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Abstract

We use K-theory to prove an isomorphism theorem for a large class of generalised Bunce–Deddens algebras constructed by Kribs and Solel from a directed graph E and a sequence ω of positive integers. In particular, we compute the torsion-free component of the K0-group for a class of generalised Bunce–Deddens algebras to show that supernatural numbers are a complete invariant for this class.

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THE CLASSIFICATION OF SOME GENERALISED BUNCE–DEDDENS ALGEBRAS

JAMES ROUT

ABSTRACT. We use K-theory to prove an isomorphism theorem for a large class of generalised Bunce–Deddens algebras constructed by Kribs and Solel from a directed graph E and a sequence ω of positive integers. In particular, we compute the torsion-free component of the K_0-group for a class of generalised Bunce–Deddens algebras to show that supernatural numbers are a complete invariant for this class.

1. Introduction

In [7] Kribs and Solel introduced a family of direct limit C^*-algebras constructed from directed graphs E and sequences $\omega = (n_k)_{k=1}^{\infty}$ of natural numbers such that $n_k | n_{k+1}$ for all $k \in \mathbb{N}$. They called these C^*-algebras generalised Bunce–Deddens algebras. The graph E consisting of a single vertex connected by a single loop-edge generates the classical Bunce–Deddens algebras.

Supernatural numbers have been used to classify UHF algebras ([4, Theorem 1.12]) and the classical Bunce–Deddens algebras ([1, Theorem 3.7] and [2, Theorem 4]). Kribs showed in [6, Theorem 5.1] that the generalised Bunce–Deddens algebras corresponding to the graph B_N consisting of a single vertex with N loops, are classified by their associated supernatural numbers in the sense that $C^*(B_N, \omega) \cong C^*(B_N, \omega')$ if and only if $[\omega] = [\omega']$. The special case $N = 1$ is Bunce and Deddens' theorem. Kribs and Solel later showed in [7, Theorem 7.5] that the generalised Bunce–Deddens algebras corresponding to the simple cycle with j edges, are classified by their associated supernatural numbers; again the special case $j = 1$ is the original result of Bunce and Deddens. Kribs and Solel asked in [7, Remark 7.7] for what class of graphs E a similar classification theorem could be obtained. Here we prove that such a theorem can be obtained for the class of generalised Bunce–Deddens algebras corresponding to a given strongly connected finite directed graph E such that 1 is an eigenvalue of the vertex matrix, and the only roots of unity that are eigenvalues are the $\mathcal{P}_E$-th roots of unity, where $\mathcal{P}_E$ is the period of the graph E.

In [10, Proposition 3.11] it was shown that if $[\omega] = [\omega']$ then $C^*(E, \omega) \cong C^*(E, \omega')$ for row-finite directed graphs E with no sinks or sources. The main result of this article (Theorem 6.1) shows that if $C^*(E, \omega) \cong C^*(E, \omega')$ then $[\omega] = [\omega']$ for strongly connected finite directed graphs E such that 1 is an eigenvalue of A_E^t and such that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. We prove this by studying the torsion-free component of $K_0(C^*(E, \omega))$; we assume that 1 is an eigenvalue of A_E^t to ensure that this is nontrivial. The Perron–Frobenius theorem (see [3, Theorem 8.2.1])

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says that if 1 is an eigenvalue of A_E^t, then the $\mathcal{P}_E$-th roots of unity are also eigenvalues of A_E^t. The hypothesis that these are the only roots of unity that are eigenvalues of A_E^t is nontrivial. The *nonnegative inverse eigenvalue problem* asks which sets of n complex numbers $\lambda_1, \dots, \lambda_n$ occur as the eigenvalues of some $n \times n$ nonnegative matrix. Deep results of [5] regarding this problem show that it is possible for any collection of roots of unity to appear as eigenvalues of a nonnegative matrix.

If 1 is not an eigenvalue of A_E^t, then $K_0(C^*(E,\omega))$ is purely torsion and another argument (perhaps along the lines of [6, Theorem 5.1]) will be needed. We have not addressed that case in this article.

We begin in Section 3 with some calculations for the sums of powers of matrices and about cokernels. We show that the matrix $\sum_{i=0}^{n_k/l-1}(A_E^{il})^t$, where $l := \lim_{j\to\infty}\gcd(\mathcal{P}_E, n_j)$ and $\gcd(\mathcal{P}_E, n_k) = l$, is invertible if the only eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity (Lemma 3.2). We recall the equivalence relation $\sim_l$ on E^0 established in [10, Lemma 4.2] to show that $\operatorname{coker}(1 - A_E^l)^t \cong \bigoplus_{i=1}^l \operatorname{coker}(1 - A_E^t)$ (Corollary 3.6).

In Section 4 we compute $K_1(C^*(E,\omega))$ for strongly connected finite directed graphs E such that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. We show that the torsion-free component is isomorphic to l copies of $K_1(C^*(E))$ (Theorem 4.1). We do this by showing that $\ker(1 - A_E(n))^t \cong \ker(1 - A_E^n)^t$ for $n \geq 1$ (Lemma 4.2), and by showing that $K_1(C^*(E(n_k)) \to K_1(C^*(E(n_{k+1}))$ induces the identity map on $\ker(1 - A_E^l)^t \cong \bigoplus_{i=1}^l \ker(1 - A_E^t)$ for all k such that $\gcd(\mathcal{P}_E, n_k) = l$.

In Section 5 we compute the torsion-free component of $K_0(C^*(E,\omega))$ for strongly connected finite directed graphs E such that 1 is an eigenvalue of A_E^t and such that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. We show that this group is isomorphic to l copies of the torsion-free component of $K_0(C^*(E))$ adjoined the supernatural number $[\omega]$ associated to ω (Theorem 5.3). We do this by showing that $\operatorname{coker}(1 - A_E(n)^t) \cong \operatorname{coker}(1 - A_E^n)^t$ for $n \geq 1$ (Lemma 5.5), and by showing that the map $K_0(C^*(E(n_k)) \to K_0(C^*(E(n_{k+1}))$ induces the multiplication by n_{k+1}/n_k map on $\operatorname{coker}(1 - A_E^l)^t \cong \bigoplus_{i=1}^l \operatorname{coker}(1 - A_E^t)$ modulo torsion (Proposition 5.15).

Finally, in Section 6 we prove that if $C^*(E,\omega) \cong C^*(E,\omega')$, then $[\omega] = [\omega']$ for strongly connected finite directed graphs E such that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity (Theorem 6.1). We prove this by recovering the supernatural number $[\omega]$ associated to ω from the torsion-free component of $K_0(C^*(E,\omega))$ (Theorem 6.3).

2. Background

2.1. **Directed graphs and their C^*-algebras.** We use the convention for graph C^*-algebras appearing in Raeburn's book [9]. So if $E = (E^0, E^1, r, s)$ is a directed graph, then a path in E is a word $\mu = e_1 \dots e_n$ in E^1 such that $s(e_i) = r(e_{i+1})$ for all i, and we write $r(\mu) = r(e_1)$, $s(\mu) = s(e_n)$, and $|\mu| = n$. As usual, we denote by E^* the collection of paths of finite length, and $E^n := \{\mu \in E^* : |\mu| = n\}$; we also write $E^{<n} := \{\mu \in E^* : |\mu| < n\}$. We borrow the convention from the higher-rank graph literature in which we write, for example vE^* for $\{\mu \in E^* : r(\mu) = v\}$, and vE^1w for $\{e \in E^1 : r(e) = v \text{ and } s(e) = w\}$. The vertex matrix of E is then the $E^0 \times E^0$ integer matrix with $A_E(v,w) = |vE^1w|$.

We say that E is *finite* if E^0 is finite, that E is *row-finite* if vE^1 is finite for all $v \in E^0$, and that E has no sources if each vE^1 is nonempty. A directed graph is *strongly connected* if for every pair of vertices $v, w \in E^0$, there exists $\mu \in E^* \backslash E^0$ such that

$r(\mu) = v$ and $s(\mu) = w$. The vertex matrix A_E is irreducible if and only if the graph E is strongly connected. The *period* $\mathcal{P}_E$ of a strongly connected directed graph E is given by $\mathcal{P}_E = \gcd\{|\mu| : \mu \in E^*, r(\mu) = s(\mu)\}$ (see for example [8, Section 6] with $k = 1$). The group $\mathcal{P}_E\mathbb{Z}$ is then equal to the subgroup generated by $\{|\mu| : \mu \in vE^*v\}$ for any vertex v of E, and so is equal to $\{|\mu| - |\nu| : \mu, \nu \in vE^*v\}$ for any v.

If E is finite or row-finite and has no sources, then a *Cuntz–Krieger E-family* in a C^*-algebra A is a pair (s, p), where $s = \{s_e : e \in E^1\} \subseteq A$ is a collection of partial isometries and $p = \{p_v : v \in E^0\} \subseteq A$ is a set of mutually orthogonal projections such that $s_e^* s_e = p_{s(e)}$ for all $e \in E^1$, and $p_v = \sum_{e \in vE^1} s_e s_e^*$ for all $v \in E^0$.

The *graph algebra* $C^*(E)$ is the universal C^*-algebra generated by a Cuntz–Krieger E-family [9, Proposition 1.21].

Theorem 7.1 of [9] says that the K-theory of $C^*(E)$ is given by

$$K_1(C^*(E)) \cong \ker(1 - A_E^t), \qquad \text{and} \qquad K_0(C^*(E)) \cong \operatorname{coker}(1 - A_E^t).$$

2.2. **Multiplicative sequences and supernatural numbers.** A *multiplicative sequence* is a sequence $\omega = (n_k)_{k=1}^\infty$ of natural numbers with $n_k | n_{k+1}$ for all $k \in \mathbb{N}$. We say that a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$ divides a multiplicative sequence $\omega' = (m_j)_{j=1}^\infty$, and write $\omega | \omega'$, if for each $k \in \mathbb{N}$ there exists $j(k) \in \mathbb{N}$ such that $n_k | m_{j(k)}$. Define an equivalence relation $\sim$ on $\{(n_k)_{k=1}^\infty : n_k | n_{k+1} \text{ for all } k\}$ by $\omega \sim \omega'$ if $\omega | \omega'$ and $\omega | \omega'$. The *supernatural number* $[\omega]$ associated to ω is the collection $[\omega] := \{\omega' : \omega | \omega' \text{ and } \omega' | \omega\}$.

2.3. **Generalised Bunce–Deddens algebras.** Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sources, and fix $n \geq 1$. Define sets

$$E(n)^0 := E^{<n} \qquad \text{and} \qquad E(n)^1 := \{(e, \mu) : e \in E^1, \mu \in s(e)E^{<n}\},$$

and maps

$$s_n(e, \mu) := \mu \qquad \text{and} \qquad r_n(e, \mu) = \begin{cases} e\mu & \text{if } |\mu| < n - 1 \\ r(e) & \text{if } |\mu| = n - 1. \end{cases}$$

Then $E(n) = (E(n)^0, E(n)^1, r_n, s_n)$ is a row-finite directed graph with no sources. For $\mu \in E^*$, we write $[\mu]_n$ for the unique element of $E^{<n}$ such that $\mu = [\mu]_n \mu'$ for some μ' with $|\mu'| \in n\mathbb{N}$; we think of $[\mu]_n$ as the residue of μ modulo n.

By Theorem 3.4 and Proposition 3.6 of [10] there exist injective homomorphisms $\tilde{j}_{n,mn} : C^*(E(n)) \to C^*(E(mn))$ such that

$$\tilde{j}_{n,mn}(s_{n,(e,\mu)}) = \sum_{\tau \in s(e)E^{<mn}, [\tau]_n = \mu} s_{mn,(e,\tau)}, \quad \text{and} \quad \tilde{j}_{n,mn}(p_{n,\nu}) = \sum_{\tau \in E^{<mn}, [\tau]_n = \nu} p_{mn,\tau},$$

for $n, m \in \mathbb{N}$ and $e \in E^1$, $\mu \in s(e)E^{<n}$ and $\nu \in E^{<n}$.

Kribs and Solel define the generalised Bunce–Deddens algebra associated to a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$ by

$$C^*(E, \omega) := \varinjlim (C^*(E(n_k)), \tilde{j}_{n_k, n_{k+1}}).$$

3. Applications of Perron-Frobenius theory

In this section we analyse the invertibility of the $|E^0| \times |E^0|$ matrix $\sum_{i=0}^{(n_k/l)-1} (A_E^{il})^t$, where $l = \gcd(\mathcal{P}_E, \omega) := \lim_{j \to \infty} \gcd(\mathcal{P}_E, n_j)$ and k is such that $\gcd(\mathcal{P}_E, n_k) = l$. We also show that $\operatorname{coker}(1 - A_E^l)^t$ is isomorphic to l copies of $\operatorname{coker}(1 - A_E^t)$. These results will be very useful when we compute the $K_1(C^*(E,\omega))$ in Section 4 and the torsion-free component of the $K_0(C^*(E,\omega))$ in Section 5.

