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Reconstruction algorithms for electrical impedance tomography

Z. Q. Chen

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RECONSTRUCTION ALGORITHMS
FOR
ELECTRICAL IMPEDANCE
TOMOGRAPHY

A Thesis submitted in fulfilment of the
requirements for the award of the degree

Doctor of Philosophy

from

The University of Wollongong

by

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Department of Electrical and Computer
Engineering

1990
Abstract

Electrical Impedance Tomography (EIT) is a new imaging technique which produces the conductivity distribution of material within an object from electrical measurement made at the object's boundary. EIT has applications in biomedical and geophysical engineering and various industrial applications are also under investigation.

This thesis is a study of reconstruction algorithm of EIT and involves an examination of nonlinear reconstruction algorithms as well as the mathematical derivation of linear reconstruction techniques.

An iterative algorithm is derived using linear network theory and finite element modelling of an continuous object. Relationships and equivalences between a number of iterative algorithms are investigated. The algorithm is tested on both networks and continuous objects, using simulated and measured data. A modified algorithm is then developed to reduce the data errors caused by the finite element modelling and the electrode contact impedance. Meaningful results are obtained from the algorithm.

A linear integral equation reconstruction method is developed from a linearized Poisson's equation. Properties of the integral equation are investigated and a numerical algorithm is derived. The relationship between the integral equation method, Radon's back-projection method [Herman and Natterer, 1981] and Barber-Brown's filtered back-projection method [Barber and Brown, 1986] is studied. Useful reconstructions are obtained from simulated and measured data, and problems associated with the integral equation method are discussed.

An error analysis of the linearized Poisson's equation is carried out with an emphasis on the applications to EIT. Three types of error are discussed and re-scaling error is found to be dominant. It is also found that the commonly used small perturbation assumption often fails. Case studies and numerical tests agree with the conclusions of the error study.
The dominance of the re-scaling error allows the reconstruction of relative conductivity distribution using linear algorithms. This provides a novel understanding of the mechanism of linear reconstruction and can be useful in the future development of algorithms.
Acknowledgement

I wish to express my sincere gratitude to my supervisor, Associate Professor F. J. Paoloni, for all the support, encouragement, and guidance he has provided throughout the course of this study.

I also wish to offer special thanks to Dr. A. D. Seagar of Division of Radio Physics, CSIRO, Australia, with whom I have had helpful discussions.

My gratitude extends to the staff at the Department of Electrical and Computer Engineering and workshop for their support and interest in my research. In particular I wish to thank Dr. M. A. Magdy for the valuable discussions with him.

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List of Symbols

**English alphabet**

- $A_a$ ---- incidence matrix.
- $A$ ---- reduced incidence matrix.
- AIR ---- Accessible node to Inaccessible node Ratio.
- CT ---- Computerized Tomography.
- $e$ ---- voltage source vector.
- EIT ---- Electrical Impedance Tomography.
- $F(c)$ ---- object function in the iterative reconstruction algorithm.
- $i$ ---- branch current vector.
- $i$ ---- component current vector.
- $I(\phi)$ ---- functional in a finite element analysis.
- $I_s$ ---- the singular value of the integral equation.
- $j$ ---- current source vector.
- $j_n$ ---- equivalent nodal current source vector.
- $J_F$ ---- Jacobian matrix in the iterative reconstruction algorithm.
- $k(\cdot,\cdot)$ ---- kernel of the integral equation.
- $[K^e]$ ---- element matrix in finite element analysis.
- $K$ ---- system matrix of a global equation in finite element analysis.
- LSVM ---- Least Square Voltage Matching algorithm.
- MB ---- number of measurable branch voltages.
- NB ---- number of branches in a network.
- NE ---- number of elements in a finite element analysis.
- NN ---- number of nodes in a network.
- NP ---- number of boundary current patterns or distributions.
- PFEM ---- Piecewise Finite Element Modelling.
PLSVM ------ Partial Least Square Voltage Matching algorithm.
Q ------ parameter for measuring the quality of a reconstruction.
r ------ log-resistance (r = - log σ).
S ------ a parameter for measuring the smoothness of a kernel.
SNR ------ Signal to Noise Ratio.
T ------ matrix connecting discrete conductivity in a finite element model and component values in the equivalent network.
v ------ branch voltage vector.
v ------ component voltage vector.
v_n ------ nodal voltage vector.
w ------ a complex variable defining a u - v plane (w = u + jv).
Y ------ component characteristics matrix.
Y_n ------ nodal-admittance matrix.
z ------ a complex variable defining a x - y plane (z = x + jy).
Z(σ) ------ transfer impedance in Geselowitz's sensitivity theory.

Greek alphabet

δ(·) ------ the delta function.
ω_u, ω_v ------ variables in frequency domain.
ϕ ------ total potential distribution.
ϕ_o ------ potential distribution in a uniform object.
ϕ_p ------ perturbed potential distribution.
ϕ(e) ------ potential distribution inside a finite element.
σ ------ conductivity distribution.
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Chapter 1

Introduction
Information about physical objects is often obtained from a measurement set-up outlined in Fig. 1.1, where an incident signal $\psi_i$ from a source impinges upon and interacts with an object of unknown physical property $\Gamma$ over a region $\Omega$, giving rise to a perturbed signal $\psi_p$. The total signal $\psi = \psi_i + \psi_p$ contains all the observable information about the object and is measured with a detector. According to the specific object being observed and the particular information desired about the object, different signal sources may be chosen for the measurement system.

![Diagram of a general system for probing properties of an object.](image)

**Fig. 1.1** A general system for probing properties of an object.

Physical properties of an object are usually represented by parameters and, in many situations, the measurements themselves do not directly present the desired information about the object. Therefore it is necessary to process the measured data to obtain the wanted parameter distributions. Data processing is usually classified into two categories: the forward problem and the inverse problem. The forward problem involves finding $\psi$ for given $\psi_i$ and $\Gamma$, and the inverse problem involves finding $\Gamma$ when $\psi_i$ and $\psi$ are known or partly known.

The inverse calculation is mathematically more difficult to solve than the forward calculation. This is because mathematical descriptions of physical phenomena are normally based on forward problems and inverse calculations are usually solved by
using the equations for forward problems. Whilst the majority of mathematical
equations for forward calculations have been studied extensively, it is not clear, in
many cases, how to manipulate such descriptions to solve inverse problems.

Existence, uniqueness, and stability are three major topics in the solution of
mathematical equations for physical phenomena. For many forward calculations,
theoretical conclusions about existence, uniqueness, and stability of the solution have
been available, whilst for inverse problems, theoretical study of the uniqueness and
stability of solution is normally difficult, even though the existence of solution is
known physically.

1.1 Computerized Tomography

Computerized Tomography (CT) is a process that generates a two-dimensional
image of the cross-section distribution of a certain parameter of an object using the
measured data along the boundary of the object. CT is a non-invasive imaging tool
because no measurement is carried out inside the object during the imaging process.
This useful feature has found many applications in various areas, particularly in
medical diagnosis, where patient welfare is of paramount importance, and in
geophysical prospecting, where underground measurement is often unfeasible or not
affordable.

In addition to two-dimensional imaging, CT is also a useful tool for non-
invasive reconstruction of three dimensional imaging, because any three dimensional
image can be built up by stacking two-dimensional images.

Many CT systems have been developed during past two decades and, among
them, X-ray CT, Ultrasonic B-scan, and Nuclear Magnet Resonance (NMR) are the
most widely used commercial CT systems. Achievements of the CT systems have
brought about remarkable revolution in medical diagnosis and had a profound impact
on all branches of the applied and basic sciences.
The major distinctions between existing CT systems are that different CTs use different signal sources and provide different information about the objects being observed. A comparison of the signal source and parameters being imaged in some CT system is listed in Table 1.1.

<table>
<thead>
<tr>
<th>NAME OF CT SYSTEM</th>
<th>INCIDENT SIGNAL</th>
<th>PARAMETERS BEING IMAGED</th>
</tr>
</thead>
<tbody>
<tr>
<td>X-ray CT</td>
<td>X-ray</td>
<td>Attenuation or absorption</td>
</tr>
<tr>
<td>B-scan</td>
<td>sonic wave</td>
<td>refractive index or impedance</td>
</tr>
<tr>
<td>NMR</td>
<td>magnetic field</td>
<td>nuclear spins</td>
</tr>
<tr>
<td>Microwave CT</td>
<td>microwave</td>
<td>permittivity and conductivity</td>
</tr>
<tr>
<td>Impedance imaging</td>
<td>current</td>
<td>conductivity or resistivity</td>
</tr>
</tbody>
</table>

Table 1.1 A brief comparison between various CT systems.

Image quality of today's X-ray CT, ultrasonic B-scan, and NMR is satisfactory for most practical applications. Cost for the commercial CT systems was very high in 1970s but has since dropped substantially. Nevertheless they are still expensive and cost from a few thousand up to a few hundred thousand dollars, due to the high complexity of the systems.

The success of today's commercial CT systems lies on the mathematical foundation of the back-projection theory established by Austrian mathematician Radon in 1919. The theory reveals that if the line projections, or line integrals, of the unknown distribution of a function are known at all the angles, then the distribution of the function can be reconstructed by back-projection processing without error. Furthermore, the reconstruction has an explicit linear mathematical expression so that the existence, uniqueness and stability of the reconstruction can be well established. Back projection theory is also referred to as Radon's transformation.
A basic requirement of the back-projection theory is that the measured data should be in a form of line integral of the function to be reconstructed. This requirement is satisfied by X-ray CT where signals travel in a straight ray path. Radon's back-projection theory can also be modified and used in other CT systems such as ultrasonic B-scan, NMR, microwave, etc., where scattering signals by an object are measured and used in image reconstructions.

The quality of a reconstructed image is mainly determined by factors such as how close the measured data is to a precise line integral of an unknown function, signal to noise ratio of the measured data, and errors in numerical back-projection calculation, etc..

