A study of wind stress and boundary effects on sea level and coastal currents

A. L. Worthy

University of Wollongong

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A Study of Wind Stress and Boundary Effects on Sea Level and Coastal Currents

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A.L. Worthy
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A.L. Worthy

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To my mother and family
ABSTRACT

This thesis includes both a discussion of free and forced topographic trapped edge waves over various continental shelf profiles, and an examination of the boundary effects on sea level and coastal current movement through a strait or channel.

Using the linearized long wave theory and the methods of complex analysis, edge wave solutions are analytically determined over a linear and exponential, finite and semi-infinite, continental shelf profile. The effects of both pressure and wind stress forcing of edge waves is also discussed. An analysis of wind generated Class I and Class II edge waves is included, along with a comparison of the analytical results with the physical data obtained from current meter recordings taken along the south eastern coast of Australia. It is shown that the analytical results compare favourably with the physical results.

A discussion of the applications of various open boundary methods, including the relatively new Characteristic method, to current flow through a strait or channel is included so that the methods can be compared when applied to the Strait of Belle Isle, Canada. A numerical model of the Strait of Belle Isle is used to examine the effect various boundary conditions have on current flow through a strait or channel. The results obtained by using the Characteristic method, the gradient method, and specifying sea level values along the open boundary are compared and it is shown that the Characteristic method gives what could be considered very 'realistic' results.
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1.1 INTRODUCTION

Wave motion can be defined as a shape or a pattern that travels through mediums such as air or water. Waves in the ocean and sound in the air are typical examples. In this thesis, we will be concerned primarily with the physical phenomenon which affects ocean waves.

There are two types of forces that generate and/or distort ocean waves, namely, primary and secondary forces. Primary forces or generating forces are gravitation, wind stress, atmospheric pressure and seismic activity. The secondary forces are forces which distort the wave due to the motion of the wave. Such forces are the Coriolis force due to rotation and friction which acts in the opposite direction of the wave motion.

Ocean waves can be divided into various categories depending on the waves wavelength ($\lambda$), period ($\tau$), the effect of boundaries, such as topography and the primary and/or secondary forces that generate and/or distort the wave.

The main categories of ocean waves as described by Pond and Pickard to categorize waves with respect to the primary and secondary forces are the following:

(1) Tidal waves which are generated by the minor gravitational imbalances due to the attraction of the sun and the moon. The periods of Tidal waves are multiples of 12.5hrs. The wavelength is in the order of thousands of kilometres.

(2) Planetary or Rossby waves which are generated by changes in potential vorticity due to the changes in depth or latitude and/or time variations in the geostrophic wind and possibly
due to the baroclinic and/or instabilities of the ocean. They have very long periods of the order of a 100 days and wavelengths running into thousands of kilometres.

(3) Gravity waves which are generated by forces such as the switching motion of the wind stress and atmospheric pressure, for example, Kelvin waves. Also, gravity waves are affected by the earth’s rotation and can be both internal and surface waves. Their periods range from minutes to hours and their wavelength is similar to that of Rossby waves.

(4) Tsunamis are generated by seismic disturbances such as an earthquake on the seabed or a large landslide into the sea. Their periods are from around 15mins to an hour or so, and their wavelength is similar to that of gravity waves. The destructive nature of the tsunami is felt only at the coastline.

(5) Internal waves which are generated by secondary forces such as current shear, density structures and surface disturbances. Periods and wavelengths, of internal waves are similar to those of gravity waves.

(6) Swell, wind waves and ripples which are due basically to the interaction of the air and sea, such as wind stress. The periods range from 30 seconds for swell waves down to about .1 of a second for ripples.

Waves can also be categorized in terms of their relative height or wave steepness. One such measure is the ratio of the wave amplitude, $\zeta$ to wavelength, $\lambda$. Deep water waves, such as, swell and wind waves have the property that $\lambda$ is less than twice the depth of the fluid, $h$. The speed, $C$, of deep water waves depends on $\lambda$ and is given by $C = (g\lambda)^{1/2}/2\pi$. These waves are called dispersive waves. Shallow-water waves have the property that $\lambda > 20h$. In other words, shallow-water waves have wavelengths much greater than the depth of the fluid. The speed of the shallow-water wave is dependent on the depth of the fluid and is given by $C = (gh)^{1/2}$. 
The waves associated with homogenous density structure are classified as barotropic. If the fluid has a non-homogeneous density structure then the fluid is called baroclinic and generates internal waves.

The categorization of ocean waves due to boundary effects is important. Waves affected by bottom topography are called topographic waves. Edge waves are one example of topographically coastally trapped waves which can be affected by the rotation of the earth. These waves, called quasigeostrophic (Class II edge waves), have low frequencies and propagate in only one direction along the coast. An example of Class II edge wave is the shelf wave which is influenced by both topography and wind stress. Inertiogravitational (Class I) edge waves are those waves which are not affected by the earth’s rotation and which have a high frequency and can propagate in either direction along a coastline.

In this thesis we will be looking at waves which are approximated by shallow-water theory. The ocean being considered to be barotropic and the waves being influenced by topography, wind stress and/or bottom friction.

An introduction to the mathematical formulae which will be used throughout the thesis is given in Chapter 2. The boundary conditions in the vertical direction are predetermined due to the ocean being barotropic. However, the kinematic boundary conditions are analytically determined for a free or rigid boundary. Also, included in Chapter 2 is the differential equation for freely propagating edge waves with variable depth which will be referred to in later chapters.

A general discussion of edge waves on a sloping shelf is found in Chapter 3. In particular, Section 3.2 examines freely propagating Class I and Class II barotropically trapped edge waves in detail. The evidence for the existence of such waves is discussed in Section 3.3. In Section 3.4, the forcing of Class I edge waves due to pressure and wind stress is introduced along with reasons why the wind stress term is dominant over the pressure term in the equations of motion, when considering the effects of southerly 'busters'. As a result of this discussion, the effects of
wind stress on Class I edge waves on a sloping shelf are analytically determined for periods of an hour or less in Section 3.5. The effects of various wind stress models are analysed in Section 3.6. It is shown in Section 3.7 that the analytic results obtained compare favourably with direct measurements when the east coast off Sydney, Australia, is approximated as a sloping shelf.

A discussion on the effects of topography on Class II edge waves, in particular the convex exponential shelf, is presented in Chapter 4. The convex exponential shelf will be either truncated or semi-infinite in extent. It is shown that the lower frequency edge waves are affected by bottom topography. The continental shelf profile off the east coast of Australia is used as an example to compare the analytic results to direct measurement at Port Kembla Harbour near Sydney, N.S.W., Australia. In Section 4.2, freely propagating Class II edge waves, in particular shelf waves, over a convex exponential shelf using appropriate boundary conditions are discussed. Also, the dispersion relation is determined for both the truncated and semi-infinite exponential shelf profile. Using Fourier transforms, Sections 4.3 and 4.4 analytically determine the effects of wind forcing on Class II waves over an exponential shelf profile. The results that are obtained theoretically in Section 4.5 are shown to be of the same order of magnitude as direct measurements that were made at Port Kembla Harbour.

In Chapter 5, an examination of Class I edge waves is given with a convex exponential shelf used as the shelf model. A general discussion on Class I edge waves as well as the effects of wind stress, pressure and bottom topography is given in Section 5.1. Freely propagating Class I waves over an exponential shelf are discussed in Section 5.2. An analytical examination of wind forcing Class I edge waves over the exponential shelf, using Fourier methods, is done in Section 5.3 and 5.4. The significance of the results obtained in Section 5.5 are discussed in Section 5.6. It is found that the periods obtained theoretically approximate periods obtain from direct measurement at Port Kembla Harbour. A comparison of the results obtained by using the sloping shelf profile in Chapter 3, and the exponential shelf profile in Chapter 5 is included. It is found that the results agree quite closely except for periods close to 60 mins.
The effects of fluid movement on vorticity changes is discussed in detail in Chapter 6. In particular, the effects of friction on vorticity changes in a channel or strait are discussed in Sections 6.1 and 6.2. To study the effects of friction on vorticity, data obtained from the Strait of Belle Isle, which is between Newfoundland and Canada, is used as an example. Hence, in Section 6.2 a discussion is given on the Strait of Belle Isle. Numerical techniques are used in Section 6.4 to model the Strait of Belle Isle. The model shows that appropriate boundary conditions are required at the open boundaries. To determine these boundary conditions both the method of characteristics and the gradient method are used and compared. A discussion of results is found in Section 6.5. It is found that due to the topography and bottom friction, vorticity decreases with a possible spin down time of around 5 - 8 hours. As well, the flow through the Strait follows lines of depth.

The appendices, which are referred to in the appropriate Chapters, include extra mathematical detail which would otherwise detract from the analytical examination under discussion.
CHAPTER TWO

2.1 BASIC MATHEMATICAL EQUATIONS

Consider a right handed co-ordinate system (z vertical) which is uniformly rotating with an angular velocity \( \Omega \). If \( \mathbf{r} = (x, y, z) \) is the position vector of a particle, then \( \mathbf{u} = (u, v, w) \) is the velocity of the particle, \( p \) the pressure and \( \rho \) the density are functions of \( \mathbf{r} \) and the time \( t \).

In this rotating or relative frame of reference, the velocity \( \mathbf{u} \) is related to the absolute or inertial velocity \( \mathbf{u}^a \) by the equation

\[
\mathbf{u}^a = \mathbf{u} + \mathbf{\Omega} \times \mathbf{r}.
\]

Let \( z \) denote the distance measured vertically upwards from the undisturbed sea surface and \( R \) the mean radius of the Earth measured from the centre of the Earth to the undisturbed surface. The distance of the particle from the Earth’s centre is given by:

\[
|\mathbf{r}| = R + z.
\]

Clearly, the position of the undisturbed sea is given by \( z = 0 \).

The Eulerian equations for the motion of an incompressible, non-viscous fluid in a relative frame of reference, may be written as:

\[
\frac{D\mathbf{u}}{Dt} + 2\mathbf{\Omega} \times \mathbf{u} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \mathbf{G} - \frac{1}{\rho} \nabla p + \frac{\mathbf{F}}{\rho} \tag{2.1}
\]

where

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla.
\]
\( \mathbf{G} = (0, 0, -g) \), \( g \) is the acceleration due to gravity, and \( \mathbf{F} = (F_1, F_2, 0) \) is the sum of all other forces per unit volume acting on the fluid.

The magnitude of the gravitational term \( \mathbf{G} \) is greater than the centrifugal term \( \Omega \times (\Omega \times \mathbf{r}) \) throughout the ocean. Hence, the centrifugal force will be neglected. The effects of rotation will be manifested through the Coriolis term \( 2\Omega \times \mathbf{u} \).

The components of \( \Omega \) in the relative frame of reference are given by :-

\[
\Omega = (0, \Omega \sin \theta, \Omega \cos \theta)
\]

where \( \Omega = |\Omega| \) and \( \theta \) is the co-latitude angle.

Therefore (2.1) becomes

\[
\frac{D\mathbf{u}}{Dt} + 2\Omega \times \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{G} + \frac{\mathbf{F}}{\rho} \tag{2.2}
\]

where

\[
2\Omega \times \mathbf{u} = (2\Omega(\omega \sin \theta - \nu \cos \theta), 2\Omega \nu \cos \theta, -2\Omega \nu \sin \theta) \tag{2.3}
\]

### 2.2 LONG WAVE THEORY

Since long wave theory is applied throughout this thesis, it is assumed that vertical acceleration is negligible, that is

\[
w < < u, v
\]

and \(-2\Omega \nu \sin \theta\) is small compared to gravity in (2.2). The Coriolis term (2.3) then becomes:-

\[
2\Omega \times \mathbf{u} = (-fv, fu, 0) \tag{2.4}
\]
where \( f = 2\Omega \cos \theta \) and is known as the Coriolis parameter.

**Definition 2.1.** If \( \Omega^{-1}, L \) and \( U \) are typical time, length and velocity scales then the **Rossby Number**, \( R_o \), is defined by \( R_o = \frac{U}{\Omega L} \).

When considering small scale motions, the curvature of the earth and latitude dependence can be ignored as in the derivation of (2.4). However, when the motions are not small, the Coriolis parameter should be replaced by a linear approximation, that is,

\[
f = f_0 + \beta y
\]

where \( f_0 = 2\Omega \cos \theta \) and \( \beta = (f_y)_\theta = \frac{2\Omega \sin \theta}{R} \).

The \( \beta \)-effect can be neglected at mid-latitudes that is, \( \theta = \pm \frac{\pi}{4} \). At mid-latitudes, \( f_0 = 10^{-4} \) rad/sec and \( \beta = 1.6 \times 10^{-11} m^{-1} \) rad/sec.

For shallow water theory to apply, it is also assumed that the wave height \( \zeta \) is small compared with the depth \( h \) of the ocean. That is,

\[
\zeta < < h
\]

and further that the Rossby number \( R_o \) is small, so that the non-linear term in (2.2) is negligible. Therefore,

\[
\frac{D u}{D t} \approx \frac{\partial u}{\partial t} = u_t
\]

The equation of motion (2.2) can now be written as:

\[
u_t - f v = -\frac{1}{\rho} p_x + \frac{F_1}{\rho}
\]

\[
v_t + f u = -\frac{1}{\rho} p_y + \frac{F_2}{\rho}
\]

and

\[
0 = -g - \frac{1}{\rho} p_z
\]
Continuity of stress and displacement is required at the disturbed surface, i.e. 
\( z = \zeta(x, y, t) \). Hence,

\[ p_a = p_{\text{atmosphere}} \quad \text{at} \quad z = \zeta \quad (2.10) \]

and

\[ \frac{Dz}{Dt} = \frac{D\zeta}{Dt} \quad \text{at} \quad z = \zeta \]

or

\[ w = \zeta_t + u\zeta_x + v\zeta_y \quad \text{at} \quad z = \zeta \quad (2.11) \]

Integrating (2.9) and using (2.10) yields the pressure at any point within the fluid as

\[ p = p_a + \rho g (\zeta - z) \quad (2.12) \]

Substituting (2.12) into (2.8) eliminates pressure and gives the two dimensional shallow water equations of motion on the rotating earth:

\[ u_t - f v = -g \zeta_x - \frac{1}{\rho} (p_a)_x + \frac{F_1}{\rho} \]

\[ v_t + f u = -g \zeta_y - \frac{1}{\rho} (p_a)_y + \frac{F_2}{\rho} \quad (2.13) \]

When \( p_a = \text{constant} \), (2.13) becomes:

\[ u_t - f v = -g \zeta_x + \frac{F_1}{\rho} \]

\[ v_t + f u = -g \zeta_y + \frac{F_2}{\rho} \quad (2.14) \]
2.3 THE CONTINUITY EQUATION

The conservation of mass is expressed by:

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \quad \text{(2.15)}
\]

Since the fluid is incompressible, the density cannot change along a particle path. Therefore, (2.15) becomes the continuity equation:

\[
\nabla \cdot \mathbf{u} = 0
\]

or

\[
u_x + v_y + w_z = 0 \quad \text{(2.16)}
\]

Since the horizontal components of barotropic long waves are independent of depth, (2.16) can be vertically integrated over the entire water column from \( z = -h(x,y) \) to \( z = \zeta \). Thus,

\[
\int_{-h}^{\zeta} (u_x + v_y + w_z) \, dz = 0 \quad \text{(2.17)}
\]

Now

\[
\int_{-h}^{\zeta} u_x \, dz = \{ u(\zeta + h) \}_x - u\{ \zeta_x + h_x \} \quad \text{(2.18)}
\]

where \( u \) is independent of \( z \). Similarly,

\[
\int_{-h}^{\zeta} v_y \, dz = \{ v(\zeta + h) \}_y - v\{ \zeta_y + h_y \} \quad \text{(2.19)}
\]

where \( v \) is independent of \( z \). The boundary condition at \( z = -h(x,y) \) is

\[
w = -uh_x - vh_y \quad \text{at} \quad z = -h \quad \text{(2.20)}
\]
(2.20) is the assumption that there is no flow through the bottom of the ocean. Using
(2.11), (2.18), (2.19) and (2.20) in (2.17) the non-linear continuity equation becomes
\[{ u(\zeta + h )}_x + { v(\zeta + h )}_y = -\zeta_t \ . \quad (2.21)\]

Using the assumption (2.6), (2.21) can be further simplified, and becomes
\{(hu)_x + (hv)_y = -\zeta_t \ . \quad (2.22)\]

Therefore, the linearised long wave equations are given by either (2.13) and (2.22) or
(2.14) and (2.22) if \( p_a \) = constant.

2.4 COMBINED DIFFERENTIAL EQUATION FOR WAVE HEIGHT \( \zeta \)

The linearised long wave equations (2.14) and (2.22) constitute a system of equations for
\( u, v \) and \( \zeta \) which satisfy all vertical boundary conditions. It follows using the equations of
motion (2.14) that
\[u_{tt} + f^2 u = -g(f\zeta_y + \zeta_{xt}) + \frac{(F_1)_t + fF_2}{\rho} \ . \quad (2.23)\]
\[v_{tt} + f^2 v = g(f\zeta_x - \zeta_{yt}) + \frac{(F_2)_t - fF_1}{\rho} \ . \quad (2.23)\]

Substituting (2.23) into the continuity equation (2.22) yields the differential equation for
\( \zeta \), namely,
\[\nabla^2 - \frac{1}{gh} (\frac{\partial^2}{\partial t^2} + f^2) + \frac{1}{h} \nabla h \cdot \nabla \zeta + \frac{f}{h} k \cdot \nabla h \times \nabla \zeta = -\frac{1}{gh\rho} W \quad (2.24)\]

where
\[\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + 0k \]
The vertical boundary conditions, as discussed in Section 2.2 theory, have been utilised in the evaluation of (2.17). However, the kinematic boundary condition depends on whether the boundary is free or fixed.

Boundary conditions for free boundaries require that:

(a) the wave height, $\zeta$, is continuous

and then if $n$ is a vector normal to the boundary,

(b) the normal component of flux $h u \cdot n$ is continuous,

For rigid or fixed boundaries the boundary conditions are:

(a) the wave height, $\zeta$, is finite

and

(b) the normal component of mass flow is zero, that is,

$$h u \cdot n = 0.$$ (2.28)

An example of a rigid boundary is the coast line (that is, $x = 0$). However, when considering a sloping boundary at the coast, the boundary condition for the wave height on the
coast is
\[ \zeta \text{ is finite at } x = 0 \quad . \quad (2.29) \]

The near shore surf zone does not necessarily conform to the shallow water theory condition (2.6) i.e. \( \zeta = 0 \) at \( h = 0 \). However, if the differential equation (2.24) has a regular singular point, then the solution, \( \zeta \), of (2.24) remains valid (at least mathematically) in a neighbourhood of \( x = 0 \). Also, the surf zone physically has finite wave height, hence (2.29) is taken as the boundary condition for the off-shore sloping coastline.

If the wave oscillation is caused by the wave being trapped on the continental shelf the wave is called a trapped wave. If the wave oscillation is not trapped then the wave is called a leaky wave.

There are two possible boundary conditions at an infinite distance away from the coastline depending on whether the wave is trapped or leaky. Trapped waves will be considered throughout this thesis. The appropriate boundary condition for trapped waves at an infinite distance away from the coastline is given by

\[ \zeta = 0 \quad \text{at infinity} \quad . \quad (2.30) \]

An alternative form of this off-shore boundary condition for a finite distance away from the coastline will be derived later. The appropriate boundary condition is found in (2.39).
2.6 FREELY PROPAGATING EDGE WAVES WITH VARIABLE DEPTH.

Edge waves can be defined as coastally trapped waves for which the motion normal to the coast is oscillatory and the wave propagates with long shore wave number $s$. As well, the offshore wavelength increases with distance away from the coast while the amplitude decreases.

Since the equations of motion (2.13) are invariant under a horizontal rotation of the co-ordinate axis, the $y$ axis can be chosen parallel to the coast.

To find the freely propagating edge waves, it is assumed that the topographic variation over the continental shelf will be normal to the coast i.e. let

$$h = h(x), \quad 0 \leq x < \infty \quad (2.31)$$

where, $x = 0$ corresponds to the coastline. Also, let

$$\zeta = Z(x)e^{i(x-y-s)} \quad , \quad s > 0 \quad (2.32)$$

and

$$F_1 = F_2 = 0 \quad (2.33)$$

for freely propagating waves.

If $\sigma > 0$ then the simple harmonic motion, defined in (2.32), is in the positive $y$ direction and if $\sigma < 0$ then the motion is in the opposite direction.

Substitution of (2.31), (2.32) and (2.33) into (2.24) gives the differential equation

$$\frac{d}{dx} \left[ h \frac{dZ}{dx} \right] + \left[ \frac{\sigma^2 - f^2}{g} - s^2 h - \frac{s f \, dh}{\sigma \, dx} \right] Z = 0 \quad (2.34)$$
which, together with the appropriate boundary conditions governs the fluid motion over the continental shelf. It should be noted that $Z$ is a function of $x$ only.

The coastal (fixed) boundary conditions are obtained by substituting (2.32) into (2.27) and (2.28), which gives

$$Z \text{ finite at } x = 0 \quad (2.35)$$

$$h \left[ \frac{d}{dx} - \frac{sf}{\sigma} \right] Z = 0 \quad \text{at } x = 0 \quad (2.36)$$

The second condition (2.36) ensures that there is no mass flux through the coastal boundary. Condition (2.36) is valid for all $x$ provided that if $h = 0$ at $x = 0$ (for example, the sloping shelf $h = \alpha x$) then $Z$ is differentiable at $x = 0$, that is

$$| Z'(x) | \text{ is finite at } x = 0 \quad (2.37)$$

(2.37) implies that the wave height is continuous and the sea surface (as well as its velocity components) is well behaved at the coastline.

For coastally trapped edge waves on the continental shelf, the (free) boundary condition depends on the depth profile of the shelf chosen. That is,

1) If the shelf profile is regarded as semi-infinite in extent, then (2.30) is sufficient. Thus, substituting (2.32) into (2.30) gives:

$$Z \to 0 \quad \text{as } x \to \infty \quad (2.38)$$

2) If the shelf profile is truncated at a distance $x = L$ by an ocean of constant depth $H$, an alternative form of (2.30) can be obtained.

Caldwell, Cutchin and Longuet-Higgins (1972) obtained the boundary condition

$$\frac{dZ}{dx} + rZ = 0 \quad \text{at } x = L \quad (2.39)$$
where

\[ r^2 = s^2 + \frac{f^2 - \sigma^2}{gH} > 0 \quad (2.40) \]

The long wave equations (2.14) and (2.22) along with the appropriate boundary conditions will be used in the remainder of this thesis to study the wind effects on edge waves.
CHAPTER THREE

EDGE WAVES ON A SLOPING SHELF

3.1 INTRODUCTION

In 1846 Stokes found, using the homogeneous wave equation with a uniformly sloping shelf, that edge waves which could propagate in either direction along the coast were generated. He showed that, provided the slope of the shelf was small, the speed of the wave was slower than the deep water gravity wave. Also, Stokes was able to show that the energy of the wave was confined to within one wave length off-shore. Eckart (1951) showed that this edge wave was the fundamental mode of a whole series of modes whose energy was trapped against the coast. Ursell (1952) was able to verify Eckart’s result experimentally without using long wave approximations.

The effects of Coriolis on edge waves using a sloping shelf has been studied by authors such as Reid (1958), Kaijura (1958), Johns (1965) and Saint-Guilly (1968). For example, Reid showed in 1958 that rotation gives rise to edge waves which have slightly different phase speeds for left and right bounded edge waves. He defined these waves as being 'quasi-geostrophic', that is, waves which can propagate only in one direction depending on which hemisphere the fluid is in. Mysak (1968) categorized these quasi-geostrophic or low frequency waves as Class II waves. This meant that the frequency of quasi-geostrophic waves, $\sigma$, is much lower than the Coriolis parameter, that is, $\sigma << f$. Consequently, Class I waves were defined as long as gravity waves for which $\sigma > f$. Hence, rotation plays only a modifying role in the generation of Class I waves.
This chapter will begin with the development of the solution and dispersion relation between frequency and wave number for freely propagating Class I and Class II barotropically trapped edge waves over a sloping shelf, as discussed by LeBlonde and Mysak (1978). These solutions will be used later to examine the effects of wind forcing on Class I waves.

### 3.2 UNFORCED EDGE WAVES ON A SLOPING SHELF

The topography of the continental shelf will be taken to be:

\[ h = \alpha x , \quad 0 \leq x < \infty , \quad (3.1) \]

where \( \alpha \) is the slope of the shelf. \( \alpha \) is taken to be sufficiently large so that the \( \beta \)-effect is negligible. Hence, the Coriolis parameter \( f \) will be regarded as a constant.

On substitution of (3.1) into (2.34), an equation for freely propagating edge waves over a variable topography, edge waves of the form given in (2.32) can be determined by solving the following differential equation

\[
\frac{d}{dx} \left[ x \frac{dZ}{dx} \right] + \left[ k - s^2 x \right] Z = 0 , \quad (3.2)
\]

where

\[
k = \frac{\sigma^2 - f^2}{\alpha g} - \frac{sf}{\sigma} , \quad (3.3)
\]

along with the boundary conditions (2.35), (2.37) and (2.38) that is

\[ Z \quad \text{and} \quad |Z'| \quad \text{finite at} \quad x = 0 , \quad (3.4) \]

and

\[ Z \to 0 \quad \text{as} \quad x \to \infty . \quad (3.5) \]
For $Z$ to satisfy (3.2) and the boundary conditions (3.4) and (3.5), it is found using Laguerre polynomials, that

$$Z = \sum_{n=0}^{\infty} e^{-sx} L_n(2sx)$$

(3.6)

and

$$k = (2n + 1)s.$$  

(3.7)

Substituting (3.3) into (3.7), gives

$$\sigma^3 - \left[ f^2 + (2n + 1)\alpha gs \right] \sigma - f s \alpha g = 0 \quad n = 0, 1, 2, \ldots$$  

(3.8)

Equation (3.8) is called the dispersion equation. It can be seen in (3.8) that for each wave number $s > 0$, there exists 3 values of the frequency $\sigma$ for each value of $n$.

The set of frequencies for each value of $n$ will be denoted by $\sigma_{in}(s)$ where $i = 1, 2, 3$. Since (3.8) is a cubic polynomial in $\sigma$, it can be shown that the roots are real since

$$\sum_{i=1}^{3} \sigma_{in} = 0 \quad \text{and} \quad \prod_{i=1}^{3} \sigma_{in} = f s \alpha g.$$  

(3.9)

The properties of the frequencies expressed in (3.9) imply that if $f < 0$, then one root, say $\sigma_{1n}$, is positive and the other two roots are negative. Thus, in the Southern Hemisphere the edge wave with frequency $\sigma_{1n}$ will move in the positive $y$ direction and hence will propagate with the rigid boundary (or coast) on the left hand side called a left bounded wave. The edge waves with the frequencies being negative are called right bounded waves.

When $f = 0$, (3.8) reduces to

$$\sigma \left( \sigma^2 - (2n + 1)\alpha gs \right) = 0$$  

(3.10)

which is Eckart's solution. The root $\sigma = 0$ of (3.10) is recognized as a Class II edge wave mode trapped by rotation whereas the roots

$$\sigma = \pm \left( (2n + 1)\alpha gs \right)^{1/2}$$
are identified as Class I edge wave modes modified by rotation.

A graph of the dispersion equation (3.8) for \( i = 1, 2, 3 \) can be found in Figure 3.1, where \( f 
eq 0 \) and \( \alpha = 4 \times 10^{-3} \). Graphs of the fundamental modes are found in Figure 3.1(a) and a graph of the \( n=1,2,3 \) modes are found in Figure 3.1(b). Also, graphs using the first three modes of the dispersion relation described in (3.10), where \( f = 0 \), are found in Figure 3.2. In Figure 3.2(a), \( \alpha = 4 \times 10^{-3} \) and in Figure 3.2(b) various values of \( \alpha \) are used.

For \( n = 0 \), (3.8) becomes

\[
\sigma^3 - \left[ f^2 + \alpha g s \right] \sigma - fs \alpha g = 0 .
\]

Further this equation is written as:

\[
\left( \sigma_{10} - \left( \frac{1}{2} + a \right) f \right) \left( \sigma_{20} - \left( \frac{1}{2} - a \right) f \right) \left( \sigma_{30} + f \right) = 0
\]

(3.11)

where

\[
a = \left( \frac{1}{4} + \frac{g \alpha s}{f^2} \right)^{1/2} .
\]

The speed of the edge wave with frequency \( \sigma_{10} \) is \( \left( \frac{1}{2} + a \right) f \), whereas for \( \sigma_{20} \) the speed of the wave is \( \left( \frac{1}{2} - a \right) f \). If \( f < 0 \) then the right bounded wave (i.e. wave with frequency \( \sigma_{10} \)) moves faster than the left bounded wave. However, the magnitude of group velocity, \( \left| \frac{d\sigma}{ds} \right| \), is the same for both the left and right bounded edge wave. Therefore, the edge wave roots exhibit rotational splitting of frequency. Also as \( s \to 0 \), \( \sigma_{10} \to f \), whereas \( \sigma_{20} \to 0 \). The third root \( \sigma_{30} = -f \) of equation (3.11) corresponds to an inertial oscillation of infinite wavelength.