Lemma 3.1. *For each $n \geq 1$, let R_n be the polynomial over $\mathbb{C}$ given by $R_n(x) = \sum_{i=0}^{n-1} x^i$. The roots of R_n are the n-th roots of unity excluding 1.*

Proof. We have $(1-x)R_n(x) = 1 - x^n$, so the roots of $(1-x)R_n$ are the n-th roots of unity. The only root of $1-x$ is 1, so every nth root of unity other than 1 is itself a root of R_n. Since the degree of R_n is $n-1$, these are all the roots of R_n. $\square$

Lemma 3.2. *Let E be a strongly connected finite directed graph, let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence, and let $l = \gcd(\mathcal{P}_E, \omega)$. Then $\mathcal{P}_E/l$ and n_k/l are coprime for all k such that $\gcd(\mathcal{P}_E, n_k) = l$. Hence, if the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity, then $0 \notin \sigma\big(R_{n_k/l}(A_E^l)^t\big)$ for k such that $\gcd(\mathcal{P}_E, n_k) = l$.*

Proof. Suppose for contradiction that $k \geq K$ and that $\mathcal{P}_E/l$ is not coprime to n_k/l. Say $p \neq 1$ satisfies $p|(\mathcal{P}_E/l)$ and $p|(n_k/l)$. Then $pl|\mathcal{P}_E$ and $pl|n_k$. This implies that $pl \leq l$, which is a contradiction.

For the second statement, we have

$$\sigma((A_E^l)^t) \cap \mathbb{T} = \{e^{(2\pi i j/P_E)l} : j \in \mathbb{N} \cup \{0\}\} = \{e^{2\pi i j/(\mathcal{P}_E/l)} : j \in \mathbb{N} \cup \{0\}\},$$

by the spectral mapping theorem. By Lemma 3.1, the roots of $R_{n_k/l}$ are the n_k/l-th roots of unity. Since $\gcd(\mathcal{P}_E/l, n_k/l) = 1$, we have that $e^{2\pi j i/(n_k/l)} \notin \sigma((A_E^l)^t)$ for any $j \in \mathbb{N} \cup \{0\}$. So $0 \notin \sigma(R_{n_k/l}(A_E^l)^t)$. $\square$

Lemma 3.3. *Let E be a strongly connected finite directed graph. Then $A_E^t\delta_v + \operatorname{Im}(1 - A_E^t) = \delta_v + \operatorname{Im}(1 - A_E^t)$ for all $v \in E^0$.*

Proof. Fix $v \in E^0$. We have that $\delta_v - A_E^t\delta_v = (1 - A_E^t)\delta_v \in \operatorname{Im}(1 - A_E^t)$, so $A_E^t\delta_v + \operatorname{Im}(1 - A_E^t) = \delta_v + \operatorname{Im}(1 - A_E^t)$. $\square$

We now show that $\operatorname{coker}(1 - A_E^l)^t \cong \bigoplus_{i=1}^{l} \operatorname{coker}(1 - A_E^t)$. By [10, Lemma 4.2] there is an equivalence relation $\sim_l$ on E^0 such that $v \sim_l w$ if and only if $|\lambda| \in l\mathbb{Z}$ for all $\lambda \in vE^*w$. We enumerate the equivalence classes for $\sim_l$. Fix $v \in E^0$, and let $\Lambda_0 = [v]$. Now iteratively fix $e \in E^1$ with $r(e) \in \Lambda_i$ and let $\Lambda_{i+1} = [s(e)]$, where addition in the subscript is modulo l. Then $\Lambda_0, \ldots, \Lambda_{l-1}$ is an enumeration of the equivalence classes in $E^0/\sim_l$.

Lemma 3.4. *Let E be a strongly connected finite directed graph. Let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence, and let $l := \gcd(P_E, \omega)$. There is an isomorphism*

$$\Theta : \operatorname{coker}(1 - A_E^l)^t \to \bigoplus_{i=0}^{l-1} \mathbb{Z}^{\Lambda_i}/(1 - A_E^l)^t\mathbb{Z}^{\Lambda_i}$$

satisfying

$$\Theta(\delta_v + \operatorname{Im}(1 - A_E^l)^t) = (0, \ldots, 0, \delta_v + (1 - A_E^l)^t\mathbb{Z}^{\Lambda_j}, 0, \ldots, 0),$$

where $v \in \Lambda_j$ for some $0 \leq j \leq l-1$ and $\delta_v + (1 - A_E^l)^t\mathbb{Z}^{\Lambda_j}$ appears in the j-th position.

Proof. Fix $0 \leq j \leq l-1$, and $v \in \Lambda_j$. Since $E^0 = \bigsqcup_{i=0}^{l-1} \Lambda_i$, there is an isomorphism $\theta : \mathbb{Z}^{E^0} \to \bigoplus_{i=0}^{l-1} \mathbb{Z}^{\Lambda_i}$ such that $\theta(\delta_v) = (0, \ldots, 0, \delta_v, 0, \ldots, 0)$, where δ_v is in the j-th position.

Our choice of $\Lambda_0, \dots, \Lambda_{l-1}$ ensures that $(A_E^l)^t\delta_v = \sum_{w\in E^0} |vE^l w|\delta_w \in \mathbb{Z}^{\Lambda_j}$ and so $(1 - A_E^l)^t\delta_v \in \mathbb{Z}^{\Lambda_j}$. Hence $\theta((1-A_E^l)^t\delta_v) = (0,\dots,0,(1-A_E^l)^t\delta_v, 0,\dots,0) \in \bigoplus_{i=0}^{l-1}(1-A_E^l)^t\mathbb{Z}^{\Lambda_i}$. Therefore θ descends to an isomorphism $\Theta : \operatorname{coker}(1 - A_E^l)^t \to \bigoplus_{i=0}^{l-1} \mathbb{Z}^{\Lambda_i}/(1 - A_E^l)^t\mathbb{Z}^{\Lambda_i}$ satisfying the desired formula. □

Lemma 3.5. *Let E be a strongly connected finite directed graph. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, and let $l = \gcd(P_E, \omega)$. For each $0 \le j \le l-1$, there is an isomorphism $\Phi_j : \mathbb{Z}^{\Lambda_j}/(1 - A_E^l)^t\mathbb{Z}^{\Lambda_j} \to \mathbb{Z}^{E^0}/\operatorname{Im}(1 - A_E)^t$ satisfying*

$$\Phi_j(\delta_v + (1 - A_E^l)^t\mathbb{Z}^{\Lambda_j}) = \delta_v + (1 - A_E^t)\mathbb{Z}^{E^0},$$

for some $v \in \Lambda_j$.

Proof. Fix $0 \le j \le l-1$. The formula $(1 - A_E^l)^t = (1 - A_E^t)\big(\sum_{i=0}^{l-1}(A_E^i)^t\big)$ shows that $\operatorname{Im}(1 - A_E^l)^t \subseteq \operatorname{Im}(1 - A_E^t)$. Since $(1 - A_E^l)^t\mathbb{Z}^{\Lambda_j} \subseteq \operatorname{Im}(1 - A_E^l)^t$, it follows that the map $\mathbb{Z}^{\Lambda_j} \to \mathbb{Z}^{E^0}$ given by $\delta_v \mapsto \delta_v$ for $v \in \Lambda_j$, descends to a homomorphism $\Phi_j : \mathbb{Z}^{\Lambda_j}/(1-A_E^l)^t\mathbb{Z}^{\Lambda_j} \to \mathbb{Z}^{E^0}/\operatorname{Im}(1-A_E)^t$ satisfying $\Phi_j(\delta_v+(1-A_E^l)^t\mathbb{Z}^{\Lambda^j}) = \delta_v+\operatorname{Im}(1-A_E^t)$, for $v \in \Lambda_j$.

We must show that Φ_j is an isomorphism. To see that Φ_j is surjective, fix $0 \le k \le l-1$ and $v \in \Lambda_k$. Then $(A_E^{j-k})^t\delta_v \in \mathbb{Z}^{\Lambda_j}$ and

$$\delta_v + \operatorname{Im}(1 - A_E^t) = (A_E^{j-k})^t\delta_v + \operatorname{Im}(1 - A_E^t) = \Phi_j\big((A_E^{j-k})^t\delta_v + (1 - A_E^l)^t\mathbb{Z}^{\Lambda_j}\big).$$

To see that Φ_j is injective, fix $a = \sum_{v\in\Lambda_j} a_v\delta_v \in \mathbb{Z}^{\Lambda_j}$ such that $\Phi_j(a+(1-A_E^l)^t\mathbb{Z}^{\Lambda_j}) = 0$. That is, $a \in \operatorname{Im}(1 - A_E^t)$. Say $a = (1 - A_E^t)b$ where $b = \sum_{w\in E^0} b_w\delta_w$. Let $b_k := b|_{\Lambda_k} = \sum_{w\in\Lambda_k} b_w\delta_w$ for each $0 \le k \le l-1$. Since $a \in \mathbb{Z}^{\Lambda_j}$, we have $0 = a|_{\Lambda_k} = ((1 - A_E^t)b)|_{\Lambda_k} = b_k - A_E^t b_{k-1}$, for all $0 \le k \le l-1$, $k \ne j$, where subtraction in the subscript is modulo l. Therefore $b_k = (A_E^t)^{k-j}b_j$ for each $0 \le k \le l-1$, $k \ne j$, where subtraction in the superscript is modulo l. Hence

$$\begin{aligned} a &= (1 - A_E^t)b = (1 - A_E^t)(b_0 + \cdots + b_{l-1}) \\ &= (1 - A_E^t)\Big(\sum_{k=0}^{l-1}(A_E^k)^t\Big)b_j = (1 - A_E^l)^t b_j \in (1 - A_E^l)^t\mathbb{Z}^{\Lambda_j}. \end{aligned}$$ □

Corollary 3.6. *Let E be a strongly connected finite directed graph. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, and let $l = \gcd(P_E, \omega)$. There is an isomorphism $\rho : \operatorname{coker}(1 - A_E^l)^t \to \bigoplus_{i=1}^l \operatorname{coker}(1 - A_E^t)$ satisfying*

$$\rho\big(\delta_v + \operatorname{Im}(1 - A_E^l)^t\big) = \big(0,\dots,0,\delta_v + \operatorname{Im}(1 - A_E^t),0,\dots,0\big),$$

where $v \in \Lambda_j$ for some $0 \le j \le l-1$ and $\delta_v + \operatorname{Im}(1 - A_E^t)$ appears in the j-th position.

Proof. Define $\rho := \big(\bigoplus_{i=0}^{l-1}\Phi_i\big) \circ \Theta$. It follows from Lemma 3.5 and Lemma 3.6 that ρ is an isomorphism that satisfies the desired formula. □

4. Computing $K_1(C^*(E,\omega))$

In this section we compute $K_1(C^*(E,\omega))$ where E is a strongly connected finite graph E such that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity, and ω is a multiplicative sequence. The main result of this section is the following.

Theorem 4.1. *Let E be a strongly connected finite graph and suppose that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence and let $l := \gcd(\mathcal{P}_E, \omega)$. Then*

$$K_1\big(C^*(E,\omega)\big) = \bigoplus_{i=1}^{l} \ker(1 - A_E^t).$$

To prove Theorem 4.1 we need a series of results. We begin by studying $\ker(1 - A_{E(n)}^t)$ for $n \geq 1$.

Let $\{\delta_v : v \in E^0\}$ be the generators of $\mathbb{Z}^{E^0}$ and let $\{\delta_{\mu,n} : \mu \in E^{<n}\}$ be the generators of $\mathbb{Z}^{E^{<n}}$. For $0 \leq k \leq n-1$ and $a = \sum_{\mu \in E^{<n}} a_\mu \delta_{\mu,n} \in \mathbb{Z}^{E^{<n}}$, we define $a_k := \sum_{\mu \in E^k} a_\mu \delta_{\mu,n} \in \mathbb{Z}^{E^{<n}}$ and $a|_{\mathbb{Z}^{E^k}} := \sum_{\mu \in E^k} a_\mu \delta_{\mu,k} \in \mathbb{Z}^{E^k}$. For $b = \sum_{v \in E^0} b_v \delta_v \in \mathbb{Z}^{E^0}$, we define $\iota_n(b) := \sum_{v \in E^0} b_v \delta_{v,n} \in \mathbb{Z}^{E^{<n}}$.

Lemma 4.2. *Let E be a row-finite directed graph with no sources and let $n \geq 1$. There is an isomorphism $\psi_n : \ker(1 - A_{E(n)}^t) \to \ker(1 - A_E^n)^t$ satisfying $\psi_n(a) = a|_{E^0}$ for $a \in \ker(1 - A_{E(n)}^t)$.*

Proof. Define $\psi_n : \ker(1 - A_{E(n)}^t) \to \mathbb{Z}^{E^0}$ by $\psi_n(a) = a|_{\mathbb{Z}^{E^0}}$ for $a \in (1 - A_{E(n)}^t)$. We check that $\psi_n(\ker(1 - A_{E(n)}^t) \subseteq \ker(1 - A_E^n)^t$. Let $a \in \ker(1 - A_{E(n)}^t)$. Then

$$(1 - A_E^n)^t(\psi_n(a)) = (1 - A_E^n)^t(a|_{\mathbb{Z}^{E^0}}) = ((1 - A_{E(n)}^n)^t a_0)|_{\mathbb{Z}^{E^0}} = 0.$$

So $\psi_n(a) \in \ker(1 - A_E^n)^t$, and hence ψ_n descends to a homomorphism $\ker(1 - A_{E(n)}^t) \to \ker(1 - A_E^n)^t$ which we also label ψ_n.

