1973

Some aspects of inverter-fed induction motors

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Recommended Citation
SOME ASPECTS OF
INVERTER-FED INDUCTION MOTORS

Thesis for the Degree of
Doctor of Philosophy

submitted by


Department of Electrical Engineering
Wollongong University College
The University of New South Wales

June 1973
ACKNOWLEDGEMENTS

I take this opportunity of placing on record my thanks to my supervisor, Professor R. E. Vowels, Pro Vice-Chancellor, The University of New South Wales, for his advice, support and encouragement and for the time spent in discussions on thesis content.

I also wish to record my thanks to Professor C.A.M. Gray, Warden, Wollongong University College and Professor B. H. Smith, Head, Department of Electrical Engineering for their support in providing all facilities needed and for their active interest in my work.

W. Charlton
SUMMARY

A set of equations is obtained in terms of 4 machine parameters, stator voltages and currents and transformed rotor currents. The transformation is extended to include non-quadrature stator windings.

A solution procedure in terms of a transition matrix is obtained for any operating condition where the supply voltage waveform is piece-wise constant. Steady state current and torque waveforms can be evaluated as well as constant speed transients.

General expressions are obtained for the system matrix eigenvalues. A transition matrix determination using the eigenvalues and system matrix is developed and this shows how the eigenvalues influence the "electrical" modes.

The solution procedure derived is in principle an exact one. An illustrative computer program to implement the required calculations for any given motor parameters is included and computed waveforms are compared with experimental results.

The matrix approach based on the transition matrix is extended to the case where applied voltage waveforms are parts of sine waves. A homogeneous equation form simplifying the solution structure is obtained by defining pseudo state variables.
This method is applied to solve the equations of a quasi two-phase induction motor in which one winding voltage is a chopped sine wave. Computed current waveforms are compared with experimental results.

From the non-linear describing equations transformed to a synchronously rotating co-ordinate system, a set of piece-wise linear equations is obtained which approximates the system's behaviour for small variations about a given operating point. The Jacobian coefficient matrices of partial derivatives govern the region of validity.

Utilizing earlier work, a time domain solution of the speed response to a small step in frequency is obtained. The derivative of this step response is the weighting function. In the s-domain, a corresponding transfer function is derived and evaluated for the very small slip condition. An approximation to the speed-frequency transfer function is derived for "high" inertia loads and a method of estimating the "high" criterion is suggested.

An "on line" technique for measuring motor-load small signal weighting functions using pseudo-random binary signals is presented and for the inverter-controlled motor, measured results are compared with corresponding step response records and sinusoidal excitation model predictions.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>1</td>
</tr>
<tr>
<td>SUMMARY</td>
<td>2</td>
</tr>
<tr>
<td>NOTATION AND LIST OF PRINCIPAL SYMBOLS</td>
<td>8</td>
</tr>
<tr>
<td>CHAPTER 1: INTRODUCTION</td>
<td>11</td>
</tr>
<tr>
<td>1.1 Motivation and area of investigation</td>
<td>11</td>
</tr>
<tr>
<td>1.2 Review of reported work in the investigation area</td>
<td>12</td>
</tr>
<tr>
<td>1.3 Discussion on research objectives</td>
<td>14</td>
</tr>
<tr>
<td>CHAPTER 2: DERIVATION OF INDUCTION MACHINE EQUATIONS</td>
<td>17</td>
</tr>
<tr>
<td>2.1 Modelling the induction motor</td>
<td>17</td>
</tr>
<tr>
<td>2.2 The idealised cage rotor machine</td>
<td>17</td>
</tr>
<tr>
<td>2.3 Practical machines</td>
<td>18</td>
</tr>
<tr>
<td>2.4 Model for three-phase cage rotor machine</td>
<td>19</td>
</tr>
<tr>
<td>CHAPTER 3: STATE VARIABLE FORMULATION AND METHODS OF SOLUTION</td>
<td>27</td>
</tr>
<tr>
<td>3.1 State descriptions</td>
<td>27</td>
</tr>
<tr>
<td>3.2 Normal form equations for 3-phase, cage induction motor</td>
<td>27</td>
</tr>
<tr>
<td>3.3 Evaluating the transition matrix</td>
<td>36</td>
</tr>
</tbody>
</table>
CHAPTER 4: MODEL PREDICTIONS AND EXPERIMENTAL RESULTS

4.1.1 Experimental equipment 71
4.1.2 Computer solutions 72
4.2 Three phase motor - predictions and performance 74

4.2.1 The experimental motor 74
4.2.2 The computer program 76
4.2.3 Comparison of predicted and measured values 78

4.3 Quasi two-phase motor - predictions and performance 109

4.3.1 The experimental motor 109
4.3.2 The computer program 110
4.3.3 Comparison of predicted and measured values 114

CHAPTER 5: TRANSFER AND WEIGHTING FUNCTION DESCRIPTIONS FOR INDUCTION MACHINES

5.1 Inverter-fed induction motor as control loop element 128
5.2.1 Linearisation of motor equations 179
5.2.2 Transformations and derivation of piece-wise linear equations 132
5.3 Weighting and transfer functions relating frequency and speed changes 145
5.4 Approximations for high inertia loads 161
5.5 Measurement of weighting and transfer functions 166
5.5.1 General description of the method 166
5.5.2 Outline of theoretical basis for the method 169
5.5.3 Experimental results 173

CHAPTER 6: CONCLUSIONS 214
6.1 Discussion on analytical methods for current and torque waveforms 215
6.2 Discussion on transfer and weighting functions 217

BIBLIOGRAPHY 220
APPENDIX 1: Rotating real type transformations for arbitrary winding displacements 226
APPENDIX 2: Motor parameters from test 234
APPENDIX 3: Sinusoidal steady state torque 237
APPENDIX 4: Evaluation of system eigenvalues 239
APPENDIX 5: Computer programs 246
NOTATION AND LIST OF PRINCIPAL SYMBOLS

(i) Matrices and vectors are designated by a ~ symbol under an upper case letter or by enclosure in square brackets.

(ii) Unit matrix is written U or \([U]\)

(iii) Transposition is designated by subscript \(t\)

(iv) A superior dot indicates differentiation with respect to time

(v) Phasors and complex impedances are indicated by a bar over an upper case letter.

(vi) Symbols and meaning:

Superscript or subscript \(s, a, b, c\) - stator quantities

Superscript or subscript \(r, \alpha, \beta\) - rotor quantities

Subscripts \(x\) and \(y\) - transformed rotor variables

\(i\) - current

\(v\) - voltage

\(l\) - self inductance

\(m\) - mutual inductance

\(r\) - resistance

\(q\) - machine parameter (composite)

\(n\) - pole pairs
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>( \frac{d}{dt} ) operation</td>
</tr>
<tr>
<td>( f )</td>
<td>coefficient of viscous friction</td>
</tr>
<tr>
<td>( T_e )</td>
<td>electrical torque developed</td>
</tr>
<tr>
<td>( T_L )</td>
<td>load torque</td>
</tr>
<tr>
<td>( J )</td>
<td>moment of inertia</td>
</tr>
<tr>
<td>( t ) and ( \Delta t )</td>
<td>time and time interval</td>
</tr>
<tr>
<td>( K_T )</td>
<td>load torque coefficient</td>
</tr>
<tr>
<td>( h )</td>
<td>system weighting function</td>
</tr>
<tr>
<td>( s )</td>
<td>per unit slip or Laplace variable</td>
</tr>
<tr>
<td>( p_1, p_2 ) etc</td>
<td>elements of defined vector ( P )</td>
</tr>
<tr>
<td>( z_1, z_2 ) etc</td>
<td>elements of defined vector ( Z )</td>
</tr>
<tr>
<td>( f_1 ) and ( f_2 )</td>
<td>defined composite parameters</td>
</tr>
<tr>
<td>( \mu )</td>
<td>machine parameter</td>
</tr>
<tr>
<td>( \theta )</td>
<td>angular measure - radians</td>
</tr>
<tr>
<td>( \tau )</td>
<td>time constant or variable of integration</td>
</tr>
<tr>
<td>( \omega )</td>
<td>angular frequency - radians/second</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>defined as ( S\omega_s )</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>defined composite parameter</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>eigenvalue</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>constant or coefficient</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>eigenvalue real part or arbitrary phase angle</td>
</tr>
<tr>
<td>( \beta )</td>
<td>eigenvalue imaginary part</td>
</tr>
<tr>
<td>( \phi )</td>
<td>time varying angle</td>
</tr>
</tbody>
</table>
\( \Phi \) or \( \Phi(t) \) - arbitrary fundamental or transition matrix

\( \psi \) - winding separation angle

\( \Phi_{ij}(t) \) - submatrix of transition matrix

\( n \) and \( k \) - defined components of eigenvalues

\( R_{xx} \) - autocorrelation function

\( R_{xy} \) - cross correlation function

\( J_1 \) and \( J_2 \) - Jacobian matrices

\( T \) and \( H \) - transformation matrices

\( \tilde{A} \) - abbreviation for \( A(\theta) \)

\( B, G(\theta), P, G, L(\theta) \) - coefficient matrices

\( K \) - arbitrary matrix

\( \tilde{X} \) - general symbol for state vector

\( x \) - general symbol for state variable

\( V, I \) - voltage and current vectors (used with subscripts)

\( D, E \) - coefficient matrices

\( W_{\text{fld}} \) - magnetic field energy
CHAPTER 1 INTRODUCTION

Since the advent of semiconductor solid state switching devices there has been a continuing interest in their application to electrical machines, particularly as a means of providing a variable frequency power source.

In broad terms the interest has been manifest in two areas: the development of new or improved configurations, particularly in connection with the switching or commutation of currents; the development of new or improved methods of mathematical analysis appropriate to conditions obtaining with switched operation of machines.

The work presented in this thesis is principally concerned with the latter aspect.

1.1 Motivation and area of investigation

A characteristic feature of semiconductor-controlled machines is the discontinuous applied voltage waveforms that occur. However in the majority of applications these discontinuous waveforms are either,

(i) piece-wise constant as is the case with inverters supplied from a smooth d.c. source. or

(ii) portions of sine waves. This is so in many voltage control schemes or in other forms such as cycloconverters.
Because of the many possible waveform patterns that may be produced even within the descriptive scope of (i) and (ii) above, it is desirable to have a method of solution for currents and torque that embraces all waveforms within the categories described above and perhaps with the potential to extend to other types of waveform.

Such a method should, if possible be an "exact" one and lend itself readily to computer solutions.

A further need that arises, particularly in connection with adjustable frequency induction motors used in speed controlled systems, is for a dynamic description of the machine-load system such as a transfer function. Because of the non-linear nature of the system, "large signal" transfer functions will not be obtainable and behaviour will only be linear for small variations about a steady-state operating point. However, "small signal" transfer or weighting function descriptions are useful in analysis and design of the overall system of which the motor and load forms a part.

1.2 Review of reported work in the investigation area

The possibilities of static switching devices as a means to obtaining a variable frequency voltage source aroused interest long before the advent of semiconductor
devices. Alexanderson and Mittag [37] in 1934 reported on a 400 H.P. variable speed motor controlled by thyratrons.

Probably the earliest approach to the steady state analysis of such non-sinusoidal excitations was the truncated Fourier series approximation and this has become known in the literature as the classical method.

In more recent years, several other methods of steady state analysis have been published. Sabbagh and Shewan [38] used instantaneous symmetrical components for the case of piece-wise constant waveforms. Time domain analysis based on linear 2-axis theory was employed by Ward et al [39]. Other papers have dealt with time domain digital methods [40, 41] and, for steady state analysis, an extension of the Fourier series approximation with harmonic transformations involving a multiple reference frame concept [42].

Much less information has been published on system descriptions for induction motors that might be useful for design purposes in such as speed-controlled processes.

Some related analysis has been presented in the context of induction motor stability under varying frequency and voltage conditions [28, 30, 31, 32].

West, Jayawant et al have investigated induction motors in control systems [25, 43] but the analysis was based on constant current operation. The system description
sought was in the form of transfer functions relating torque to frequency and experimental information included frequency response tests.

1.3 Discussion on research objectives

In this thesis the machines considered are squirrel cage induction motors and the approach adopted in seeking analytical methods is generally called the state variable description.

In the field of systems analysis a great deal of work has been done in establishing results for the solution of normal form simultaneous equations [9].

Rotating machine equations, which are well suited to linear transformations and reduction to normal form, are typical of the equations of systems analysis and may be formulated in terms of state variables defined on a state vector space.

The generality and flexibility of state variable methods commends it as a potential analysis procedure. In particular, for linear, constant coefficient equations, the translating properties of a transition matrix are well suited to discontinuous type supply voltages.

For control applications it is desirable to have some form of "system" description for the machine-load unit. In
the work reported here, the principal analytical and experimental "small signal" transfer and weighting functions sought are those connecting frequency variations with speed variations. This objective differs from some of the work reported in section 1.2 in which frequency-torque transfer functions were sought [25,43].

The reason for choosing the frequency-speed relationship is that even if a torque-frequency transfer function for the motor were available it could not be utilised in the loop description without a corresponding full description of the motor-plus-load moment of inertia and all the resisting torques.

A formulation of the transfer function relating input frequency variations with shaft speed variations automatically includes the inertias and torques that comprise the motor-load system. Within this description, the developed torque would exist as an intermediate variable and need not be known in systems where speed is the controlled variable.

In the work connected with conventional three-phase induction motors, a supply voltage waveform is assumed and no special reference is made to the laboratory inverter used. This inverter is supplied from a motor-generator set, has feedback diodes and a continuous train of pulses to the gates
of the conducting thyristors. Oscilloscope records of line voltage conditions under load are given in the appropriate section of the text.
CHAPTER 2. DERIVATION OF INDUCTION MACHINE EQUATIONS

2.1 Modelling the Induction Motor

Of necessity, workable mathematical models of induction motors only approximately describe the physical object. Depending on the nature of the predictions required from the model, varying degrees of approximation may be acceptable. In certain studies, neglect of iron saturation, rotor resistance variation with frequency, or harmonic flux and current generation may not greatly affect the accuracy of the model predictions. At the same time, some aspects of the machine's performance may be strongly affected by these factors.

2.2 The Idealised Cage Rotor Machine

In the idealised machine the stator winding currents each are assumed to produce a sinusoidally distributed radial flux density in the air gap. Alternatively, in this idealised machine it is convenient to think of the stator-produced, radially distributed air gap m.m.f. as varying sinusoidally with angular position. Similarly the idealised rotor cage winding carries bar currents that produce a sinusoidally distributed air gap m.m.f. and a correspondingly distributed air gap flux density.

In deriving a mathematical model of the idealised 3-phase machine it is usual [1,2,3] to consider the rotor
m.m.f. as being produced by symmetrical 3-phase windings having the property of sinusoidal m.m.f. distribution. However, for the model assumed, the rotor produced m.m.f. or air gap flux density is uniquely represented by suitable currents in two assumed "basis" windings. This may be verified by noting that the air gap flux density for a 2-pole machine may be represented by a vector in the plane perpendicular to the rotor shaft. Such a 2-dimensional space is spanned by two basis vectors and correspondingly any such rotor-produced flux density is equivalent to that produced by suitable currents in two assumed "basis" windings on the rotor. However, it may be shown (App.I) that if the two "basis" winding axes are not orthogonal, rotating real type transformations generate time-varying rotor resistances.

2.3 Practical Machines

A characteristic of the idealised windings described in Section 2.2 is that the mutual inductance between a pair of windings varies cosinusoidally with the angle of displacement between axes. As is well known [4] practical stator and rotor windings generate space harmonics of flux density and as a consequence mutual inductance variations do not, in general, follow the cosine law. For such a system, a rotating real type transformation is not able simultaneously to render the describing equations time-invariant [5].
When the stator and rotor windings both independently produce harmonic poles, multiple interactions occur and may give rise to line current harmonics and unrelated frequencies [6,7]. However if squirrel cage windings have a relatively large number of bars per pole, the rotor currents react on the stator at essentially line frequency.

Provided stator winding and permeance harmonics are not large, for many types of investigation the simplification resulting from an assumption of ideal windings compensates for any loss in accuracy of the model. In this context it has been claimed [7] that in practical machines the influence of m.m.f. harmonics on winding inductances is negligible.

In this thesis, squirrel cage rotor windings are represented by two orthogonal, ideal windings. Similarly, stator windings are treated as ideal.

2.4 Model for Three-phase Cage Rotor Machine

As discussed in the previous section, the effect of the rotor cage is represented by two short circuited windings in quadrature on the rotor. Using the concept of coupled circuits in relative motion, the five Kirchhoff voltage law equations for the three stator and two rotor
windings are, in partitioned vector matrix form,

\[
\begin{bmatrix}
\dot{V}^S \\
\dot{V}^R
\end{bmatrix} = \begin{bmatrix}
R^S & 0 \\
0 & R^R
\end{bmatrix} \begin{bmatrix}
\dot{I}^S \\
\dot{I}^R
\end{bmatrix} + \frac{d}{dt} \begin{bmatrix}
L^S & -L(\theta) \\
-L(\theta)^T & L^R
\end{bmatrix} \begin{bmatrix}
\dot{I}^S \\
\dot{I}^R
\end{bmatrix} \quad \ldots \ldots (2.1)
\]

where

\[
\begin{align*}
\dot{V}^S &= \begin{bmatrix} v_a & v_b & v_c \end{bmatrix}_t \\
\dot{V}^R &= \begin{bmatrix} 0 & 0 \end{bmatrix}_t \\
\dot{I}^S &= \begin{bmatrix} i_a & i_b & i_c \end{bmatrix}_t \\
\dot{I}^R &= \begin{bmatrix} i_\alpha & i_\beta \end{bmatrix}_t \\
L^S &= L_s \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{2} & 1 & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \\
L^R &= L_r \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix}
\end{align*}
\]

\[
L(\theta) = m \begin{bmatrix} \sin n\theta & \cos n\theta \\
\sin(n\theta-120^\circ) & \cos(n\theta-120^\circ) \\
\sin(n\theta+120^\circ) & \cos(n\theta+120^\circ) \end{bmatrix} \quad \ldots \ldots (2.2)
\]

\[
\begin{align*}
R^S &= r_s \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix} \\
R^R &= r_r \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix}
\end{align*}
\]
and

\[ l_s = \text{stator phase self inductance} \]
\[ l_r = \text{rotor phase self inductance} \]
\[ m = \text{maximum value of mutual inductance between stator and rotor windings} \]
\[ n = \text{number of pole pairs} \]
\[ \theta = \text{angular displacement in radians} \]

As with such as magnetic saturation, this approximate model neglects end leakage in respect to mutual inductance relations between phases. From basic energy conversion considerations [8] an expression for the generated torque may be found from field energy changes as,

\[ T_e = -\frac{\partial W_{\text{fld}}(\theta,i)}{\partial \theta} \quad \ldots (2.3) \]

where

\[ W_{\text{fld}}(\theta,i) = I_s L(\theta) I_r \]

Introducing a transformation of the rotor currents,

let

\[ I' = H(\theta) I_r \quad \ldots (2.4) \]

where

\[ I' = \begin{bmatrix} i_x \\ i_y \end{bmatrix} \]

and

\[ H(\theta) = \frac{r_r}{m} \begin{bmatrix} \cos n\theta - \sin n\theta \\ \sin n\theta \cos n\theta \end{bmatrix} \]
Applying equation (2.4) to equation (2.1)

\[-L(\theta)I^r = -L(\theta)H(\theta)^{-1}I^s\]

\[
\begin{bmatrix}
0 & -1 \\
\sqrt{3} & \frac{1}{2} \\
\sqrt{3} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
i_x \\
i_y
\end{bmatrix}
\text{where } \mu = \frac{m^2}{r_T}
\]

Using \( p = \frac{d}{dt} \)

\[
\left\{ \frac{1}{r_T} H(\theta)p \{ L^r H(\theta)^{-1} I^s \} \right\} = \tau_r \begin{bmatrix} p & n\dot{\theta} \\ -n\dot{\theta} & p \end{bmatrix} \begin{bmatrix} i_x \\ i_y \end{bmatrix}, \quad \tau_r = \frac{1}{r_T}
\]

\[
-\frac{m}{r_T} H(\theta)p \{ L(\theta) I^s \} = \begin{bmatrix} -n\dot{\theta} & \left( \frac{1}{2} n\dot{\theta} + \frac{\sqrt{3}}{2} p \right) & \left( \frac{1}{2} n\dot{\theta} - \frac{\sqrt{3}}{2} p \right) & \left[ i_a \right] \\
-p & \left( \frac{\sqrt{3}}{2} n\dot{\theta} \right) & \left( \frac{\sqrt{3}}{2} n\dot{\theta} + \frac{1}{2} p \right) & \left[ i_b \right] 
\end{bmatrix}
\]

and the transformed set of volt-amp equations for the motor becomes,

\[
\begin{bmatrix}
v_a \\
v_b \\
v_c \\
o \\
o
\end{bmatrix} = \begin{bmatrix}
(r_s + p l_s) & -\frac{1}{2} p l_s & -\frac{1}{2} p l_s & 0 & -\mu p \\
-\frac{1}{2} p l_s & (r_s + p l_s) & -\frac{1}{2} p l_s & \frac{\sqrt{3}}{2} \mu p & \frac{1}{2} \mu p \\
-\frac{1}{2} p l_s & -\frac{1}{2} p l_s & (r_s + p l_s) & -\frac{\sqrt{3}}{2} \mu p & \frac{1}{2} \mu p \\
0 & -n\dot{\theta} & \left( \frac{1}{2} n\dot{\theta} + \frac{\sqrt{3}}{2} p \right) & \left( \frac{1}{2} n\dot{\theta} - \frac{\sqrt{3}}{2} p \right) & \left( 1 + p \tau_r \right) n\dot{\theta} \tau_r \\
0 & -p & \left( -\frac{\sqrt{3}}{2} n\dot{\theta} + \frac{1}{2} p \right) & \left( \frac{\sqrt{3}}{2} n\dot{\theta} + \frac{1}{2} p \right) & -n\dot{\theta} \tau_r \left( 1 + p \tau_r \right)
\end{bmatrix} \begin{bmatrix}
i_a \\
i_b \\
i_c \\
i_x \\
i_y
\end{bmatrix}
\]

\[\ldots (2.5)\]
Similarly, using the transformation equation (2.4) in equation (2.3)

\[ T_e = -n\mu \left[ \begin{array}{ccc} i_a & i_b & i_c \\ \end{array} \right] \left[ \begin{array}{c} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{array} \right] \left[ \begin{array}{c} i_x \\ \sqrt{\frac{3}{2}} i_y \\ \sqrt{\frac{3}{2}} \end{array} \right] \] ....(2.6)

A case of major practical interest is when the constraint \( i_a + i_b + i_c = 0 \) may be used. Then, eliminating \( i_c \), the equations become,

\[
\begin{bmatrix}
 v_a \\
 v_b \\
 0 \\
 0 \\
 0
\end{bmatrix} = \begin{bmatrix}
 (r_s + \frac{3}{2} l_s p) & 0 & 0 & -\mu p & i_a \\
 0 & (r_s + \frac{3}{2} l_s p) & \sqrt{\frac{3}{2}} \mu p & \frac{1}{2} \mu p & i_b \\
 -\frac{3}{2} n \dot{\theta} + \sqrt{\frac{3}{2}} p & \sqrt{3} p & (1 + \mu \tau_r) & n \dot{\theta} \tau_r & i_x \\
 -\frac{3}{2} n \dot{\theta} - \frac{3}{2} p & -\sqrt{3} n \dot{\theta} & -n \dot{\theta} \tau_r & (1 + \mu \tau_r) & i_y
\end{bmatrix} \]

....(2.7)

\[ T_e = -n\mu (\frac{3}{2} i_x i_a + \sqrt{\frac{3}{2}} i_y i_a + \sqrt{3} i_i i_b) \] ....(2.8)

Assuming a load torque \( T_L \), moment of inertia of rotor and load, \( J \) and viscous friction with coefficient \( f \), the torque balance equation is,

\[ T_e - T_L = J\ddot{\theta} + f\dot{\theta} \] ....(2.9)
Equations (2.5) to (2.9) provide a mathematical model of the machine-load system in terms of four machine "electrical" parameters and two system "mechanical" parameters. In App.2, it is shown that these parameters may be found from direct tests. Thus the analysis methods apply to existing machines or to the case where the parameters are obtained from design data.

2.5 The Sinusoidal Steady State

Based on equation (2.7), let the supply voltage be balanced sinusoids of angular frequency $\omega_s$.

Using phasors and impedance notation,

$$\bar{V}_b = \bar{V}_a / -120^\circ \text{ volts}$$
$$\bar{I}_b = \bar{I}_a / -120^\circ \text{ amps}$$
$$\bar{Z}_s = r_s + j\frac{3}{2}\omega_s l_s \text{ ohms}$$

It may be shown that for these assumed conditions, $\bar{I}_y = -j\bar{I}_x$ and the set of four equations reduces to two independent equations i.e.

$$\bar{V}_a = \bar{Z}_s \bar{I}_a - \omega_s \nu \bar{I}_x$$
$$0 = \frac{3}{2}(\omega_s - n\dot{\theta})\bar{I}_a + (1 + j\tau_r(\omega_s - n\dot{\theta}))\bar{I}_x$$
Putting \( \omega_s - n \dot{\theta} = \sigma \)

\[
\bar{V}_a = \bar{Z}_s \bar{I}_a - \omega_s \mu \bar{I}_x
\]

\[
0 = \frac{3}{2} \sigma \bar{I}_a + (1+j \tau_r \sigma) \bar{I}_x
\]

.... (2.10)

and

\[
\bar{I}_a = \text{stator phase current}
\]

\[
= \frac{\bar{V}_a (1+j \tau_r \sigma)}{\bar{Z}(1+j \tau_r \sigma) + \frac{3}{2} \omega_s \mu \sigma}
\]

.... (2.11)

\[
\bar{I}_a = \frac{|\bar{V}_a| \left[1 + \tau_r^2 \sigma^2 \right]^{1/2}}{\left\{ [\frac{r_s}{\omega_s} - \omega_s \Delta S]^2 + \left[ \frac{3}{2} l_s + r_s \tau_r S \right]^2 \right\}^{1/2}}
\]

where \( \Delta = \frac{3}{2} (l_s \tau_r - \mu) \)

Introducing the conventional slip symbol \( S = \frac{\sigma}{\omega_s} \)

\[
|\bar{I}_a| = |\bar{V}_a| \left\{ \frac{(\omega_s^{-2} + \tau_r^2 S^2)}{\left[ \frac{r_s}{\omega_s} - \omega_s \Delta S \right]^2 + \left[ \frac{3}{2} l_s + r_s \tau_r S \right]^2} \right\}^{1/2}
\]

.... (2.12)

In a similar manner,

\[
\bar{I}_x = \frac{\bar{V}_a \left( \frac{3}{2} S \right)}{\left[ \frac{r_s}{\omega_s} - \omega_s \Delta S \right] + j \left[ \frac{3}{2} l_s + r_s \tau_r S \right]}
\]

.... (2.13)
It can be shown (Appendix 3) that the corresponding expression for developed torque is

\[ T_e(\text{sinus}) = -nu \text{Re}\{3 \overline{\overline{I}}_a \overline{I}_x^*\} \] ....(2.14)

where Re means "real part"

and from (2.12), (2.13) and (2.14)

\[ T_e(\text{sinus}) = \frac{9}{2} \frac{nu}{\omega_s} |V_a|^2 \frac{r_s}{(\frac{r_s}{\omega_s} - \omega_s \Delta S)^2 + (\frac{3}{2} \frac{1}{s} + r_s \tau_r S)^2} \] ....(2.15)

Equations (2.12) and (2.15) give the sinusoidal, steady state current and corresponding torque for any supply frequency \( \omega_s \) and with the motor operating at slip S relative to that frequency.
CHAPTER 3. STATE VARIABLE FORMULATION AND METHODS OF SOLUTION

3.1 State Descriptions

In the case considered where the machines operate in a switched mode, the winding applied voltages generally are discontinuous but the waveforms may be described in terms of piece-wise continuous sections.

For such conditions, a formulation and solution in terms of state variables [9] is most convenient since the system state is fully described at every instant and the state transition matrix connects successive states.

As is known from system and circuit theory, [10] the machine winding currents and rotor angular velocity qualify as state variables for the machine-load system and these arise naturally in the system description.

The method is general in application and in the succeeding sections it is applied to the conventional 3-phase cage motor as well as to a non-conventional quasi 2-phase induction motor.

3.2 Normal form equations for 3-phase, cage induction motor

The state variable normal form is

$$\frac{d}{dt} X(t) = f(X(t), W(t)) \quad \ldots (3.1)$$

Where $X$ is the state vector and $W$ the control or forcing vector.
For the case considered, the vector function $\tilde{f}$ is obtained in terms of the machine and system parameters.

