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Zhuo Wang
University of Wollongong, zw213@uow.edu.au

Timothy McCarthy
University of Wollongong, timmc@uow.edu.au

M Neaz Sheikh
University of Wollongong, msheikh@uow.edu.au

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Taut-slack Algorithm for Analyzing the Geometric Nonlinearity of Cable Structures

Zhuo Wang
Faculty of Engineering, University of Wollongong
Wollongong, NSW, Australia

Timothy McCarthy
Faculty of Engineering, University of Wollongong
Wollongong, NSW, Australia

M.Neaz Sheikh
Faculty of Engineering, University of Wollongong
Wollongong, NSW, Australia

ABSTRACT

This paper presents an improved small strain elastic finite element formulation to analyze static multi-component mooring cable problems. The inherent catenary profile of a cable subjected to self-weight and other loads can be solved quickly with the flexibility iteration approach coupled with the 'Taut-slack' algorithm. This new algorithm improves the stability of the Newton-Raphson solution process. The results for an example problem have been found to be consistent with those from OrcaFlex.

KEY WORDS: Catenary Cables; Newton-Raphson; Static analysis; Taut-slack.

NOMENCLATURE

H: the length of the horizontal projection of a cable
L: the stressed length of a cable
L_u: the unstressed length of a cable
P_1: the horizontal component of tension at node i
P_2: the vertical component of tension at node i
P_3: the horizontal component of tension at node j
P_4: the vertical component of tension at node j
V: the length of the vertical projection of a cable
w: the unit weight of a cable

INTRODUCTION

Cable structures such as mooring lines are subjected to large deformation due to their high flexibility. Since the behavior of mooring lines is significantly different from that of solid structures, there exist a large number of approaches for the analysis of this highly non-linear system. Energy based dynamic relaxation approach introduced by Lewis (1984) and stiffness matrix method by Krishna (1978) are examples. A comprehensive review of current existing techniques for the analysis of cable structures can be found in Kwan (1998).

A literature review on the analysis of cable structures reveals that modeling of an individual cable or cable system is challenging, because they are highly non-linear (Matulea, et al, 2008).

When the displacements of cable structures are not very large and the geometry of the system is well defined even at the initial design phase it is common to discretize the cable to bar-like elements and solve from numerical analysis from algebraic equations (Peyrot, 1979, Silva, et al, 2000). In terms of the geometric profile, however, the bar-like elements do not represent the real world and require a large number of elements. According to Irvine (1992), the non-linear stress strain relationships introduce a catenary shape to the hanging cable under its self-weight. The complete catenary geometry of a multi-component mooring line is determined by a procedure named flexibility iteration. This iteration approach was first suggested by O'Brien (1964 and 1967).

In order to better analyze the multi-component catenary mooring line, this flexibility iteration method has been modified by including a 'Taut-slack' algorithm in combination with Newton-Raphson method. This overcomes the discontinuity in the solution space when a cable element transitions from a slack state to a taut state and vice versa. The developed 'Taut-slack' algorithm ensures the convergence in the situations. The improved methodology presented in this paper is able to predict the final geometry of the cable, internal forces vector of the cable elements, and its tangent stiffness matrix. The required inputs are the given original length of the cable at unstressed condition, gravity load, elastic modulus, cross-sectional area and positions of its corresponding end points which are commonly known beforehand.

The approach presented herein is derived from the exact analytical solution based on O'Brien (1967). It is assumed that the stretching of the cables is purely elastic and axial and has no bending stiffness. Details of the analytical solution process can be found in the Appendix.

Compared to other approaches such as bar elements, cables can be divided to fewer segments when subjected to distributed load such as ocean current. Hence, the current solution method requires less computational effort and achieves fast convergence.