Define $\varphi_n : \ker(1 - A_E^n)^t \to \mathbb{Z}^{E^{<n}}$ by $\varphi_n(b) = \sum_{i=0}^{n-1} (A_{E(n)}^t)^i(\iota_n(b))$ for $b \in \ker(1 - A_E^n)^t$. We check $\varphi_n(\ker(1 - A_E^n)^t) \subseteq \ker(1 - A_{E(n)}^t)$. Let $b \in \ker(1 - A_E^n)^t$. Then

$$\begin{aligned}
(1 - A_{E(n)}^t)(\varphi_n(b)) &= (1 - A_{E(n)}^t)\Big(\sum_{i=0}^{n-1} (A_{E(n)}^i)^t(\iota_n(b))\Big) \\
&= (1 - A_{E(n)}^n)^t(\iota_n(b)) \\
&= \iota_n\big((1 - A_{E(n)}^n)^t(b)\big) = 0.
\end{aligned}$$

So $\varphi_n(b) \in \ker(1 - A_{E(n)}^t)$, and hence φ_n descends to a homomorphism $\ker(1 - A_E^n)^t \to \ker(1 - A_{E(n)}^t)$ which we also label φ_n.

We check that φ_n is an inverse for ψ_n. Let $a \in \ker(1 - A_{E(n)}^t)$. Fix $k < n$. We have

$$0 = (1 - A_{E(n)}^t)(a_k) = \begin{cases} a_k - A_{E(n)}^t(a_{k+1}) & \text{if } k \neq n-1 \\ a|_{E^{n-1}} - A_{E(n)}^t(a_0) & \text{if } k = n-1. \end{cases}$$

So $a_{n-1} = A_{E(n)}^t(a_0)$. Then $a_{n-2} = A_{E(n)}^t(a_{n-1}) = (A_{E(n)}^2)^t(a_0)$. Repeating this step yields $a_{n-i} = (A_{E(n)}^i)^t(a_0)$ for $i < n$. Since $a_0 = \iota_n(a|_{\mathbb{Z}^{E^0}})$, we have $\varphi_n(\psi_n(a)) = \varphi_n(a|_{\mathbb{Z}^{E^0}}) = \sum_{i=0}^{n-1} (A_{E(n)}^i)^t a_0 = a$.

Now, we check that ψ_n is an inverse for φ_n. Let $v \in E^0$ and $0 \leq i < n$. Repeated applications of (5.1) shows that $\big(A^i_{E(n)}\big)^t \delta_{v,n} \in \operatorname{span}\{\delta_{\mu,n} : \mu \in E^{n-i}\}$. Thus

$$\big(\big(A^i_{E(n)}\big)^t \delta_{v,n}\big)\big|_{\mathbb{Z}E^0} = \begin{cases} \delta_v & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Now, let $b \in \ker(1 - A^n_E)^t$. By (4.1), we have

$$\psi_n(\varphi_n(b)) = \Big(\sum_{i=0}^{n-1} (A^i_{E(n)})^t(\iota_n(b))\Big)\Big|_{\mathbb{Z}E^0} = b. \qquad \square$$

Suppose E is a row-finite directed graph with no sources. Define the skew-product graph $E \times_1 \mathbb{Z}$ as the graph with edge set $(E \times_1 \mathbb{Z})^1 = E^1 \times \mathbb{Z}$ and vertex set $(E \times_1 \mathbb{Z})^0 = E^0 \times \mathbb{Z}$ and range and source maps defined by

$$r(e,k) = (r(e), k-1) \text{ and } s(e,k) = (s(e),k).$$

For each $n \geq 1$, we denote by $s_{n,((e,\mu),k)}$ and $p_{n,(\mu,k)}$ the generators of $C^*(E(n) \times_1 \mathbb{Z})$. Proposition 6.7 of [9] gives a natural action $\beta_{E(n)}$ of $\mathbb{Z}$ on $C^*(E(n) \times_1 \mathbb{Z})$ such that $(\beta_{E(n)})_m(s_{n,((e,\mu),k)}) = s_{n,((e,\mu),k+l)}$. By [9, Lemma 7.10] there is an isomorphism $\phi_{E(n)}$ of $C^*(E \times_1 \mathbb{Z})$ onto the crossed product $C^*(E(n)) \rtimes \mathbb{T}$ such that $\phi_{E(n)} \circ (\beta_{E(n)})_m = \hat{\gamma}^n_m \circ \phi_{E(n)}$, where $\hat{\gamma}^n$ is the dual of the gauge action γ^n of $C^*(E(n))$.

Lemma 4.3. *Let E be a row-finite directed graph with no sources, and let $n, m \in \mathbb{N}$. There is a homomorphism $i_{n,mn} : C^*(E(n) \times_1 \mathbb{Z}) \to C^*(E(mn) \times_1 \mathbb{Z})$ such that*

$$i_{n,mn}(s_{n,((e,\mu),1)}) = \sum_{\tau \in s(e)E^{<mn}, [\tau]_n = \mu} s_{mn,((e,\tau),1)} \quad \textit{and}$$
$$i_{n,mn}(p_{n,(\mu,1)}) = \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} p_{mn,(\tau,1)},$$

for all $n \geq 1$.

Proof. Let $(i_{C^*(E(n))}, i_{\mathbb{T}})$ be the universal covariant representation of $(C^*(E(n)), \mathbb{T}, \gamma^n)$. Recall the injective homomorphism $\tilde{j}_{n,mn} : C^*(E(n)) \to C^*(E(mn))$. We show that $\tilde{j}_{n,mn}$ is $\mathbb{T}$-equivariant. For $e \in E^1$ and $\mu \in s(e)E^{<n}$ and $z \in \mathbb{T}$, we have

$$\begin{aligned} \tilde{j}_{n,mn}(\gamma^n_z(s_{n,(e,\mu)}) &= \tilde{j}_{n,mn}(z s_{n,(e,\mu)}) \\ &= \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} z s_{mn,(e,\tau)} = \gamma^{mn}_z(\tilde{j}_{n,mn}(s_{n,(e,\mu)})), \end{aligned}$$

and similarly for $\mu \in E^{<n}$, $\tilde{j}_{n,mn}(\gamma^n_z(p_{n,\mu})) = (\gamma^{mn}_z(\tilde{j}_{n,mn}(p_{n,\mu}))$.

By [12, Corollary 2.48] there is a homomorphism $\tilde{j}_{n,mn} \times 1 : C^*(E(n)) \rtimes \mathbb{T} \to C^*(E(mn)) \rtimes \mathbb{T}$ satisfying

$$(\tilde{j}_{n,mn} \times 1)(i_{C^*(E(n))}(a) i_{\mathbb{T}}(z)) = i_{C^*(E(mn))}(\tilde{j}_{n,mn}(a)) i_{\mathbb{T}}(z)$$

for all $a \in C^*(E(n))$ and $z \in \mathbb{T}$.

Define $i_{n,mn} := \phi_{E(mn)}^{-1} \circ (\tilde{j}_{n,mn} \times 1) \circ \phi_{E(n)}$. Let $((e,\mu),1) \in E(n)^1 \times_1 \mathbb{Z}$ and let $f_1(z) = z$ for $z \in \mathbb{T}$. We calculate

$$\begin{aligned}\big(\phi_{E(mn)}^{-1} \circ (\tilde{j}_{n,mn} \times 1) \circ \phi_{E(n)}\big)(s_{n,((e,\mu),1)}) &= \phi_{E(n)}^{-1}\big((\tilde{j}_{n,mn} \times 1)(i_A(s_{(n,(e,\mu))})i_{\mathbb{T}}(f_1))\big) \\ &= \phi_{E(n)}^{-1}\Big(i_A\Big(\sum_{\tau \in s(e)E^{<mn}, [\tau]_n = \mu} s_{mn,(e,\tau)}\Big)i_{\mathbb{T}}(f_1)\Big) \\ &= \sum_{\tau \in s(e)E^{<mn}, [\tau]_n = \mu} s_{mn,((e,\tau),1)}.\end{aligned}$$

Similarly, for $(\mu, 1) \in E(n)^0 \times_1 \mathbb{Z}$, we have

$$\begin{aligned}\big(\phi_{E(mn)}^{-1} \circ (\tilde{j}_{n,mn} \times \mathrm{id}) \circ \phi_{E(n)}\big)(p_{n,(\mu,1)}) &= \phi_{E(n)}^{-1}\big((\tilde{j}_{n,mn} \times \mathrm{id})(i_A(p_{n,\mu})i_{\mathbb{T}}(f_1))\big) \\ &= \phi_{E(n)}^{-1}\Big(i_A\Big(\sum_{\tau \in E^{<mn}, [\tau]_n = \mu} p_{mn,\tau}\Big)i_{\mathbb{T}}(f_1)\Big) \\ &= \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} p_{mn,(\tau,1)}.\end{aligned}$$

□

Proposition 4.4. *Let E be a row-finite directed graph with no sources and let $n, m \in \mathbb{N}$. There are isomorphisms $K_1(C^*(E(n))) \to \ker(1 - A_E^n)^t$ and $K_1(C^*(E(mn))) \to \ker(1 - A_E^{mn})^t$ such that the following diagram commutes.*

$$\begin{array}{ccc} K_1(C^*(E(n))) & \xrightarrow{K_1(\tilde{j}_{n,mn})} & K_1(C^*(E(mn))) \\ \downarrow & & \downarrow \\ \ker(1 - A_E^n)^t & \xrightarrow{x \mapsto x} & \ker(1 - A_E^{mn})^t \end{array}$$

Proof. The naturality of the Pimsner–Voiculescu diagram gives the following commutative diagram (see [9, Lemma 7.12]).

$$\begin{array}{ccc} K_1(C^*(E(n))) & \xrightarrow{K_1(\tilde{j}_{n,mn})} & K_1(C^*(E(mn))) \\ \downarrow & & \downarrow \\ \ker(1 - (\beta_{E(n)})_*^{-1}) & \longrightarrow & \ker(1 - (\beta_{E(mn)})_*^{-1}) \end{array}$$

By [9, Lemma 7.13] there is an injection $\sigma_n : \mathbb{Z}^{E^{<n}} \to K_0(C^*(E(n) \times_1 \mathbb{Z}))$ satisfying $\sigma_n(\delta_{\mu,n}) = [p_{n,(\mu,1)}]_0$. Define $\phi_{n,mn} : \mathbb{Z}^{E^{<n}} \to \mathbb{Z}^{E^{<mn}}$ by $\phi_{n,mn}(\delta_{\mu,n}) = \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} \delta_{\tau,mn}$ for $\mu \in E^{<n}$. We claim that the following diagram commutes.

$$\begin{array}{ccc} K_0(C^*(E(n) \times_1 \mathbb{Z})) & \xrightarrow{K_0(i_{n,mn})} & K_0(C^*(E(mn) \times_1 \mathbb{Z})) \\ {\scriptstyle \sigma_n}\uparrow & & \uparrow{\scriptstyle \sigma_{mn}} \\ \mathbb{Z}^{E^{<n}} & \xrightarrow{\phi_{n,mn}} & \mathbb{Z}^{E^{<mn}} \end{array} \tag{4.2}$$

To prove this claim, fix $\mu \in E^{<n}$. Then

$$\begin{aligned}(\sigma_{mn}^{-1} \circ K_0(i_{n,mn}) \circ \sigma_n)(\delta_{\mu,n}) &= (\sigma_{mn}^{-1} \circ K_0(i_{n,mn}))([p_{(n,\mu,1)}]_0) \\ &= \sigma_{mn}^{-1}([i_{n,mn}(p_{(n,\mu,1)})]_0) \\ &= \sigma_{mn}^{-1}\Big(\sum_{\tau \in E^{<mn}, [\tau]_n = \mu} [p_{(mn,\tau,1)}]_0\Big) \\ &= \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} \delta_{\tau,mn} \\ &= \phi_{n,mn}(\delta_{\mu,n}).\end{aligned}$$

It follows from [9, Theorem 7.16] that σ_n restricts to an isomorphism of $\ker(1 - A^t_{E(n)})$ onto $\ker(1 - (\beta_{E(n)})_*^{-1})$. Restricting diagram 4.2 to the subgroups $\ker(1 - (\beta_{E(n)})_*^{-1}) \subseteq K_0(C^*(E(n)) \times_1 \mathbb{Z}))$ and $\ker(1 - A^t_{E(n)}) \subseteq \mathbb{Z}^{E^{<n}}$ yields the following commuting diagram.

$$\begin{array}{ccc} \ker(1 - (\beta_{E(n)})_*^{-1}) & \xrightarrow{K_0(i_{n,mn})} & \ker(1 - (\beta_{E(mn)})_*^{-1}) \\ \downarrow & & \downarrow \\ \ker(1 - A^t_{E(n)}) & \xrightarrow{\phi_{n,mn}} & \ker(1 - A^t_{E(mn)}) \end{array}$$