Equation (2.7) may be re-arranged to form,

$$\dot{V} = P \dot{I} + G(\dot{\theta}) I$$

where

$$\dot{I} = \frac{d}{dt} I$$

$$P = \begin{bmatrix} \frac{3}{2} I_s & 0 & 0 & -\mu \\ 0 & \frac{3}{2} I_s & \sqrt{3} \mu & \frac{1}{2} \mu \\ \sqrt{3} \mu & \sqrt{3} & \tau_r & 0 \\ -\frac{3}{2} & 0 & 0 & \tau_r \end{bmatrix}$$

$$G(\dot{\theta}) = \begin{bmatrix} r_s & 0 & 0 & 0 \\ 0 & r_s & 0 & 0 \\ -\frac{3}{2} n \dot{\theta} & 0 & 1 & \tau_r n \dot{\theta} \\ -\frac{3}{2} n \dot{\theta} & -\sqrt{3} n \dot{\theta} & -\tau_r n \dot{\theta} & 1 \end{bmatrix}$$

Hence

$$\dot{I} = A(\dot{\theta}) \dot{I} + BV$$
where

\[ B = p^{-1} \]

\[
\begin{bmatrix}
\tau_r & 0 & 0 & \mu \\
0 & \tau_r & -\sqrt{3/2}\mu & -\frac{1}{2}\mu \\
-\sqrt{3/2} & -\sqrt{3} & \frac{3}{2}l_s & 0 \\
\frac{3}{2} & 0 & 0 & \frac{3}{2}l_s
\end{bmatrix}
\]

\[ A(\dot{\theta}) = -p^{-1}G(\dot{\theta}) \]

\[
\begin{bmatrix}
(-r_s\tau_r + \frac{3}{2}un_\theta) & \sqrt{3}un_\theta & \tau_r un_\theta & -\mu \\
-\sqrt{3}un_\theta & -(r_s\tau_r + \frac{3}{2}un_\theta) & \frac{\mu}{2}(\sqrt{3}-\tau_r n_\theta) & \frac{\mu}{2}(1+\sqrt{3}\tau_r n_\theta) \\
(\frac{3}{2}r_s + \frac{3}{2}l_s n_\theta) & \sqrt{3}r_s & -\frac{3}{2}l_s & -\frac{3}{2}l_s \tau_r n_\theta \\
-\frac{3}{2}(r_s + \frac{3}{2}l_s n_\theta) & \frac{3\sqrt{3}}{2}l_s n_\theta & \frac{3}{2}l_s \tau_r n_\theta & -\frac{3}{2}l_s
\end{bmatrix}
\]

\[ \Delta = \frac{3}{2}(l_s \tau_r - \mu) \]

Similarly equations (2.8) and (2.9) may be written

\[
\frac{d}{dt} \cdot \dot{\theta} = -\frac{T}{J} - \frac{f_\theta}{J} - \frac{n_{\mu}}{J} [\frac{3}{2}a_i x + \sqrt{2}(i_a + 2i_b)i_y] \]

\[ \text{.... (3.4)} \]

When written in expanded form, equations (3.3) and (3.4) become equation (3.5) which is in the form of equation (3.1). In this description, the state vector is,

\[ \chi = [i_a i_b i_x i_y] \]
\[
\frac{d}{dt} i_a = \frac{1}{\Delta} \left\{ \tau_r v_a + \left( \frac{3}{2} \mu n \hat{\theta}_r - r_s \tau_r \right) i_a + \sqrt{3} \mu n \hat{\theta}_b + \tau_r \mu n \hat{\theta}_i x - \mu i_y \right\}
\]

\[
\frac{d}{dt} i_b = \frac{1}{\Delta} \left\{ \tau_r v_b - \sqrt{3} \mu n \hat{\theta}_a - (r_s \tau_r + \frac{3}{2} \mu n \hat{\theta}) i_b + \left( \frac{3}{2} \mu - \frac{1}{2} \mu n \hat{\theta} \right) i_x + \mu \left( \frac{3}{2} \tau_r n \hat{\theta} + \frac{1}{2} \right) i_y \right\}
\]

\[
\frac{d}{dt} i_x = \frac{1}{\Delta} \left\{ -\frac{3}{2} v_a + \sqrt{3} v_b \right\} + \left( \frac{3}{2} r_s + \frac{3}{4} l n \hat{\theta} \right) i_a + \sqrt{3} r_s i_b - \frac{3}{2} l i_x - \frac{3}{2} l r n \hat{\theta} i_y \right\}
\]

\[
\frac{d}{dt} i_y = \frac{1}{\Delta} \left\{ \frac{3}{2} v_a + \left( \frac{3}{4} l n \hat{\theta} - \frac{3}{2} r_s \right) i_a + \frac{3}{2} l n \hat{\theta} i_b + \frac{3}{2} l r n \hat{\theta} i_x - \frac{3}{2} l i_y \right\}
\]

\[
\frac{d}{dt} \hat{\theta} = -\frac{T}{J} - \frac{f \hat{\theta}}{J} - \frac{n u}{J} \left[ \frac{3}{2} i_a i_x + \frac{\sqrt{3}}{2} \left( i_a + 2i_b \right) i_y \right]
\]

\[\ldots(3.5)\]

As written, equation (3.5) assumes ideal voltage sources for \( v_a \) and \( v_b \). However, if their output "impedance" is not zero, such "impedances" are readily included in the describing equations.
Since $X$ is a state vector, knowledge of $X(t_0)$ and integration of equation (3.5) will yield $X(t)$ for all $t>t_0$. Although the equations are non-linear, based on the normal form equation (3.1), an approximation to the trajectory in state space i.e. currents and velocity as functions of time, may be found using a simple step-by-step method in which a time increment $\Delta t$ is chosen sufficiently small for $\dot{X}$ to remain approximately constant over the interval.

When operation at constant speed is considered, equation (3.5) reduces to a 4th order linear, time-invariant set i.e. $A(\theta)$ and $B$ in equation (3.3a) are constant matrices. A solution to this form of equation may be found in terms of a transition matrix $e^{At}$ [11] i.e.

$$I(t) = e^{At} I(0) + \int_0^t e^{A(t-\tau)} BV(\tau) d\tau$$

The transition matrix is a fundamental matrix having the particular property that for a homogeneous system,

$$X(t) = e^{A(t-t_1)} X(t_1) \quad t>t_1 \quad \ldots (3.6)$$

for any initial state $X(t_1)$ at time $t_1$.

This property makes the state variable description well suited to analysis of motors operated in a switched mode such that the supply waveform is discontinuous but with piece-wise continuous elements.
A common case is the inverter-fed motor where the applied voltage is discontinuous and piece-wise constant. The writer has shown two relatively simple procedures for finding the currents (and hence torque) under such conditions [12, 13]. In the following, the second and conceptually simpler derivation is given.

From equation (3.2)

\[ V = P_i + G(\dot{\theta})I \]

and

\[ I = -P^{-1}G(\dot{\theta})I + P^{-1}V \]

or

\[ I = A I + P^{-1}V \] \hspace{1cm} \ldots (3.7)

where

\[ A = -P^{-1}G(\dot{\theta}) \]

For constant \( \dot{\theta} \), i.e. steady state conditions, the non-homogeneous equation (3.7) has the solution

\[ I(t) = e^{At}I(0) + e^{At} \int_0^t e^{-A\tau}P^{-1}V(\tau) d\tau \] \hspace{1cm} \ldots (3.8)

Since the rotor voltage variables in \( V \) are zero (i.e. shorted windings) under the conditions postulated, \( V \) is a piece-wise constant vector.
Over the interval for which \( V \) is constant, equation (3.8) may be simplified by evaluating the integral to yield

\[
I(t) = e^{-\lambda t}I(0) + [e^{-\lambda t} - U] A^{-1}P^{-1}V
\]

(3.9)

where \( U \) is unit matrix

Also,

\[
A^{-1}p^{-1} = G(\dot{\theta})^{-1}pp^{-1} = G(\dot{\theta})^{-1}
\]

Hence

\[
I(t) = e^{-\lambda t}I(0) + (U - e^{-\lambda t})G(\dot{\theta})^{-1}V
\]

(3.10)

To apply equation (3.10), it is necessary to find the initial current \( I(0) \). By way of illustrating how this may be done, consider a common case where the phase voltage is a stepped waveform having six piece-wise constant steps (of equal time duration) per period.

Suppose that the waveform discontinuities occur at times \( t_0, t_1, t_2, t_3 \) etc and that the voltage vector over the interval \( t_0 - t_1 \) is \( V_0 \), over \( t_1 - t_2 \) is \( V_1 \), over \( t_2 - t_3 \) is \( V_2 \) etc.

Let the current vector at time \( t_0 \) be \( I(0) \) and the time interval \( (t_n - t_{n-1}) \) be \( \Delta t \).

At the end of the first time interval i.e. at time \( t_1 \)

\[
I(t_1) = e^{\lambda \Delta t}I(0) + (U - e^{\lambda \Delta t})G(\dot{\theta})^{-1}V_0
\]

(3.11)
Using the transition matrix property, \( \dot{I}(t_1) \) then becomes the initial state for the next time interval such that:

\[
\dot{I}(t_2) = e^{A\Delta t} \dot{I}(t_1) + (U - e^{A\Delta t}) G(\dot{\theta})^{-1} V_1
\]

\[
= e^{A\Delta t} (e^{A\Delta t} \dot{I}(0) + (U - e^{A\Delta t}) G(\dot{\theta})^{-1} V_0)
\]

\[
+ (U - e^{A\Delta t}) G(\dot{\theta})^{-1} V_1
\]

\[
\ldots (3.12)
\]

Similarly,

\[
\dot{I}(t_3) = e^{A\Delta t} \dot{I}(t_2) + (U - e^{A\Delta t}) G(\dot{\theta})^{-1} V_2
\]

Since

\[
(e^{A\Delta t})^2 = e^{A2\Delta t}
\]

\[
\dot{I}(t_3) = e^{A2\Delta t} (e^{A\Delta t} \dot{I}(0) + (U - e^{A\Delta t}) G(\dot{\theta})^{-1} V_0)
\]

\[
+ e^{A\Delta t} (U - e^{A\Delta t}) G(\dot{\theta})^{-1} V_1
\]

\[
+ (U - e^{A\Delta t}) G(\dot{\theta})^{-1} V_2
\]

\[
\ldots (3.13)
\]

For the assumed voltage waveform, \( t_0 - t_3 \) is a half period and from the expected symmetry,

\[
\dot{I}(t_3) = -\dot{I}(0)
\]

\[
\ldots (3.14)
\]

When this result is substituted in equation (3.13), one obtains
\[
I(0) = (e^{A_3 A t_+ U})^{-1} [ (e^{A_3 A t_+ e^{A_2 A t}}) G(\dot{\theta})^{-1} V_0 \\
+ (e^{A_2 A t} - e^{A_1 A t}) G(\dot{\theta})^{-1} V_1 \\
+ (e^{A_1 A t} - U) G(\dot{\theta})^{-1} V_2 ]
\]

The elements of the above equations are the transition matrix (to be discussed in the next section) the applied voltage vector \(V\) and the matrix \(G(\theta)^{-1}\).

The matrix \(G(\theta)\) is defined by equation (3.2c) and the inverse is readily found to be,

\[
G(\theta)^{-1} = \begin{bmatrix}
\frac{1}{r_s} & 0 & 0 & 0 \\
0 & \frac{1}{r_s} & 0 & 0 \\
\frac{n_0^\dot{\theta}}{r_s} (\frac{\sqrt{3}}{2 + 3/2}\tau_r n_0^\dot{\theta}) & -\frac{\sqrt{3}\tau_r (n_0^\dot{\theta})^2}{f r_s} & \frac{1}{f} & -\theta r n_0^\dot{\theta} \\
\frac{n_0^\dot{\theta}}{r_s} (\frac{\sqrt{3}}{2} + 3/2\tau_r n_0^\dot{\theta}) & \frac{\sqrt{3}n_0^\dot{\theta}}{f r_s} & \frac{\tau_r n_0^\dot{\theta}}{f} & \frac{1}{f}
\end{bmatrix}
\]

where \(f = 1 + (\tau_r n_0^\dot{\theta})^2\)

Using \(I(0)\) as found from equation (3.15), equation (3.10) and subsequent equations may be solved successively to determine the currents in the intervals \(t_0-t_1, t_1-t_2, t_2-t_3\) etc.
In summary, the method developed finds the steady state current waveforms for all windings by successively evaluating an equation of the form of (3.10). The key element is the transition matrix which, as well as providing the link between successive states in equation (3.10), also provides the means for finding the required initial vector \( \mathbf{I}(0) \) as expressed in equation (3.15). With full information on all currents, equation (2.8) yields the torque as a function of time.

An alternative method that would apply to the system under discussion has been published by the writer in relation to synchronous machines [14].

The method obtains a solution of the equations by transformation to canonical form and expansion in terms of eigenvectors.

3.3 Evaluating the transition Matrix

In the last section it was shown that the motor currents (and hence torque) could be found as functions of time, given the initial currents, stator applied voltages and the state transition matrix. In particular, for the case where the applied voltages are piece-wise constant, equations (3.10) and (3.15) apply.

As stated, the solution for linear systems depends on the state transition matrix which may be expressed as an
infinite power series, i.e.

\[ e^{Kt} = U + Kt + \sum_{j=2}^{\infty} \frac{K^j t^j}{j!} \ldots (3.16) \]

and approximated by truncation of the series.

Many methods are available for finding closed form expressions for the transition matrix. Most of these require evaluation of the system eigenvalues and the method to be derived and used in subsequent work is based on the Cayley-Hamilton theorem.

For a system \( \dot{X} = KX \) let the characteristic equation of \( K \) be,

\[ p(\lambda) = 0 \]

Then by the Cayley-Hamilton theorem,

\[ p(K) = 0 \ldots (3.17) \]

If \( K \) is of order \( n \), from equation (3.17)

\[ K^n = -a_{n-1}K^{n-1} - a_{n-2}K^{n-2} - \ldots - a_0U \ldots (3.18) \]

Using this result, any \( K^m, m > n \) may be expressed as a polynomial in \( K \) with highest degree \( (n-1) \). This must be true of \( e^{Kt} \) and so,

\[ e^{Kt} = b_0U + b_1K + b_2K^2 + \ldots + b_{n-1}K^{n-1} \ldots (3.19) \]

where \( n \) is the order of \( K \).
Reverting to the motor under consideration, \( A(\dot{\theta}) \) in equation (3.3) is of order 4 and so the transition matrix corresponding is

\[ e^{A_t} = \gamma_0 + \gamma_1 A + \gamma_2 A^2 + \gamma_3 A^3 \quad \ldots (3.20) \]

If \( \lambda_i \) is a solution of the characteristic equation \( p(\lambda) = 0 \) i.e. \( \lambda_i \) is an eigenvalue of \( A \), then,

\[ e^{\lambda_i t} = \gamma_0 + \gamma_1 \lambda_i + \gamma_2 \lambda_i^2 + \gamma_3 \lambda_i^3 \]

Assuming that the \( \lambda_i \) are distinct (this aspect is discussed later) one obtains

\[
\begin{bmatrix}
\gamma_0 \\
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\end{bmatrix} = [E]^{-1} \begin{bmatrix}
e^{\lambda_1 t} \\
e^{\lambda_2 t} \\
e^{\lambda_3 t} \\
e^{\lambda_4 t} \\
\end{bmatrix} \quad \ldots (3.21)
\]

where

\[
E = \begin{bmatrix}
1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\
1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\
1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 \\
1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 \\
\end{bmatrix} \quad \ldots (3.22)
\]
Thus the form of $e^{At}$ depends on the eigenvalues of $A$.

For large order systems, numerical methods may be used to find computer solutions of the system eigenvalues. However it is much more informative and computationally simpler and faster if solutions can be presented in terms of machine parameters.

This has been done for any motor described by equations (3.3) and details of the method are given in Appendix 4.

To simplify equations, let

$$a = \frac{\frac{3}{2}l_s}{\Delta}$$

$$b = \frac{\frac{r_s}{s}r_s}{\Delta}$$

$$c = \frac{r_s}{\Delta}$$

Then the characteristic equation for $A$ may be written

$$p(\lambda) = Q(\lambda)Q_2(\lambda) = 0$$

where (Appendix 4)

$$Q_1(\lambda) = [\lambda + \frac{1}{2}(a+b) - k \cos \frac{n}{2}]^2 + [\frac{1}{2}n\delta - k \sin \frac{n}{2}]^2$$

$$Q_2(\lambda) = [\lambda + \frac{1}{2}(a-b) + k \cos \frac{n}{2}]^2 + [\frac{1}{2}n\delta + k \sin \frac{n}{2}]^2$$

$$...(3.23)$$
and
\[ \tan \eta = \frac{2n\dot{\theta}(a-b)}{(a+b)^2 - 4c-n^2}\delta^2 \] .... (3.24)

\[ k = \left\{ [(a+b)^2 - 4c-n^2\delta^2]^2 + \left[ \frac{n\dot{\theta}(a-b)}{2} \right]^2 \right\}^{1/4} \] .... (3.25)

Equation (3.23) defines the eigenvalues of matrix A and as is discussed in the next section, for all practical purposes the four eigenvalues may be written as

\[ \lambda_1 = \alpha_1(\dot{\theta}) + j\beta_1(\dot{\theta}) = \alpha_1 + j\beta_1 \]

\[ \lambda_2 = \lambda_1^* \]

\[ \lambda_3 = \alpha_2(\dot{\theta}) + j\beta_2(\dot{\theta}) = \alpha_2 + j\beta_2 \] .... (3.26)

\[ \lambda_4 = \lambda_3^* \]

with the range of both \( \beta \)'s including zero as a limit.

e.g. for \( \dot{\theta} \neq 0 \), \( Q_1(\lambda) \) yields

\[ \lambda_{1,2} = (\mp (a+b)+k \cos \frac{n}{2}) \pm j(\mp n\dot{\theta}-k \sin \frac{n}{2}) \] .... (3.26a)

Since the transition matrix must be real, and in view of equations, (3.20), (3.21) and (3.26),

\[ e^{At} = e^{\alpha_1^t} \cos \beta_1 t[C_1] + e^{\alpha_1^t} \sin \beta_1 t[C_2] \]

\[ + e^{\alpha_2^t} \cos \beta_2 t[C_3] + e^{\alpha_2^t} \sin \beta_2 t[C_4] \] .... (3.27)
where the matrices $C_1, C_2, C_3, C_4$ are constant for constant $\dot{\theta}$.

A property of fundamental matrices is that

$$\dot{\phi} = A\phi$$

where $\phi$ is a fundamental matrix and $A$ is the system matrix (in this case $\phi = e^{At}$ and $\dot{\phi} = Ae^{At}$).

Then, from equation (3.27)

$$\phi \big|_{t=0} = \tilde{u} = C_1 + C_3$$

$$\dot{\phi} \big|_{t=0} = \tilde{\dot{u}} = A\phi \big|_{t=0} = A = \alpha_1 \dot{C}_1 + \beta_1 \dot{C}_2 + \alpha_2 \dot{C}_3 + \beta_2 \dot{C}_4$$

Similarly, one obtains

$$A^2 = (\alpha_1^2 - \beta_1^2)C_1 + 2\alpha_1 \beta_1 C_2 + (\alpha_2^2 - \beta_2^2)C_3 + 2\alpha_2 \beta_2 C_4$$

$$A^3 = \alpha_1 (\alpha_1^2 - 3\beta_1^2)C_1 + \beta_1 (3\alpha_1^2 - \beta_1^2)C_2$$

$$+ \alpha_2 (\alpha_2^2 - 3\beta_2^2)C_3 + \beta_2 (3\alpha_2^2 - \beta_2^2)C_4$$

....(3.28a)
Putting

\[
\begin{align*}
d_{11} &= d_{13} = 1 & d_{12} &= d_{14} = 0 \\
d_{21} &= \alpha_1 & d_{22} &= \beta_1 \\
d_{23} &= \alpha_2 & d_{24} &= \beta_2 \\
d_{31} &= (\alpha_1^2 - \beta_1^2) & d_{32} &= 2\alpha_1\beta_1 \\
d_{33} &= (\alpha_2^2 - \beta_2^2) & d_{34} &= 2\alpha_2\beta_2 \\
d_{41} &= \alpha_1(\alpha_1^2 - 3\beta_1^2) & d_{42} &= \beta_1(2\alpha_1^2 - \beta_1^2) \\
d_{43} &= \alpha_2(\alpha_2^2 - 3\beta_2^2) & d_{44} &= \beta_2(3\alpha_2^2 - \beta_2^2)
\end{align*}
\]  

\ldots (3.28b)

\[
\begin{align*}
\mathbf{u} &= d_{11}c_1 + d_{12}c_2 + d_{13}c_3 + d_{14}c_4 \\
\mathbf{a} &= f_{21}c_1 + d_{22}c_2 + d_{23}c_3 + d_{24}c_4 \\
\mathbf{a}^2 &= d_{31}c_1 + d_{32}c_2 + d_{33}c_3 + d_{34}c_4 \\
\mathbf{a}^3 &= d_{41}c_1 + d_{42}c_2 + d_{43}c_3 + d_{44}c_4 
\end{align*}
\]  

\ldots (3.28c)

Equation (3.28) may be written

\[
\begin{bmatrix}
\mathbf{u} \\
\mathbf{a} \\
\mathbf{a}^2 \\
\mathbf{a}^3
\end{bmatrix}
= D \otimes \mathbf{u}
\]

where \( \mathbf{u} \) is a 4 x 4 unit matrix, \( D \) is the matrix \( d_{ij} \) and \( D \otimes \mathbf{u} \) defines a Kronecker product. [15]
Hence,

\[
\begin{bmatrix}
\tilde{c}_1 \\
\tilde{c}_2 \\
\tilde{c}_3 \\
\tilde{c}_4
\end{bmatrix} = \begin{bmatrix}
\tilde{D}^{-1} \
\mathbf{X} 
\end{bmatrix} \begin{bmatrix}
\mathbf{U} \\
\mathbf{A} \\
\mathbf{A}^2 \\
\mathbf{A}^3
\end{bmatrix}
\]

...(3.29)

and the results of equation (3.29) together with equations (3.23) and (3.26), when substituted into equation (3.27) yield the state transition matrix.

Collecting together the procedures derived in this section, the method of finding the transition matrix for any motor whose parameters are given consists of the following steps,

(i) For the specified speed, calculate the real and imaginary parts of the eigenvalues (equations 3.23, 24, 25 and A4.10).  

(ii) Using the results of (i) calculate the elements of matrix \(\tilde{D}\), equation (3.28b) and evaluate the inverse \(\tilde{D}^{-1}\).  

(iii) Substitute numerical values for parameters in matrix \(\tilde{A}\) (equation 3.3c) and then calculate \(\tilde{A}^2\) and \(\tilde{A}^3\).
(iv) With $D^{-1}$ and the powers of $A$ previously found, use equation (3.29) to find matrices $C_1$, $C_2$, $C_3$, $C_4$.

(v) For $\dot{\theta} \neq 0$ the transition matrix is given explicitly by equation (3.27) i.e.

$$e^{At} = e^{\alpha_1 t} \cos \beta_1 t [C_1] + e^{\alpha_1 t} \sin \beta_1 t [C_2]$$

$$+ e^{\alpha_2 t} \cos \beta_2 t [C_3] + e^{\alpha_2 t} \sin \beta_2 t [C_4]$$

The eigenvalue real and imaginary parts $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, are obtained from (i) and the matrices $C_1$, $C_2$, $C_3$, $C_4$ from (iv).

Although the above form for $e^{At}$ is not true for $\dot{\theta} = 0$, in practice this is no limitation since $\dot{\theta}$ may be made to approach zero as closely as desired.

3.4 The System Matrix Eigenvalues

Expressions for the system matrix (matrix $\Lambda$) eigenvalues are derived in Appendix 4 and stated in section 3.3, equations (3.23), (3.24) and (3.25).

Inspection of these expressions shows in a qualitative way, how the eigenvalues vary with rotor speed.

It is readily shown that $(a+b)^2 - 4c > 0$ and hence when the rotor is stationary, $\eta = 0$ and the eigenvalues are real, occurring as two double roots.

Similarly, as $\eta \dot{\theta}$ becomes very large, $\eta + \pi$, $\cos \left( \frac{\eta}{2} \right) \to 0$, ...
\[
\sin \left( \frac{n}{2} \right) = 1 \text{ and } k = \frac{1}{2} n^2 \theta. \text{ Thus by this and similar reasoning it may be concluded that except at the limit } \theta = 0, \text{ the eigenvalues are distinct (an assumption used in section 3.3) and complex.}
\]

For some motor designs
\[
T_r = \frac{3}{2} l_s \frac{s}{r_s}
\]

i.e. \( a = b \)

For the condition \( a = b \)

\[
k = \left| \left[ a^2 - c - \left( \frac{n \theta}{2} \right)^2 \right] \right|^{\frac{1}{2}}
\]

and

\[
\tan \eta = \frac{0}{(a^2 - c) - \left( \frac{n \theta}{2} \right)^2}
\]

Hence for the condition \( a = b \), \( \eta \) takes on only two values, \( 0^\circ \) or \( 180^\circ \) and the discontinuity in \( \eta \) occurs at a rotor speed such that

\[
\left( \frac{n \theta}{2} \right)^2 = (a^2 - c)
\]

(i) Speed range \( n \theta < 2\sqrt{a^2 - c} \)

Within this speed range, \( \eta = 0, \)

\[
\cos \left( \frac{n}{2} \right) = 1, \quad \sin \left( \frac{n}{2} \right) = 0
\]

Hence for both pairs of eigenvalues, the imaginary part is \( \pm \frac{n \theta}{2} \).
Similarly, the real part of one pair of eigenvalues is 
\((a+k)\) and for the other pair \((a-k)\).

Since within this speed range \(\left(\frac{n_0}{2}\right)^2 < (a^2-c)\),

\[
k^2 = (a^2-c) - \left(\frac{n_0}{2}\right)^2
\]

and

\[
[a-(a \pm k)]^2 + \left(\frac{n_0}{2}\right)^2 = k^2 + \left(\frac{n_0}{2}\right)^2
\]

\[
= (a^2-c) - \left(\frac{n_0}{2}\right)^2 + \left(\frac{n_0}{2}\right)^2
\]

\[
= (a^2-c)
\]

\[
= \text{a constant}
\]

Hence it follows that over the speed range specified the loci of the four eigenvalues in the complex plane combine to form a circle, radius \(\left[a^2-c\right]^\frac{1}{2}\) and centre at \((-a+j0)\).

(ii) Speed range \(n_0 > 2\sqrt{a^2-c}\)

Over this speed range, \(\eta = 180^\circ\),

\[
\cos\left(\frac{\eta}{2}\right) = 0, \quad \sin\left(\frac{\eta}{2}\right) = 1
\]

Thus

\[
\lambda_{1,2} = a \pm j\left(\frac{n_0}{2} + k\right)
\]

\[
\lambda_{3,4} = a \pm j\left(\frac{n_0}{2} - k\right)
\]

Also, it is seen that when \(\left(\frac{n_0}{2}\right)^2 >> (a^2-c)\)

\[
k + \frac{n_0}{2}
\]
Figure 3.4.1 Eigenvalue loci for case $a = b$
i.e. the imaginary parts approach \( n \hat{\omega} \) or 0 as \( n \hat{\omega} \) becomes very large.

The loci for this speed range are then relatively simple. The four loci combine to form a straight line parallel to the imaginary axes and passing through the centre of the circular loci. Figure 3.4.1 shows the complex plane loci of the four eigenvalues when machine parameters are such that \( a = b \). By implication the motor parameters are assumed to be constants, independent of frequency.

Eigenvalue loci for four squirrel cage motors are shown in Figures 3.4.2 to 3.4.5. Many motors were tested and all showed the same general loci structure, recognisably similar in form to the special case of \( a = b \), but with relatively large numerical deviations. In Figures 3.4.2 to 3.4.5, only the upper half plane loci have been plotted since the lower half plane is the mirror image. Speed in electrical radians per second is included as a running parameter.

In equation (3.27) the transition matrix is expressed in terms of the eigenvalues of \( \tilde{A} \). Further, equations (3.10) through to (3.15) are expressions (in terms of the transition matrix) for the current segments in response to the stepped and piece-wise continuous voltage waveforms.

Thus knowledge of the eigenvalues of \( \tilde{A} \) at any given speed gives some insight into the expected shape of the
Figure 3.4.2 Eigenvalue loci for 5 H.P., 415 Volt, 4 pole, S.C. motor.
Figure 3.4.3  Eigenvalue loci for 7.5 H.P., 415 Volt, 4 pole motor
Figure 3.4.4 Eigenvalue loci for
30 H.P., 415 Volt, 4 pole motor
Figure 3.4.5 Eigenvalue loci for 100 H.P., 415 Volt, 4 pole motor
Figure 3.5.1  Principle of quasi 2-phase induction motor
current waveforms at that speed.

3.5 State Equations for Quasi 2-Phase Induction Motor

In earlier sections of this chapter it was shown that a state variable formulation is well suited to conditions where the applied voltages are only piece-wise continuous. To further illustrate the method, in this section a non-conventional, quasi 2-phase cage rotor induction motor [16] is analysed.

The essential principle of this motor is that of two-phase stator windings supplied from a single phase supply, the required voltage phase displacement being achieved by "chopping" the supply voltage with thyristor switches.

Figure (3.5.1) indicates in diagrammatic form, the general principle of the motor. In the most general case, the angular displacement between the two stator windings is arbitrary and the two windings are dissimilar in electrical properties.

If the cage rotor is, as previously, assumed to be represented by two, balanced, orthogonal windings, the transformations and transformed equations obtained in App.1 apply to this motor.