CABLE ELEMENT FORMULATION - SLACK

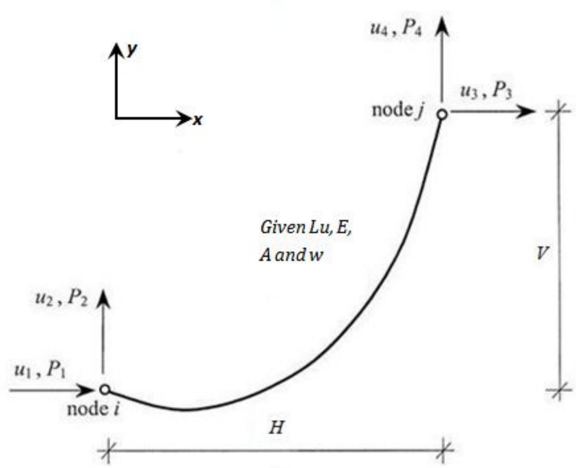

Figure 1 Catenary Cable Element

Consider the elastic cable element shown in Fig. 1 which is naturally suspended under gravity in a vertical plane. According to Irvine (1992) it has an equilibrium catenary profile under gravity load (self-weight) which satisfies

$$L^2 = V^2 + H^2 \frac{\sinh^2 \lambda}{\lambda^2} \tag{1}$$

where

$$\lambda = \frac{w|H|}{2|P_1|} \tag{2}$$

$$P_2 = \frac{w}{2}\left[-V\frac{\cosh\lambda}{\sinh\lambda} + L\right] \tag{3}$$

Geometrical relationships integrated along the projections are shown as follows (Huang, 1992 and Chucheepsakul, 1995)

$$H = -P_1\left(\frac{L_u}{EA} + \frac{1}{w}\ln\frac{P_4 + T_j}{T_i - P_2}\right) \tag{4}$$

$$V = \frac{1}{2EAw}\left(T_j^2 - T_i^2\right) + \frac{T_j - T_i}{w} \tag{5}$$

where T_i and T_j are the cable tensions of the element at nodes i and j respectively. P and T are related by the following equations:

$$P_4 = wL_u - P_2 \tag{6}$$

$$P_3 = -P_1 \tag{7}$$

$$T_i = \sqrt{P_1^2 + P_2^2} \tag{8}$$

$$T_j = \sqrt{P_3^2 + P_4^2} \tag{9}$$

The expressions for horizontal and vertical projections H and V have been written for small changes in terms of P_1 and P_2 only by their first order differentials as (Jaymaraman, et al, 1981):

$$dH = \left(\frac{\partial H}{\partial P_1}\right)dP_1 + \left(\frac{\partial H}{\partial P_2}\right)dP_2 \tag{10}$$

$$dV = \left(\frac{\partial V}{\partial P_1}\right)dP_1 + \left(\frac{\partial V}{\partial P_2}\right)dP_2 \tag{11}$$

Rewriting Eq. 10 and Eq. 11 in a matrix notation,

$$\begin{Bmatrix} dH \\ dV \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial H}{\partial P_1} & \dfrac{\partial H}{\partial P_2} \\ \dfrac{\partial V}{\partial P_1} & \dfrac{\partial V}{\partial P_2} \end{bmatrix} \begin{Bmatrix} dP_1 \\ dP_2 \end{Bmatrix} = F \begin{Bmatrix} dP_1 \\ dP_2 \end{Bmatrix} \tag{12}$$

where F is the incremental flexibility matrix and is equal to the inverse of the stiffness matrix K:

$$K = F^{-1} = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \tag{13}$$

When comparing Eqs. 10~13 with Eq. 4 and Eq. 5, to ensure the matrix is invertible, it must have a non-zero determinant. Hence, Eqs. 14~16 are obtained by taking partial derivatives.

$$k_1 = -\frac{1}{\det F}\left(\frac{L_u}{EA} + \frac{1}{w}\left(\frac{P_4}{T_j} + \frac{P_2}{T_i}\right)\right) \tag{14}$$

$$k_2 = k_3 = -\frac{1}{\det F}\left(\frac{P_1}{w}\left(\frac{1}{T_j} - \frac{1}{T_i}\right)\right) \tag{15}$$

$$k_4 = \frac{1}{\det F}\left(\frac{H}{P_1} + \frac{1}{w}\left(\frac{P_4}{T_j} + \frac{P_2}{T_i}\right)\right) \tag{16}$$

where the determinant is given by

$$\det F = \left(-\frac{L_u}{EA} - \frac{1}{w}\left(\frac{P_4}{T_j} + \frac{P_2}{T_i}\right)\right) \times \left(\frac{H}{P_1} + \frac{1}{w}\left(\frac{P_4}{T_j} + \frac{P_2}{T_i}\right)\right) - \left(\frac{P_1}{w}\left(\frac{1}{T_j} - \frac{1}{T_i}\right)\right)^2 \tag{17}$$

The idea of the flexibility iteration method starts with an initial estimation of horizontal and vertical projections H and V, respectively.