Now, we claim that the following diagram commutes.

$$\begin{array}{ccc} \ker(1 - A^t_{E(n)}) & \xrightarrow{\phi_{n,mn}} & \ker(1 - A^t_{E(mn)}) \\ \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_{mn}} \\ \ker(1 - A^n_E)^t & \xrightarrow{x \mapsto x} & \ker(1 - A^{mn}_E)^t \end{array}$$

To prove this claim, fix $x \in \ker(1 - A^t_{E(n)})$. Then

$$\psi_{mn}(\phi_{n,mn}(x)) = \psi_{mn}\Big(\sum_{\mu \in E^{<n}} x_\mu \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} \delta_{\tau,mn}\Big) = \sum_{v \in E^0} x_v \delta_v = \psi_n(x).$$

Combining the preceding commutative diagrams gives the desired commutative diagram. □

Proof of Theorem 4.1. By [11, Theorem 6.3.2], we have

$$K_1\big(C^*(E,\omega)\big) \cong \varinjlim \big(K_1(C^*(E(n_k)), K_1(j_{n_k,n_{k+1}})\big).$$

By Proposition 4.4, we have

$$\big(\varinjlim K_1(C^*(E(n_k)), K_1(j_{n_k,n_{k+1}})\big) \cong \varinjlim \big(\ker(1 - A^{n_k}_E)^t, x \mapsto x\big).$$

By Lemma 3.2 the matrix $\sum_{j=0}^{n_k/l-1} (A^{jl}_E)^t$ is invertible for k such that $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$. So

$$\ker(1 - A^{n_k}_E)^t = \ker\Big(\Big(\sum_{j=0}^{n_k/l-1} (A^{jl}_E)^t\Big)(1 - A^l_E)^t\Big) = \ker(1 - A^l_E)^t,$$

for k such that $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$. Hence

$$\big(\ker(1 - A^{n_k}_E)^t, x \mapsto x\big) \cong \ker(1 - A^l_E)^t.$$

Combining the previous three isomorphisms gives an isomorphism

$$K_1(C^*(E,\omega)) \cong \ker(1-A_E^l)^t.$$

Now, $\ker(1-A_E^l)^t \cong \mathbb{Z}^r$, where $r = \operatorname{rank}\operatorname{coker}(1-A_E^l)^t = l \cdot \operatorname{rank}\operatorname{coker}(1-A_E^t)$ by Corollary 3.6. So $\ker(1-A_E^l)^t \cong \bigoplus_{i=1}^{l} \ker(1-A_E^l)^t$, giving the result. □

5. Computing the torsion-free component of $K_0(C^*(E,\omega))$

In this section we calculate the torsion-free component of $K_0(C^*(E,\omega))$. We will use this group in Section 6 to recover the supernatural number $[\omega]$ associated to ω. In order to state the main theorem of this section, we need the following lemma.

Lemma 5.1. *Let A be a free abelian group and let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence. Define an equivalence relation $\sim$ on $A \times \mathbb{N}$, by $(a,j) \sim (a',j')$ if*

$$\frac{\max\{n_j, n_{j'}\}}{n_j} a = \frac{\max\{n_j, n_{j'}\}}{n_{j'}} a',$$

and define

$$A\left[\frac{1}{\omega}\right] := \{(a,j) : a \in A, j \in \mathbb{N}\} / \sim .$$

Then $A\left[\frac{1}{\omega}\right]$ is a torsion-free abelian group under the operation

$$[(a,j)] + [(a',j')] = \begin{cases} [((n_{j'}/n_j) \cdot a + a', j')] & \text{if } j' \geq j \\ [(a + (n_j/n_{j'}) \cdot a', j)] & \text{if } j \geq j'. \end{cases}$$

Moreover, $\operatorname{rank} A\left[\frac{1}{\omega}\right] = \operatorname{rank} A$.

Proof. Closure, associativity, and commutativity follow easily since A is abelian. Let 0 be the identity element of A. Then $[(0,i)] + [(a,i)] = [(0+a,i)] = [(a,i)]$ so $[(0,i)]$ is an identity for $A\left[\frac{1}{\omega}\right]$. Fix $a \in A$ and let $-a$ be the inverse. Then $[(a,i)] + [(-a,i)] = [(a-a,i)] = [(0,i)]$, so $[(-a,i)]$ is an inverse for $[(a,i)]$.

If $k \cdot [(a,i)] = [(0,i)]$, then $[(k \cdot a, i)] = [(0,i)]$, so $k \cdot a = 0$ forcing $a = 0$ since A is free abelian. To see that $\operatorname{rank} A\left[\frac{1}{\omega}\right] = \operatorname{rank} A$, let $\{a_\alpha\}$ be a maximal linearly independent subset of A. Suppose $[(0,i)] = \sum_\alpha c_\alpha \cdot [(a_\alpha, i)]$ for $c_\alpha \in \mathbb{N}$ with all but finitely many nonzero. Then $[(0,i)] = \sum_\alpha [(c_\alpha \cdot a_\alpha, i)] = [(\sum_\alpha c_\alpha \cdot a_\alpha, i)]$, so $0 = \sum_\alpha c_\alpha \cdot a_\alpha$, and since $\{a_i\}$ is linearly independent, $c_\alpha = 0$ for all α. Hence $\{[(a_\alpha, i)]\}$ is a linearly independent subgroup of $A\left[\frac{1}{\omega}\right]$. To see that it is maximal, take $c \in \mathbb{N}$ and $b \in A$. Then $\sum_\alpha c_\alpha [(a_\alpha, i)] + c[(b,i)] = [(\sum_\alpha c_\alpha \cdot a_\alpha + c \cdot b, i)] = [(0,i)]$, by the maximality of $\{a_\alpha\}$. □

Remark 5.2. We have $A\left[\frac{1}{\omega}\right] \cong A \bigotimes \mathbb{Z}\left[\frac{1}{\omega}\right]$ via the map $[a,j] \to a \bigotimes \frac{1}{n_j}$. We will regard the elements $[(a,j)]$ as formal fractions and write a/n_j for $[(a,j)]$.

We now state the main theorem of this section about the torsion-free component of $K_0(C^*(E,\omega))$. Recall that the torsion subgroup of an abelian group A consists of the nonzero elements of A which have finite order.

Theorem 5.3. *Let E be a strongly connected finite directed graph. Let $\mathcal{P}_E$ denote the period of E, and let $l = \gcd(\mathcal{P}_E, \omega)$. Suppose 1 is an eigenvalue of A_E^t and that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^{\infty}$*

be a multiplicative sequence. Let tor_E *denote the torsion subgroup of* $K_0(C^*(E))$*, and* $tor_{(E,\omega)}$ *the torsion subgroup of* $K_0(C^*(E,\omega))$*. There is an isomorphism*

$$\Psi : K_0(C^*(E,\omega))/\,\mathrm{tor}_{(E,\omega)} \to \bigoplus_{i=1}^{l} \big(K_0(C^*(E))/\,\mathrm{tor}_E\big)\Big[\frac{1}{\omega}\Big]$$

satisfying

$$\Psi([1_{C^*(E,\omega)}]_0 + \mathrm{tor}_{E,\omega}) = ([1_{C^*(E)}]_0 + \mathrm{tor}_E, \dots, [1_{C^*(E)}]_0 + \mathrm{tor}_E).$$

To prove Theorem 5.3 we need a series of lemmas. We begin by studying $K_0(C^*(E(n)) \cong \mathrm{coker}(1 - A^t_{E(n)})$ for $n \geq 1$.

Lemma 5.4. *Let* E *be a row-finite directed graph with no sources and let* $n \geq 1$*. Then*

$$A^t_{E(n)}\delta_{\mu,n} = \begin{cases} \delta_{\mu_2\dots\mu_{|\mu|},n} & \text{if } \mu \in E^{<n}\backslash E^0 \\ \sum_{\lambda\in\mu E^n} \delta_{\lambda_2\dots\lambda_n,n} & \text{if } \mu \in E^0. \end{cases} \tag{5.1}$$

Moreover, $\delta_{\mu,n} - \delta_{s(\mu),n} \in \mathrm{Im}(1 - A^t_{E(n)})$ *for each* $\mu \in E^{<n}$.

Proof. Let $\mu \in E^n \setminus E^0$. We calculate

$$\begin{aligned} A^t_{E(n)}\delta_{\mu,n} &= \sum_{\nu\in E^{<n}} A^t_{E(n)}(\nu,\mu)\delta_{\nu,n} = \sum_{\nu\in E^{<n}} |\mu E(n)^1\nu|\delta_{\nu,n} \\ &= \sum_{\nu\in E^{<n}, e\in E^1 r(\nu), [e\nu]_n=\mu} \delta_{\nu,n} = \delta_{\mu_2\dots\mu_{|\mu|},n}. \end{aligned}$$

Let $\mu \in E^0$. Then

$$A^t_{E(n)}\delta_{\mu,n} = \sum_{\nu\in E^{<n}} A^t_{E(n)}(\nu,\mu)\delta_{\nu,n} = \sum_{\nu\in E^{<n}} |\mu E(n)^1\nu|\delta_{\nu,n} = \sum_{\lambda\in\mu E^n} \delta_{\lambda_2\dots\lambda_n,n}.$$

The final statement clearly holds when $\mu \in E^0$, so let $\mu \in E^{<n}\backslash E^0$. Repeated applications of the first case of (5.1) give $(A^{|\mu|}_{E(n)})^t\delta_{\mu,n} = \delta_{s(\mu),n}$, so $\delta_{\mu,n} - \delta_{s(\mu),n} = (1 - A^{|\mu|}_{E(n)})^t\delta_{\mu,n} \in \mathrm{Im}(1 - A^t_{E(n)})$. □

Lemma 5.5. *Let* E *be a row-finite directed graph with no sources and let* $n \geq 1$*. There is an isomorphism* $\psi_n : \mathrm{coker}(1 - A^n_E)^t \to \mathrm{coker}(1 - A^t_{E(n)})$ *satisfying* $\psi_n(\delta_v + \mathrm{Im}(1 - A^n_E)^t) = \delta_{v,n} + \mathrm{Im}(1 - A^t_{E(n)})$ *for* $v \in E^0$.

Proof. Define a map $\psi_n : \mathbb{Z}^{E^0} \to \mathrm{coker}(1 - A^t_{E(n)})$ by $\psi_n(\delta_v) = \delta_{v,n} + \mathrm{Im}(1 - A^t_{E(n)})$. We show that $\psi_n(\mathrm{Im}(1 - A^n_E)^t) \subseteq \mathrm{Im}(1 - A^t_{E(n)})$. Let $v \in E^0$. Repeated applications of (5.1) give $(A^n_{E(n)})^t\delta_{v,n} = \sum_{\lambda\in vE^n} \delta_{s(\lambda),n} = \sum_{w\in E^0} |vE^n w|\delta_{w,n} = \sum_{w\in E^0}(A^n_E)^t(w,v)\delta_w = \psi_n\big((A^n_E)^t\delta_v\big)$, so

$$\psi_n\big((1 - A^n_E)^t\delta_v\big) = (1 - A^n_{E(n)})^t\delta_{v,n} + \mathrm{Im}(1 - A^n_{E(n)})^t = \mathrm{Im}(1 - A^n_{E(n)})^t \subseteq \mathrm{Im}(1 - A^t_{E(n)}).$$

Thus ψ_n descends to a homomorphism $\mathrm{coker}(1 - A^n_E)^t \to \mathrm{coker}(1 - A^t_{E(n)})$, which we also label by ψ_n, satisfying $\psi_n(\delta_v + \mathrm{Im}(1 - A^n_E)^t) = \delta_{v,n} + \mathrm{Im}(1 - A^t_{E(n)})$ for $v \in E^0$.

Define a map $\varphi_n : \mathbb{Z}^{E^{<n}} \to \operatorname{coker}(1 - A_E^n)^t$ by $\varphi_n(\delta_{\mu,n}) = \delta_{s(\mu)} + \operatorname{Im}(1 - A_E^n)^t$. We show that $\varphi_n(\operatorname{Im}(1 - A_{E(n)}^t) = \operatorname{Im}(1 - A_E^n)^t$. Take $(1 - A_{E(n)}^t)\delta_{\mu,n} \in \operatorname{Im}(1 - A_{E(n)}^t)$. If $\mu \in E^{<n} \backslash E^0$, then

$$\varphi_n((1 - A_{E(n)}^t)\delta_{\mu,n}) = \varphi_n(\delta_{\mu,n} - \delta_{\mu_2\dots\mu_{|\mu|},n}) = \delta_{s(\mu)} - \delta_{s(\mu)} + \operatorname{Im}(1 - A_E^n)^t = \operatorname{Im}(1 - A_E^n)^t,$$

by the first case of (5.1). If $\mu \in E^0$, then applying the second case of (5.1) at the first equality, we have

$$\begin{aligned}
\varphi_n\big(\delta_{v,n} - A_{E(n)}^t\delta_{v,n}\big) &= \varphi_n\big(\delta_{v,n} - \sum_{\lambda \in vE^n} \delta_{\lambda_2\dots\lambda_n,n}\big) \\
&= \delta_v - \sum_{\lambda \in vE^n} \delta_{s(\lambda)} + \operatorname{Im}(1 - A_E^n)^t \\
&= \delta_v - \sum_{w \in E^0} |vE^n w|\delta_w + \operatorname{Im}(1 - A_E^n)^t \\
&= \delta_v - \sum_{w \in E^0} (A_E^n)^t(w, v)\delta_w + \operatorname{Im}(1 - A_E^n)^t \\
&= (1 - A_E^n)^t\delta_v + \operatorname{Im}(1 - A_E^n)^t \\
&= \operatorname{Im}(1 - A_E^n)^t.
\end{aligned}$$

Thus φ_n descends to a homomorphism $\operatorname{coker}(1 - A_{E(n)}^t) \to \operatorname{coker}(1 - A_E^n)^t$, which we also label φ_n.