For higher modes \( n \geq 1 \), (3.8) can be rewritten as

\[
\left( \sigma_{1n} - \left( \frac{1}{2} + a_n f \right) \right) \left( \sigma_{2n} - \left( \frac{1}{2} - a_n f \right) \right) \left( \sigma_{3n} + f \right) = 0
\]

(3.12)

where

\[
a_n = \left( \frac{1}{4} + \frac{(2n + 1)g \alpha s}{f^2} \right)^{1/2} .
\]
FIGURE 3.1(a)

FIGURE 3.1(b)

FIGURE 3.1 The dispersion relation for edge and quasi-geostrophic waves on a semi-infinite sloping beach of slope $\alpha$ and $f \neq 0$. (a) The fundamental mode ($n=0$) of the solution to (3.8). (b) The $n = 1, 2, 3$ modes of the solution to (3.8).
FIGURE 3.2 Stokes edge wave solution in the absence of rotation (f=0), for $\sigma<0$ and $\sigma>0$.
(a) The first 3 fundamental modes of the solution to (3.10) for $\alpha = 4 \times 10^{-3}$.
(b) The $n=1$ mode of the solution to (3.10), using shelf parameters $\alpha_1 = 4 \times 10^{-3}$ and $\alpha_2 = 6.25 \times 10^{-3}$.
As $s \to 0$, it can be seen from (3.12) that $\sigma_{1n} \to f$, $\sigma_{2n} \to -f$ and $\sigma_{3n} \to 0$. So for higher modes, $\sigma_{1n}$ and $\sigma_{2n}$ are the modes for Class I edge waves and are asymmetric about $\sigma = 0$. As $s \to 0$, all modes have the property that $\sigma_{1n} \to f$ and $\sigma_{2n} \to -f$. The Class II edge wave mode is defined by $\sigma_{3n}$ which shows that $\sigma_{3n} \to 0$ as $s \to 0$ and $|\sigma_{3n}/f| \ll 1$ for all $s$. As a consequence the Class II edge wave has a large vorticity (refer to Chapter 6) compared to the Class I wave.

3.3 EVIDENCE OF EDGE WAVES

Sea level records from the east coast of the United States were examined by Munk et al (1956) during the passage of hurricanes and storms. They found that the fundamental edge wave mode was being excited by these storms. These fundamental edge waves have a typical amplitude of 1 metre with period around 6hrs and a very large wavelength. As a consequence, both Greenspan (1956) with $f = 0$ and Kaijura (1958) with $f \neq 0$, produced mathematical calculations using an idealized model for a pressure forcing to simulate edge waves being excited by storms.

In Australia, extensive pioneering work was done by Hamon (e.g. Hamon (1958), (1962), Hamon and Stacey (1960) and Hamon and Grieg (1972)). Hamon carried out investigations into the effects of weather systems on Class II edge waves. Further detailed analysis has been done by such authors as Robinson (1964), Adams and Buchwald (1969), Gill and Schumann (1974), Allen (1976) and Clarke (1977). Also, Freeland et al (1986) were involved in collecting physical data from off the east coast of Australia. They found that the physical data, although influenced by eddies from the east Australian current, showed a clear separation of the first three edge wave modes over the range of frequencies appropriate to the weather forcing.
Spectral analysis techniques can be used on physical data to detect the existence and the characteristics of Class II trapped edge waves. New and improved spectral analysis techniques to analyse current and sea level data have been introduced by such authors as Hsieh (1982), Freeland et al. (1986) and Huntley (1988). However, Huntley (1988) concluded that further improvements are required into the spectral analysis of data, in particular, the coupling between edge wave modes.

Long waves with very high frequency and large amplitude were observed in Jervis Bay, N.S.W., Australia by Clarke, Keane and O'Halloran (1968). Based on these observations, Buchwald and de Szoeke (1973) analytically obtained the same period and amplitude using a pressure front along a step shelf, provided the speed of the disturbance was between the speed of the long wave on and off the continental shelf. Further, there is evidence that on the California coast, U.S.A. a continuum of edge wave noise is always present in the period ranging from 10 to 30 minutes. Munk et al (1956) suggested that these waves may be generated by atmospheric internal gravity waves. Earthquake generated long waves, called Tsunamis, incident on the continental shelf have been shown by Kaijura (1963), Aida (1967), Fuller and Mysak (1977) to excite edge waves.

Bowen (1969) and Bowen and Inman (1971) have shown the behaviour of edge waves is of fundamental importance to beach erosion and sedimentation.

The generation of high frequency Class I edge waves using the pressure as the forcing function has been examined by Greenspan (1956). Using an idealized model for the pressure distribution, Greenspan (1956) showed that there was a resurgence motion which consisted of an infinite number of edge waves modes.

Viera and Buchwald (1982) used a pressure distribution acting on waves over a truncated exponential shaped continental shelf. However, they found that for the pressure model used, the
theoretical front speeds were far greater than actual front speeds measured along the coast of Australia.

In the next section wind generated Class I waves will be analytically examined over a sloping semi-infinite shelf, as discussed by Worthy (1982). It will be shown that the wind stress forcing function does not have to be restricted to a single model, such as, the square wave forcing model used by Adams and Buchwald (1969).

3.4 WIND FORCED HIGH FREQUENCY EDGE WAVES

Using wave height recordings outside Port Kembla Harbour, N.S.W., Australia, Clarke (1979) has observed that certain long waves with periods ranging from an hour and less are excited more than others when storm fronts coupled with very strong southerly winds called southerly 'busters' move in a northerly direction along the east coast of Australia. The shallow water equations (2.13) and the continuity equation (2.22) will be used to analytically model such phenomena.

Baines (1980) discussed the movement of southerly 'busters' along the east coast of Australia. He showed, using isobaric graphs showing a storm front associated with southerlies, that the isobars have a tendency to align themselves parallel to the mountain range which is situated near the east coast of Australia. Hence, a tunnelling effect is created on the coast and consequently there is an increase in wind speed. In (1986), Holland and Leslie investigated the generation and intensity of the southerly 'buster' by numerical means. Since isobars align themselves parallel to the coast, the change in pressure is about 1 to 2 mb or 1 to 2 dyn/cm$^3$ over a distance of 200 to 800 kms. Taking an average distance of 250 kms per mb change and a continental shelf slope, $\alpha$, off the east coast of Australia as $6.25 \times 10^{-3}$, the pressure gradient, $\alpha p_x$, is of the order $10^{-8}$ dyn/cm$^3$. Furthermore, if the wind stress divided by $\rho$, the density of
water, is taken to be 5 dyn/cm² over a wind fetch of 50km, then the wind stress gradient has an order of magnitude less than the wind stress term. Hence, the pressure term will not be included in the following analysis.

The Coriolis parameter $f$ plays only a modifying role for Class I waves. If the Rossby number $R_0$ is small, then $f$ can be neglected. Let $f = 0$, then the shallow water wave equations are:

$$
\frac{\partial u}{\partial t} = -g \frac{\partial \zeta}{\partial x} + \frac{X}{h},
$$

$$
\frac{\partial v}{\partial t} = -g \frac{\partial \zeta}{\partial y} + \frac{Y}{h},
$$

(3.13)

where $X$ and $Y$ are the wind stress terms (divided by the density of water) in the $x$ and $y$ directions, respectively. It will be assumed that the main forcing mechanism is the southerly 'buster' winds and hence the wind stress in the off-shore direction is negligible. This assumption is supported by experimentation done by Schwing et al (1983), who stated that cross-shore currents are not coherent with cross-shore winds. Also, the longshore component of wind stress, $\tau_x = \frac{Y}{h}$ can impart the necessary 'barotropic' vorticity to generate edge waves. The vorticity equation being

$$
\frac{\partial}{\partial t} (\xi) = \frac{1}{\rho h} (\text{Curl } \tau)_z
$$

where $\xi = (\text{Curl } U)_z$ and $\tau$ is the wind stress vector.

The equation of continuity is:

$$
\frac{\partial}{\partial x} (hu) + \frac{\partial}{\partial y} (hv) = -\frac{\partial \zeta}{\partial t}
$$

(3.14)

The elimination of $u$ and $v$ from (3.13) and (3.14) yields

$$
g h \nabla^2 \zeta + g \zeta \frac{dh}{dx} - \zeta_{tt} = Y_y
$$

(3.15)
Since only the near shore part of the continental shelf influences Class I edge waves because they rapidly decay away from shore, the linear shelf profile (3.1) is a good approximation. The appropriate boundary conditions given (3.1) are (2.29) and (2.30), respectively.

To find the solution $\zeta$ of (3.15), Laplace and Fourier transforms of (3.15) with respect to $t$ and $y$ are taken in turn. Therefore, it is required that

$$\zeta = \zeta_t = 0 \text{ at } t = 0,$$

(3.16)

$$\zeta \to 0 \text{ for } |y| \to \infty.$$  

Let the one sided Laplace transform of $\zeta(x,y,t)$ be given by:

$$\bar{\zeta}(x,y,a) = L(\zeta) = \int_0^\infty e^{-at} \zeta(x,y,t) \, dt$$

(3.17)

and the Fourier transform of (3.17) be represented by

$$\zeta^*(x,s,a) = F(\bar{\zeta}) = \int_{-\infty}^\infty e^{-isy} \bar{\zeta}(x,y,a) \, dy$$

(3.18)

Taking the Laplace and Fourier transforms of (3.15) with respect to $t$ and $y$, in turn, using (3.17), (3.18) and (3.16) gives

$$uf''' + (1-u)f' - \left[ .5 + \frac{\omega^2}{2s* \alpha g} \right] f = R$$

(3.19)

where

$$s* = \left( s^2 + \varepsilon^2 \right)^{1/2},$$

(3.20)

$$u = 2s* x,$$

(3.21)

$$R = \frac{iY^* e^{u/2}}{2s* \alpha g},$$

$$Y^* = F\left[ L(Y_x) \right] = i\sigma F(\bar{Y}),$$

(3.22)

$$\zeta^* = e^{-u/2}f(u,s,\omega)$$

(3.23)
where the dashes represent derivatives with respect to $u$.

Using the boundary conditions (2.29), (2.30) and (3.23), $f(u,s,\omega)$ satisfies $|f(0,s,\omega)| < \infty$ and $f(u,s,\omega) \to 0$ as $u \to \infty$.

Difficulties arise with the Fourier inversion of (3.23) therefore $s^*$ is used in (3.21) instead of $s$ (as discussed by Greenspan (1956)). After taking the inverse Laplace and Fourier transforms of the solution to (3.19), $\epsilon$ is allowed to tend to zero to obtain the solution of (3.15). The function $s^*$ has two branches at $\pm \epsilon$ and extending to $\pm i\infty$ respectively. The Reimann sheet is chosen so that $s^* \geq 0$ on the entire real axis.

The technique used by Greenspan (1956) will be used to solve (3.19) by expanding $f$ in terms of Laguerre polynomials.

Thus, let

$$f = f(u,s,\omega) = \sum_{n=0}^{\infty} A_n L_n(u)$$

and

$$R = \sum_{n=0}^{\infty} B_n L_n(u)$$

where

$$L_n(u) = \frac{e^u}{n!} \frac{d^n}{du^n} (u^n e^{-u})$$

is the Laguerre polynomial of degree $n$.

The coefficient $A_n$ and $B_n$ are determined by substituting (3.24) and (3.25) into (3.19) and using the orthogonality condition:

$$\int_0^{\infty} e^{-u} L_m(u)L_n(u) du = \delta_{mn}$$

where $\delta_{mn}$ is the Kronecker Delta function.
Hence,

\[ A_n = \frac{-B_n}{a_n^2 + \omega^2} \]  

(3.26)

where

\[ B_n = \int_0^\infty R e^{-u} L_n(u) \, du \]

and

\[ a_n^2 = s^* \alpha g (2n + 1) \]  

(3.27)

On substitution of (3.26) and (3.24) into (3.23) gives

\[ \zeta* = -\sum_{n=0}^{\infty} \left( \int_0^\infty R e^{-u} L_n(u) \, du \right) \frac{L_n(u)}{a_n^2 + \omega^2} \]  

(3.28)

The wave height, \( \zeta \), obtained by inverting the Fourier and Laplace transforms, is given by

\[ \zeta = \frac{1}{4\pi^2i} \int_{-\infty}^{\infty} e^{is} \int_{\gamma-i\infty}^{\gamma+i\infty} \zeta* \, e^{os} \, ds \]  

(3.29)

The path of integration in the \( s \)-plane for the inverse Fourier transform lies within a strip containing the real axis. The function \( s* \) in (3.20) has been defined so that the inversion contour can pass between the two branch points \( \pm i\epsilon \) without crossing a branch line.

The evaluation of (3.28) is dependent on the model of the wind stress. In Section 3.5 analytical results are obtained using various wind stress models.
3.5 WIND STRESS MODEL

The wind stress model will be represented by an envelope of impulses with various strengths with each impulse, propagating along the coast. Therefore, a unit of impulse will be represented as:

\[ Y = Y_0(\eta) \delta(y - Vt - \eta) H(t) \]  

where \( \delta(y - Vt - \eta) \) is the Dirac Delta function, \( H(t) \) is the unit step function and \( Y_0(\eta) \) is the amplitude of the wind stress.

Using (3.30), the total shape of the wind stress is given by

\[ Y_T = \int_{-\infty}^{\infty} Y_0(\eta) \delta(y - Vt - \eta) H(t) \, d\eta \]  

The propagating characteristic of the wind stress in (3.30) could be used to describe strong gusts of wind travelling parallel to the coast, a situation which occurs along the east coast of Australia. Adams and Buchwald (1969) used a model similar to (3.30) to describe the effects of a wind stress motion which was oscillatory with respect to time rather than being a progressive front moving along the coast.

Using (3.22) and (3.30), \( Y^* \) can be found and on substitution into (3.28), gives

\[ \zeta^* = Y_0(\eta) \sum_{n=0}^{\infty} \frac{s^n s C_n \, e^{-i\eta}}{(\omega - i s V)(\omega^2 + a_n^2)} \]  

where

\[ C_n = 2i(-1)^{n+1} e^{-u/2} L_n(u) \]  

The inverse Laplace transform of \( \zeta^* \) is given by

\[ \zeta = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \zeta^* e^{\omega t} \, d\omega \]
where $\gamma$ is chosen to ensure the convergence and $\zeta^*$ is as given in (3.32).

It is found that $\zeta^*$ has simple poles at $\omega = -isV$ and $\omega = \pm ia_n$. Upon performing the integration it is found that for $t > 0$,

$$
\zeta^* = Y_0(\eta) \sum_{n=0}^{\infty} s C_n e^{-is\eta} \left( \frac{e^{-isVt}}{a_n^2 - s^2V^2} - \frac{1}{2a_n} \left( \frac{e^{-ia_nt}}{(a_n - sV)} + \frac{e^{ia_nt}}{(a_n + sV)} \right) \right)
$$

and $\zeta^* = 0$ for $t < 0$.

As a function of $s$, $\zeta^*$ has branch points at $s = \pm i\epsilon$. Consequently, the wave height, $\zeta$, can be obtained from (3.29) and is given by

$$
\zeta = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta^* e^{is\eta} d\eta = \frac{1}{2\pi} Y_0(\eta) \sum_{n=0}^{\infty} C_n \left( I_1^n + I_2^n + I_3^n \right),
$$

where

$$
z = y - Vt,
$$

$$
I_1^n = \int_{\Gamma} \frac{se^{-is(\eta - z)}}{a_n^2 - s^2V^2} d\eta = \int_{\Gamma} F(s, t) d\eta,
$$

$$
I_2^n = -\int_{\Gamma} \frac{se^{-is(\eta - z)} - ia_nt}{2a_n(a_n - sV)} d\eta,
$$

$$
I_3^n = -\int_{\Gamma} \frac{se^{-is(\eta - z)} + ia_nt}{2a_n(a_n + sV)} d\eta
$$

and $\Gamma$ is an appropriate contour which encloses the poles of the integrand and excludes the branch points.
FIGURE 3.3 The most suitable contour for the Fourier inversion of $I_n^1$, found in (3.34), where $s_0 = \frac{\alpha g}{\sqrt{\bar{z}}}$.
The poles of the integrand $I_n^1$ are located at $s^2V^2 - a_n^2 = 0$, that is,

$$s_n = \pm \frac{\alpha g(2n + 1)}{V^2} \quad n = 0, 1, 2, ...$$

(3.36)

for $\epsilon = 0$. After the substitution of $s = Re^{i\theta}$ into the integrand of $I_n^1$ it can be seen that

$$|I_n^1| \leq R \int F(Re^{i\theta}, t) |e^{-R\sin(\eta - z)}| d\theta$$

(3.37)

where

$$|F(Re^{i\theta}, s)| = O\left(\frac{1}{R}\right).$$

Hence from (3.37), the most suitable contour for $\Gamma$ is the one that goes below the singularities in (3.36), whereby, $z > \eta$, so that $I_n^1$ is convergent when $R \to \infty$. Thus, the contour $\Gamma$ is closed in the upper half plane, by quadrants $\Gamma_1$, $\Gamma'_1$, the two sides of the branch line $\Gamma_b$ and the circular path, $\Gamma_c$, about the branch point, as shown in Figure 3.3. For $z < \eta$ the contour $\Gamma$ is closed in the lower half plane. Therefore, the poles of $I_n^1$ contribute only when $z > \eta$. From (3.37), it can be seen that the integrals over connecting quadrants, for $I_n^1$, would tend to zero as $|R| \to \infty$. Also, integrals around the small circle about the branch point tends to zero as the radius $|s| \to 0$. The two integrals about the branch points need to be computed. Therefore,

$$I_n^1 = 2\pi i \sum \text{Residues} - 2i \int_0^\infty \frac{b_n}{R^2V^4 + b_n^2} e^{R(\eta - z)} dR$$

$$= -4\pi i \frac{\cos s_n(\eta - z)}{V^2} + \frac{2is_n^2}{V^2} \left\{ c[i[s_n(\eta - z)]\sin s_n(\eta - z) \right\}$$

(3.38)

$$- si[s_n(\eta - z)]\cos s_n(\eta - z) \right\}$$

where $b_n = \alpha g(2n + 1)$, $s_n$ is defined in (3.36), $ci$ and $si$ are the cosine and the sine integrals defined in Gradshteyn and Ryzhik for $\eta < z$, respectively.

Asymptotically for large $t$ and $y$ values, the integrals around the branch line tend to zero provided $y < Vt$. 
In Appendix 3A, it is shown that $l_n^2$ and $l_n^3$ yield a cancelling wave equivalent to (3.38) for $y < \frac{\lambda}{2} V_t$ and $\epsilon \to 0$. Hence, the resurgence motion is given asymptotically by

$$
\zeta \sim \frac{-4Y_0(\eta)}{V^2} \sum_{n=0}^{\infty} C_n \cos s_n(\eta - z)
$$

(3.39)

provided \( \frac{\lambda}{2} V_t < y < V_t \). Otherwise, $\zeta$ is zero.

Therefore, the real part of the wave height $\zeta$ in (3.39) due to a unit of impulse is given by

$$
\zeta = \frac{8Y_0(\eta)}{V^2} \sum_{n=0}^{\infty} (-1)^{n+1} D_n \cos[s_n(z - \eta)] H(z - \eta)
$$

(3.40)

where

$$
D_n = e^{-\mu/2} L_n(u).
$$

Hence, the total wave height, $\zeta_T$, is given by

$$
\zeta_T = \int_{-\infty}^{\infty} \zeta \, d\eta.
$$

(3.41)

3.6 RESULTS

Different shapes of the wind stress model $Y_T$, can be obtained by varying $Y_0(\eta)$ in (3.30). Figures 3.4 and 3.5 show the particular shapes of the amplitude, $Y_0(\eta)$, which will be discussed in this section.

(a) The first model is a rectangular shaped forcing function

$$
Y_0(\eta) = \begin{cases} 
Y_0 & |\eta| \leq L \\
0 & \text{o.w.}
\end{cases}
$$

(3.42)
FIGURE 3.4(a) Wind stress model defined in (3.42), where $L = 12 \text{kms}$. (b) Wind stress model defined in (3.45) where $\gamma = 1.4 \times 10^{-4} \text{kms}^{-1}$ with $L = 8.5 \text{kms}$ and $\gamma = 1.0 \times 10^{-4} \text{kms}^{-1}$ with $L = 13 \text{kms}$. 
Figure 3.4(a) shows the graph of (3.42) for \( L = 12 \text{kms} \).

The \( Y_T \) obtained by substituting (3.42) in (3.31) is different from \( Y_T \) of Adams and Buchwald (1969) since the wind stress is progressive rather than oscillatory in time. However, both wind stress models have a constant amplitude for \( Y_T \) over the wind fetch (-L,L).

Substituting (3.40) and (3.42) into (3.41) gives

\[
\zeta_T = \frac{8Y_0}{V^2} \sum_{n=0}^{\infty} (-1)^n D_n \sin nL \cos n^2 \frac{n}{s_n} 
\]

for \( z \geq L \).

The maximum wave height occurs at the coast (\( u = 0 \)) and also when \( z = 4L \), where \( L = (2m + 1)\pi/2s_0 \) for \( m = 0, 1, 2, \ldots \). Using these values in (3.43) the total wave height, for \( m = 0 \), is

\[
\zeta_T = \frac{-8Y_0}{V^2} \sum_{n=0}^{\infty} \frac{1}{s_n}.
\]

It can be easily seen that (3.44) is a divergent series, so that the wind stress model chosen is not suitable to describe Class I edge waves on a sloping shelf using boundary conditions (2.29) and (2.30). This may be due to the sharp changes in the wind stress profile or the exclusion of the friction terms in (3.13). However, using a smoother wind stress model yields (b) which is a more appropriate model.

(b) Consider the model given by

\[
Y_0(\eta) = \begin{cases} 
Y_0 e^{-\gamma |\eta|} & |\eta| \leq L \\
0 & \text{o.w.}
\end{cases}
\]

where \( \gamma > 0 \). For this particular wind stress model \( Y_0(\eta) \) tends to a peak with value of \( Y_0 \) as \( \eta \to 0 \) and at the end of the fetch \( Y_0(\eta) \) tends to zero as \( \eta \to \pm \infty \). How rapidly the curve (wind stress amplitude) increases to this peak and, consequently, how rapidly the curve dies
at a distance of away depends on the value of $\gamma$. The larger the value of $\gamma$ the sharper $Y_0(\eta)$
becomes as $\eta \to 0$. This behaviour is displayed in Figure 3.4(b).

Using (3.40) and (3.45) in (3.41) and evaluating $\zeta_T$ at the coast (i.e. $u=0$ ) it is found
that

$$
\zeta = \frac{8Y_0}{V^2} \sum_{n=0}^{\infty} \frac{(-1)^n \left[ \gamma - \gamma L (s_n \sin nL - \gamma \cos nL) \right] \cos n^2}{\left( s_n^2 + \gamma^2 \right)}
.$$  \hfill (3.46)

The value of $\alpha = 6.25 \times 10^{-3}$ is taken to be an approximation to the slope of the shelf off
the east coast of Australia. Using this value of $\alpha$, the depth is about 200m $32\text{kms}$ off-shore.
The values $\alpha = 4 \times 10^{-3}$ and $\alpha = 5 \times 10^{-3}$ (Greenspan (1956) ) will also be used for the slope
of the shelf off the east coast of Australia and the results compared in Table 3.1.

Wind gusts with speeds of about 35 knots travel parallel to the east coast of Australia ( Buchwald and De Szoeke (1973) ), called southerly 'busters', and also along the west coast of
the United States ( Munk et al (1956) ). The progress of these southerlies behave like cold air
fronts moving along the coast, whereby, the speed of the atmospheric front is approximately the
same as the wind speed. Therefore, it is assumed that the wind speed, $V$ is the same as the
speed of the front. Hence, $V$ can be determined by

$$
Y_0 = c \rho V^2 \quad (\text{Krauss (1973)})
,$$

where $c = 1.2 \times 10^{-3}$ ( Hasse (1968) ) is the drag co-efficient, $\rho = 1.225 \times 10^{-3} \text{ g cm}^{-3}$ is the
density for dry air at the sea surface at $15^\circ C$ and the wind speed $V$ is measured in anemometer
height so that $V = 18.443 \text{ms}^{-1}$.

The period, $\tau_n$ and wavelength, $\lambda_n$ of the progressing wave in (3.43) can be determined
using (3.36) and (3.43). Therefore,

$$
\tau_n = \frac{2\pi V}{\alpha g (2n + 1)}
,$$

$$
\lambda_n = V \tau_n
.$$  \hfill (3.47)
for $n = 0, 1, 2, \ldots$.

As $\gamma \to 0$, the wind stress model in (3.45) reduces to the constant wind stress in (3.42). Similarly, the wave height $\zeta_T$ in (3.46) reduces to the wave height in (3.43) corresponding to the constant wind stress model. Therefore, care has to be taken in choosing values of $L$ and $\gamma$, so that the series in (3.46) is convergent. Consequently, we will discuss those values of $L$ and $\gamma$ which on substitution into (3.46) produce a convergent series. If $s_0L = \pi$, (3.46) reduces to

$$
\zeta_T = \frac{8Y_0}{V^2} \sum_{n=0}^{\infty} \frac{(-1)^n \gamma \left(1 + e^{-\gamma L}\right)}{\left(s_n^2 + \gamma^2\right)}.
$$

(3.48)

The dominant mode for high frequency waves is usually the first mode (denoted by $\zeta_0$). Hence, $\zeta_0$ from (3.48) is maximized with respect to $\gamma$. Values of $\gamma$ are obtained by solving the equation (3.49) for $x = \gamma L$.

$$
\frac{d\zeta_0}{d\gamma} = e^{-x} \left\{ a^2 (1-x) - x^2 (1+x) \right\} + a^2 - x^2 = 0.
$$

(3.49)

Since $\frac{d\zeta_0}{d\gamma}$ is strictly decreasing function of $x$ in $(0, \infty)$, only one value of $x$ is obtained from (3.49).

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\zeta_T$</th>
<th>$\gamma$</th>
<th>$\tau_0, \tau_1, \tau_2$</th>
<th>$\lambda_0, \lambda_1, \lambda_2$</th>
<th>$2L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 10^{-3}$</td>
<td>2.52</td>
<td>$9.6 \times 10^{-5}$</td>
<td>49, 16.4, 9.8</td>
<td>55, 18, 11</td>
<td>54.5</td>
</tr>
<tr>
<td>$5 \times 10^{-3}$</td>
<td>1.85</td>
<td>$1.2 \times 10^{-4}$</td>
<td>47, 15.6, 9.4</td>
<td>44, 15, 8.8</td>
<td>43.6</td>
</tr>
<tr>
<td>$6.25 \times 10^{-3}$</td>
<td>1.61</td>
<td>$1.5 \times 10^{-4}$</td>
<td>32, 10.5, 6.4</td>
<td>35, 11, 7</td>
<td>34.9</td>
</tr>
</tbody>
</table>

**TABLE 3.1.** Summary of values used to obtain the wave height, $\zeta_T$ using slopes $\alpha$ in (3.48).
FIGURE 3.5(a)

Wind stress model defined in (3.50), where, (a) $\beta L = J \pi$ ($J = 1, 2$), $s_0 L = \frac{\pi}{2}$ and (b) $s_0 L = \frac{K \pi}{2}$ ($K = 1, 3$), $\beta L = \pi$. 

FIGURE 3.5(b)
Table 3.1 shows the amplitude $\zeta_T$ (cms) for (3.48) using the root of (3.49). Also, included in Table 3.1 is the period (mins), wavelength (kms), wind fetch (kms) and $\gamma m^{-1}$ corresponding to shelf slopes, $\alpha$.

(c) Consider the sinusoidal wind stress model of the form:

$$Y_0(\eta) = \begin{cases} Y_0 \sin(\beta \eta) & |\eta| \leq L \\ 0 & o.w. \end{cases}$$

(3.50)

where $\beta > 0$. This particular wind stress model behaves like a switching motion of the wind in general, that is, wind is blowing in one direction at one instance in time then changing direction in another instance in time. Also, as the value of $\beta$ increases then so does the period of oscillation in the wind stress model and therefore the switching motion increases, as seen in Figure 3.5.

Substituting (3.50) into (3.40) and again evaluating $\zeta_T$ at the coast it is found that:

$$\zeta_T = \frac{8Y_0}{V^2} \sum_{n=0}^{\infty} (-1)^n D_n l(z)$$

(3.51)

where

$$l(z) = \frac{\beta \cos \beta L \sin nL - s_n \sin \beta L \cos nL}{\beta^2 - s_n^2} \sin nz$$

As for the wind stress model (b) parameters, care has to be taken in choosing these parameters so that $\zeta_T$ in (3.51) contains a convergent series. If $s_0z = \pi/2$, and $s_0L = (2m + 1)\pi/2$, $m = 1, 2,...$ then (3.51) becomes:

$$\zeta_T = \frac{16Y_0}{V^2} \sum_{n=0}^{\infty} (-1)^n \beta \cos \beta L \frac{\sin nz}{\beta^2 - s_n^2}$$

(3.52)

The maximum value of (3.52) is obtained when $\beta L = m\pi$, $m = 0, 1, 2,...$. Table 3.2 shows the amplitude $\zeta_T$ (cms), wind fetch ($2L$) for (3.52) using various values of $m$ for the 3 different shelf slopes, the period and wavelengths of the edge waves are the same as those calculated in Table 3.1, using (3.47).
<table>
<thead>
<tr>
<th>$K$</th>
<th>$\alpha$</th>
<th>$2L$</th>
<th>$J = 1, J = 2, J = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4 \times 10^{-3}$</td>
<td>27.2</td>
<td>20.4, 13.6, 17.7</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$5 \times 10^{-3}$</td>
<td>21.8</td>
<td>16.3, 10.8, 14.1</td>
</tr>
<tr>
<td></td>
<td>$6.25 \times 10^{-3}$</td>
<td>18</td>
<td>13.1, 8.7, 11</td>
</tr>
<tr>
<td>Dom. mode</td>
<td></td>
<td>0 &gt; 1, 1 &gt; 2 &gt; 0, 2 &gt; 3 &gt; 1 &gt; 0</td>
<td></td>
</tr>
<tr>
<td>$4 \times 10^{-3}$</td>
<td>81.8</td>
<td>23.3, 37.9, 20.3</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$5 \times 10^{-3}$</td>
<td>65.4</td>
<td>18.6, 30.3, 16.2</td>
</tr>
<tr>
<td></td>
<td>$6.25 \times 10^{-3}$</td>
<td>52</td>
<td>14.9, 24.26, 13</td>
</tr>
<tr>
<td>Dom. mode</td>
<td></td>
<td>0, 0, 0</td>
<td></td>
</tr>
<tr>
<td>$4 \times 10^{-3}$</td>
<td>136.3</td>
<td>9, 43.8, 58.1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$5 \times 10^{-3}$</td>
<td>109</td>
<td>7, 35, 46</td>
</tr>
<tr>
<td></td>
<td>$6.25 \times 10^{-3}$</td>
<td>87</td>
<td>5.8, 28.0, 37.21</td>
</tr>
<tr>
<td>Dom. mode</td>
<td></td>
<td>0, 0, 0</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 3.2.** Summary of results by using various of $K$ ( $s_0L = K\pi/2$ ) and $J$ ( $\beta L = J\pi$ ) in (3.52).
If \( s_0 z = s_0 L = \pi/2 \) and \( \beta L = \pi \) then (3.52) becomes:

\[
\zeta_T = \frac{32Y_0}{\alpha g} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4 - (2n + 1)^2}.
\]  

(3.53)

It can be easily seen, from (3.53), that the first two modes (i.e. \( n = 0, 1 \)) are dominant, that is, the first two modes have the same order of magnitude. Table 3.2 also shows the dominant edge wave modes for the various values of \( s_0 L \) and \( \beta L \). Hence, excitation of certain edge wave modes are apparent under the right conditions, namely, wind fetch length and wind stress parameters.