Then, the differences between the actual projections and the estimated projections are minimized until a tolerable error is found. In order to initialize the loop, reasonable estimations of P_1 and P_2 are required to ensure the convergence. The value for the horizontal component of the tension can be obtained from Eq. 1 by substituting the stretched length L with the original cable length L_u. Keeping the first two terms of the series expansion of $(\sinh^2\lambda)/\lambda^2$, one can get an expression for λ as

$$\lambda = \left(6\sqrt{\frac{L_u^2 - V^2}{H^2}} - 6 \right)^{1/2} \tag{18}$$

Details of the derivation of Eq. 18 can be found in the Appendix.

By substituting Eq. 18 into Eq. 2 and rearranging, an approximation of P_1 can be estimated. Likewise, substituting Eq. 18 to Eq. 3, P_2 can be found directly. Karoumi (1998) demonstrated that, with these initial values, convergence is achieved rapidly, generally within four to five iterations.

CABLE ELEMENT FORMULATION - TAUT

If the Eq. 18 does not have a real root, this may indicate a taut cable. That is a cable whose unstretched length is less than the distance between its current ends. The initial position has a situation where L_u is shorter than the distance between nodes i and j, following assumptions in Peyrot (1979). Since λ is about four times the sag to span ratio for horizontal span, a conservative estimate of sag to span ratio of five percent can be assumed. Therefore, an initial estimation value of 0.2 for λ can be applied in cases where the cable has a stretched and taut position. If the initial cable arrangement is vertical or near vertical, a large value of λ is applied (10^6) in order to stabilize the iterations.

Summarizing the implementation process of the flexibility iterations method above, the initial components of tension force P_1 and P_2 are evaluated at the first stage. Then, cable projections H and V are obtained. The misclosure vector based on actual projections and the estimated projections $\{\Delta H, \Delta V\}^T$ can then be calculated. Corrections to the initial estimation of forces are available through computed misclosure vector as:

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \end{Bmatrix} = K \begin{Bmatrix} \Delta H \\ \Delta V \end{Bmatrix} \tag{19}$$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}^{i+1} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}^{i} + \begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \end{Bmatrix} \tag{20}$$

If the geometry of the whole cable is to be determined, coordinates for a number of points along the cable need to be computed. This process becomes very simple because both P_1 and P_2 are known after a few iterations. By substituting all the necessary values into Eq. 4 and Eq. 5, the corresponding positions of each component can be calculated and therefore, the cable profile is obtained.

NEWTON-RAPHSON IN MULTI-COMPONENT CABLES

Suspended cables subjected to its self-weight can be determined efficiently by the approach introduced in previous sections. However, when the cable has multi-component constitutions and/or varying applied external distributed loads, the cable profile does not stick to its natural catenary shape (self-weight only). The entire cable is then assembled from the individual stiffness matrices to form a system for which the equilibrium can be found by adopting Newton-Raphson non-linear approach.

Since the cable is subdivided into components by nodes, the element tangent stiffness matrix K_t for the cable component can be obtained in terms of the four nodal degrees of freedom as ($k_2 = k_3$)

$$K_t = \begin{bmatrix} -k_1 & -k_2 & k_1 & k_2 \\ -k_3 & -k_4 & k_3 & k_4 \\ k_1 & k_2 & -k_1 & -k_2 \\ k_3 & k_4 & -k_3 & -k_4 \end{bmatrix} \tag{21}$$

Likewise, from Eq. 19, the element tangent stiffness matrix K_t relates the incremental element force vector and the incremental displacement vector through the Hooke's law

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = K_t \begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \\ \Delta u_4 \end{Bmatrix} \tag{22}$$

The flow chart of the calculation process is shown in Fig. 2. The allowable error (TE) is assumed as 10^{-5} in the programming. In Fig. 2, each component of the cable is calculated through the flexibility iteration approach initially, and then the global tangent stiffness matrix of the structure is formed for the Newton iterations.