1.2 Electrical Impedance Tomography

Electrical Impedance Tomography (EIT), also called "impedance imaging", "impedance camera", "applied potential tomography", applies low frequency currents [Seagar, 1983] to a conducting object and measures resulting potentials on the boundary of the object. The measured boundary potentials are then used to generate a two-dimensional image of the internal conductivity distribution of the object. According to the duality between current and voltage, one could also apply potential to the object and reconstruct an image of the conductivity using the resulting boundary current.

EIT is an imaging tool that uses nonionizing radiation. The imaging system can be made at low cost (on the order of thousands of dollars), relatively high speed (real-time imaging in many applications), simple in configuration, and operated with no harm to patients when used in medical practices.

A basic assumption in EIT is that the conductivity distribution is time-invariant. If the conductivity of an object varies with time slowly, it is assumed that boundary data collection is fast enough so that the slow time varying conductivity can be treated as stationary during the data acquisition. Reconstruction of a time-variant conductivity
distribution is much more difficult than that of time-invariant conductivity and nearly all research on EIT is restricted to the time-invariant case.

At low frequency, displacement current and skin effect can be ignored so that only conduction current travels inside the object. The distribution of the conduction current and the potential are determined by the conductivity distribution of the object, and this provides the possibility for estimating the conductivity distribution from measurement of boundary potentials.

While the back-projection theory forms a basis for most CT systems, it does not apply, strictly speaking, to EIT. This is because in EIT, current immediately diverges when leaving a source and travels in every possible path inside the object. Therefore the measured boundary potentials cannot be represented as line integrals of conductivity distribution which is desired by the back-projection theory.

Relationships between potential, current and conductivity distributions are governed by Poisson's equation, which exhibits a nonlinear relation between conductivity and voltage or current. The nonlinear relationship and the limited data available from boundary measurement make EIT a difficult task.

In addition to reconstruction difficulties, accurate measurements of boundary potentials are also very important to EIT. Reliable measurement is difficult to obtain for biomedical applications because of uncertain contact impedance at the electrode-tissue interface. A great deal of effort has been devoted to overcoming the measurement error caused by contact impedance and achievements have been reported [Brown and Seagar, 1987, Newell et al, 1988].

1.3 Applications of Electrical Impedance Tomography

Electrical Impedance Tomography has found major applications in biomedical and geophysical engineering. Other potential industrial applications have yet to be explored. In all the cases, conductivity distribution is used to infer some physical properties of an object.
A. Geophysical Applications

In geophysics, materials exhibit significant differences in conductivity $\sigma$, permittivity and permeability. Various minerals found in the earth allow metallic conduction (e.g. native minerals, $\sigma = 10^4$ to $10^7$ Sm), semiconductor conduction (e.g. sulphides, $\sigma = 10^4$ to $10^{-4}$ Sm), and solid electrolytic conduction (e.g. silicates, $\sigma < 10^{-4}$ Sm). In addition, for most rocks near the surface of the earth, conduction is due to salts dissolved in ground water permeating the pores in the rock, and the resulting conductivity is usually between $10^{-3}$ and $10^{-1}$ Sm. The large range of the conductivity infers that a conductivity image can be useful for geophysical prospecting.

Low frequency current probing has been used for years as a useful and common aid to geophysical prospecting and has been successful in profiling the lateral variations of the earth's electrical characteristics. However, the profiling problem is virtually a one dimensional problem as opposed to the imaging problem, which aims at the electrical characteristics of the earth in at least two dimensions.

In laboratory tests of electrical characteristics of mineral, ore is usually cut into samples of regular shape before testing. The cutting procedure applies mechanical tension to the samples and causes deformation of the composition of the mineral. The deformation will then lead to distortions of the original electrical properties of the mineral. As a non-invasive imaging technique, application of EIT to the laboratory test of mineral has been proposed and investigated by several research groups [Dine and Lytle, 1981, Yorkey et al, 1987].

B. Biomedical Applications

In biomedicine, electrical conduction in living organisms is due to the movement of ions in solution. Within tissues the movement of ions is restricted by cellular membranes, so that different tissues have different conductivities.

There is much literature on the electrical conductivity of biological tissues and organs. Some of them are listed in Table 1.2. The measured conductivities have many
discrepancies in reported values because of the complexity of the tissue composition, difficulties in making in vivo measurements, as well as the problems in preserving tissue for in vitro measurement.

The results in Table 1.2 demonstrate a large range of conductivity among different tissues. Even in soft tissues, the variation of conductivity is substantial. This establishes the possibility to non-invasively monitor biomedical object by imaging the internal conductivity of the object.

In medical application of EIT, intensity and frequency of the current applied to an object should be carefully chosen so that the artificial electrical potentials generated within the body by the boundary current do not interfere with the normal function of the body. The intensity and frequency of the current are determined by the consideration of patient's safety and comfort, the signal to noise ratio requirement in data acquisition, as well as the quasi-static field requirement. Experiments on humans and other animals indicate that the threshold current required to elicit observable effects, such as slowing of the heart, ventricular fibrillation and sensation, increases with the frequency of the applied current.

Electrical currents which do interfere with body functions can also induce beneficial effects. A large current pulse through the heart depolarises all ventricular tissue simultaneously allowing a heart undergoing ventricular fibrillation to return to normal synchronous operation. Electrodes placed near nerves can be used to block the propagation of action potentials and induce an anaesthesia-like state.

When measuring the boundary potential of a biological material, chemical effects at the electrode-electrolyte (tissue) interface have to be considered. A transfer of ions from an electrode into solution creates a static voltage which in equilibrium balances the tendency of further ions to enter the solution. This ionic reaction appears electrically to have both a capacitive and resistive impedance [Baker 1971]. It is important, therefore, to carefully prepare the electrode-tissue interface to reduce the impedance when low frequency current is used. When measuring high frequency
potentials the capacitive component is negligible and electrode preparation is less important [Nowotny 1980].

<table>
<thead>
<tr>
<th>Tissue</th>
<th>Conductivity (Sm)</th>
<th>Reference</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cerebrospinal fluid</td>
<td>1.538</td>
<td>Geddes (1967)</td>
<td></td>
</tr>
<tr>
<td>Plasma</td>
<td>1.515</td>
<td>Geddes (1967)</td>
<td></td>
</tr>
<tr>
<td>Blood</td>
<td>0.667</td>
<td>Shimazu (1981)</td>
<td>50% Haematocrit</td>
</tr>
<tr>
<td>Liver</td>
<td>0.286</td>
<td>Tucker (1982)</td>
<td>in vivo perfused canine tissue</td>
</tr>
<tr>
<td></td>
<td>0.182</td>
<td>Tucker (1982)</td>
<td>in vivo ligated canine tissue</td>
</tr>
<tr>
<td>Human arm</td>
<td>0.417 (longitudinal)</td>
<td>Burger (1961)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.148 (transverse)</td>
<td>Burger (1961)</td>
<td></td>
</tr>
<tr>
<td>Skeletal muscle</td>
<td>0.8 (longitudinal)</td>
<td>Burger (1961)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.056 (transverse)</td>
<td>Burger (1961)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.667 (longitudinal)</td>
<td>Rush (1963)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.044 (transverse)</td>
<td>Rush (1963)</td>
<td></td>
</tr>
<tr>
<td>Cardiac muscle</td>
<td>0.625 (longitudinal)</td>
<td>Plonsey (1982)</td>
<td>Calculated interstitial value</td>
</tr>
<tr>
<td></td>
<td>0.236 (transverse)</td>
<td>Plonsey (1982)</td>
<td>Calculated interstitial value</td>
</tr>
<tr>
<td></td>
<td>0.147 (longitudinal)</td>
<td>Plonsey (1982)</td>
<td>Calculated intracellular value</td>
</tr>
<tr>
<td></td>
<td>0.019 (transverse)</td>
<td>Plonsey (1982)</td>
<td>Calculated intracellular value</td>
</tr>
<tr>
<td>Neural tissue</td>
<td>0.172</td>
<td>Geddes (1967)</td>
<td>Average results for rabbit, cow and pig</td>
</tr>
<tr>
<td>- grey matter</td>
<td>0.352</td>
<td>Geddes (1967)</td>
<td>Rabbit tissue</td>
</tr>
<tr>
<td>- white matter</td>
<td>0.147</td>
<td>Geddes (1967)</td>
<td>Rabbit tissue</td>
</tr>
<tr>
<td>Lung</td>
<td>0.042 - 0.138</td>
<td>Witsoe (1967)</td>
<td>Range from expiration to inspiration</td>
</tr>
<tr>
<td></td>
<td>0.048 ± 17%</td>
<td>Rush (1963)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.089</td>
<td>Schwan (1956)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.078</td>
<td>Geddes (1967)</td>
<td>Average of many published results</td>
</tr>
<tr>
<td>Fat</td>
<td>0.049</td>
<td>Kaufman (1943)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>Rush (1963)</td>
<td>Average of many animal results</td>
</tr>
<tr>
<td></td>
<td>0.038</td>
<td>Geddes (1967)</td>
<td></td>
</tr>
<tr>
<td>Bone</td>
<td>0.006</td>
<td>Reddy (1984)</td>
<td>Wet bovine bone</td>
</tr>
</tbody>
</table>

Table 1.2 A list of published conductivity values for mammalian tissue.
A number of applications of EIT in biomedical engineering have been attempted in recent years. Possible applications include diagnosis and monitoring of gastric emptying, lung ventilation, pulmonary edema, cerebral ventricular hemorrhage, etc. Useful images have been reported by a few groups [Barber and Brown, 1983, 1986, Kim et al, 1983].

C. Other Industrial Applications

Many industrial systems require continuous and non-invasive monitoring techniques. EIT is certainly a possible technique to this purpose. Examples of the possible industrial applications include monitoring of fluids mixing in process control, measurement of flow in pipes, quality control of castings, etc. Feasibility of the above applications have yet to be fully explored.