As a specific example, consider an 'n' pole-pair machine having a displacement of 90° electrical between stator windings. Thus, with \( \psi = 90° \), App.1 equations (A1.14) and
(A1.15) become,

\[ V^s = \left[ \begin{array}{cc} r_{s1} & 0 \\ 0 & r_{s2} \end{array} \right] I^s + \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1_{s2} \end{array} \right] P_1^s - \left[ \begin{array}{cc} \mu_1 & 0 \\ 0 & \mu_2 \end{array} \right] P_1' \]

\[ \begin{aligned}
\begin{bmatrix} 0 \\ \tau_r \end{bmatrix} &= \begin{bmatrix} 1 & q \tau_r n^\theta \\ -\frac{1}{q} \tau_r n^\theta & 1 \\ \end{bmatrix} I' + \begin{bmatrix} \tau_r & 0 \\ 0 & \tau_r \end{bmatrix} P_1' \\
- n^\theta \begin{bmatrix} 0 & q \\ -\frac{1}{q} & 0 \\ \end{bmatrix} I^s - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \end{bmatrix} P_1^s \\
\end{aligned} \]  

where \( q = \sqrt{\frac{\mu_2}{\mu_1}} \)

As in section 2.4, the developed torque may be found from

\[ T_e = -\frac{\partial W_{fld}(\theta)}{\partial \theta} \]

\[ W_{fld}(\theta) = I_s L(\theta) I_r \]

From Appendix 1, equation (A1.11) and modifying the equation to allow for \( n \) pole pairs and a displacement \( \psi \) of 90°,
\[ L(\theta) = \begin{bmatrix} m_1 \cos(n\theta) - m_1 \sin(n\theta) \\ m_2 \sin(n\theta) & m_2 \cos(n\theta) \end{bmatrix} \]

\[ T_e = -I_s \left( \frac{\partial}{\partial \theta} L(\theta) \right) I^R \]

and using the transformation

\[ I^R = H_3 I' \]

where \( H_3 \) is given by equation (A1.12)

\[ T_e = I_s \left( \frac{\partial}{\partial \theta} L(\theta) \right) H_3 I' \]

Putting \( I_s = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \)

\( I' = \begin{bmatrix} i_x \\ i_y \end{bmatrix} \)

\[ T_e = -n \sqrt{\mu_1 \mu_2} (i_2 i_x - i_1 i_y) \]

where \( \mu_1 = \frac{m_1^2}{r} \) and \( \mu_2 = \frac{m_2^2}{r} \)

In a manner similar to that used in section 3.2, equations (3.30) and (3.31) could be combined and re-arranged to give an equation of the form

\[ \dot{\tilde{z}} = A\tilde{z} + B\tilde{v} \quad \ldots \ldots (3.32) \]
which has a solution,

\[ x(t) = e^{At}x(0) + \int_0^t e^{A(t-\lambda)} Bv(\lambda) d\lambda \quad \ldots (3.33) \]

However, this solution requires that \( x(0) \) be known and also, the two non-zero elements of \( V \) i.e. \( v_1 \) and \( v_2 \), are sinusoids or portions of sinusoids having a common angular frequency \( \omega \) radians per second.

This situation is somewhat similar to the 3-phase motor case previously discussed but with the complication that the applied voltage vector \( \tilde{V}(t) \), in addition to being discontinuous is also time-varying on each interval.

To simplify the equations and also provide a means for determining \( x(0) \), advantage may be taken of the form of \( V \) to convert equation (3.32) to a homogeneous one. This may be done by first noting that \( v_1 \) and \( v_2 \) satisfy the differential equations

\[ p^2 v_1 + \omega^2 v_1 = 0 \]
\[ p^2 v_2 + \omega^2 v_2 = 0 \quad \ldots (3.34a) \]

Next, define new variables

\[ v_1 = x_1 \quad v_2 = x_3 \]
\[ px_1 = \dot{x}_1 = x_2 \]
\[ px_2 = \dot{x}_3 = x_4 \quad \ldots (3.34b) \]
Hence
\[ p^2 v_1 = \dot{x}_2 = -\omega^2 x_1 \]
\[ p^2 v_2 = \dot{x}_4 = -\omega^2 x_3 \]

Then, if
\[ \dot{X} = [i_1 \ i_2 \ i_x \ i_y \ x_1 \ x_2 \ x_3 \ x_4]_t \]
the describing equation may now be written,
\[ [0] = DX + EX \]
or
\[ \dot{X} = -D^{-1} EX \]
i.e.
\[ \dot{X} = AX \]

Where, in partitioned form
\[ D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \]
\[ E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \]
The 4 x 4 submatrices are

$$D_{11} = \begin{bmatrix} 1_{s1} & 0 & -\mu_1 & 0 \\ 0 & 1_{s2} & 0 & -\mu_2 \\ -1 & 0 & \tau_r & 0 \\ 0 & -1 & 0 & \tau_r \end{bmatrix}$$

$$D_{21} = D_{12} = \text{null matrix}$$

$$D_{22} = -U$$ \quad (unit matrix)

$$E_{11} = \begin{bmatrix} r_{s1} & 0 & 0 & 0 \\ 0 & r_{s2} & 0 & 0 \\ 0 & -qn\dot{\theta} & 1 & q\tau_r n\dot{\theta} \\ \frac{1}{qn\dot{\theta}} & 0 & -\frac{1}{q\tau_r n\dot{\theta}} & 1 \end{bmatrix}$$

where $q = \sqrt{\frac{\mu_2}{\mu_1}}$

$$E_{12} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
In partitioned form, the inverse of $D$ is

$$D^{-1} = \begin{bmatrix}
\begin{array}{cc}
D_{11}^{-1} & 0 \\
0 & -U
\end{array}
\end{bmatrix}$$

with

$$D_{11}^{-1} = \begin{bmatrix}
\frac{\tau_r}{f_1} & 0 & \frac{\mu_1}{f_1} & 0 \\
0 & \frac{\tau_r}{f_2} & 0 & \frac{\mu_2}{f_2} \\
\frac{1}{f_1} & 0 & \frac{l_{s1}}{f_1} & 0 \\
0 & \frac{1}{f_2} & 0 & \frac{l_{s2}}{f_2}
\end{bmatrix}$$

and

$$f_1 = (\tau_r l_{s1} - \mu_1)$$

$$f_2 = (\tau_r l_{s2} - \mu_2)$$
Hence, from

\[-D^{-1}E = A = \begin{bmatrix} \sim_{11} & \sim_{12} \\ \sim_{21} & \sim_{22} \end{bmatrix}\]

\[
\sim_{11} = -D^{-1}E_{11}
\]

\[
\sim_{12} = -D^{-1}E_{12}
\]

\[
\sim_{21} = 0
\]

\[
\sim_{22} = E_{22}
\]

i.e.

\[
\sim_{11} = \begin{bmatrix}
-\frac{\tau r s_1}{f_1} & \frac{\mu_1 q n\dot{\theta}}{f_1} & -\frac{\mu_1}{f_1} & -\frac{\mu_1 q r r n\dot{\theta}}{f_1} \\
-\frac{\mu_2 n\dot{\theta}}{q f_2} & -\frac{\tau r s_2}{f_2} & \frac{\mu_2 r r n\dot{\theta}}{q f_2} & -\frac{\mu_2}{f_2} \\
-\frac{r s_1}{f_1} & \frac{1 s_1 q n\dot{\theta}}{f_1} & -\frac{1 s_1}{f_1} & -\frac{1 s_1 q r r n\dot{\theta}}{f_1} \\
-\frac{1 s_2 n\dot{\theta}}{q f_2} & -\frac{r s_2}{f_2} & \frac{1 s_2 r r n\dot{\theta}}{q f_2} & -\frac{1 s_2}{f_2}
\end{bmatrix}
\]
Thus the elements of $\mathbf{A}$ are known and the problem is reduced to solving the homogeneous equation,

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$$

subject to given initial states

$$\mathbf{X}(t)|_{t=0} = \mathbf{X}(0)$$

The solution of equation (3.36) is, as indicated in equation (3.6),

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}(0)$$

$$\ldots (3.37)$$
Figure 3.5.2  Winding voltage waveforms
As in section 3.2 it is necessary to establish the initial conditions, preferably at some switching instant. With this information and the transition matrix \( e^{At} \), the steady state winding current waveforms and the developed torque can be computed for any motor speed.

The applied stator winding voltages \( v_1 \) and \( v_2 \) are illustrated in Figure (3.5.2) for an arbitrary switching delay \( t_d \). The switching instants \( t_0 \) and \( t_2 \) are separated in time by one half period of the supply voltage. At the switching instants \( t_0 \) and \( t_2 \), \( v_1 \) and \( \frac{dv_1}{dt} \) are discontinuous and at the zero crossing time \( t_1 \), \( v_1 \) changes direction and \( \frac{dv_1}{dt} \) is discontinuous.

Since \( v_1 \) and \( \frac{dv_1}{dt} \) are elements of the vector \( \dot{X} \), it is necessary to distinguish between the values of \( \dot{X} \) at a small increment of time before and a small increment of time after the switching or transition instant i.e. to distinguish between \( X(t_0^-) \) and \( X(t_0^+) \), \( \dot{X}(t_1^-) \) and \( \dot{X}(t_1^+) \), \( \dot{X}(t_2^-) \) and \( \dot{X}(t_2^+) \).

In one possible method to find the initial condition vector \( \dot{X}(0) \) it is convenient to write,

\[
\dot{X} = \dot{X}_i + \dot{X}_v
\]

where

\[
\dot{X}_i = \begin{bmatrix} i_1 & i_2 & i_x & i_y & 0 & 0 & 0 & 0 \end{bmatrix}_t
\]

\[
\dot{X}_v = \begin{bmatrix} 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 & x_4 \end{bmatrix}_t
\]
Suppose it is required to find the initial conditions corresponding to \( t_0^+ \).

Then

\[
X(t) = e^{A(t-t_0)} X(t_0^+) \quad t > t_0 \quad \ldots (3.38)
\]

and

\[
X(t_1^-) = e^{A(t_1-t_0)} X(t_0^+)
\]

Because of the physical constraints on currents in practical inductive circuits, \( \dot{X}_i \) is continuous and

\[
\dot{X}_i(t_1^-) = \dot{X}_i(t_1^+)
\]

However, \( \dot{X}_v \) is not continuous and

\[
\dot{X}_v(t_1^-) \neq \dot{X}_v(t_1^+)
\]

Similarly, \( e^{At} \) is continuous and there is no need to distinguish between \( t_1^- \) and \( t_1^+ \) in that term,

Hence,

\[
X(t_1^+) = X(t_1^-) - \dot{X}_v(t_0^+) + \dot{X}_v(t_1^+)
\]

\[
= e^{A(t_1-t_0)} X(t_0^+) - \dot{X}_v(t_0^+) + \dot{X}_v(t_1^+)
\]

\[
\ldots (3.39)
\]

Over the next time interval \( t > t_1 \),

\[
X(t) = e^{A(t-t_1)} X(t_1^+)
\]

\[
\ldots (3.40)
\]
Using the waveform symmetries,

\[
((I_1 + I_2 \vec{X} - (I_1 + I_2) \vec{Y})] + \\
[((I_1)^1 \vec{X} - (I_1)^1 \vec{X})] (I_2 - Z_2) \vec{V}^\theta [\tilde{\eta} + (O_1 - Z_2) \vec{V}^\theta] = (I_1 + I_2) \vec{X} 
\]
\[ \frac{(z_1)\tilde{\Lambda}_X - (0_1)\tilde{\Lambda}_X}{(\tilde{\Lambda}_X - 0_1)\tilde{\Lambda}_X} = \frac{(-z_1)\tilde{\Lambda}_X}{(\tilde{\Lambda}_X - 0_1)\tilde{\Lambda}_X} = \frac{(z_1)\tilde{\Lambda}_X + (0_1)\tilde{\Lambda}_X}{(\tilde{\Lambda}_X - 0_1)\tilde{\Lambda}_X} = \frac{(-z_1)\tilde{\Lambda}_X}{(\tilde{\Lambda}_X - 0_1)\tilde{\Lambda}_X} \]

and

\[ \frac{(z_1)\tilde{\Lambda}_X - (0_1)\tilde{\Lambda}_X}{(\tilde{\Lambda}_X - 0_1)\tilde{\Lambda}_X} = \frac{(-z_1)\tilde{\Lambda}_X}{(\tilde{\Lambda}_X - 0_1)\tilde{\Lambda}_X} = \frac{(z_1)\tilde{\Lambda}_X + (0_1)\tilde{\Lambda}_X}{(\tilde{\Lambda}_X - 0_1)\tilde{\Lambda}_X} = \frac{(-z_1)\tilde{\Lambda}_X}{(\tilde{\Lambda}_X - 0_1)\tilde{\Lambda}_X} \]
The right hand side of equation (3.41) consists of the transition matrix and the (known) forcing function part of \( \dot{X}(t) \). Thus the initial condition vector \( \dot{X}(t_0^+) \) can be found and used in equation (3.38) to solve for \( \dot{X}(t) \) over the interval \( t_0 \) to \( t_1 \), and in which interval \( \dot{X}(t) \) is continuous. In a similar manner, using equations (3.39) and (3.40), \( \dot{X}(t) \) can be calculated over the interval \( t_1 \) to \( t_2 \).

Successive application of this procedure enables \( \dot{X}(t) \) and hence \( \dot{X}_1(t) \) to be determined over a full period. Knowing the currents as functions of time, the corresponding values of torque can also be calculated. It is also evident that the method as presented above is quite independent of the initial time chosen.

Since in the vector \( \dot{X}(t) \) only the currents \( i_1, i_2, i_x, i_y \) are really independent variables, an alternative method which uses lower order matrices may have some computational advantage.

Referring to equation (3.37), let

\[
e^{At} = \hat{\phi}(t)
\]

and partition the equation

\[
\begin{bmatrix}
\dot{X}_1(t) \\
\dot{X}_2(t)
\end{bmatrix} = \begin{bmatrix}
\phi_{11}(t) & \phi_{12}(t) \\
\phi_{21}(t) & \phi_{22}(t)
\end{bmatrix} \begin{bmatrix}
X_1(0) \\
X_2(0)
\end{bmatrix}
\]

\[
\ldots (3.42)
\]
where

\[
\begin{align*}
\mathbf{x}_1 &= [i_1 \ i_2 \ i_x \ i_y]_t \\
\mathbf{x}_2 &= [x_1 \ x_2 \ x_3 \ x_4]_t
\end{align*}
\]

Using the same switching time notation and analogous to equation (3.38)

\[\mathbf{x}(t) = \phi(t-t_0)\mathbf{x}(t_0^+) \quad t > t_0\]

In the partitioned form of equation (3.42) one obtains

\[\mathbf{x}_1(t) = \phi(t-t_0)\mathbf{x}_1(t_0^+)+\phi_{12}(t-t_0)\mathbf{x}_2(t_0^+) \quad \ldots (3.43)\]

At the switching time \(t_1\),

\[\mathbf{x}(t_1^+) = \mathbf{x}(t_1^-) = \phi_{11}(t_1-t_0)\mathbf{x}_1(t_0^+)+\phi_{12}(t_1-t_0)\mathbf{x}_2(t_0^+) \quad \ldots (3.44)\]

The new initial condition for the next time interval \(t_1\) to \(t_2\) is

\[
\begin{bmatrix}
\mathbf{x}(t_1^+) \\
\mathbf{x}(t_1^+)
\end{bmatrix}
\begin{bmatrix}
\phi_{11}(t_1-t_0)\mathbf{x}_1(t_0^+)+\phi_{12}(t_1-t_0)\mathbf{x}_2(t_0^+) \\
\mathbf{x}_2(t_1^+)
\end{bmatrix}
\]

For time \(t > t_1\)

\[\mathbf{x}(t) = \phi(t-t_1)\mathbf{x}(t_1^+)\]
and when $t = t_2$

$$X_1(t_2^+) = X_1(t_2^-) = \phi_{11}(t_2-t_1)X_1(t_1^+) + \phi_{12}(t_2-t_1)X_2(t_1^+)$$

Substituting for $X_1(t_1^+)$ from equation (3.44)

$$X_1(t_2^+) = \phi_{11}(t_2-t_1) [\phi_{11}(t_1-t_0)X_1(t_0^+) + \phi_{12}(t_1-t_0)X_2(t_0^+)]$$

$$+ \phi_{12}(t_2-t_1)X_2(t_1^+)$$

$$= \phi_{11}(t_2-t_0)X_1(t_0^+) + \phi_{11}(t_2-t_1)\phi_{12}(t_1-t_0)X_2(t_0^+)$$

$$+ \phi_{12}(t_2-t_1)X_2(t_1^+)$$

... (3.45)

From the waveform symmetry,

$$X_1(t_2^+) = -X_1(t_0^+)$$

and substituting this in equation (3.45)

$$[0] = [\phi_{11}(t_2-t_0) + \phi_{12}(t_2-t_1)X_1(t_0^+)$$

$$+ \phi_{11}(t_2-t_1)\phi_{12}(t_1-t_0)X_2(t_0^+)$$

$$+ \phi_{12}(t_2-t_1)X_2(t_1^+)$$
\[
X_1(t_0^+) = -\left[\phi_{11}(t_2-t_0) + \phi_{12}(t_1-t_0)X_2(t_0^+)\right]^{-1}\left(\phi_{11}(t_2-t_1)\phi_{12}(t_1-t_0)X_2(t_0^+) + \phi_{12}(t_2-t_1)X_2(t_1^+)\right)
\] ....(3.46)

In a manner similar to the previous method, the initial condition vector \(X_1(t_0^+)\) obtained from equation (3.46) is used in equation (3.43) to find \(X_1(t)\) in the interval \(t_0\) to \(t_1\). The procedure is repeated for the next interval, \(t_1\) to \(t_2\), using \(X(t_1^+)\) from equation (3.44) as the initial condition vector.

Inspection of equations (3.42), (3.43) and (3.46) shows that the second method only requires evaluation of the submatrices \(\phi_{11}\) and \(\phi_{12}\).
4.1.1 Experimental Equipment

To test the models and analytical methods developed in previous chapters, a conventional, 3-phase, 4-pole, 5HP, squirrel cage induction motor and a rewound, 1/2HP, single-phase induction motor were used as test machines.

For each machine, model parameters were determined by the methods described in App.2. Model parameters were also obtained for a range of 3-phase, cage induction motors, partly to assess typical parameter variation with variation in frequency and voltage and partly to gain information on eigenvalue loci for various machines (see Figures 3.4.2, 3, 4,5).

Laboratory measurements of winding currents were obtained with current probes (Tektronix P6021 and P6004). For the 3-phase motor, an ASEA Torductor was used to measure shaft torque.

Experimental results were recorded with a multiple trace ultraviolet (U/V) chart recorder and a Polaroid camera.

For most purposes the U/V recorder is adequate and the 15cm chart width permits multiple trace records of sufficient size. However, it does have limited resolution at higher frequencies because of the limiting maximum chart speed and galvanometer frequency response. On the other hand, the
Polaroid photograph, while limited in size can use the flexibility of the oscilloscope display to provide greater waveform detail where needed.

4.1.2 Computer Solutions

In Chapter 3, solution methods were developed to solve the machine mathematical model for steady state winding currents and torque.

These analytical methods yield explicit expressions for the winding currents (e.g. equation (3.10) combined with equation (3.15) and in principle the currents may be found "exactly" by evaluating the appropriate equation. However, evaluating the equation elements, even for one single time instant requires many arithmetic operations including matrix inversion and it is convenient to combine all required operations into a computer program. Such a program can then provide numerical values for plotting steady state currents and torques as functions of time.

It is pertinent to point out that the "exact" steady state solutions for currents and torque as found by these methods could also provide the initial states needed for transient response predictions using numerical integration procedures such as the Euler or Runge-Kutta methods. Such numerical methods are, by their nature, subject to cumulative errors and starting from correct initial conditions not only minimizes possible error by reducing the necessary integration
time, but also avoids additional (extraneous) transients.

The computer programs (App.5) were written in Fortran II language for use with the College's IBM 1620 Computer which was the computer readily available to the writer. As a consequence of this language's restriction on certain variables it was not always possible to use model symbols as program variables. A list identifying relevant computer symbols is given with each program.

The main purpose of this phase of the work was to implement the mathematical analysis and obtain results for comparison with laboratory measurements. It is not claimed that the programs (even allowing for 1620 limitations) are in the most efficient form. They are presented essentially as initially written and include, in some cases, intermediate output to monitor the progress of execution, a useful piece of information with comparatively slow computers.

Programs involving vector-matrix manipulations such as those in App.5 could be structurally simplified if written for a computer having matrix interpretive facilities.
4.2 Three Phase Motor - Predictions and Performance

4.2.1 The Experimental Motor

A mathematical model for the three-phase cage rotor induction motor (derived in section 2.4) requires four "electrical" parameters to characterise the machine under constant speed conditions. Additional "mechanical" parameters, descriptive of the rotating system are needed for the case where the speed is varying.

Such a model does not take account of the power losses in the magnetic circuit iron. In addition, the "electrical" parameters may be influenced by the level of magnetic saturation of the iron and the frequency of the supply voltage.

The experimental motor used for tests was a commercially made, 4 pole machine, rated at 5 h.p., 415V. Using the methods described in App.2, measurements to determine the parameters were made for a range of supply frequencies from 20 to 50 Hz and at 100 and 80 per cent of rated volts on no-load. One could obtain the parameter sets corresponding to specified operating conditions. However, since the parameters of the test machine did not vary too widely, for the purpose of assessing the model, the policy adopted was to assign to the model a mean value of each parameter derived from the range of tests. The model and motor comparisons
Figure 4.2 Assumed voltage waveforms
(a) Line voltage
(b) Phase to neutral voltage
were then made at various frequencies and flux density levels extending over a range approximating that of the tests.

To measure shaft torque, an ASEA torductor was coupled between the motor and load. This device is capable of displaying torque variations of frequencies up to 250 Hz. However, owing to an unresolved magnetic circuit problem, the torque output signal was strongly modulated at multiples of shaft frequency. While not affecting the mean value, this modulation tended to obscure wanted signals.

Motor parameters:

\[ r_s = 1.29 \text{ ohms} \quad l_s = 0.273 \text{ H} \]
\[ \mu = 0.083 \quad \tau_r = 0.32 \text{ secs} \]

4.2.2 The Computer Program

The methods of analysis developed in chapter 3 lead to expressions for the steady state winding currents and torque. The "core" of the solution is the transition matrix. Once this is found and manipulated to obtain relevant initial conditions, all current and torque waveform detail can be calculated using the translating property of this matrix together with input voltage information.

The operations leading to the end result of steady state current and torque waveforms are three part,
(i) Finding the transition matrix

(ii) Finding the initial conditions

(iii) Using (i) and (ii) to obtain currents and torques as functions of time. This part uses the assembled results of equations (3.10), (3.15), and (3.27).

In principle, each step may be evaluated "exactly" but involves many arithmetic operations. Computer program 1 was written to perform the sequence of calculations. The program accepts as input data, a set of machine parameters, the fundamental period of the applied voltage, the rotor speed and the applied voltages. Although program 1 as written assumes an applied voltage waveform of the type described in section 3.2 and illustrated in Figure 4.2, it is only a minor alteration to cope with any other piece-wise constant waveform. No doubt the program could be improved in form and generalised to cope with arbitrary piece-wise constant waveforms but the intention of this phase of the work is merely to show how the results could be obtained and to get those results. In section 3.3 the transition matrix was shown to be dependent on the system matrix eigenvalues and some insight into how the transition matrix will vary with time is gained by inspection of the eigenvalues. In the program, the eigenvalues are evaluated from literal coefficient equations developed in the text. Thus the influence of motor
parameters on the transition matrix time variations and hence indirectly on current waveforms may be estimated.

A listing of program 1 is given in Appendix 5, together with identification information on program variables. A matrix inversion subroutine MATINV is used in the program but since inversion is a fairly common matrix operation, the subprogram is not listed.

4.2.3 Comparison of Predicted and Measured Values

(a) Current Waveforms

The predicted values of current and torque were obtained using a single set of parameters. As explained previously these are mean value parameters obtained from a series of tests at different flux density levels and frequencies. Correspondingly the tests have been made for a range of applied voltages and supply frequencies. The upper value of 250 volts d.c. is a consequence of the rating limitation of the electronic contactor used to protect the inverter. The analysis assumes piece-wise constant applied voltage waveforms and Figures 4.2.1, and 4.2.2, show actual line voltage conditions occurring for two different applied voltages and frequencies.

In Figure 4.2.4 the measured current displayed in Figures 4.2.3 and 4.2.17 is re-plotted on the same axes as the predicted result. The photographic record indicates a peak to peak current value of 22 amps compared with 23 amps
for the computed curve. However, bearing in mind measuring and display instrument accuracies, the essential comparison is in the relative waveforms. In this regard, checks show that a 4.5% expansion (increase in amplitude) of the measured curve would cause it to practically coincide with predicted curve over the full period.

For the remaining comparisons, measured and predicted results are presented in adjacent positions to assist in comparisons. For reasons of convenience in reproduction, drawings, charts and photographs were kept on separate pages.

Predicted results have been plotted from a computer printed output. A sample of this output is given in App.5, the sample being the data used for plotting Figure 4.2.4.

Figures 4.2.17 to 4.2.20 are a selection of chart records included to complement the photographs.

In the figure captions, the following symbols are used,

\[ V_{dc} \]

refers to the inverter supply voltage. It is also the amplitude of the line voltage.

\[ f_s \]

is the fundamental frequency of the stator supply voltage. It is also \( \frac{1}{6} \)th of the inverter switching frequency.

\[ s_f \]

is the per unit rotor slip measured with respect to the synchronous speed corresponding to \( f_s \).
Predicted and experimentally determined currents are compared for a range of fundamental frequencies from 55 Hz down to 17 Hz. Similarly, supply voltages and hence flux density levels vary from slightly above rated values for the test motor, as in Figures 4.2.5 and 4.2.6, down to well below rated values as in Figures 4.2.7 and 4.2.8.

Figures 4.2.15 and 4.2.16 show results for locked rotor and light running. All other tests are for loaded conditions at speeds corresponding to slips in the range up to 10%.

In comparing recorded and predicted currents it should be noted that oscilloscope display photographs are not always symmetrical about the nominal zero axis. This arose as a consequence of a small amount of low frequency "jitter" in the display.

Generally, the agreement between measured and predicted values is good. However, it is to be expected that some motors would show a greater parameter variation over such a test range and the use of a single set of mean values would not necessarily yield this degree of prediction accuracy.
Figure 4.2.1  Inverter line voltage and current.  
Supply frequency, 50 Hz.  Slip, .04. 
Scales:  1 division = 4 amps.  
1 division = 100 volts.
Figure 4.2.2  Inverter line voltage and current.
Supply frequency, 25 Hz.  Slip, .035.
Scales: 1 division = 100 volts.
1 division = 4 amps.
Figure 4.2.3  
Stator phase current

$V_{dc} = 240 \text{ Volts, } f_s = 40 \text{ Hz, } S_f = 0.097$

Scale: 1 division = 4 amps.
Figure 4.2.4 Measured and predicted current

\( V_{dc} = 240 \text{ Volts}, f_s = 40 \text{ Hz}, S_f = 0.097 \)

--- Computed  --- Measured
Figure 4.2.5  Stator phase current

$V_{dc} = 250$ Volts, $f_s = 26.7$ Hz, $S_f = .08$

Scale: 1 cm. = 3 amps.
Figure 4.2.6  Predicted stator current
\[ V_{dc} = 250 \text{ Volts}, \quad f_s = 26.7 \text{ Hz}, \quad S_f = 0.08 \]
Figure 4.2.7: Sator Phase Currents.

Scales: 1 division = 7 amps.

Lower: A V = 240 volts, f = 60 Hz, S = 0.09

Upper: A V = 240 volts, f = 60 Hz, S = 0.06
Figure 4.2.8 Predicted stator currents
(a) $V_{dc} = 240 \text{ volts}, f_s = 55 \text{ Hz}, S_f = \cdot06$
(b) $V_{dc} = 240 \text{ volts}, f_s = 50 \text{ Hz}, S_f = \cdot061$
Figure 4.2.9  Stator phase currents

Upper:  $V_{dc} = 240 \text{ Volts}, \quad f = 41.7 \text{ Hz}, \quad S_f = 0.038$

Lower:  $V_{dc} = 216 \text{ Volts}, \quad f = 55 \text{ Hz}, \quad S_f = 0.072$

Scales: Upper - 1 division = 2 amps;
        Lower - 1 division = 4 amps.
Figure 4.2.10  Predicted stator currents

(a) $V_{dc} = 240$ volts, $f_s = 41.7$ Hz, $S_f = 0.038$

(b) $V_{dc} = 216$ volts, $f_s = 35$ Hz, $S_f = 0.072$
Figure 4.2.11  Stator phase currents.
Upper: $V_{dc} = 204$ Volts, $f_s = 30$ Hz, $S_f = .053$
Lower: $V_{dc} = 180$ Volts, $f_s = 25$ Hz, $S_f = .079$
Scales: 1 division = 2 amps.
Figure 4.2.12 Predicted stator currents

(a) $V_{dc} = 204$ volts, $f_s = 30$ Hz, $S_f = 0.058$

(b) $V_{dc} = 180$ volts, $f_s = 25$ Hz, $S_f = 0.079$
Figure 4.2.13  Stator phase current
$V_{dc} = 105$ Volts, $f_s = 17$ Hz, $S_f = 0.104$
Scale: 1 cm = 2 amps.
Figure 4.2.14  Predicted stator currents

$V_{dc} = 105$ volts, $f_s = 17$ Hz, $S_f = 0.104$
Figure 4.2.15  Stator phase currents

Upper:  $V_{dc} = 112$ Volts, $f_s = 50$ Hz, locked rotor

Lower:  $V_{dc} = 240$ Volts, $f_s = 40$ Hz, $S_f = .007$

Scales:  Upper - 1 division = 3 amps;
         Lower - 1 division = 1 amp.
Figure 4.2.16 Predicted stator currents
(a) $V_{dc} = 112$ volts, $f_s = 50$ Hz, $S_f = 1.0$
(b) $V_{dc} = 240$ volts, $f_s = 40$ Hz, $S_f = 0.007$
Figure 4.2.17    Stator phase current
$V_{dc} = 240$ Volts, $f_s = 40$ Hz, $S_f = 0.097$
Scale: 1 cm = 2 amps.
Figure 4.2.18  Stator phase current

$V_{dc} = 240$ Volts, $f_s = 55$ Hz, $S_f = .06$

Scale: 1 cm = 2 amps.
Figure 4.2.19  Stator phase current

$V_{dc} = 204$ Volts, $f_s = 30$ Hz, $S_f = 0.058$

Scale: 1 cm = 2 amps.
Figure 4.2.20  Stator phase current

\[ V_{dc} = 132 \text{ Volts}, \quad f_s = 20 \text{ Hz}, \quad S_f = 0.09 \]

Scale: 1 cm = 2 amps.
(b) **Torque Measurements and Waveforms**

Figures 4.2.21, 22, show model predictions for motor torque at various speeds and loads. The common characteristic is a dominant pulsating component at 6 times the fundamental supply frequency.