3.7 DISCUSSION

Asymptotic results have been shown in Table 3.1 and Table 3.2 for large values of \( y \) and \( t \) and various sloping beach profiles using two different wind stress models. The first wind stress model (a), shown in Figure 3.4(a), is constant wind stress over the wind fetch, \( 2L \). However, \( \zeta_T \) in (3.43) is found to have a divergent series. The second wind stress model (b) rises to a peak within the region \( 2L \), and the sharpness of the peak depends on the wind stress parameter, \( \gamma \). Model (c) wind stress is sinusoidal and relates to the switching motion of wind, that is, the wind moving in one direction then suddenly changing to another direction. Hence, it can be seen that a variety of wind stress shapes can be used in (3.6) to model storm fronts.

The choice of the wind stress parameters is important to ensure that the wave height \( \zeta_T \) is finite. For instance, the parameter(s) used in the wind stress models (b) and (c) have to be chosen so that the series solution in (3.40) is convergent. It is found that using model (a) in (3.40) results in a divergent series. This suggests that the sloping beach profile does not support high frequency edge waves which are generated by a constant wind stress. Using model (b),
it is found that if the wind fetch, $2L$, is chosen carefully, (3.40) will result in an absolutely convergent series.

The periods and wavelengths calculated in Table 3.1 are the same as those obtained for the wave heights in Table 3.2.

It should be noted, particularly for the shelf slope $\alpha = 6.25 \times 10^{-3}$, that the wind fetch lengths are to be within the required limits so that the wind stress term is of an order of magnitude greater than the pressure gradient, i.e. the fetch length should be of the order $\leq 50$ kms. This ensures that the wind stress term is the dominant term for southerly 'busters'. This is illustrated in Table 3.1. However, Table 3.2 attempted to show the trend in the various values of $s_0L$ and $\beta L$ rather than choosing the particular wind fetch that best suited southerly 'busters'. For $\alpha = 6.25 \times 10^{-3}$, there are two choices of fetch length, namely, 18 kms and 52 kms. Also, Table 3.2 shows that modes other than the zero mode for Class I edge waves on a sloping shelf are excited when $K = 1$.

To check the results found in Table 3.1 and 3.2, physical data was obtained from a current meter. Recordings were taken at 3min. intervals from outside Port Kembla Harbour, on the south coast of N.S.W. Australia. Figure 3.6 shows a power spectrum with predicted tide variations removed from a current meter recording starting at 2200hrs on 24th October, 1975 and finishing at about 1200hrs on 25th October, 1975. During this time interval, the Bureau of Meterology of Australia (1975) recorded that there was local storm activity in the Wollongong - Port Kembla Harbour region. The horizontal axis in Figure 3.6 is scaled to 900 min/$\tau$ where $\tau$ is the period and the vertical axis being a logarithmic scale of the power. Dominant periods of Figure 3.6 are at 69.2, 52.9, 37.5, 26.4, 18, 13.4 and 10.8 mins which can be compared to the periods found in Table 3.1. Also, Clarke (1979) has carried out an analysis of current meter readings from Wallaga Lake, N.S.W. Australia, showing that significant wave energy packets are concentrated at periods of 52, 33, 22.5, 14.8 and 10.9 mins. It can be seen that the periods 31 and 10 mins, from Table 3.1 for $\alpha = 6.25 \times 10^{-3}$, deviate slightly from the physical results reported by...
FIGURE 3.6 A power spectrum for a current meter recording starting at 2200hrs on 24/10/75 and ending at 1200hrs on 25/10/75 with predicted tide variations removed. Vertical axis is a logarithmic scale of power and the horizontal axis is scaled to 900min./r.
Clarke. Indeed, these two calculated periods also approximate closely the periods of waves with significant wave amplitudes recorded outside Port Kembla Harbour, N.S.W. Australia (Clarke (1979)), following days of strong southerly winds.

Results in Table 3.2 indicate significant wave amplitudes for all values of the shelf parameter \( \alpha \). A maximum amplitude of 24.26 cms is obtained for \( \alpha = 6.25 \times 10^{-3} \), where the fetch length is around 52 kms (this fetch length being within the limits for a southerly 'buster'). The speed of the wave generated is around 18 cms\(^{-1}\). The results in Table 3.2 strongly support the observations of pressure and current recordings obtained during storm activity near Sydney, N.S.W., Australia, which were made by Middleton, Cahill and Hsieh (1986). Their observations suggested the existence of edge waves, during storm activity, in the range of 40s to 17mins with amplitudes of approximately 20cms and velocities of around 10cms\(^{-1}\).

Worthy (1984(a)) used a semi-infinite shelf profile to model the continental shelf off the east coast of Australia. In Chapter 5, the results obtained from the semi-infinite shelf profile will be compared to the results obtained in this chapter for a sloping beach profile using \( \alpha = 6.25 \times 10^{-3} \).
CHAPTER FOUR

CLASS II EDGE WAVES ON A CONVEX EXPONENTIAL SHELF

4.1 INTRODUCTION

The importance of the effect of the bottom topography on ocean currents was first studied by Ekman (1923) who showed that the geostrophic current in the Northern Hemisphere deflected to the right when passing over a bottom contour where the depth of the ocean decreases and to the left where the depth of the ocean increases. The deflections being the opposite in the Southern Hemisphere. The magnitude of deflections of a current is strongest either when the current passes over the shallowest part of a submarine ridge or when the current passes over a trough of the deepest water.

Using a non-homogeneous ocean model, Sverdrup (1941) was able to make some modifications to Ekman’s results. He showed that the geostrophic currents have zero deflection at the top of a submarine ridge instead of having the largest deflection as predicted by Ekman. Using an inhomogeneous and viscous ocean model, Neumann (1960) was able to produce both Ekman and Sverdrup types.

In the development of edge wave theory, authors such as Stokes (1846), Ekman (1923), Sverdrup (1941) and Reid (1958) used simplified models for the continental shelf region. The simplest model for the continental shelf, which is the step shelf profile, either finite or semi-infinite in extent, was used by Snodgrass et al (1962), Aida (1967) and Larsen (1969). The
sloping shelf model is also a simplified model of the continental shelf, and, was used by Ursell (1952), Reid (1958) and Robinson (1964) as a shelf profile to demonstrate properties of waves in the ocean.

Attempts at improved shape modelling of the continental shelf were given by Hidaka (1935) and Olson (1952), who considered standing waves on a semi-infinite sheet. Their depth profiles had the advantage of allowing for variations in depth at the coastline as well as for the initial slope at the shore. It was Robinson’s (1964) article which prompted extensive investigation into new ways of modelling of the continental shelf regions.

In 1968, Mysak discussed the effects that shelf width and shelf truncation had on ocean waves and introduced the concept of Class I and Class II edge waves. Mysak showed, in particular, that the shelf width played an increasingly important role in the determination of the wave modes as the longshore wave length increased beyond the shelf width.

The depth profile of the continental shelf used by Ball (1967) was of the form

$$h(x) = h_0 \left(1 - e^{-ax}\right). \quad (4.1)$$

Huntley and Bowen (1973) showed that this profile was found to be quite useful for modelling some near shore beach profiles.

Also, by using the exponential profile (4.1), Ball was able to obtain Kelvin waves and Reid’s edge waves as limiting cases of the exponential profile.

An important development in the theory of edge waves was the use of an exponentially varying depth profile,

$$h(x) = h_0 e^{ax}, \quad (4.2)$$

used by Buchwald and Adams (1968) in the discussion of non-divergent shelf waves (Class II waves). They showed that the shelf profile, (4.2), was a good approximation to the
continental shelf off the east coast of Australia. Consequently, the analytic results were reinforced, physically, from data obtained from measurements along the east coast of Australia. Buchwald and Adams (1968), also discussed the importance of wind stress as the principal driving mechanism of shelf waves. Further, Adams and Buchwald (1969) showed that the correlation between sea level and pressure changes analysed by Hamon (1962, 1963) were due to the correlation between changes in wind stress and changes in pressure.

Gill and Schumann (1974) used the simple technique of the method of characteristics to solve the wave equation and to verify the results of Adams and Buchwald in 1968 and 1969. Gill and Schumann also considered the effects of wind stress variations on Class II waves which propagate along the coast and their results compared favourably to the data measurements made by Moores and Smith (1968) on sea level variations on the coast of Oregon.

Clarke and Louis (1975) presented a full analysis of Buchwald and Adams profile, in which they discussed the effects of shelf truncation and the perturbation of shelf parameters on both classes of edge waves. Most results agree with those obtained by Mysak, although it was found that the semi-infinite exponential shelf did not support quasi-geostrophic (Class II) edge waves. Mæland (1983) supported this result. He showed, using a sloping shelf profile and Laplace transforms, that the infinite shelf did not support discrete poles and hence did not support Class II edge waves.

Instead of obtaining particular wave solutions for a given shelf profile, Shaw and Paskausky (1981) discussed what topographies would support trapped wave solutions. They found a family of three topographies which would support trapped edge wave solutions. One shelf profile being concave upward and the other two profiles being concave downward. In the limiting case, the concave upward shelf profile becomes the same profile (4.1), as that used by Ball (1967).
In Chapter 4, there is a re-examination of the work done by Adams and Buchwald (1968) using Fourier methods. In Sections 4.2 and 4.3, respectively, the propagation of free and forced Class II edge waves over the convex exponential profile which was used by Adams and Buchwald (1968) to model the east coast of Australia will be discussed. However, in Section 4.2.1, the continental shelf profile will be truncated at the edge of the shelf. The semi-infinite continental shelf will be discussed in Section 4.2.2.

4.2 FREELY PROPAGATING CLASS II EDGE WAVES

As discussed in Chapter 2, edge waves along a continental shelf are considered to be standing waves normal to the coast and propagating along shore with wave number $s$. Then the wave height, $\zeta(x)$, has the same form as (2.32) and satisfies the differential equation (2.34).

To solve (2.34) using the appropriate boundary conditions, a specific continental shelf profile is required for (2.31). Two shelf profiles will be considered, namely, the truncated and semi-infinite exponential shelf.

4.2.1 THE TRUNCATED EXPONENTIAL SHELF

Let the continental shelf profile have the form:-

$$h = \begin{cases} 
  h_0e^{ax} & 0 \leq x \leq L \\
  H & x > L 
\end{cases}$$

(4.3)
where $h_0$ is the depth of the ocean at the coast, $H$ is the depth of the ocean at the edge of the continental shelf and $a$ is the shelf parameter determined by the values of $h_0$ and $H$.

Since the shelf profile is fixed at the coast and truncated at the edge of the shelf then the appropriate boundary conditions for trapped edge waves associated with (2.35) will be (2.36) and (2.39).

Using (4.3), the transformations

$$X = e^{-ax/2}$$

and

$$Z = X^{-1}F$$

(2.34) becomes

$$X^2 \frac{d^2Z}{dX^2} + X \frac{dZ}{dX} + \left( \kappa^2 X^2 - \nu^2 \right) Z = 0$$

where

$$\kappa^2 = 4 \frac{\sigma^2 - f^2}{a^2 gh_0}$$

and

$$\nu^2 = \frac{4s^2}{a^2} + \frac{4sf}{a\sigma} + 1$$

Consequently, the general solution of (2.34) is

$$F = AXJ_\nu(\kappa X) + BXY_\nu(\kappa X)$$

where $Y_\nu(\kappa X)$ is the Bessel function of the first or second kind depending on whether $\nu$ is an integer or not. It should be noted that the constants $A$ and $B$ of (4.9) are taken to be complex conjugates ($C \pm iD$), so that a real solution to (4.9) exists.
The general solution (4.9) is of particular importance when $\kappa^2 < 0$ and $\nu^2 < 0$, since the solution corresponds to continental shelf waves.

Hence, continental shelf waves have the form

$$ F = C \left[ \int_0^{\infty} e^{\mu \cos(\theta) \cosh(\delta t) \sinh(\delta \pi)} d\theta - \sinh(\delta \pi) \int_0^{\infty} e^{-\mu \cosh(t) \sin(\delta t)} dt \right] $$

$$ + D \left[ \sinh(\delta \pi) \int_0^{\infty} e^{-\mu \cosh(t) \cos(\delta t)} dt \right] $$

(4.10)

where $\mu^2 = -\kappa^2$ and $\delta^2 = -\nu^2$. By neglecting the horizontal divergence (i.e. $\zeta_\nu$) in the continuity equation (2.22), Buchwald and Adams (1968) obtained a simplified general solution of (4.10) which is of the form

$$ F = \begin{cases} 
  e^{-ax/2}(A \cos(\xi x) + B \sin(\xi x)) & x < L \\
  C e^{-ax} & x \geq L
\end{cases} $$

(4.11)

where

$$ \frac{\xi^2}{a^2} = \frac{-s f}{a \sigma} - \frac{s^2}{a^2} - \frac{1}{4} > 0 $$

After substituting (4.3) and using the transformation (4.4), the boundary conditions (2.35) and (2.36) become

$$ \frac{F}{|F|} \text{ finite at } X = 1 $$

(4.12)

$$ X \frac{dF}{dX} + \frac{2sf}{a \sigma} F = 0 , \quad X = 1 $$

at the coast and

$$ X \frac{dF}{dX} - \frac{2r}{a} F = 0 , \quad X = \Delta $$

(4.13)
at the edge of the continental, where
\[ \Delta = e^{-aL/2} \]  
(4.14)

and \( r \) is defined in (2.40).

Let
\[ \alpha_i = \kappa J_i'(\kappa) + \left( 1 + \frac{2sf}{a\sigma} \right) J_i(\kappa), \]
(4.15)
\[ \beta_i = \kappa \Delta J_i'(\kappa \Delta) + \left( 1 - \frac{2r}{a} \right) J_i(\kappa \Delta) \]
where \( i = 1, 2 \) and \( J_1(X) = J_\nu(X) \) and \( J_2(X) = Y_\nu(X) \).

The dispersion relation obtained after applying the boundary conditions (4.12) and (4.13) to the general solution (4.10) has the form
\[ \alpha_1 \beta_2 - \alpha_2 \beta_1 = 0, \]
(4.16)

where \( \alpha_i \) and \( \beta_i, \ i = 1, 2 \) are defined in (4.15). After applying the boundary conditions (2.37) and (2.39) to (4.11), the dispersion relation for continental shelf waves corresponding to (4.11) is
\[ \tan(\xi L) = -\frac{\xi}{s + a/2} \]  
(4.17)

where \( \xi \) is defined in (4.11). The dispersion relation in (4.17) is similar to that derived by Adams and Buchwald (1968), namely,
\[ \tan(\xi L) = -\frac{\xi}{b} \]  
(4.18)

where \( b = 2a \). Since the boundary conditions defined in (4.12) and (4.13) are slightly different from those used by Adams and Buchwald (1968), there is a correction term in the denominator on the right hand side of (4.17).
Using the boundary conditions (4.12) and (4.13), Clarke and Louis (1975) have shown that for given values of $\lambda$, the values of $\mathcal{F}$ obtained from the dispersion relation (4.17) for the general solution (4.10) and the dispersion relation (4.18) obtained by Adams and Buchwald (1968) agree within three significant figures. Consequently, the omission by Buchwald and Adams (1968) of horizontal divergence for low frequency quasi-geostrophic edge waves, where $\varepsilon = \frac{f^2L^2}{8\bar{h}} \ll 1$, (where $\bar{h}$ = average depth of the ocean) neglect was valid. However, in experimental laboratory studies, Caldwell, Cutchin and Longuett-Higgins (1972) showed that the neglect of horizontal divergence caused large percentage deviations in their dispersion analysis. In the laboratory models, where (4.17) is not satisfied, the dispersive properties of Class II edge waves are given by (4.16).

The wave height, $\zeta$, for freely propagating edge waves at the coast (i.e. $X = 1$) has the form:

$$\zeta = \frac{A}{\alpha_2} \{ \alpha_2 J_\nu(\kappa) - \alpha_1 J_{-\nu}(\kappa) \} \cos(\sigma y - \omega t)$$

which is a progressive wave travelling along the coastline in the positive $y$ direction when $\sigma > 0$ and the negative $y$ direction when $\sigma < 0$.

### 4.2.2 SEMI-INFINITE EXPONENTIAL SHELF

If a semi-infinite shelf profile is used, i.e.

$$h = h_0 e^{ax}, \quad 0 \leq x < \infty,$$

instead of the truncated shelf given by (4.3), the general solution obtained in (4.9) is still valid. For coastally trapped waves over a semi-infinite shelf profile, the boundary condition (2.38) must be used.

Upon substituting (4.19) and using the transformation (4.4), (2.38) becomes

\[ F \to 0 \quad as \quad X \to 0 \quad . \tag{4.20} \]

Applying the boundary condition (4.20) to the general solution (4.9) it is found that because of the behaviour of \( J_{-\nu}(X) \) at \( X = 0 \), \( B \) must be zero whenever \( \nu^2 \leq 1 \). For all other values of \( \nu \), (4.20) is already satisfied, which means that the boundary condition (4.20) is redundant. An analysis of this problem by Louis (1975) showed that the following boundary conditions, for trapped waves, are required in the deep ocean; namely

\[ hF \quad \text{and} \quad h \frac{dF}{dx} \to 0 \quad \text{as} \quad x \to \infty \quad . \]

or

\[ X^{-2}F \quad \text{and} \quad X^{-2} \frac{dF}{dx} \to 0 \quad , \quad X \to 0 \quad . \tag{4.21} \]

Using the transformation (4.4), Louis (1975) showed that the only trapped edge waves on the semi-infinite shelf are the inertiogravitational waves (i.e. Class I high frequency waves where \( \sigma^2 > f^2 \)). These waves satisfy the dispersion relation

\[ \alpha_1 = \kappa J_\nu(\kappa) + \left( 1 + \frac{2sf}{\alpha \sigma} \right) J'_\nu(\kappa) = 0 \tag{4.22} \]

and have a long wavelength cut-off given by \( \nu > 1 \) i.e. \( s/a > f/\sigma \).

Figure 4.1(a) shows the behaviour of the first three modes for the inertiogravitational edge waves described in (4.22), as described by Clarke and Louis (1975). Figure 4.1(b) shows the behaviour of the fundamental mode of (4.22) using various shelf parameters.

Figure 4.2 shows the dispersion curves for inertiogravitational edge waves which satisfy (4.16), as discussed by Reid (1958), and the high frequency cut-off defined by (2.40), using parameters appropriate to the east coast of Australia near Sydney. As discussed in Chapter
FIGURE 4.1(a) Graph of the inertiogravitational edge wave dispersion relation defined in (4.22), using the semi-infinite shelf profile (4.19), (a) for the first 3 modes with shelf parameters $h_0 = 70\, m$, $H = 5\, kms$ and $L = 80\, kms$. (b) First mode of (4.22) using shelf parameters $h_0 = 70\, m$, $H = 5\, kms$ and $L = 40\, kms$ and $L = 80\, kms$. 
FIGURE 4.2 Graph of the inertiogravitational edge wave dispersion relation defined in (4.16), using the truncated shelf profile (4.3), for the first 3 modes with shelf parameters $h_0 = 70\text{m}$, $H = 5\text{kms}$ and $L = 80\text{kms}$. The high frequency edge wave cut-off is shown by the line $r^2 = 0$. 
3, these Class II edge waves possess the characteristic Coriolis split in frequency for waves travelling up the coast versus waves travelling down the coast.

4.3 WIND INDUCED CLASS II EDGE WAVES

The linearized equations of motion and continuity equation, using shallow water theory, are derived in Sections 2.2 and 2.3, respectively. Neglecting frictional forces, then (2.13) becomes

\[ u_t - f v = -g P_x + \frac{\tau_x}{\rho h} \]

\[ v_t + f u = -g P_y + \frac{\tau_y}{\rho h} \]

where \( \tau_x \) and \( \tau_y \) are the components of wind stress acting on the surface due to geostrophic wind, \( \rho \) is the density of water and

\[ P = \zeta + \frac{p_a}{\rho} \]

is the non-isostatic part of sea level, \( p_a \) represents the atmospheric pressure. Buchwald and Adams (1968) have shown that the Coriolis term can be taken to be constant provided the time and length scales are less than 10 days and 100kms respectively. The wave motion, as discussed by Gill and Schumann (1974) can be considered horizontally non-divergent provided \( \epsilon = \frac{f^2 L^2}{gh} \ll 1 \) holds. Hence, the last term in (2.22) can be neglected. It is also assumed that the waves under consideration are generated by the longshore component of wind stress in the shelf region only. Hence, assuming a harmonic variation in time i.e. \( e^{-i\sigma t} \) for all dependent variables, the equations of motion in (4.23) reduce to
\[ i \sigma u + fv = gP_x \]  
\[ -i \sigma v + fu = -gP_y + \frac{\tau^y}{h} \]  

(4.24)

(\text{where} \ \tau^y \ \text{is divided by} \ p \ \text{along with the continuity equation:})

\[ (hu)_x + (hv)_y = 0 \]  

(4.25)

The time reduced velocities from (4.24) are

\[ u = \frac{1}{\sigma^2 - f^2} \left\{ -i \sigma gP_x + gfp_y - \frac{f\tau^y}{h} \right\} \]  

(4.26)

\[ v = \frac{1}{\sigma^2 - f^2} \left\{ -i \sigma gP_y - gfp_x + \frac{i \sigma \tau^y}{h} \right\} \]  

Substituting (4.26) into (4.25) gives:

\[ (hP_x)_x - \frac{f}{i \sigma} P_y \frac{dP}{dx} + hP_{yy} = \frac{\tau^y}{g} \]  

(4.27)

Let

\[ F\{P\} = \overline{P} = \int_{-\infty}^{\infty} e^{-isyP} \, dy \]  

(4.28)

and

\[ F\{\tau^y\} = T \]

then the Fourier transform of (4.27) is

\[ \frac{d}{dx} \left( h \frac{d\overline{P}}{dx} \right) - \left( \frac{fs \, dh}{\sigma \, dx} + hs^2 \right) \overline{P} = \frac{is}{g} T \]  

(4.29)

Only a truncated shelf profile supports trapped edge waves (Louis (1975) and Mæland (1983)). Therefore, consider the shelf profile described in (4.3). The general solution to (4.29)
is obtained by substituting (4.3) into (4.29). Hence, (4.29) reduces to

\[
\frac{d^2 \overline{P}}{dx^2} + a \frac{d \overline{P}}{dx} - \gamma \overline{P} = \frac{isT}{gh_0} e^{-ax},
\]

(4.30)

where

\[
\gamma = -sa\left(\frac{f}{\sigma} + \frac{s}{\bar{a}}\right)
\]

for \(0 < x < L\).

The general solution to (4.30) is

\[
\overline{P} = e^{-ax/2}(A e^{avx/2} + B e^{-avx/2}) - \frac{isT e^{-ax}}{gh_0 \gamma},
\]

(4.31)

where \(v^2 = 1 + \frac{4\gamma}{a^2}\) is defined in (4.8).

Using (2.35) and (2.36) the boundary condition at the coast \((x = 0)\) becomes, after substituting (4.3) and using (4.28),

\[
\frac{d \overline{P}}{dx} - \frac{sf}{\sigma} \overline{P} = 0, \quad \text{at} \quad x = 0.
\]

(4.32)

However, at the edge of the continental shelf the boundary condition is

\[
\frac{d \overline{P}}{dx} + s \overline{P} = 0 \quad \text{at} \quad x = L.
\]

(4.33)

The procedure used to obtain the boundary condition in (4.33) is found in Appendix 4A.

By substituting (4.31) into (4.32) and (4.33), the constants \(A\) and \(B\) are determined by the matrix equation:

\[
\begin{bmatrix}
  m(v) - \frac{sf}{\sigma} & m(-v) - \frac{sf}{\sigma} \\
  (m(v) + s)e^{m(v)L} & (m(-v) + s)e^{m(-v)L}
\end{bmatrix}
\begin{bmatrix}
  A \\
  B
\end{bmatrix} = -
\begin{bmatrix}
  R_1 \\
  R_2
\end{bmatrix}
\]

(4.34)
\[ m(\nu) = \frac{a}{2}(\nu - 1) + \nu + \frac{2sf}{a\sigma} \]

and

\[ R_1 = -\frac{isT}{g\nu h_0 y} \left( a + \frac{sf}{\sigma} \right) , \]

\[ R_2 = -\frac{isT}{g\nu h_0 y} (a - s)e^{-aL} . \]

It should be noted that the determinant of the matrix co-efficients in (4.34) is the dispersion relation for freely propagating edge waves as obtained in (4.17) provided \( \nu \) is purely imaginary.

If the wind stress is known, \( P \) can be determined by evaluating

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} P e^{is(y-\eta)} \, ds . \]  

(4.35)

**4.4 GENERAL SOLUTION**

The wind stress model which is assumed to be a function of \( y \) alone, will be represented by an envelope of forces with various strengths located at different points. A unit of impulse will be represented by:-

\[ \tau = \tau_0(\eta) \delta(y - \eta) \]  

(4.36)

where \( \delta(y - \eta) \) is the Dirac delta function, so that the total wind stress is given by:-

\[ \tau_T = \int_{-\infty}^{\infty} \tau \, d\eta . \]  

(4.37)

It should be noted that \( \tau \) is a function of \( e^{-i\sigma t} \).
Using (4.36), $T$ can be found from (4.37) and on substitution into (4.32), $\overline{P}(x = 0)$ becomes:

$$
\overline{P} = iW \left\{ \frac{\left( \frac{a}{2}(a - s) e^{-aL/2} + (1 + \frac{f}{\sigma}) s D_s(s, \sigma) - (a + \frac{if}{\sigma}) \alpha_1(s, \sigma) \right) \tau_0(\eta)}{s \left( \frac{f}{\sigma} + \frac{f}{\sigma} \right) D_s(s, \sigma)} \right\}, \quad (4.38)
$$

where

$$
W = \frac{-1}{agh_0(1 + \frac{f}{\sigma})}, \quad (4.39)
$$

$$
\alpha_1(s, \sigma) = (s - \frac{a}{2}) \sinh \left( \frac{aL \nu}{2} \right) + \frac{av}{2} \cosh \left( \frac{aL \nu}{2} \right),
$$

and

$$
D_s(s, \sigma) = (s + \frac{a}{2}) \sinh \left( \frac{aL \nu}{2} \right) + \frac{av}{2} \cosh \left( \frac{aL \nu}{2} \right). \quad (4.40)
$$

Using contour integration to evaluate (4.35), we let

$$
l = \int_\Gamma \overline{P} e^{is(y-\eta)} \, ds \quad (4.41)
$$

where $\overline{P}$ is defined in (4.38) and $\Gamma$ is an appropriate contour.

To evaluate (4.41), the branch points and poles of the integrand must be determined. Firstly, branch points of the integrand are located at $\nu = 0$. Therefore, the two branch points, $b_1$ and $b_2$, are given by

$$
b_1 = \frac{a}{2} \left\{ -\frac{f}{\sigma} + \sqrt{\frac{f^2}{\sigma^2} - 1} \right\}
$$

and

$$
b_2 = \frac{a}{2} \left\{ -\frac{f}{\sigma} - \sqrt{\frac{f^2}{\sigma^2} - 1} \right\}.
$$

If $a > 0$, then both branch points are positive when $\frac{f}{\sigma} < 0$ and negative when $\frac{f}{\sigma} > 0$, while if $a < 0$ the reverse inequalities hold. The branch points lie on the real axis.
when considering sub-inertial frequencies i.e., \( \sigma^2 < f^2 \). Since shelf waves have sub-inertial frequencies and travel in a northerly direction on the east coast of Australia (Hamon (1966)), \( \sigma^2 \) will be considered to be less than \( f^2 \). \( \sigma \) will be taken to be positive. Since \( f \) is negative in Australian latitudes, i.e. in the Southern Hemisphere it will be assumed that in the remainder of this Section that \( \sigma \) is positive and \( f \) is negative.

Secondly, the poles of the integrand in (4.41) are located at \( s = 0 \),

\[
\frac{s}{a} = -\frac{f}{\sigma} \tag{4.42}
\]

and

\[
D_s(s, \sigma) = 0 \tag{4.43}
\]

where \( D_s(s, \sigma) \) is defined in (4.40).