1.4 Achievements and Advancing in Electrical Impedance Tomography

1.4.1 Theoretical Studies

In the past few years, research effort has been devoted to certain important topics, such as spatial resolution of an imaging system, sensitivity of boundary potential to the change of conductivity in an object, the optimal boundary current distribution for best distinguishing conductivity change of an object, etc.. Some theoretical results have been obtained based on the theoretical potential solution of a simple concentric circular object, as shown in Fig.1.4.1 [Seagar 1984, 1985, 1987a, Issacson 1986]. The results are useful in predicting image quality, in posing specification for hardware system design, as well as in understanding unique features of EIT.
Spatial resolution and conductivity resolution are two important parameters for quality measurement of impedance images. Based on the analytical solution of boundary potential for the object of Fig.1.4.1, Seagar et al. [1981, 1984, 1985, 1987a] found that parameters such as spatial resolution, conductivity resolution, conductivity contrast, sensitivity, the number of electrodes, measurement noise, etc., are interrelated. An improvement in one parameter will be balanced by improvements in other parameters. Quantitative relationships among the parameters are available in their papers.

Using conformal mapping, Seagar also argued that as the electrode number N increases without limit, the spatial and conductivity resolution do not increase without limit. In other words, beyond some value of N no more useful information about the central region of the object can be obtained from boundary measurement.

Isaacson [1986] used the same object in Fig.1.4.1 to reveal the relationship between distinguishability of conductivities and boundary current distributions. He has shown that an optimal boundary current distribution exists which gives rise to the best distinguishability for a given object. He also stressed the importance of using the optimal boundary current distributions in a practical imaging system with limited precision.
Studies of Seagar et al [1985] and Murai et al [1985] showed that boundary potentials are less sensitive to conductivity change in the inner part than that in the outer part of an object. This is because in the imaging system, the boundary potential distribution is mainly determined by the current distribution close to the boundary. Conductivity changes in the inner part of the object only cause significant current change to its neighbouring current distribution but have little influence on the current elsewhere. Thus the boundary potential will be less affected by inner conductivity changes.

The research results have inferred that EIT is intrinsically an imaging tool with low spatial resolution and non-uniform sensitivity to conductivity changes at different location within an object.

1.4.2 Reconstruction Algorithms

Success of EIT depends, to a large extent, on reconstruction algorithms that extract conductivity distribution information from measured boundary potentials.

Nearly all the existing reconstruction algorithms for EIT are developed from Poisson's equation. In general, given a conductivity distribution $\sigma$ and boundary current distribution $J$, there is a forward transformation $T$ which relates boundary potential distribution $v$ with the $\sigma$ and $J$:

$$v = T(\sigma, J). \quad (1.4.1)$$

The image reconstruction problem is to find an inverse transformation of $T$ such that

$$\sigma = T^{-1}(v, J) \quad (1.4.2)$$
In practice the $\sigma$ and $v$ are usually replaced by vectors of finite dimension. The solvability of the equation (1.4.1) then requires that the intrinsic dimensionality of $v$ is at least equal to that of $\sigma$.

For EIT, the transformation $T$ is normally nonlinear and therefore iterative procedures are required in finding the inversion of the $T$. In some cases, however, equation (1.4.1) can be approximated by a linear mapping which greatly simplifies the reconstruction calculation, but also introduces errors due to the linearization procedure.

Various reconstruction algorithms have been developed in past years [Pryce 1979, Dines 1981, Barber 1983, Seagar 1983, Kim 1983, Murai 1985, Yorkey 1987, Newell 1988, Hua 1990]. The algorithms can be grouped into two categories: iterative reconstruction algorithms and linear reconstruction algorithms. A brief review is given below.

**Iterative reconstruction algorithms**

Iterative algorithms reconstruct impedance images by solving a set of nonlinear equations. The nonlinear equations are obtained from Poisson's equation or other approaches.

In Dines and Lytle's resistive network method [1981], a conductive continuum is modelled by a resistive network. An iterative algorithm is then developed based on electrical network theory. In the imaging system, voltages are applied to the boundary electrodes on the object and the current flowing between electrode pairs is measured. Some simulation results and error analysis have been reported but it is not clear under what condition a continuous conductive object can be replaced by a network without admitting significant errors.

Kim et al [1983] have developed an imaging system in which current is passed to a grid of boundary electrodes and resulting boundary voltages are measured. In their iterative algorithm, the finite element method is used to solve Poisson's equation in order to obtain the voltage distribution for an assumed conductivity distribution. Difference between measured and calculated voltages are used as an amendment to the
assumed conductivity distribution at each iteration step. The amendment is based on a sensitivity matrix relating the change in current at an electrode to a small change in resistance of one element in the finite element model. This results in a slow rate of convergence. Some simulation results are also presented in their publication.

Seagar [1983] has used an iterative algorithm to reconstruct a circularly concentric distribution of conductivity. His algorithm was derived by minimizing a functional obtained from an analytic solution to this problem. Simulation and experimental results have been presented but the application of the algorithm is limited due to the special structure of the object.

Starting from Geselowitz's sensitivity integral that relates changes in boundary transfer impedance and changes in conductivity within an object, Murai et al [1985] developed an iterative algorithm. The algorithm used the finite element method to discretize the integral and its equivalence to a network problem was also mentioned. Some interesting simulations have been carried out but no application using measured data is reported.

Yorkey et al [1987] presented an iterative algorithm which is similar to Kim's algorithm except that the reconstruction matrix at each iteration step has taken into account the conductivity changes in all the elements instead of a single element. Moreover, Yorkey tested six different reconstruction algorithms (both iterative and linear) using simulation data. He suggested the use of the iterative Newton-Raphson algorithm for solving the nonlinear reconstruction problem due to certain advantages of the method. However, his conclusions were based on the simulations and none of the tests were performed with measured data.

Hua [1990] has reported an iterative algorithm which includes the use of optimal boundary current at each iteration step to gain the best distinguishability. He also used the regularization method in the algorithm to improve the stability of reconstructions. Advantages of the algorithm have been demonstrated in the reported reconstructions using simulated data and measured data from phantoms.
Iterative reconstruction approach is regarded as a method that may possibly produce authentic reconstructions of an object's conductivity distributions. This is because iterative algorithms obtained from the Poisson's equation are based on a precise description of interaction between conductivity distribution and potentials. However, because iterative algorithms solve a set of nonlinear equations that have non-unique solutions, then from the theory of nonlinear equations, the convergence of an iterative algorithm is determined by the starting solution. Thus the theoretical advantage of iterative algorithms can rarely be achieved due to the difficulty in selecting correct starting solution, and risk exists for iterative algorithms to converge to false reconstructions.

**Linear reconstruction algorithms**

Poisson's equation is a nonlinear equation in terms of conductivity because potential is a function of the conductivity. Under certain approximation, however, the Poisson's equation can be linearized and linear reconstruction algorithms have been developed.

From the linearized Poisson's equation, Barber and Brown [1983] developed a linear reconstruction algorithm. They assumed that the Radon's back-projection theory can be modified and applied to the electrical impedance tomography. In the algorithm, boundary potential differences between neighbouring electrodes are filtered, weighted, and projected back into the object domain along curved equipotential lines of an uniform object. The algorithm, although approximate, has been able to produce useful images using both *in vitro* and *in vivo* data. Due to the one-step procedure, the algorithm has a fast reconstruction speed.

A major drawback of Barber - Brown's filtered back-projection method is that it lacks rigorous mathematical support and this makes further improvement of the algorithm difficult.

Newell et al [1988] have used a one-step reconstruction algorithm in their experiments. The algorithm is virtually a Newton-Raphson iterative reconstruction
procedure except that the procedure is started from a uniform distribution of conductivity and terminated at the first step of iteration. The algorithm has been employed to process the measured data from a cylindrical phantom consisting of a saline bath and a number of conducting targets placed in the bath. Successful reconstructions of the testing phantom were presented but no reconstruction using in vivo data was reported.

Since the algorithm is essentially an iterative algorithm, it still suffers the problem of starting solution selection and it is not clear whether the algorithm can produce a faithful image of practical objects.

1.4.3 Data Acquisition System

The basic task of a data acquisition system is to apply a desired boundary current distribution to an object and collect boundary potential distribution under the control of a computer. Quality of measured potential data directly affects the quality of the reconstructed image. The precision of measured data depends on many factors and different applications of EIT require different measurement precision. Considerations related to the design of data acquisition system are reviewed below.

Electrodes

Electrodes are used to apply current to objects and sense the resulting boundary potentials. In order to obtain an authentic potential distribution, the size of electrodes should be small enough so that the perturbation on the potential distribution by the electrodes becomes ignorable. In the case that large electrodes are used, proper reconstruction algorithms should be used which takes into account the influence of the electrodes to the potential distribution.

In real applications of EIT, particularly in biomedical applications, uncertain contact impedance between electrode and object often turns out to be a serious problem. Four-electrode measurement has been developed to reduce the error caused
by the uncertain contact impedance and satisfactory results have been reported [Barber 1984, Plonsey 1982, Brown 1983]. Even in the four-electrode system, contact impedance is still noticeable. In some reconstruction approaches, the contact impedances were found to be extremely harmful and this needs to be compensated by various techniques. A common practice is to represent the contact impedances by certain equivalent electrical circuits of the electrodes and then include them into reconstruction algorithms [Cromwell 1980, Plonsey 1982, Cheng 1989, Hua 1990].

Spacing between electrodes is another factor to be considered. Electrode system with large spacing gives less data for reconstruction, and detail of potentials between electrodes is also lost. Reducing electrode spacing can increase the amount of data but stray capacitances between electrodes may increase. The stray capacitances may cause significant error to the measured data and eventually defeat the merit of densing the electrodes. In reported practical hardware systems [Brown 1987, Newell 1988], the number of electrodes has been limited to 32.

**Boundary current distributions**

Potential distribution over the object is dependent on boundary potential and current conditions. For a given conductivity distribution, linear relation exists between boundary current or potential condition and the resulting internal potential distribution. One of the advantages of the linear system is that superposition principle applies.