In section 4.2.1, mention was made of the noise in the output of the torque measuring instrument. One consequence of this was that for mean value determinations (read from an indicating instrument) precise location of the zero was difficult to establish. A further consequence was that the output waveform of the torque signal was so noise contaminated that it was not possible visually to identify the torque ripple predicted by the model. Figure 4.2.23 shows a typical torque versus time signal and the spectral complexity of the signal is readily evident.

The problem here is to extract a wanted signal from a noisy environment. Since the torque ripple dominant frequency is predicted to be at 6 times supply frequency i.e. at inverter switching frequency, a cross correlation \[20\] between the output of the square wave, inverter-switching oscillator and the torque signal should detect any torque signal component of that frequency. The relevant theoretical basis for this is developed in a closely related context in Chapter 5.
Figures 4.2.24 and 25 display the results of the cross correlation operation and the resulting cross correlation functions show that the predicted torque ripple is present in every case tested. The plots of Figure 4.2.25 were obtained by an X-Y recorder plotting from discrete points on the waveform. This accounts for the lack of smoothness in the curve.

The measured values of average torque included in Figures 4.2.21, and 22 show some variations from the predicted value. However, in view of the torductor zero location problem, these variations are not necessarily significant.
Figure 4.2.21 Motor torque

- Computed
- Measured mean

(a) $V_{dc} = 250$ volts, $f_s = 26.7$ Hz
(b) $V_{dc} = 240$ volts, $f_s = 40$ Hz
(c) $V_{dc} = 180$ volts, $f_s = 25$ Hz
Figure 4.2.22 Motor torque

- Computed -- measured

(a) $V_{dc} = 132$ volts, 20 Hz
(b) $V_{dc} = 204$ volts, 30 Hz
(c) $V_{dc} = 240$ volts, 50 Hz
Figure 4.2.23  Torductor output signal (lower trace)
\( f_s = 20 \text{ Hz}, \ S_f = .05, \ \text{chart speed} = 125 \text{ cm/sec}. \)
Upper trace: repetition rate = 12 (rotor r.p.s.)
Figure 4.2.24 Cross correlation functions.
Confirming existence of predicted torque ripple
Upper: $f_s = 20\ Hz$  Lower: $f_s = 33.33\ Hz$
Time scales: 1 division = 2 milliseconds
Figure 4.2.25  Cross correlation functions.
Confirming existence of predicted torque ripple.
Upper:  \( f_s = 41.67 \) Hz;  Lower:  \( f_s = 55 \) Hz
Time scales:  1 inch = 2 milliseconds.
Figure 4.3 Circuit to realise the general principle of Figure 3.5.1.
4.3 QUASI TWO PHASE INDUCTION MOTOR.

PREDICTIONS AND PERFORMANCE

4.3.1 The Experimental Motor

The theoretical analysis of section 3.5 is based on the general principle indicated in Figure 3.5.1. One possible winding configuration approximately implementing the principle is shown in Figure 4.3, an advantage of the arrangement being its relative simplicity compared to commutation circuit complexity in a direct implementation. The diodes $D_1$ and $D_2$ connected to the ends of the centre-tapped winding prevent the capacitor $C$ interacting with the winding except during the short commutation interval.

If the distributed centre-tapped winding has $2N_2$ effective total turns, the m.m.f. produced is $N_2(i_{21}+i_{22})$ where the positive current directions are as in Figure 4.3. This m.m.f. is the same as that produced by an equivalent current $i_2 = i_{21}+i_{22}$ in a single winding having $N_2$ effective turns and the same space distribution. Thus the current sum $i_{21}+i_{22}$ in the circuit of Figure 4.3 should approximate the current predicted for the circuit of Figure 3.5-1 if the two arrangements have the same parameters.

The parameters of a small, 4 pole experimental machine, wound as in Figure 4.3 were determined by the methods given in App.2. Since the two halves of the centre tapped winding are only nominally identical a mean value of the parameters for the two halves was assigned to the equivalent winding.
Motor parameter values:

\[ r_{s1} = r_{s2} = 12.5 \text{ ohms} \]
\[ l_{s1} = 0.61 \text{ Hy} \quad l_{s2} = 0.56 \text{ Hy} \]
\[ \mu_1 = 0.031 \quad \mu_2 = 0.030 \]
\[ \tau_r = 0.057 \text{ seconds} \]

4.3.2 The Computer Program

Computer program 2 (App.5) implements the analytical method developed in section 3.5 and finds current and torque waveforms for any specified steady state condition.

In this case, the coefficient matrix \( A \) in equation (3.36) is of order eight and as an alternative to evaluating the transition matrix via eigenvalues, as in section 3.3, an approximate value is obtained by truncating the series form.

It was stated in section 3.3 that the transition matrix may be expressed as an infinite matrix series,

\[ e^{Kt} = U + Kt + \frac{1}{2!} K^2 t^2 + \frac{1}{3!} K^3 t^3 + \ldots \]

\[ = \sum_{m=0}^{\infty} \frac{(Kt)^m}{m!} \ldots (3.16) \]
This may also be written

\[ e^{-Kt} = \left( \sum_{m=0}^{m=n-1} \frac{(Kt)^m}{m!} \right) + \phi_n \]

\[ = \phi_n + R_n \]

Where \( R_n \) is the remainder matrix after \( n \) terms of the series.

Although a finite number of terms of the series approximates \( e^{-Kt} \) with an error depending on the remainder terms, this error may be made small by taking sufficient terms.

Several procedures have been proposed for finding a \( \phi_n \) approximation to \( e^{-Kt} \) that is accurate in all elements to within some specified number of significant digits [17,18]. However, when the transition matrix is to be manipulated the accuracy required is not clear.

As an alternative scheme for determining when to terminate the series, subroutine TMSS, a listing of which is included in Appendix 5, was written to evaluate the transition matrix to within the desired accuracy by computing and adding terms of the series until a specified criterion has been met.

The method employed is to compute simultaneously the sum of the first 'n' terms of the series for \( e^{-Kt} \) and \( e^{-K(-t)} \). The product of these approximate values for \( e^{-Kt} \) and \( e^{-K(-t)} \) is
then formed and compared with unit matrix. If any element of the product formed differs from its counterpart in the unit matrix by more than a prescribed amount, a new approximation to \( e^{Kt} \) and \( e^{K(-t)} \) is obtained by adding the \((n+1)\)th term of the series and repeating the test procedure. When every element in the product \((\text{approx.} e^{Kt})(\text{approx.} e^{K(-t)})\) differs from its corresponding element in the unit matrix by less than a specified amount, the series is terminated. To save computing time, the subroutining TMSS also searches for zeros before forming products.

As would apply in any finite series approximation scheme, choice of a criterion (in this case allowable deviation from unit matrix) is somewhat arbitrary. In these calculations, the deviation was set at \(10^{-5}\) and with this value the terms of the series used varied from 6 to 19, depending on the magnitude of \(t\). Because of the long execution times on a slow computer (IBM 1620) experimentation was not convenient but brief tests with criterion variations suggest that the deviation allowed probably could have been set much larger (resulting in fewer series terms and shorter computation times) without significant loss of accuracy in the resulting transition matrix.

Although the method of finding the transition matrix as presented above may be convenient when the system matrix order is large, relative to the method of section 3.3,
Figure 4.3.1 Verification of program 2
Predicted currents for balanced, 2-phase operation
240 Volts per phase. Slip 0.047
important information is not revealed. In the latter methods, the eigenvalues and their associated eigenvectors can be used to depict the system modes [14,19] and thus give valuable insight into motor behaviour.

4.3.3 Comparison of Predicted and Measured Values

The homogeneous equation (3.36) was obtained by introducing an augmented state vector, equation (3.35).

To test the analytical method and the computer program it was assumed that the motor of Figure 3.5.1 had identical stator windings with parameters approximating those of the experimental machine. Further it was assumed that the stator winding voltages were equal amplitude sinusoids with one quarter period phase displacement. This is the arrangement of a balanced, two-phase motor, and in the steady state one expects balanced two-phase currents in the windings.

Figures 4.3.1 shows the currents computed by program 2 for the arrangement described above when the parameters are,

\[ r_s = 12.5 \text{ ohms} \]
\[ l_s = 0.58 \text{ Hy} \]
\[ \tau = 0.057 \text{ seconds} \]
\[ \mu = 0.03 \]
\[ \text{slip} = 0.045 \]
The predicted equal amplitude currents, sinusoidal and 90° phase shifted are in agreement with expected results for this arrangement. Similarly, the predicted torque, constant at 2.656 newton metres, also agrees with expectations and these results may be taken as verifying the analytical methods and computer program 2.

In Figures 4.3.2 to 4.3.7 for three points in the speed range, currents predicted using program 2 are compared with photographs of measured currents. Figures 4.3.8, 9, 10 are recorder charts for the same conditions. For the predicted and experimental results, the switching instant was set at maximum and minimum on the voltage wave, i.e. \( t_d = 0 \) in Figure 3.5.2. Such a choice was only one of convenience and any other phase displacement could have been used.

For the experimental results, the current \( i_{21} + i_{22} \) was obtained by summing the two half winding currents in a current probe. As can be seen in the records, there is some asymmetry between the two halves of the circuit. Since the winding is essentially bifilar in construction it would suggest that the asymmetry arises in the semi-conductor elements.

At speeds zero and 905 RPM the measured and predicted currents are, as regards waveform, in moderately good agreement except in the region around changes in sign. This is to be expected since, in the experimental machine, this region is
influenced by the commutation operation.

For speeds above about 1000 RPM the computer solution predicts that the current $i_2$ will change sign during each half cycle. This may also be interpreted as an increasing third harmonic component of current $i_2$. However, in the experimental machine, because of the diodes $D_1$ and $D_2$, the currents cannot reverse and so for that portion of the half cycle, the current will be constrained to be zero. This situation is shown in Figures 4.3.6, 7, 10 for a speed of 1450 RPM. Because of the constraint in the experimental machine not imposed on the model, no correlation is expected between measured and computed currents.

The motor torque at standstill was measured at half rated volts. From Figure 4.3.11 the motor average torque at standstill and with 120 volts applied (parameters adjusted for the increase in temperature) may be calculated to be 1.33 newton metres. The measured value was 1.16 newton metres, approximately 13% below the predicted value.

One factor which would tend to lower the measured value of output torque is the reduction in effective voltage applied to winding 2 due to the voltage drops in the four series semi-conductor elements i.e. three diodes plus one S.C.R.

No torque measurements were made with the motor running.
It is worth reiterating that the experimental machine was wound with stator winding axes displaced by 90° electrical and the switching instants adjusted so that the fundamental component of phase voltages was displaced by 90°. However, the analysis of section 3.6 and App.1 is in no way restricted to these conditions and space and time displacements of other than 90° would be analysed in precisely the same manner.

Although the purpose of section 4.3 was to test the application of a particular analytical method, for the sake of completeness it should be said that the experimental machine used performs much better when the diodes $D_1$ and $D_2$ are removed. With this circuit arrangement the capacitor is free to interact with the windings at all times in the cycles. However, the circuit description must now take account of the capacitor and the model previously developed is no longer valid.

A new model fully accounting for the divided nature of one winding and the coupling effects of the capacitor could be obtained and except for the added complexity of a higher order system, precisely the same analytical methods apply.
Figure 4.3.2 Predicted currents at standstill
Supply voltage, 48 at 50 Hz.
Figure 4.3.3  Motor currents at standstill.
Supply voltage, 48 at 50 Hz.
Upper: Current $i_1$
Lower: Current $i_{21} + i_{22}$
Scales: 1 division = 1 amp.
Figure 4.3.4 Predicted currents at 905 r.p.m.
Supply Voltage, 48 at 50 Hz
Figure 4.3.5  Motor currents at speed 905 r.p.m.
Supply voltage, 48 at 50 Hz.
Upper:  Current $i_1$.  Scale: 1 division = 1 amp
Lower:  Current $i_{21} + i_{22}$.  Scale: 1 div. = 0.5 amp.
Figure 4.3.6 Predicted currents at 1450 r.p.m
Supply voltage 80 at 50 Hz
Figure 4.3.7  Motor currents at speed 1450 r.p.m.
Supply voltage, 80 at 50 Hz.
Upper: Current $i_1$
Lower: Current $i_{21} + i_{22}$
Scales: 1 division = .5 amp.
Figure 4.3.8 Motor currents at standstill.
Supply voltage, 48 at 50 Hz.
Upper: Current $i_1$
Lower: Current $i_{21} + i_{22}$
Scales: 1 cm. = 1 amp.
Figure 4.3.9  Motor currents at speed 905 r.p.m.
Supply voltage, 48 at 50 Hz.
Upper:  Current $i_1$
Lower:  Current $i_{21} + i_{22}$
Scales:  1 cm. = 1 amp.
Figure 4.3.10  Motor currents at speed 1450 r.p.m.
Supply voltage, 80 at 50 Hz.
Upper:  Current $i_{21} + i_{22}$
Lower:  Current $i_1$
Scales:  1 cm. = .8 amp.
Figure 4.3.11 Predicted motor torque
Supply voltage, 120 at 50 Hz
(a) at standstill. (b) at 905 r.p.m.
5.1 Inverter-fed induction motor as control loop element

The induction motor property of speed dependence on stator supply frequency makes the inverter-fed induction motor suitable for variable speed drive applications [21]. Supply frequency variations are easily effected by varying the frequency of the oscillator controlling the inverter gate circuits.

A particular application of the variable speed property of the inverter-fed induction motor could be as the controlled element in a closed loop, speed controlled system [22]. In such systems, the supply frequency may be changed very rapidly because the changes occur at low signal power levels. Further, with such as voltage-to-frequency circuits, the supply frequency may be made proportional to a signal voltage and again, rapid changes are possible at the low signal power levels.

In the type of speed control system described above, the induction motor is a control element. Designing such a system to meet some specified performance criterion requires knowledge i.e. a description of all the control loop elements. If all the control loop elements are linear, the system may be designed and performance for any input predicted
using well established linear system methods.

For linear systems analysis, the most common methods of element description are the transfer function (the complex frequency domain description) and the weighting function (time domain description). These two "input-output" type descriptions are of course, related as a Laplace transform pair [23].

While weighting or transfer functions relating supply voltage, load torque or supply frequency with current, motor speed or developed torque are all of potential interest, within the context of this chapter, main interest is centred on input-output relations connecting frequency changes with speed changes.

Some writers [24,25] have sought transfer functions connecting supply frequency with developed torque. In many cases, such as that described above, where shaft speed is the variable of interest, the system input and output are effectively the supply frequency and the shaft/load speed.

5.2.1 Linearisation of motor equations

The equations developed in section 3.2 e.g. equation (3.5) or (3.2) and (3.3) for the constant speed case, utilise a basis for the vector space (or co-ordinate reference frame) that is well suited to analysis when the stator voltages are non-sinusoidal. With this basis the
stator voltages and currents are not transformed.

The equations of motion for the motor such as equation (3.5) form a non-linear set. While a general analytical solution is not possible it is often of interest to investigate how the system behaves in response to small variations about a steady state operating point. With this approach it is possible to get a set of linear differential equations that approximately represent the system behaviour in a "small signal" sense.

To get such a set of equations, consider the $i^{th}$ equation in a set such as (3.5) or, in terms of the symbols of equation (3.1), an equation of the form

$$\frac{d}{dt} x_i = f_i(X, W)$$

Taking the total differential,

$$\delta \left( \frac{d}{dt} x_i \right) = \frac{d}{dt} (\delta x_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \delta x_j + \sum_{j=1}^{m} \frac{\partial f_i}{\partial w_j} \delta w_j \quad \ldots \quad (5.1a)$$

The partial derivatives $\frac{\partial f_i}{\partial x_j}$ and $\frac{\partial f_i}{\partial w_j}$ are to be taken at the reference values of $X$ and $W$ e.g. at $X_0$ and $W_0$.

When equation (5.1a) is obtained for all $f_i$, $i = 1, \ldots, n$, the result can be assembled in matrix form,
\[
\frac{d}{dt} \{ \delta \dot{x} \} = J_1 \delta \dot{x} + J_2 \delta \dot{w} 
\] .... (5.1b)

The result shown in equation (5.1b) is usually derived in a somewhat different manner and the matrices \( J_1 \) and \( J_2 \) are then referred to as Jacobian matrices [26].

For the case where

\[
\dot{x} = [i_a \ i_b \ i_x \ i_y \ \dot{\theta}]_t 
\]

\[
\dot{w} = [v_a \ v_b \ T_L]_t 
\]

\[
J_1 = \begin{bmatrix}
\frac{\partial f_1}{\partial i_a} & \frac{\partial f_1}{\partial i_b} & \frac{\partial f_1}{\partial i_x} & \frac{\partial f_1}{\partial i_y} & \frac{\partial f_1}{\partial \theta} \\
\frac{\partial f_5}{\partial i_a} & \frac{\partial f_5}{\partial i_b} & \frac{\partial f_5}{\partial i_x} & \frac{\partial f_5}{\partial i_y} & \frac{\partial f_5}{\partial \theta} \\
\end{bmatrix} 
\] ....(5.2)

\[
J_2 = \begin{bmatrix}
\frac{\partial f_1}{\partial v_a} & \frac{\partial f_1}{\partial v_b} & \frac{\partial f_1}{\partial T_L} \\
\frac{\partial f_5}{\partial v_a} & \frac{\partial f_5}{\partial v_b} & \frac{\partial f_5}{\partial T_L} \\
\end{bmatrix} 
\]
The partial derivatives indicated in $J_1$ and $J_2$ are understood to be evaluated at the reference values of the vectors $\tilde{X}$ and $\tilde{W}$.

If the operations indicated in equation (5.2) are applied to equation (3.5), equations of the form of (5.1) will result. Although linear, these will have time-varying coefficients as a consequence of time-varying steady state voltages and currents.

As was recognised by Kron [27] provided the supply voltages are balanced sinusoids, the above difficulty can be avoided by employing a transformation to a new set of variables such that voltages and currents are constant in the steady state.

With these (state) variables, the matrices $J_1$ and $J_2$ are constant and equation (5.1) is a linear one with constant coefficients.

5.2.2 Transformations and derivation of piece-wise linear equations

For the cage rotor induction motor, the basis vector transformation to the so-called "synchronously rotating reference frame" is most easily visualised as two successive transformations. One to refer rotor currents to a reference frame stationary with respect to the stator and the second to rotate this reference frame at synchronous speed.
When these two transformations (say $T_1$ and $T_2$) are combined so that

$$T = T_1 T_2$$

and then this is applied to the basic coupled circuit equation (2.1) one obtains

$$V'' = (T R T)^{-1} I'' + T \frac{d}{dt} \{L T^{-1} I''\} \quad \ldots (5.3a)$$

i.e.

$$V = \begin{bmatrix} C_1 \\ \cdots \\ C_2 \end{bmatrix} I'' + \begin{bmatrix} \cdots \end{bmatrix} \frac{d}{dt} I'' \quad \ldots (5.3b)$$

If the elements of $V = [v_a v_b v_c 0 0]^T$ are single frequency sinusoids and $T$ is appropriately chosen, $V''$ is a constant vector and for constant speed ($\dot{\theta}$) and angular frequency ($\omega$) the steady state components of $I''$ are constants.

In partitioned form, the transformation matrix may be written,

$$T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \ldots (5.4)$$

where

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\sin \phi & \cos(\phi-120^\circ) & \cos(\phi+120^\circ) \\ -\sin(\phi-120) & -\sin(\phi-120) & -\sin(\phi+120) \end{bmatrix} \quad \ldots (5.5)$$
\[ \mathbf{U}_2 = \frac{r}{m} \begin{bmatrix} \cos(\phi-n\theta) & \sin(\phi-n\theta) \\ -\sin(\phi-n\theta) & \cos(\phi-n\theta) \end{bmatrix} \] ....(5.6)

and \( \phi = (\omega_s t - \alpha) \)

In section 2.4, equation (2.5) was obtained by using the first transformation (referred to above at \( \mathbf{T}_1 \)) where, in partitioned form,

\[ \mathbf{T}_1 = \begin{bmatrix} \mathbf{U} & 0 \\ 0 & \mathbf{H}(\theta) \end{bmatrix} \] ....(5.7)

\( \mathbf{H}(\theta) \) being as defined in that section and \( \mathbf{U} \) is unit matrix, order 3.

Equation (2.5) may be rearranged in the form

\[ \mathbf{V} = \mathbf{G}_1 \mathbf{I}' + \mathbf{G}_2 \mathbf{p} \mathbf{I}' \] ....(5.8)

where

\[ \mathbf{V} = [\mathbf{V}_a \mathbf{V}_b \mathbf{V}_c 0 0]_t \]

\[ \mathbf{I}' = [i_a i_b i_c i_x i_y]_t \]
Effecting the second transformation, (referred to above as \( T_2 \))

\[
\mathbf{I}'' = T_2 \mathbf{I}'
\]

\[
= \begin{bmatrix} i_0 & i_{s1} & i_{s2} & i_r1 & i_r2 \end{bmatrix}_t \quad \ldots \text{(5.9)}
\]

\[
\mathbf{V}'' = T_2 \mathbf{V}
\]

\[
= \begin{bmatrix} v_0 & v_{s1} & v_{s2} & 0 & 0 \end{bmatrix}_t
\]

where

\[
T_2 = \begin{bmatrix} H_1 & 0 \\ 0 & H_3 \end{bmatrix} \quad \ldots \text{(5.10)}
\]

\( H_1 \) has been previously defined

and

\[
H_3 = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \quad \ldots \text{(5.11)}
\]

Thus equation (5.8) becomes

\[
\mathbf{V}'' = (T_2 \mathbf{G}_1 T_2^{-1}) \mathbf{I}'' + T_2 \mathbf{G}_2 \mathbf{P}(T_2^{-1} \mathbf{I}'')
\]
or, when expanded,

\[ \tilde{v}'' = \{T_2 G_1 T_2^{-1} + T_2 G_2 (p T_2^{-1})\} \tilde{v}'' + \{T_2 G_2 T_2^{-1}\} p \tilde{v}'' \]

i.e.

\[ \tilde{v}'' = \tilde{G}_1 \tilde{v}'' + \tilde{G}_2 p \tilde{v}'' \]  

\[ \ldots (5.12) \]

Since

\[ \tilde{v}'' = T_2 \tilde{v} \]

\[ v_0 = \frac{1}{\sqrt{3}} (v_a + v_b + v_c) \]

\[ v_{s1} = \sqrt{\frac{2}{3}} (v_a \cos \phi + v_b \cos (\phi-120^\circ) + v_c \cos (\phi+120^\circ)) \]

\[ v_{s2} = \sqrt{\frac{2}{3}} (-v_a \sin \phi - v_b \sin (\phi-120^\circ) - v_c \sin (\phi+120^\circ)) \]

and when \( \phi = \omega_s t - \alpha \)

\[ v_{s1} = \sqrt{\frac{3}{2}} V_m \sin \alpha \]  

\[ v_{s2} = -\sqrt{\frac{3}{2}} V_m \cos \alpha \]  

\[ \ldots (5.13) \]
The phase angle $\alpha^\circ$ is arbitrary. If it is chosen to be $90^\circ$, then $v_{s1} = \sqrt{\frac{3}{2}} V_m$ and $v_{s2} = 0$. Similarly, $v_{s1}$ and $v_{s2}$ may be adjusted to set any required current to zero in the steady state.

When the transformation $T_2$ is applied to equation (2.5) and manipulated as described above, one obtains

$$v'' = G_{1}' \, i'' + G_{2}' \, p \, i''$$

...(5.12)

where, for balanced conditions ($i_0 = 0$)

$$G_{1}' = \begin{bmatrix}
    r_s & -\frac{3}{2} l_s \omega_s & -\sqrt{\frac{3}{2}} \mu \omega_s & 0 \\
    \frac{3}{2} \omega_s l_s & r_s & 0 & -\sqrt{\frac{3}{2}} \mu \omega_s \\
    \sqrt{\frac{3}{2}} \sigma & 0 & 1 & -\tau \sigma \\
    0 & \sqrt{\frac{3}{2}} \sigma & \tau \sigma & 1
\end{bmatrix}$$

as before, $\sigma = \omega_s - n \theta$
Similarly, using the transformation

\[ \tilde{z}' = T_2 \tilde{z}' \]

in equation (2.6) one obtains the expression for torque in terms of \( I'' \) elements i.e.

\[ T_e = -\sqrt{\frac{3}{2}} n \mu (i_{s1} i_{r1} + i_{s2} i_{r2}) \quad \text{(5.14)} \]

In a manner similar to that of section 3.2, equation (5.12) can be re-arranged into the form

\[ p \tilde{z}'' = A' \tilde{z}'' + B' \tilde{v}'' \quad \text{(5.15a)} \]
\[ \Delta' = \frac{1}{\Delta} \]

\[
\begin{bmatrix}
-\tau_r r_s & (\omega_s \Delta + \frac{3}{2} \mu n \dot{\theta}) & \sqrt{\frac{3}{2}} \mu \tau_r n \dot{\theta} & -\sqrt{\frac{3}{2}} \mu_s \\
-(\omega_s \Delta + \frac{3}{2} \mu n \dot{\theta}) & -\tau_r r_s & \sqrt{\frac{3}{2}} \mu_s & \sqrt{\frac{3}{2}} \mu \tau_r n \dot{\theta} \\
\frac{3\sqrt{3}}{2\sqrt{2}} l_s n \dot{\theta} & \sqrt{\frac{3}{2}} r_s & -\frac{3}{2} l_s & (\omega_s \Delta - \frac{3}{2} \mu l_s \tau n \dot{\theta}) \\
-\sqrt{\frac{3}{2}} r_s & \frac{3\sqrt{3}}{2\sqrt{2}} l_s n \dot{\theta} & -(\omega_s \Delta - \frac{3}{2} \mu l_s \tau n \dot{\theta}) & -\frac{3}{2} l_s
\end{bmatrix}
\]

\[ \Delta = \frac{3}{2} (l_s \tau - \mu) \]  

\[ \ldots (5.15b) \]
In equations (3.4) and (3.5) the load torque is written in symbolic form as $T_L$. However, in many cases the motor load has a known speed-torque characteristic. For example, provided the system remains unchanged, in turbo-pumps the load varies as $\dot{\theta}^2$ while in positive displacement pumps, torque is practically constant. Similar load torque-speed relations are well established for rolling mills, wire drawing, paper making processes etc.

For these reasons, in the following development load torque is expressed as an arbitrary function of speed i.e.

$$T_L = K_T f(\dot{\theta})$$

and the torque balance equation becomes

$$\frac{d}{dt} \dot{\theta} = -\sqrt{\frac{3}{2}} \frac{nU}{J} (i_{s1} i_{r1} + i_{s2} i_{r2}) - \frac{1}{J} K_T f(\dot{\theta}) \quad (5.16)$$
Within equation (5.16), the $K_T f(\dot{\theta})$ term can also be thought of as including friction and windage torques arising in the motor.

Equations (5.15) and (5.16) together are the synchronously rotating reference frame counterpart of equation (3.5). With sinusoidal applied voltages in the actual machine, the currents and voltages of equation (5.15) are constant in the steady state.

