the set of known exact solutions of the nonlinear diffusion-convection equation by relying on the systematic symmetry analysis of the general class of equation in one, two and three spatial dimensions. In addition, we have exploited an \textit{ad hoc} method developed by Philip and Knight [65] to obtain an infinite family of solutions for special cases of the diffusion-convection equation in two and three dimensions. However, we have also noted the current high level of interest in generalised symmetry methods, many of which at this stage appear to be \textit{ad hoc}. We believe that the Richards equation should be reconsidered in the future, as new more general systematic methods become established.
Appendix A

We wish to find the set of determining equations for the diffusion equation (2.10). We have defined the function $F$ as

$$F = u_t - u_{xx} - u_{yy}.$$ 

In this example, we have one dependent variable $u$ and three independent variables $x$, $y$ and $t$. Our governing equation is a second order equation, so $k = 2$. Hence we will need to construct the twice extended generator $\Gamma^{(2)}$, that is,

$$\Gamma^{(2)} = \frac{\partial}{\partial x} X + \frac{\partial}{\partial y} Y + \frac{\partial}{\partial t} T + \frac{\partial}{\partial u} U + \frac{\partial}{\partial u_x} U_x + \frac{\partial}{\partial u_y} U_y + \frac{\partial}{\partial u_t} U_t$$

and the requirement is that

$$\Gamma^{(2)} F |_{F=0} = 0$$

be satisfied. Hence we obtain

$$U_t^{(1)} - U_{xx}^{(2)} - U_{yy}^{(2)} |_{F=0} = 0,$$

where

$$U_t^{(1)} = \frac{\partial U}{\partial t} + \left[ \frac{\partial U}{\partial u} - \frac{\partial T}{\partial t} \right] u_t - \frac{\partial X}{\partial t} u_x - \frac{\partial Y}{\partial t} u_y$$

$$- \frac{\partial T}{\partial u} (u_t)^2 - \frac{\partial X}{\partial u} u_x u_t - \frac{\partial Y}{\partial u} u_y u_t,$$
\[ U_{xx}^{(2)} = \frac{\partial^2 U}{\partial x^2} + \left[ 2 \frac{\partial^2 U}{\partial x \partial u} - \frac{\partial^2 X}{\partial x^2} \right] u_x - \frac{\partial^2 Y}{\partial x^2} u_y - \frac{\partial^2 T}{\partial x^2} u_t + \left[ \frac{\partial U}{\partial u} - 2 \frac{\partial X}{\partial u} \right] u_{xx} - 2 \frac{\partial Y}{\partial u} u_{xy} - 2 \frac{\partial T}{\partial u} u_{xt} \]

\[ + \left[ \frac{\partial^2 U}{\partial u^2} - 2 \frac{\partial X}{\partial u} \frac{\partial X}{\partial u} \right] (u_x)^2 - 2 \frac{\partial^2 Y}{\partial x \partial u} u_x u_y - 2 \frac{\partial^2 T}{\partial x \partial u} u_x u_t \]

\[ - \frac{\partial^2 X}{\partial u^2} (u_x)^3 - \frac{\partial^2 Y}{\partial u^2} (u_x)^2 u_y - \frac{\partial^2 T}{\partial u^2} (u_x)^2 u_t \]

\[ - 3 \frac{\partial X}{\partial u} u_x u_{xx} - \frac{\partial Y}{\partial u} u_y u_{xx} - \frac{\partial T}{\partial u} u_t u_{xx} \]

\[ - 2 \frac{\partial Y}{\partial u} u_x u_{xy} - 2 \frac{\partial T}{\partial u} u_x u_{xt} \]

and

\[ U_{yy}^{(2)} = \frac{\partial^2 U}{\partial y^2} + \left[ 2 \frac{\partial^2 U}{\partial y \partial u} - \frac{\partial^2 Y}{\partial y^2} \right] u_y - \frac{\partial^2 X}{\partial y^2} u_x - \frac{\partial^2 T}{\partial y^2} u_t + \left[ \frac{\partial U}{\partial u} - 2 \frac{\partial Y}{\partial u} \right] u_{yy} - 2 \frac{\partial X}{\partial u} u_{xy} - 2 \frac{\partial T}{\partial u} u_{yt} \]

\[ + \left[ \frac{\partial^2 U}{\partial u^2} - 2 \frac{\partial Y}{\partial u} \frac{\partial Y}{\partial u} \right] (u_y)^2 - 2 \frac{\partial^2 X}{\partial y \partial u} u_y u_x - 2 \frac{\partial^2 T}{\partial y \partial u} u_y u_t \]

\[ - \frac{\partial^2 Y}{\partial u^2} (u_y)^3 - \frac{\partial^2 X}{\partial u^2} u_x (u_y)^2 - \frac{\partial^2 T}{\partial u^2} (u_y)^2 \]

\[ - 3 \frac{\partial Y}{\partial u} u_y u_{yy} - \frac{\partial X}{\partial u} u_x u_{yy} - \frac{\partial T}{\partial u} u_t u_{yy} \]

\[ - 2 \frac{\partial X}{\partial u} u_y u_{xy} - 2 \frac{\partial T}{\partial u} u_y u_{yt} \].
We use the condition $F = 0$ to eliminate all terms involving $u_{yy}$. Hence our expression becomes

$$
\Gamma^{(2)} F \bigg|_{F=0} = \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} + \left[ -2 \frac{\partial^2 U}{\partial x \partial u} - \frac{\partial U}{\partial t} + \frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} \right] u_x
$$