When \( v^2 \geq 0 \), the roots of (4.43) are at \( v = 0 \), which corresponds to the branch points of the integrand in (4.41). Consequently, no solution exists when \( v^2 \geq 0 \). However, when \( v^2 < 0 \), (4.43) reduces to

\[
tan\left(\frac{aL\Omega}{2}\right) = \frac{-\Omega}{1 + 2s/a} \tag{4.44}
\]

where \( \Omega^2 = -v^2 \). By letting \( \xi = a\Omega/2 \), (4.44) becomes the dispersion relation for the freely propagating continental shelf waves as described in Section 4.2. Using (4.44), it is found that if \( a > 0 \) and \( \frac{f}{\sigma} < 0 \) then the roots, \( s \), of (4.43) are all positive. If, however, \( a < 0 \) then the roots of (4.43) are all negative. The inequalities are reverse when \( \frac{f}{\sigma} > 0 \).

By letting \( b = (s + \frac{\xi}{2})L \) in (4.44), it can be easily seen from Figure 4.3(a) that there exists a root \( \xi_n \) in each interval

\[
(n - \frac{1}{2})\pi < \xi_nL < n\pi \quad .
\]
FIGURE 4.3(a)

Graph shows that the roots of \( \tan X = -X/b \) are within the intervals \( \left( \frac{2n-1}{2} \pi, \frac{2n+1}{2} \pi \right), \ n = 1, 2, 3, 4 \).

FIGURE 4.3(b)

The first 3 modes of the dispersion equation defined in (4.44), using Buchwald and Adams (1968) shelf parameters \( h_0 = 23 \text{m}, \ H = 5 \text{kms} \) and \( L = 50 \text{kms} \).
where \( n = 1, 2, \ldots \) and that

\[
\xi_nL \to \left( n - \frac{1}{2} \right)\pi \quad \text{as} \quad n \to \infty .
\]

Let \( s = s_n, \quad n = 1, 2, 3, \ldots \) be the roots of (4.44), then Figure 4.3(b) shows the dispersion relationship between \( \frac{s}{\sigma} \) and \( \frac{\sigma}{f} \) for the first three modes. Using the same shelf parameters, \( h_0 = 23 \text{m}, \quad L = 80 \text{km} \) and \( H = 5 \text{kms} \), as used by Adams and Buchwald (1966), then \( a \) will be considered to be positive. Consequently, if it is assumed that \( f < 0 \) and \( \sigma > 0 \) then all roots are real and lie on the negative real axis.

Also, Figure 4.3(b) shows that there is a frequency cut-off for each mode. For example, the first mode has a cut-off at approximately \( .63f/s^{-1} \), the second mode at \( .35f/s^{-1} \) and the third mode at \( .25f/s^{-1} \). It should be noted that for particular frequencies in Figure 4.3(b), there are two values (wave numbers) for each mode. The shorter wavelength waves, i.e., \( sL \gg 1 \), have the group velocity of each mode in the opposite direction to the phase velocity. The group velocity changes sign by passing through zero at an intermediate wave number. This result reinforces the work done by Huthnance (1975) who showed that if \( h'(x)/h(x) \) is bounded, for any shelf profile, then the phase velocity, \( c \), and the group velocity, \( c_g \), are of opposite sign for some \( s \). When \( c_g = 0 \), then the wave does not propagate and, consequently, resonant behaviour is likely to occur at those wave numbers which are associated with zero group velocity. Cutchin and Smith (1973) observed low frequency oscillation on the Oregon shelf with a shelf profile similar to that of (4.3), and have shown that there is a correlation between peaks on an energy spectrum with frequencies near \( c_g = 0 \).

Buchwald (1977) addressed the case of the diffraction of shelf waves by an irregular coast line and discussed that the shorter wave length shelf wave may be the source by which wave energy can be scattered backwards. Consequently, the scattering of shelf waves by topography could be a mechanism through which energy can be transferred between different scales of motion and in different directions. Using the same exponential shelf as Buchwald (1977), Chao
et al (1979) concurred with Buchwald's view by showing that a continuous spectrum of shelf waves propagate over an irregular topography scattering backwards towards the higher wave number end of the spectrum.

Since the branch points and poles of the integrand in (4.41) lie on the real axis, the radiation condition is used. Thus, $\sigma$ is replaced by $\sigma - i \varepsilon$, so that the edge wave decreases in time as $\varepsilon$ tends to infinity. The effect of using the radiation condition is to ensure that there are only outgoing waves and thus the branch points and poles are removed from the real axis. Consequently, the contour $\Gamma$ for the integral $I$, in (4.41), can incorporate the real axis. Once a solution is obtained, $\varepsilon$ is then allowed to tend to zero.

To evaluate (4.41), it is necessary to move the poles from the real axis. Consider, the poles, $s_n$, obtained from (4.43). It follows that for each $s$,

$$dD_s(s, \sigma) = \frac{\partial D_s}{\partial s} ds + \frac{\partial D_s}{\partial \sigma} d\sigma = 0.$$ (4.45)

Note that the group velocity for the shelf waves can be obtained from (4.45). Since $d\sigma = -i \varepsilon$, the dissipation of poles is found from the sign of $ds$, where

$$ds = i \varepsilon \left( \frac{\partial D_s}{\partial \sigma} \right) \left( \frac{\partial D_s}{\partial s} \right).$$ (4.46)

Values of (4.46) were computed using $|a| = 6.73 \times 10^{-5}$, $L = 80kms$ and $f = -8.133 \times 10^{-5}$, which are comparable with the values used by Adams and Buchwald (1968). Letting $\nu^2 = -\Omega^2$, it was found that the poles of (4.43) with negative group velocity move to the lower half plane, while the poles with positive or zero group velocity move to the upper half plane.

A similar procedure is used to remove the pole obtained from (4.42) and the branch points $b_1$ and $b_2$ from the real axis. Using $a < 0$ and $\frac{f}{\sigma} < 0$, it can be shown that the pole in (4.42) and $b_1$ move to the upper half $s$ plane, whereas, $b_2$ moves to the lower half plane. Figure 4.4
shows the displaced poles and the appropriate contour $\Gamma$ which closes the upper and lower half plane. When $y < \eta$, $\Gamma$ is the contour which closes the lower half plane by $\Gamma_1$ and $\Gamma_1'$, the two sides of the branch line $\Gamma_b$ and the circular path, $\Gamma_c$, about the branch point $b_2$. For $y > \eta$, the contour $\Gamma$ is closed in the upper half plane by $\Gamma_2$ and $\Gamma_2'$, the two sides of the branch line $\Gamma_b'$ and the circular path $\Gamma_c'$.

The poles $s_n, \quad n = 1, 2, \ldots$, with negative group velocity contribute to $I$, only for $y < \eta$. Appendix 4B, shows that the integrals over connecting regions tend to zero as $|s| \to \infty$ and the integral around the small circle about the branch point tends to zero as the radius $|s| \to 0$. The two integrals along the branch lines add to zero.

Therefore, (4.35) becomes

$$P = i \sum \text{Residues} .$$

Hence,

$$P = -W \left\{ \sum_{n=1}^{N} R_n e^{i s_n(y-\eta)} \right\} \tau_0(\eta) H(\eta - y) , \quad (4.47)$$

where $R_n$ is the residue from the pole $s = s_n$ which was obtained from (4.44) and $N$ is an integer. Therefore,

$$R_n = \frac{F_n}{s_n \left( \frac{a^2}{\sigma} + \frac{4}{\sigma} \int D_3(s, \sigma) \right) \big|_{s=s_n} } , \quad (4.48)$$

where

$$F_n = \left( \frac{a \Omega}{2} (a - s_n) e^{-aL/2} - (a + \frac{s_n f}{\sigma}) \alpha_1(s_n, \sigma) \right) .$$

and $\alpha_1(s_n, \sigma)$ is defined in (4.39).

Now,

$$P_T = \int_{-\infty}^{\infty} P \, d\eta . \quad (4.49)$$
Figure 4.4: The most suitable contour for the Fourier inversion of (4.41), where the branch points are given by

\[ b_1 = \frac{a}{2} \left( -\frac{I}{\sigma} + \sqrt{\frac{I^2}{\sigma^2} - 1} \right), \]

\[ b_2 = \frac{a}{2} \left( -\frac{I}{\sigma} - \sqrt{\frac{I^2}{\sigma^2} - 1} \right) \]

and the poles, \( s_n (n=1,2,3) \), are defined in (4.44).
4.5 RESULTS AND DISCUSSION

Varying shapes of the wind stress function can be used in (4.36) to calculate (4.49).

(a) Consider the wind stress model of the form

$$\tau_0(y) = \begin{cases} Y_0 & -L \leq y \leq 0 \\ 0 & \text{o.w.} \end{cases}$$

(4.50)

where $Y_0$ is the constant amplitude of the wind stress and $L$ is the wind fetch. This wind stress is a rectangular function or square wave forcing and is similar to that used by Adams and Buchwald (1969). The graph of (4.50) can be found in Figure 3.4(a). Substituting (4.50) into (4.49) gives

$$P_T = -Y_0 W \int_{-L}^{y} \sum_{n=1}^{N} R_n e^{i\alpha_n(y-\eta)} d\eta .$$

(4.51)

Recall that the solution $P_T$ is a function of $e^{-i\sigma t}$. Therefore, the real part of (4.51) including the time variation is :-

$$P_T^y = 2Y_0 W \sum_{n=1}^{N} C_n \sin(s_n z) \cos(s_n z - \sigma t)$$

(4.52)

where

$$C_n = \frac{R_n}{s_n} ,$$

$R_n$ is defined in (4.48) and $z = \frac{y}{2}$. Note that (4.52) can be regarded as a progressive wave whose amplitude is modulated in time.

The approximate values of the wave number and coefficients $C_n$ of (4.52) for the first three modes are given in Table 4.1 for the given value of $X = \frac{\phi}{\lambda} = 7$.

The results in Table 4.1 agree, within two significant figures, with the results obtained by Adams and Buchwald (1968) for the wave numbers for the first three modes. Hence, it can
be deduced that the extra term in the dispersion relation of (4.44) is simply a correction term, as discussed by Louis (1975). Further, Table 4.1 shows that the second and third modes have amplitudes which are small enough to be neglected. Therefore, (4.52) becomes:

$$p_T^y = \frac{2Y_0 C_1}{agk_0 \left( 1 + \frac{f}{\sigma} \right)} \sin \left( \frac{s_L y}{2} \right) \cos \left( \frac{s_L y}{2} - \sigma t \right).$$  \hfill (4.53)

| MODE | $\Omega_n$ | $L$ | $s_n L$ | $|C_n|$ |
|------|-----------|-----|---------|-------|
| 1    | 2.46      | -0.35 |         | 718.  |
| 2    | 5.31      | -0.96 |         | 5.21  |
| 3    | 8.38      | -2.18 |         | 1.0   |

**TABLE 4.1** Approximate values of the wave number $s_n$ obtained from (4.44) and the corresponding amplitude $C_n$ for the first three modes.
From (4.53), the maximum amplitude reached by the first mode is:–

\[
\text{Amplitude} = \left| \frac{2C_1 Y_0}{a g h_0 \left( 1 + \frac{f}{\sigma} \right)} \right|. \tag{4.54}
\]

A value of \( Y_0 \) is required to determine the amplitude in (4.54). Using a geostrophic wind with speed of 2.6m/sec, \( Y_0 \) is approximately equal to .1 dyne/cm^2, which is a reasonable estimate for a longshore breeze. At latitude 35°S on the east coast of Australia the value of \( f \) is about \(-8.33 \times 10^{-5}\) sec\(^{-1}\). Hence, given that \( X = \frac{\phi}{f} = 7 \), (which is equivalent to a period of 6 days) the amplitude in (4.53) is approximately equal to 15.7cms. The amplitude obtained for the first mode is of the same order of magnitude of typical sea level variations found in measurements made by Mysak and Hamon (1969) and Caldwell et al (1972). Although, different boundary conditions were used, the amplitude is marginally different from that obtained by Adams and Buchwald (1968) using the same shelf parameters.

Table 4.2 shows the amplitudes obtained in (4.54) by varying the initial depth, \( h_0 \), of the ocean at the coast. Also, Table 4.2 shows that a decrease in the initial depth at the coast increases the amplitude of the waves.

In Table 4.3 shows how the amplitude in (4.54) changes when the shelf width, rather than the initial depth, is varied. It can be seen that as the shelf width increases the amplitude of the wave increases.

The maximum amplitude of (4.53) is reached when \( y = \pi/s_1 \), which is a distance of about 700kms from \( y = 0 \).
<table>
<thead>
<tr>
<th>$h_0$(m)</th>
<th>Amp(cms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>15.7</td>
</tr>
<tr>
<td>45</td>
<td>.5</td>
</tr>
<tr>
<td>70</td>
<td>.2</td>
</tr>
</tbody>
</table>

**TABLE 4.2** The amplitude from (4.54) for the first mode using various values of $h_0$ with $L = 80$ kms and $Y_0 = .1$ dyne/cm$^2$. 
TABLE 4.3 The amplitude for the first mode using (4.54), \( h_0 = 23 \text{m} \) and \( Y_0 = 0.1 \text{dyne/cm}^2 \).

(b) Consider the wind stress model of the form

\[
\tau_0(y) = \begin{cases} 
  Y_0 e^{iky} & -L \leq y \leq 0 \\
  0 & \text{o.w.}
\end{cases}
\]  

(4.55)

Combined with the \( e^{-i\alpha t} \) factor, the wind stress model in (4.55) is a progressive wave with wave number \( k \) and with the same frequency, \( \sigma \), as \( P \). Model (b) is similar to that used by Gill and Schumann (1974), and is a more realistic model of wind stress then that given by a constant wind stress in Model (a).

Substituting (4.55) into (4.47) and using (4.49) gives:-
\[ P_T^y = -2Y_0W \sum_{n=1}^{N} \frac{R_n}{(s_n - k)} \sin(s_n - k)z \{ \cos[(s_n - k)z - \sigma t] \} \]  \hspace{1cm} (4.56)

which is a progressive wave whose amplitude is modulated with time. When the forcing system moves with the free-wave speed, resonance occurs. Therefore, in the limiting case as \( k \to s_n \), (4.56) becomes

\[ P_T^y = -Y_0W \sum_{n=1}^{N} R_n \cos \sigma t \]  

This equation implies that the amplitude grows indefinitely as \( |y| \to \infty \), i.e. at large distances from the forcing. This appears to be a resonance effect.

Assuming that the first mode of (4.56) is the most significant mode then \( P_T^y \) can be approximated by

\[ P_T^y = \frac{2Y_0R_1 \sin((s_1 - k)z) \cos((s_1 - k)z - \sigma t)}{agh_0(1 + \frac{f}{\sigma})(s_1 - k)} \]  \hspace{1cm} (4.57)

Resonance occurs when \( k = s_1 \). Hence, in the limiting case when \( k = s_1 \), the maximum amplitude of the first mode, (4.57), is

\[ \text{Amplitude} = \left| \frac{Y_0C_1\pi}{agh_0(1 + \frac{f}{\sigma})} \right| \]  \hspace{1cm} (4.58)

where \( y = -L = \frac{a}{h_1} \) is used. The calculated values of the amplitude in (4.58) are determined by using \( Y_0 = .1 \) dyne/cm\(^2\) and \( X = \frac{Q}{f} = 7 \), are displayed in Table 4.4 and Table 4.5, for various shelf parameters.
<table>
<thead>
<tr>
<th>$h_0$(m)</th>
<th>Amp(cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>24.6</td>
</tr>
<tr>
<td>45</td>
<td>.78</td>
</tr>
<tr>
<td>70</td>
<td>.31</td>
</tr>
</tbody>
</table>

**TABLE 4.4** The amplitude from (4.58) using various values of $h_0$ with $L=80$ kms and $Y_0=1$ dyne/cm².

Using $k = s_1$, it is assumed that the wind stress is in resonance with the first mode of the freely propagating wave. If resonance does not occur, that is, $k \rightarrow s_1$ then the wind stress defined in (4.55) would behave as a moderating factor to the amplitude of the progressive wave in (4.57).
Table 4.5 The amplitude for the first mode using \((4.58)\), \(h_0=23\text{m}\) and \(Y_0=.1\text{ dyne/cm}^2\).

Comparing the results obtained in Tables 4.4 and 4.5 for the wind stresses used, it can be seen that both generate substantial amplitudes which are marginally greater than those produced by Adams and Buchwald (1968).

Chapter 5 examines high frequency (Class I) edge waves using wind stress as the forcing mechanism.
CHAPTER FIVE

CLASS I EDGE WAVES ON A CONVEX EXPONENTIAL SHELF

5.1 INTRODUCTION

The effects of a sloping shelf and wind forcing on Class I edge waves were discussed in Chapter 3. The sloping shelf profile is a valid approximation for most continental shelves when studying the propagation of edge waves with periods within the order of 10 mins or less. An example of edge waves with periods within 10 min or less are those long waves observed by Clarke, Keane and O’Halloran (1968) in Jervis Bay, N.S.W. Australia.

However, the topography of shelf plays an increasingly important role for waves with periods greater than 10 mins. Thus, for periods greater than 10 mins, a discussion on wave phenomena is required using a more accurate approximation for the continental shelf profile. For instance, Buchwald and Adams (1968) have shown that the east coast of Australia is suitably approximated by a convex exponential shelf. Hence, Viera and Buchwald (1982) and Worthy (1984(a) and (b)) used this convex exponential shelf profile in their discussion of Class I edge waves. In addition, authors, such as Clarke (1973), examined the phase and group velocities of high frequency waves using 3 different shelf profiles, that is, linear, concave and convex exponential shelf profiles. In particular, Clarke and Louis (1975) examined freely propagating Class I waves over the convex exponential shelf profile which was described by Buchwald and Adams (1968). Louis (1975), discussed the effects of shelf truncation on Class I waves. He showed that shelf truncation had little effect on Class I waves. In Section 5.2 of this
chapter, freely propagating Class I edge wave over a convex exponential shelf both truncated and semi-infinite in extent, will be discussed.

The generation of high frequency edge waves (Class I waves) by pressure disturbances has been examined by authors such as Greenspan (1956), Buchwald and de Szoeke (1973), Viera and Buchwald (1982) and experimental work by Middleton, Cahill and Hsieh (1986).

Using a sloping shelf, Greenspan (1956) showed that a resurgence motion occurred in the generation of Class I edge waves by pressure disturbances. The resurgence motion consisted of an infinite number of edge wave modes. The pressure disturbance assumed by Greenspan bears little resemblance to physical pressure fronts.

Using a travelling atmospheric pressure disturbance of constant speed in the longshore direction, modelled as a Delta function, Buchwald and de Szoeke (1973) considered a step shelf and obtained analytically periods and amplitudes for Class I waves. These Class I waves were similar to the very high frequency waves observed by Clarke, Keane and O'Halloran (1968). Buchwald and de Szoeke (1973) found that the amplitudes generated by the pressure disturbance may be appreciably larger than the amplitude of the pressure disturbance. Also, the analytical solution depended on the speed of the pressure disturbance being bounded by the speed of the long waves on the shelf and the speed of the waves in the ocean.

Using a convex exponential shaped shelf profile, Viera and Buchwald (1982) found that in the wake of pressure disturbances, having very high speeds, edge waves may be generated which have amplitudes appreciably larger than what would be expected by direct measurement.

In Sections 5.2 to 5.5, a general discussion of the propagation of free and forced Class I edge waves over a convex exponential continental shelf will be examined. Since Class I waves are supported by the truncated and semi-infinite shelf profile, then both profiles will
be considered in Sections 5.2 to 5.5. Finally, using these profiles, a discussion on the results generated from the previous sections is given in Section 5.6.

5.2 FREELY PROPAGATING CLASS I WAVES

An exponential continental shelf profile was used in Chapter 4 to discuss the propagation of free edge waves. The discussion was divided into 2 parts depending on whether the shelf profile was truncated or semi-infinite in extent. It was shown that both shelf profiles, (4.3) and (4.19), supported Class I waves.

A dispersion relation of the form (4.16) was found using the shelf profile (4.3). Upon using the shelf profile (4.19) a less complicated expression for the dispersion relation was obtained in (4.22).

Figure 4.1 (a) shows the first three modes of the dispersion relation described in (4.22) using (4.19). Whereas, Figure 4.1(b) shows the first mode for various shelf parameters.

Figure 4.2 shows the first three modes for the dispersion relation defined in (4.16) using the shelf profile (4.3) as well as the various cut-off corresponding to Class II edge waves.

The theory in Chapter 4, Section 4.2 will be used to discuss Class I edge waves. However, since the Coriolis parameter \( f \) only plays, in this Chapter, a modifying role for Class I waves, \( f \) will be neglected. Consequently, some equations from Chapter 4, Section 4.2 will be rewritten without the Coriolis term.

Consider a continental shelf bordered by an indefinitely long coastline depth profile in the off-shore direction (the x-axis) as given by (4.3). The differential equation for the forcing
of Class I edge waves by wind stress over any shelf profile is given by (3.15). For freely propagating waves (3.15) reduces to

\[ g h \nabla^2 \zeta + g \zeta_x \frac{dh}{dx} - \zeta_{tt} = 0. \]  

As discussed in Chapter 2 and 3 freely propagating edge waves along a continental shelf are considered to be standing waves normal to the coast and propagating along the coast with wave number \( s \). Hence, it is assumed that there is a simple harmonic wave propagating along the coast, where \( \sigma > 0 \) is in a positive direction, so that (2.32) is valid.

Substituting (2.32), (4.3) and using the transformations (4.4) and (4.5), (5.1) becomes (4.6). The general solution of (4.6) takes the form of (4.9), where

\[ \kappa^2 = \frac{4 \sigma^2}{a^2 gh_0} \]  

and

\[ \nu^2 = \frac{4 \sigma^2}{a^2} + 1 \quad , \quad \nu^2 > 0 \]  

The boundary conditions at the coast are that the wave height is finite and the velocity is zero normal to the coast. Therefore, after substituting (4.3) and using the transformation (4.4), the coastal boundary conditions (2.35) and (2.36) become:

\[ Z \text{ finite at } X = 1 \]  

and

\[ \chi \frac{dF}{dX} = 0 \text{ at } X = 1 \]  

At the edge of the continental shelf the boundary condition, for the truncated shelf profile (4.3), is (4.13), where \( \Delta \) is defined in (4.14) and

\[ r^2 = s^2 - \frac{\sigma^2}{gh} > 0 \]
Clarke and Louis (1975) has shown that the criterion for trapped waves using the truncated shelf profile (4.3) is $r^2 > 0$. The solutions of (5.1), using (4.3), for $r^2 \leq 0$ are called leaky waves.

After applying the boundary conditions (5.4) and (5.5) to the general solution (4.9) a dispersion relation is obtained which has the form

$$\alpha_1 \beta_2 - \alpha_2 \beta_1 = 0 \quad , \quad (5.6)$$

where

$$\alpha_i = \kappa J_i'(\kappa) + J_i(\kappa) \quad \quad (5.7)$$

and $i = 1, 2$ then $J_1(X) = J_\nu(X)$ and $J_2(X) = Y_\nu(X)$.

The wave height, $\zeta$, for freely propagating edge waves at the coast (i.e. $X = 1$) is of the form given in (4.19), that is,

$$\zeta = \frac{A}{\alpha_2} (\alpha_2 J_\nu(\kappa) - \alpha_1 J_{-\nu}(\kappa)) \cos(sy - \sigma t) \quad (5.8)$$

The dispersion relation in (5.6) is similar to that obtained in (4.16). It should be noted that the equations from (5.2) to (5.6) exclude the Coriolis parameter.

The boundary conditions remain unchanged when the Coriolis term is neglected and the semi-infinite shelf of the form (4.19) is being used. Hence, the boundary conditions are (4.20) for $\nu^2 \leq 1$ and (4.21) for $\nu^2 > 1$.

In general, the cases where $\sigma^2 < 0$ in (5.2) and/or $\nu^2 \leq 1$ in (5.3) produce leaky waves. Therefore, the edge waves being considered in this chapter are trapped waves which satisfy the criteria $\nu^2 > 1$ and $\sigma^2 > 0$. Hence, trapped edge waves on the semi-infinite shelf which satisfy the criteria $\nu^2 > 1$ and $\sigma > 0$ have a dispersion relation of the form:-

$$\alpha_1 = \kappa J'_\nu(\kappa) + J_\nu(\kappa) = 0 \quad . \quad (5.9)$$
Figure 4.1(a) and 4.2 are the graphs of the dispersion relations (5.6) and (5.9) except that the neglect of the Coriolis parameter has changed the high frequency cut-off slightly. The high frequency cut-off for (5.6) and (5.9) is defined by (5.5), that is, $s^2 > \sigma^2 / gH$ and $s^2 > 0$.

5.3 EFFECTS OF WIND STRESS ON HIGH FREQUENCY EDGE WAVES

Chapter 3 of this thesis examined the effects of various types of stress on Class I edge waves over a sloping shelf. However, the effect of wind stress on Class I edge waves has not been examined closely. Worthy (1984(a) and (b)) discussed the generation of edge waves by wind stress, using both a truncated and semi-infinite convex exponential shelf. These results will now be derived analytically.

Within the earth’s mid-latitudes, the effects of the Coriolis force on high frequency edge waves with periods less than an hour is minimal. Thus, in the following analysis of high frequency edge waves the Coriolis force $f$ as well as all frictional forces will be neglected. It will further be assumed that the effects of the off-shore wind stress are negligible.

Finally, under the conditions associated with a southerly 'buster', the pressure gradient term in the equations of motion is less significant than the wind stress term. Consequently, the pressure term $\phi$ will be neglected. However, it should be noted that when considering hurricane activity the dominant forcing mechanism is pressure. For example, Buchwald and de Szoekoe (1973) used the travelling pressure disturbance in discussing the generation of very high frequency edge waves over a continental step shelf profile.

The elimination of $u$ and $v$ from (3.13) and (3.14) yields (3.15), i.e.,

$$ gh \nabla^2 \zeta + g \zeta_x \frac{dh}{dx} - \zeta_{tt} = Y_y. $$

(5.10)
The right hand side of (5.10) is the term associated with the forcing function which, in this case, is the wind stress.

To determine the effects of wind stress, in particular southerly 'busters', on Class I edge waves, a model of the continental shelf off the east coast of Australia will be used. The most suitable shelf model to describe the east coast of Australia is the exponential shelf profile used by Adams and Buchwald (1968). Further, the exponential shelf will be divided into two types of profiles, namely, the truncated shelf as defined by (4.3) and the semi-infinite as defined by (4.19).

Consider the truncated exponential shelf profile, defined by (4.3). The boundary condition at the coast is that the velocity is zero normal to the coast, that is,

\[ \zeta_x = 0 \quad \text{at} \quad x = 0 \]  \hspace{1cm} (5.11)

Louis (1975) has shown that at the edge of the shelf

\[ \zeta_x + r \zeta = 0 \quad \text{at} \quad x = L \]  \hspace{1cm} (5.12)

where \( r \) is defined in (5.5).

The criterion for trapped waves using the truncated shelf profile in (4.3) is \( r^2 > 0 \). Solutions of (5.10) for \( r^2 \leq 0 \) are not considered.

5.4 FORCING MECHANISM

The longshore wind stress will be modelled by the rectangular wave distribution used by Worthy (1984(a)) and is similar to that used by Adams and Buchwald (1969). Hence,
\[ Y = Y_0 \{ H(y + y_0) - H(y - y_0) \} e^{-i\sigma t} \]  

(5.13)

where \( H(y) \) is the Heaviside unit function, \( \sigma \) is the frequency and \( y_0 \) is a constant related to the wind fetch. \( Y_0 \) is the amplitude of the wind stress which will be taken to be constant. The term \( e^{-i\sigma t} \) in (5.13) ensures the periodic shearing motion, which is primarily responsible for the generation of edge waves we are considering.

Taking the Fourier transform of (5.10) gives

\[ \frac{d^2}{dX^2} \left( X^{-1} \frac{d\xi}{dX} \right) + \left( \frac{4\sigma^2}{a^2gh_0} - \frac{4s^2X^2}{a^2} \right) \xi = \frac{4\overline{W}}{a^2gh_0} \]  

(5.14)

Equations (5.11), (5.12) and (5.14) gives

\[ \frac{d\xi}{dX} = 0 \text{ at } X = 1 \]  

(5.15)

\[ X \frac{d\zeta}{dX} - \frac{2\sigma \xi}{a} = 0 \text{ at } X = \Delta = e^{-aL/2} \]  

(5.16)

where

\[ \zeta = \zeta e^{-i\sigma t} \]  

(5.17)

\[ \overline{W} = F \{ Y_y e^{i\sigma t} \} = 2iY_0s \sin(sy_0)e^{i\sigma t} \]  

(5.18)

and \( X \) is defined in (4.4).

In deriving equations (5.14) to (5.17), it is assumed that both \( \zeta \) and \( (\zeta)_{y} \to 0 \) as \( y \to \pm \infty \).

Anticipating difficulties with the Fourier inversion of \( \zeta \), \( \sigma \) is replaced by \( \sigma + i\varepsilon \), where \( \varepsilon > 0 \). This ensures that the radiation condition for outward travelling waves as \( y \to \pm \infty \) is satisfied. The wave height \( \zeta \) is then found by letting \( \varepsilon \to 0 \). Those solutions which do not satisfy \( \zeta \to 0 \) as \( y \to \pm \infty \) are discarded.