TAUT SLACK ALGORITHM IN THE FLEXIBILITY ITERATION METHOD

The advantages of applying the flexibility iteration are the rapid converging speed (Karoumi, 1998) and the natural catenary built component which resembles the real behavior. However, this flexibility iteration approach does not always converge when looped in the Newton iteration where there are the multi-component cables. The reason for the divergence is because the flexibility iteration approach can only work in a smooth and continuous solution surface. When spikes or discontinuities occur, even a reasonably good initial estimation may still lead to instability in the solution space or a complete failure of the iteration.

When a function has a discontinuous domain or spikes in a range, Newton's method has its own limitations. In that case, Andreu et al. (2006) suggest using bisection approach in element resolution scheme for the sake of stability and accuracy. Nevertheless, the method converges linearly and is very slow.

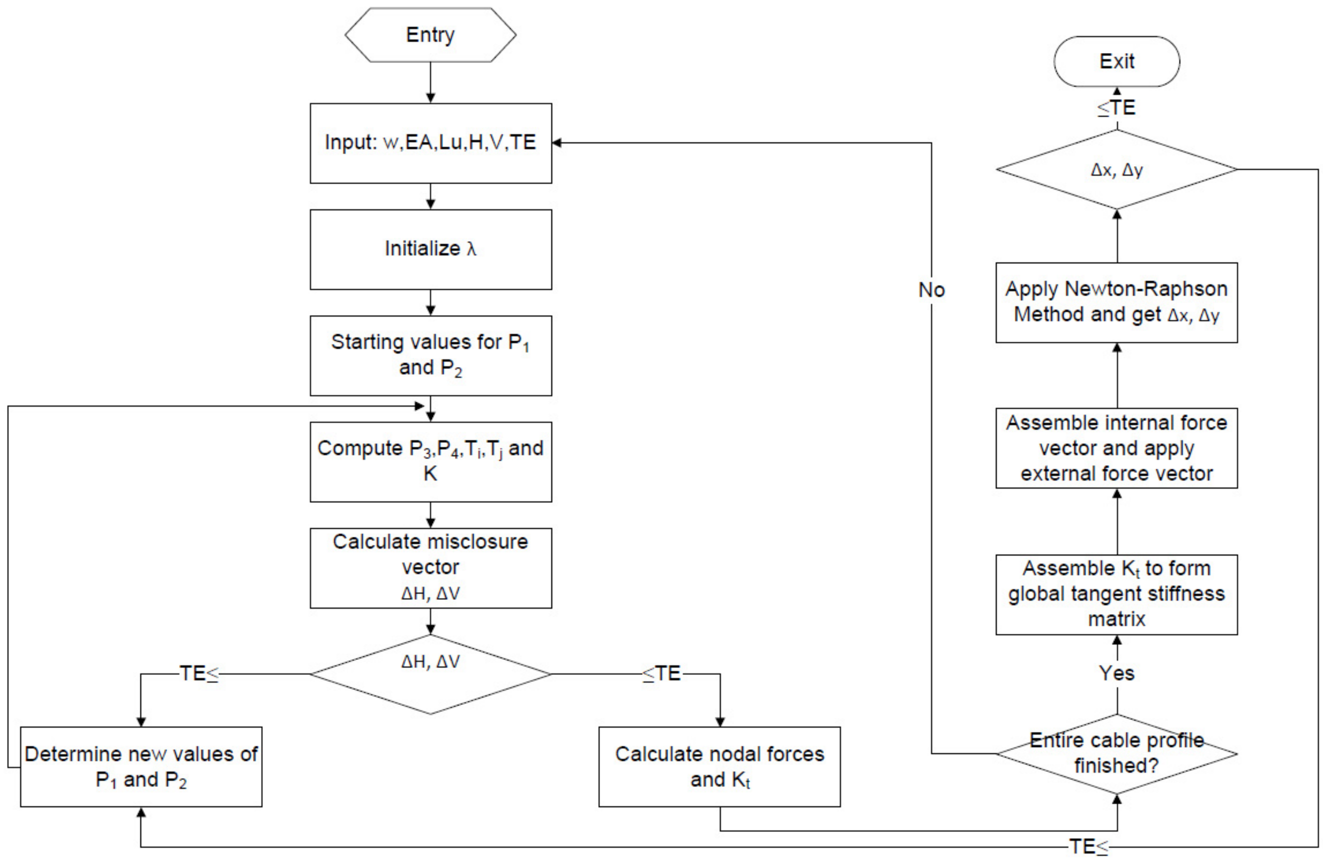

Figure 2 Numerical modeling flow chart of multi-component catenary cable

For example, a multi-component cable has a taut component with an initial estimation value of 0.2 for λ as suggested by Peyrot (1979). This taut component means that the unstressed length L_u is shorter than the distance between nodes i and j which are the end points. The flexibility iteration approach searches for the equilibrium based on the initial estimation of λ until the equilibrium position found. However, it is possible that the stressed length of the cable is long enough to reach equilibrium due to the self-weight stretch when hanging in its working condition.

Another possibility may occur in looping the multi-component cable with Newton-Raphson method. Assuming one has a slack component with an estimation value of λ based on Eq. 18. As the end positions of the cable component keep changing in the Newton iterations, it is highly likely that at an intermediate step that the cable component can become taut. The flexibility iteration approach, nevertheless, keeps searching for the equilibrium in slack range, which results divergence of the approach.

As mentioned above, the flexibility iteration approach cannot always guarantee a convergence when applied in multi-component cables. To improve the stability, it needs an algorithm to smooth the calculation process from taut to slack and vice versa. At the occurrence of divergence, a switch has been placed in the calculation. The function of the switch starts to take action when it detects instability. It terminates the on-going calculation and then assigns a new initial estimation that is always in the opposite range of the previous to re-run the simulation. For