It was found that different boundary current distributions, or current patterns, lead to different sensitivity of the resulting boundary potentials to the change of conductivity. Issacson [1986] proposed a concept of using optimal boundary current distribution to achieve best distinguishability in electrical impedance tomography. He also demonstrated the existence of the optimal boundary current in a simple case of concentric object and provided mathematical formula for general situations.

Seagar [1983, 1987a] has studied different boundary current distributions and found that various boundary current distributions, or patterns, exhibit different features. He discussed three basic boundary current patterns, i.e. trigonometry current
pattern, adjacent electrodes injection, and opposite electrodes injection, as shown in Fig.1.4.2. Whilst trigonometry pattern gives highest distinguishability, the opposite injection pattern produces a potential that is more sensitive to the conductivity changes in the central part of an object than others. Using Fourier analysis, Seagar found that the adjacent injection pattern may result in a boundary potential with nearly uniform signal-to-noise ratio (SNR) for each frequency component of the potential (boundary potential generated by other current patterns has much lower SNR for high frequency component than low frequency component). Thus information stored in high spatial frequency components of the boundary potential can be extracted to the greatest extent.

Due to uncertain contact impedance of electrodes, potentials on current driving electrodes contain errors. It will be beneficial if those potentials can be avoided in reconstruction algorithms. On the other hand, eliminating potentials on driving electrodes reduces the amount of data for reconstruction. Thus, the number of electrodes dedicated for current injection should be kept to a minimum in order to obtain the maximum amount of data from the remaining electrodes. This is achieved in single pair driving schemes, as shown in Fig.1.4.2 (a) and (b).

The aforementioned three current patterns spanned three bases and any arbitrary current pattern can be represented as a sum of one of the patterns [Seagar 1987a, Cheng 1990]. In addition, each of the three current patterns can be expressed in terms of others. For an linear object, i.e. boundary current and resulting potential has a linear relationship, the relationships for current patterns can also be applied to the resulting boundary potential. Therefore the potential distribution corresponding to one boundary current pattern can be represented in terms of potentials of other boundary current patterns by superposition. However, there is a difference between the measured boundary potential and the potential by superposition of measured potentials for other current patterns because of measurement noise.
It is also important to note that the linear relationship is only valid between potential and current. The relation between potential and conductivity, or the relation between current and conductivity is nonlinear. Therefore different current patterns can lead to a potential containing different amounts of information about the conductivity distribution, and therefore have different effects in reconstructions. Fig. 1.4.3 illustrates the relationship amongst conductivity, current, and potential.
Data collecting

In a data acquisition system consisting of $N$ boundary electrodes, there are at most $\frac{N(N-1)}{2}$ independent measurements according to superposition and reciprocity principles. These independent measurements can be obtained by applying $N-1$ boundary current distributions to an object and measuring the resulting potentials. The current distributions must be applied serially whereas the resulting potentials may be measured serially or in parallel. If serial measurements are used, then there is a sequence of $\frac{N(N-1)}{2}$ measurements to be collected. If parallel measurements are used there are $N-1$ sequences of up to $N-1$ measurements to be collected, resulting in a faster collection speed but higher cost and complexity of the system.

Two sets of multiplexer are normally required by the data collection system. One is for current injection and the other is for potential measurement. The multiplexing has two functions: switching among all the electrodes; and multiplexing with respect to time. When conductivity is time-invariable, boundary potential is static and the multiplexing simply switches electrodes to data collection circuits. In this case, multiplexing does not affect signal to noise ratio (SNR) of the data on electrodes. For time-variable conductivity, boundary potential is a function of time. Therefore the multiplexing is both in space and time. In this case, multiplexing with respect to time will deteriorate the original SNR of potential data on electrodes. Seagar [1987b] has discussed effects of the multiplexing in serial and parallel data collection, and found that parallel data collection receives less SNR deterioration than serial one.

Source Frequency and sampling frequency

In practice, ac current sources are commonly used in EIT. dc current is avoided because of the electro-chemical reaction at the the electrode-object interface which causes error to potential measurement. The frequency of the ac current sources has an upper bound that ensures a quasi-static field condition [Seagar, 1983]. The lower bound of the source frequency is determined by the applications and speed of data collection. In biomedical applications, the frequency should be high enough so
that nerves will not produce pain during data acquisition. Also, for a given data collection speed, there is a limit on the current source frequency which allows adequate time for current and resulting potentials to reach a steady state before data collection takes place.

For a time-variant object, the lower bound of the source frequency is related to the bandwidth of potential signals, the sampling rate, and serial or parallel data collection [Seagar 1987b]. Generally, a higher value of lower bound of frequency will be resulted in this case.

**Examples of data acquisition systems**

Two data acquisition systems are reviewed here. One is the system used in Sheffield University, England[Brown 1987], and the other is in the Rensselaer Polytechnic Institute, New York, USA [Newell 1988].

The Sheffield data collection system was designed to enable measurement of boundary potential to a precision of 0.1%. The system used 16 electrodes which were addressed through four 16 channel multiplexers. A master clock running at 820 kHz was used to generate a 51 kHz sine wave by clocking out 16 values per cycle from an EPROM which fed a DA convertor. The 51 kHz sine wave was applied to a VI convertor with an output of 5 mA p-p. This current was applied to adjacent electrode pairs via two of the four multiplexers. The remaining two multiplexers were used to select adjacent pairs of electrodes and passed the signals to a differential amplifier. The signals were then sent to a phase sensitive detector (PSD) which selected only in-phase signals, i.e. resistive signals. The output of the PSD was taken to a sample and hold (SH) circuit prior to a 12 bit AD conversion. Fig.1.4.4 is a block diagram of the system.

The multiplexer addresses for current driving and potential measurement were held in EPROM and clocked out in a total time of 79 ms, resulting in 380μs per point. This speed of data collection allowed up to 12 sets of data to be collected per second. In data collection, no electrode was used for both current driving and potential
recording simultaneously. Transformer isolation was used at the input circuit of the differential amplifier both to minimise common mode signals generated at the record electrodes and for electrical safety. Common mode feedback is adopted to reduce the corrupting influence caused by stray capacitance of electrodes.

The system has been used to collect data from both phantoms and humans. Some successful images have been reconstructed from the measured data. Possibilities and limitations of extending the system to a 32 electrode system, or even to a 128 electrode system, were discussed by Brown and Seagar [1987].

![Block diagram of the Sheffield data collection system.](image)

**Fig. 1.4.4** A block diagram of the Sheffield data collection system.

The 32 electrodes data collection system constructed in the Rensselaer Polytechnic Institute was designed for "maximum flexibility with a minimum of special-purpose hardware." The unique feature of the system is an array of 32 current generators, each has a separately programmable output level. The frequency of the current sources is 15 kHz and the maximum load for the current generators is 1000 Ω at 5 mA. The boundary potentials are fed to a voltmeter through a multiplexer. The
voltmeter consists of a high-pass filter \((f_{oc} = 1.9 \text{ kHz})\) followed by a synchronous full-wave demodulator. A two-stage low-pass filter having cutoff frequencies of 177 Hz and 1770 Hz then produces a \(\pm 5\text{V}\) dc signal proportional to the electrode voltage amplitude. This voltage is sampled by a 16-bit AD converter.

In the Rensselaer system, wide electrodes are used instead of thin electrodes. Modelling of the electrodes was studied by Cheng [1989]. The system has been used to confirm Issacson's conclusions about distinguishability and optimal current pattern. Some reconstructions using measured data by the system are also reported.

![Block Diagram](image)

Fig. 1.4.5 A block diagram of the Rensselaer data collection system.

1.5 Objectives, Thesis Review and Publications

The reconstruction technique is a central subject in the study of electrical impedance tomography and various reconstruction algorithms have been developed in past years. However, many important properties of the existing reconstruction
algorithms still remain unclear and further investigation and improvement of the algorithms are necessary. Furthermore, novel reconstruction approaches are highly desired for improving the resolution of electrical impedance tomography.

Many published iterative algorithms were developed based on different theories. Murai's algorithm, for instance, was derived from Geselowitz's sensitivity theory but Yorkey's algorithm was developed from a finite element modelling. It is, therefore, necessary to explore the relationship between various iterative algorithms.

The convergence behaviour is an important issue to iterative algorithms. In the absence of a priori information, iterative algorithms are usually started from a uniform distribution. The convergence of iterative algorithms under uniform starting distributions has yet to be fully investigated.

The accuracy of the forward calculation in iterative algorithms is essential to the quality of reconstructions. Most forward calculation involves finite element analysis but the problems associated the accuracy of the finite element analysis in iterative reconstruction approach has not received adequate attention. Therefore, study on the relationship between reconstruction accuracy and finite element analysis accuracy under different boundary current injection scheme is required.

The filtered back-projection reconstruction is one of the major linear approaches to electrical impedance tomography and its usefulness has been proved by clinical applications. However, the approach does not have a firm mathematical foundation and a study of the method using integral equation may lead to a more precise mathematical description and improved techniques to linear reconstruction approaches.

The linear algorithm is based on a linearized Poisson's equation. The validity of the linearized Poisson's equation requires that the condition of small perturbation be satisfied. This condition has yet been examined in many reported reconstructions. Additionally, the linearization of Poisson's equation will inevitably introduce errors.
An investigation on the errors is, therefore, desired which is important to the development of the linear reconstruction approach.

This thesis is dedicated to the study, analysis and development of reconstruction algorithms for electrical impedance tomography. In chapter 2, an iterative reconstruction algorithm is derived based on Poisson's equation, finite element method and electrical network theory. The nonlinearity of the reconstruction problem is addressed with the help of network theory. Equivalence between the algorithm and many reported iterative algorithms is discussed. Tests are carried out to investigate the effectiveness of the algorithm under the condition of uniform starting distributions. A modified iterative algorithm is developed to overcome modelling error and contact impedance error. Encouraging results are obtained from the modified algorithm.