For ease of reference it is convenient to regard equations (5.15) and (5.16) together as being of the general form,

$$\dot{X} = f(X, W) \quad \cdots (5.17)$$

Before obtaining the set of "small signal" linear differential equations as described in section 5.2.1 some attention must be given to the form of the vector $\delta W$ in equation (5.1b)

Since $T_L = K_T f(\dot{\theta})$

$$\delta T_L = \frac{\partial T_L}{\partial K_T} \delta K_T + K_T \frac{\partial T_L}{\partial \dot{\theta}} \delta \dot{\theta}$$

$$= f(\dot{\theta}) \delta K_T + K_T f'(\dot{\theta}) \delta \dot{\theta} \quad \cdots (5.18)$$

where $f'(\dot{\theta}) = \frac{\partial f(\dot{\theta})}{\partial \dot{\theta}}$
Similarly, since a transfer function relating say \( \delta \dot{\theta} \) and \( \delta \omega \) is required, \( \omega \) must be regarded as an element of the vector \( W \). When the operations indicated in equations (5.1), (5.2) are applied to equation (5.17), one obtains,
The notation $f'(\dot{\theta})^\circ$ implies $\frac{\partial f(\dot{\theta})}{\partial \dot{\theta}}$
\[-\sqrt{\frac{3}{2}} \frac{\mu}{\Delta} \quad \sqrt{\frac{3}{2}} \frac{\mu n}{\Delta} (\sqrt{\frac{3}{2}} i_{s2} + \tau_{r} i_{r1})\]

\[\sqrt{\frac{3}{2}} \frac{\mu \tau_{r} n \theta^{\circ}}{\Delta} \quad -\sqrt{\frac{3}{2}} \frac{\mu n}{\Delta} (\sqrt{\frac{3}{2}} i_{s1} - \tau_{r} i_{r2})\]

\[(\omega_s - \frac{3}{2} \frac{1_s \tau_{r} n \theta^{\circ}}{\Delta}) \quad \frac{3}{2} \frac{n 1_s}{\Delta} (\sqrt{\frac{3}{2}} i_{s1} - \tau i_{r2})\]

\[-\frac{3}{2} \frac{1_s}{\Delta} \quad \frac{3}{2} \frac{n 1_s}{\Delta} (\sqrt{\frac{3}{2}} i_{s1}^{\circ} + \tau i_{r1})\]

\[-\sqrt{\frac{3}{2}} \frac{n u i_{s2}^{\circ}}{J} \quad -\frac{K_T}{J} f'(\theta)^{\circ}\]

\[
\ldots\ldots(5.19)
\]

evaluated at \(\dot{\theta} = \dot{\theta}^{\circ}\)
Thus equations (5.19), (5.20) and (5.21) are the elements of the linear differential equation,

\[ \frac{d}{dt} \{ \delta \bar{X} \} = J_a \delta \bar{X} + J_b \delta \bar{W} \]  

\[ \text{....(5.22)} \]
5.3 **Weighting and transfer functions relating frequency and speed changes**

Equation (5.22) describes the response of the motor to small variations in supply frequency, voltage amplitude and load torque magnitude. As is clear from the derivation, the accuracy of the description depends on the size of the disturbances about the reference values.

For any disturbance $\delta \omega$, the form of the system response is governed by its natural modes [19] and the response may be described in terms of these modes and the vector $J_B \delta \omega$. [14].

Let $\delta \omega$ be a constant vector i.e. the elements $\delta \omega_s$, $\delta \nu_s$, etc are constants.

As indicated in section 3.2 the general solution of equation (5.22) is,

$$
\delta X(t) = e^{-a} \delta X(0) + e^{-a} \int_0^t e^{-a \tau} J_B \delta \omega(\tau) d\tau
$$

but by definition $\delta X(0) = 0$ and the results of section 3.2 yield the solution

$$
\delta X(t) = [e^{-a} + U] J_a^{-1} J_b \delta \omega
$$

... (5.23)
Two methods of evaluating the transition matrix $J_a e^t$ have been described in sections 3.3 and 4.3.2. For any given operating point, the vector $(J_{a b}^{-1} J_b \delta \omega)$ is a constant and with the transition matrix, equation (5.23) describes the motor's incremental response (in terms of currents, speed and implicitly, developed torque) to any combination of incremental changes in voltage amplitude, supply frequency and load torque.

As a specific case, consider the motor's response to a change in frequency. This may be implemented at constant voltage or at constant flux density. In the latter case, for operation above 5 or 10 Hz, constant flux density is approximately maintained if voltage is proportional to frequency.

Thus for a frequency change at constant flux density, $\delta \omega$ becomes

$$\delta \omega = [\delta \omega_s (\gamma \sin \alpha) \delta \omega_s (-\gamma \cos \alpha) \delta \omega_s 0]_t \quad \ldots (5.24a)$$

where $\gamma$ is a constant such that

$$\sqrt{\frac{3}{2}} V_m = \gamma \omega_s$$

The fact that $\delta K_T$ is zero does not imply that the load torque is constant. Torque changes will depend on the nature of $K_T f(\theta)$.
\( \delta W \) can be written

\[
\delta W = \begin{bmatrix} 1 & \gamma \sin \alpha & \gamma \cos \alpha & 0 \end{bmatrix}_t \delta \omega_s
\]

\[= \delta y \delta \omega_s \tag{5.24b} \]

If \( Z = [z_1 \ z_2 \ z_3 \ z_4 \ z_5]_t \)

and

\[Z = (J_a^{-1} J_b \delta Y) \tag{5.24c} \]

equation (5.23) becomes

\[\delta X(t) = [e^{-a} + U] Z \delta \omega_s \tag{5.25} \]

Let the elements of the transition matrix \( e^{-a} \)

then, from equation (5.25)

\[
\delta x_1(t) = \delta i_{s1}(t) = \{(\phi_{11} + 1)z_1 + \phi_{12}z_2 + \phi_{13}z_3 + \phi_{14}z_4 + \phi_{15}z_5\} \delta \omega_s
\]

\[
\delta x_2(t) = \delta i_{s2}(t) = {\cdots}
\]

\[
\delta x_5(t) = \delta i_{s5}(t) = \{(\phi_{51}z_1 + \phi_{52}z_2 + \phi_{53}z_3 + \phi_{54}z_4 + (\phi_{55} + 1)z_5)\} \delta \omega_s
\]
\( \delta x_5(t) \) is \( \delta \dot{\theta}(t) \) and the last term is a "time domain" relationship between a step change in frequency at constant flux density and the change in speed produced.

The expression represented by

\[
\frac{\delta \dot{\theta}(t)}{\delta \omega_s}
\]

is the integral of the system weighting function.

If the frequency change is to be at constant voltage, \( \delta Y \) can be adjusted by setting \( \gamma \) to zero in that vector.

As was shown in section 3.3, the form of the transition matrix elements \( \phi_{ij} \) depends on the eigenvalues of \( J_a \). Typically, since \( J_a \) is of order 5,

\[
\phi_{ij} = a_{ij} e^{\lambda_1 t} + b_{ij} e^{\lambda_2 t} + c_{ij} e^{\lambda_3 t} + d_{ij} e^{\lambda_4 t} + f_{ij} e^{\lambda_5 t} \quad (5.26a)
\]

Some of the \( \lambda_i \) may be complex and as shown by a number of researchers, these may, for some combinations of voltage and frequency, have very small negative real parts or even positive real parts [28, 29, 30, 31, 32].

The latter case is of course, an unstable condition and the terms in equation (5.26a) which had \( \lambda_i \) with positive real parts would increase exponentially with time.

Suitable methods and procedures for obtaining \( \delta x(t) \) and hence \( \delta \dot{\theta}(t) \) have already been derived and illustrated in Chapters 3 and 4.
An alternative approach is to express relationships in the complex frequency domain.

Transforming equation (5.22) and recalling that

\[ \delta X(0) = 0 \]

\[ \delta X(S) = \begin{bmatrix} S & -J \end{bmatrix} \begin{bmatrix} J \end{bmatrix} \delta W(S) \]

\[ = G_1(S) \delta W(S) \]

\[ = G_1(S) \delta W(S) \]

where \[ G_1(S) = \begin{bmatrix} S & -J \end{bmatrix} \begin{bmatrix} J \end{bmatrix} \]

is a transfer matrix relating "output" vector \( \delta X(S) \) with "input" vector \( \delta W(S) \).

For the relationships described by equations (5.24a) and (5.24b)

\[ \delta X(s) = \begin{bmatrix} S & -J \end{bmatrix} \begin{bmatrix} J \end{bmatrix} \delta W(s) \]

\[ = \begin{bmatrix} S & -J \end{bmatrix} \begin{bmatrix} J \end{bmatrix} \delta W(s) \]

i.e. \[ \delta X(s) = G_2(s)[1 1 1 1 1] \delta W(s) \]

\[ = G_2(s)[1 1 1 1 1] \delta W(s) \]

\[ G_2(s) \] is a 5th order matrix.

Let the general element be \( g_{ij} \).
Then part of equation (5.28) may be written

\[ \delta x_5(s) = \delta \hat{\theta}(s) = \{g_{51}(s) + g_{52}(s) + g_{53}(s) + g_{54}(s) + g_{55}(s)\} \delta \omega_s(s) \]

\[ = g_5(s) \delta \omega_s(s) \quad \text{.....(5.29)} \]

or \[ g_5(s) = \frac{\delta \hat{\theta}(s)}{\delta \omega_s(s)} \quad \text{.....(5.30)} \]

is the transfer function relating speed variations and frequency variations. Constant flux density or constant voltage conditions are provided for in the vector \( \delta \gamma \).

The product \( \mathbf{J} \delta \gamma \) is a vector derived from equations (5.20) and (5.24b). Let this vector be

\[ \mathbf{z} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 \end{bmatrix}^t \]

Then \( g_5(s) \) may be written

\[ g_5(s) = \begin{bmatrix} \frac{\Delta_{15}(s)}{\Delta(s)} & \frac{\Delta_{25}(s)}{\Delta(s)} & \frac{\Delta_{35}(s)}{\Delta(s)} & \frac{\Delta_{45}(s)}{\Delta(s)} & \frac{\Delta_{55}(s)}{\Delta(s)} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} \quad \text{.....(5.31)} \]
where \[ \Delta(s) = \text{det.} [SU-J_a]^{-1} \]

\[ \Delta_i(s) \text{ are cofactors of } [SU-J_a]^{-1} \]

\[ i = 1 \ldots 5 \]

From equation (5.31)

\[ g_5(s) = \sum_{i=1}^{5} p_i \frac{\Delta_i(s)}{\Delta(s)} \]

or

\[ g_5(s) = \frac{n(s)}{d(s)} \]

...(5.32)

where \( n(s) \) is a 4th order and \( d(s) \) a 5th order polynomial in \( s \).

While the basic calculations are simple, their extent is such that evaluating \( g_5(s) \) for the most general case and with literal coefficients is a formidable task. Several writers have published expressions for polynomials derived following certain simplifying assumptions [26, 29].

In reference [29] mention is made of a simplification in calculation effected by setting \( v_{s1} = 0 \). However a greater simplification can be made by adjusting \( v_{s1} \) and \( v_{s2} \) such that (say) \( i_{s2} = 0 \). That this can be done is seen
from equations (5.9) and (5.13) where it is shown that the actual stator current amplitude is \( \sqrt{i_{s1}^2 + i_{s2}^2} \) and \( v_{s1} = -v_{s2} \tan \alpha \), so that since \( \tan \alpha \) can vary from \(-\infty\) to \(+\infty\), \( v_{s1} \) and \( v_{s2} \) can take on any relative values.

To obtain a simplified form of the transfer function \( g_5(s) \), the case of very light load will be considered.

At very light load, \( i_{r1} \) and \( i_{r2} \) approach zero. Under these conditions, if \( v_{s2} = \frac{3}{2} v_{s1} \) then \( i_{s2} = 0 \) and \( i_{s1} = \frac{v_{s1}}{r_s} \).

When \( \Delta_a \) is evaluated for these conditions the resulting polynomial in \( s \) i.e. \( \Delta(s) \) is as shown in equation (5.33) and (5.34). Except for the restriction to very small slips, no simplifying approximations have been made.

\[
s^5 + a_4 s^4 + (a_3 + b_3 \xi) s^3 + a_2 + b_2 \xi) s^2 + (a_1 + b_1 \xi) s + b_0 \xi \quad \ldots (5.33)
\]

where \( S \) is the Laplace variable

and

\[
\xi = \frac{\frac{3}{2} u (n v_{s1})^2}{r_s^2 \Delta J}
\]

\[
a_1 = \omega_s^2 \left( \frac{r_s \tau_r}{\Delta} \right)^2 + \left( \frac{r_s}{\Delta} \right)^2
\]
\[ a_2 = \omega_s^2 \left[ \frac{2r_s \tau_r}{\Delta} + \frac{2r_s}{\Delta} \left( \frac{3}{2} \frac{l_s}{s} + \frac{r_s \tau_r}{\Delta} \right) \right] \]

\[ a_3 = \omega_s^2 + \frac{r_s \tau_r}{\Delta} + \left( \frac{3}{2} \frac{l_s}{s} \right)^2 + \frac{2r_s}{\Delta} \]

\[ a_4 = \frac{2r_s \tau_r}{\Delta} + \frac{3l_s}{s} \]

\[ b_0 = \frac{r_s^2 + \left( \frac{3}{2} \frac{l_s}{s} \omega_s \right)^2}{\Delta} \]

\[ b_1 = \left( \frac{3}{2} \frac{l_s}{s} \omega_s^2 + \frac{3l_s r_s}{\Delta} + \frac{r_s^2 \tau_r}{\Delta} \right) \]

\[ b_2 = r_s \left( \frac{3}{2} (\tau_r r_s + \frac{3}{2} l_s) \right)^{\frac{1}{2}} \frac{l_s}{s} \]

\[ b_3 = \frac{3}{2} l_s \]

\[ \Delta = \frac{3}{2} (l_s \tau_r - \mu) \]

where \[ \Delta = \frac{3}{2} (l_s \tau_r - \mu) \]

and since, as derived, \( i_{s2} = 0 \),

\[ v_{s1} = \frac{\sqrt{\frac{1}{2} v_m r_s}}{\left[ r_s^2 + \left( \frac{3}{2} \frac{l_s}{s} \omega_s \right)^2 \right]^{\frac{1}{2}}} \]
In expanded form, equation (5.31) becomes,

\[ \gamma_5(s) = \frac{\Delta_{15}(s)p_1 + \Delta_{25}(s)p_2 + \Delta_{35}(s)p_3 + \Delta_{45}(s)p_4 + \Delta_{55}(s)p_5}{\Delta(s)} \]

\[(5.35)\]

From the product \( J_0 \delta Y \), the vector \( P \) can be found. In the case being considered, since \( i_{s2} \) is to be held at zero, it is readily found from solving equation (5.12) for the steady state conditions required that,

\[ v_{s1} = \sqrt{\frac{3}{2}} V_m \sin \alpha = \frac{\sqrt{\frac{3}{2}} V_m r_s}{[r_s^2 + (\frac{3}{2} s \omega_s)^2]^\frac{1}{2}} \]

\[ v_{s2} = -\sqrt{\frac{3}{2}} V_m \cos \alpha = \frac{-\sqrt{\frac{3}{2}} V_m (\frac{3}{2} s \omega_s)}{[r_s^2 (\frac{3}{2} s \omega_s)^2]^\frac{1}{2}} \]

satisfies this condition.

The vector of forcing increments \( \delta \omega \), equation (5.21) has terms \( \delta v_{s1} \) and \( \delta v_{s2} \).

Putting \( \sqrt{\frac{3}{2}} V_m = \gamma \omega_s \),

\[ \delta v_{s1} = \frac{(r_s \omega_s) \delta \gamma}{[r_s^2 + (\frac{3}{2} s \omega_s)^2]^\frac{1}{2}} + \frac{(r_s^3 \gamma) \delta \omega_s}{[r_s^2 + (\frac{3}{2} s \omega_s)^2]^\frac{3}{2}} \]
\[ \delta v_{s2} = \frac{-\left(\frac{3}{2}s^2 \omega s^2\right) \delta \gamma}{[r_s^2 + \left(\frac{3}{2}s^2 \omega s^2\right)]^2} - \frac{\frac{3}{2}s^2 \gamma \omega s \left[2r_s^2 + \left(\frac{3}{2}s^2 \omega s^2\right)\right] \delta \omega s}{[r_s^2 + \left(\frac{3}{2}s^2 \omega s^2\right)]^2} \]

\[ \text{(5.36)} \]

The cofactors in equation (5.35) are,

\[ \Delta_{15}(s) = \left(\frac{-\frac{3}{2}m \gamma \omega s \sin \alpha}{r_s \Delta J}\right)s^2 + \frac{\frac{3}{2}m \gamma \omega s \sin \alpha}{r_s \Delta J} \left[ r_s \omega s - \frac{3}{2}s \frac{1}{\Delta} \left(\frac{3}{2}s + \tau r_s\right)\right]s \]

\[ \Delta_{25}(s) = \left(\frac{-\frac{3}{2}m \gamma \omega s \sin \alpha}{r_s \Delta J}\right)s^2 + \frac{\frac{3}{2}m \gamma \omega s \sin \alpha}{r_s \Delta J} \left[ \frac{r_s^2}{\Delta} + \frac{3}{2}s \frac{r_s}{\Delta} + \frac{3}{2}s^2 \omega s^2\right]s \]
\[ \Delta_{35} = \left( -\frac{\sqrt{3} \mu \gamma \omega_s \sin \alpha}{r_s J} \right) S^3 \]

\[ -\frac{\sqrt{3} \mu \gamma \omega_s \sin \alpha}{r_s J} \left( \frac{2\tau}{\Delta} \frac{r_s + \frac{3}{2} l_s}{\Delta} \right) S^2 \]

\[ -\frac{\sqrt{3} \mu \gamma \omega_s \sin \alpha}{r_s J} \left[ \frac{r_s + \frac{3}{2} \mu \omega_s}{\Delta} + \frac{\tau r_s}{\Delta^2} \left( \frac{r_s}{\Delta} + \frac{3}{4} l_s \omega_s \right) \right] S \]

\[ -\frac{\sqrt{3} \tau}{r_s J \Delta^2} \left( r_s^2 + \frac{3}{4} l_s^2 \omega_s^2 \right) \]

\[ \Delta_{45} = -\frac{\sqrt{3} \mu \omega_s^2 y \sin \alpha}{r_s J} \left( 1 - \frac{3}{2} \frac{l_s}{\Delta} \right) S^2 \]

\[ -\frac{\sqrt{3} \mu \gamma \omega_s \sin \alpha}{r_s J} \left[ \frac{3}{2} \frac{l_s \tau r_s \omega}{\Delta^2} + \frac{3}{4} \frac{l_s \mu \omega}{\Delta} - \frac{\tau r_s \omega}{\Delta} \right] S \]
\[ \Delta_{55} = S^4 + \left( \frac{2r_s \tau + 31}{\Delta} \right) S^3 \]

\[ + \left[ \omega_s^2 \frac{2r_s}{\Delta} + \frac{1}{\Delta^2} \left( 31_s \tau r_s + \frac{3}{4} s^2 + r_s^2 \tau r^2 \right) \right] S^2 \]

\[ + \left( \frac{31_s \omega_s^2}{\Delta} + \frac{2 \tau r_s^2 + 31_s r_s}{\Delta^2} - \frac{9 \tau^2 l_s \mu_s}{\Delta^3} \right) S \]

\[ + \left( - \frac{r_s^2 + 2 \frac{1}{4} s^2 \omega_s^2}{\Delta^2} \right) \]

... (5.37)

For frequencies above about 5 Hz, constant air gap flux density is approximately maintained if \( \gamma \) is constant. Under these conditions, \( \delta \gamma = 0 \) and \( \delta \gamma \) becomes

\[ \left[ 1 \frac{r_s^3 \gamma}{Z^3} - \frac{31_s \omega_s \gamma (r_s^2 + Z^2)}{Z^3} \right]_t \]

... (5.38)

where \( Z^2 = [r_s^2 + (\frac{3}{2} s^2 \omega_s)^2] \)

and it is assumed that the torque constant \( K_T \) is unchanging.

When the transfer function equation (5.35) is assembled from equations (5.36) and (5.38), application of the Final value Theorem shows that, provided the system is stable, the
Figure 5.3.1 Stability boundaries

5 H.P. Test motor at very small slip
steady state response to a small step change in frequency is given by

\[
\frac{\delta \hat{\delta}}{\delta \omega_s} \bigg|_{t \to \infty} = \frac{1}{n} \quad \ldots (5.39)
\]

For the postulated conditions, the result expressed by equation (5.39) is that obtained from solving equation (5.22). This provides a partial check on the correctness of the derived transfer functions.

Mention has been made several times of the condition "provided the system is stable". This is determined by the location, in the complex plane, of the poles of equation (5.35). Stability boundaries may be fixed by any test e.g. Routh, which determines the conditions under which the denominator of equation (5.35) ceases to be a Hurwitz polynomial. Figure 5.3.1 shows the stability boundaries for the 5 HP test motor when operating at very small slips and with two values of J, one including the load and the other for the motor alone.

Although the emphasis in this section has been on weighting or transfer functions connecting speed and input frequency variations, as seen from equations (5.21), (5.23) and (5.27) this is only one possible input-output pair.
For example, the analysis also covers the current or speed response to load torque variations or to supply voltage fluctuations.
5.4 Approximations for high inertia loads

In section 3.4, typical eigenvalue plots for constant speed operation indicate that "electrical transients" might be expected to decay with time constants of the order of 10 to 30 milliseconds. If the load inertia is such that negligible speed change can occur in this order of time, an approximate analysis may be made by assuming that the developed torque is that pertaining to steady state operation at that speed.

Re-writing the sinusoidal steady state torque equation (2.15)

\[ T_c = \frac{\frac{3}{2} n \mu (\omega - n^0)}{\sqrt{[r_s - \Delta \omega_s (\omega - n^0)]^2 + \left[\frac{3}{2} \frac{1}{s} \omega_s + r_s (\omega_s - n^0)\right]^2}} \]  

(5.42)

Two possible conditions for frequency change are, as previously discussed,

\[ |V_a| \text{ constant} \]

or \[ |V_a| = \gamma \omega_s \]

where \( \gamma \) is a constant.
In either case, for any given machine

\[ T_e = T_e(\omega_s, \dot{\theta}) \]

and hence

\[ \delta T_e = \frac{\partial T_e}{\partial \omega_s} \delta \omega_s + \frac{\partial T_e}{\partial \dot{\theta}} \delta \dot{\theta} \] \hspace{1cm} ...(5.43)

The partial derivatives are understood to be evaluated at the reference values of \( \omega_s \) and \( \dot{\theta} \) about which the variation in \( T_e \) is to be obtained.

Assuming that the load torque plus friction and windage losses can be accounted for by a general term

\[ T_L = K_T f(\dot{\theta}) \]

the equation of motional variation can be written

\[ \delta T_e = J \frac{d}{dt} \delta \dot{\theta} + K_T f'(\dot{\theta}) \delta \ddot{\theta} \]

i.e.

\[ \left( \frac{\partial T_e}{\partial \omega_s} \right)_0 \delta \omega_s = J \frac{d}{dt} \delta \dot{\theta} + (K_T f'(\dot{\theta}) - \frac{\partial T_e}{\partial \dot{\theta}})_0 \delta \dot{\theta} \]

where \( (\ )_0 \) imples "evaluated at the reference value".
Converting to conventional transfer function notation

$$\frac{\delta \dot{\theta}(s)}{\delta \omega_s(s)} = G(s) = \frac{\frac{1}{J} \frac{\partial T_e}{\partial \omega_s}}{s + \frac{1}{J} (K_T f'(\dot{\theta}) - \frac{\partial T_e}{\partial \dot{\theta}})} \quad \ldots(5.44)$$

On this analysis, the speed response time constant is,

$$\frac{J}{(K_T f'(\dot{\theta}) - \frac{\partial T_e}{\partial \dot{\theta}})} \quad \ldots(5.45)$$

Over the slip range from zero slip to maximum torque, $\frac{\partial T_e}{\partial \dot{\theta}}$ is negative.

Clearly, equation (5.44) is at best only one of the five modes predicted by such as equation (5.32). However, with high inertia loads it may approximate the dominant mode in the speed response to frequency change and in consequence give a simple approximation to the more involved expression.

To get some measure of the partial derivative terms, consider the case of normal working slips. Within this range and neglecting the effect of $r_s$, 
\[ T_e = \frac{\frac{3n\mu(\omega_s - n\dot{\theta})|V_a|^2}{\frac{2}{s}s^2 \omega_s^2}}{\frac{2}{s}s^2 \omega_s^2} \]

\[ T_e = \frac{2n\mu(\omega_s - n\dot{\theta})|V_a|^2}{\frac{1}{s}s^2 \omega_s^2} \]

If operation at approximately constant flux density is assumed

\[ |V_a| = \gamma \omega_s \]

and

\[ T_e = \frac{2n\mu(\omega_s - n\dot{\theta})\gamma^2}{\frac{1}{s}s^2} \]

Hence

\[ \frac{\partial T_e}{\partial \omega_s} = \frac{2n\mu\gamma^2}{\frac{1}{s}s^2} \]

\[ \frac{\partial T_e}{\partial \dot{\theta}} = \frac{-2n^2\mu\gamma^2}{\frac{1}{s}s^2} \]

Taking values for the 5 HP motor given in section 4.2.1 and letting \( \gamma = 0.6 \),

\[ \frac{\partial T_e}{\partial \dot{\theta}} = -3.2 \]
Neglecting any change in load torque and assuming motor plus load inertia $J = 0.3$, the time constant, equation (5.45) yields approximately 100 milliseconds.

Clearly, the time constant is sensitive to $\gamma$ since this appears as a squared term.
5.5 Measurement of Weighting and Transfer Functions

5.5.1 General description of the method

As well as confirming theoretical work, it may be useful on occasions (for example, with an existing installation) to determine transfer or weighting functions directly by experiment. This approach could also have advantages in avoiding difficult measurements of inertia and loss characteristics as well as determining load torque functional forms.

Several researchers in this field have sought this information using frequency response methods. However, it is demonstrated in this section that correlation methods using the statistical properties of a class of perturbing signal are well suited to the problem in that a range of results can be obtained quickly, simply and even while major plant is in operation. A further and important advantage is that the results may be obtained in the presence of high levels of noise signals which could, in this context, include normal plant operating signals.

In section 5.3 expressions were derived for the small signal transfer function relating input frequency to shaft speed. An expression was also derived for the small signal step response which is the integral of the impulse response. These two results have the same system information
content in that the Laplace transform of the step response is \( \left( \frac{1}{s} \times \text{transfer function} \right) \). Also, as was stated previously, the transfer and weighting functions form a Laplace transform pair.

The principle of the method used in this section is such as to find a weighting function description of the system between the given "input" and "output". Bearing in mind that the information sought is the "small signal" weighting function about a given reference value, the input in this case is the variation in the supply frequency \( \delta \omega \).

The fundamental of the supply frequency \( \omega \) is governed by the inverter-switching oscillator which operates, in this case, at a frequency \( 6\omega \). If the oscillator frequency is external-voltage-controlled about a reference value \( \omega_0 \) by a voltage \( v_c \), then for all practical purposes,

\[
\delta \omega = k v_c,
\]

where \( k \) is some constant.

The two weighting functions examined in this section were frequency-speed and frequency-shaft torque. The output signal in the former case was obtained from an optical tachometer, the signal average value varying linearly with speed within the set speed range. In the latter case, the
output signal was obtained from the ASEA Torsiometer which has previously been discussed.

When the frequency-perturbing signal $v_c$ is chosen to be a pseudo random binary coded signal (p.r.b.s.) the weighting function is found directly by cross correlating the input p.r.b.s. with the output signal. This cross correlation operation can be done off-line by computer from chart records or directly on-line if a suitable correlator is available.

A system weighting function is the appropriate time domain description to use with a convolution integral for finding system output in response to an arbitrary input. Should a transfer function be required for frequency domain calculations, the operation of converting a weighting function to a transfer function is most conveniently carried out by a suitable computer program. Depending on the complexity of the weighting function waveform or the number of calculations required, it may be advantageous to incorporate Fast Fourier transform methods into the program.

A simple program to convert weighting functions to transfer functions is given in Appendix 5.
5.2.2 Outline of theoretical basis for the method.

Since the theoretical basis for the method as well as practical applications have been described in the literature [33, 34, 35, 36] only a brief review of the basic principles will be given.

Consider a linear system with input signal $x(t)$ and output or response signal $y(t)$. Then,

$$y(t) = \int_{0}^{\infty} x(t-\lambda) h(\lambda) \, d\lambda \quad \ldots (5.46)$$

where $h(t)$ is the system weighting function.

The time auto correlation function for $x(t)$ is,

$$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t-\tau)dt \quad \ldots (5.47)$$

Similarly, for signals $x(t)$ and $y(t)$, the time cross correlation function is

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t+\tau)dt \quad \ldots (5.48)$$

Combining equations (5.46), (5.47) and (5.48) and simplifying leads to

$$R_{xy}(\tau) = \int_{0}^{\infty} R_{xx}(\tau-\lambda) h(\lambda) \, d\lambda \quad \ldots (5.49)$$
The idealised random signal known as white noise has an auto correlation function

\[ R_{xx}(\tau) \text{ (white noise)} = S_0 \delta(\tau) \quad \cdots (5.50) \]

where \( S_0 \) is a constant. (Actually, the constant value of the power spectral density).

Substituting this last result in equation (5.49) and simplifying leads to the result,

\[ h(\tau) = \frac{R_{xy}(\tau)}{S_0} \quad \cdots (5.51) \]

i.e. showing that if the input signal is white noise, the weighting function is proportional to the cross correlation function.

In most practical measurement situations, extraneous noise signals will be present. This problem in connection with the ASEA Torsio meter was discussed in section 4.2.3 where the wanted torque variation signals were much smaller than the added noise signals.