instance, when an initial trial set of tension force components, obtained from the slack condition, fails in convergence, the switch terminates its calculation, and assigns a new trial set of values from the taut condition. The application of this switch ensures that the fast convergence of flexibility iteration approach. This works well even if in an inferior value of λ is chosen initially. This switching of initial conditions in the calculation is the 'taut-slack' algorithm. An example demonstrating the application of the 'taut-slack' algorithm is outlined in the following section.

EXAMPLE OF THE TAUT SLACK ALGORITHM

The main application of the taut-slack algorithm is in the numerical solution process of multi-component cables using Newton-Raphson method. The divergence always occurs in the vicinity of boundary between taut and slack during flexibility iteration. Therefore, it is rare to see this occurring by using the flexibility iteration for a single component cable. However, it is common during Newton-Raphson numerical iteration as cable components have been frequently changing positions during iterations. The cable in the following example is a middle component of a mooring cable of total length 100 meters. The length of the component is one-third of the total length.

This is an efficient example requiring the use of taut-slack algorithm in applying the flexibility iteration approach. The example is chosen to demonstrate that running the original flexibility iteration alone with Newton-Raphson method would result in divergence of the calculation and reach no solution to the question. This example has been incorporated in MATLAB code and results are compared with simulation from OrcaFlex (2005).

Fig. 3 shows a component from a normal cable with unstressed length of 33.3333m and axial stiffness EA = 1.3×10^9 N. The horizontal and vertical projections of this component are 8.1476m and 32.3358m respectively. An initial estimation of λ equal to 0.2 has been considered for the iterations. However, the flexibility iteration does not converge with this λ value and results are not available by using this approach. This is because during the iterations, this part of the cable becomes taut. When the taut-slack algorithm detects the divergence, it re-assigns a value 0.05 to λ as per Eq. 18. As to the flexibility iteration approach, the λ value claims that the cable component in a state of slack instead of taut. It can be seen in Fig. 3 that all the lengths have been kept four significant figures after the decimal point. If simulation is carried out without taut-slack algorithm, the overall response of the cable is failure due to divergence in the second component. Results are summarized in Table 1 for comparison.