In chapter 3, a linear integral equation reconstruction approach is derived from linearized Poisson's equation, conformal transformation and Fourier transformation. Properties of the integral equation are discussed and a numerical algorithm is developed. The relationship between the integral equation method, Radon's back-projection theory and Barber-Brown's filtered back-projection method is discussed. Reconstructions of both concentric and offset objects using simulated and measured data are presented. Problems associated with the integral equation method are discussed.

Chapter 4 is an error analysis of the linearized Poisson's equation with the emphasis on the application to the linear reconstruction algorithms in electrical impedance tomography. The study has found that the small perturbation assumption commonly used in linear reconstruction algorithms is violated in majority of practical situations. Nevertheless, the linear algorithms are still able to reconstruct useful relative conductivity images because of the dominance of the re-scaling error. This study provides a novel explanation to the validity of many linear algorithms.
In chapter 5, conclusions are drawn based on the research work in previous chapters. Suggestions on future development of reconstruction algorithms for electrical impedance tomography are also addressed.

Publications arising from the thesis:


Chapter 2

Iterative Reconstruction Algorithms
2.1 Introduction

Most iterative reconstruction algorithms for electrical impedance tomography solve a set of nonlinear equations which are usually obtained from Poisson's equation using the finite element or finite difference modellings.

Theoretically, iterative reconstruction approaches have been regarded as being able to produce accurate images provided that the size of elements are infinitely small, numerical calculations have no error, and the measured data is free from noise. Practically, quality of the reconstructed images depend on many factors such as modelling error, numerical error, measurement error, boundary current injection, starting distribution of iteration, etc..

Many iterative reconstruction algorithms have been reported. Dines and Lytle [1981] developed an iterative algorithm based on circuit modelling of continuous objects. Kim et al.[1983] developed an algorithm that used the finite element method to calculate a perturbation matrix which related an exit current change to a resistivity change. The resistivity distribution is then reconstructed from the measured currents using the matrix. Murai and Kagawa [1985] used the Geselowiz sensitivity distribution to calculate a sensitivity matrix that related the measured voltages to the conductivity distribution to be reconstructed. Wexler et al.[1985] solved Poisson's equation with Neumann and Dirichlet boundary conditions using the finite element method. The current difference resulting from the two boundary conditions was then minimized to yield the resistivity distribution. Yorkey et al.[1987] developed an iterative algorithm using finite element modelling and the Newton-Raphson method. Hua et al.[1990] modified the Yorkey's algorithm by using a complete finite element model that includes the contact impedance of the electrodes, and optimal boundary currents in the iterations.

Whilst most of the iterative algorithms were tested using simulated data, few have used measured data. Although extensive work has been carried out, many aspects of iterative reconstruction still remain unclear. Therefore further effort is required in order to apply the approaches to real situations.
In this chapter, the following aspects of iterative reconstruction approaches are investigated:

(i) The relationship and equivalence between the algorithms derived from an electrical network modelling and that derived from a finite element modelling.

(ii) Convergence behaviour of the iterative reconstruction algorithms using uniform starting distributions.

(iii) Problems associated with measured data processing and modification to the algorithms.

The arrangement of the chapter is as follows: section 2.2 is a brief review of the relevant finite element analysis theory and linear electrical network analysis theory, which is in preparation for the derivation of an iterative reconstruction algorithm in the following sections. In section 2.3, an iterative reconstruction algorithm, called the Least Square Voltage Matching (LSVM) algorithm, is developed. The relations between the LSVM algorithm and other algorithms are discussed in section 2.4. The LSVM algorithm is then tested on network and continuous objects using simulated and measured data. In section 2.6, a modified LSVM algorithm is developed to overcome problems associated with the application of the LSVM algorithm. The modified algorithm is tested using measured data from continuous objects.

2.2 Review of Finite Element Analysis and Nodal Linear Network Analysis

In this section, finite element analysis of static electrical potential and nodal linear network analysis will be briefly reviewed. The review is a mathematical preparation for the development and investigation of iterative reconstruction algorithms for electrical impedance tomography. More details about the theories can be found in textbooks.
2.2.1 Solving Static Electrical Potential by Finite Element Method

The finite element method is one of the most flexible tools for solving a wide range of engineering problems such as electromagnetic field distribution, structural analysis, deformation of solids, transfer of heat, flow of fluids, etc. The major advantage of method is that it can be used to solve almost any engineering problem for which a differential equation can be written. Also, it can be applied to systems with almost any geometric configuration or boundary conditions. Thus when analytic techniques of solving differential equations become unfeasible, the finite element method, or some other numerical method, can be used to obtain a solution.

A. Formulation of the finite element analysis of static electrical potential in a two-dimensional conducting medium

Solving an engineering problem by the finite element method follows an orderly step-by-step process, as shown in Table 2.1. Details concerning general theory and implementation of the finite element method can be found in various text books [Silvester 1986, Allaire 1985, Davis 1980, Huebner 1975].

<table>
<thead>
<tr>
<th>Step</th>
<th>Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Formulate governing equations and boundary conditions.</td>
</tr>
<tr>
<td>2.</td>
<td>Divide analysis domain into finite elements.</td>
</tr>
<tr>
<td>3.</td>
<td>Select interpolation functions.</td>
</tr>
<tr>
<td>5.</td>
<td>Assemble global equations.</td>
</tr>
<tr>
<td>7.</td>
<td>Verification of solution.</td>
</tr>
</tbody>
</table>

Table 2.1 Procedures of Finite Element Method
The static electrical potential inside a two-dimensional conducting medium \( D \) is governed by Poisson's equation

\[ \nabla (\sigma \nabla \phi) = 0 \quad (2.2.1a) \]

where \( \phi \) and \( \sigma \) are the potential and conductivity distribution, respectively. On the boundary of the domain \( D \), denoted by \( \bar{D} \), the following general boundary conditions are given (see Fig.2.2.1):

\[ \phi = f_1(x,y) \quad \text{on } \bar{D}_1 \quad (2.2.1b) \]

and

\[ \sigma \frac{\partial \phi}{\partial n} = f_2(x,y) \quad \text{on } \bar{D}_2 \quad (2.2.1c) \]

where \( \bar{D} = \bar{D}_1 + \bar{D}_2 \), \( \frac{\partial \phi}{\partial n} \) is the normal gradient of potential \( \phi \), \( f_1(x,y) \) and \( f_2(x,y) \) are known functions.

**Fig.2.2.1** A diagram for the boundary problem (2.2.1).

In finite element analysis, the domain \( D \) has to be divided into elements. Although many different element shapes can be used, we will restrict ourself to
triangular elements with nodes at the vertices. Formulation of finite element equations with more complicated elements can be performed in an analogous manner.

For the chosen triangular element, various interpolation functions are available to satisfy different requirements of practical engineering problems. The electrical potential is continuous inside the object and boundary conditions at a discontinuity require that the tangent electric field and the normal electrical displacement must be continuous. The simplest interpolation function that meets the requirements is a two-dimensional linear function, expressed by

$$\phi(e)(x,y) = \beta_1(e) + \beta_2(e) x + \beta_3(e) y,$$  \hspace{1cm} (2.2.2)

where the superscript \(e\) stands for the \(e\)'th element. The constants \(\beta_1(e), \beta_2(e)\) and \(\beta_3(e)\) can be expressed in terms of the triangular element's vertex coordinates \((x_1, y_1), (x_2, y_2), (x_3, y_3)\), as well as the potential values on the vertices of the element, \(\phi_1, \phi_2, \phi_3\), as shown in Fig.2.2.2.

\[\text{Fig.2.2.2} \quad \text{Triangle element for deriving the element equations.}\]

With known \(\beta_1(e), \beta_2(e), \beta_3(e)\) and some simple mathematical re-arrangement, equation (2.2.2) can be written in a more systematic and explicit form:

$$\phi(e)(x,y) = \left[ N^{(e)} \right]^t \left[ \phi^{(e)} \right] = N_1 \phi_1 + N_2 \phi_2 + N_3 \phi_3$$  \hspace{1cm} (2.2.3)
where

\[
\begin{bmatrix}
\phi^e
\end{bmatrix} = 
\begin{bmatrix}
\phi_1, \phi_2, \phi_3
\end{bmatrix}^t \tag{2.2.4}
\]

\[
\begin{bmatrix}
N^e
\end{bmatrix} = 
\begin{bmatrix}
N_1, N_2, N_3
\end{bmatrix}^t \tag{2.2.5}
\]

and

\[
N_i = \frac{a_i + b_i x + c_i y}{2\Delta^e}, \quad i = 1, 2, 3 \tag{2.2.6}
\]

The constants \( a_i, b_i, c_i \) and \( \Delta^e \) are determined by the node coordinates and expressed as

\[
\begin{align*}
    a_1 &= x_2 y_3 - x_3 y_2, \quad b_1 = y_2 - y_3, \quad c_1 = x_3 - x_2, \\
    a_2 &= x_3 y_1 - x_1 y_3, \quad b_2 = y_3 - y_1, \quad c_2 = x_1 - x_3, \\
    a_3 &= x_1 y_2 - x_2 y_1, \quad b_3 = y_1 - y_2, \quad c_3 = x_2 - x_1, \tag{2.2.7}
\end{align*}
\]

\[
\Delta^e = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \text{area of the triangle} \tag{2.2.8}
\]

With the chosen element and interpolation functions, finite element equations can be established using either the variational method or the Galerkin's method [Huebner 1975]. Here, for simplicity, the element equations based on the function (2.2.3) will be obtained by the variational method.