To show that ψ_n is an isomorphism, we show that ψ_n and φ_n are mutually inverse. Let $\mu \in E^{<n}$. Then

$$\begin{aligned}
\psi_n(\varphi_n(\delta_{\mu,n} + \operatorname{Im}(1 - A_{E(n)}^t))) &= \varphi_n(\delta_{s(\mu)} + \operatorname{Im}(1 - A_E^n)^t) \\
&= \delta_{s(\mu),n} + \operatorname{Im}(1 - A_{E(n)}^t) = \delta_{\mu,n} + \operatorname{Im}(1 - A_{E(n)}^t)
\end{aligned}$$

by Lemma 5.4, so $\psi_n \circ \varphi_n$ is the identity on $\operatorname{coker}(1 - A_{E(n)}^t)$. Now, let $v \in E^0$. Then

$$\varphi_n(\psi_n(\delta_v + \operatorname{Im}(1 - A_E^n)^t)) = \varphi_n(\delta_{v,n} + \operatorname{Im}(1 - A_{E(n)}^t)) = \delta_v + \operatorname{Im}(1 - A_E^n)^t,$$

so $\varphi_n \circ \psi_n$ is the identity on $\operatorname{coker}(1 - A_E^n)^t$. □

Remark 5.6. For each $n \geq 1$, let $\sigma_n : \operatorname{coker}(1 - A_{E(n)}^t) \to K_0(C^*(E(n)))$ be the isomorphism of [9, Theorem 7.16]. Looking into the proof of [9, Theorem 7.1] shows that this isomorphism is given by $\sigma_n(\delta_{\mu,n} + \operatorname{Im}(1 - A_{E(n)}^t)) = [p_{\mu,n}]_0$ for $\mu \in E^{<n}$. So $\sigma_n \circ \psi_n : \operatorname{coker}(1 - A_E^n)^t \to K_0(C^*(E(n)))$ is an isomorphism satisfying $(\sigma_n \circ \psi_n)\big(\delta_v + \operatorname{Im}(1 - A_E^n)^t\big) = [p_{v,n}]_0$ for $v \in E^0$. By Lemma 5.4, we have $[p_{\mu,n}]_0 - [p_{v,n}]_0 = \sigma_n(\delta_{\mu,n} - \delta_{v,n} + \operatorname{Im}(1 - A_{E(n)}^t)) = 0$ for any $\mu \in E^{<n}v$. So $(\sigma_n \circ \psi_n)\big(\delta_v + \operatorname{Im}(1 - A_E^n)^t\big) = [p_{\mu,n}]_0$ for any $\mu \in E^{<n}v$.

Lemma 5.7. *Let E be a row-finite directed graph with no sources and let $n, m \in \mathbb{N}$. The following diagram commutes.*

$$\begin{tikzcd}
\mathbb{Z}^{E^0} \arrow[r, "\sum_{i=0}^{m-1}(A_E^{in})^t"] \arrow[d] & \mathbb{Z}^{E^0} \arrow[d] \\
\operatorname{coker}(1-A_E^n)^t \arrow[d, "\sigma_n\circ\psi_n"] & \operatorname{coker}(1-A_E^{mn})^t \arrow[d, "\sigma_{mn}\circ\psi_{mn}"] \\
K_0(C^*(E(n))) \arrow[r, "K_0(\tilde{j}_{n,mn})"] & K_0(C^*(E(mn)))
\end{tikzcd}$$

Proof. Let $\eta_n := \sigma_n \circ \psi_n$, and fix $v \in E^0$. Then

$$\begin{aligned}K_0(\tilde{j}_{n,mn})(\eta_n)(\delta_v + \operatorname{Im}(1-A_E^n)^t)) &= K_0(\tilde{j}_{n,mn})([p_{v,n}]_0)\\ &= \sum_{\mu\in vE^{<mn}, |\mu|\in n\mathbb{N}} [p_{\mu,mn}]_0 = \sum_{i=0}^{m-1}\sum_{\mu\in vE^{in}} [p_{\mu,mn}]_0.\end{aligned}$$

Now, by Remark 5.6, we have

$$\begin{aligned}(\eta_{mn})\Big(\sum_{i=0}^{m-1}(A_E^{in})^t\delta_v + \operatorname{Im}(1-A_E^{mn})^t\Big) &= \eta_{mn}\Big(\sum_{i=0}^{m-1}\sum_{\mu\in vE^{in}} \delta_{s(\mu)} + \operatorname{Im}(1-A_E^{mn})^t\Big)\\ &= \sum_{i=0}^{m-1}\sum_{\mu\in vE^{in}} (\eta_{mn})(\delta_{s(\mu)} + \operatorname{Im}(1-A_E^{mn})^t)\\ &= \sum_{i=0}^{m-1}\sum_{\mu\in vE^{in}} [p_{s(\mu),mn}]_0 = \sum_{i=0}^{m-1}\sum_{\mu\in vE^{in}} [p_{\mu,mn}]_0. \qquad \square\end{aligned}$$

Corollary 5.8. *Let E be a row-finite directed graph with no sources and let $n, m \in \mathbb{N}$. There exists a homomorphism $\phi_{n,mn} : \operatorname{coker}(1-A_E^n)^t \to \operatorname{coker}(1-A_E^{mn})^t$ satisfying $\phi_{n,mn}(\delta_v + \operatorname{Im}(1-A_E^n)^t) = \sum_{\mu\in vE^{<mn},|\mu|\in n\mathbb{N}} \delta_{s(\mu),mn} + \operatorname{Im}(1-A_E^{mn})^t$ for $v \in E^0$.*

Proof. Let $\eta_n := \sigma_n \circ \psi_n$, and define $\phi_{n,mn} : \operatorname{coker}(1-A_E^n)^t \to \operatorname{coker}(1-A_E^{mn})^t$ by $\phi_{n,mn} := \eta_{mn}^{-1} \circ K_0(\tilde{j}_{n,mn}) \circ \eta_n$. Let $v \in E^0$. By Remark 5.6, we have

$$\begin{aligned}\phi_{n,mn}(\delta_v + \operatorname{Im}(1-A_E^n)^t) &= (\eta_{mn}^{-1} \circ K_0(\tilde{j}_{n,mn}) \circ \eta_n)(\delta_v + \operatorname{Im}(1-A_E^n)^t)\\ &= (\eta_{mn}^{-1} \circ K_0(\tilde{j}_{n,mn}))([p_{s(\mu),n}]_0)\\ &= \eta_{mn}^{-1}\Big(\sum_{\mu\in vE^{<mn},|\mu|\in n\mathbb{N}} [p_{\mu,mn}]_0\Big)\\ &= \eta_{mn}^{-1}\Big(\sum_{\mu\in vE^{<mn},|\mu|\in n\mathbb{N}} [p_{s(\mu),mn}]_0\Big)\\ &= \sum_{\mu\in vE^{<mn},|\mu|\in n\mathbb{N}} \delta_{s(\mu),mn} + \operatorname{Im}(1-A_E^{mn})^t. \qquad \square\end{aligned}$$

We now look at direct limits of quotients of abelian groups by their torsion subgroups. We seek to apply the following result to the sequence $(\operatorname{coker}(1-A_E^{n_k})^t, \phi_{n_k,n_{k+1}})_{k=1}^\infty$.

Lemma 5.9. *Let $(G_k, \phi_{k,k+1})$ be a directed system of abelian groups. Let $\operatorname{tor}_k := \operatorname{tor}(G_k)$ for each $k \in \mathbb{N}$, and $\operatorname{tor}_\infty := \operatorname{tor}(\varinjlim G_k)$. For each k there exists a homomorphism*

$\tilde{\phi}_{k,k+1} : G_k/\operatorname{tor}_k \to G_{k+1}/\operatorname{tor}_{k+1}$ *such that* $\tilde{\phi}_{k,k+1}(g+\operatorname{tor}_k) = \phi_{k,k+1}(g)+\operatorname{tor}_{k+1}$. *Moreover, there is an isomorphism*

$$\tilde{q}_\infty : \varinjlim(G_k, \phi_{k,k+1})/\operatorname{tor}_\infty \to \varinjlim(G_k/\operatorname{tor}_k, \tilde{\phi}_{k,k+1})$$

such that $\tilde{q}_\infty(\phi_{k,\infty}(g) + \operatorname{tor}_\infty) = \tilde{\phi}_{k,\infty}(g + \operatorname{tor}_k)$.

Proof. Write $Q_k := G_k/\operatorname{tor}_k$. For each $k \in \mathbb{N}$, let $q_k : G_k \to Q_k$ be the quotient map. Let $r \in \operatorname{tor}_k$. Then there exists $n \geq 1$ such that $nr = 0$, and then $n\phi_{k,k+1}(r) = \phi_{k,k+1}(nr) = 0$. So $\phi_{k,k+1}(\operatorname{tor}_k) \subseteq \operatorname{tor}_{k+1}$, and hence $q_{k+1} \circ \phi_{k,k+1}$ descends to a homomorphism $\tilde{\phi}_{k,k+1} : Q_k \to Q_{k+1}$ such that $\tilde{\phi}_{k,k+1}(g + \operatorname{tor}_k) = \phi_{k,k+1}(g) + \operatorname{tor}_{k+1}$ for all $g \in G_k$. So

$$(\tilde{\phi}_{k+1,\infty} \circ q_{k+1}) \circ \phi_{k,k+1} = \tilde{\phi}_{k+1,\infty} \circ \tilde{\phi}_{k,k+1} \circ q_k = \tilde{\phi}_{k,\infty} \circ q_k$$

for all $k \in \mathbb{N}$. Therefore the universal property of $\varinjlim(G_k, \phi_k)$ gives a homomorphism $q_\infty : \varinjlim(G_k, \phi_{k,k+1}) \to \varinjlim(Q_k, \tilde{\phi}_{k,k+1})$ satisfing $q_\infty \circ \phi_{k,\infty} = \tilde{\phi}_{k,\infty} \circ q_k$.

We show that q_∞ descends to a homomorphism satisfying the desired formula. Let $p \in \operatorname{tor}_\infty$. Then there exists $r \in G_k$ and $n \geq 1$ such that $0 = np = n\phi_{k,\infty}(r) = \phi_{k,\infty}(nr)$. By [11, Proposition 6.2.5(ii)] we have $\ker \phi_{k,\infty} = \bigcup_{m\geq 0} \ker \phi_{k,k+m}$, so there exists $m \geq 0$ such that $0 = \phi_{k,k+m}(nr) = n\phi_{k,k+m}(r)$, giving $\phi_{k,k+m}(r) \in \operatorname{tor}_{k+m}$. Therefore $q_\infty(p) = q_\infty(\phi_{k,\infty}(r)) = q_\infty(\phi_{k+m,\infty}(\phi_{k,k+m}(r))) = \tilde{\phi}_{k+m,\infty}(q_{k+m}(\phi_{k,k+m}(r))) = 0$. So $q_\infty(\operatorname{tor}_\infty) \subseteq \{0\}$, and hence q_∞ descends to a homomorphism $\tilde{q}_\infty : \varinjlim(G_k, \phi_{k,k+1})/\operatorname{tor}_\infty \to \varinjlim(Q_k, \tilde{\phi}_{k,k+1})$ satisfying

$$\tilde{q}_\infty(\phi_{k,\infty}(g) + \operatorname{tor}_\infty) = q_\infty(\phi_{k,\infty}(g)) = \tilde{\phi}_{k,\infty}(q_k(g)) = \tilde{\phi}_{k,\infty}(g + \operatorname{tor}_k)$$

for all $g \in G_k$.