The general solution of (5.14) with \( \sigma \) replaced by \( \sigma + i\varepsilon \) is

\[ \zeta = (A + u_1)J_{\nu}(\kappa X) + (B + u_2)XJ_{-\nu}(\kappa X) \]  

(5.19)
where
\[ \kappa = \frac{2(\sigma + i\epsilon)}{a\sqrt{gh_0}} , \] (5.20)

\[ a^2 gh_0 \sin(n\pi)u_1 = 2\pi \bar{W} \int_0^X J_{-\nu}(\kappa X') \, dX' \, , \]

\[ a^2 gh_0 \sin(n\pi)u_2 = -2\pi \bar{W} \int_0^X J_{\nu}(\kappa X') \, dX' \, . \]

Also, \( v \) is defined in (5.3) and is not an integer.

Applying the boundary conditions (5.15) and (5.16) to (5.19) and also evaluating \( \zeta \) at the coast, i.e. \( x = 0 \) \( (X = 1) \), it is found that

\[ \zeta(X = 1) = \frac{4\bar{W}\eta}{a^2 gh_0 \kappa (\alpha_1\beta_2 - \alpha_2\beta_1)} \] (5.21)

and

\[ \eta = \int_{\kappa\Delta}^X [\beta_1 J_{-\nu}(z) - \beta_2 J_{\nu}(z)] \, dz \, . \]

where \( \alpha_i \) and \( \beta_i , i = 1,2 \) are defined in (5.7).

Therefore, \( \zeta* \) is given by

\[ \zeta* = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta e^{isy} \, ds \, . \] (5.22)

To determine \( \zeta* \), let

\[ I = \int_C \zeta e^{isy} \, ds \] (5.23)

where the contour \( C \) is appropriately chosen after consideration of the branch points and poles of \( \zeta \) in (5.23).
5.5 GENERAL SOLUTION

5.5.1 TRUNCATED EXPONENTIAL SHELF

The integrand in (5.23) has branch points at \( \nu = 0 \) (\( s = \pm ia/2 \)) and at \( r = 0 \) (\( s = \pm(\sigma + i\epsilon)/(\sqrt{gH}) \)). The high frequency cut-off for edge waves corresponds to the branch points at \( r = 0 \). The replacement of \( \sigma \) by \( \sigma + i\epsilon \) has displaced the poles of \( \zeta \) from the real axis. The direction of displacement of the poles is determined by the following method.

Let \( s = s_n, \ n = \pm 1, \pm 2, \ldots \) be the roots of

\[
\alpha = \alpha(\sigma, s) = 0.
\]  

(5.24)

The roots of (5.24) are the poles of the integrand in (5.23). Equation (5.24) is called the dispersion relation for the forced high frequency edge waves under consideration. Also, (5.24) is the dispersion relation obtained by Clarke and Louis (1975) for freely propagating high frequency edge waves on the truncated convex exponential shelf with the Coriolis parameter set equal to zero. That is, the dispersion relation in (5.24) is the same as the dispersion relation for freely propagating waves in (5.6). It follows from (5.24) that

\[
d\alpha = \frac{\partial \alpha}{\partial \sigma} d\sigma + \frac{\partial \alpha}{\partial s} ds = 0.
\]

Since \( d\sigma = i\epsilon \), the displacement of poles from the real axis is determined by using

\[
ds = \frac{i\epsilon v}{sgh_o \kappa} \frac{\partial \alpha}{\partial \kappa} \frac{\partial \alpha}{\partial v}.
\]

Assuming \( \kappa \) is positive and given \( v > 0 \) (from (5.20)), the displacement of the poles depends on the sign of
It can be seen that \( \frac{\partial \alpha}{\partial s} \frac{\partial s}{\partial \nu} \) is an even function of \( s \). Consequently, the poles are dispersed evenly into upper and lower half planes. Numerical calculations of \( \frac{\partial \alpha}{\partial s} \) and \( \frac{\partial \alpha}{\partial \nu} \) are required to determined the sign of (5.25).

The appropriate contour \( C \) in (5.23) is illustrated in Figure 5.1. The path of integration is \( \Gamma_1 \) for \( y > y_0 \), the lower half plane, and the path of integration is \( \Gamma_2 \), the upper half plane, when \( y \leq -y_0 \).

Appendix 5A shows that by using Jordan’s Lemma the contributions from the path \( \Gamma_1 \) approaches zero when \( y > y_0 \). The branch line path \( \Gamma_1' \) has a cancelling effect. Therefore, (5.23) can now be replaced by \( 2\pi i \) times the sum of the residues in the appropriate half plane. Hence, for \( y > y_0 \) the resultant solution from (5.17) is

\[
\zeta = \sum_{n=1}^{\infty} \gamma_n \cos (s_n - \sigma t) ,
\]

where

\[
\gamma_n = \frac{2Y_0\nu_n \eta (s_n) \sin (s_n y_0)}{\kappa g h_0 s_n \frac{d}{\partial \nu} (\alpha_1 \beta_2 - \gamma_2 \beta_1) |_{s=s_n}} .
\]

A similar result is obtained for \( y \leq -y_0 \). The amplitude of the wave height in (5.26) is given by \( |\gamma_n| \).

A further discussion of these results will be given in Section 5.6.
FIGURE 5.1 The most suitable contour for the Fourier inversion of (5.23), where the poles, $s_\eta(n=1,2,3)$, are defined in (5.24).
5.5.2 SEMI-INFINITE EXPONENTIAL SHELF.

Consider the semi-infinite shelf profile described in (4.19). The boundary conditions associated with the semi-infinite shelf are that the velocity is zero normal to the coast as defined in (5.11) and that

\[ \zeta \to 0 \quad as \quad x \to \infty \quad . \tag{5.28} \]

The boundary condition in (5.28) is the criterion for trapped waves on the semi-infinite shelf described in (4.19). However, as discussed in Section 4.2.2, (5.28) is insufficient. A stronger condition is required for trapped waves and is defined in (4.21).

Buchwald and de Szoeke (1973) included a sample of anemograph recordings from Port Kembla and Mascot, N.S.W. on 15 May 1968, showing the change in wind direction when a typical local storm front is followed by a southerly gale. It is this switching motion that is responsible for the generation of edge waves. In order to simulate these changing wind directions, the wind stress used in (5.13) is taken to be a simple oscillatory function. Using the wind stress model in (5.13) means that comparisons can be made concerning the generation of Class I waves on the truncated exponential shelf (4.3) and on the semi-infinite exponential shelf (4.19).

Using the Fourier transform techniques, the resultant equations in this case will be the same as (5.14) to (5.18) except that the off-shore boundary condition (5.16) is replaced by

\[ X^{-2\zeta} \to 0 \quad \text{and} \quad X^{-1} \frac{d\zeta}{dX} \to 0 \quad as \quad X \to 0 \quad . \tag{5.29} \]

By assuming that the horizontal transport is zero at infinity in (5.29), the solution is restricted to \( \nu^2 \geq 1 \). Applying the boundary condition (5.29) to (5.19) sets \( B = 0 \). In the case of \( B = 0 \) and \( \nu^2 = 1 \) there is finite, nonzero, transport at infinity. When \( B = 0 \) and \( \nu^2 = 1 \) then
the solutions of (5.19) are regarded as 'leaky', as discussed by Larsen (1969). From (5.3), it is found that \( s = 0 \) constitutes the leaky line. However, since (5.19) must still be satisfied there are only a discrete number of leaky modes possible. Leaky modes will not be treated here. The solution to (5.19) can now be written as

\[
\zeta = (A + u_1)XJ_{\nu}(\kappa X) + u_2 XJ_{-\nu}(\kappa X), \quad \nu^2 \geq 1 .
\]  

(5.30)

After using the boundary condition (5.15) and evaluating the wave height \( \zeta \) at the coast, that is, at \( X = 1 \), (5.30) becomes

\[
\zeta = \frac{8iY_0\eta \sin(sy_0)}{a^2 \varepsilon h_0 \kappa \alpha_1} ,
\]  

(5.31)

where

\[
\eta = \int_0^X J_{\nu}(z) \, dz
\]

and \( \alpha_1 \) is defined in (5.7). Therefore, \( \zeta^* \) is given by (5.22).

To determine \( \zeta^* \), contour integration is again used (see Section 5.5.1) in the form of (5.23) where the contour is appropriately chosen after consideration of the poles and branch points of \( \zeta \). The replacement of \( \sigma \) by \( \sigma + i\varepsilon \) in (5.14) has displaced the poles of \( \zeta \) from the real axis. The displacement of the poles is determined by the following procedure.

Let \( s = s_n \), \( n = \pm 1, \pm 2, \ldots \) be the roots of

\[
\alpha_1 = \alpha_1(\sigma, s) = 0 .
\]  

(5.32)

The roots of (5.32) are the poles of the integrand in (5.23). Equation (5.32) is called the dispersion relation for the forced high frequency edge waves under consideration. Also, (5.32) is the dispersion relation obtained by Clarke and Louis (1975) for freely propagating high frequency edge waves on the semi-infinite convex exponential shelf. That is, the dispersion
relation in (5.32) is the same as the dispersion relation for freely propagating waves in Section 5.2. It follows from (5.32) that 
\[ d\alpha = \frac{\partial \alpha}{\partial \sigma} d\sigma + \frac{\partial \alpha}{\partial s} ds = 0. \]
Since \( d\sigma = i\epsilon \), the displacement of poles from the real axis is determined by using the sign of \( S\frac{\partial \alpha}{\partial \sigma} \). It is found that the positive poles move to the upper half-plane and the negative poles into the lower half-plane in the complex \( s \)-plane. When \( s = 0 \), the pole moves to \(-i\infty\). The branch points are at \( v = 0 \), that is \( s = \pm ia/2 \).

The appropriate contour \( C \) is illustrated in Figure 5.2. The path of integration for (5.23) is \( \Gamma_1 \) when \( y \geq y_0 \), and \( \Gamma_2 \) for \( y \leq -y_0 \).

Appendix 5B shows by Jordan's Lemma that the contributions of the paths \( \Gamma_1 \) and \( \Gamma_2 \) approach zero as \( R \to \infty \). Therefore, the integral (5.23) can now be replaced by \( 2\pi i \) times the sum of the residues in the appropriate half-plane. Using (5.17) and (5.22), (5.31) becomes

\[ \zeta = \sum_{n=1}^{\infty} \gamma_n \cos(s_n y + \alpha t), \tag{5.33} \]

for \( y \geq y_0 \), where

\[ \gamma_n = \frac{2y_0 \eta_n \sin(s_n y_0)}{\kappa gh_0 \xi_n \left( \frac{\partial \alpha}{\partial \nu} \right)_{s=s_n}}. \tag{5.34} \]

A similar result is obtained from (5.31) for \( y \leq -y_0 \). After substitution of (5.34) into (5.33), (5.33) becomes the sum of two waves, one generated by a delta function in the wind stress at \(-y_0\) and the other at \( y_0 \). The sum produces an amplitude-modulated wave which gives a frequency-dependent amplification. The wave-height amplitude for the \( n \)th mode is given by \( |\gamma_n| \).
FIGURE 5.2 The most suitable contour for the Fourier inversion of (5.32), where the poles, $s_n (n=1,2,3)$, are defined in (5.23).
5.6 RESULTS AND DISCUSSION

Typical values used by Clarke and Louis (1975) for the east coast of Australia were $h_0 = 70m$, $a = 5.33 \times 10^{-5} m^{-1}$, $L = 80kms$, for the width of the continental shelf. Strong southerly winds with speeds around 35 knots have been observed along the eastern coast of Australia. These southerlies are progressive with respect to time, whereas, the model considered here assumes the motion is oscillatory. To obtain the wave height $\zeta$ in (5.10), the method of solution is simplified by using (5.13) as the wind stress model. The wind stress parameters used by Worthy (1984 (a),(b)) were $Y_0 = .5N/m^2$ and the wind fetch $2y_0 = 200kms$, where the storm front is taken to be the same as the wind speed. These shelf and wind stress parameters will be used in Section 5.6 to determine the analytical results for the theory in this chapter.

For the truncated shelf, the first three terms of the series in (5.26) are used to calculate the wave height $\zeta$ at the coast. The first term of (5.26) is the most significant, whereas, the next terms in the series are basically correction terms. Figure 5.2 show the resulting amplitude (cms) plotted against frequency ($sec^{-1}$). Similarly, the first three terms in (5.33) are used to calculate the wave height at the coast for the semi-infinite shelf.

Table 5.1 shows the periods and amplitude of significant long wave disturbances for the truncated shelf which are shown in Figure 5.3. Similarly, Table 5.2 shows the significant long wave disturbances for the semi-infinite shelf profile.

There is a correspondence between the period and amplitudes obtained in Table 5.1 and the results for the semi-infinite continental shelf obtained in Table 5.2, (Worthy (1984(a) and (b)). The amplitudes and periods are similar except for the 65min period and amplitude. This discrepancy may be due to the high frequency cut-off experienced by waves on a truncated shelf and/or the erratic behaviour of the Bessel function when $v$ is close to an integer. Further, the
FIGURE 5.3 Graph of frequency vs amplitude for Class I edge waves using the truncated exponential shelf, (4.1), and wind stress profile defined in (5.13).
amplitude calculated for the 25min period wave is quite different from the amplitude calculated for the 25min period wave using a semi-infinite profile.

<table>
<thead>
<tr>
<th>PERIOD(mins)</th>
<th>65</th>
<th>41</th>
<th>32</th>
<th>26</th>
<th>22</th>
<th>18</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMPLITUDE(cms)</td>
<td>7.5</td>
<td>1.2</td>
<td>.87</td>
<td>.68</td>
<td>.55</td>
<td>.47</td>
<td>.47</td>
</tr>
</tbody>
</table>

**TABLE 5.1.** Summary of significant long wave disturbances obtained from (5.26), using the rectangular wind stress profile in (5.13).

It can be seen that shelf truncation has little effect on the forcing of high frequency edge waves with periods less than an hour. However, waves with periods near an hour, for example, the wave with period of 65min, are affected by the shelf truncation. Consequently, the truncation of the shelf profile plays an important role for longer period waves. Also, the shape of the shelf profile plays a major role in the generation of edge waves. This is evident in the comparison of results in Chapter 3 and 5. In Chapter 3, the linearization of the continental shelf for high frequency edge waves was discussed. A comparison of the results in Table 3.1, where \( \alpha = 6.15 \times 10^{-3} \) (equivalent shelf parameter to the east coast of Australia), and Table 5.1 shows that trapped edge waves of periods greater than 35mins are not generated on a sloping shelf profile. Also, the amplitude of waves found in Tables 3.1 and 3.2 are at least an order of magnitude greater than the amplitude of waves, with equivalent period, that are found in Table 5.1. As a result the shelf shape modifies the natural frequency of oscillation and the amplitude is influenced by the nature of the forcing used.
It can be shown that the inclusion of the forcing term on the right hand side of (5.12), and hence (5.16), has little effect on the results obtained in Table 5.1.

<table>
<thead>
<tr>
<th>PERIOD (mins)</th>
<th>60</th>
<th>42</th>
<th>32</th>
<th>26</th>
<th>21</th>
<th>18</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMPLITUDE (cms)</td>
<td>4.4</td>
<td>1.3</td>
<td>0.9</td>
<td>1.6</td>
<td>0.8</td>
<td>0.7</td>
<td>0.6</td>
</tr>
</tbody>
</table>

**TABLE 5.2.** Summary of significant long wave disturbances obtained from (5.33), using the rectangular wind stress profile in (5.13).

As discussed by Worthy (1984(a) and (b)), the periods obtained in Table 5.1 closely approximate the periods obtained by Clarke (1979) using Fourier analysis techniques on current meter recordings that were obtained outside Port Kembla Harbour, N.S.W. Australia on days which had gusty southerly winds. On current meter recordings, following days of gusty southerly winds, the periods of significant long wave disturbances were identifiable with the calculated periods found in Table 5.1.

Also, from Table 5.1 it appears that the periods closely approximate the harmonics of two hours.

The results in Table 5.1 are obtained by using a rectangular wave distribution for the wind stress profile. Suppose that a Gaussian wave distribution of the form:

\[ Y = Y_0 e^{-(b^2 y^2 + i \sigma t)} , \quad |y| < \infty, \quad (5.35) \]
is used for the truncated shelf defined in (4.3) instead of the rectangular wave distribution. Then (5.18) will become,

$$\bar{W} = F \{ Y_y e^{i\sigma t} \} = \frac{i\sqrt{\pi} Y_0 e^{-s^2/(4b^2)+i\sigma t}}{b}.$$ 

Consequently, (5.27) becomes

$$\gamma_n = \frac{-Y_0 \nu_{n} \eta(s_n) e^{-s^2/(4b^2)}}{gh_0b \frac{d}{dy} (\gamma_1 \beta_2 - \gamma_2 \beta_1 )}.$$  \hspace{1cm} (5.36)

Due to a smoother forcing profile, the wind stress profile in (5.35) ensures that the first mode is the dominant mode. Therefore, only the first term is calculated using (5.36). A summary of results can be obtained provided reasonable values of the wind stress parameter $b$ are obtained. The shelf parameters and $Y_0$ will be chosen to be the same as those used in calculating the values in Table 5.1.

The maximum amplitude for $\gamma_1$ is obtained when $b = s_1/\sqrt{2}$, for each wave number $s_1$. However, only one value of $b$ can be chosen at any one time. Therefore, the values of $b$ were chosen to correspond to different orders of magnitude of the wave number, $s_1$, i.e. $b = O(10^{-5}), O(10^{-4}), O(10^{-3})$. Using $b = O(10^{-4})$, Table 5.3 shows the dominant amplitudes and corresponding amplitude of (5.33) using (5.36), and wave number along with the maximum amplitude.

For the truncated shelf profile in (4.3), the most significant wave amplitude, in Table 5.3 is one that corresponds to the 65 min wave period when $b = O(10^{-5})$. This is the same wave period obtained from Table 5.1 using the rectangular wave distribution as the wind stress profile. Also, as the order of magnitude of $b$ becomes larger the harmonics of around two hours appear, although the amplitudes are relatively small. With the exception of the 54min wave period, these harmonics closely approximate those found in Table 5.1.
TABLE 5.3 Summary of significant long wave disturbances obtained from (5.33) using the Gaussian wave distribution as the wind stress profile in (5.35), where $b = O(10^{-4})$.

The most significant difference in the two wind stress profiles used in (5.13) and (5.35) is that the Gaussian wave distribution has no dependence on wind fetch, whereas, the rectangular wave distribution has a high dependence on wind fetch via the term $\sin(s_n y_0)$. That is, the wind fetch dependence of (5.36) is due to the wind stress profile (5.35) which determines the natural edge wave modes to be excited. Whereas, the $\sin(s_n y_0)$ dependence in (5.34) is caused by the sharp edges in the forcing function (5.13). If the wind fetch is relatively small, say $O(10)$, then it would be appropriate to say that high frequency edge waves with periods around the hour or less would be affected by such a fetch length. Therefore, the wind stress profile defined in (5.13) is an appropriate wind stress model to generate Class I trapped edge waves over an exponential continental shelf.
6.1 THE VORTICITY EQUATION

Vorticity as defined by Pond and Pickard is the measure of rotation of a body of fluid and is directly related to the 'velocity shear'. The vertical component of the relative vorticity in a horizontal plane (x,y) is given by

\[ \xi = (\text{Curl} \mathbf{U})_z = (\nabla \times \mathbf{U})_z \]

\[ = \frac{\partial y}{\partial x} - \frac{\partial u}{\partial y} \]

Note that \( U \) is the horizontal component of the velocity \( u \). If \( \nabla \times U \) equals zero then there is no rotation of the fluid.

The Coriolis parameter defined in (2.5), also known as the planetary vorticity, is defined as \( f = 2x \) angular velocity = \( 2\Omega \cos \theta \). Hence, a stationary body of water relative to the earth possesses a planetary vorticity which varies with the latitude of its position on the earth's surface. In particular, a body of water has zero planetary vorticity at the equator. The planetary vorticity of a body of water decreases as the location varies in the direction of the South Pole, with a minimum of \(-2\Omega\) at the South Pole and increases as the location varies in the direction of the North Pole with a maximum of \(2\Omega\) at the North Pole.
Absolute vorticity is defined as the sum of the relative and planetary vorticities, $\xi + f$. Provided there is no friction and external forcing the principle of conservation of absolute vorticity must be conserved. The conservation of absolute vorticity is illustrated in the following example. Consider the set of barotropic equations of motion in (2.2). After excluding friction and external forces (2.2) becomes

$$\frac{Du}{Dt} - fV = -\frac{1}{\rho} \frac{\partial p}{\partial x} \tag{6.1}$$
$$\frac{Dv}{Dt} +fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} \tag{6.2}$$

After cross differentiation the two equations in (6.1) and eliminating the pressure term, the following equation results:

$$\frac{D(\xi + f)}{Dt} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(\xi + f)$$

$$= -(\xi + f) \nabla \cdot U \tag{6.2}$$

The term $\nabla \cdot U$ measures the divergence of the horizontal flow, so that, if $\nabla \cdot U$ is positive then the horizontal flow tends to diverge. If, however, $\nabla \cdot U$ is negative then the horizontal flow tends to converge.

If the scale motion is not small then $f$, in (6.1) and (6.2), is defined by (2.5). Hence,

$$\frac{Df}{Dt} = \nu \beta \tag{6.2}$$

otherwise $\frac{Df}{Dt} = 0$.

(6.2) describes the principle of conservation of absolute vorticity in a uniformly rotating, barotropic and frictionless fluid.

If $\nabla \cdot U > 0$ (divergent flow) then from (6.2) the magnitude of the absolute vorticity, $|\xi + f|$, tends to increase with time. Whereas, if $\nabla \cdot U < 0$ (convergent flow) then the
magnitude of absolute vorticity decreases with time. Since $\xi + f$ could be positive or negative then the magnitude of $\xi + f$ must be considered. Since $\frac{\xi}{f} = \frac{\nu_f}{\eta} = R_0$ the Rossby number, which is 'usually' small, then the magnitude of the planetary vorticity is 'generally' larger than the relative vorticity, i.e. $|\gamma| > \xi$.

Physically, (6.2) can be explained by considering a cylinder of water of small height in a shallow ocean which is initially at rest relative to the earth's latitude. Hence, the parcel of fluid has only planetary vorticity, $f$. If the cylinder of water is in the Southern Hemisphere and moves in a lateral direction over a sill, then to satisfy the conservation of volume the parcel of water becomes squashed in length and stretched in width. In addition, the fluid moves away from the axis of rotation. Therefore, the flow is divergent and hence, in the Southern Hemisphere, the absolute vorticity decreases from $f$ to $f - |\xi|$. These observations imply that the moment of inertia increases and therefore, to conserve angular momentum, the angular velocity decreases, where

$$\text{Angular Momentum} = \text{Moment of Inertia} \times \text{Angular Velocity}$$

The decrease in absolute vorticity and the consequent increase in the moment of inertia of the cylinder of water can be compared to a rotating ice skater who extends his arms. As a result of the increase in the momentum of inertia, the spin of the ice skater is slowed down (a decrease in absolute vorticity).

If, however, the cylinder of water moves over a trough in the ocean then, to satisfy the conservation of volume, the parcel of water becomes elongated in height and smaller in width. Further, the fluid moves toward the axis of rotation. Therefore, the flow is convergent and hence, in the Southern Hemisphere, the absolute vorticity increases from $f$ to $\xi - |\gamma|$. In addition, angular velocity increases. The increase in absolute vorticity can be compared to a rotating ice skater who moves his arms to his sides which causes the ice skater to spin faster.
Consider the two dimensional vertically integrated equation of continuity which is defined in (2.20). The continuity equation can be rewritten as:

\[ \nabla \cdot \mathbf{U} = -\frac{1}{h} \frac{\partial}{\partial t} \mathbf{V} \cdot \mathbf{h} \]  

(6.3)

where \( h \gg \) wave height. The resultant equation after substituting (6.3) into (6.2) gives the constraint

\[ \frac{d}{dt} \left( \frac{\xi + f}{h} \right) = 0 \]

that is,

\[ \frac{\xi + f}{h} = \text{constant} \]  

(6.4)

For a uniformly rotating, barotropic and frictionless ocean, the potential vorticity is defined by (6.4). Further, (6.4) implies that if the flow is steady then the potential vorticity is conserved along its streamlines.

In the case of no friction and external forcing, the relative vorticity changes can be predicted by using (6.4). For example,

(1) consider a column of water in the Southern Hemisphere and suppose the depth \( h \) is constant.

(a) If the column of water moves laterally, \( f \) remains unchanged and therefore by (6.4) the relative vorticity, \( \xi \), is unchanged.

(b) If the column of water moves longitudinally towards the South Pole then \( f \) decreases and hence \( \xi \) increases. That is, the column of water obtains a more positive (anticlockwise) rotation.

(c) If on the other hand, the column of water moves towards the North Pole then \( \xi \) will decrease. Hence, the column of water acquires a more clockwise (negative) rotation.
(2) Consider a column of water in the Southern Hemisphere and suppose $h$ decreases, the potential vorticity, $\xi + f$, is positive and decreasing, and $\xi > f$. From (6.4),

(a) If the column of water moves laterally, then by (6.4), $\xi$ decreases.

(b) If the column of water moves longitudinally towards the South Pole then $f$ decreases. However, the behaviour of $\xi$ cannot be determined from (6.4).

(c) If on the other hand the column of water moves towards the North Pole, then $f$ increases positively and $\xi$ decreases. Thus, the column of water obtains a negative (clockwise) rotation.

(3) Consider a column of water in the Southern Hemisphere and suppose $h$ increases and initially $\xi + f$ is negative.

(a) If there is lateral movement of the water column then it implies that $\xi$ is negative increasing. Hence, the water column acquires more clockwise rotation.

(b) If there is longitudinal movement towards the South Pole then $\xi$ increases positively. That is, the water column acquires a more positive (anticlockwise) rotation.

(c) If on the other hand the water column moves towards the North Pole then the behaviour of $\xi$ cannot be determined from (6.4).

(4) Consider a column of water in the Southern Hemisphere and suppose $h$ increases and $\xi + f$ is initially negative.

(a) If the water column moves laterally then $\xi$ negatively decreases, that is, the column of water acquires a more negative rotation or clockwise rotation.
(b) If there is longitudinal movement towards the South Pole then the behaviour of $\xi$ cannot be determined from (6.4).

(c) If on the other hand the water column moves towards the North Pole then $\xi$ decreases, that is, $\xi$ is negatively increasing or the relative vorticity acquires clockwise rotation.

When considering large scale processes, relative vorticity, $\xi$, is negligible compared to planetary vorticity, $f$. Hence, in large scale processes, the potential vorticity can be rewritten as:

$$\frac{\xi + f}{h} = \frac{f}{h} = \text{constant} \quad (6.5)$$

From (6.5), the deflection of large scale current flows can be predicted when passing over various bottom topographies. Current deflection by bottom topography is called topographic steering. For instance, if a current flow moves over a sill in the ocean, then $h$ decreases and consequently, to ensure that (6.5) holds $f$ decreases. Therefore, the current flow deflects towards the equator. If, however, the current flow moves over a trough, then $h$ increases and consequently $f$ increases. Hence, the current flow deflects towards the North Pole. These deflections of current flow are independent of which hemisphere the current is in.

### 6.2 VORTICITY CHANGES IN A FRICTIONAL CHANNEL

Consider the frictional effects on vorticity in a channel where $x$ is the direction of current flow and $y$ is the direction across the channel. Bottom friction is one particular type of friction that effects flow through a channel. The direct effect of molecular viscousity in the $z$ (vertical) direction will be regarded as minimal. Therefore, only the shearing stresses acting across horizontal planes need to be considered. The shearing stresses per unit area across a surface
perpendicular to the \( z \) direction will be defined to have components \( \tau_x \) and \( \tau_y \) in the \( x \) and \( y \) direction, respectively. The stress components are taken to be positive if water above the particular area acts on the water below it in the positive \( x \) or \( y \) direction.

Hence, the force per unit mass acting on an element of water has components

\[
\frac{1}{\rho} \frac{\partial}{\partial z} \tau_x \quad \text{and} \quad \frac{1}{\rho} \frac{\partial}{\partial z} \tau_y
\]

in the \( x \) and \( y \) directions respectively.

Including these frictional forces and the advective terms, the depth average equations of motion are:

\[
\begin{align*}
\frac{Du}{Dt} - f v &= -g \frac{\partial \zeta}{\partial x} + \frac{(\tau_x)_s - (\tau_x)_b}{\rho h}, \\
\frac{Dv}{Dt} + f u &= -g \frac{\partial \zeta}{\partial y} + \frac{(\tau_y)_s - (\tau_y)_b}{\rho h}
\end{align*}
\]  

(6.6)

where \( \zeta \ll h \), and \( s \) and \( b \) represent the surface and bottom stresses of the fluid.

Assuming that the effects of the wind are negligible, the resultant bottom stress \( \tau_b \) can be related to the bottom current \( u_b \) by

\[ \tau_b = k \rho U_b u_b \]

where \( k \) is the co-efficient of friction or drag co-efficient, \( \rho \) is the density of water and \( u_b \) is usually measured about 1m above the bottom. Note that \( \tau_b \) is assumed to be in the same direction as \( u_b \).

If the components of \( u_b \) are taken to be \( u_b \) and \( v_b \) in the \( x \) and \( y \) directions, respectively, then

\[ U_b = \sqrt{u_b^2 + v_b^2} \]
Therefore,

\[ (\tau_x)_b = k \rho U_b u_b \quad \text{and} \quad (\tau_y)_b = k \rho U_b v_b \]

Hence, (6.6) becomes

\[
\frac{Du}{Dt} - fv = -g \frac{\partial \zeta}{\partial x} - \lambda u_b ,
\]

\[
\frac{Dv}{Dt} + fu = -g \frac{\partial \zeta}{\partial y} - \lambda v_b ,
\]

where \( \lambda = \frac{kU_b}{h} \). The spin down time of a parcel of fluid is defined as \( \lambda^{-1} \).