The variation method replaces the solution of a boundary problem by a functional minimization problem. It has been proven that a function \( \phi(x,y) \) satisfying boundary problem (2.2.1) also minimizes the functional
\[ I(\phi) = \iint_{D} \left\{ \frac{\sigma}{2} \left[ \left( \frac{\partial \phi}{\partial x} \right)^2 + \left( \frac{\partial \phi}{\partial y} \right)^2 \right] \right\} dx \, dy - \int_{\partial D} f_2 \phi \, ds \quad (2.2.9) \]

where \( s \) is the coordinate along the boundary of \( D \). Integrating equation (2.2.9) element by element and substituting the interpolation function (2.2.3) into equation (2.2.9), we obtain, for each element, a discretized functional \( I(\phi^{(e)}) \) in terms of the discrete nodal potentials. Requiring that \( I(\phi^{(e)}) \) be a minimum is equivalent to requiring that

\[ \frac{\partial I(\phi^{(e)})}{\partial \phi_i} = 0, \quad i = 1, 2, 3 . \quad (2.2.10) \]

By utilizing equation (2.2.3) we obtain

\[ 0 = \iint_{D^{(e)}} c \left( \left[ \frac{\partial N_i}{\partial x} \right]^T \frac{\partial N_i}{\partial x} + \left[ \frac{\partial N_i}{\partial y} \right]^T \frac{\partial N_i}{\partial y} \right) \phi \, dx \, dy \]

\[ - \iint_{\partial D_{2}^{(e)}} f_2 N_i \, ds, \quad i = 1, 2, 3 \quad (2.2.11) \]

or, in a matrix notation,

\[ \begin{bmatrix} K^{(e)} \end{bmatrix} \begin{bmatrix} \phi^{(e)} \end{bmatrix} = \begin{bmatrix} F^{(e)} \end{bmatrix}, \quad (2.2.12) \]

where the elements of matrix \( [K^{(e)}] \) and vector \( [F^{(e)}] \) are given by

\[ k_{ij}^{(e)} = \iint_{D^{(e)}} c \left\{ \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right\} dx \, dy , \quad (2.2.13a) \]

\[ f_i = \iint_{\partial D_2^{(e)}} f_2 N_i \, ds . \quad i, j = 1, 2, 3 . \quad (2.2.13b) \]
Equation (2.2.12) is a general representation of element equations for the finite element analysis of two-dimensional static potential distribution problems, although it is derived from a triangulation with linear interpolation function. The equation requires information about the conductivity distribution of the continuum, boundary conditions of the potential, element shape, node coordinates, as well as expression of the interpolation function. It should be pointed out that, since the functional (2.2.9) contains only first-order derivatives of the potential, the interpolation functions need only guarantee the continuity of the potential across interelement boundaries. In addition, only boundary condition (2.2.1c) has appeared in the above equations. The boundary condition (2.2.1b) will be imposed after all the element equations are assembled into a set of global equations.

Substituting the linear interpolation function (2.2.6) into the element equations (2.2.12), and assuming conductivity \( \sigma(e) \) to be a constant within each element (i.e. the continuous conductivity is approximated by a piece-wise constant distribution), we obtain

\[
\begin{align*}
  k_{ij}(e) &= \sigma(e) g_{ij}(e) \\
  g_{ij}(e) &= \frac{b_i b_j + c_i c_j}{4 \Delta(e)} , \quad i, j = 1, 2, 3
\end{align*}
\]

(2.2.14) (2.2.15)

where \( b_i, b_j \) and \( c_i, c_j \) are given in equation (2.2.7). From equation (2.2.15) the following properties of the \( k_{ij}(e) \) are found:

\[
\begin{align*}
  k_{11} &= -k_{12} - k_{13}, \\
  k_{22} &= -k_{21} - k_{23}, \\
  k_{33} &= -k_{31} - k_{32}, \\
  k_{ij} &= k_{ji}.
\end{align*}
\]

(2.2.16a) (2.2.16b)
After all the element equations are known individually, they are assembled into a set of global equations before numerical solution takes place. The global equations are established by combining all the element equations and have the following form:

\[
\frac{\partial I(\phi)}{\partial \phi_i} = 0 = \sum_{e=1}^{NE} \left\{ - \int_{D_2(e)} f_2 N_i \, ds \right\} \\
+ \int_{D(e)} \sigma \left( \frac{\partial N_i}{\partial x} \frac{\partial N_i}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_i}{\partial y} \right) \phi \, dx \, dy \tag{2.2.17}
\]

\[i = 1, 2, \ldots, NN\]

where \(NE\) is the number of elements and \(NN\) is the number of nodes excluding the reference node (or datum node), respectively. The resulting global equations form a set of linear equations and are usually expressed in a form of

\[
K \phi = f \tag{2.2.18}
\]

where \(\phi\) and \(f\) are \(NN \times 1\) column vectors, and \(K\) is a \(NN \times NN\) square matrix assembled from element equations (2.2.12).

**B. Properties of the finite element method relevant to the electrical impedance tomography**

The following properties are useful in the application of the finite element method to electrical impedance tomography:

1. In electrical impedance tomography, the object under test can have either a continuous conductivity distribution, or a distribution with certain discontinuities. For
both cases, the finite element method is valid, and the resulting equations for the finite element analysis has the same expression as shown in equation (2.2.12). Thus, reconstruction algorithms based on the finite element method are applicable to almost all the practical situations.

(2) The finite element method approximates a real, continuous potential by a discrete potential distribution with finite freedom. Convergence of the discrete potential to the real potential mainly depends on the size of the elements, the smoothness of the boundary condition, and the numerical accuracy of the computer used.

Generally speaking, the smaller the element size, the more accurate the finite element solution. However, reducing element size increases the number of unknowns, or the dimension of the equation (2.2.18). Extensive studies have indicated that as the dimension of the system matrix $K$ increases, the condition of the matrix degrades which may lead to an unstable or inaccurate solution. Additionally, the computational time and cost increases dramatically as the dimension of the system matrix expands.

The boundary conditions on either voltage or current also play an important part in the convergence of the finite element solution. When the boundary condition has a smooth distribution along the boundary, the finite element analysis converges quickly, and the error between the finite element solution and the real distribution can be estimated theoretically in terms of element size [Blum et al 1986, Lin and Zhu, 1986, Rannacher and Scott, 1982]. For the non-smooth boundary condition, particularly the pulse-like boundary condition, there is no general conclusion about the convergence of the finite element solution.

(3) The system matrix $K$ is a symmetric, positive definite and banded matrix. These properties simplify the finite element analysis, reduce computational expense, and improve numerical accuracy.
2.2.2 Nodal Analysis of Linear Electrical Networks

Any network of lumped components obeys three laws: the Kirchhoff voltage law (KVL), the Kirchhoff current law (KCL), and the components' law (branch characteristics). The first two laws, KVL and KCL, are linear algebraic constraints on branch voltages and currents, arising from interconnection of branches, and are independent of the branch characteristics.

A. Basic mathematical relations

Computer aided nodal voltage analysis of linear electrical networks requires that the physical interconnection of branches in the network be expressed in a way suitable for storing in a digital computer memory. A usual practice is to use an incidence matrix representation.

The incidence matrix is established as follows. A directed graph associated with the given network is drawn according to the following rule: replace each two-terminal component by a line segment, called a branch, with an arrow in the same direction as the assumed positive current through that branch. This arrow also serves as the branch voltage reference: the + voltage terminal is assumed to be at the tail of the arrow. Fig.2.2.3 shows an example of a network and associated directed graph.

Based on the directed graph, a matrix called the node-branch incidence matrix, or simply the incidence matrix, can be established. A precise definition for the incidence matrix is given below:

Definition: For a directed graph with \( NN+1 \) nodes and \( NB \) branches, the incidence matrix is an \( (NN+1) \times NB \) matrix \( A_a = [a_{ij}] \) where

\[
a_{ij} = 1 \text{ if branch } j \text{ is incident at node } i, \text{ and the arrow is pointing away from node } i.
\]

\[
a_{ij} = -1 \text{ if branch } j \text{ is incident at node } i, \text{ and the arrow is pointing toward}
\]
node i.

\[ a_{ij} = 0 \] if branch j is not incident at node i.

Fig. 2.2.3 A network and its associated directed graph.

As an example, for the directed graph in Fig. 2.2.3 (b), its incidence matrix is found to be

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
-1 & 1 & 0 & 0 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & -1 & 1
\end{bmatrix}
\]

The incidence matrix contains all the information about the topology of the circuit, and it can be easily stored in the computer's memory.

The row vectors of the incidence matrix \( A_a \) are not linearly independent because a zero vector will result if all the row vectors of \( A_a \) are added up. By deleting any one
row from $A_a$, a new matrix $A$ is obtained. The matrix $A$ is called reduced incidence matrix, and all row vectors in $A$ are linearly independent. Using the reduced incidence matrix, the following useful equations can be obtained:

(i) Matrix representation of the KCL:

$$A \mathbf{i} = 0,$$  \hspace{1cm} (2.2.19)

where $\mathbf{i}$ is branch current vector defined as

$$\mathbf{i} = [i_1, i_2, \ldots, i_{NB}]^t.$$  \hspace{1cm} (2.2.20)

(ii) Node-branch voltage transformation:

$$\mathbf{v} = A^t \mathbf{v_n},$$  \hspace{1cm} (2.2.21)

where the node voltage vector $\mathbf{v_n}$ and the branch voltage vector $\mathbf{v}$ are defined as

$$\mathbf{v_n} = [v_{n1}, v_{n2}, \ldots, v_{nNN}]^t,$$  \hspace{1cm} (2.2.22)

and

$$\mathbf{v} = [v_1, v_2, \ldots, v_{NB}]^t,$$  \hspace{1cm} (2.2.23)

respectively. Since network analysis involves matrix inversion and the number of nodes in a network is always less than the number of branches, it is beneficial to solve the node voltages first and then obtain the branch voltages by the transformation (2.2.21). In this way, a large amount of computation can be saved.
B. Nodal linear network analysis

In linear network analysis, it is common practice to consider each branch as a composite branch. The composite branch consists of a two-terminal component $y_k$, an independent voltage source with terminal voltage $e_k$, and an independent current source with terminal current $j_k$, as shown in Fig.2.2.4. The two-terminal component $y_k$ can be a linear resistor, capacitor, or inductor.