It remains to show that $\tilde{q}_\infty$ is an isomorphism. We do this by finding an inverse. As in the first paragraph, we find that $\phi_{k,\infty}(\operatorname{tor}_k) \subseteq \operatorname{tor}_\infty$ since $\phi_{k,\infty}$ is a homomorphism. Therefore $\phi_{k,\infty}$ descends to a homomorphism $\psi_{k,\infty} : Q_k \to \varinjlim(G_k, \phi_{k,k+1})/\operatorname{tor}_\infty$ satisfying $\psi_{k,\infty}(g + \operatorname{tor}_k) = \phi_{k,\infty}(g) + \operatorname{tor}_\infty$ for each $g \in G_k$. We have

$$\begin{aligned}\psi_{k+1,\infty}(\tilde{\phi}_{k,k+1}(g + \operatorname{tor}_k)) &= \psi_{k+1,\infty}(\phi_{k,k+1}(g) + \operatorname{tor}_{k+1}) = \phi_{k+1,\infty}(\phi_{k,k+1}(g)) + \operatorname{tor}_\infty \\ &= \phi_{k,\infty}(g) + \operatorname{tor}_\infty = \psi_{k,\infty}(g + \operatorname{tor}_k).\end{aligned}$$

So $\psi_{k+1,\infty} \circ \tilde{\phi}_{k,k+1} = \psi_{k,\infty}$, and hence the universal property of $\varinjlim(Q_k, \tilde{\phi}_k)$ gives a homomorphism $\psi : \varinjlim(Q_k, \tilde{\phi}_k) \to \varinjlim(G_k, \phi_{k,k+1})/\operatorname{tor}_\infty$ satisfying $\psi(\tilde{\phi}_{k,\infty}(g + \operatorname{tor}_k)) = \phi_{k,\infty}(g) + \operatorname{tor}_\infty$ for all $g \in G_k$.

We check that ψ is an inverse for $\tilde{q}_\infty$. Let $g \in G_k$. Then

$$\begin{aligned}\tilde{q}_\infty(\psi(\tilde{\phi}_{k,\infty}(g + \operatorname{tor}_k))) &= \tilde{q}_\infty(\phi_{k,\infty}(g) + \operatorname{tor}_\infty) = q_\infty(\phi_{k,\infty}(g)) \\ &= \tilde{\phi}_{k,\infty}(q_k(g)) = \tilde{\phi}_{k,\infty}(g + \operatorname{tor}_k).\end{aligned}$$

We also have

$$\begin{aligned}\psi(\tilde{q}_\infty(\phi_{k,\infty}(g) + \operatorname{tor}_\infty)) &= \psi(q_\infty(\phi_{k,\infty}(g)) \\ &= \psi(\tilde{\phi}_{k,\infty}(q_k(g))) = \phi_{k,\infty}(g) + \operatorname{tor}_\infty .\end{aligned}$$

So $\tilde{q}_\infty \circ \psi$ is the identity on $\tilde{\phi}_{k,\infty}(Q_k)$ and $\psi \circ \tilde{q}_\infty$ is the identity on $\phi_{k,\infty}(G_k)/\operatorname{tor}_\infty$, and hence by continuity, ψ and q_∞ are mutually inverse. $\square$

For $n \geq 1$, the torsion subgroup of $\operatorname{coker}(1 - A_E^n)^t$ is

$$\{a + \operatorname{Im}(1 - A_E^n)^t : a \in \mathbb{Z}^{E^0}, ma \in \operatorname{Im}(1 - A_E^n)^t \text{ for some } m \in \mathbb{N}\}.$$

Define

$$T_n := \{a \in \mathbb{Z}^{E^0} : ma \in \operatorname{Im}(1 - A_E^n)^t \text{ for some } m \in \mathbb{N}\}.$$

So $T_n = q_n^{-1}(\operatorname{tor}_n)$ where $q_n : \mathbb{Z}^{E^0} \to \operatorname{coker}(1 - A_E^n)^t$ is the quotient map.

Proposition 5.10. *Let E be a strongly connected finite directed graph. Suppose 1 is an eigenvalue of A_E^t and that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence and let $l := \gcd(P_E, \omega)$. Then $T_{n_k} = T_l$ for all k such that $\gcd(\mathcal{P}_E, n_k) = l$.*

Proof. Fix $k \geq K$. Let $C := \sum_{i=0}^{n_k/l-1} (A_E^{il})^t$. We have

$$(1 - A_E^{n_k})^t = (1 - A_E^l)^t \Big(\sum_{i=0}^{n_k/l-1} (A_E^{il})^t \Big) = (1 - A_E^l)^t C. \tag{5.2}$$

So $\operatorname{Im}(1 - A_E^{n_k})^t \subseteq \operatorname{Im}(1 - A_E^l)^t$. Now take $x \in T_{n_k}$. Then there exists $m \in \mathbb{N}$ such that $mx \in \operatorname{Im}(1 - A_E^{n_k})^t \subseteq \operatorname{Im}(1 - A_E^l)^t$. Hence $T_{n_k} \subseteq T_l$.

For the reverse inclusion, take $x \in T_l$. Then $mx = (1 - A_E^l)^t y$, for some $m \in \mathbb{N}$ and $y \in \mathbb{Z}^{E^0}$. Equation (5.2) gives

$$\begin{aligned}(m \det C)x &= (\det C)(1 - A_E^l)^t y = (1 - A_E^l)^t C (\det C) C^{-1} y \\ &= (1 - A_E^{n_k})^t (\det C) C^{-1} y \in \operatorname{Im}(1 - A_E^{n_k})^t.\end{aligned}$$

By Lemma 3.2 $\det C \neq 0$, so $T_l \subseteq T_{n_k}$. □

Remark 5.11. If we could compute $\det C$, we could compute $\det(1 - A_E^{n_k})^t$. Then (when 1 is not an eigenvalue), we could calculate $|K_0(C^*(E, n_k))|$ and try to use Kribs' argument for [6, Theorem 5.1] to prove a classification result for the generalised Bunce–Deddens algebras constructed from a finite strongly connected graph whose vertex matrix does not have eigenvalue 1.

Lemma 5.12. *Let E be a strongly connected finite directed graph. Suppose 1 is an eigenvalue of A_E^t and that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, and let $l := \gcd(P_E, \omega)$. For each k such that $\gcd(\mathcal{P}_E, n_k) = l$, there is an isomorphism $\tau : \operatorname{coker}(1 - A_E^{n_k})^t / \operatorname{tor}_{n_k} \to \mathbb{Z}^{E^0}/T_l$ satisfying*

$$\tau(a + \operatorname{Im}(1 - A_E^{n_k})^t + \operatorname{tor}_{n_k}) = a + T_l \tag{5.3}$$

for $a \in \mathbb{Z}^{E^0}$.

Proof. To see that the formula (5.3) is well-defined, suppose $(a + \operatorname{Im}(1 - A_E^{n_k})^t) + \operatorname{tor}_{n_k} = (b + \operatorname{Im}(1 - A_E^{n_k})^t) + \operatorname{tor}_{n_k}$, where $a, b \in \mathbb{Z}^{E^0}$. Then $a + \operatorname{Im}(1 - A_E^{n_k})^t = b + \operatorname{Im}(1 - A_E^{n_k})^t + t$, where $t \in \operatorname{tor}_{n_k}$, that is, $t = c + \operatorname{Im}(1 - A_E^{n_k})^t$ for some $c \in T_{n_k}$. Then $a - b - c \in \operatorname{Im}(1 - A_E^{n_k})^t \subseteq \operatorname{Im}(1 - A_E^l)^t \subseteq T_l$. By Proposition 5.10, $c \in T_l$, so $a - b \in T_l$. So there is a map τ satisfying (5.3).

The map τ is clearly a surjective group homomorphism. To see that it is injective, suppose $a + T_l = b + T_l$ for $a, b \in \mathbb{Z}^{E^0}$. We have $a = b + c$, for some $c \in T_l$, and hence $a + \operatorname{Im}(1 - A_E^{n_k})^t = b + c + \operatorname{Im}(1 - A_E^{n_k})^t$. So $a + \operatorname{Im}(1 - A_E^{n_k})^t = b + \operatorname{Im}(1 - A_E^{n_k})^t +$

$c + \operatorname{Im}(1 - A_E^{n_k})^t$. By Proposition 5.10, $c \in T_{n_k}$. Therefore $a + \operatorname{Im}(1 - A_E^{n_k})^t + \operatorname{tor}_{n_k} = b + \operatorname{Im}(1 - A_E^{n_k})^t + \operatorname{tor}_{n_k}$. □

Corollary 5.13. *Let E be a strongly connected finite directed graph. Suppose that 1 is an eigenvalue of A_E^t and that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, let $l := \gcd(P_E, \omega)$. For each k such that $\gcd(\mathcal{P}_E, n_k) = l$, there is an isomorphism $\theta_{n_k} : \operatorname{coker}(1 - A_E^{n_k})^t / \operatorname{tor}_{n_k} \to \operatorname{coker}(1 - A_E^l)^t / \operatorname{tor}_l$ given by $\theta_{n_k}\big((a + \operatorname{Im}(1 - A_E^{n_k})^t) + \operatorname{tor}_{n_k}\big) = (a + \operatorname{Im}(1 - A_E^l)^t) + \operatorname{tor}_l$ for $a \in \mathbb{Z}^{E^0}$.*

Proof. Fix k such that $\gcd(\mathcal{P}_E, n_k) = l$. The previous Lemma gives an isomorphism $\operatorname{coker}(1 - A_E^{n_k})^t / \operatorname{tor}_{n_k} \to \mathbb{Z}^{E^0}/T_l$ satisfying $(a + \operatorname{Im}(1 - A_E^{n_k})^t) + \operatorname{tor}_{n_k} \mapsto a + T_l$, where $a \in \mathbb{Z}^{E^0}$. The result follows since $\mathbb{Z}^{E^0}/T_l$ is isomorphic to $\operatorname{coker}(1 - A_E^l)^t / \operatorname{tor}_l$ via $a + T_l \mapsto a + \operatorname{Im}(1 - A_E^l)^t + \operatorname{tor}_l$. We take θ_{n_k} to be the composition of these isomorphisms. □

We give another description of the torsion-free abelian group $A\big[\frac{1}{\omega}\big]$ of Lemma 5.1.

Lemma 5.14. *Let A be a free abelian group and let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, and let $m_k := n_{k+1}/n_k$ for all $k \in \mathbb{N}$. Define maps $M_k : A \to A$ by $M_k(a) = m_k \cdot a$, and let $M_{k,\infty}$ be the natural map $A \to \varinjlim(A, M_k)$. There is an isomorphism $\phi : \varinjlim(A, M_k) \cong A\big[\frac{1}{\omega}\big]$ satisfying $\phi(M_{k,\infty}(a)) = a/n_k$, for each $k \in \mathbb{N}$ and $a \in A$.*

Proof. Fix $k \in \mathbb{N}$. Define $j_{k,\infty} : A \to A\big[\frac{1}{\omega}\big]$ by $j_{k,\infty}(a) = a/n_k$ for $a \in \mathbb{Z}$. This $j_{k,\infty}$ is a homomorphism by definition of the operation on $A\big[\frac{1}{\omega}\big]$. We calculate $j_{k+1,\infty}(M_k(a)) = (m_k \cdot a)/n_{k+1} = (n_{k+1}/n_k) \cdot (a/n_{k+1}) = a/n_k = j_{k,\infty}(a)$. So the universal property of $\varinjlim(A, M_k)$ induces a homomorphism ϕ satisfying the desired formula. It remains to check that ϕ is an isomorphism. To see that ϕ is injective, fix $a \in A$ such that $\phi(M_{k,\infty}(a)) = 0$. Then $a/n_k = 0$, so $a = 0$. To see that ϕ is surjective, fix $a/n_k \in A\Big[\frac{1}{\omega}\Big]$. Then $\phi(M_{k,\infty}(a)) = a/n_k$. □

Proposition 5.15. *Let E be a strongly connected finite directed graph. Suppose that 1 is an eigenvalue of A_E^t and that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a mulitiplicative sequence, and let $l := \gcd(P_E, \omega)$. Fix K such that $\gcd(P_E, n_K) = l$, and define $\omega' := (n'_k)_{k=1}^\infty$ where $n'_1 = l$ and $n'_k = n_{K+k-1}$ for $k \geq 2$. For each $k \geq 1$, the map $\phi_{n'_k, n'_{k+1}}$ descends to a map $\tilde{\phi}_{n'_k, n'_{k+1}}$ such that the following diagram commutes.*

$$\begin{array}{ccc}
\operatorname{coker}(1 - A_E^{n'_k})^t / \operatorname{tor}_{n'_k} & \xrightarrow{\tilde{\phi}_{n'_k, n'_{k+1}}} & \operatorname{coker}(1 - A_E^{n'_{k+1}})^t / \operatorname{tor}_{n'_{k+1}} \\
\Big\downarrow \theta_{n'_k} & & \Big\downarrow \theta_{n'_{k+1}} \\
\operatorname{coker}(1 - A_E^l)^t / \operatorname{tor}_l & \xrightarrow{M'_k} & \operatorname{coker}(1 - A_E^l)^t / \operatorname{tor}_l
\end{array}$$

Proof. Fix $k \geq 1$. Applying the first assertion of Lemma 5.9 we see that $\phi_{n'_k, n'_{k+1}}$ descends to a homomorphism $\tilde{\phi}_{n'_k, n'_{k+1}} : \operatorname{coker}(1 - A_E^{n'_k})^t / \operatorname{tor}_{n'_k} \to \operatorname{coker}(1 - A_E^{n'_{k+1}})^t / \operatorname{tor}_{n'_{k+1}}$, satisfying $\tilde{\phi}_{n'_k, n'_{k+1}}(g + \operatorname{tor}_{n'_k}) = \phi_{n'_k, n'_{k+1}}(g) + \operatorname{tor}_{n'_{k+1}}$.