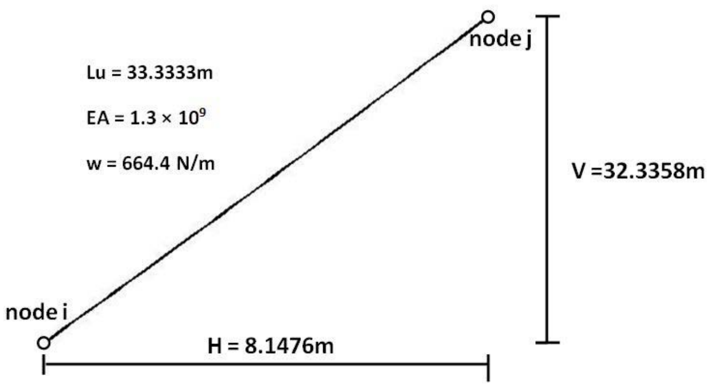

Figure 3 A Catenary Cable Component

Table 1. Tensions comparison with and without taut-slack algorithm in the flexibility iteration approach for the cable component

	With 'Taut-slack' algorithm	Without 'Taut-slack' algorithm	OrcaFlex	Differences (%)
Top tension (kN)	530.42	divergence	530.52	0.02
End tension (kN)	508.94	divergence	508.94	0

It is clear that in the example demonstrated here, the *taut-slack* algorithm improves the stability of the flexibility iteration approach for convergence. Meanwhile, it retains the advantages of the flexibility iteration approach, such as fast convergence and good accuracy. With the taut-slack algorithm, multi-component cable simulations can be easily accomplished in the Newton-Raphson iterations without significant increase of computation cost.

CONCLUSION

A catenary curved element that included self-weight calculation has been presented for the analysis of cable structures. The analysis is based on a flexibility iteration procedure that computes the stiffness matrices and corresponding forces. An extension of applying this approach to the multi-component cable analysis can be smoothly incorporated by the Newton-Raphson iterations. The '*taut-slack*' algorithm has been used to ensure the stability of the calculation, and results can be achieved in any situation of the problem regardless of the accuracy of the initial estimation. The example demonstrated the feasibility and reliability of the analysis, and the potential application in offshore mooring problems.

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APPENDIX

The derivation of the basic equation of a suspended cable is as follows: T is defined as the tension in the cable and dy/dx is the sine of the angle subtended to the horizontal by the tangent profile. The vertical equilibrium from Figure A1 gives

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) = -w \qquad \text{(A1)}$$

Horizontal equilibrium when free hanging cable results in

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0 \qquad \text{(A2)}$$

Integrate Eq. A2 along the cable length s

$$T\frac{dx}{ds} = P_H \qquad \text{(A3)}$$

where P_H is the horizontal component of cable tension which corresponds to P_1 and P_3 in Figure 1.

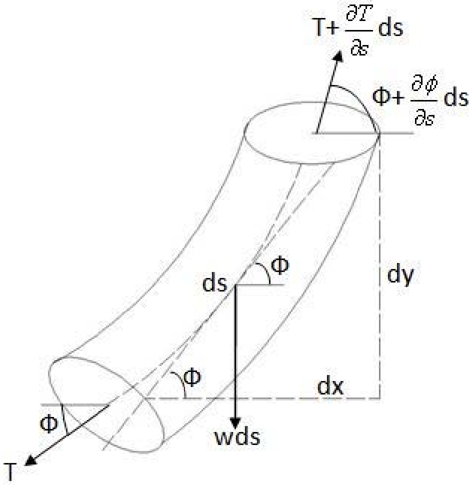

Fig. A1. An infinitesimal cable element

Now,

$$\frac{dy}{ds}=\frac{dy}{dx}\cdot\frac{dx}{ds} \tag{A4}$$

Substituting Eq. A4 to Eq. A1 and get

$$\frac{d}{ds}\left(T\cdot\frac{dy}{dx}\cdot\frac{dx}{ds}\right)=-w \tag{A5}$$

Rearrange Eq. A3 to get

$$\frac{dx}{ds}=\frac{P_H}{T} \tag{A6}$$

Substituting Eq. A6 into Eq. A5 and rearranging gives

$$\Rightarrow \frac{d}{ds}\left(P_H\frac{dy}{dx}\right)=-w$$

$$\Rightarrow \frac{d}{dx}\frac{dx}{ds}\left(P_H\frac{dy}{dx}\right)=-w$$

Therefore, the classical differential equation of a cable subject to its own weight can be obtained in Eq. A7.