*Fig.2.2.4 Structure of k'th composite branch in a network.*

Based on the composite branch, the following vector equations are obtained:

$$ v = v - e, \quad (2.2.24) $$

$$ i = i - j, \quad (2.2.25) $$

where vectors $v$ and $v$ are given in equation (2.2.22) and (2.2.23), vectors $e$, $i$ and $j$ are defined as follows:

$$ e = [ e_1, e_2, \ldots, e_{NB} ]^t, \quad (2.2.26) $$
\[ i = [i_1, i_2, \ldots, i_{NB}]^t, \quad (2.2.27) \]

\[ j = [j_1, j_2, \ldots, j_{NB}]^t. \quad (2.2.28) \]

Substituting equation (2.2.25) into equation (2.2.19) yields

\[ A_i = A_j. \quad (2.2.29) \]

Note that the components in a linear network can be characterised by the equation

\[ i = Y v \quad (2.2.30) \]

where \( Y \) is a \( NB \times NB \) matrix. If the network does not have any controlled source and mutual inductance, then \( Y = \text{diag} (y_1, y_2, \ldots, y_N) \). Substituting equation (2.2.30) into equation (2.2.29) and using equation (2.2.24) yields

\[ A Y v = A (j - Y e). \quad (2.2.31) \]

By introducing the node-branch transformation (2.2.21) into equation (2.2.31) we obtain

\[ Y_n v_n = j_n \quad (2.2.32) \]

where

\[ Y_n = A Y A^t \quad (2.2.33) \]

is called the node-admittance matrix, and
\[ j_n = A (j - Ye) \]  \hspace{1cm} (2.2.34)

is called the equivalent nodal current source vector. If the network has no voltage source, i.e. \( e = [0] \), then \( v = v \) and equation (2.2.34) simplifies to

\[ j_n = A j . \]  \hspace{1cm} (2.2.35)

In electrical impedance tomography, the resulting electrical network from finite element analysis is a resistive network with current source only.

Solving equation (2.2.32) yields

\[ v_n = Y_n^{-1} j_n . \]  \hspace{1cm} (2.2.36)

Once \( v_n \) is found, \( v \) can be obtained from equation (2.2.21), and \( v \) and \( i \) can be computed from equations (2.2.24) and (2.2.30), respectively. The above procedure is usually referred to as nodal analysis.

For a linear network consisting of passive elements, the matrix \( Y_n \) is a \( NN \times NN \) symmetric, positive definite matrix. The elements of the node-admittance matrix satisfy the following equation

\[ y_{ii} (> 0) = - \sum_{\substack{i=1 \atop j \neq i}}^{NN} y_{ij} . \]  \hspace{1cm} (2.2.37)

For large network, \( Y_n \) is usually a sparse matrix. By proper arrangement of the number of nodes, \( Y_n \) can be a banded matrix.
2.2.3 Finite Element Analysis and Nodal Analysis

Finite element analysis of static electrical potential in a conducting medium with linear interpolation function is equivalent to a linear resistive network analysis problem. This is demonstrated below.

Consider a triangular element in Fig.2.2.5 (a) with linear interpolation function and constant conductivity within the element. The finite element equations are given by equations (2.2.12), (2.2.14), and (2.2.15).

![Fig.2.2.5](a) Triangular element.  (b) its equivalent circuit.

For a circuit shown in Fig.2.2.5 (b), its node-admittance matrix is given by

\[
Y_n = \begin{bmatrix}
  y_{12} + y_{31} & -y_{12} & -y_{31} \\
  -y_{12} & y_{12} + y_{23} & -y_{23} \\
  -y_{31} & -y_{23} & y_{31} + y_{23}
\end{bmatrix}, \tag{2.2.38}
\]

where

\[
y_{12} = \sigma^{(e)} | g_{12} | \\
y_{23} = \sigma^{(e)} | g_{23} | \\
y_{31} = \sigma^{(e)} | g_{31} | \\
i, j = 1, 2, 3. \tag{2.2.39}
\]
Thus, the element equations of a single triangular element with linear interpolation is equivalent to the nodal analysis of a linear resistive network given in Fig.2.2.5 (b).

When two finite elements are interconnected, an equivalent network may be constructed by assembling the equivalent networks of each single element. Fig.2.2.6 shows the situation where two triangular elements are interconnected. It is seen that the component between nodes 2 and 4 is composed of conductances on the neighbouring boundary of each equivalent network, as shown in Fig.2.2.5 (b). Application of the principle to a general finite element mesh is straightforward. In this thesis, the network generated from a finite element mesh using linear interpolation functions is called the equivalent network for that mesh.

![Fig.2.2.6 Connection of two elements and the equivalent circuit.](image)

Electrical impedance tomography aims to produce an image of the conductivity distribution. In the equivalent network of each single finite element, the three conductances are associated with the same conductivity variable $\sigma^{(e)}$, as seen in equation (2.2.39), and it is necessary to establish a relationship between the component values in an equivalent network and the piecewise distribution of conductivity. Consider the network in Fig.2.2.6. The relationship between the component and the circuit and the conductivity values in each element is found to be
\[ y = T \sigma \]  \hspace{1cm} (2.2.40)

where

\[ y = [y_{12}, y_{23}, y_{34}, y_{41}, y_{24}]^T \]  \hspace{1cm} (2.2.41)

\[ \sigma = [\sigma^{(1)}, \sigma^{(2)}]^T \]  \hspace{1cm} (2.2.42)

\[
T = \begin{bmatrix}
\sigma^{(1)} \\
g_{12} & 0 \\
0 & g^{(2)}_{23} \\
0 & g^{(2)}_{34} \\
g_{41} & 0 \\
g_{24} & g_{24}^{(2)} \\
\end{bmatrix}
\]  \hspace{1cm} (2.2.43)

Element of matrix \( T \) can be evaluated from equation (2.2.15).

In practice, after an equivalent network of \( NB \) branches is established from a finite element mesh of \( NE \) elements, it is always possible to find a \( NB \times NE \) matrix \( T \) which connects the component vector \( y \) with the discrete conductivity vector \( \sigma \). Since the \( NB \) and \( NE \) has the relationship:

\[ NB = 2NE + 1 \]  \hspace{1cm} (2.2.44)

for a triangular element, the matrix \( T \) is a tall matrix. Thus if the matrix \( T \) is of rank \( NE \), a unique solution \( \sigma \) can be obtained from
\[ \sigma = (T^\top T)^{-1} T^\top y. \quad (2.2.45) \]

From the structure of the equivalent network, it is seen that each branch has at most two components connected in parallel from neighbouring finite elements, and therefore the rows of matrix \( T \) have at most two nonzero terms. Moreover, it is easy to see that each column of the matrix has at most three nonzero terms for a triangular element. Thus, according to the composition of the vector \( y \), it is known that any two rows of \( T \) are linearly independent. This implies that the matrix \( T \) is of rank \( NE \) and therefore the unknown vector \( \sigma \) can be uniquely solved from equation (2.2.45).

The unique connection between a discrete distribution of conductivity in finite element analysis and the component distribution in the equivalent network allows us to reconstruct the discrete conductivity \( \sigma \) in two ways: one is to reconstruct the vector \( \sigma \) directly and the other is to reconstruct the vector \( y \) first and then solve for the vector \( \sigma \) in terms of the known vector \( y \). The former is normally used in practical algorithms whereas the latter is useful in investigation of the relationship between various reconstruction algorithms.

Since techniques for computer aided analysis of linear electrical networks have been well developed, and there is increasing interests in inverse network problems such as circuit diagnosis, system identification, etc., it is often beneficial to relate the finite element analysis problem with an equivalent electrical network analysis problem. This equivalence provides an alternative to the derivation of the iterative algorithms for the electrical impedance tomography.

2.3 Least Square Voltages Matching Reconstruction Algorithm (LSVM)

The finite element method provides a suitable mathematical ground for the numerical solution of static electrical potentials in a conducting medium and many
reconstruction algorithms have been already developed [Murai, 1985, Yorkey, 1987, Newell, 1988, Hua, 1990]. In this section, an iterative reconstruction algorithm will be derived based on nodal network analysis theory and an improved Newton-Raphson method. Due to the equivalence between finite element analysis and nodal linear network analysis, the resulting iterative algorithm is mathematically equivalent to algorithms developed from the finite element method. However, the network derivation approach reveals some properties of the algorithm that have not been addressed elsewhere, such as the nonlinearity of the reconstruction equations and the condition that the reconstruction equations become linear equations. The system matrix of the iterative algorithm, which plays an important role in the reconstruction, is constructed by using an adjoint network method which simplifies the process of updating the system matrix at each iteration step.

2.3.1 A General Formulation

In an electrical network, assume that the interconnection of all the branches is given, as well as the position and value of all independent current and voltage sources. We wish to calculate the component values for the network from the measured node or branch voltages.

Suppose we re-write the component characteristic equation as

\[ i = V y \] (2.3.1)

where \( i \) is defined in (2.2.27), and \( V \) and \( y \) are defined as

\[ V = \text{diag} ( v_1, v_2, \ldots, v_b ), \] (2.3.2)

\[ y = [ y_1, y_2, \ldots, y_b ]^\top. \] (2.3.3)
Substituting (2.3.1) into Kirchhoff's current law (2.2.29) yields

\[ H y = j_n \]  \hspace{1cm} (2.3.4)

where

\[ H = AV \]  \hspace{1cm} (2.3.5)

\[ j_n = Aj \]  \hspace{1cm} (2.3.6)

Since the topology of the network and values of independent sources are known for the inverse problem, \( j_n \) is a known vector and matrix \( H \) is constructed from the branch voltages in the network. Equation (2.3.4) establishes a useful relationship for reconstruction of component values of a network, where matrix \( H \) plays an important role in relating the unknown component vector \( y \) and the known vector \( j_n \).