Define $B_k := \sum_{i=0}^{m'_k - 1} (A_E^{in'_k} - 1)^t$. Note that $B_k + m'_k 1 = \sum_{i=0}^{m'_k - 1} (A_E^{in'_k})^t$. We have that $(A_E^{in'_k} - 1)^t = (A_E^{n'_k} - 1)^t (\sum_{j=0}^{i-1} (A_E^{jn'_k})^t)$, so $\operatorname{Im} B_k \subseteq \operatorname{Im}(1 - A_E^{n'_k})^t \subseteq T_{n'_k} = T_l$ by Lemma 5.10. Thus $\operatorname{Im} B_k + \operatorname{Im}(1 - A_E^l)^t \subseteq \operatorname{tor}_l$.

Fix $x \in \mathbb{Z}^{E^0}$. By the preceding paragraph, we have

$$\begin{aligned}
\theta_{n'_{k+1}}\big(\tilde{\phi}_{n'_k,n'_{k+1}}\big(x + \operatorname{Im}(1 - A_E^{n'_k})^t + \operatorname{tor}_{n'_k}\big)\big) &= \theta_{n'_{k+1}}\big(\phi_{n'_k,n'_{k+1}}\big(x + \operatorname{Im}(1 - A_E^{n'_k})^t\big) + \operatorname{tor}_{n'_{k+1}}\big) \\
&= \sum_{i=0}^{m'_k - 1} (A_E^{in'_k})^t x + \operatorname{Im}(1 - A_E^l)^t + \operatorname{tor}_l \\
&= (B_k + m'_k 1)(x) + \operatorname{Im}(1 - A_E^l)^t + \operatorname{tor}_l \\
&= (m'_k 1)(x) + \operatorname{Im}(1 - A_E^l)^t + \operatorname{tor}_l \\
&= M'_k(x + \operatorname{Im}(1 - A_E^l)^t + \operatorname{tor}_l) \\
&= (M'_k \circ \theta_{n'_k})(x + \operatorname{Im}(1 - A_E^{n'_k})^t + \operatorname{tor}_{n'_k}). \qquad \square
\end{aligned}$$

Recall the isomorphism $\rho : \operatorname{coker}(1 - A_E^l)^t \to \bigoplus_{i=1}^l \operatorname{coker}(1 - A_E^t)$ of Lemma 3.6 satisfying

$$\rho\big(\delta_v + \operatorname{Im}(1 - A_E^l)^t\big) = \big(0, \dots, \delta_v + \operatorname{Im}(1 - A_E^t), \dots, 0\big),$$

where $v \in \Lambda_j$ for some $0 \le j \le l-1$, and $\delta_v + \operatorname{Im}(1 - A_E^t)$ appears in the j-th position.

Lemma 5.16. *Let E be a strongly connected finite directed graph. Suppose that 1 is an eigenvalue of A_E^t and that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, and let $l = \gcd(P_E, \omega)$. There is an isomorphism $\psi : K_0(C^*(E(l))) \to \bigoplus_{i=1}^l K_0(C^*(E))$ such that the following diagram commutes.*

$$\begin{array}{ccc}
\operatorname{coker}(1 - A_E^l)^t & \xrightarrow{\rho} & \bigoplus_{i=1}^l \operatorname{coker}(1 - A_E^t) \\
\Big\downarrow{\scriptstyle \sigma_l \circ \psi_l} & & \Big\downarrow{\scriptstyle \bigoplus_{i=1}^l \sigma_1} \\
K_0(C^*(E(l))) & \xrightarrow{\psi} & \bigoplus_{i=1}^l K_0(C^*(E))
\end{array}$$

Moreover, $\psi\Big(\sum_{\mu \in E^{<l}} [p_{s(\mu),l}]_0\Big) = ([1_{C^*(E)}]_0, \dots, [1_{C^*(E)}]_0)$.

Proof. We define $\psi := (\bigoplus_{i=1}^l \sigma_1) \circ \rho \circ (\sigma_l \circ \psi_l)^{-1}$. Since $\rho, \sigma_l \circ \psi_l$, and σ_1 are all isomorphisms, so is ψ.

We now show that ψ satisfies the second statement. Fix $0 \le i \le l-1$, and $v \in \Lambda_i$. Using Lemma 3.3 at the second equality, we have

$$\begin{aligned}
\rho\Big(\sum_{j=0}^{l-1} (A_E^j)^t \delta_v + \operatorname{Im}(1 - A_E^l)^t\Big) &= \Big((A_E^{l-i})^t \delta_v + \operatorname{Im}(1 - A_E^t), \dots, (A_E^{l-i-1})^t \delta_v + \operatorname{Im}(1 - A_E^t)\Big) \\
&= \Big(\delta_v + \operatorname{Im}(1 - A_E^t), \dots, \delta_v + \operatorname{Im}(1 - A_E^t)\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\psi\Big(\sum_{\mu\in E^{<l}}[p_{s(\mu),l}]_0\Big) &= \Big(\Big(\bigoplus_{i=1}^{l}\sigma_1\Big)\circ\rho\circ(\sigma_l\circ\psi_l)^{-1}\Big)\Big(\sum_{\mu\in E^{<l}}[p_{s(\mu),l}]_0\Big)\\
&= \Big(\Big(\bigoplus_{i=1}^{l}\sigma_1\Big)\circ\rho\Big)\Big(\sum_{\mu\in E^{<l}}\delta_{s(\mu)}+\mathrm{Im}(1-A_E^l)^t\Big)\Big)\\
&= \Big(\Big(\bigoplus_{i=1}^{l}\sigma_1\Big)\circ\rho\Big)\Big(\sum_{v\in E^0}\sum_{j=0}^{l-1}(A_E^j)^t\delta_v+\mathrm{Im}(1-A_E^l)^t\Big)\\
&= \Big(\bigoplus_{i=1}^{l}\sigma_1\Big)\Big(\sum_{v\in E^0}\delta_v+\mathrm{Im}(1-A_E^t),\dots,\sum_{v\in E^0}\delta_v+\mathrm{Im}(1-A_E^t)\Big)\\
&= ([1_{C^*(E)}]_0,\dots,[1_{C^*(E)}]_0). \qquad\square
\end{aligned}$$

Proof of Theorem 5.3. Fix K such that $\gcd(P_E,n_K)=\gcd(P_E,\omega)$, and let $\omega'=(n'_k)_{k=1}^\infty$ where $n'_1=l$ and $n'_k=n_{K+k-1}$ for $k\ge 2$. Let $m'_k=n'_{k+1}/n'_k$. Since $[\omega]=[\omega']$, we have a unital isomorphism $C^*(E,\omega)\cong C^*(E,\omega')$ by [10, Proposition 3.11]. Hence

$$\big(K_0(C^*(E,\omega)),[1_{C^*(E,\omega)}]\big)\cong\big(K_0(C^*(E,\omega')),[1_{C^*(E,\omega')}]\big).$$

So it suffices to prove the theorem for ω'.

Let $\mathrm{tor}_{\omega'}:=\mathrm{tor}\big(\varinjlim\big(K_0(C^*(E(n'_k))),K_0(j_{n'_k,n'_{k+1}})\big)\big)$. By [11, Theorem 6.3.2] there is an isomorphism

$$K_0(C^*(E,\omega'))\cong\varinjlim\Big(K_0(C^*(E(n'_k))),K_0(j_{n'_k,n'_{k+1}})\Big)$$

satisfying

$$[1_{C^*(E,\omega')}]\mapsto K_0(j_{n'_1,\infty})\Big(\sum_{\mu\in E^{<n'_1}}[p_{\mu,n'_1}]_0\Big).$$

This isomorphism descends to an isomorphism

$$K_0(C^*(E,\omega'))/\,\mathrm{tor}_{(E,\omega')}\cong\varinjlim\Big(K_0(C^*(E(n'_k))),K_0(j_{n'_k,n'_{k+1}})\Big)/\,\mathrm{tor}_{\omega'}$$

satisfying

$$[1_{C^*(E,\omega')}]_0+\mathrm{tor}_{(E,\omega')}\mapsto K_0(j_{n'_1,\infty})\Big(\sum_{\mu\in E^{<n'_1}}[p_{\mu,n'_1}]_0\Big)+\mathrm{tor}_{\omega'}\,.$$

Let $x:=\sum_{\mu\in E^{<n'_1}}\delta_{s(\mu)}\in\mathbb{Z}^{E^0}$, and let $\mathrm{tor}_\infty:=\mathrm{tor}\big(\varinjlim\big(\mathrm{coker}(1-A_E^{n'_k})^t,\phi_{n'_k,n'_{k+1}}\big)\big)$. The isomorphisms $(\sigma_{n'_k}\circ\psi_{n'_k})^{-1}$ discussed in Remark 5.6 induce an isomorphism

$$\varinjlim\big(K_0(C^*(E(n'_k))),K_0(j_{n'_k,n'_{k+1}})\big)/\,\mathrm{tor}_{\omega'}\cong\varinjlim(\mathrm{coker}(1-A_E^{n'_k})^t,\phi_{n'_k,n'_{k+1}})/\,\mathrm{tor}_\infty$$

satisfying

$$\begin{aligned}
K_0(j_{n'_1,\infty})\Big(\sum_{\mu\in E^{<n'_1}}[p_{\mu,n'_1}]_0\Big)+\mathrm{tor}_{\omega'} &\mapsto \phi_{n'_1,\infty}\Big((\sigma_{n'_1}\circ\psi_{n'_1})^{-1}\Big(\sum_{\mu\in E^{<n'_1}}[p_{\mu,n'_1}]_0\Big)\Big)+\mathrm{tor}_\infty\\
&= \phi_{n'_1,\infty}(x+\mathrm{Im}(1-A_E^{n'_1})^t)+\mathrm{tor}_\infty\,.
\end{aligned}$$

By Lemma 5.9 there is an isomorphism

$$\varinjlim(\operatorname{coker}(1-A_E^{n'_k})^t, \phi_{n'_k,n'_{k+1}})/\operatorname{tor}_\infty \cong \varinjlim(\operatorname{coker}(1-A_E^{n'_k})^t/\operatorname{tor}_{n'_k}, \tilde{\phi}_{n'_k,n'_{k+1}})$$

satisfying $\phi_{n'_1,\infty}(x+\operatorname{Im}(1-A_E^{n'_1})^t)+\operatorname{tor}_\infty \mapsto \tilde{\phi}_{n'_1,\infty}(x+\operatorname{Im}(1-A_E^{n'_1})^t+\operatorname{tor}_{n'_1})$.

By Proposition 5.15 there is an isomorphism

$$\varinjlim(\operatorname{coker}(1-A_E^{n'_k})^t/\operatorname{tor}_{n'_k}, \tilde{\phi}_{n'_k,n'_{k+1}}) \cong \varinjlim(\operatorname{coker}(1-A_E^{l})^t/\operatorname{tor}_{l}, M_{n'_k})$$

satisfying $\tilde{\phi}_{n'_1,\infty}(x+\operatorname{Im}(1-A_E^{n'_1})^t+\operatorname{tor}_{n'_1}) \mapsto M_{n'_1,\infty}(x+\operatorname{Im}(1-A_E^{l})^t+\operatorname{tor}_{l})$.

By Lemma 5.14 there is an isomorphism

$$\varinjlim(\operatorname{coker}(1-A_E^{l})^t/\operatorname{tor}_{l}, M_{n'_k}) \cong \big(\operatorname{coker}(1-A_E^{l})^t/\operatorname{tor}_{l}\big)\Big[\frac{1}{\omega'}\Big]$$

satisfying $m_{n'_1,\infty}(x+\operatorname{Im}(1-A_E^{l})^t+\operatorname{tor}_{l}) \mapsto (x+\operatorname{Im}(1-A_E^{l})^t+\operatorname{tor}_{l})/n'_1$.