Suppose that \( \lambda = \text{constant} \) in (6.7). Differentiating the first equation of (6.7) by \( y \) and the second equation by \( x \) and then subtracting the resulting equation gives

\[
\frac{d(\xi + f)}{dt} + (\nabla \cdot U)(\xi + f) = -\lambda \xi .
\]

Using (6.3), (6.8) becomes

\[
(U \cdot \nabla)(\frac{\xi + f}{h}) + \lambda \left( \frac{\xi + f}{h} \right) = \frac{\lambda f}{h} .
\]

where \( \frac{d}{dt} \left( \frac{\xi + f}{h} \right) \) is assumed to be small compared to the other terms in (6.8). That is, the change in the potential vorticity with respect to the time scale is regarded as small compared to the change in length scales. Since the transport is in the \( x \) direction it is assumed that the velocity in the \( y \) direction is zero. Therefore, \( U = (u, 0) \).

Hence, (6.9) becomes

\[
\left( \frac{d}{dx} + \alpha \right) \left( \frac{\xi + f}{h} \right) = \alpha \frac{f}{h} 
\]

where \( \alpha = \frac{\lambda}{h} \).
Integrating (6.10) with respect to $x$ gives
\[
\frac{\xi + f}{h} = \int_{-\infty}^{x} \frac{\alpha f}{h} e^{\alpha(x' - x)} dx'.
\tag{6.11}
\]

From (6.11), it can be seen that in a frictional channel the potential vorticity $\frac{\xi + f}{h}$ is a weighted average of $\frac{f}{h}$ over a distance that has an order of magnitude similar to $\alpha^{-1}$. That is,
\[
\left(\frac{\xi + f}{h}\right)_{\text{local}} = \left(\frac{f}{h}\right)_{\alpha^{-1}}.
\tag{6.12}
\]

Hence, (6.12) gives a simple approximation to the potential vorticity in a frictional channel. However, the exclusion of various terms from the equations of motion affects the flow through the channel, in particular, mass transport. Terms such as, the variation of $U$ and $h$ in the $y$ direction, the variation of $\lambda$, as well as some non-linear terms, affect the flow through a channel. To include these terms in (6.6) would imply that a numerical model is needed. The Strait of Belle Isle will be used as an example of a numerical model. Section 6.3 will include a discussion on the research that has been done on the Strait of Belle Isle, its importance as well as the reasons why it should be used as a typical model for a barotropic, bottom friction affected channel.

6.3 THE STRAIT OF BELLE ISLE

Figure 6.1 shows the general location of the Strait of Belle Isle. The Strait runs from south-west to north-east between the coast of Labrador and Newfoundland, Canada. Its length is approximately 100km with an average depth of 70m. The narrowest width of the channel is about 18kms.

Since the Strait is a major shipping lane, the study of the flow through the Strait of Belle Isle and, hence, the knowledge of surface flow through the Strait is important for safe
FIGURE 6.1 The general location of the Strait of Belle Isle. Its length is approximately 100km long with an average depth of 70m. The narrowest width of the channel is about 18kms.
navigation. Ice flows from the arctic and sub-arctic bring nutrient and micro-organisms into the Strait on which fish feed. Therefore, the Strait is also important to the fishing industry of Canada. In particular, very cold water from the arctic region penetrates the warmer water on the Newfoundland shelf. The cod fish lives and breeds in water around 2 - 4°C which is usually trapped under the insurgence of cold water. The thickness of the cold water column affects the number of fish that are caught by fishing vessels in any one season. Figure 6.2 gives an idea of the problem associated with the cold water insurgence.

Also, the Strait of Belle Isle can be used as a model for flow through straits or channels which are inaccessible, such as those channels found in the arctic and sub-arctic regions.

The earliest study of the Strait of Belle Isle was done by Dawson (1907, 1913) who described the general current as primarily tidal. However, Dawson found that there appeared to be an overbalance of flow through the Strait depending in the barometric pressure. Bailey (1958) confirmed the observations of Dawson by noting the behaviour of flow through the Strait during the passage of a hurricane in 1953. Bailey also stated that the Labrador current diverted into the Strait. A study of the flow of ice sheets also gives evidence that the Labrador current moves through the Strait of Belle Isle. Figure 6.3 shows the movement of a buoy attached to an ice sheet which moved along the Labrador coast inward to the Strait of Belle Isle. Studying the behaviour of free floating ice sheets gives some evidence of the direction of flow of the Labrador current. It can be seen in Figure 6.3 that the buoy attached to an ice sheet near the Labrador coast moves through the Strait of Belle Isle. However, it should be mentioned that there were strong north westerly winds associated with the buoy drift.

Huntsman et al (1954) looked at the general oceanography of the Strait and observed that the easterly currents were stronger on the southside of the channel with speeds of 15km/day and the westerly currents stronger on the northside of the channel with speeds around 17km/day.
FIGURE 6.2 Very cold water from the arctic region penetrates onto the Newfoundland shelf in between warmer water. Cod fish live and breed in water around 2 - 4° which is usually trapped under the insurgence of cold water. The above figure depicts the problem for the fishing industry of Canada associated with the cold water insurgence.
FIGURE 6.3 Evidence of the Labrador current moving through the Strait of Belle Isle by the tracking of a buoy attached to an ice sheet of 1 metre thickness.
After the analysis of a 50 day data set, Farquharson and Bailey (1966) described the tidal regime and noted that there appeared to be a surface outflow of around $7.1 \times 10^4 \text{m}^3/\text{sec}$ on the northside of the Strait leading into the Gulf of St. Lawrence and a compensating inflow from the Labrador coast and the southside of the Gulf of St. Lawrence. Farquharson and Bailey also found a tendency for the sea level slope across the Strait to have the sign that would be expected if the daily mean sea currents were in geostrophic balance.

In (1981), Garrett and Petrie reanalyzed the data set used by Farquharson and Bailey (1966). Garrett and Petrie confirmed the findings of Farquharson and Bailey (1966) concerning sea level. They also stated that the flow through the Strait could be due to sea level differences between opposite ends of the Strait. Meteorological forcing was cited as a possible reason for the difference in the sea level across the Strait. Using a regression model which included acceleration and bottom friction, Garrett and Petrie (1981) obtained a spin down time of 1.1 days for the flow through the Strait.

A data set of 18 months was collected from the Strait of Belle Isle and was analytically examined by Garrett and Toulaney (1981). They looked at periods of 2 days and 1 month. From the analysis Garrett and Toulaney found that the surface flow was coherent with large scale atmospheric pressure gradient and stated that large scale wind fields could be the cause of sea level differences at the ends of the Strait which drives flow through the Strait.

Garrett and Toulaney (1982) used regression analysis and came to the conclusion that the sea level acts as an inverted barometer to the atmospheric pressure and has a frequency dependent response to the wind.

After allowing for the thickness of the bottom Ekman layer compared to the model used by Garrett and Petrie (1981), Toulaney et al (1987) examined a data set of 84 days and found a frequency dependence of near bottom to near shore flows. The model used by Toulaney et al best accounted for frequency dependent amplitude of the spin down time of around 4 hrs.
Using a spin down time of 4-6 hours, a comparison was made between the model predictions of Toulaney et al and real data for
\[
\frac{(P_s - P_b)}{\Delta P}
\]
where \( P \) represents pressure, \( s \) and \( b \) represents the surface and bottom respectively. Good agreement was obtained provided the drag co-efficient for friction, \( C_D \) is about \( 5 \times 10^{-3} \).

### 6.4 NUMERICAL MODEL OF THE STRAIT OF BELLE ISLE

During the winter months, that is, January to May, the Strait of Belle Isle becomes almost uniform in its density structure. Hence, it is assumed for the numerical model that the Strait of Belle Isle is barotropic. The Strait also has a very strong across Strait velocity shear of about \( 5 \times 10^{-4} \) sec. Further, the current has an average flow of about \( .62 \text{m/s} \). The transport through the Strait toward the Gulf of St. Lawrence is about \( .6 \text{Sv} \).

#### 6.4.1 BASIC EQUATIONS

The depth averaged equations of motion that were used for the numerical model are:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = f v - g \frac{\partial \zeta}{\partial x} - P u ,
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -f u - g \frac{\partial \zeta}{\partial y} - P v
\]

and the equation for continuity being
\[
\frac{\partial}{\partial x}(Hu) + \frac{\partial}{\partial y}(Hv) = -\frac{\partial \zeta}{\partial t},
\]

where
\[
P = \frac{C_D(\frac{U}{H} + U_0)}{1/2},
\]

\[
H = h + \zeta.
\]

Using the finite difference method on a domain of 102km x 231kms and a grid size of 3km square, the equations (6.13) and (6.14) are discretized in a semi-implicit manner using a Richardson lattice (see Figure 6.4).

Using the Richardson lattice and (6.13) an expression for \( u_{i,j}^{t+\Delta t} \) is given by
\[
\frac{u_{i,j}^{t+\Delta t} - u_{i,j}^{t}}{\Delta t} = \frac{u_{i,j}^{t}}{2\Delta x}\left[u_{i+1,j}^{t+\Delta t} - u_{i,j}^{t} + u_{i,j}^{t+\Delta t} - u_{i-1,j}^{t+\Delta t}\right]
- \frac{1}{4\Delta x}\left[v_{i,j-1}^{t+\Delta t} + v_{i+1,j-1}^{t+\Delta t}\right]\left[u_{i,j-1}^{t+\Delta t} - u_{i,j}^{t+\Delta t}\right] + \left[v_{i+1,j}^{t} + v_{i+1,j+1}^{t}\right]\left[u_{i,j}^{t+\Delta t} - u_{i,j+1}^{t+\Delta t}\right]
+ fr_{i,j} - C_D u_{i,j}^{t+\Delta t}\left(\frac{\left[v_{i,j}^{t+\Delta t} + v_{i+1,j}^{t+\Delta t}\right]}{d_{i,j}} + U_0\right) - g \frac{\left[\zeta_{i,j}^{t+\Delta t} - \zeta_{i,j}^{t}\right]}{\Delta x},
\]

where the advective term derivatives and Coriolis are centred in time and
\[
r_{i,j} = \frac{1}{4}\left[v_{i,j-1}^{t+\Delta t} + v_{i+1,j-1}^{t+\Delta t} + v_{i,j}^{t} + v_{i+1,j}^{t}\right].
\]

The procedure for the finite difference scheme is to find \( \zeta(t+\Delta t) \), which is obtained from the continuity equation (6.14), then a row of \( v \)'s is calculated which is used to find \( v(t+\Delta t) \). Thus, a row of \( u \)'s is calculated and used to obtain a value for \( u(t+\Delta t) \).

The advective terms have been removed from the boundary. Therefore, the interior elevation points adjacent to the two open grid squares are smoothed using
\[
\zeta_{i,j} = \frac{1}{8}\left(\zeta_{i-1,j} + \zeta_{i+1,j} + \zeta_{i,j-1} + \zeta_{i,j+1} + 4\zeta_{i,j}\right).
\]
Richardson Lattice

\[\zeta_{i,j-1} \quad v_{i-1,j-1} \quad v_{i,j-1} \quad v_{i+1,j-1} \quad \zeta_{i-1,j} \quad u_{i-1,j} \quad \zeta_{i,j} \quad u_{i,j} \quad \zeta_{i+1,j} \quad v_{i-1,j} \quad v_{i,j} \quad v_{i+1,j}\]

FIGURE 6.4 The Strait of Belle Isle depicting the orientation of the \( x \) and \( y \) co-ordinate axes, open boundaries \( A, B, C \) and \( D \) and the Richardson Lattice associated with the numerical model of the Strait.
From Figure 6.4 there are four open boundaries. To represent pressure differences between the northern side of the Strait and the Gulf of St. Lawrence, the sea level is raised at the open boundary adjoining the Labrador coast. Several types of boundary conditions are used to determine a reasonable estimate of the flow through the Strait. A more realistic approach in determining the open boundary conditions when compared to, the use of a specific sea level and/or the gradient method, is the Characteristic method.

6.4.2 THE CHARACTERISTIC METHOD ON THE OPEN BOUNDARY

Rearranging the equations in (6.13) and (6.14) gives

\[ u_t + g \zeta_x = f v - P u = F_1 \quad , \]

\[ v_t = -g \zeta_y - f u - P v = F_2 \quad , \]

\[ H u_x + \zeta_t = -(H v)_y = F_3 \quad . \]

Consider an (right hand) open boundary at \( x = L \), say. Noting that each wave has associated with it a forward and backward characteristic, then Figure 6.5 shows the characteristics associated with a wave at the boundary. To determine the compatibility equations, the procedure is to multiply (6.18) by \( \pm c \) and add the new equation to (6.16) to give

\[ (u + c \zeta)_t + c_0 (u + c \zeta)_x = F_1 + cF_3 \quad , \]

\[ (u - c \zeta)_t - c_0 (u - c \zeta)_x = F_1 - cF_3 \quad . \]

respectively, where
FIGURE 6.5 The forward, $x - c_0t = \text{const.}$, and the backward, $x + c_0t = \text{const.}$, characteristic of a wave at an open boundary $x = L$. 
\[ c_0 = \sqrt{\frac{gH}{H}} \]

\[ c = \sqrt{\frac{g}{H}}. \]

Hence, (6.17) and (6.19) can be rewritten so as to form the compatibility equations, that is,

\[ \frac{D_1}{D_1^t} (u + c\zeta) = F_1 + cF_3 \quad \text{along} \quad \frac{D_1x}{D_1^t} = c_0, \]

\[ \frac{D_2}{D_2^t} (u - c\zeta) = F_1 - cF_3 \quad \text{along} \quad \frac{D_2x}{D_2^t} = -c_0, \tag{6.20} \]

\[ \frac{D_3}{D_3^t} \nu = F_2 \quad \text{along} \quad \frac{D_3x}{D_3^t} = 0 \]

where \( \frac{D_3x}{D_3^t} \) refers to the slope of the incoming characteristic wave.

Recall that \( f(x - ct) \) is a wave travelling in the positive \( x \) direction. To minimize reflections at the boundary the slope of the incoming characteristic is forced equal to zero, that is, \( \frac{D_3x}{D_3^t} \) is set equal to zero. Hence, (6.20) reduces to

\[ (u + c\zeta)_t + c_0(u + c\zeta)_x = F_1 + cF_3, \]

\[ (u - c\zeta)_t = F_1 - cF_3, \tag{6.21} \]

\[ \nu_t = F_2. \]

Solving (6.21) for \( \zeta_t \) and \( u_t \), it follows that
Equations (6.22) are to be satisfied at the boundary, that is, at \( x = L \).

The time rate of change for the velocity in the \( x \)-direction is given by (6.22) and forms the analytic basis for the Characteristic method as described by Roed and Cooper (1987) for the open boundary \( x = L \).

For the numerical model, the derivatives are replaced by a one-sided difference scheme. That is, from (6.22) \( u_t \) becomes

\[
\frac{u_{i,j}^{t+\Delta t} - u_{i,j}^t}{\Delta t} = -\frac{1}{2} \left[ v_{i,j}^t + v_{i,j-1}^t \right] - C_D u_{i,j}^{t+\Delta t} \left( \frac{1}{4} \left( v_{i,j}^t + v_{i,j-1}^t \right)^2 + \left( u_{i,j}^t \right)^2 \right)^{1/2} + U_0 \]

\[
- \frac{c_0}{2\Delta x} \left[ u_{i,j}^t - u_{i-1,j}^t + c \left( \zeta_{i,j}^t - \zeta_{i-1,j}^t \right) \right].
\]

A similar procedure using (6.16) and (6.17) can be followed to obtain the compatibility equations for a (left hand) open boundary at, say \( y = -B \). Hence, the compatibility equations are given by
\[ \zeta_t = F_3 - \frac{c_0}{2c} (v - c \zeta)_y , \]

\[ u_t = F_1 , \]  

\[ v_t = F_2 + \frac{c_0}{2} (v - c \zeta)_y . \]  

Similarly, (6.23) gives the rate of change of the velocity in the \( y \)-direction which is used for the Characteristic method at the boundary \( y = -B \).

6.4.3 OPEN BOUNDARY MODELS

As determined by Toulaney et al (1987), the drag co-efficient, \( C_D \), is taken to be \( 5 \times 10^{-3} \) with a background bottom current of \( U_0 = 30 \text{ cm/sec} \). Real depth values are used for the numerical depth for the Strait of Belle Isle.

A non-zero decreasing value of sea level from the coastline is used at the open boundary \( A \) in Figure 6.4 for most of the numerical models, in this section, in order to ensure that the Labrador current is depicted as the source of inflow through the Strait of Belle Isle (Bailey (1958)) and to ensure there is a pressure difference between the two ends of the Strait.

Several numerical runs were made using various open boundary conditions. The boundary conditions used were a mixture of the characteristic method, gradient method and/or a specified sea level. In some of the numerical models discussed, in this Section, the Characteristic method is applied to the open boundaries labelled \( A \), \( C \) and \( D \) in Figure 6.4.
MODEL 1. The sea level is set equal to zero along the boundaries $A$ and $B$ except for the first three grid points along boundary $A$ near the Labrador coast. The sea level is made positive and decreasing away from the coast at the three particular boundary points on boundary $A$. Along boundary $C$, the characteristic method is used to determine the boundary condition for the velocity in the $x$-direction (i.e. $u$-velocity). The gradient method is used along boundary $D$ so that $v_y$ is set equal to zero.

MODEL 2. Using a positive value at the Labrador coast, the sea level is decreased to zero along boundary $A$ to the end of boundary $B$. The boundary conditions for $C$ and $D$ are the same as for Model 1.

MODEL 3. The sea level is decreased from a positive value to zero from the Labrador coast on boundary $A$ along boundary $B$ to the end of boundary $C$. The gradient method is used on boundary $D$ to set $v_y$ and $\zeta_y$ equal to zero.

MODEL 4. A sea level condition is used mainly on the boundaries. That is, the sea level is set equal to zero along the boundaries $A$ and $B$ except for the first three grid points which are set to be positive and decreasing as in Model 1. Also, the gradient method is used on boundary $C$ to set $\zeta_x$ equal to zero. Along boundary $D$, the gradient method is used to set $\zeta_y$ equal to zero. However, the Sommerfeld radiation condition, $v = -\sqrt{\frac{g}{H}} \zeta$, is used from grid points 17 to 28 to ensure no reflection occurs at the boundary. Greenberg (1977) describes the use of the Sommerfeld radiation condition.

MODEL 5. Along boundaries $A$ and $C$, the method of characteristics is used to determine the boundary condition for the $u$-velocity. Along boundary $B$, the sea level is set equal to zero. However, along boundary $D$ the sea level is decreased from, say, 10cm to -10cm so that the average sea level is zero.
MODEL 6. Boundaries $A$, $B$ and $C$ are similar to those used in Model 5. boundary $D$ ensures a sea level difference at the ends of the Strait.

All numerical iterations took about 10hrs of run time to settle down to a steady state which is equivalent to 10days. Also, tidal oscillations are excluded.

Other numerical runs were performed, in particular, the characteristic method at the open boundaries $C$ and $D$. However, instabilities arose with a period oscillation of about three days.

6.5 RESULTS AND DISCUSSION

Figures 6.6 to 6.11 show the results for the Models 1 to 6 after steady state has been reached. In each of the Figures, (a) and (b) show respectively the mean velocity direction and mass transport.

It can be seen that numerous open boundary condition can be used for the numerical model of the Strait of Belle Isle. However, not all models generate flow through the Strait as described by Garrett and Petrie (1981). Models 1 and 2 do not generate such flow through the Strait but rather the flow of the Labrador current by-passes the Strait. Both Models 1 and 2 have similar boundary conditions except for the boundary condition along boundary $B$. Model 1 shows a more streamline effect of the Labrador current passing the Strait of Belle Isle. This is basically due to the open boundary condition.

The results of Model 3, as displayed in Figure 6.8(a) and (b), show a strong southward flow through the Strait coming primarily from the Labrador current. As well, some flow into the Strait is experienced on the south side of Belle Isle. This could be is due to geostrophic
effects. The large amount of mass transport along boundary $C$ is due to the restriction of the sea level on the boundary. There are distinct vorticity changes and, using the bathymetry overlay for Figure 6.8(b), it can be seen that the flow through the Strait follows the isobaths corresponding to greatest depth.

A strong Labrador current is displayed in Figure 6.9 for Model 4 indicating a small amount of the Labrador current rotates and moves through the Strait. However, this model uses the radiation condition to satisfy the physics of the Strait of Belle Isle, which is not as physically viable as the characteristic method and/or the gradient method.

Setting the sea level to be zero on boundary $B$ and having a mean sea level zero on boundary $D$ as well as using the characteristic method on the other boundaries allows the numerical model to choose the direction of flow through the Strait. This is done in Model 5. Hence the results of Model 5, found in Figure 6.10, show that there is flow through the Strait from north to south with fluid being drawn from fluid south of Belle Isle. Using the bathymetry overlay, it can be seen that fluid flow through the Strait is similar to Model 4. That is, the flow through the Strait follows the isobaths of greatest depth. The vortex motion at boundary $D$ in Model 5, is due to the restriction in sea level.

A comparison of Model 5 with Model 6 can be done since both use the method of characteristics on boundaries $A$ and $C$ and zero sea level an boundary $B$. The difference being the sea level condition at the southern end, that is, boundary $D$. As suggested by Garrett and Toulaney (1981), the sea level on boundary $D$ is specified as negative to model pressure differences between the two ends of the Strait. Consequently, the pressure differences ensure a set down of fluid into the Gulf of St. Lawrence. The difference between the numerical models 5 and 6 can be seen in Figures 6.10 and 6.11. Model 6 shows a stronger flow through the coast which also follows the largest isobaths. Once again, fluid is drawn from the south side of Belle Isle rather than directly from the Labrador current. Also, it appears that the direction of flow for Models 5 and 6 are in the opposite direction to Model 4. This could be due to the Labrador
current being specified in Model 4 but not in Models 5 and 6. However, Models 4, 5 and 6 generate the same type of flow through the Strait. As well, models 4 and 6 show the same flow into the Gulf of St. Lawrence suggesting that if considering only the flow in the Strait of Belle Isle the boundary condition on the north side of the Strait is not an important factor.

Models 3, 4 and 6, which use the characteristic method to determine the open boundary condition for the $u$-velocity generate similar flow patterns and changes in vorticity. The flow in each case generally following bathymetry. Model 4 shows the Labrador current does divert into the Strait as discussed by Bailey (1958). However, it is assumed by Garrett and Petrie (1981) that the Labrador current moves directly into the Strait of Belle Isle as shown by Model 4. It is possible that the Labrador current is diverted further downstream as found in Models 3 and 6 due to a probable sea level difference between the north and south side of Belle Isle. No model shows the strong velocity shear as reported by Garrett and Petrie (1981).

Vorticity changes are experienced by all models, in particular, Models 4, 5 and 6. As shown by the overlay on Figure 6.8, there are large variations in the bottom topography near and along the Strait of Belle Isle. As a result the variation in relative vorticity is affected by the bottom topography more strongly than the Coriolis ($f$) value. However, it can be seen that the southward movement of the Labrador current decreases $f$, which means a decrease in relative vorticity, that is, a clockwise (negative) rotation is experienced as suggested by Models 2, 3 and 4, in Figures 6.7 to 6.9. As a consequence of the clockwise rotation, there is some flow toward the Strait. Since the topography changes throughout the Strait there is also a change in the potential vorticity.

In Models 5 and 6, there is a northward current flow. Consequently, the Coriolis value increases and thus, as seen in Figures 6.10 to 6.11 the relative vorticity becomes a more anticlockwise (negative) rotation. In these models, the flow once again moves toward the Strait and is influence by the general topography. As a consequence, it can be seen that the
characteristic method is a useful method for determining open boundary conditions whether it be a velocity condition or sea level condition.

Physically, every wave has two characteristics, a forward and backward characteristic, hence the characteristic method is a more realistic approach in determining boundary conditions. Therefore, the characteristic method is more physically correct than specifying sea level values that are basically unknown quantities or using the radiation condition on some grid points to satisfy the physics of the Strait of Belle Isle.

In conclusion, Models 3 to 6 all show the behaviour that Garrett and Petrie (1981) found in their analytic results of the Strait of Belle Isle. That is, all models showed a strong flow through the Strait of Belle Isle as well as the flow following the isobathymetry of the Strait. Models 3 and 4 show the presence of the Labrador current which Bailey (1958) has shown to exist and most importantly Models 3 to 6 showed a strong cross strait shear as discussed by Garrett and Petrie (1981). Therefore, all boundary methods illustrate some aspects of the observed flow through the Strait of Belle Isle, but none completely describes the observed flow. However, no one type of boundary condition by itself reproduces the observed flow through the Strait of Belle Isle.
FIGURE 6.6 Model 1 numerical results using zero sea level along boundaries A and B except for the first 3 grid points along boundary A (near Labrador coast). The sea level is made positive and decreasing away from the coast at the 3 grid points. Along boundary C the characteristic method is used for the u-velocity. The gradient method, $u_y=0$, is used along boundary D.
FIGURE 6.7(a) Averaged Flow Velocity Vectors

FIGURE 6.7(b) Averaged Mass transport

FIGURE 6.7 Model 2 numerical results using a positive decreasing sea level along boundary A to the end of boundary B. The boundary conditions for C and D are the same as for Model 1.
FIGURE 6.8(a) Averaged Flow Velocity Vectors

FIGURE 6.8(b) Averaged Mass transport

FIGURE 6.8 Model 3 numerical results using a positive decreasing sea level along boundary A and B to the end of boundary C. The gradient method is used on boundary D, that is, $v_y=0$ and $\zeta_x=0$. 
FIGURE 6.8(a) Averaged Flow Velocity Vectors

FIGURE 6.8(b) Averaged Mass transport

FIGURE 6.8 Model 3 numerical results using a positive decreasing sea level along boundary A and B to the end of boundary C. The gradient method is used on boundary D, that is, $v_y=0$ and $\zeta_y=0$. 
FIGURE 6.9(a) Averaged Flow Velocity Vectors

FIGURE 6.9(b) Averaged Mass transport

FIGURE 6.9 Model 4 numerical results using zero sea level along boundaries A and B except at the first 3 grid points along boundary A (near Labrador coast). The sea level is made positive and decreasing away from the coast at the 3 grid points. The gradient method is used on boundaries C, to set $\zeta_x=0$, and D, to set $\zeta_y=0$. Also, on boundary D, the Sommerfeld radiation condition is used along grid points 17 to 28.
FIGURE 6.10 Model 5 numerical results using zero sea level along boundary B. Boundaries A and C use the method of characteristics to determine the condition for the u-velocity. Along boundary D, the sea level is decreased from 10cm to -10cm starting from the coast of Quebec to the coast of Newfoundland.
FIGURE 6.11(a) Averaged Flow Velocity Vectors

FIGURE 6.11(b) Averaged Mass transport

FIGURE 6.11 Model 6 numerical results using zero sea level along boundary B. Boundaries A and C use the method of characteristics to determine the condition for the u-velocity. Along boundary D, the sea level is set equal to -10cm.
APPENDIX 3A

Consider the integral \( I_n^2 \), which is defined in (3.35). The method of steepest descent is used to obtain an asymptotic solution for \( \zeta \) in (3.33) for large values of \( t \) and \( y \) compared to \( \eta \). Note that \( z > \eta \) is required for \( I_n^1 \), defined in (3.34), to converge in the upper half \( s \)-plane. Therefore, \( e^{-i\pi \eta} \) is included in the slowly varying function \( \phi(s) \) defined below. Let \( \beta = \frac{y}{r} > 0 \), then

\[
I_n^2 = \int f_2 ds,
\]

\[
= \int \phi(s) e^{it\chi(s)} ds
\]

where \( b_n = \alpha g(2n + 1) \) and

\[
\phi(s) = \frac{se^{-i\pi \eta}}{2a_n(a_n - s\nu)}
\]

\[
\chi(s) = \beta s - \sqrt{s b_n}
\]

The saddle point is determined by the condition

\[
\chi'(\nu) = 0
\]

which is

\[
\nu = \frac{b_n}{4\beta^2}
\]

provided that \( \beta > 0 \), that is \( y > 0 \). Hence, from Whitman it can be shown that
The poles of $f_2$ in (3.A.1) are the same as the positive poles of $I_n^1$ which are defined in (3.40). Let the poles of $I_n^2$ be represented by $s_r$. Hence,

$$I_n^2 = 2\pi i \sum \text{Residues} + \int_{\Gamma_{S,D.}} f_2 \, ds + \int_{\Gamma_b} f_2 \, ds$$  \hspace{1cm} (3.A.4)$$

where $\Gamma_{S,D.}$ and $\Gamma_b$ are the respective paths along the steepest descent and branch line. The poles of $f_2$ are within the contour only when $s_r < \nu$ which means that $y < \frac{1}{2}Vt$.

By letting $s = \xi + i\mu$ and

$$\text{Im} \chi(s) = \text{Im} \chi(\nu) \quad .$$

then the path of steepest descent is the parabola,

$$\xi^2 - \left( \frac{b_n}{4\beta^2} \right)^2 = -\frac{b_n \mu}{2\beta^2} \quad .$$

Figure 3.A.1 shows the conversion of Figure 3.1 into the path of steepest descent. The branch line, $\Gamma_b$, in the upper half $s$ plane can now be rotated by an angle of $\frac{3\pi}{4\nu}$. Hence for large $t$, the integral about the branch line $\Gamma_b$ is negligible compared to the integral along the steepest descent in the neighbourhood of the saddle point, $\nu$, and the residue at the pole, $s = s_r$. Therefore, (3.A.4) becomes

$$I_n^2 \sim 2\pi i \, \frac{e^{-i\epsilon}(\eta - \zeta)}{\nu^2} - \frac{1}{2} \sqrt{\frac{\pi}{b_n \nu}} \frac{e^{i(\pi/4 - b_n \epsilon)(\eta - \nu t)}}{(\nu - \nu t/2)} H(\nu - \beta) \quad .$$
FIGURE 3.A.1 The conversion of the contour found in Figure 3.1 using the method of steepest decent, where $s_0 = \frac{a_0}{V^2}$ and $v = \frac{b_m}{4\beta^2}$. 
Asymptotically, for large values of $y$ and $t$ and $0 < y < \frac{1}{2} V t$ then

\[ I_n^2 = \frac{2 \pi i}{\nu^2} e^{-i s_n (\eta - z)} \]

which cancels the residue contribution of the pole $s_r$ in $I_n^1$ of (3.39). For $y > \frac{V t}{2}$, $I_n^2 = 0$.