$$P_H\frac{d^2y}{dx^2}+w\frac{ds}{dx}=0 \tag{A7}$$

Solve the differential Eq. A7 as follows. First, because the geometric constraint must be satisfied, namely,

$$\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2=1 \tag{A8}$$

The governing differential Eq. A7 now takes the form as substituted in Eq. A8

$$P_H\frac{d^2y}{dx^2}+w\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} \tag{A9}$$

From the following identity

$$1+\sinh^2 t=\cosh^2 t \tag{A10}$$

and letting

$$\frac{dy}{dx}=\sinh t \tag{A11}$$

Substitute Eq. A11 to Eq. A9 to give

$$\Rightarrow P_H\frac{d}{dt}(\sinh t)\cdot\frac{dt}{dx}+w\cosh t=0$$

$$\Rightarrow P_H\cdot\cosh t\cdot\frac{dt}{dx}+w\cosh t=0$$

$$\Rightarrow P_H\frac{dt}{dx}+w=0$$

Integrating the above expression results in

$$t=-\frac{w}{P_H}x+\phi \tag{A12}$$

Substitute Eq. A11 to Eq. A10 and integrate

$$\Rightarrow \frac{dy}{dx}=-\sinh\left(\frac{w}{P_H}x-\phi\right)$$

$$\Rightarrow y=-\frac{w}{P_H}\cosh\left(\frac{w}{P_H}x-\phi\right)+Constant$$

Given the boundary conditions

$$x=0, y=0$$
$$x=H, y=V \tag{A13}$$

Therefore, the constant of integration can be found by considering the boundary conditions: $Constant=\frac{P_H}{w}\cos\phi$ and

$$y=\frac{P_H}{w}\left[\cosh\phi-\cosh\left(\frac{wx}{P_H}-\phi\right)\right] \tag{A14}$$

$$V=\frac{P_H}{w}\left[\cosh\phi-\cosh\left(\frac{wH}{P_H}-\phi\right)\right] \tag{A15}$$

where $\phi = \sinh^{-1}\left[\frac{\lambda\left(\frac{H}{V}\right)}{\sinh\lambda}\right] + \lambda$ are calculated by utilizing the trigonometry identity

$$\cosh a - \cosh b = 2\sinh\frac{a+b}{2}\sinh\frac{a-b}{2}.$$

The coefficient λ is given by Eq. 2. The process of solution of Eq. A7 has been accomplished. To obtain the length of the cable, one can take integration along x

$$L = \int_0^H \frac{ds}{dx}\cdot dx = \int_0^H \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} dx = \int_0^H \cosh t \cdot dx$$

Substitute Eq. A12 to the above expression

$$L = \int_0^H \cosh\left(\frac{wx}{P_H}-\phi\right)\cdot dx$$

$$\Rightarrow L = \frac{H}{w}\left[\sinh\left(\frac{wx}{P_H}-\phi\right)+\sinh\phi\right]$$

Therefore,

$$L = \frac{2H}{w}\sinh\lambda \times \cosh(\phi-\lambda) \quad \text{(A16)}$$

Rearranging (Eq. A15)^2 – (Eq. A14)^2,

$$L^2 - V^2 = \frac{4H^2}{w^2}\sinh^2\lambda\cdot\cosh^2(\phi-\lambda)$$

$$-\frac{4H^2}{w^2}\sinh^2\lambda\cdot\sinh^2(\phi-\lambda)$$

and simplifying using Eq. A10 gives Eq. A17

$$L^2 - V^2 = \frac{H^2\sinh^2\lambda}{\lambda^2} \quad \text{(A17)}$$

Using series expansion on the right hand side or Eq. A17,

$$\left(\frac{\sinh\lambda}{\lambda}\right)^2 = \left(\frac{\lambda+\frac{\lambda^3}{6}+\text{K}}{\lambda}\right)^2 \approx \left(1+\frac{\lambda^2}{6}\right)^2$$

$$= 1+\frac{\lambda^2}{3}+\frac{\lambda^4}{36}$$

Ignoring the higher order part and simplify to get Eq. A18.

$$1+\frac{\lambda^2}{3} = \frac{L^2-V^2}{H^2} \quad \text{(A18)}$$

Further simplifying this equation results in

$$\lambda = \left(6\sqrt{\frac{L_u^2-V^2}{H^2}}-6\right)^{1/2} \quad \text{(A19)}$$