If an additive noise signal \( n(t) \) is injected into the system at some point \( n \) such that the weighting function connecting system input point \( n \) with the output is \( h_n(t) \)
then the cross correlation between the input signal \( x(t) \) and the new value of output \( y(t) \) is,

\[
R_{xt}(\tau) = \int_{-\infty}^{\infty} R_{xx}(\tau-\lambda)R(\lambda)d\lambda + \int_{0}^{\infty} R_{xn}(\tau-\lambda)h_{n}(\lambda)d\lambda
\]

...(5.52)

where \( R_{xn}(\tau) \) is the cross correlation relating \( x(t) \) and \( n(t) \). If \( x(t) \) and \( n(t) \) are statistically independent, then,

\[
R_{xn}(\tau) = \bar{x}\bar{n} = \text{product of mean values.}
\]

For this commonly occurring case, equation (5.52) simplifies to

\[
R_{xy}(\tau) = S_{o}h(\tau) + C
\]

...(5.53)

In addition, if either of the mean values \( \bar{x} \) or \( \bar{n} \) are zero, \( C \) in equation (5.53) is zero and equation (5.51) holds just as though the noise were not present.

From a practical point of view, white noise is not very satisfactory, principally because of the long averaging times needed to reduce statistical errors. However, the p.r.b.s. has an auto correlation function that approximates to an impulse at the origin as well as having other desirable
properties.

The p.r.b.s. can be generated rather simply using a feedback shift register and exclusive OR gates. Because of its binary nature it greatly simplifiers the cross correlation operation.

In summary, the principle advantages of the method are,

(i) The low level of perturbing signal required. A practical implication of this is that tests may be carried out while plant is in normal operation.

(ii) The relative immunity of the method to noise contaminated signals. In practice the noise signal amplitude may be many times greater than the wanted signal.

(iii) The short time involved in getting a weighting function curve. Typically for a motor and with "on-line" cross correlation, of the order of a half minute.
5.5.3 Experimental Results

5.5.3 (a) Aims and methods of experimental work

In section 5.3, theoretical expressions were developed for "small signal" transfer and weighting functions relating speed variation response to input frequency variations.

As verification of these results, the speed response to chosen input functions could have been calculated and compared with measured responses. However, as an alternative approach and to illustrate the convenience of the cross correlation method, measured weighting functions have been compared with step responses, with some reference to the high inertia load approximation and direct computation from the non-linear describing equations.

Details of the measurements made and results obtained are set out in Table 5.1.

The analytical methods developed in Chapter 5 assume balanced sinusoidal voltages applied to the stator. However, the experiments were conducted with an inverter supply having the waveform shown in Figure 4.2. For this phase voltage waveform, the nth harmonic amplitude is \(2V_{dc}/n\pi\), with even and triplen values of \(n\) excluded. Using this result and the expression for steady state torque, it
<table>
<thead>
<tr>
<th>System Input variable</th>
<th>Input Signal waveform</th>
<th>Systems</th>
<th>Output signal</th>
<th>Information obtained</th>
<th>Record</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>step function</td>
<td>motor + load</td>
<td>speed variation</td>
<td>speed response</td>
<td>Chart</td>
</tr>
<tr>
<td>frequency</td>
<td>p.r.b.s.</td>
<td>motor + load</td>
<td>speed variation</td>
<td>weighting function</td>
<td>photograph</td>
</tr>
<tr>
<td>frequency</td>
<td>p.r.b.s.</td>
<td>motor + load</td>
<td>shaft torque variation</td>
<td>weighting function</td>
<td>photograph</td>
</tr>
<tr>
<td>frequency</td>
<td>step function</td>
<td>motor only</td>
<td>speed variation</td>
<td>step response</td>
<td>chart</td>
</tr>
<tr>
<td>frequency</td>
<td>p.r.b.s.</td>
<td>motor only</td>
<td>speed variation</td>
<td>weighting function</td>
<td>photograph</td>
</tr>
</tbody>
</table>
can be shown that at least for the steady state, the average torque is essentially that due to the fundamental frequency component.

The experimental machine used was the same 5 HP, 4 pole motor described in Chapter 4. Measurements were made with a coupled load and also with the motor alone.

Speed was measured with an optical tachometer, speed variations within the set range being proportional to the mean value of the output signal. The primary speed signal had a fundamental frequency pulsation at 12 times shaft revs/second. A simple, passive L.P. filter was used to reduce the fundamental frequency component and a series buck/boost d.c. voltage was added to maintain approximately zero mean at the input to the u/v recorder and the correlator. Chart record ripple is the amplitude-reduced fundamental frequency component of the speed signal.

Since the pseudo-random test signal only approximates white noise, the weighting functions obtained are subject to a small error, particularly near the time origin. Also since the machine weighting functions are in general complicated functions (a fifth order system) lower order approximations are more easily seen and fitted to the integral of the weighting function curve, which is the step response prediction.
The frequency steps used to obtain chart records were rather large for the "small signal" linear approximation but this was necessary to get a readable chart record in relation to the signal ripple amplitude. For the correlation tests, frequency deviations were much smaller, generally in the range 1 to 2%. With these tests, the clock period (minimum time of frequency deviation from mean) was, for most tests, either 10 or 33 milliseconds. From these figures it can be seen why "on line" correlation tests have negligible effect on normal operation.

All photographic and chart records together with some plotted curves are assembled together at the end of this chapter.
5.5.3 (b) Expected and measured speed responses - "high" inertia load

From a series of retardation tests the moment of inertia of the motor plus load was estimated to be 0.33 kg m².

Figures 5.5.1 to 5.5.5 are chart records of speed variation responses to step changes in input frequency. These are for a selection of load, voltage and base frequency conditions. Owing to the use of a sign inverting amplifier, in some cases an increase in speed appears as an increase in negative voltage at the recorder. Also, the chart records displayed are limited in length but in every case the charts have been cut so that the start and finish represent steady state values.

Figures 5.5.6 to 5.5.11 are photographic records of weighting functions for a similar range of operating conditions.

Based on the analysis of section 5.4 for high inertia loads, the expected responses are exponential with time constants ranging from 80 to 300 milliseconds depending on voltage and frequency.

Reference to the eigenvalue plot of Figure 3.4.2 shows that the damping time constants of the "electrical" transients are 15 milliseconds (both pairs) for speeds above
about 140 electrical radians/second. At speeds below this one of the damping terms increases its time constant, reaching 40 ms at about 100 electrical radians/second.

In terms of these figures and depending on voltage it is to be expected that the section 5.4 approximation may not be very good at frequencies below about 25 Hz. Figures 5.5.11 and 5.5.7 (upper) bear out this prediction.

Generally it was found that for frequencies above about 25 Hz, the step response was approximately that predicted by the weighting function. Also, the time constant of the approximating exponential was in reasonably good agreement with the calculated "high inertia load" value.

As specific examples, Figure 5.5.1 and 5.5.6 are for similar operating conditions: The step response approximates an exponential rise with time constant 80 ms. The weighting function (upper photo) yields approximately 80 ms and the lower photo, more easily interpreted after integration, yields approximately 90 ms. The calculated value was 85 ms.

Similarly, Figure 5.5.5 indicates a time constant of 400 ms, compared with Figure 5.5.10 (upper) at approximately 300 ms. The calculated value was 330 ms.
Figure 5.5.10 lower shows in addition a lightly damped periodic component with period near 45 m sec.
5.5.3 (c) Measured shaft torque responses

Using the output of the coupled torque-measuring torsiometer, weighting functions were obtained connecting input frequency variations with shaft torque variations. Some typical results are shown in Figures 5.5.12 and 5.5.13. In the latter figure, where mean load torque was much higher, very clear cut waveforms show, in both photographs, a lightly damped, 20 millisecond period oscillation.

As the weighting functions displayed in Figure 5.5.13 were unexpected results, program 5, based on the solution principles outlined in section 3.2, was written to solve the non-linear equations set out in equation (3.5) and with inverter supply waveforms. The required initial steady state values of currents and speed were obtained from program 1. The resulting solutions of torque variations following a frequency step (both up and down) are shown in Figures 5.5.26 to 5.5.29. In no case is there any suggestion of oscillation in the torque. (The characteristic torque ripple is at very much higher frequencies and has been "smoothed out" of the graphs).

It seems highly likely that the explanation of the apparent discrepancy is that in the motor-load combination, no account was taken of the second order mechanical system
resulting from the load moment of inertia $J_L$ on the end of a shaft of length $L$ and torsional compliance $K_S$. In the experimental arrangement, owing to the integral torsimeter, the effective shaft length $L$ was approximately 0.75 metres including couplings.

Because of the three different shaft sizes plus the flexible couplings involved, an accurate solution would be difficult. However, by making a number of simplifying approximations, a natural period of 40 m secs was calculated for the shaft-load inertia system and this is in the area of that indicated by the weighting function.

It was observed that the speed signal during and following a frequency step frequently showed a small modulation at a period near 20 m secs. This effect, quite noticeable in Figure 5.5.5, is to be expected with the torque oscillation observed.

An important aspect of the measured weighting function is that it reflects the true system, i.e. it measures (in this case) the weighting function connecting frequency variations and shaft torque variations. If the shaft torque is influenced by the interactions between the mechanical load and the connecting shaft, then this is reflected in the measurement.

A listing of program 5 is included in Appendix 5.
5.5.3 (d) **Measured speed response - motor only.**

For this series of tests the load was uncoupled. With various voltage and frequency conditions, both weighting functions and step responses were recorded. Also, by way of illustrating an alternative approach, the information of the weighting function in Figure 5.5.16 (lower) was used to generate the corresponding transfer function and the plot of this is shown in Figure 5.5.18.

The photographs in Figures 5.5.14 to 5.5.17 are weighting functions connecting speed and frequency for the motor only. Figures 5.5.19 to 5.5.25 are chart records of speed responses to step change in input frequency. It should be noted that amplifier sign inversion causes increased negative voltage for increase in speed.

Generally, the step response record confirms the weighting function information. For example, the weighting function of Figure 5.5.14 predicts a step response with a small overshoot and (after integration) a 100% rise time of approximately 80 m secs. This expectation is confirmed by Figure 5.5.20 which is the step response at the same voltage and frequency.

Similarly, Figure 5.5.17, upper, weighting function predicts a lightly damped oscillatory step response with
period 135 m secs. This is confirmed by Figure 5.5.25 which is the record of a step response at the same voltage and frequency.

Figure 5.5.16, upper, also predicts an oscillatory step response, this time with period 100 m secs. Step response Figure 5.5.23 again agrees closely with this prediction.

As has been mentioned previously, the weighting function and transfer function are related as a transform pair. To illustrate how the transfer function may be found from the weighting function, a simple program was written to achieve this. Figure 5.5.18 displays the modulus and argument information for a transfer function corresponding to the weighting function of Figure 5.5.16 (lower).

The modulus plot of Figure 5.5.18 shows a break frequency at approximately 65 radians/second and a slope of -27 db per octave. The argument is approximately 184° at the break frequency and the modulus peak is 6.5 db. All of this information suggests a close approximation to a 4th order system and accordingly the transfer function is approximately,

\[ F(s) = \frac{65^4}{(s^2 + 45.5s + 65^2)^2} \]
The degree of damping and the oscillation frequency are similar to that expected from the weighting function. However, the fact that the system is approximately 4th order is not so easily detected from Figure 5.5.16.

A listing of the program used is given in Appendix 5 as program 4.
Figure 5.5.1 Speed response to frequency step.

\[ J = 0.33 \text{ kg m}^2 \]
\[ \text{Chart speed 50 cm/sec.} \quad T_1 = 11 \text{ N.M.} \]
\[ V_{dc} = 240 \text{ v.} \quad f_s = 25 \text{ to } 26.5 \text{ Hz.} \]
Figure 5.5.2 Speed response to frequency step.

\[ J = 0.33 \text{ kg m}^2 \]  
Chart speed 50 cm/sec. \( T_1 = 11 \text{ N.M.} \)

\[ V_{dc} = 220 \text{ v.} \]  
\( f_s = 25 \text{ to } 27 \text{ Hz.} \)
Figure 5.5.3  Speed response to frequency step.

$J = 0.33 \text{ kg m}^2$. Chart speed 25 cm/sec.  $T_1 = 9 \text{ N.M.}$

$V_{dc} = 220 \text{ v.}$.  $f_s = 42 \text{ to } 45 \text{ Hz.}$
Figure 5.5.4  Speed response to frequency step.

\[ J = 0.33 \text{ kg m}^2 \]  \quad \text{Chart speed 25 cm/sec.}  \quad \text{\( T_1 = 9 \) N.M.}

\[ V_{dc} = 200 \text{ v.} \quad \text{\( f_s = 35 \) to 38 Hz.} \]
Figure 5.5.5  Speed response to frequency step.

$J=0.33 \text{ kg m}^2$.  Chart speed 25 cm/sec.  $T_1 = 9 \text{ N.M.}$

$V_{dc} = 242 \text{ V}$.  $f_s = 50 \text{ to } 53.5 \text{ Hz.}$
Figure 5.5.6 Frequency-speed weighting function. $J_f = 33 \text{ kg m}^2$.

Upper: $V_{dc} = 240 \text{ v}$, $f_s = 25 \text{ Hz}$, $T_1 = 17 \text{ N.M.}$, 1 div. = 100 msecs.

Lower: $V_{dc} = 240 \text{ v}$, $f_s = 25 \text{ Hz}$, $T_1 = 17 \text{ N.M.}$, 1 div. = 33 msecs.
Figure 5.5.7 Frequency-speed weighting function. $J = 33 \text{ kg m}^2$.

Upper: $V_{dc} = 240 \text{ v}, f_s = 25 \text{ Hz}, T = 10 \text{ N.M.}, 1 \text{ div.} = 30 \text{ msecs.}$

Lower: $V_{dc} = 200 \text{ v}, f_s = 25 \text{ Hz}, T = 10 \text{ M.M.}, 1 \text{ div.} = 30 \text{ msecs.}$
Figure 5.5.8 Frequency-speed weighting function. $J=33 \text{ kg m}^2$.

Upper: $V_{dc}=240 \text{ v}, f_s=33 \text{ Hz}, T_1=9 \text{ N.M.}, 1 \text{ div.}=100 \text{ msecs.}$

Lower: $V_{dc}=200 \text{ v}, f_s=35 \text{ Hz}, T_1=11 \text{ N.M.}, 1 \text{ div.}=33 \text{ msecs.}$
Figure 5.5.9 Frequency-speed weighting function. $J = 0.33 \text{ kg m}^2$.

Upper: $V_{dc} = 220 \text{ V}, f_s = 41 \text{ Hz}, T_1 = 3 \text{ N.M., 1 div. = 100 msecs.}$

Lower: $V_{dc} = 250 \text{ V}, f_s = 46 \text{ Hz}, T_1 = 5 \text{ N.M., 1 div. = 100 msecs.}$
Figure 5.5.10 Frequency—speed weighting function. J = 33 kg m².

Upper: \( V_{dc} = 250 \text{ v}, \ f_s = 50 \text{ Hz}, \ T_1 = 2.5 \text{ N.M.}, \ 1 \text{ div.} = 100 \text{ msecs.} \)

Lower: \( V_{dc} = 240 \text{ v}, \ f_s = 50 \text{ Hz}, \ T_1 = 10 \text{ N.M.}, \ 1 \text{ div.} = 33 \text{ msecs.} \)
Figure 5.5.11 Frequency-speed weighting function. $J = 0.33 \text{ kg m}^2$.

Upper: $V_{dc} = 220 \text{ V}, f_s = 20 \text{ Hz}, T_1 = 2.5 \text{ N.M.}, 1 \text{ div.} = 30 \text{ msecs.}$

Lower: $V_{dc} = 200 \text{ V}, f_s = 20 \text{ Hz}, T_1 = 3 \text{ N.M.}, 1 \text{ div.} = 30 \text{ msecs.}$
Figure 5.5.12 Frequency-shaft torque weighting function.

\[ J = 0.33 \text{ kg m}^2 \]

Upper: \[ V_{dc} = 240 \text{ v}, f_s = 40 \text{ Hz}, T_1 = 2.5 \text{ N.M.}, \text{ 1 div.} = 30 \text{ msecs.} \]

Lower: \[ V_{dc} = 240 \text{ v}, f_s = 25 \text{ Hz}, T_1 = 2.5 \text{ N.M.}, \text{ 1 div.} = 24 \text{ msecs.} \]
Figure 5.5.13 Frequency-shaft torque weighting function.

\[ J = 3.3 \text{ kg m}^2 \]

Upper: \( V_{dc} = 240 \text{ v}, \ f = 50 \text{ Hz}, \ T_1 = 13 \text{ N.M.}, \ 1 \text{ div.} = 20 \text{ msecs.} \)

Lower: \( V_{dc} = 240 \text{ v}, \ f = 30 \text{ Hz}, \ T_1 = 10 \text{ N.M.}, \ 1 \text{ div.} = 10 \text{ msecs.} \)
Figure 5.5.14 Frequency-speed weighting function. Kotor only.

Upper: \( V_{dc} = 240 \, \text{V}, \, f_s = 55 \, \text{Hz}, \, 1 \, \text{div.} = 33 \, \text{msecs.} \)

Lower: \( V_{dc} = 240 \, \text{V}, \, f_s = 40 \, \text{Hz}, \, 1 \, \text{div.} = 33 \, \text{msecs.} \)
Figure 5.5.15  Frequency-speed weighting functions. Motor only.

Upper: $V_{dc} = 240 \text{ v}, f_s = 35 \text{ Hz}, \text{ 1 div.} = 33 \text{ msecs.}$

Lower: $V_{dc} = 160 \text{ v}, f_s = 35 \text{ Hz}, \text{ 1 div.} = 33 \text{ msecs.}$
Figure 5.5.16 Frequency-speed weighting functions. Motor only.

Upper: \( V_{dc} = 240 \text{ v} \), \( f_s = 25 \text{ Hz} \), 1 div. = 33 msecs.

Lower: \( V_{dc} = 100 \text{ v} \), \( f_s = 25 \text{ Hz} \), 1 div. = 33 msecs.
Figure 5.5.17 Frequency-speed weighting functions. Motor only.

Upper: $V_{dc} = 100 \, \text{v}$, $f_s = 20 \, \text{Hz}$, 1 div. = 33 msecs.

Lower: $V_{dc} = 120 \, \text{v}$, $f_s = 10 \, \text{Hz}$, 1 div. = 33 msecs.
Figure 5.18: Transfer function data calculated from Figure 5.5, 16 degrees.
Figure 5.5.19  Speed response to frequency step. Motor only

\[ V_{dc} = 240 \, \text{v}, \quad f_s = 55 \text{ to } 53.5 \, \text{Hz}, \quad \text{Chart speed} = 25 \, \text{cm/sec}. \]
Figure 5.5.20 Speed response to frequency step. Motor only

\[ V_{dc} = 240 \text{ v}, \quad f_s = 40 \text{ to } 38.5 \text{ Hz}, \quad \text{Chart speed}=25 \text{ cm/sec.} \]
Figure 5.5.21  Speed response to frequency step. Motor only

\[ V_{dc} = 240 \text{ v}, \quad f_s = 35 \text{ to } 33.5 \text{ Hz}, \quad \text{Chart speed} = 25 \text{ cm/sec}. \]
Figure 5.5.22  Speed response to frequency step. Motor only

$V_{dc} = 160 \text{v}$,  $f_s = 33 \text{ to } 33.5 \text{ Hz}$,  Chart speed = 25 cm/sec.
Figure 5.5.23  Speed response to frequency step. Motor only

\[ V_{dc} = 240 \text{ v}, \quad f_s = 25 \text{ to } 23.5 \text{ Hz}, \quad \text{Chart speed} = 25 \text{ cm/sec}. \]
Figure 5.5.24  Speed response to frequency step. Motor only

\[ V_{dc} = 160 \text{ v}, \quad f_s = 25 \text{ to } 23.5 \text{ Hz}, \quad \text{Chart speed}=25 \text{ c/sec.} \]
Figure 5.5.25  Speed response to frequency step. Motor only

$V_{dc} = 160\, \text{v}$, $f_s = 20 \text{ to } 18.5\, \text{Hz}$, Chart speed = 25 cm/sec.
Figure 5.5.26

$J = 0.33 \text{ kgm}^2$
Computed torque response

$V_{dc} = 240 \text{ V} \quad f_s = 40 \text{ to } 38.7 \text{ Hz}$
Figure S.5.27 Computed torque response

\[ J = 0.33 \text{ kgm}^2 \quad V_{dc} = 240 \text{ V} \quad f_s = 40 \text{ to } 41.7 \text{ Hz} \]
Figure 5.5.28  Computed torque response

$J = 0.33 \text{ kgm}^2$  $V_{dc} = 240\text{ V}$  $f_s = 50\text{ to } 46.3\text{ Hz}$
Figure 5.5.29

\[ J = 0.33 \text{ kgm}^2 \]
Computed torque response

$V_{dc} = 240 \text{ V}$

$f_s = 50 \text{ to } 53.2 \text{ Hz}$
CHAPTER 6  CONCLUSIONS

Broadly, the research objectives were

(a) To establish, using time domain matrix methods, analytical and computational procedures for determining motor current and torque waveforms when the applied voltage waveforms are,
   (i) discontinuous but piece-wise constant, or
   (ii) discontinuous portions of sine waves.

(b) To formulate "small signal" linear equations and from them derive input-output relationships for the motor-load system. The principal input-output pair to be frequency/speed variations formulated in terms of weighting functions or transfer functions.

(c) To investigate the feasibility and accuracy of an "on line" method of measuring motor-load system weighting functions using a low energy, wide band perturbing signal of such nature that normal operation of the plant may be maintained during tests.

These objectives have been met in that the analytical methods sought have been developed and for most cases, satisfactorily substantiated, while the proposed experimental method of motor-load system weighting function determination has been shown to be practicable and verified as giving reasonably good prediction of the system response under
varying operating conditions.

6.1 Discussion on the analytical methods for current and torque waveforms

Analytical methods have been developed for determining current and torque waveforms when the stator applied voltages are either piece-wise constant or portions of sine waves, these being the two most common types of waveform encountered with switched mode operation of induction motors.

In principle the method yields exact solutions for constant speed operation and hence provides steady-state current and torque waveform details. In addition and because of the transition matrix properties, the method can also provide information on responses to certain types of aperiodic phenomena. For example, under conditions of motor plus load inertia and transient time interval such that changes in shaft speed are negligible or occur slowly, current and torque waveforms following a voltage disturbance are obtained in precisely the same manner as are the steady state waveforms.

A further use of these equations is in providing exact initial conditions (i.e. the value of all currents at a given time instant) for motor-load simulations to observe the response to such as step changes in supply frequency.
Program 5 and related Figures 5.5.26 to 5.5.29 are examples of such computations requiring correct initial conditions.

In deriving a closed form solution for the transition matrix, expressions were obtained for the eigenvalues in terms of machine parameters and rotor speed. This provides additional information regarding the electrical characteristics of the machine and may be used for example, as in section 5.4 in estimating limits for the "high inertia load" approximation. Knowledge of the eigenvalues can also be used to anticipate the structure of the transition matrix, as is shown in section 3.3.

For the three phase induction motor, current and torque waveforms predicted by the computed solutions of the motor model are in good agreement with the recorded waveforms for the same operating conditions. Because of high noise levels in the torque measuring device output signal, a direct comparison of predicted and measured torque waveform was not possible. However, using cross correlation it was confirmed that the fundamental component of the predicted torque ripple was present in the measured signal.

The agreement between predicted and measured current waveforms for the quasi two-phase motor, while reasonable was not as good as for the 3 phase motor. The computer program to evaluate current was verified and shown to be
accurate by computing a response of known form. The discrepancy was expected and arises because the experimental machine only approximates the model used.

An experimental machine to more nearly match the model or alternatively a model more accurately representing the experimental machine used would both have been possible but at the cost of considerably increased complexity in machine switching circuits or in the model.

6.2 Discussion on transfer and weighting functions

Using a "small signal" linearised set of equations, transfer function and integral of weighting function relations were obtained connecting variations in input frequency and speed. Even when operation was restricted to very small slips, the derived transfer function, which describes a 5th order system, tends to be cumbersome, particularly as the dominant modes vary with total moment of inertia, supply frequency and applied voltage.

For the case where the motor plus load inertia is "high" a first order approximation to the transfer function connecting frequency and speed variations was derived. This also is restricted to small slips but "small" in this case includes up to at least full load slip. Comparison of the "high inertia" transfer function approximation with recorded
speed responses to step inputs in frequency showed that the approximation was quite good if the "mechanical" time constant of the 1st order approximation was at least five times the largest "electrical" time constant. By electrical time constant is meant the time constant associated with a damped sinusoidal term, the values being obtained from the eigenvalue plot, Figure 3.4.2 for the speed under consideration.

Weighting functions connecting speed and input frequency variations were in good agreement with the predicted and measured step response for "high inertia" loads. This comparison was based on the indicated time constant.

Similarly for the "motor only" inertia, the weighting function information appears to agree well with the measured step response. Although the step response is the integral of the weighting function, useful direct comparisons can usually be made without carrying out the integration. Over the range of the tests, the weighting function predictions of step response rise time, oscillation frequency and damping are well supported by the recorded step responses.

Generally the weighting functions obtained have been shown to identify the motor-load system quite well as regards "small signal" behaviour. The information may be obtained quickly and at such low levels as to leave the
An important aspect of this identification measurement is that it obtains information on the actual system and not on an approximating model. This feature is brought out in connection with the shaft torque-input frequency weighting functions obtained for a high inertia load. The discrepancy found between the computed system step response and the measured shaft torque weighting function (apparently due to an additional 2nd order mechanical system of shaft plus load inertia) emphasises that the weighting function measurements record the actual system.

If the system description is required as a transfer function, this may be derived from the weighting function. A simple computer program to effect this, together with an example has been given to illustrate one possible method of conversion.

For any given motor and connected load inertia, stability boundaries can be determined to delineate the regions of stable operation. This information for the experimental motor operating at very small slip has been presented in Figure 5.3.1.
BIBLIOGRAPHY


Figure A1.1  Winding axes
a and b - stator winding axes
α and β - rotor winding axes
APPENDIX 1  TRANSFORMATIONS

A1.1 Rotating real type transformations for arbitrary winding displacements

Consider the case of a two pole machine having two windings on the stator and two on the rotor, the former having their axes displaced by $\psi_1^\circ$ and the latter by $\psi_2^\circ$. Coil axes configurations are indicated in Fig. A1.1.

Using the notation of equation (2.1) but writing stator and rotor equations separately,

\[ V^S = R^S I^S + pL^SS I^S - p\{L(\theta) I^R\} \quad \ldots (A1.1) \]

where

\[ L^SS = I_s \begin{bmatrix} 1 & \cos\psi_1 \\ \cos\psi_1 & 1 \end{bmatrix} \]

\[ L(\theta) = m \begin{bmatrix} \cos\theta & \cos(\psi_2 + \theta) \\ \cos(\psi_1 - \theta) & \cos(\psi_2 + \theta - \psi_1) \end{bmatrix} \]

\[ R^S = r_s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Similarly,

\[ V^r = R^r I^r + pL^rr I^r - p\{L(\theta) I^S\} \quad \ldots (A1.2) \]

where

\[ L^rr = I_r \begin{bmatrix} 1 & \cos\psi_2 \\ \cos\psi_2 & 1 \end{bmatrix} \]

\[ R^r = r_r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
Figure A1.2  Non-orthogonal basis vectors.
Derivation of rotation transformation
When the basis vectors for a 2-dimensional space are not orthogonal, the transformation to rotate them may be determined from consideration of Figure A1.2. This leads to the result,

\[
\begin{bmatrix}
  x_2 \\
  y_2
\end{bmatrix} = 
\begin{pmatrix}
  H_1 & H_2 \\
  \tilde{H}_1 & \tilde{H}_2
\end{pmatrix}
\begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix} = 
\begin{pmatrix}
  \tilde{H}_1 & \tilde{H}_1^{-1}
\end{pmatrix}
\begin{bmatrix}
  x_1 \\
  y_1
\end{bmatrix}
\]

where

\[
\tilde{H}_1 = \begin{bmatrix}
  1 & \cos \psi \\
  \cos \psi & 1
\end{bmatrix}
\]

\[\tilde{H}_2 = \begin{bmatrix}
  \cos \theta & \cos(\psi + \theta) \\
  \cos(\psi - \theta) & \cos \theta
\end{bmatrix}
\]  

Define also

\[
\tilde{H}_v = \tilde{H}_1 \tilde{H}_2^{-1}
\]

**Case 1**

As a first case, let \( \psi_1 = 90^\circ \), \( \psi_2 = \psi \) and introduce transformations such that,

\[
I^r = \tilde{H}_1 I \\
Y^r = \tilde{H}_v Y
\]

In the stator equations (A1.1)

\[
L(\theta) \tilde{H}_1 = \begin{bmatrix}
  1 & \cos \psi \\
  0 & \sin \psi
\end{bmatrix}
\]
Similarly, in the rotor equations,

\[ H_v^{-1} p \{ L(\theta) I^s \} = H_v^{-1} [(p L(\theta)_t I^s + L(\theta)_t (p I^s))] \]

and

\[ H_v^{-1} (p L(\theta)_t) = m \dot{\theta} \begin{bmatrix} 0 & 1 \\ \sin \psi & \cos \psi \end{bmatrix} \]

\[ H_v^{-1} L(\theta)_t = m \begin{bmatrix} 1 & 0 \\ \cos \psi & \sin \psi \end{bmatrix} \]

\[ H_v^{-1} p \{ L_{rr} I^s \} = H_v^{-1} L_{rr} p \{ H_i I' \} \]

\[ = l_r \begin{bmatrix} 1 & \cos \psi \\ \cos \psi & 1 \end{bmatrix} + l_r \dot{\theta} \begin{bmatrix} 0 & \sin \psi \\ -\sin \psi & 0 \end{bmatrix} \]

However, the remaining term

\[ H_v^{-1} R_i = R_r H_v^{-1} \]

\[ = \frac{r_r}{\sin^2 \psi} \begin{bmatrix} 1 - \frac{1}{2} \cos 2\theta - \frac{1}{2} \cos(\theta + 2 \psi) & 2 \cos \psi \sin^2 \theta \\ 2 \cos \psi \sin^2 \theta & 1 - \frac{1}{2} \cos 2\theta - \frac{1}{2} \cos(\theta + 2 \psi) \end{bmatrix} \]

which is \( \theta \) and hence time dependent unless \( \psi = 90^\circ \) or \( r_r = 0 \).