Similarly, it can be shown that $I_n^3$ cancels the residue contribution of the negative pole, $s_l$ in $I_n^1$.

Therefore, for large $y$ and $t$

\[
\zeta \sim 0 \quad y > V t ,
\]

\[
- \frac{4 Y_0(\eta)}{\nu^2} \sum_{n=0}^{\infty} C_n \cos s_n (\eta - z) \quad \frac{1}{2} V t < y < V t ,
\]

\[
- 0 \quad y < \frac{1}{2} V t .
\]
The boundary condition at the edge of the continental shelf is that

(a) The wave height be continuous, and

(b) \( h \mathbf{u} \cdot \mathbf{n} \) is also continuous.

Hence,

\[
    h \left[ -i \sigma g \frac{\partial P_s}{\partial x} + \frac{g}{i} \frac{\partial P_s}{\partial y} - f \tau y \right] = H \left[ -i \sigma g \frac{\partial P_0}{\partial x} + \frac{g}{i} \frac{\partial P_0}{\partial y} - f \tau y \right]
\]

where the subscripts \( s \) and \( 0 \) represent the shelf and ocean respectively. That is,

\[
    \frac{\partial P_s}{\partial x} - \frac{f}{i \sigma} \frac{\partial P_s}{\partial y} - \frac{f}{i \sigma g} \tau y = \frac{H}{h} \left[ \frac{\partial P_0}{\partial x} + \frac{f}{i \sigma} \frac{\partial P_0}{\partial y} + \frac{f}{i \sigma g} \tau y \right].
\]

Taking Fourier transforms and using (4.28) gives:

\[
    \frac{d \bar{P}_s}{dx} - \frac{sf}{i \sigma} \bar{P}_s = \frac{H}{h} \left[ \frac{d \bar{P}_0}{dx} - \frac{fs}{i \sigma} \bar{P}_0 + \frac{ft}{i \sigma g} \right] - \frac{ft}{i \sigma g}. \tag{4A.1}
\]

\( P_0 \) can be determined by solving (4.29), whereby \( h = H \), a constant. Also, it is assumed that the effects of longshore wind stress is significant only on the continental shelf. Hence, (4.29) reduces to

\[
    \frac{d^2 \bar{P}_0}{dx^2} - s^2 \bar{P}_0 = 0 \quad \text{at} \quad x = L
\]

where

\[
    \bar{P}_0 = A e^{-sx}, \tag{4A.2}
\]

\[
    \frac{d \bar{P}_0}{dx} = -s \bar{P}_0.
\]
Substituting (4.A.2) into (4.A.1) gives

\[
\frac{d\bar{P}_s}{dx} - \frac{s}{\sigma}\bar{P}_s = -\frac{H}{h}\left[\frac{f}{\sigma} + 1\right]s\bar{P}_0 + \frac{fT}{\sigma g} - \frac{fT}{\sigma g}.
\]  

(4.A.3)

Let \( h \to H \), then \( \bar{P}_s \to \bar{P}_0 \) so that (4.A.3) becomes

\[
\frac{d\bar{P}_s}{dx} + s\bar{P}_s = 0 \quad \text{at} \quad x = L.
\]

Recall that \( X = e^{-ax/2} \) from (4.4), then the boundary condition at the edge of the shelf becomes

\[
X\frac{d\bar{P}_s}{dX} - \frac{2s}{a}\bar{P}_s = 0 \quad \text{at} \quad X = \Delta,
\]

where \( \Delta \) is defined in (4.14).
From (4.41),

\[ I = \int_{\Gamma} \overline{P} e^{is(y-\eta)} \, ds \quad (4.B.1) \]

where \( \overline{P} \) is a function of \( s \) and \( \Gamma \) is the appropriate contour found in Figure 4.4.

In lower half \( s \)-plane (4.B.1) becomes:

\[ I = \left( \int_{\Gamma_1} + \int_{\Gamma_b} + \int_{\Gamma_c} + \int_{\Gamma_1'} \right) \overline{P} e^{is(y-\eta)} \, ds \quad (4.B.2) \]

where \( \Gamma_1 \) and \( \Gamma_1' \) form the outer semi-circle, \( \Gamma_b \) and \( \Gamma_c \) form the two branch lines and the circular path about the branch point \( b_2 \) respectively.

Consider the semi-circle formed by the contours \( \Gamma_1 \) and \( \Gamma_1' \). \( \theta \) is the angle at which the branch line intersects the semi-circle. Let \( s = Re^{i\theta} \), then from (4.B.2)

\[ I_1 = \left( \int_{\Gamma_1} + \int_{\Gamma_1'} \right) \overline{P} e^{is(y-\eta)} \, ds \]

\[ = \left( \int_{0}^{\theta'} + \int_{\theta'}^{\pi} \right) \overline{P}(Re^{i\theta}) e^{i(Re^{i\theta})(y-\eta)} Rie^{i\theta} \, d\theta \]

\[ = \int_{0}^{\pi} \overline{P}(Re^{i\theta}) e^{i(Re^{i\theta})(y-\eta)} Rie^{i\theta} \, d\theta \]

Hence,

\[ |I_1| \leq \int_{0}^{\pi} R |\overline{P}(Re^{i\theta})| e^{-R\sin\theta(y-\eta)} \, d\theta \]

When \( -\pi \leq \theta \leq 0 \), \( \sin\theta < 0 \). This implies that if \( R \to \infty \), then \( e^{-R\sin\theta(y-\eta)} \) tends to zero provided \( y < \eta \).
If $|\bar{P}(Re^{i\theta})| \leq \frac{M}{R^{k+\epsilon}}$, where $k$ is some positive number or zero then $|I_1| \to 0$ as $R \to \infty$. Now

$$|\bar{P}(Re^{i\theta})| \leq \frac{|\tau_0(\eta)|}{\alpha \gamma h_0(1 + \frac{f}{\sigma})} |F_n(Re^{i\theta}, \sigma)|$$

where

$$F_n(s, \sigma) = \frac{\alpha \gamma (a - s)e^{-aL/2} + \left(1 + \frac{f}{\sigma}\right)sD_s(s, \sigma) - \left(a + \frac{sf}{\sigma}\right)\alpha_1(s, \sigma)}{s\left(\frac{f}{a} + \frac{f}{\sigma}\right)D_s(s, \sigma)}$$

Since

$$\nu = O(R)$$
$$\alpha_1 = O(Re^{aLR/2})$$
$$D_s(s, \sigma) = O(Re^{aLR/2})$$

then $F_n(Re^{i\theta}, \sigma) = O\left(\frac{1}{R}\right)$.

Hence,

$$|\bar{P}(Re^{i\theta})| \leq O\left(\frac{1}{R}\right)$$

which is the criteria for $|I_1|$ to converge to zero, as $R \to \infty$, provided that $y < \eta$.

Consider the integral

$$I_2 = \int_{\Gamma_b} \bar{P} e^{is(y-\eta)} ds$$

$$= \left(\int_{\Gamma_{b_1}} + \int_{\Gamma_{b_2}}\right) \bar{P} e^{is(y-\eta)} ds$$

where $\Gamma_{b_1}$ and $\Gamma_{b_2}$ are the two branch lines of $\Gamma_b$. Let $s = b_2 + Re^{3i\pi/2}$, where $b_2 = \frac{-f}{\sigma} - \sqrt{\frac{f^2}{\sigma^2} + 1}$,
for the branch line $\Gamma_{b_2}$ and where $R$ goes from $-\infty$ to $\varepsilon_a$.

Hence the first integral in $I_2$ becomes

$$\int_{b_2} P e^{is(y-\eta)} ds = \int_{-\infty}^{\varepsilon_a} P(b_2 + Re^{3i\pi/2}) e^{i(b_2 + Re^{3i\pi/2})(y-\eta)} e^{3i\pi/2} dR$$

where

$$P(b_2 + Re^{3i\pi/2}) = iW \left[ F_n(b_2 + e^{3i\pi/2}) \right] \tau_0(\eta) ,$$

where $\varepsilon_a$ is the radius of the contour $\Gamma_c$.

Let $s = b_2 + Re^{-\pi/2}$, for the branch line $\Gamma_{b_2}$. Hence, the second integral in $I_2$ becomes

$$\int_{b_2'} P e^{is(y-\eta)} ds = \int_{-\varepsilon_a}^{\infty} P(b_2 + Re^{-i\pi/2}) e^{i(b_2 + Re^{-i\pi/2})e^{-i\pi/2} dR} ,$$

where

$$P(b_1 + Re^{-3i\pi/2}) = iW \left[ F_n(b_2 + e^{-i\pi/2}) \right] \tau_0(\eta) ,$$

Since $b_2 + e^{3i\pi/2} = b_2 + e^{-i\pi/2}$ and $P$ is an even function of $\nu$, then it can be deduced that

$$\int_{\Gamma_{b_2}} P e^{is(y-\eta)} ds = -\int_{\Gamma_{b_2}} P e^{is(y-\eta)} ds .$$

That is, $I_2 = 0$.

The last integral to be considered in (4.B.2) is $I_c$, where $c$ is the circle around the branch point $b_2$. Let

$$s = b_2 + \varepsilon_a e^{i\theta} , \quad \text{where} \quad \frac{-\pi}{2} \leq \theta \leq \frac{3\pi}{2} .$$

Then

$$\int_{\Gamma_c} P e^{is(y-\eta)} ds = \int_{-\pi/2}^{3\pi/2} P(b_2 + \varepsilon_a e^{i\theta}) e_{a\theta} e^{i\theta} d\theta$$
where
\[ \overline{P}(b_2 + \varepsilon_a e^{i\theta}) = iWF_n(b_2 + \varepsilon_a e^{i\theta})\tau_0(\eta) . \]

Now,
\[ \lim_{\varepsilon_a \to 0} F_n(b_2 + \varepsilon_a e^{i\theta}) = \frac{0}{0} \]

\[ L' I = 1 . \]

Therefore,
\[ \lim_{\varepsilon_a \to 0} \int_{-\pi/2}^{\pi/2} \overline{P}(b_2 + \varepsilon_a e^{i\theta}) e_a i e^{i\theta} d\theta = 0 . \]

Consequently, (4.B.1) becomes:
\[ \int_\Gamma \overline{P} e^{isy} ds = i\pi \sum \text{Residues} . \]
From (5.23),
\[ I = \int_C \zeta e^{isy} ds \]  
(5.A.1)

where \( \zeta \) is a function of \( s \) which is defined in (5.21) and \( C \) is the appropriate contour as found in Figure 5.1 for \( y \geq y_0 \). That is, \( C \) is the contour found in the lower half \( s \)-plane.

Therefore, (5.A.1) becomes
\[ I = \left( \int_{\Gamma_1} + \int_{\Gamma_b} + \int_C + \int_{\Gamma_1^*} + \int_{\Gamma_b^*} + \int_{c^*} + \int_{\Gamma_1^{**}} \right) \zeta e^{isy} ds \]  
(5.A.2)

Consider the contours \( \Gamma_1, \Gamma_1^* \) and \( \Gamma_1^{**} \). Let \( s = Re^{i\theta} \), then
\[ I_R = \left( \int_{\Gamma_1} + \int_{\Gamma_1^*} + \int_{\Gamma_1^{**}} \right) \zeta e^{isy} ds \]
\[ = \left( \int_0^{\pi/2} + \int_{-\pi/2}^{\theta^*} + \int_{\theta^*}^\pi \right) e^{iRe^{i\theta}} iRe^{i\theta} d\theta \]

where \( \theta^* \) is the appropriate angle from the origin to the corresponding branch cut \( \Gamma_b^* \).

If \( |\zeta(R e^{i\theta})| \leq \frac{M}{R^{k+1}} \), where \( k \geq 0 \) then as \( R \to \infty \), \( I_R \to 0 \).

It can be shown from the power series for the Bessel function that
\[ J_\nu(\zeta) = \frac{e^\nu (\zeta)^\nu}{\sqrt{2\pi\nu} \nu} \left[ 1 + \frac{a_1}{\nu} + \frac{a_2}{\nu^2} + \ldots \right] \]

for large values of \( \nu \) (Copson). Since,
\[ \nu = O(R) \quad \text{and} \quad r = O(R) \]
then it can be seen that
\[
\frac{\eta}{\alpha_1 \beta_2 - \alpha_2 \beta_1} = O\left(\frac{1}{R^2}\right)
\]

Therefore, \(|\zeta| = O\left(\frac{1}{R^2}\right)|\), as required for convergence.

Noting from (5.21), that \(\zeta\) is a function of \(e^{iy_0}\) then
\[
|I_R| \leq \int_{\Re} e^{-R \sin \theta (y - y_0)} d\theta.
\]

As \(R \to \infty\), then \(|I_R| \to 0\), provided that \(y \geq y_0\).

Consider the integral
\[
I_b = \left(\int_{\Gamma_{b_1}} + \int_{\Gamma_{b_1'}}\right) e^{isy} ds,
\]
where \(\Gamma_{b_1}\) and \(\Gamma_{b_1'}\) are the branch lines along the branch cut \(\Gamma_b\) \((v^2 = 0)\). Let \(b_1 = \frac{\pi i}{2}\) and substitute
\[
s = b_1 + Re^{-i\pi/2}
\]
in the branch line integral \(\Gamma_{b_1}\), where \(\epsilon_a = b_1 + \epsilon\) and \(R\) goes from \(-\infty\) to \(-\epsilon_a\).

Hence, the first integral in (5.A.3) becomes
\[
I_{\Gamma_{b_1}} = i e^{-\pi/2} \int_{-\infty}^{\epsilon_a} \zeta(b_1 + Re^{-i\pi/2}) e^{i(R e^{b_1} + e^{-i\pi/2})} dR.
\]

Similarly, substitute
\[
s = b_1 + Re^{3i\pi/2}
\]
in the integral \(\Gamma_{b_1'}\). \(R\) goes from \(-\epsilon_a\) to \(-\infty\). Hence, the second integral of \(I_b\), in (5.A.3), becomes
\[
I_{\Gamma_{b_1'}} = i e^{3i\pi/2} \int_{-\epsilon_a}^{R} \zeta(b_1 + Re^{3i\pi/2}) e^{i(R e^{b_1} + e^{3i\pi/2})} dR.
\]
Since $e^{-i\pi/2} = e^{-3i\pi/2}$ and $\zeta$ is an even function of $v$ then

$$I_{\Gamma_{\beta}} + I_{\Gamma_{\beta}'} = 0 \quad (5.A.4)$$

Consider the integral

$$I_{b'} = \left( \int_{\Gamma_{b_2}} + \int_{\Gamma_{b_2}'} \right) \zeta e^{isy} ds \quad (5.A.5)$$

where $\Gamma_{b_2}$ and $\Gamma_{b_2}'$ are the branch lines along the branch cut $\Gamma_b \ (r^2 = 0)$.

Let $b_2 = -\frac{\sigma + i\varepsilon}{\sqrt{gH}}$ and $s = b_2 + e^{-i\pi/2}$ along the branch line $\Gamma_{b_2}$, where $\varepsilon_a = b_2 + \varepsilon$ and $R$ goes from $-\infty$ to $-\varepsilon_a$. Hence, the first integral in (5.A.5) becomes

$$I_{\Gamma_{b_2}} = ie^{-i\pi/2} \int_{-\infty}^{-\varepsilon_a} \zeta (b_2 + Re^{-i\pi/2})e^{i(b_2 + Re^{-i\pi/2})y} dR$$

Similarly, let $s = b_2 + Re^{3i\pi/2}$ along the branch line $\Gamma_{b_2}'$, where $R$ goes from $-\varepsilon_a$ to $-\infty$. Hence, it can be shown that the second integral of $I_{b'}$ is equivalent to the first integral in (5.A.5). That is,

$$I_{b'} = 2e^{ib_2Y} \int_{-\varepsilon_a}^{-\infty} \frac{C(b_2 - iR)(J_{-\nu}(\kappa\Delta)\alpha_1 - \alpha_2^* J_{-\nu}(\kappa\Delta))}{(\alpha_1\beta_2 - \alpha_2^* \beta_1)(\alpha_1^* \beta_2^* - \alpha_2^* \beta_1^*)} e^{Ry} dR \quad (5.A.6)$$

where $r^* = R^2 + 2b_2 Y$,

$$C(b_2 - iR) = \frac{8WnR^*}{a^3gh_0\kappa}$$

$$\beta_i = \kappa\Delta J_{-\nu}(\kappa\Delta) + (1 - \frac{2ir^*}{a}) J_{\nu}(\kappa\Delta)$$

$$\beta_i^* = \kappa\Delta J_{-\nu}(\kappa\Delta) + (1 + \frac{2ir^*}{a}) J_{\nu}(\kappa\Delta)$$

It can be shown from the power series for the Bessel function that

$$J_{\nu}(\kappa) \sim \sqrt{\frac{2}{\pi\kappa}} \cos(\kappa - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)$$
for large values of $\kappa$ (Abramowitz and Stegun). Hence, the integrand of (5.A.6) is of the order $e^{R(y-y_0)}$, for large values of $\kappa$. As a result, the integral in (5.A.6) is a decreasing function of $R$, provided that $y \geq y_0$. Hence, it is assumed that the integral $I_b'$ in (5.A.6) is negligible compared to the sum of the residues of (5.A.1), provided $y \geq y_0$. From (5.A.2), consider the integral $I_c$ where $c$ is the circle around the branch point $b_1 = -\frac{i}{2}$. Let $s = b_1 + \varepsilon e^{i\theta}$ then

$$I_c = i\varepsilon \int_0^{2\pi} \zeta(b_1 + \varepsilon e^{i\theta}) e^{i(b_1 + \varepsilon e^{i\theta})} e^{i\theta} d\theta.$$  \hspace{1cm} (5.A.7)

From (5.A.7), $\zeta(b_1 + \varepsilon e^{i\theta})$ is finite as $\varepsilon \to 0$. Therefore, $I_c \to 0$ as $\varepsilon \to 0$. Similarly, let $s = b_2 + e^{i\theta}$ in the integral $I_c'$ of (5.A.2). It can be shown that $I_c' \to 0$ as $\varepsilon \to 0$.

Therefore, using the results of (5.A.4), (5.A.5), (5.A.7) and the asymptotic result of (5.A.6) for large values of $\kappa$, let $R \to \infty$ and $\varepsilon \to 0$, then (5.A.1) becomes

$$\int_C \zeta e^{isy} ds = \pi i \sum \text{Residues}$$

provided that $y \geq y_0$. 
From (5.23),
\[ I = \int_C \zeta e^{isy} \, ds \]  \hspace{1cm} (5.B.1)

where \( \zeta \) is a function of \( s \), which is defined in (5.31). \( C \) is the appropriate contour, as found in Figure 5.3, for \( y > y_0 \). That is, \( C \) is the contour found in the lower half \( s \)-plane.

Therefore, (5.B.1) becomes
\[ I = \left( \int_{\Gamma_1} + \int_{\Gamma_b} + \int_c + \int_{\Gamma_1'} \right) \zeta e^{isy} \, ds \]  \hspace{1cm} (5.B.2)

Consider the contours \( \Gamma_1 \) and \( \Gamma_1' \). Let \( s = Re^{i\theta} \), then
\[
I_R = \left( \int_{\Gamma_1} + \int_{\Gamma_1'} \right) \zeta e^{isy} \, ds \\
= \left( \int_{0}^{\pi/2} + \int_{-\pi/2}^{-\pi} \right) \zeta e^{iR e^{i\theta} y} iRe^{i\theta} \, d\theta
\]

If \( |\zeta(R e^{i\theta})| \leq \frac{M}{R^{k+1}} \), where \( k \geq 0 \) then as \( R \to \infty \), \( I_R \to 0 \).

In Copson, it is found that for large values of \( \nu \)
\[ J_\nu(\kappa) = \frac{e^\nu \left( \frac{a}{2} \right)^\nu}{\sqrt{2\pi \nu} \nu^\nu} \left[ 1 + \frac{a_1}{\nu} + \frac{a_2}{\nu^2} + \cdots \right] \]

Also,
\[ \nu = O(R) \] 
\[ r = O(R) \] and 
\[ \frac{\eta}{\alpha_1} = O\left( \frac{1}{R} \right) \]

Hence, \( |\zeta| = O\left( \frac{1}{R} \right) \) as required for convergence.
Noting from (5.31), that $\zeta$ is a function of $e^{i\alpha_0}$, then

$$|I_R| \leq \frac{2\pi}{\pi} e^{-R \sin \theta (y - y_0)} d\theta .$$

As $R \to \infty$, $|I_R| \to 0$, provided that $y \geq y_0$.

Consider the integral

$$I_b = \left( \int_{\Gamma_{b_1}} + \int_{\Gamma_{b_i}} \right) \zeta e^{i\alpha_0} ds ,$$  \hspace{1cm} (5.B.3)

where $\Gamma_{b_1}$ and $\Gamma_{b_i}$ are the branch lines along the branch cut $\Gamma_b$, when $v^2 = 0$.

Let $b_1 = -\frac{i\alpha_0}{2}$ and $s = b_1 + Re^{i\pi/2}$ along the branch line $\Gamma_{b_1}$, where $\varepsilon_0 = b_1 + \varepsilon$ and $R$ go from $-\infty$ to $-\varepsilon_0$. Hence, the first integral in (5.B.3) becomes

$$I_{\Gamma_{b_1}} = \int_{-\varepsilon_0}^{\varepsilon_0} \zeta (b_1 + Re^{i\theta}) e^{i(Re^{b_1} + e^{i\theta})} Re^{i\theta} dR .$$

Similarly, let $s = b_1 + Re^{i\pi/2}$ for the branch line $\Gamma_{b_{i+1}}$, where $R$ goes from $-\varepsilon_0$ to $-R$. Hence, the second integral in (5.B.3) can be shown to be equivalent to $I_{\Gamma_{b_{i+1}}}$. Therefore,

$$I_b = ie^{ib_1} \int_{-\varepsilon_0}^{\infty} C(b_1 - iR) \left( \frac{\eta \alpha^*_1 - \eta^*_\alpha_1}{\alpha_1 \alpha^*_1} \right) e^{Ry} dR$$  \hspace{1cm} (5.A.4)

where $\nu^2 = \frac{4}{a^2}(R^2 + 2ib_1 R),

\begin{align*}
C( b_1 - iR ) &= -\frac{8iY_0 \sin(b_1 - iR)y_0}{a^2 y_0} ,
\eta &= \int_{0}^{\kappa} J_{i\nu}(z) dz ,
\alpha_1 &= \kappa J'_{i\nu}(\kappa) + J_{i\nu}(\kappa) .
\end{align*}$
and $\eta^*$ and $\alpha_1^*$ are the complex conjugates of $\eta$ and $\alpha_1$, respectively, with respect to $\nu$. It can be shown from the power series for the Bessel function that

$$J_\nu(\kappa) \sim \sqrt{\frac{2}{\pi \kappa}} \cos(\kappa - \frac{1}{2} \sqrt{\pi} - \frac{1}{4} \pi)$$

for large values of $\kappa$ (Abramowitz and Stegun). Hence, the integrand of (5.A.4) is of the order $e^{R(y-y_0)}$, for large values of $\kappa$. As a result, the integral in (5.A.4) is a decreasing function of $R$, provided that $y > y_0$. Hence, it is assumed that the integral $I_b'$ in (5.A.4) is negligible compared to the sum of the residues of (5.A.1), provided $y > y_0$.

Consider the integral $I_c$ in (5.B.2), where $c$ is the integral around the branch point $b_1 = -\frac{i\eta}{2}$. Let $s = b_1 + \epsilon e^{i\theta}$ then substitute into $I_c$. Hence,

$$I_c = i\epsilon \int_0^{2\pi} \zeta(b_1 + \epsilon e^{i\theta}) e^{i(b_1 + \epsilon e^{i\theta})} e^{i\theta} d\theta. \quad (5.B.5)$$

From (5.B.5), $\zeta(b_1 + \epsilon e^{i\theta})$ is finite as $\epsilon \to 0$. Therefore, $I_c \to 0$ as $\epsilon \to 0$.

Using the above results, let $R \to \infty$ and $\epsilon \to 0$ then (5.B.1) becomes

$$\int_C \zeta e^{isy} ds = \pi i \sum \text{Residues}$$

provided that $y > y_0$. 


Dawson, W.B. (1907). The Currents in Belle Isle Strait. Dept. of Marine and Fisheries, Ottawa, Ont., Canada.


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All inquiries should be addressed to
The Editor-in-Chief, CSIRO
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Wind-generated, High-frequency Edge Waves

A. L. Worthy

Department of Mathematics, University of Wollongong, Wollongong, N.S.W. 2500.

Abstract

The effect of longshore wind stress on high-frequency edge waves, over a convex, exponential, semi-infinite sheet, is determined analytically. Comparison is made with physical data measured at Port Kembla Harbour, on the eastern coast of Australia. Good agreement is obtained for the amplitudes of the waves, as well as for their periods.

Introduction

For an homogeneous rotating fluid, Mysak (1968) classified edge waves into two types: class I waves were high-frequency waves with periods of less than 2 h, and class II waves were high-frequency waves with periods of about 1 day or more.

The propagation of class II waves over various shelf profiles has been examined by, for example, Reid (1958), Mysak (1968), Buchwald and Adams (1968) and Clarke and Louis (1975). Extensive pioneering work by B. V. Hamon (e.g. Hamon 1958, 1962, 1966; Hamon and Stacey 1960; Hamon and Grieg 1972) has been carried out on the effects of weather systems on class II edge waves in Australia. Further detailed analysis has been done by such authors as Robinson (1964), Adams and Buchwald (1969), Gill and Schumann (1974), Allen (1976) and Clarke (1977).

Freely propagating class I waves have been examined by Reid (1958), Clarke (1973) and Louis (1975). In 1956, Greenspan examined the effect of a pressure system, such as a hurricane, on class I waves over a linearized ocean depth. He found that there was a resurgent wave motion, which consisted of an infinite number of edge-wave modes. In 1968, Clarke et al. observed the presence of long waves with very high frequency and large amplitude in Jervis Bay, N.S.W. As a result, Buchwald and de Szoeko (1973) showed analytically that the same period and amplitude, but perhaps not the same duration, are produced from a pressure front along a step shelf, provided the speed of the disturbance is between the speed of the long waves on the shelf and the speed of the long waves of the ocean.

Buchwald and de Szoeko (1973) considered a step profile for the eastern coast of Australia, which was suitable for studying waves of very high frequency (with periods of 3 min), which are only affected by the first kilometre of the shelf. For the larger period waves to be considered in this study, the effect extends to the order of 20 km (Louis 1975). Thus, the exponential shelf is more applicable.

Viera and Buchwald (1982) examined class I waves over an exponential shelf generated by a travelling pressure disturbance. They showed that a travelling pressure disturbance may be followed by a wake of comparatively large edge waves and illustrated this phenomenon by using parameters appropriate to the eastern coast of Australia.
In this paper, the effects of a longshore wind stress on class I waves is examined. The profile of the shelf is taken as an exponential, which is similar to the shelf profile off the eastern coast of Australia (see, for example, Buchwald and Adams 1968). This particular profile is used to illustrate the long-wave disturbances, observed by Clarke (unpublished data) in Port Kembla Harbour, N.S.W. The periods of these disturbances range from 1 h to less than 1 h on days following strong southerly winds called southerly ‘busters’.

The Equations

Choosing a right-handed coordinate system with z vertically up and x normal to the coast, the equations of motion and continuity in the linearized, shallow-water wave theory are:

\[ u_t = -g \frac{\partial \xi}{\partial x}, \]
\[ v_t = -g \frac{\partial \xi}{\partial t} + Y/h, \]
\[ (hu)_x + (hv)_y = -Y_t, \]

where \( u, v \) are the depth-averaged components of velocity in the \( x, y \) directions, respectively; \( \xi \) is the surface elevation; \( Y \) is the longshore wind stress acting on the surface divided by the density \( \rho_w \) of water; and \( h \) is the depth.

Wind-stress forcing is preferred to pressure forcing on the basis that for long shallow fronts, such as southerly ‘busters’, the pressure may be less than the wind stress if the wind stress is associated with strong wind gusts. The isobars from these southerly ‘busters’ have a tendency to align themselves parallel to the coast, as discussed by such authors as Hunt (1894), Gentilli (1969) and Baines (1980). Consequently, the change in pressure is small compared to the wind-stress term. Hence, the pressure term has not been included in the above equations. It should be noted, however, when considering the generation of high-frequency waves due to hurricanes, that the dominant force associated with the hurricanes is the pressure. This has been demonstrated by Le Blond and Mysak (1978).

Also in the above equations, the Coriolis force has been neglected since its effect on high-frequency edge waves of periods less than 1 h is minimal. The effect of offshore wind stress is minimal and so it is also neglected. All frictional forces are neglected.