The isomorphism $\eta_l := \sigma_l\circ\psi_l : \operatorname{coker}(1-A_E^l)^t \to K_0(C^*(E(l)))$ of Remark 5.6 descends to an isomorphism $\tilde{\eta}_l : \operatorname{coker}(1-A_E^l)^t/\operatorname{tor}_l \to K_0(C^*(E(l)))/\operatorname{tor}_{E(l)}$. This $\tilde{\eta}_l$ induces an isomorphism

$$\big(\operatorname{coker}(1-A_E^{l})^t/\operatorname{tor}_{l}\big)\Big[\frac{1}{\omega'}\Big] \cong \big(K_0(C^*(E(l)))/\operatorname{tor}_{E(l)}\big)\Big[\frac{1}{\omega'}\Big],$$

satisfying

$$\begin{aligned}\big(x+\operatorname{Im}(1-A_E^{l})^t+\operatorname{tor}_{l}\big)/n'_1 &\mapsto \tilde{\eta}_l(x+\operatorname{Im}(1-A_E^{l})^t+\operatorname{tor}_{l})/n'_1\\ &= \Big(\sum_{\mu\in E^{<l}}[p_{s(\mu),l}]_0+\operatorname{tor}_{E(l)}\Big)/n'_1.\end{aligned}$$

The isomorphism of Lemma 5.16 descends to an isomorphism $K_0(C^*(E(l)))/\operatorname{tor}_{E(l)} \to \bigoplus_{i=1}^l K_0(C^*(E))/\operatorname{tor}_E$, and this induces an isomorphism

$$\big(K_0(C^*(E(l)))/\operatorname{tor}_{E(l)}\big)\Big[\frac{1}{\omega'}\Big] \cong \Big(\bigoplus_{i=1}^l K_0(C^*(E))/\operatorname{tor}_E\Big)\Big[\frac{1}{\omega'}\Big],$$

satisfying

$$\begin{aligned}\Big(\sum_{\mu\in E^{<l}}[p_{s(\mu),l}]_0+\operatorname{tor}_{E(l)}\Big)/n'_1 &\mapsto \tilde{\psi}\Big(\sum_{\mu\in E^{<l}}[p_{s(\mu),l}]_0+\operatorname{tor}_{E(l)}\Big)/n'_1\\ &= ([1_{C^*(E)}]_0+\operatorname{tor}_E,\dots,[1_{C^*(E)}]_0+\operatorname{tor}_E)/l,\end{aligned}$$

since $n'_1 = l$.

Composing the isomorphisms of the previous seven paragraphs gives an isomorphism

$$\Psi : K_0(C^*(E,\omega'))/\operatorname{tor}_{(E,\omega')} \to \bigoplus_{i=1}^l \big(K_0(C^*(E))/\operatorname{tor}_E\big)\Big[\frac{1}{\omega'}\Big]$$

satisfying $\Psi([1_{C^*(E,\omega')}]) = ([1_{C^*(E)}]_0+\operatorname{tor}_E,\dots,[1_{C^*(E)}]_0+\operatorname{tor}_E)/l$. □

Remark 5.17. In the proof of Theorem 5.3, we needed to apply Lemma 5.16 to relate the torsion-free component of $K_0(C^*(E(l)))$ back to the torsion-free component of $K_0(C^*(E))$. This uses Corollary 3.6, which requires Lemma 3.5, where it is crucial that the power of A_E^t in the term $(1-A_E^l)^t$ matches the number of equivalence classes for the equivalence relation $\sim_l$. We also needed to apply Corollary 5.13 to obtain an isomorphism between the torsion-free component of $K_0(C^*(E(l)))$ and the torsion-free component of $K_0(C^*(E(n_k)))$ for all k such that $\gcd(\mathcal{P}_E, n_k) = l$. This uses Lemma 5.10 which depends on Lemma 3.2 explaining why we require that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity.

6. Classification of $C^*(E,\omega)$

In this section we use Theorem 5.3 to prove the following isomorphism theorem.

Theorem 6.1. *Fix a strongly connected finite directed graph E. Let $\omega = (n_k)_{k=1}^\infty$ and $\omega' = (n'_k)_{k=1}^\infty$ be multiplicative sequences. Suppose 1 is an eigenvalue of A_E^t and that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. Then $C^*(E,\omega) \cong C^*(E,\omega')$ if and only if $[\omega] = [\omega']$.*

To prove this theorem we need some preliminary results.

Lemma 6.2. *Let $D \subseteq \mathbb{N}$. Suppose $|D| = m$ for some $1 \le m \le \infty$, enumerate D in increasing order, $(d_1, d_2, \dots, d_m)$, and define a nondcreasing sequence $\operatorname{lcm}(D)$ by*

$$\operatorname{lcm}(D) := (d_1, \operatorname{lcm}(d_1,d_2), \operatorname{lcm}(d_1,d_2,d_3), \dots \operatorname{lcm}(d_1,d_2,\dots,d_m), \operatorname{lcm}(d_1,d_2,\dots,d_m), \dots).$$

Then $\operatorname{lcm}(D)$ is a multiplicative sequence such that $d_k | \operatorname{lcm}(D)$ for all $1 \le k \le m$. Moreover, if $\omega = (n_k)_{k=1}^\infty$ is another multiplicative sequence such that $d_k|\omega$ for all $1 \le k \le m$, then $[\operatorname{lcm}(D)]$ divides $[\omega]$.

Proof. Clearly $\operatorname{lcm}(D)_k | \operatorname{lcm}(D)_{k+1}$ for each $k \ge 1$. It is also clear that, for each $1 \le k \le m$, $d_k | \operatorname{lcm}(D)_l$ for all $l \ge k$, and so $d_k | \operatorname{lcm}(D)$.

For the final statement, fix ω such that $d_k|\omega$ for each $1 \le k \le m$. For each $1 \le k \le m$, there exist natural numbers $l_1, \dots, l_k$ such that $d_1|n_{l_1}, \dots, d_k|n_{l_k}$. Let $l(k) = \max\{l_1, \dots, l_k\}$. Then $d_i | n_{l(k)}$ for each $1 \le i \le k$, so $\operatorname{lcm}(d_1, \dots, d_k) | n_{l(k)}$. □

If A is a free abelian group, $a \in A$ and $n \ge 1$, we write $n|a$ if there exists $a' \in A$ such that $na' = a$.

Theorem 6.3. *Fix a strongly connected finite directed graph E, and a generalised Bunce–Deddens algebra $C^*(E,\omega)$. Suppose that the only roots of unity that are eigenvalues of A_E^t are the $\mathcal{P}_E$-th roots of unity. Set*

$$D := \{n \ge 1 : n | \big([1_{C^*(E,\omega)}]_0 + \operatorname{tor}_{(E,\omega)}\big) \in K_0(C^*(E,\omega))/\operatorname{tor}_{(E,\omega)}\}$$

and let

$$d := \operatorname{lcm}\{n \ge 1 : n | \big([1_{C^*(E)}]_0 + \operatorname{tor}_E\big) \in K_0(C^*(E))/\operatorname{tor}_E\}.$$

Then $[\omega] = [l \cdot \operatorname{lcm}(D)]/d$.

Proof. There is an isomorphism $\theta : K_0(C^*(E))/\operatorname{tor}_E \to \mathbb{Z}^N$, where $N = \operatorname{rank} K_0(C^*(E))$. Let $(u_1, \dots, u_N) := \theta([1_{C^*(E)}]_0 + \operatorname{tor}_E) \in \mathbb{Z}^N$.

We claim that $\gcd(u_1, \dots, u_N) = d$. Let $e_1, \dots, e_N$ be the generators of $\mathbb{Z}^N$, and let $n \ge 1$ such that $n|u_i$ for each $1 \le i \le N$. Then n divides $\sum_{i=1}^N u_i \theta^{-1}(e_i) = \theta^{-1}(u_1, \dots, u_N) = [1_{C^*(E)}]_0 + \operatorname{tor}_E$. So $n|d$, and hence $\gcd(u_1, \dots, u_N)|d$.

Now, fix $n \ge 1$ such that $n|([1_{C^*(E)}]_0 + \operatorname{tor}_E)$. Then there exists $a \in K_0(C^*(E))$ such that $na + \operatorname{tor}_E = [1_{C^*(E)}]_0 + \operatorname{tor}_E$. We then have that $n\theta(a + \operatorname{tor}_E) = (u_1, \dots, u_N)$. So n is a common divisor of $u_1, \dots, u_N$, and hence $n | \gcd(u_1, \dots, u_N)$. So $\gcd(u_1, \dots, u_N)$ is a common multiple of $\{n \ge 1 : n | \big([1_{C^*(E)}]_0 + \operatorname{tor}_E\big) \in K_0(C^*(E))/\operatorname{tor}_E\}$, giving $d | \gcd(u_1, \dots, u_N)$, and so $\gcd(u_1, \dots, u_N) = d$.

Next we claim that for $n \geq 1$, we have $n|\operatorname{lcm}(D)$ if and only if $n \in D$. If $n \in D$, it is clear that $n|\operatorname{lcm}(D)$. For the other direction, suppose $n|\operatorname{lcm}(D)$. Then there is an $i \geq 1$ such that $n|\operatorname{lcm}(d_1, \dots, d_i)$. Since $d_1, \dots, d_i \in D$, we have that $\operatorname{lcm}(d_1, \dots, d_i)$ divides $[1_{C^*(E,\omega)}]_0 + \operatorname{tor}_{(E,\omega)}$, and so $n \in D$.

We now show that $[\operatorname{lcm}(D)]$ divides $[d\omega/l]$. Fix $n \geq 1$. Then

$$\begin{aligned} n \in D &\iff n|\big([1_{C^*(E,\omega)}]_0 + \operatorname{tor}_{(E,\omega)}\big) \in K_0(C^*(E,\omega))/\operatorname{tor}_{(E,\omega)} \\ &\iff n|\big([1_{C^*(E))}]_0 + \operatorname{tor}_E, \dots, [1_{C^*(E)}]_0 + \operatorname{tor}_E\big)/l \in \bigoplus_{i=1}^{l} \Big(K_0(C^*(E))/\operatorname{tor}_E\Big)\Big[\frac{1}{\omega}\Big] \\ &\iff n|\big([1_{C^*(E)}]_0 + \operatorname{tor}_E\big)/l \in \Big(K_0(C^*(E))/\operatorname{tor}_E\Big)\Big[\frac{1}{\omega}\Big] \\ &\iff n|(u_1, \dots, u_N)/l \in \bigoplus_{i=1}^{N} \mathbb{Z}\Big[\frac{1}{\omega}\Big] \\ &\iff n|(d/l) \in \mathbb{Z}\Big[\frac{1}{\omega}\Big] \\ &\iff n|1 \in \mathbb{Z}\Big[\frac{1}{(d\omega)/l}\Big] \\ &\iff n|(d\omega/l). \end{aligned}$$

Hence $n|d\omega$ for all $n \in D$, and so $[\operatorname{lcm}(D)]$ divides $[d\omega/l]$ by Lemma 6.2.

To see that $[d\omega/l]$ divides $[\operatorname{lcm}(D)]$, fix $k \geq 1$. We have that $n_k|1 \in \mathbb{Z}\big[\frac{1}{\omega}\big]$, so $(dn_k/l)|(d/l) \in \mathbb{Z}\big[\frac{1}{\omega}\big]$. The above string of implications gives us $(dn_k/l)|\operatorname{lcm}(D)$ for each $k \geq 1$, so $[d\omega/l]$ divides $[\operatorname{lcm}(D)]$, and the result follows. □

We now prove Theorem 6.1.

Proof of Theorem 6.1. Suppose that $[\omega] = [\omega']$. Then $C^*(E,\omega) \cong C^*(E,\omega')$ by [10, Proposition 3.11].

Now suppose that $C^*(E,\omega) \cong C^*(E,\omega')$. Let $l = \gcd(\mathcal{P}_E, \omega)$ and $l' = \gcd(\mathcal{P}_E, \omega)$. Since $C^*(E,\omega) \cong C^*(E,\omega')$, the number of summands in Theorem 5.3 must be equal, so $l = l'$.

Let d be as in Theorem 6.3. Let

$$D := \{n \geq 1 : n|\big([1_{C^*(E,\omega)}]_0 + \operatorname{tor}_{(E,\omega)}\big) \in K_0(C^*(E,\omega))/\operatorname{tor}_{(E,\omega)}\}$$

and let

$$D' := \{n \geq 1 : n|\big([1_{C^*(E,\omega')}]_0 + \operatorname{tor}_{(E,\omega')}\big) \in K_0(C^*(E,\omega'))/\operatorname{tor}_{(E,\omega')}\}.$$

Fix $n \geq 1$. Since $C^*(E,\omega) \cong C^*(E,\omega')$, we have that n divides $[1_{C^*(E,\omega)}]_0 + \operatorname{tor}_{(E,\omega)}$ precisely when n divides $[1_{C^*(E,\omega')}]_0 + \operatorname{tor}_{(E,\omega')}$, so $D = D'$. By Theorem 6.3 we have that $[\omega] = [l \cdot \operatorname{lcm}(D)]/d = [l \cdot \operatorname{lcm}(D')]/d = [\omega']$. □

Remark 6.4. Theorem 6.1 says that for a given graph E and $[\omega] \neq [\omega']$, we have $C^*(E,\omega) \neq C^*(E,\omega')$. One might ask whether this can be extended to say that given graphs E and F and given $[\omega] \neq [\omega']$, we must have $C^*(E,\omega) \neq C^*(F,\omega')$. The following example demonstrates that the answer is no. Let C_1 be the graph consisting of a single vertex connected by a single loop and let C_3 be the graph with three vertices connected by a single cycle. Let $\omega = (3, 6, 12, 24, \dots)$ and let $\omega' = (1, 2, 4, 8, 16, \dots)$. Note that $\omega = 3\,\omega'$.

Since $C_1(3) = C_3$, we have that $C^*(C_1, \omega) \cong C^*(C_3, \omega')$. This illustrates why Theorem 6.1 applies only to generalised Bunce–Deddens algebras constructed from the same graph.

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