Case 2

In this case, let \( \psi_2 = \psi = 90^\circ \) and \( \psi_1 \) be any arbitrary angle greater than zero.
Then
\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]
define
\[
\pi = \frac{\mathbf{r}_x}{m} \cdot \pi_1 \pi_2
\]
and
\[
\pi' = \pi \mathbf{r}, \quad \nu' = \pi \nu, \quad \pi = \pi_1 \pi_2
\]
\[
\pi' = \pi \mathbf{r} \quad \nu' = \pi \nu
\]
or
\[
\mathbf{r} = \pi^{-1} \pi', \quad \nu = \pi^{-1} \nu
\]
\[
\Rightarrow \text{equation (A1.5)}
\]

When this substitution is effected in equation (A1.1)
\[
\mathbf{L}(\theta) \pi^{-1} = \mu \begin{bmatrix} 1 & 0 \\ \cos \psi_1 & \sin \psi_1 \end{bmatrix}
\]
where
\[
\mu = \frac{m^2}{R^2}
\]
Similarly when \(\pi\) is substituted for in the rotor equations and then they are pre-multiplied by \(\pi\)
\[
\pi \mathbf{L}(\theta) \pi = \pi (p\mathbf{L}(\theta) \pi) + \pi \mathbf{L}(\theta) (p\pi)
\]
\[
= \mathbf{r} \pi \left[ \begin{array}{cccc}
0 & \sin \psi_1 \\
-1 & -\cos \psi_1
\end{array} \right] \pi \mathbf{r} + \frac{\mathbf{r}}{m} \left[ \begin{array}{cccc}
1 & \cos \psi_1 \\
0 & \sin \psi_1
\end{array} \right] \pi \mathbf{r}^S
\]
\[
\Rightarrow \text{equation (A1.7)}
\]
\[ L_{rr}^{-1} I' = L_{rr}^{-1} (p_{rr}) I' + L_{rr}^{-1} (p_{rr}) I' \]

\[ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} I' + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} p_{rr} I' \]

Since \( V' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \), the transformed equations can then be written

\[ V^S = R^S I^S + pL^S I^S - \mu \begin{bmatrix} 1 & 0 \\ \cos \psi_1 & \sin \psi_1 \end{bmatrix} [p_{rr}] \] ....(A1.8)

\[ [0] = \begin{bmatrix} 1 & \dot{\theta}_{rr} \\ -\dot{\theta}_{rr} & 1 \end{bmatrix} [I'] + \begin{bmatrix} \tau_r & 0 \\ 0 & \tau_r \end{bmatrix} [p_{rr}] \]

\[ -\dot{\theta} \begin{bmatrix} 0 & \sin \psi_1 \\ -1 & -\cos \psi_1 \end{bmatrix} [I^S] - \begin{bmatrix} 1 & \cos \psi_1 \\ 0 & \sin \psi_1 \end{bmatrix} [p_{rr}^S] \] ....(A1.9)

**Case 3**

In the motor considered in section 3.5 the winding configuration is that of case 2 with the additional feature that the two stator windings are dissimilar. For this condition, equations (A1.1) and (A1.2) are modified i.e.

\[ \psi_2 = 90^\circ \quad \text{and} \]

\[ L^{ss} = \begin{bmatrix} l_{s1} & m_{12} \cos \psi \\ m_{12} \cos \psi & l_{s2} \end{bmatrix} \] ....(A1.10)
Define

\[
\begin{pmatrix}
\frac{m_1}{r} \cos \theta & \frac{m_2}{r} \sin \theta \\
\frac{-m_1}{r} \sin \theta & \frac{m_2}{r} \cos \theta
\end{pmatrix}
\]

and let

\[
I_r^r = \lll_3 I'
\]

\[
V_r^r = \lll_3 V'
\]
Carrying out the same operations that lead to equations (A1.6) to (A1.9) one obtains,

\[ v^s = \begin{bmatrix} r_{s1} & 0 \\ 0 & r_{s2} \end{bmatrix} [I^s] + \begin{bmatrix} 1_{s1} & m_{12}\cos\psi \\ m_{12}\cos\psi & 1_{s2} \end{bmatrix} p^I_s \]

\[ - \begin{bmatrix} \mu_1 & 0 \\ \mu_2\cos\psi & \mu_2\sin\psi \end{bmatrix} p^I_\dot{s} \]

\[ [0] = \begin{bmatrix} 1 & q\tau_r \dot{\theta} \\ - \frac{1}{q}\tau_r \dot{\theta} & 1 \end{bmatrix} [I^s] + \begin{bmatrix} \tau_r & 0 \\ 0 & \tau_r \end{bmatrix} [I^s] - \dot{\theta} \begin{bmatrix} 0 & q\sin\psi \\ -\frac{1}{q} & -\cos\psi \end{bmatrix} [I^s] - \begin{bmatrix} 1 & q\cos\psi \\ 0 & \sin\psi \end{bmatrix} [p^I_s] \]

where

\[ \mu_1 = \frac{m_1^2}{r^2} \]

\[ \mu_2 = \frac{m_2^2}{r^2} \]

\[ q = \sqrt{\frac{\mu_2}{\mu_1}} \]
APPENDIX 2  MOTOR PARAMETERS FROM TEST

A2.1 Three phase, cage rotor induction motor

From equations (2.10) when \( n\dot{\theta} = \omega_s \) and balanced, three phase sinusoidal voltages are applied,

\[
\overline{I}_a = \frac{\overline{V}_a}{Z_s}
\]

or \( Z_s = r_s + j\frac{3}{2} \omega_s l_s = \frac{\overline{V}_a}{\overline{I}_a}, \quad \sigma = 0 \quad \ldots (A2.1) \)

The model as derived does not provide for iron losses but these will appear in input measurements at \( \sigma = 0 \).

Stator resistance \( r_s \) may be estimated from d.c. resistance measurements and since this is normally very small compared to the iron loss equivalent resistor \( R_i \), it is reasonable to assume that at \( \sigma = 0 \), the motor load per phase is effectively \( R_i \) in parallel with \( \frac{3}{2} l_s \).

On this basis, from measured power and volt-amps at \( \sigma = 0 \), the power factor \( \cos \theta \) is determined and

\[
\frac{3}{2} \omega_s l_s = \frac{\text{(volts/phase)}}{\text{(amps/phase) \sin \theta}} \quad \ldots (A2.2)
\]

Thus \( r_s \) and \( l_s \) are known.

When \( \dot{\theta} = 0, \sigma = \omega_s \) and from equations (2.10),

\[
\frac{\overline{V}_a}{\overline{I}_a} - \frac{\overline{Z}_s}{\overline{Z}_s} = \frac{\frac{3}{2} \mu \omega_s^2}{1+j \omega_s \tau_r} \quad \dot{\theta} = 0 \quad \ldots (A2.3)
\]
In this measurement at $\dot{\theta} = 0$, the influence of the iron losses may be neglected and from the measured power and volt-amps input one may obtain,

$$\bar{Z}_{bl} = \frac{\bar{V}_a/\bar{I}_a}{\bar{I}_a} \bigg|_{\dot{\theta} = 0}$$

Substituted into equation (A2.3), this yields

$$1 + j\omega_s \tau_r = \frac{\frac{3}{2}\mu \omega_s^2}{\bar{Z}_{bl} - \bar{Z}_s}$$

or

$$\frac{1}{\mu} + j\omega_s \frac{\tau_r}{\mu} = \frac{\frac{3}{2}\omega_s^2}{\bar{Z}_{bl} - \bar{Z}_s} \quad \ldots (A2.4)$$

Since $\omega_s$, $\bar{Z}_{bl}$ and $\bar{Z}_s$ are all known, equation (A2.4) may be solved for $\mu$ and $\tau_r$.

As would be expected, the parameters do, in general, vary with applied voltage (flux densities in the iron) and frequency. For any given motor, such variations may be estimated by carrying out the above tests for a range of voltages and frequencies.

A2.2 Quasi two phase motor

For this machine, there are seven parameters to find. Two, the stator resistances, may, as before, be estimated from d.c. resistance measurements.

For the case where the two stator windings are displaced
by 90° electrical ($\Phi = 90°$) the following test procedure may be used.

Let $n\dot{\Phi} = \omega_s$

Apply sinusoidal voltages to the two stator windings such that the phase displacement between the two currents is 90° and the magnitudes adjusted so that the nett power inputs (gross less winding $I^2r_s$ losses) are equal. Under these conditions, the air gap flux density is approximately balanced and the rotor winding currents approach zero. Then, from equations (A2.1, A2.2) and in a similar manner to section A2.1, $I_{s1}$ and $I_{s2}$ can be obtained.

Hence

$$Z_{s1} = r_{s1} + j\omega_s I_{s1}$$

and

$$Z_{s2} = r_{s2} + j\omega_s I_{s2}$$

...(A2.5)

At $\dot{\Phi} = 0$, the two stator phases are not coupled and the method of section A2.1 can be used to find $Z_{b11}$ and $Z_{b12}$. Then, as before it is seen that,

$$\frac{1}{\mu_1} + j\omega_s \frac{\tau_r}{\mu_1} = \frac{\omega_s^2}{Z_{b11} - Z_{s1}}$$

...(A2.6a)

$$\frac{1}{\mu_2} + j\omega_s \frac{\tau_r}{\mu_2} = \frac{\omega_s^2}{Z_{b12} - Z_{s2}}$$

...(A2.6b)

Thus from equations (A2.6), $\mu_1$, $\mu_2$ and $\tau_r$ can be found.
APPENDIX 3  TORQUE

A3.1 Sinusoidal steady state torque

From equation (2.8) the developed torque expression is

$$T_e = -n\mu\left[\frac{3}{2} i_x + \sqrt{3} i_y + \sqrt{3} i_y \right] \quad \text{...(A3.1)}$$

Under the balanced 3 phase conditions assumed,

$$i_a = I_1 \sin \omega_s t$$
$$i_b = I_1 \sin (\omega_s t - 120^\circ)$$
$$i_x = I_2 \sin (\omega_s t + \phi)$$
$$i_y = I_2 \sin (\omega_s t + \phi - 90^\circ) \quad \text{...(A3.2)}$$

When the equations of (A3.2) are substituted into equation (A3.1), the sinusoidal steady state torque is

$$T_e (\text{sinus.}) = -n\mu\frac{3}{2} I_1 I_2 \cos \phi$$

$$= -n\mu 3I_1 (\text{r.m.s.})I_2 (\text{r.m.s.}) \cos \phi \quad \text{...(A3.3)}$$

where $\phi$ is the phase displacement between $i_a$ and $i_x$.

When expressed in phasor notation, such a relation may be written

$$I_1 (\text{r.m.s.})I_2 (\text{r.m.s.}) \cos \phi = \text{Re}\{I_a I_x^*\}$$

Where Re means "real part of" and the asterisk denotes
"complex conjugate of". Hence, under sinusoidal steady state conditions, the torque is given by

\[ T_e(\text{sinus}) = -3n \mu \text{Re}\{ \bar{T}_a \bar{T}_x^* \} \]

...(A3.4)
APPENDIX 4 EVALUATION OF SYSTEM EIGENVALUES

In Chapter 3, the normal form motor equation (3.3a) is written,

\[ \dot{\tilde{z}} = A\tilde{z} + Bv \]

It is required to find the eigenvalues of matrix \( \tilde{A} \) i.e. the roots of the polynomial

\[ \left| \tilde{A} - \lambda \tilde{z} \right| = 0 \]

...(A4.1)

where \( \tilde{A} \) is described in equation (3.3c).

However, a simpler operation results if (A4.1) is expanded using equation (3.2).

\[ \left| \tilde{A} - \lambda \tilde{z} \right| = \left| -\tilde{P}^{-1}(\tilde{G} - \lambda \tilde{z}) \right| \]

\[ = \left| -\tilde{P}^{-1}(\tilde{G} + \lambda \tilde{P}) \right| \]

\[ = \left| -\tilde{P}^{-1} \right| \left| (\tilde{G} + \lambda \tilde{P}) \right| = 0 \]

i.e. \( p(\lambda) = \left| \tilde{G} + \lambda \tilde{P} \right| = 0 \)

...(A4.2)
and

\[
\begin{pmatrix}
(r_s + \lambda \frac{3}{2} l_s) & 0 & 0 & -\mu \lambda \\
0 & (r_s + \lambda \frac{3}{2} l_s) & \sqrt{3} \mu \lambda & \frac{1}{2} \mu \lambda \\
(-\frac{3}{2} n \dot{\theta} + \lambda \sqrt{3}) & \sqrt{3} \lambda & (1 + \lambda \tau_r) & \tau_r n \dot{\theta} \\
(-\sqrt{3} n \dot{\theta} - \frac{3}{2} \lambda) & -\sqrt{3} n \dot{\theta} & -\tau_r n \dot{\theta} & (1 + \lambda \tau_r)
\end{pmatrix}
\]

\[G + \lambda P = \text{...}(A4.3)\]

The presence of the zeros in this matrix makes the determinant much easier to evaluate than would be the case with equation (A4.1).

Expansion of equation (A4.2) yields the characteristic polynomial,

\[
p(\lambda) = \lambda^4 + \lambda^3 \left(\frac{2r_s \tau_r}{\Delta} + \frac{3 l_s}{\Delta}\right) + \lambda^2 \left(n^2 \dot{\theta}^2 + \left(\frac{r_s \tau_r}{\Delta} + \frac{3 l_s}{\Delta}\right)^2 + \frac{2r_s}{\Delta}\right) + \lambda \left(n^2 \dot{\theta}^2 \left(\frac{2r_s \tau_r}{\Delta} + \frac{2r_s}{\Delta} \left(\frac{3 l_s}{\Delta} + \frac{r_s \tau_r}{\Delta}\right)\right)\right) + n^2 \dot{\theta}^2 \left(\frac{r_s \tau_r}{\Delta}\right)^2 + \left(\frac{r_s}{\Delta}\right)^2 = 0 \quad \text{...}(A4.4)\]

where \[\Delta = \frac{3}{2} (l_s \tau_r - \mu)\]
Equation (A4.4) is a quartic with literal coefficients. To find the roots of this equation, let the general form of the quartic be

\[ \lambda^4 + p\lambda^3 + q\lambda^2 + r\lambda + t = 0 \] ....(A4.5)

Adding \( k\lambda^2 \) and \(-k\lambda^2\)

\[ \lambda^4 + [p\lambda+k]\lambda^2 + [(q-k)\lambda^2+r\lambda + t] = 0 \]

This last equation can be evaluated as a quadratic in \( \lambda^2 \) with bracketed terms treated as coefficient and constant.

This yields

\[ \lambda^2 = \frac{-(p\lambda+k) \pm \sqrt{[\lambda^2(p^2+4k-4q) + \lambda(2pk-4r) + k^2-4t]^2}}{2} \]

For the expression under the square root sign to be a perfect square,

\[ 2[p^2 + 4k - 4q]^{\frac{1}{2}} [k^2 - 4t]^{\frac{1}{2}} = 2pk - 4r \]

or

\[ k^3 - qk^2 + (pr - 4t)k + 4qt - p^2t - r^2 = 0 \] ....(A4.7)

Letting

\[ \frac{\frac{3}{2}l_s}{\Delta} = a \]

\[ \frac{r_s r}{\Delta} = b \]

\[ \frac{r_s}{\Delta} = c \]
and equating coefficients in equations (A4.4) and (A4.5), equation (A4.7) may be written in factored form

\[ [k-2c][k^2-(a+b)^2+n^2\delta^2]k+2c(a+b)^2 + n^2\delta^2(4ab-2c)-4c^2 ] = 0 \]

and \( k = 2c \) is a solution. Substituting for \( p, q, r, t \) and \( k \) in equation (A4.6)

\[ \lambda^2 = \frac{-[2(a+b)\lambda+2c] \pm [-2n\delta^2(\lambda+b)^2]^{\frac{1}{2}}}{2} \]

i.e.

\[ \lambda^2 = [-(a+b) + jn\delta]\lambda - c + jn\delta b \quad \text{.....(A4.8a)} \]

\[ \lambda^2 = [-(a+b) - jn\delta]\lambda - c - jn\delta b \quad \text{.....(A4.8b)} \]

\[ \therefore \] The four roots of \( p(\lambda) \) are

\[ \lambda = -(\frac{a+b}{2} - \frac{jn\delta}{2}) + [\frac{(a+b)^2 - n^2\delta^2 - 4c}{4} - \frac{jn\delta(a-b)}{2}]^{\frac{1}{2}} \]

\[ \lambda = -(\frac{a+b}{2} + \frac{jn\delta}{2}) + [\frac{(a+b)^2 - n^2\delta^2 - 4c}{4} + \frac{jn\delta(a-b)}{2}]^{\frac{1}{2}} \]

\[ \text{.....(A4.9)} \]

Referring to equations (A4.9), the parts under the square root sign may be written as

\[ M \ e^{-jn} \quad \text{or} \quad M \ e^{jn} \]
where

\[
\text{modulus } M = \left| \left( \frac{(a+b)^2 - n^2 \phi^2 - 4c}{4} \right)^2 + \left( \frac{n \phi(a-b)}{2} \right)^2 \right|^{\frac{1}{2}}
\]

and

\[
\text{argument } \eta = \frac{2n \phi(a-b)}{(a+b)^2 - n^2 \phi^2 - 4c}
\]

Hence

\[
[M e^{-j\eta}]^{\frac{1}{2}} = \pm M \left\{ \cos \frac{\eta}{2} - j \sin \frac{\eta}{2} \right\}
\]

\[
[M e^{j\eta}]^{\frac{1}{2}} = \pm M \left\{ \cos \frac{\eta}{2} + j \sin \frac{\eta}{2} \right\}
\]

Putting \( k = M \) (Equation 3.25) and re-writing equation (A4.9) in terms of \( k \) and \( \eta \), the four roots become,

\[
\lambda = -\left( \frac{a+b}{2} + k \cos \frac{\eta}{2} \right) + j\left( \frac{n \phi}{2} + k \sin \frac{\eta}{2} \right) \quad \ldots \quad (A4.10a)
\]

\[
\lambda = -\left( \frac{a+b}{2} + k \cos \frac{\eta}{2} \right) - j\left( \frac{n \phi}{2} + k \sin \frac{\eta}{2} \right)
\]

\[
\lambda = -\left( \frac{a+b}{2} - k \cos \frac{\eta}{2} \right) + j\left( \frac{n \phi}{2} - k \sin \frac{\eta}{2} \right) \quad \ldots \quad (A4.10b)
\]

\[
\lambda = -\left( \frac{a+b}{2} - k \cos \frac{\eta}{2} \right) - j\left( \frac{n \phi}{2} - k \sin \frac{\eta}{2} \right)
\]

These four roots correspond to the quadratic terms in equation (3.25).
For any value of \( \dot{\theta} > 0 \) the 4 eigenvalues of matrix \( \Lambda \) are two pairs of complex conjugates. As shown in section 3.3, these eigenvalues determine the angular frequency and damping of the damped sinusoid terms that make up the elements of the transition matrix. This connection is summarised in equations (3.26) and (3.27).

In analysing machines, it is sometimes assumed that the stator resistance is small and may be neglected. In the above analysis, if the stator resistance \( r_s \rightarrow 0 \), then \( b = c = 0 \) and it may be deduced that

\[
k = \frac{1}{2} [a^2 + n^2 \dot{\theta}^2]^{\frac{1}{2}}
\]

\[
\cos \frac{n}{2} = a [a^2 + n^2 \dot{\theta}^2]^{-\frac{1}{2}}
\]

\[
\sin \frac{n}{2} = n \dot{\theta} [a^2 + n^2 \dot{\theta}^2]^{-\frac{1}{2}}
\]

Thus

\[
k \cos \frac{n}{2} = \frac{1}{2} a
\]

\[
k \sin \frac{n}{2} = \frac{1}{2} n \dot{\theta}
\]

The eigenvalues are now

\[
\lambda_{1,2} = -a \pm jn \dot{\theta}
\]

\[
\lambda_{3,4} = 0
\]
This result is in agreement with the rather curious symmetry of the eigenvalues of $\tilde{A}$ indicated in equations (A4.10a, b). For the two complex conjugate pairs, the sum of the real parts is $(a+b)$, independent of frequency and the sum of the imaginary parts is $n\phi$, independent of motor parameters.

Figure 3.4.5 shows an eigenvalue plot that approaches the loci predicted for a motor having "very small" stator phase resistance.
APPENDIX 5  COMPUTER PROGRAMS

(a) Program 1 - Three phase motor - calculation of current plus torque

(b) Program 2 - Quasi two-phase motor - current plus torque

(c) Program 3 - Subroutine TMSS

(d) Program 4 - Transfer function data

(e) Program 5 - Non-linear equations

(f) Output from Program 1
## IDENTIFICATION OF COMPUTER SYMBOLS

### PROGRAM 1

<table>
<thead>
<tr>
<th>Computer program symbols not internally defined</th>
<th>Text meaning or use</th>
</tr>
</thead>
<tbody>
<tr>
<td>SL</td>
<td>Stator phase inductance ( l_s )</td>
</tr>
<tr>
<td>RS</td>
<td>Stator phase resistance ( r_s )</td>
</tr>
<tr>
<td>BA</td>
<td>Motor parameter ( \mu )</td>
</tr>
<tr>
<td>TA</td>
<td>Rotor winding time constant ( \tau )</td>
</tr>
<tr>
<td>DF</td>
<td>Speed in electrical rad/sec ( n\dot{\theta} )</td>
</tr>
<tr>
<td>TI</td>
<td>Time interval ( \Delta t )</td>
</tr>
<tr>
<td>A(I,J)</td>
<td>Matrix ( \bar{A} = \bar{A}(\dot{\theta}) )</td>
</tr>
<tr>
<td>C(I,J)</td>
<td>Matrix ( \bar{C} )</td>
</tr>
<tr>
<td>TM(I,J)</td>
<td>Transition matrix</td>
</tr>
<tr>
<td>A1 and A2</td>
<td>Eigenvalue real parts ( \alpha_1 ) and ( \alpha_2 )</td>
</tr>
<tr>
<td>B1 and B2</td>
<td>Eigenvalue imaginary parts ( \beta_1 ) and ( \beta_2 )</td>
</tr>
<tr>
<td>VA1 VB1</td>
<td>Stator phase voltages over time interval ( \Delta t )</td>
</tr>
<tr>
<td>VA2 VB2</td>
<td>( G(\dot{\theta})^{-1}v_1 )</td>
</tr>
<tr>
<td>GV1(J)</td>
<td>Instantaneous developed torque</td>
</tr>
<tr>
<td>TORK</td>
<td>Subroutine to invert a matrix</td>
</tr>
<tr>
<td>MATINV</td>
<td>Unit matrix</td>
</tr>
<tr>
<td>U(I,J)</td>
<td>current vector</td>
</tr>
<tr>
<td>AT1</td>
<td></td>
</tr>
<tr>
<td>AT2</td>
<td></td>
</tr>
<tr>
<td>AT3</td>
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</tbody>
</table>
PROGRAM 1
THREE PHASE MOTOR - CALCULATION OF CURRENT + TORQUE

INPUT - MOTOR PARAMETERS, SPEED, SUPPLY FREQUENCY, VOLTAGE

DIMENSION GV1 (4), GV2 (4), GV3 (4), VF (4), AO (4), AT1 (4), AT2 (4), AT3 (4)

DIMENSION TD1 (4, 4), TM3 (4, 4), TMU (4, 4), TD3 (4, 4), TD2 (4, 4)

DIMENSION A (4, 4), S (4, 4), T (4, 4), F (4, 4)

DIMENSION C (4, 4), W (4, 4), X (4, 4), Y (4, 4), Z (4, 4)

DIMENSION TM (4, 4), E (4, 4), U (4, 4)

READ 31, RS, TA, BA, SL

31 FORMAT (4F10.5)

READ 32, DF, TI

32 FORMAT (2F10.5)

PUNCH 34

34 FORMAT (//15HPARAMETERS USED//)

PUNCH 33, RS, TA, BA, SL, DF, TI

33 FORMAT (6X, 1HR, 9X, 1HT, 9X, 1HB, 9X, 1HS, 9X, 2HDF, 9X, 2HTI/6F1C.5)

SQ = SQRTF (3.)

DELTA = 1.5 * (SL * TA - BA)

D = 1.0 / DELTA

A (1, 1) = (.5 * SQ * BA * DF - RS * TA) * D
A (1, 2) = SQ * BA * DF * D
A (1, 3) = TA * BA * DF * D
A (1, 4) = -BA * D
A (2, 1) = -SQ * BA * DF * D
A (2, 2) = (.5 * SQ * BA * DF + RS * TA) * D
A (2, 3) = (.5 * SQ * BA - 5 * BA * TA * DF) * D
A (2, 4) = (.5 * BA + .5 * SQ * TA * BA * DF) * D
A (3, 1) = (.5 * SQ * RS + 2.25 * SL * DF) * D
A (3, 2) = SQ * RS * D
A (3, 3) = -1.5 * SL * D
A (3, 4) = 1.5 * SL * TA * DF * D
A (4, 1) = 1.5 * (.5 * SQ * SL * DF - RS) * D
A (4, 2) = 1.5 * SQ * SL * DF * D
A (4, 3) = 1.5 * SL * TA * DF * D
A (4, 4) = -1.5 * SL * D

DO 1 I = 1, A

1 E (I, J) = A (I, J)

DO 2 I = 1, A

2 S (I, J) = S (I, J) + A (I, K) * E (K, J)

DO 3 I = 1, A

3 T (I, J) = T (I, J) + S (I, K) * A (K, J)

DO 5 I = 1, A

5 U (I, J) = 0.
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<th>1</th>
<th>1</th>
</tr>
</thead>
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</tr>
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</tr>
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</tr>
<tr>
<td>B</td>
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</tr>
<tr>
<td>Z</td>
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</table>
250

\[ C_2 = \expf(A_2 \cdot T_1) \cdot \cosf(B_2 \cdot T_1) \]
\[ S_2 = \expf(A_2 \cdot T_1) \cdot \sinf(B_2 \cdot T_1) \]
\[ C_{12} = \expf(A_1 \cdot 2 \cdot T_1) \cdot \cosf(B_1 \cdot 2 \cdot T_1) \]
\[ S_{12} = \expf(A_1 \cdot 2 \cdot T_1) \cdot \sinf(B_1 \cdot 2 \cdot T_1) \]
\[ C_{22} = \expf(A_2 \cdot 2 \cdot T_1) \cdot \cosf(B_2 \cdot 2 \cdot T_1) \]
\[ S_{22} = \expf(A_2 \cdot 2 \cdot T_1) \cdot \sinf(B_2 \cdot 2 \cdot T_1) \]
\[ C_{13} = \expf(A_1 \cdot 3 \cdot T_1) \cdot \cosf(B_1 \cdot 3 \cdot T_1) \]
\[ S_{13} = \expf(A_1 \cdot 3 \cdot T_1) \cdot \sinf(B_1 \cdot 3 \cdot T_1) \]
\[ C_{23} = \expf(A_2 \cdot 3 \cdot T_1) \cdot \cosf(B_2 \cdot 3 \cdot T_1) \]
\[ S_{23} = \expf(A_2 \cdot 3 \cdot T_1) \cdot \sinf(B_2 \cdot 3 \cdot T_1) \]

\[ \text{DO 101 } I = 1, 4 \]
\[ \text{DO 101 } J = 1, 4 \]

\[ 62 \text{ TM}(I, J) = C_1 \cdot W(I, J) + S_1 \cdot X(I, J) + C_2 \cdot Y(I, J) + S_2 \cdot Z(I, J) \]
\[ 101 \text{ TM3}(I, J) = C_{13} \cdot W(I, J) + S_{13} \cdot X(I, J) + C_{23} \cdot Y(I, J) + S_{23} \cdot Z(I, J) \]

\[ \text{DO 102 } I = 1, 4 \]
\[ \text{DO 102 } J = 1, 4 \]

\[ \text{TMU}(I, J) = \text{TM3}(I, J) + U(I, J) \]
\[ \text{TD3}(I, J) = (C_{13} - C_{12}) \cdot W(I, J) + (S_{13} - S_{12}) \cdot X(I, J) + (C_{23} - C_{22}) \cdot Y(I, J) \]
\[ + (S_{23} - S_{22}) \cdot Z(I, J) \]
\[ \text{TD2}(I, J) = (C_{12} - C_1) \cdot W(I, J) + (S_{12} - S_1) \cdot X(I, J) + (C_{22} - C_2) \cdot Y(I, J) \]
\[ + (S_{22} - S_2) \cdot Z(I, J) \]
\[ \text{TD1}(I, J) = \text{TM}(I, J) - U(I, J) \]

\[ \text{CALL MATINV (TMU, 4)} \]
\[ \text{DEN} = RS \cdot (1.0 + TA \cdot TA \cdot DF \cdot DF) \]
\[ G_{11} = DF \cdot (1.5 - 0.5 \cdot SQ \cdot TA \cdot DF) / DEN \]
\[ G_{12} = -SQ \cdot TA \cdot DF \cdot DF / DEN \]
\[ G_{21} = DF \cdot (1.5 \cdot TA \cdot DF + 0.5 \cdot SQ) / DEN \]
\[ G_{22} = SQ \cdot DF \cdot DF / DEN \]

\[ \text{READ 103, VA1, VB1, VA2, VB2, VA3, VB3} \]
\[ \text{103 FORMAT (GF10.5)} \]
\[ \text{PUNCH 445, VA1, VB1, VA2, VB2, VA3, VB3} \]
\[ \text{445 FORMAT (/5HA1 = F5.0, 5HB1 = F5.0, 5HA2 = F5.0, 5HB2 = F5.0, 5HA3 = F5.0, 5HB3 = F5.0)} \]
\[ G_{V1}(1) = VA1 / RS \]
\[ G_{V1}(2) = VB1 / RS \]
\[ G_{V1}(3) = G_{11} \cdot VA1 + G_{12} \cdot VB1 \]
\[ G_{V1}(4) = G_{21} \cdot VA1 + G_{22} \cdot VB1 \]
\[ G_{V2}(1) = VA2 / RS \]
\[ G_{V2}(2) = VB2 / RS \]
\[ G_{V2}(3) = G_{11} \cdot VA2 + G_{12} \cdot VB2 \]
\[ G_{V2}(4) = G_{21} \cdot VA2 + G_{22} \cdot VB2 \]
\[ G_{V3}(1) = VA3 / RS \]
\[ G_{V3}(2) = VB3 / RS \]
\[ G_{V3}(3) = G_{11} \cdot VA3 + G_{12} \cdot VB3 \]
\[ G_{V3}(4) = G_{21} \cdot VA3 + G_{22} \cdot VB3 \]

\[ \text{DO 104 } I = 1, 4 \]
\[ \text{VF}(I) = 0.0 \]
\[ \text{DO 104 } J = 1, 4 \]

\[ 104 \text{ VF}(I) = VF(I) + TD3(I, J) \cdot GV1(J) + TD2(I, J) \cdot GV2(J) + TD1(I, J) \cdot GV3(J) \]
\[ \text{DO 105 } I = 1, 4 \]
\[ A_0(1) = 0. \]

DO 105 J = 1, 4

105 \[ A_0(1) = A_0(1) + TMU(1, J) \times VF(J) \]

\[ TK = 1.5 \times A_0(1) + A_0(3) + 0.5 \times SQ \times A_0(1) + A_0(4) + SQ \times A_0(2) \times A_0(4) \]

\[ TORK = 2.0 \times BA \times TK \]

PUNCH 444, TORK, (A_0(1), J = 1, 4)

\[ 444 \text{FORMAT}(/17HCURRENT VECTOR A0//7HTORk= E18.7, //4E18.7) \]

TI = T1 / 10.