The elimination of \( u, v \) from equations (1)-(3) yields

\[ gh \nabla^2 \xi + g \frac{\partial h}{\partial x} \frac{\partial \xi}{\partial x} - \xi_t = Y_t. \]

Buchwald and Adams (1968) used a convex exponential shelf for their continental-shelf profile bordering the ocean. The same exponential-shelf profile will be used except it will be represented as a semi-infinite sheet. The choice of this shelf profile follows the work of Louis (1975) who has shown that the deep-water part of the shelf has little effect on high-frequency edge waves. Hence, the shelf profile is given by

\[ h = h_0 \exp(\alpha x), \quad 0 \leq x < \infty. \]

Buchwald and Adams (1968) used a convex exponential shelf for their continental-shelf profile bordering the ocean. The same exponential-shelf profile will be used except it will be represented as a semi-infinite sheet. The choice of this shelf profile follows the work of Louis (1975) who has shown that the deep-water part of the shelf has little effect on high-frequency edge waves. Hence, the shelf profile is given by

\[ h = h_0 \exp(\alpha x), \quad 0 \leq x < \infty. \]

The boundary condition at the coast is that the velocity is zero normal to the coast, that is,

\[ \xi_x = 0 \text{ at } x = 0, \]

and the criterion for trapped waves is given by

\[ \xi \to 0 \text{ as } x \to \infty. \]
Forcing Effect by a Longshore Wind Stress

Buchwald and de Szoeke (1973) included a sample of anemograph recordings from Port Kembla and Mascot, N.S.W. on 15 May 1968, showing the change in wind direction when a typical local storm front is followed by a southerly gale. It is this switching motion that is responsible for the generation of edge waves. In order to simulate these changing wind directions, the wind stress used in equation (8) is taken to be a simple oscillatory function. Accordingly, the rectangular wave distribution to be considered is of the form

\[ Y = Y_0 \left[ H(y+y_0) - H(y-y_0) \right] e^{-i\sigma t} \quad (8) \]

where \( H(y) \) is the Heaviside unit function, \( \sigma \) is the frequency and \( y_0 \) is the constant related to the wind fetch. \( Y_0 \) determines the amplitude of the wind stress, which is taken to be constant. This distribution is similar to that used by Adams and Buchwald (1969). The Fourier transform of equations (4) and (6)-(8) gives

\[ \mathcal{X} \mathcal{d} \frac{d}{dX} \left( \mathcal{X}^{-1} \frac{d\mathcal{F}}{dX} \right) + \left( 4\sigma^2/a^2 g h_0 - 4\sigma^2/a^2 X^2 \right) \mathcal{F} = 4\mathcal{Y}_y/a^2 gh_0, \quad (9) \]

\[ \frac{d\mathcal{F}}{dX} = 0 \text{ at } X = 1, \quad (10) \]

and

\[ \mathcal{F} \to 0 \text{ as } X \to 0 \quad (11) \]

where

\[ \mathcal{F} = \mathcal{F}_\infty(x,y)e^{-i\sigma t}, \quad (12) \]

\[ \mathcal{F} = \int_{-\infty}^{\infty} \mathcal{F}_\infty e^{-i\nu y} dy, \quad (13) \]

\[ X = e^{-a\nu^2/2}, \quad (14) \]

and \( \mathcal{Y}_y \) denotes the Fourier transform of \( Y_y \). In obtaining equations (9)-(11), it is assumed that \( \mathcal{F}_\infty \) and \( (\mathcal{F}_\infty) \to 0 \) as \( y \to \pm \infty \).

The solution of equation (9) is obtained by first replacing \( \sigma \) by \( \sigma + i\epsilon \). This ensures that the radiation condition for outward travelling waves as \( y \to \pm \infty \) is satisfied. Those solutions that do not satisfy \( \mathcal{F}_\infty \to 0 \) as \( y \to \pm \infty \) are discarded. The solution \( \mathcal{F}_\infty \) is then found by letting \( \epsilon \to 0 \).

The general solution of equation (9) with the new value of \( \sigma \) is

\[ \mathcal{F} = (A+u_1) \mathcal{X} \mathcal{J}_\nu(\kappa X) + (B+u_2) \mathcal{X} \mathcal{J}_{-\nu}(\kappa X), \quad (15) \]

where

\[ \kappa = 2(\sigma+i\epsilon)/a(gh_0)^{1/2}, \quad (16) \]

\[ \nu^2 = 4\sigma^2/a^2 + 1, \quad \nu^2 > 0, \quad (17) \]

and \( \nu \) is not an integer.

By assuming that the horizontal transport is zero at infinity (see Louis 1975), the solution is restricted to \( \nu^2 \geq 1 \). Applying the boundary condition (11) to equation (15) sets \( B = 0 \). In the case of \( B = 0 \) and \( \nu^2 = 1 \), there is finite transport at infinity. These solutions are regarded as 'leaky' as discussed by Larsen (1969). From equation (17), it is found that \( \nu = 0 \) constitutes the leaky line, however, since equation (15) still needs to be satisfied there is only a discrete number of leaky modes possible. Leaky modes will not be treated here.
The solution (15) can now be written as
\[ \bar{\psi} = (A+u_1) XJ_\nu(\kappa x) + u_2 XJ_{\nu'}(\kappa x) , \quad \nu^2 \geq 1 . \]

After using boundary condition (10) and then evaluating the wave height at the coast, that is at \( x = 1 \),
\[ \bar{\psi} = -8i Y_0 \eta \sin \nu y_0 / \alpha^2 g h_0 \kappa \alpha_1 , \quad \alpha_1 = \kappa J'_\nu(\kappa) + J_\nu(\kappa) , \]
\[ \eta = \int_0^\nu J_\nu(z) \, dz . \]

Therefore, \( \bar{\psi} \) is given by
\[ \bar{\psi} = (2\pi)^{-1} \int_{-\infty}^\infty \bar{\psi} e^{i\nu} \, d\nu . \]
To determine \( \bar{\psi} \), let \( I = \oint \bar{\psi} e^{i\nu} \, dy \),
where the contour is appropriately chosen after consideration of the poles and branch points of \( \bar{\psi} \).

Amplitude at the Coast
Replacing \( \sigma \) by \( \sigma + i\epsilon \) in equation (9) causes the poles to be removed from the real axis. The displacement of the poles is determined by the following procedure.
Let \( s = s_n, n = 0, \pm 1, \pm 2, \ldots \) be the roots of
\[ \alpha_1 = \alpha_1(\sigma, s) = 0 . \]
These roots are the poles of the integrand in equation (21). Now \( ds = i\epsilon \) and \( \frac{d\alpha_1}{d\sigma} + \frac{d\alpha_1}{ds} \, ds \), hence, the displacement of the poles is determined numerically by the sign of \( \frac{d\alpha_1}{d\sigma} \).

It is found that the positive poles move to the upper half-plane and the negative poles into the lower half-plane in the complex \( \nu \)-plane. When \( \nu = 0 \), the pole moves to \( -i\infty \). There are branch points at \( \nu = 0 \), that is, \( s = \pm i\alpha/2 \). Equation (22) is the dispersion curve for the freely propagating edge waves on the semi-infinite shelf as found by Louis (1975).
The appropriate contour \( C \) is illustrated in Fig. 1. The path of integration for equation (21) when \( y \geq y_0 \) is \( \Gamma_1 \), and when \( y \leq -y_0 \), \( \Gamma_2 \) is required.
The contributions of the paths \( \Gamma_1 \) and \( \Gamma_2 \) are vanishingly small by Jordan’s Lemma, so the integral (21) can now be replaced by \( 2\pi i \) times the sum of the residues in the appropriate half-plane, with the result, using equations (12), (17) and (20),
\[ \bar{\psi} = \sum_{n=1}^\infty \gamma_n \cos(\sigma_n y + \sigma t) , \quad \gamma_n = \frac{2 Y_0 y_n \eta_n \sin \nu \nu_0 / \kappa \epsilon h_0 \nu_n \left( \frac{\partial \alpha_1}{\partial \nu} \right)_{\nu=\nu_n} } . \]

for \( y \geq y_0 \), where
\[ \gamma_n = 2 Y_0 y_n \eta_n \sin \nu \nu_0 / \kappa \epsilon h_0 \nu_n \left( \frac{\partial \alpha_1}{\partial \nu} \right)_{\nu=\nu_n} . \]
A similar result is obtained for \( \nu < -\nu_0 \). Also, on substitution of equation (24) into equation (23), the solution (23) becomes the sum of two waves, one generated by a delta function in the wind stress at \(-\nu_0\) and the other at \(\nu_0\). The sum gives an amplitude-modulated wave and it is this amplitude-modulated wave that gives a frequency-dependent amplification. The wave-height amplitude for the \(n\)th mode is given by \( |\gamma_n| \).

![Complex s-plane, showing paths of integration, branch cuts and poles in the respective half-planes.](image)

**Results**

The eastern coast of Australia is used to illustrate the foregoing analysis. Typical values for the shelf parameters are \( h_0 = 70 \text{ m}, \omega = 5 \times 10^{-3} \text{ m}^{-1} \), which are similar to those used by Buchwald and Adams (1968). Strong southerly winds with speeds around 35 knots are observed along the eastern coast of Australia. These southerlies are progressive with respect to time, whereas the wind-stress model [equation (8)] being considered here assumes the motion is oscillatory. To obtain the wave height in equation (4), the method of solution is simplified using equation (8) as the wind-stress model. It will be assumed that the speed of the storm front is the same as the wind speed, hence \( \nu_0 = 0.5 \text{ N m}^{-2} \).

It is found that the wave height with period equal to 60 min produces the largest amplitude. To maximize this amplitude, the wind fetch is taken to be 200 km, which is approximately the distance from Eden to Sydney on the eastern coast of Australia. Using this value, the amplitude of the wave heights for various frequencies were obtained. It should be noted that changing the wind fetch does cause a change in amplitude of the wave at various frequencies.

In the following tabulation, the period and amplitude of the significant long-wave disturbances using the above parameters are given. The first three co-efficients in equation (23) were used to determine the amplitude of \( \gamma \).

<table>
<thead>
<tr>
<th>Period (min)</th>
<th>60</th>
<th>42</th>
<th>32</th>
<th>26</th>
<th>21</th>
<th>18</th>
<th>16</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude (cm)</td>
<td>4.4</td>
<td>1.3</td>
<td>0.9</td>
<td>1.6</td>
<td>0.8</td>
<td>0.7</td>
<td>0.6</td>
<td>0.5</td>
</tr>
</tbody>
</table>

During days of gusty southerly winds, wave height was measured in Port Kembla Harbour by Clarke (unpublished data). It was found that there were long-wave disturbances with periods of 56, 39, 33, 27, 17, 14.8, 10.7 min. These waves are not always present following days of gusty southerly winds nor is any one necessarily present at any given time. The periods of these measured long-wave disturbances are within 1.3 min of the calculated periods in the above tabulation, including those frequencies in the 21.14-min range. The amplitudes of the ocean disturbances typically range from 1.5 cm for the
60-min period to 0.5 cm for the 13-min period. These amplitudes are in agreement with those shown in the tabulation, except for the 60-min period. However, when the wind fetch falls below 200 km, the computed amplitude of the 60-min wave rapidly decreases to about 1.5 cm.

In the tabulation, the periods of long-wave disturbances cannot all be of freely propagating edge waves. The long-wave disturbance with period of 42 min does not correspond to the harmonics of the free wave (see Clarke 1973). This implies that the 42-min period is due to the forcing effect. Headlands may also influence the forced-wave propagation by limiting the fetch length, particularly for waves of higher frequency. At a distance of 60 km south of Port Kembla Harbour, there is a prominent headland, known as Beecroft Head, which may influence the long-wave disturbance of period 42 min, since the maximum computed amplitude of 1.2 cm is obtained when the wind fetch is at a distance of 60 km. A similar influence on the 14-min period may possibly be attributed to Bass Point, which is 12 km south of Port Kembla Harbour.

Discussion

To obtain the above results, the condition $|y| > y_0$ is required for the method of solution to exist. This means that the observation point is outside or on the fringe of the storm front. For results inside the storm front (that is $|y| \leq y_0$), a different method of solution would be required.

The linear shelf profile was used by Worthy (unpublished data) to generate wind-forced, high-frequency edge waves. When examining the generation of these waves, where the period of the wave is in the vicinity of 10 min or less, the linear shelf is an adequate approximation to the continental shelf off the eastern coast of Australia. However, when considering forced waves with periods in the range of 10–60 min, the shelf plays an important role in determining long-wave disturbances. Buchwald and Adams (1968) have shown that the exponential-shelf profile, using the shelf parameters $a = 5.33 \times 10^{-5} \text{m}^{-1}$ and $h_0 = 70 \text{m}$, is a good approximation to the continental shelf off the eastern coast of Australia. Hence, the semi-infinite exponential shelf was used. It should be noted, however, that waves with periods near 60 min are also affected by shelf truncation.

Although the wind-stress model in equation (8) is not a typical model of a storm front in that it is oscillatory, the theoretical results in the tabulation on p. 5 are in close agreement with physical results obtained at Port Kembla Harbour, N.S.W. The difference in amplitude for the 60-min period could be due to the fact that this period is near the high-frequency cut-off and consequently small errors in $\gamma$ produce large errors in $\xi$ during computation of the Bessel functions.

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References


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Wind Forcing of Edge Waves on a Truncated Exponential Shelf

Annette Lorraine Worthy

UDC 551.466.44

Summary

The effects of wind forcing on high frequency edge waves over a truncated exponential shelf is examined. Two wind stress models are used and a comparison of results is made. Also, the results are compared to those obtained for the semi-infinite continental shelf profile.

Introduction

This paper is concerned with the problem of wind forcing of waves on a truncated convex exponential continental shelf. The general theory is based on the results obtained by Worthy [1984] for wind forcing on a semi-infinite shelf. Since the results for the convex exponential shelf can be obtained by changing the boundary conditions in the general theory which describes the semi-infinite shelf model we will begin with the equations from the general theory.

General theory

Choosing a right handed co-ordinate system with z vertically up, the x axis is normal to a (locally) straight coastline. Then, the equations of motion and continuity in the linearized shallow water wave theory are

\[ u_t = -g(\zeta + \Phi)_x + X/h \]
\[ v_t = -g(\zeta + \Phi)_y + Y/h \]

and

\[ (hu)_x + (hv)_y = -\zeta \]

where \( u, v \) are the depth average components of velocity in the \( x, y \) directions respectively.
\( \zeta \) is the surface elevation, \( \Phi \) is the atmospheric pressure, \( X, Y \) are the \( x, y \) components of wind stress acting on the surface divided by the density of water and \( h \) is the depth of the ocean from the free surface.

Since the effects of the Coriolis force on high frequency edge waves with periods less than an hour, within the earth's mid-latitudes, is minimal then the Coriolis force will be neglected in the forgoing analysis. Similarly, all frictional forces will be neglected. Also, it is assumed that the effects of the off-shore wind stress are negligible.

Finally, under the conditions associated with a southerly 'buster', the pressure gradient term in the equations of motion is less significant than the wind stress term. Consequently, the pressure term \( \Phi \) will be neglected. However, it should be noted that when considering hurricane activity the dominant forcing mechanism is the pressure, for example, Buchwald and de Szoeke [1973] used the travelling pressure disturbance in discussing the generation of very high frequency edge waves over a continental step shelf profile.

The elimination of \( u \) and \( v \) from (1) and (2) yields

\[
g h \nabla^2 \zeta + g \frac{\partial h}{\partial x} \psi - \zeta v = Y_v. \tag{3}
\]

A model of a continental shelf is required which can best represent the east coast of Australia. Hence, the model developed by Buchwald and Adams [1968] will be used, that is,

\[
h = \begin{cases} 
    h_0 e^{\alpha x} & 0 \leq x \leq L \\
    H & x > L,
\end{cases} \tag{4}
\]

where \( h_0 \) is the depth at the coast, \( \alpha \) is the shelf parameter, \( L \) is the width of the continental shelf and \( H \) is the depth at the edge of the shelf.

The boundary condition at the coast is that the velocity is zero normal to the coast, that is,

\[
\zeta_x = 0 \quad \text{at} \quad x = 0. \tag{5}
\]

At the edge of the shelf, Louis [1975] has shown that

\[
\zeta_x + \alpha \zeta = 0 \quad \text{at} \quad x = L, \tag{6}
\]

where

\[
r^2 = s^2 - \frac{\sigma^2}{gh} > 0,
\]

and \( s \) being the wave number associated with \( \rho \).

The criterion for trapped waves using the truncated shelf profile in (4) is that \( r^2 > 0 \). Solutions for \( r^2 \leq 0 \) which are called leaky waves will not be considered in this paper.

**Forcing mechanism**

The longshore wind stress will be modelled by the same rectangular wave distribution used by Worthy [1984] and is similar to that used by Adams and Buchwald [1969]. Hence,

\[
Y = Y_0 \left( H(y + y_0) - H(y - y_0) \right) e^{-i\sigma}, \tag{7}
\]

where \( H(y) \) is the Heaviside unit function, \( \sigma \) is the frequency and \( y_0 \) is a constant related to the wind fetch. \( Y_0 \) is the amplitude of the wind stress which will be taken to be constant. The term \( e^{-i\sigma} \) in (7) ensures the periodic shearing motion, which is primarily responsible for the generation of edge waves we are considering.
Taking the Fourier transform of (3) gives

\[ X \frac{d}{dX} \left( X^{-1} \frac{d\xi}{dX} \right) + \left( \frac{4a^2}{a^2 gh_0} - \frac{4s^2 X^2}{a^2} \right) \xi = \frac{4\tilde{W}}{a^2 gh_0}. \] (8)

Equations (5) to (7) give

\[ \frac{d^2 \xi}{dX^2} - \gamma_0^2 \xi = 0 \quad \text{at} \quad X = 1, \]
(9)

\[ X \frac{d^2 \xi}{dX^2} - \frac{2r^2}{a} \xi = 0 \quad \text{at} \quad X = \Delta = e^{-aL/2}, \]
(10)

where

\[ \xi = \tilde{\zeta}, \quad e^{-i\sigma_1}, \]
(11)

\[ \tilde{W} = F \{ Y, e^{i\sigma_1} \} = 2iY_0 \sin(sy_0) e^{i\sigma_1} \]
(12)

and

\[ X = e^{-aL/2}. \]

In deriving equations (8) to (11), it is assumed that \( \tilde{\zeta} \) and \( (\tilde{\zeta}_n)_n \to 0 \) as \( y \to \pm \infty \).

Anticipating difficulties with the Fourier inversion of \( \tilde{\zeta} \), \( \sigma \) is replaced by \( \sigma + i\varepsilon \), where \( \varepsilon > 0 \). This ensures that the radiation condition for outward travelling waves as \( y \to \pm \infty \) is satisfied. Those solutions which do not satisfy \( \tilde{\zeta} \to 0 \) as \( y \to \pm \infty \) are discarded. The wave height \( \tilde{\zeta} \) is then found by letting \( \varepsilon \to 0 \).

The general solution of (8) with \( \sigma \) replaced by \( \sigma + i\varepsilon \) is

\[ \xi = (A + u_1)XJ_v(xX) + (B + u_2)XJ_{-v}(xX) \]
(13)

where

\[ x = \frac{2(\sigma + i\varepsilon)}{a \sqrt{gh_0}}, \]
(14)

\[ v^2 = \frac{4s^2}{a^2} + 1, \quad v^2 > 0, \]

\( J_v \), Bessel Function

\[ \frac{a^2 gh_0}{a^2 g h_0} \sin(v\pi) u_1 = 2\pi \tilde{W} X J_v(xX') dX' \]

\[ \frac{a^2 gh_0}{a^2 g h_0} \sin(v\pi) u_2 = -2\pi \tilde{W} X J_v(xX') dX' \]

\( A, B \) constants,

and \( v \) not an integer.

Applying the boundary conditions (9) and (10) to (14) and also evaluating \( \xi \) at the coast, i.e. \( x = 0(X = 1) \), it is found that

\[ \xi(X = 1) = \frac{4\tilde{W} \eta}{a^2 gh_0} \frac{1}{x(\gamma_1 \beta_2 - \gamma_2 \beta_1)} \]
(15)
where

\[ y_1 = xJ_0'(x) + J_0(x), \]
\[ y_2 = xJ_{-1}'(x) + J_{-1}(x), \]
\[ \beta_1 = x\Delta J_0'(\kappa\Delta) + (1 - \frac{2r}{a})J_0(\kappa\Delta), \]
\[ \beta_2 = x\Delta J_{-1}'(\kappa\Delta) + (1 - \frac{2r}{a})J_{-1}(\kappa\Delta) \]

and

\[ \eta = \int_{\kappa\Delta}^{\kappa} [\beta_1 J_{-1}(z) - \beta_2 J_0(z)] \, dz. \]

Therefore, \( \xi \) is given by

\[ \xi = \frac{1}{2\pi i} \int_{C} \xi e^{i\sigma} \, ds. \quad (16) \]

To determine \( \xi \), let

\[ I = \int_{C} \xi e^{i\sigma} \, ds \quad (17) \]

where the contour \( C \) is appropriately chosen after consideration of the branch points and poles of \( \xi \) in (17).

**General solution**

The integrand in (17) has branch points at \( v = 0 \) \( (s = \pm ia/2) \) and at \( r = 0 \) \( (s = \pm (\sigma + ie)/(\sqrt{gH})) \). The high frequency cut off for edge waves corresponds to the branch points at \( r = 0 \). Due to \( \sigma \) being replaced by \( \sigma + ie \) the poles of \( \xi \) are displaced from the real axis. The direction of displacement of the poles are determined by the following method.

Let \( s = s_n, n = \pm 1, \pm 2, \ldots \) be the roots of

\[ a = a(\sigma, s) = 0. \quad (18) \]

The roots of (18) are the poles of the integrand in (17). Equation (18) is called the dispersion relation for the forced high frequency edge waves under consideration. Also, (18) is the dispersion relation obtained by Louis [1975] for freely propagating high frequency edge waves on the truncated convex exponential shelf.

From (18),

\[ d\alpha = \frac{\partial \alpha}{\partial \sigma} \, d\sigma + \frac{\partial \alpha}{\partial s} \, ds = 0. \]

Since \( d\sigma = ie \), then the displacement of poles from the real axis is determined by using

\[ ds = -\frac{iev}{sgh_0 x} \frac{\partial \alpha}{\partial \sigma} \frac{\partial \alpha}{\partial x} \frac{\partial \alpha}{\partial v}. \]
Assuming $x$ is positive and given $v > 0$ (from (14)), the displacement of the poles depends on the sign of

$$-\frac{\partial \alpha}{\partial x} \bigg/ \frac{\partial \alpha}{\partial v}. \quad (19)$$

It can be shown that $\frac{\partial \alpha}{\partial x} / \frac{\partial \alpha}{\partial v}$ is an even function of $s$. Consequently, the poles are dispersed evenly into upper and lower half planes. Numerical calculations are required to determine the sign of (19).

The appropriate contour $C$ from (17) is illustrated in Figure 1. The path of integration for $y \geq y_0$ is $I_1$, the lower half plane, and when $y \leq -y_0$, the path of integration is $I_2$, the upper half plane.

By Jordan's Lemma the contributions from the path $I_1$ approaches zero when $y \geq y_0$. The branch line path $I_2$ has a cancelling effect. Therefore, (17) can now be replaced by $2\pi i$ times the sum of the residues in the appropriate half plane. Hence, for $y \geq y_0$ the resultant solution from (11) is

$$\zeta = \sum_{n=1}^{\infty} \gamma_n \cos (s_n - \sigma t), \quad (20)$$

where,

$$\gamma_n = \frac{2Y_0v_0\eta(s_n) \sin (s_n y_0)}{xgh_0s_n \frac{\partial}{\partial v}(\gamma_1 \beta_2 - \gamma_2 \beta_1) \big|_{s=s_n}}. \quad (21)$$

A similar result is obtained for $y \leq -y_0$. The amplitude of the wave height in (20) is given by $|\gamma_n|$.

**Results**

Buchwald and Adams [1968] arrived at typical values for the continental shelf parameters off the east of Australia. Those values were $h_0 = 70$ m, $a = 5.33 \times 10^{-5}$ m$^{-1}$ and the width of the continental shelf being $L = 80$ km. The wind stress parameters used by Worthy [1984] were $Y_0 = 0.5$ N/m$^2$ and the wind fetch, $2y_0 = 200$ km. The above shelf and wind stress parameters will be used in determining analytical results for the theory in this paper.

From (20), the first three terms of the series are used to calculate the wave height $\zeta$ at the coast. The first term of (20) is the most significant, whereas, the next terms in the series are basically correction terms. Figure 2 shows the resulting amplitude in cm plotted against frequency in s$^{-1}$.

Table 1 shows the periods and amplitude of significant long wave disturbances found in Figure 2.

<table>
<thead>
<tr>
<th>Period in min</th>
<th>65</th>
<th>41</th>
<th>32</th>
<th>26</th>
<th>22</th>
<th>18</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amplitude in cm</td>
<td>7.5</td>
<td>1.2</td>
<td>0.87</td>
<td>0.68</td>
<td>0.55</td>
<td>0.47</td>
<td>0.47</td>
</tr>
</tbody>
</table>
Fig. 1. Complex $s$-plane, showing paths of integration, branch cuts and poles in respective half planes.
Fig. 2. Graph of frequency versus amplitude for the truncated exponential shelf using (7) as the wind stress profile.

There is a correspondence between the period and amplitudes obtained in Table 1 and the results for the semi-infinite continental shelf obtained by Worthy [1984]. The amplitudes and periods are similar except for the 65 min period and amplitude. This discrepancy may be due to the high frequency cut-off experienced by waves on a semi-infinite shelf and/or the erratic behaviour of the Bessel function when $v$ is close to an integer. Further, the amplitude concerning the 25 min period wave is quite different compared to the semi-infinite shelf amplitude.

It can be seen that shelf truncation has little effect on the forcing of high frequency edge waves with periods less than an hour. However, waves with periods near the hour, for example the wave with period of 65 min, are affected by the shelf truncation. Consequently, the shelf profile plays an important role for longer period waves. Worthy [1982] has discussed the linearization of the continental shelf for high frequency edge waves with periods of less than 10 min.

It can be shown that the inclusion of the forcing term on the right hand side of (6), and hence (10), has little effect on the results obtained in Table 1.

As discussed by Worthy [1984], the periods obtained in Table 1 closely approximate the periods obtained by Clarke [1979] using Fourier analysis techniques on current meter recordings that were obtained outside Port Kembla Harbour, N.S.W. Australia on days which had gusty southerly winds. The periods from these recordings were not always present nor was any one present at any one time following days of gusty southerly winds.

Also, from Table 1 it appears that the periods closely approximate the harmonics of two hours.
The results in Table 1 are obtained by using a rectangular wave distribution for the wind stress profile. Suppose that a Gaussian wave distribution of the form

\[ Y = Y_0 e^{-b^2 y^2} \quad \text{for} \quad y \ll \infty, \tag{22} \]

is used instead of the rectangular wave distribution, then (12) will become,

\[ \tilde{W} = F \{ Y, e^{i\omega} \} = \frac{i \sqrt{\pi} Y_0 e^{-b^2 (i \omega)^2} e^{i\omega}}{b}. \]

Consequently, (21) becomes

\[ \gamma_n = \frac{-Y_0 v_n \eta(s_1) e^{-b^2 (i \omega)^2}}{g h e b \frac{\partial}{\partial y} (\gamma_1 \beta_1 - \gamma_2 \beta_2) \Gamma(s_1) \Gamma(n)}. \tag{23} \]

The wind stress profile in (22) ensures that the first mode is the most dominant mode. Therefore, only the first term is calculated using (23). A summary of results can be obtained provided desirable values of the wind stress parameter \( b \) are obtained. The shelf parameters and \( Y_0 \) will be chosen to be the same as those used in Table 1.

The maximum amplitude for \( \gamma_1 \) is obtained when \( b = s_1/\sqrt{3} \), for each wave number \( s_1 \). However, only one value of \( b \) can be chosen at any one time, therefore, values of \( b \) were chosen to correspond to different orders of magnitude of the wave number, \( s_1 \), i.e., \( b = 0(10^{-5}), 0(10^{-4}), 0(10^{-3}). \) Table 2 shows the dominant amplitudes with corresponding period for \( b = 0(10^{-4}) \) and wave number along with the maximum obtainable amplitude by using \( b = s_1/\sqrt{3}. \)

<table>
<thead>
<tr>
<th>Period in min</th>
<th>Amplitude in cm</th>
<th>Max. amplitude in cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>65</td>
<td>0.83</td>
<td>6.4</td>
</tr>
<tr>
<td>54</td>
<td>0.46</td>
<td>1.6</td>
</tr>
<tr>
<td>44</td>
<td>0.48</td>
<td>1.0</td>
</tr>
<tr>
<td>32</td>
<td>0.52</td>
<td>0.7</td>
</tr>
</tbody>
</table>

From Table 2, the most significant wave amplitude is the one that corresponds to the 65 min wave period when \( b = 0(10^{-4}). \) This is the same wave period obtained from Table 1 using the rectangular wave distribution as the wind stress profile. Also, as the order of magnitude of \( b \) becomes larger the harmonics of around two hours appear although the amplitudes are relative small. These harmonics closely approximate those found in Table 1, with the exception of the 54 min wave period.

The most significant difference in the two wind stress profiles used above is that the Gaussian wave distribution has no dependence on wind fetch, whereas, the rectangular wave distribution has a high dependence on wind fetch via the term \( \sin(s_1 y_0). \) If the wind fetch is relatively small, say \( 0(10), \) then it would be appropriate to say that high frequency edge waves with periods around the hour or less would be affected by such a fetch length. Hence, the wind stress profile in (7) is an appropriate wind stress model.
References


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Anschrift der Verfasserin:
A. L. Worthy
The University of Wollongong, Mathematics Department,
P. O. Box 1144, Wollongong N.S.W. 2500, Australien