$$
+ \left[ -2 \frac{\partial^2 U}{\partial y \partial u} - \frac{\partial U}{\partial t} + \frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} \right] u_y + \left[ - \frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + 2 \frac{\partial Y}{\partial y} \right] u_t
$$

$$
+ 2 \left[ \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} \right] u_{xx} + 2 \left[ \frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x} \right] u_{xy} + 2 \frac{\partial T}{\partial x} u_{xt} + 2 \frac{\partial T}{\partial y} u_{yt}
$$

$$
+ 2 \left[ \frac{\partial^2 X}{\partial x \partial u} - \frac{\partial^2 U}{\partial u^2} \right] (u_x)^2 + 2 \left[ \frac{\partial^2 X}{\partial y \partial u} + \frac{\partial^2 U}{\partial u^2} \right] u_x u_y + 2 \frac{\partial^2 T}{\partial x \partial u} u_x u_t
$$

$$
+ 2 \left[ \frac{\partial^2 Y}{\partial y \partial u} - \frac{\partial^2 U}{\partial u^2} \right] (u_y)^2 + 2 \left[ \frac{\partial^2 T}{\partial y \partial u} + \frac{\partial Y}{\partial u} \right] u_y u_t
$$

$$
+ \frac{\partial^2 X}{\partial u^2} (u_x)^3 + \frac{\partial^2 Y}{\partial u^2} (u_x)^2 u_y + \frac{\partial^2 T}{\partial u^2} (u_x)^2 u_t
$$

$$
+ \frac{\partial^2 Y}{\partial u^2} (u_y)^3 + \frac{\partial^2 X}{\partial u^2} u_x (u_y)^2 + \frac{\partial^2 T}{\partial u^2} (u_y)^2 u_t
$$

$$
+ 2 \frac{\partial X}{\partial u} u_x u_{xx} + 2 \frac{\partial Y}{\partial u} u_x u_{xy} + 2 \frac{\partial T}{\partial u} u_x u_{xt}
$$

$$
- 2 \frac{\partial Y}{\partial u} u_y u_{xx} + 2 \frac{\partial X}{\partial u} u_y u_{xy} + 2 \frac{\partial T}{\partial u} u_y u_{yt}.
$$

(A.1)

Equating the coefficients of $u_{xt}$, $u_{yt}$, $u_x u_{xx}$, $u_x u_{xy}$ and $u_x u_{xt}$ to zero we immediately obtain the results

$$
\frac{\partial T}{\partial u} = \frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial X}{\partial u} = \frac{\partial Y}{\partial u} = 0.
$$
That is

\[ T = T(t) \]
\[ \mathcal{X} = \mathcal{X}(x, y, t) \]
\[ \mathcal{Y} = \mathcal{Y}(x, y, t). \]

These first five results simplify the expression (A.1) and lead to the set of determining equations listed in (2.12). The fact that the infinitesimals \( T, \mathcal{X} \) and \( \mathcal{Y} \) do not depend on the dependent variable \( u \) is not surprising. Since (2.10) is a linear P.D.E., we could in fact have written the one-parameter group of transformations (2.11) as

\[
\begin{align*}
x^* &= x + \epsilon \mathcal{X}(x, y, t) + O(\epsilon^2) \\
y^* &= y + \epsilon \mathcal{Y}(x, y, t) + O(\epsilon^2) \\
t^* &= t + \epsilon T(x, y, t) + O(\epsilon^2) \\
u^* &= u + \epsilon \mathcal{U}(x, y, t)u + O(\epsilon^2)
\end{align*}
\]

without any loss of generality.
Appendix B

load symS

Dimsym 2.0, 1-Oct-93

Symmetry determination and linear D.E. package

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Any publication resulting from these calculations must reference this program.

*** sign already defined as operator

freeunknown(k,d);

depend d,u(1);

depend k,u(1);

% Symmetry analysis of the one-dimensional diffusion-convection equation
% D(u) and K(u) are unknown functions of the dependent variable u
% We use u=u(1), t=x(1) and z=x(2)
loaddeq( u(1,2,2) = (u(1,1) - totder(2,d)*u(1,2) + totder(2,k))/d );

% totder(i,f) gives the i(th) total derivative of f
% i.e. totder(2,d) gives the total derivative of D with respect to x(2)
mkdets(point);

*** free or special functions found when dividing by d

solvedets(std);
Appendix B

Solving equations using std algorithm.

op!intfac part a: integrals left in

\[ \text{int} \left( \frac{\text{df}(d,u(1),2)}{\text{df}(d,u(1))} \right) - \text{int} \left( \frac{\text{df}(d,u(D))}{d} \right) \]

*** free or special functions found when dividing by \( \text{df}(d,u(1)) \)

*** free or special functions found when dividing by

\[ \text{df}(d,u(1)) \]

\[ \text{df}(d,u(1),3) \cdot [\text{df}(d,u(1))]^2 \cdot d - 2 \cdot \text{df}(d,u(1),2) \cdot d^2 + \text{df}(d,u(1),2) \cdot \text{df}(d,u(1)) \]

Must have all of
\[ \text{df}(k,u(1)) \]
\[ 1 \]

linearly independent in \((u 1)\)

There are 0 equations remaining.

mkgens();

There are 2 symmetries found.

The generators of the finite algebra are:

\[ \text{Gen}(1) = \emptyset \]
\[ x(1) \]

\[ \text{Gen}(2) = \emptyset \]
\[ x(2) \]

end;
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Publications of the author


Papers 1 - 4 are included in this thesis.

Paper 1 is Chapter 6, Paper 2 is Chapter 5, Paper 3 is Chapter 3 and Paper 4 is Chapter 4.

Paper 5 is earlier work which is not included in this thesis.