DO 120 M = 1, 10

B = M

CS1 = EXPF (A1 * B * TI) \times COSF (B1 * B * TI)

SN1 = EXPF (A1 * B * TI) \times SINF (B1 * B * TI)

CS2 = EXPF (A2 * B * TI) \times COSF (B2 * B * TI)

SN2 = EXPF (A2 * B * TI) \times SINF (B2 * B * TI)

DO 110 I = 1, 4

DO 110 J = 1, 4

TM(I, J) = CS1 * W(I, J) + SN1 * X(I, J) + CS2 * Y(I, J) + SN2 * Z(I, J)

110 CONTINUE

DO 111 I = 1, 4

AT1(I) = 0.

DO 111 J = 1, 4

AT1(I) = AT1(I) + TM(I, J) \times (A_0(J) - GV1(J))

111 CONTINUE

AT1(I) = AT1(I) + GV1(I)

TK = 1.5 * AT1(1) + AT1(3) + 0.5 * SQ * AT1(1) + AT1(4) + SQ * AT1(2) \times AT1(4)

TORK = 2.0 \times BA \times TK

120 PUNCH 112, M, TORK, (AT1(I), I = 1, 4)

\[ 112 \text{FORMAT}(/22HCURRENT VECTOR AT1--M=13, //7HTORk= E18.7, //4E18.7) \]

DO 121 M = 1, 10

B = M

CS1 = EXPF (A1 * B * TI) \times COSF (B1 * B * TI)

SN1 = EXPF (A1 * B * TI) \times SINF (B1 * B * TI)

CS2 = EXPF (A2 * B * TI) \times COSF (B2 * B * TI)

SN2 = EXPF (A2 * B * TI) \times SINF (B2 * B * TI)

DO 122 I = 1, 4

DO 122 J = 1, 4

TM(I, J) = CS1 * W(I, J) + SN1 * X(I, J) + CS2 * Y(I, J) + SN2 * Z(I, J)

122 CONTINUE

DO 123 I = 1, 4

AT2(I) = 0.

DO 123 J = 1, 4

AT2(I) = AT2(I) + TM(I, J) \times (AT1(J) - GV2(J))

123 CONTINUE

AT2(I) = AT2(I) + GV2(I)

TK = 1.5 * AT2(1) + AT2(3) + 0.5 * SQ * AT2(1) + AT2(4) + SQ * AT2(2) \times AT2(4)

TORK = 2.0 \times BA \times TK

121 PUNCH 113, M, TORK, (AT2(I), I = 1, 4)

\[ 113 \text{FORMAT}(/22HCURRENT VECTOR AT2--M=13, //7HTORk= E18.7, //4E18.7) \]
DO 124 M=1,10
B=M
CS1=EXP(F(A1*B*T1))*COS(F(B1*B*T1))
SN1=EXP(F(A1*B*T1))*SIN(F(B1*B*T1))
CS2=EXP(F(A2*B*T1))*COS(F(B2*B*T1))
SN2=EXP(F(A2*B*T1))*SIN(F(B2*B*T1))
DO 125 I=1,4
DO 125 J=1,4
TM(I,J)=CS1*W(I,J)+SN1*X(I,J)+CS2*Y(I,J)+SN2*Z(I,J)
125 CONTINUE
DO 126 I=1,4
AT3(I)=0.
DO 126 J=1,4
AT3(I)=AT3(I)+TM(I,J)*(AT2(J)-GV3(J))
126 CONTINUE
DO 129 I=1,4
AT3(I)=AT3(I)+GV3(I)
TK=1.5*AT3(I)*AT3(3)+.5*SQ*AT3(I)*AT3(4)+SQ*AT3(2)*AT3(4)
TORK=2.*BA*TK
124 PUNCH 127,M,TORK,(AT3(I),I=1,4)
127 FORMAT(//22HCURRENT VECTOR AT3--M=13,//7HTORK= E18.7,//4E18.7)
GO TO 81
END
IDENTIFICATION OF COMPUTER SYMBOLS

PROGRAM 2

Computer program symbols not internally defined

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Text meaning or use</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1 and R2</td>
<td>Stator winding resistances $r_1$ and $r_2$</td>
</tr>
<tr>
<td>SL1 and SL2</td>
<td>Stator winding inductances $l_{s1}$ and $l_{s2}$</td>
</tr>
<tr>
<td>U1 and U2</td>
<td>Parameters $\mu_1$ and $\mu_2$</td>
</tr>
<tr>
<td>T</td>
<td>Rotor winding time constant $\tau$</td>
</tr>
<tr>
<td>DF</td>
<td>Speed in electrical rads/sec $n\dot{\theta}$</td>
</tr>
<tr>
<td>A(I,J)</td>
<td>Matrix $A$</td>
</tr>
<tr>
<td>TM(I,J)</td>
<td>Transition matrix</td>
</tr>
<tr>
<td>TI</td>
<td>Time interval</td>
</tr>
<tr>
<td>XX</td>
<td>Governs number of ordinates calculated over interval TI</td>
</tr>
<tr>
<td>CTU, CTO</td>
<td>Arbitrary criteria for transition matrix error. Controls number of series terms used.</td>
</tr>
<tr>
<td>XVO XV1</td>
<td>Specific values of applied voltage vector $\chi_2$</td>
</tr>
<tr>
<td>XVO2, XVL2</td>
<td></td>
</tr>
<tr>
<td>TORK and TORK 2</td>
<td>Instantaneous values of motor developed torque</td>
</tr>
<tr>
<td>MATINV</td>
<td>Subroutine to invert a matrix</td>
</tr>
<tr>
<td>TMSS</td>
<td>Subroutine to evaluate the transition matrix</td>
</tr>
</tbody>
</table>
PROGRAM 2

QUASI TWO PHASE MOTOR CURRENT + TORQUE

INPUT - MOTOR PARAMETERS, SPEED, VOLTAGE, ACCURACY CRITERION

PUNCH 1

1 FORMAT (/45HPROGRAM WH67—CURRENT + TORQUE—1 PHASE MOTOR )
DIMENSION XT2 (4), X102 (4)
DIMENSION A (8, 8), TM (8, 8), U (8, 8), CK (8, 8), XVQ2 (4), XV12 (4)
DIMENSION T11 (4, 4), TE (4, 4), T12 (4, 4), U4 (4, 4), T11S (4, 4), TP (4, 4)
DIMENSION FIU (4, 4), XVO (4), XV1 (4), XT (4), XIC (4), XI1 (4), XI2 (4)
COMMON A, T1, TL, M, TM, NN, U, BN, CTU, CTO, REF, CK

READ 2, R1, SL1, R2, SL2, U1, U2

2 FORMAT (6F10.5)
READ 3, T, DF

3 FORMAT (2F10.5)

8 READ 9, T1, CTU, CTO, REF, XX

9 FORMAT (F10.7, 2E10.4, 2F10.7)

NN = 8

TL = 1.

Q = SQRTF (U2 / U1)
F1 = T * SL1 - U1
F2 = T * SL2 - U2
F2Q = F2 * Q

A (1, 1) = -T * R1 / F1
A (1, 2) = U1 * Q * DF / F1
A (1, 3) = -U1 / F1
A (1, 4) = -U1 * Q * T * DF / F1
A (1, 5) = T / F1
A (1, 6) = 0.
A (1, 7) = 0.
A (1, 8) = 0.

A (2, 1) = -U2 * DF / F2Q
A (2, 2) = T * R2 / F2
A (2, 3) = U2 * T * DF / F2Q
A (2, 4) = -U2 / F2
A (2, 5) = 0.
A (2, 6) = 0.
A (2, 7) = T / F2
A (2, 8) = 0.

A (3, 1) = -R1 / F1
A (3, 2) = SL1 * Q * DF / F1
A (3, 3) = -SL1 / F1
A (3, 4) = -SL1 * Q * T * DF / F1
A (3, 5) = 1. / F1
A (3, 6) = 0.
A (3, 7) = 0.
A (3, 8) = 0.

A (4, 1) = -SL2 * DF / F2Q
A (4, 2) = -R2 / F2
A (4, 3) = SL2 * T * DF / F2Q
A (4, 4) = -SL2 / F2
A(4,5)=0.
A(4,6)=0.
A(4,7)=1./F2
A(4,8)=0.
DO 41=5,8
DO 4 J=1,4
A(1,J)=0.
4 CONTINUE
A(5,5)=0.
A(5,6)=1.
A(5,7)=0.
A(5,8)=0.
A(6,5)=-314.159265**2
A(6,6)=0.
A(6,7)=0.
A(6,8)=0.
A(7,5)=0.
A(7,6)=0.
A(7,7)=0.
A(7,8)=1.
A(8,5)=0.
A(8,6)=0.
A(8,7)=-314.159265**2
A(8,8)=0.
DO 16 I=1,8
DO 16 J=1,8
16 U(I,J)=0.
DO 17 I=1,8
17 U(I,I)=1.
PUNCH 11,T1,CTU,CTO,REF
11 FORMAT (8X,2HT1,10X,3HCTU,10X,3HCTO,10X,3HREF/4E14.4)
PUNCH 52,R1,SL1,R2,SL2,U1,U2
52 FORMAT (6F10.5)
PUNCH 53,T,DF
53 FORMAT (2F10.5)
PUNCH 173,((A(I,J),J=1,NN),I=1,NN)
173 FORMAT (/8HMATRIX A/(4E14.7))
CALL TMSS(NN)
PRINT 170,((TM(I,J),J=1,NN),I=1,NN)
170 FORMAT (/9HMATRIX TM/(4E14.7))
DO 29 I=1,4
DO 29 J=1,4
T11(I,J)=TM(I,J)
29 TE(I,J)=TM(I,J)
DO 30 I=1,4
DO 30 J=1,4
30 T12(I,J)=TM(I,J+4)
DO 31 I=1,4
DO 31 J=1,4
31 U4(I,J)=0.
U4(1,1)=1.
U4(2,2)=1.
U4(3,3)=1.
U4(4,4)=1.
DO 32 I=1,4
DO 32 J=1,4
T11S(I,J)=0.
TP(I,J)=0.
DO 32 K=1,4
T11S(I,J)=T11S(I,J)+T11(I,K)*TE(K,J)
32 TP(I,J)=TP(I,J)+T11(I,K)*T12(K,J)
DO 34 I=1,4
DO 34 J=1,4
34 FIU(I,J)=T11S(I,J)+U4(I,J)
CALL MATINV(FIU,4)
66 READ 50,(XVO(I),I=1,4)
READ 50,(XV1(I),I=1,4)
READ 50,(XVO2(I),I=1,4)
READ 50,(XV12(I),I=1,4)
50 FORMAT(4F10.2)
DO 33 I=1,4
XT(I)=0.
XT2(I)=0.
DO 33 J=1,4
XT2(I)=XT2(I)+TP(I,J)*XVO2(J)+T12(I,J)*XV12(J)
33 XT(I)=XT(I)+TP(I,J)*XVO(J)+T12(I,J)*XV1(J)
DO 35 I=1,4
X102(I)=0.
X10(I)=0.
DO 35 J=1,4
X102(I)=X102(I)-FIU(I,J)*XT2(J)
35 X10(I)=X10(I)-FIU(I,J)*XT(J)
TORK2=2.*SQRTF(U1*U2)*(X10(2)*X10(3)-X10(1)*X10(4))
TORK2=2.*SQRTF(U1*U2)*(X102(2)*X102(3)-X102(1)*X102(4))
PUNCH 54,(XVO(I),I=1,4)
PUNCH 54,(XV1(I),I=1,4)
PUNCH54,(XVO2(I),I=1,4)
PUNCH54,(XV12(I),I=1,4)
54 FORMAT(/*4E18.7)
PUNCH 36,(X10(I),I=1,4),TORK
36 FORMAT(/*18HCURRENT VECTOR X10/*4E18.7/*HTORK=E18.7)
PUNCH136,(X102(I),I=1,4),TORK2
136 FORMAT(/*19HCURRENT VECTOR X102/*4E18.7/*HTORK2=E18.7)
T1=T1/XX
MN=XX
DO120M=1,MN
TL=M
CALL TMSS(NN)
DO 39 I=1,4
DO 39 J=1,4
T11(I,J)=TM(I,J)
39 CONTINUE
   DO 40 I=1,4
   DO 40 J=1,4
   T12(I,J)=TM(I,J+4)
40 CONTINUE
   DO 41 I=1,4
   XI1(I)=0.
   X12(I)=0.
   DO 41 J=1,4
   XI1(I)=XI1(I)+T11(I,J)*XI0(J)+T12(I,J)*XVO(J)
   X12(I)=X12(I)+T11(I,J)*X102(J)+T12(I,J)*XVO2(J)
41 CONTINUE
   TORK=2.*SQRTF(U1*U2)*(XI1(2)*XI1(3)-XI1(1)*XI1(4))
   TORK2=2.*SQRTF(U1*U2)*(X12(2)*X12(3)-X12(1)*X12(4))
   PUNCH 42, M, TORK, (XI1(I), I=1,4)
   PUNCH 46, M, TORK2, (X12(I), I=1,4)
121 CONTINUE
42 FORMAT(//22HCURRENT VECTOR XI1—M=I3, //7HTORK= E18.7//4E18.7)
46 FORMAT(//22HCURRENT VECTOR X12—M=I3, //7HTORK2= E18.7//4E18.7)
GO TO 66
END
SUBROUTINE TMSS(NN)

EVALUATES TRANSITION MATRIX AS TRUNCATED SERIES

FIXES TERMS REQUIRED BY COMPUTING AN ERROR

DIMENSION ATN(8,8), TTN(8,8), TAN(8,8), TBN(8,8), TMN(8,8), CK(8,8)
DIMENSION AT(8,8), TT(8,8), TD(8,8), TB(8,8), TM(8,8), A(8,8), U(8,8)
COMMON A, T1, TL, M, TM, NN, U, BN, CTU, CTO, REF, CK

DO 160 I = 1, NN
DO 160 J = 1, NN
AT(1, J) = A(1, J) * T1 * TL
TT(1, J) = U(1, J)
TM(1, J) = U(1, J)

160 CONTINUE
TIN = - T1
DO 170 I = 1, NN
DO 170 J = 1, NN
ATN(1, J) = A(1, J) * TIN * TL
TTN(1, J) = U(1, J)
TMN(1, J) = U(1, J)

170 CONTINUE
BN = 0.0
BN = BN + 1.
PRINT 180, BN

180 FORMAT (/4HBN= F10.3)

IF (BN - REF) 7, 8, 8

7 CONTINUE
DO 162 I = 1, NN
DO 162 J = 1, NN
TD(1, J) = TT(1, J)

162 CONTINUE
DO 163 I = 1, NN
DO 163 J = 1, NN
TB(1, J) = 0.0
DO 163 K = 1, NN
IF (TD(1, K)) 105, 163, 105

105 IF (AT(K, J)) 101, 163, 101

101 TB(1, J) = TB(1, J) + TD(1, K) * AT(K, J) / BN

163 CONTINUE
DO 164 I = 1, NN
DO 164 J = 1, NN
TM(1, J) = TM(1, J) + TB(1, J)

164 CONTINUE
DO 165 I = 1, NN
DO 165 J = 1, NN

165 TT(1, J) = TB(1, J)
DO 172 I = 1, NN
DO 172 J = 1, NN
TAN(1, J) = TTN(1, J)

172 CONTINUE
DO 173 I = 1, NN
DO 173 J = 1, NN
TBN(I, J) = 0.0
DO 173 K = 1, NN
  IF (TAN(I, K) .GT. 106, 173, 106)
    106 IF (ATN(K, J) .GT. 102, 173, 102)
    102 TBN(I, J) = TBN(I, J) + TAN(I, K) .ATN(K, J) / BN
  CONTINUE
DO 174 I = 1, NN
  DO 174 J = 1, NN
    TBN(I, J) = TBN(I, J) + TBN(I, J)
  CONTINUE
DO 175 I = 1, NN
  DO 175 J = 1, NN
    TTN(I, J) = TBN(I, J)
  CONTINUE
DO 176 I = 1, NN
  DO 176 J = 1, NN
    CK(I, J) = .0
  CONTINUE
DO 176 K = 1, NN
  IF (TM(I, K) .GT. 104, 176, 104)
    104 IF (TMN(K, J) .GT. 103, 176, 103)
    103 CK(I, J) = CK(I, J) + TM(I, K) .TMN(K, J)
  CONTINUE
DO 4 I = 1, NN
  DO 4 J = 1, NN
    IF (I - J) 3, 6, 3
    6 IF (CTU - ABSF(1, -CK(I, J))) 5, 4, 4
    3 IF (CTO - ABSF(CK(I, J))) 5, 4, 4
  CONTINUE
8 CONTINUE
RETURN
END
BASIC PROGRAM 4

10 DIM W[50], H[255]
20 PRINT "N, M, D1"
30 INPUT N, M, D1
40 PRINT
50 PRINT "w(J)"
60 FOR I = 0 TO N
70 READ H[I]
80 NEXT I
90 LET D2 = .02
100 FOR C = 1 TO M
110 FOR J = 1 TO 5
120 LET W[J] = J*D2
130 LET Y = C
140 LET X = C
150 FOR I = 0 TO N
160 LET X = D1*H[I] + COS (I*D1*W[J]) + X
170 LET Y = D1*H[I] + SIN (I*D1*W[J]) + Y
180 NEXT I
190 LET M1 = SQRT (X*X + Y*Y)
200 LET D = 20 + LOG (M1)/2.30259
210 LET P1 = ATN (Y/X)*180/3.14159
220 IF X >= 0 GOTO 220
230 LET P1 = P1 - 180
240 PRINT W[J], M1, D, P1
250 NEXT J
260 NEXT C
270 LET D2 = D2*1C
280 NEXT C
290 GOTO 30
300 STOP
## IDENTIFICATION OF COMPUTER SYMBOLS

### PROGRAM 5

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning or use</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_1 )</td>
<td>Stator resistance ( r_s )</td>
</tr>
<tr>
<td>( P_2 )</td>
<td>Rotor time constant ( \tau_r )</td>
</tr>
<tr>
<td>( P_3 )</td>
<td>Parameter ( \mu )</td>
</tr>
<tr>
<td>( P_4 )</td>
<td>Stator inductance ( l_s )</td>
</tr>
<tr>
<td>( X_{1} )</td>
<td>Stator current ( i_a )</td>
</tr>
<tr>
<td>( X_{2} )</td>
<td>Stator current ( i_b )</td>
</tr>
<tr>
<td>( X_{3} )</td>
<td>Transformed rotor current ( i_x )</td>
</tr>
<tr>
<td>( X_{4} )</td>
<td>Transformed rotor current ( i_y )</td>
</tr>
<tr>
<td>( X_{5} )</td>
<td>Speed in electrical radians/second ( n\dot{\theta} )</td>
</tr>
<tr>
<td>( T )</td>
<td>( 1/6 )th period of ( v_a )</td>
</tr>
<tr>
<td>( X )</td>
<td>Time increments in ( 1/6 )th period</td>
</tr>
<tr>
<td>( N )</td>
<td>Pole pairs ( n )</td>
</tr>
<tr>
<td>( T_L )</td>
<td>Load torque ( T_L )</td>
</tr>
<tr>
<td>( J )</td>
<td>Moment of inertia ( J )</td>
</tr>
<tr>
<td>( K )</td>
<td>Friction coefficient ( f ).</td>
</tr>
</tbody>
</table>
BASIC PROGRAM 5

NON-LINEAR EQUATIONS

1.00 PRINT PROGRAM 5

262
PROGRAM 1 OUTPUT
DATA USED FOR FIGURE 4.2.4
CURRENT VECTOR AND TORQUE FOR 60 EQUALLY SPACED ORDINATES PER PERIOD

PARAMETERS USED

<table>
<thead>
<tr>
<th>R</th>
<th>T</th>
<th>B</th>
<th>S</th>
<th>DF</th>
<th>TI</th>
</tr>
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<tbody>
<tr>
<td>1.29200</td>
<td>.31968</td>
<td>.08314</td>
<td>.27305</td>
<td>226.94900</td>
<td>.00416</td>
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</tbody>
</table>

A1 = 8C.B1 = 8C.A2 = -8C.B2 = 16C.A3 = -16C.B3 = 8C.

CURRENT VECTOR AO
TORK = -16357.5510979E-03
11504.199112E-03 -73280.0024381E-04 26390.7275220E-04 52724.6908615E-03

CURRENT VECTOR AT1—M= 1
TORK = -14985.2514033E-03
10285.637791E-03 -46967.1116333E-04 -81645.366496E-04 46391.201384E-03

CURRENT VECTOR AT1—M= 2
TORK = -14219.9569107E-03
92078.6632216E-04 -24462.6350801E-04 -17216.132356E-03 40743.8430533E-03

CURRENT VECTOR AT1—M= 3
TORK = -13950.8335027E-03
82899.269374E-04 -55996.836311E-04 -24662.5864270E-03 35877.3995656E-03

CURRENT VECTOR AT1—M= 4
TORK = -14062.5555321E-03
75478.6505576E-04 98071.6173607E-05 -30652.9803141E-03 31872.0044038E-03
<table>
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<th>CURRENT VECTOR AT1—M= 5</th>
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<tbody>
<tr>
<td>TORK= -14438.1417329E-03</td>
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<tr>
<td>69947.8560200E-04 21959.3340914E-04 35337.3480721E-03 28793.3995483E-03</td>
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<table>
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<th>CURRENT VECTOR AT1—M= 6</th>
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<tr>
<td>TORK= -14961.4854442E-03</td>
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<tr>
<td>66409.2905634E-04 31071.8166345E-04 38865.4790898E-03 26693.2913183E-03</td>
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<table>
<thead>
<tr>
<th>CURRENT VECTOR AT1—M= 7</th>
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<tr>
<td>TORK= -15519.5854286E-03</td>
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<tr>
<td>64937.6550323E-04 37370.6739779E-04 41385.7860738E-03 25609.7954345E-03</td>
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<table>
<thead>
<tr>
<th>CURRENT VECTOR AT1—M= 8</th>
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</thead>
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<tr>
<td>TORK= -16004.4793403E-03</td>
</tr>
<tr>
<td>65581.049204E-04 41090.6420945E-04 43044.2439883E-03 25567.9631035E-03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CURRENT VECTOR AT1—M= 9</th>
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<tbody>
<tr>
<td>TORK= -16314.8899748E-03</td>
</tr>
<tr>
<td>68362.2248236E-04 42472.7783730E-04 43983.4045620E-03 26580.3797121E-03</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CURRENT VECTOR AT1—M= 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>TORK= -16357.5938119E-03</td>
</tr>
<tr>
<td>73279.9562198E-04 41762.1891925E-04 44341.4901528E-03 28647.8275919E-03</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>CURRENT VECTOR AT2—M= 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>TORK= -14985.2889091E-03</td>
</tr>
<tr>
<td>46967.0360790E-04 55889.4708110E-04 44258.3255705E-03 16124.8681694E-03</td>
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<table>
<thead>
<tr>
<th>CURRENT VECTOR AT2—M= 2</th>
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</thead>
<tbody>
<tr>
<td>TORK= -14219.9902939E-03</td>
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<tr>
<td>24462.5326111E-04 67616.2326900E-04 43893.3601046E-03 54622.6388477E-04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CURRENT VECTOR AT2—M= 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>TORK= -13950.8638312E-03</td>
</tr>
<tr>
<td>55995.5458893E-05 77299.7896654E-04 43402.1150473E-03 34197.8837622E-04</td>
</tr>
</tbody>
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CURRENT VECTOR AT2→M= 4
TORK= -14662.5839367E-03
-98073.1044285E-05 85286.0086459E-04 42928.5288433E-03 -10610.3300868E-03

CURRENT VECTOR AT2→M= 5
TORK= -14438.1689942E-03
-21959.502193E-04 91907.378637E-04 42604.5528901E-03 -16206.423652E-03

CURRENT VECTOR AT2→M= 6
TORK= -14961.516288E-03
-31072.011563E-04 97481.285775E-04 42549.860958E-03 -20311.9370585E-03

CURRENT VECTOR AT2→M= 7
TORK= -15519.6132361E-03
-37370.8724728E-04 10230.8495645E-03 42871.6688206E-03 -23036.3418815E-03

CURRENT VECTOR AT2→M= 8
TORK= -16004.5085756E-03
-41090.8519689E-04 10661.1844982E-03 43664.6589484E-03 -24493.5304703E-03

CURRENT VECTOR AT2→M= 9
TORK= -16314.9212837E-03
-42472.9970398E-04 11083.514092E-03 45011.0066237E-03 -24800.6635712E-03

CURRENT VECTOR AT2→M= 10
TORK= -16357.6277376E-03
-41762.4142495E-04 11504.2266759E-03 46980.506548E-03 -24077.0541217E-03

CURRENT VECTOR AT3→M= 1
TORK= -14985.3201925E-03
-55889.6998175E-04 10285.6610735E-03 -36093.7062931E-03 -30266.5132847E-03

CURRENT VECTOR AT3→M= 2
TORK= -14220.0196461E-03
-67616.4633411E-04 92078.8508099E-04 -26677.1264658E-03 -35281.7470213E-03
CURRENT VECTOR AT3—M = 3
TORK = -13950.8919551E-03
-77300.0197661E-04 82899.4108073E-04

CURRENT VECTOR AT3—M = 4
TORK = -14062.6115135E-03
-85286.2361227E-04 75478.7453877E-04

CURRENT VECTOR AT3—M = 5
TORK = -14438.196669E-03
-91907.6015748E-04 69947.9042469E-04

CURRENT VECTOR AT3—M = 6
TORK = -14961.5409423E-03
-97481.5023145F-04 66409.2926102E-04

CURRENT VECTOR AT3—M = 7
TORK = -15519.6428698E-03
-10230.8764169E-03 64937.6117174E-04

CURRENT VECTOR AT3—M = 8
TORK = -16004.5399555E-03
-10667.2043983E-03 65580.9624290E-04

CURRENT VECTOR AT3—M = 9
TORK = -16314.9548350E-03
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CURRENT VECTOR AT3—M = 10
TORK = -16357.6638197E-03
-11504.2442843E-03 73279.7852438E-04

MEAN TORQUE ——— 15081.5078312E-03
-18739.4103857E-03 -39297.3423307E-03
-12275.4151810E-03 -42482.4743581E-03
-72670.5819339E-04 -44999.9475884E-03
-36842.2385459E-04 -47005.3367709E-03
-14857.1511561E-04 -48646.229121E-03
-62023.9578194E-05 -50061.5684512E-03
-10274.2072682E-04 -51381.1010631E-03
-26388.2496766E-04 -52724.9223830E-03