2.3.2 A Linear Equations When All the Node Voltages Are Known

In the case that all the node voltages are measureable, matrix \( H \) is known and equation (2.3.4) is a set linear equations with respect to the unknown \( y \). However, equation (2.3.4) is usually under-determined because \( NN \) (number of the nodes excluding datum) is often less than \( NB \) (number of the branches). It is necessary, therefore, to increase the number of independent equations so that the value of components can be solved uniquely. In practice, this is achieved by using a multi-excitation procedure which applies a different set of current sources to different ports of the network one after another. For each set of current sources, the resulting node voltage distribution is normally different. Thus the number of independent node equations can be increased after the resulting node voltages for every set of current
sources are collected. In electrical impedance tomography, the sets of current sources are often called the boundary current patterns.

Denoting the node equations for i'th boundary current pattern as

$$H^i y = j^i_n, \quad i = 1, 2, \ldots, \text{NP} \quad (2.3.7)$$

where NP is the number of boundary current pattern, then by assembling all the node equations, we have

$$H y = j_n \quad (2.3.8)$$

where $H$ is a $(NN-\text{NP}) \times \text{NB}$ matrix, and $H$ and $j_n$ are composed of

$$H = A \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^\text{NP} \end{bmatrix}, \quad j_n = A \begin{bmatrix} j^1 \\ j^2 \\ \vdots \\ j^\text{NP} \end{bmatrix} \quad (2.3.9)$$

It is worthwhile noting that the incidence matrix $A$ is independent of the boundary current pattern. If $H$ is of rank $\text{NB}$, then $y$ can be uniquely solved as

$$y = (H^\text{T} H)^{-1} (H^\text{T} j_n) \quad (2.3.10)$$

The linear reconstruction algorithm has little practical value because the reconstruction problem becomes trivial if all the node voltages are measurable. However, it provides a
limit to the nonlinear reconstruction algorithms that will be derived in the following section.

2.3.3 Nonlinear Equation When Partial Node Voltages Are Known

Assume that in a network of NN nodes, only some of the node voltages are accessible so that only MB branch voltages are known. By properly labeling branch voltages of the network, equation (2.3.7) may be expressed as

\[ A_{\alpha} V^i_{\alpha} y_{\alpha} + A_{\beta} V^i_{\beta} y_{\beta} = j_n, \quad i = 1, 2, \ldots, NP \]  

(2.3.11)

where \( V^i_{\alpha} = \text{diag} (v^i_1, v^i_2, \ldots, v^i_{MB}) \) contains the known branch voltages, \( V^i_{\beta} = \text{diag} (v^i_{MB+1}, \ldots, v^i_{NB}) \) contains unknown voltages, and \( A_{\alpha}, A_{\beta}, y_{\alpha} \) and \( y_{\beta} \) are partitioned from \( A \) and \( y \) in accordance with the partition of \( V^i \). Equation (2.3.11) is a nonlinear equation because of the product of unknown vectors \( V^i_{\beta} \) and \( y_{\beta} \).

The nonlinearity of the equation (2.3.11) can be treated in two ways. If \( V^i_{\beta} \) is regarded as a variable independent of \( y \), then equation (2.3.11) becomes a quadratic nonlinear equation. The number of unknowns in the quadratic nonlinear equation is equal to \((NB - MB) + NB = 2NB - MB\), which is greater than the number of components in the network. Based on circuit theory, branch voltages (and node voltages) can have derivatives up to any order with respect to component values because the voltages may be expressed as ratios of polynomials in terms of component values of a network. Thus, if \( V^i_{\beta} \) is treated as a function of \( y \), then equation (2.3.11) will have an infinite order of nonlinearity whilst the number of unknowns remains the same as the number of components in a network. Therefore, a trade-off exists between the order of nonlinearity and the number of unknowns for the nonlinear inverse problem.
In practice, the spatial resolution of a reconstructed conductivity image is determined by the size of elements in the finite element model. Each element contains only one unknown conductivity value. The more elements a continuum is divided into, the better the resolution. However, the number of measurable branch (node) voltages, which is equal to the number of available equations for reconstructing the conductivity, is determined by the number of boundary electrodes and the number of boundary current patterns. The solvability of the reconstruction problem requires that the number of independent equations be at least the same as that of unknowns. Thus, for a system with fixed number of electrodes, $V^i_\beta$ should be treated as a function of component values so that the number of unknown can be reduced to the minimum in order to gain the maximum resolution.

As a nonlinear equation, equation (2.3.11) has non-unique solutions. However, as $MB$ approaches $NB$, the dimension of matrix $V^i_\beta$ continuously decreases. At the limit of $MB = NB$, equation (2.3.11) becomes linear which has a unique solution. Thus, it is reasonable to expected that equation (2.3.11) will have less possible solutions when more measurable voltages are used. A nonlinear equation possessing less solutions is always easier to solve than a equation of more solutions, In other words, more measureable voltages lead to more constraints on the reconstruction problem, which may help to obtain better reconstructions.

In equation (2.3.11), branch voltages were used instead of node voltages. The advantages are that component values appear explicitly in the equation, and that when all the voltages are measurable, the equation reduces to a linear one with respect to the unknown component values. The disadvantage is that the branch voltages are not as convenient to obtain as node voltages.

A more suitable equation can be obtained from equation (2.2.32). Suppose node voltages on node 1 through $NA$ ($NA < NN$) are measureable and voltages on the remaining nodes are unknown. By partitioning the node voltage vector $v_n$ into two parts: $v_\zeta = [v_{1n}, \ldots, v_{NAn}]^t$ and $v_\zeta = [v_{(NA+1)n}, \ldots, v_{NN}]^t$, and
partitioning $Y_n$ into $Y_\xi$ and $Y_\zeta$ accordingly, the nodal equation (2.2.32) can be expressed as

$$Y_\xi v_\xi + Y_\zeta v_\zeta = j_n. \quad (2.3.12)$$

Since each term in the nodal-admittance matrix $Y_n$ is a linear combination of component values of a network, then equation (2.3.12) is a quadratic nonlinear equation if $v_\zeta$ is treated as an independent variable, or an equation of infinite order of nonlinearity if $v_\zeta$ is treated as a function of component values.

### 2.3.4 A Least Square Voltage Matching Reconstruction Algorithm

In order to derive a reconstruction algorithm, we replace the vector $v_\xi$ in equation (2.3.12) by a measured node voltage vector $\tilde{v}_\xi$. We then seek a component value distribution $y$ which minimizes the equation (2.3.12) so that

$$\| Y_\xi (v_\xi - \tilde{v}_\xi) + Y_\zeta v_\zeta - j_n \|^2 = \min. \quad , \quad (2.3.14)$$

where $\| \cdot \|$ stands for a norm in $L^2$ space. Substituting equation (2.312) into (2.3.13) yields

$$\| Y_\xi (v_\xi - \tilde{v}_\xi) \|^2 = \min \quad (2.3.15)$$

which is equivalent to

$$F(y) = \frac{1}{2} \| \tilde{v}_\xi - v_\xi \|^2 = \min. \quad (2.3.15)$$

Thus the problem of determining component values in a network from measured node voltages has been converted to a Least Square Voltage Matching (LSVM) problem, or a
minimum point problem of the objective functional $F(y)$. Notice that in equation (2.3.15), vector $\tilde{v}_\xi$ consists of measured voltages, and vector $v_\xi$ consists of calculated node voltages. Thus equation (2.3.15) determines the unknown component values of a network by matching the calculated node voltages with the calculated node voltages.

Since node voltages are nonlinear functions of component values (of infinite order of nonlinearity), iterative methods are required for the solution of equation (2.3.15). There exist many methods for searching for the minima. Basic gradient methods include the steepest descent, the Newton search, the conjugate directions, the Fletcher-Reeves method, the Fletcher-Powell method, and the Modified Newton-Raphson (MNR) method (or Gaussian-Newton method).

The MNR has a second order convergence speed when the iteration gets close to a minimum point, and it only uses the first order derivatives of the functional $F(y)$ in constructing the searching direction at each iteration step. In addition, the method uses an optimal step length at each iteration in order to speed up the convergence. Therefore the MNR method is chosen to solve the LSVM problem.

Application of the MNR to equation (2.3.15) yields the following LSVM iterative reconstruction algorithm:

(1) Select a vector $y_0$ (called starting distribution, or starting point in functional analysis) to start the iteration.

(2) Calculate the searching direction vector from the equation

$$J_F \Delta y_0 = - \{ \tilde{v}_\xi - v_\xi(y_0) \}$$

(2.3.16a)

which yields
\[ \Delta y_0 = - \left( J_F^T J_F \right)^{-1} J_F^T \left( \tilde{v}_\xi - v_\xi(y_0) \right) \]  \hspace{1cm} (2.3.16b)

where \( v_\xi(y_0) \) is the node voltages obtained from nodal analysis, and the Jacobian matrix \( J_F \) is defined as

\[ J_F = \begin{bmatrix}
\frac{\partial v_{1n}}{\partial y_1} & \cdots & \frac{\partial v_{1n}}{\partial y_{NB}} \\
\cdots & \cdots & \cdots \\
\frac{\partial v_{N_{An}}}{\partial y_1} & \cdots & \frac{\partial v_{N_{An}}}{\partial y_{NB}}
\end{bmatrix} \]  \hspace{1cm} (2.3.17)

(3) Find an optimal step length \( \lambda_s \) such that \( F(y_0 + \lambda_s \Delta y_0) = \min. \), and then let

\[ y_1 = y_0 + \lambda_s \Delta y_0 \]  \hspace{1cm} (2.3.18)

(4) Replace \( y_0 \) by \( y_1 \) and repeat above procedures until at the k'th step

\[ F(y_k) = \frac{1}{2} \| \tilde{v}_\xi - v_\xi \|^2 < \varepsilon_1 \]  \hspace{1cm} (2.3.19a)

or

\[ \| \Delta y_k \| < \varepsilon_2 \]  \hspace{1cm} (2.3.19b)

is satisfied, where \( \varepsilon_1 \) and \( \varepsilon_2 \) are given positive numbers for controlling the precision of the solution, then terminate the iteration.

Equation (2.3.19a) is normally called the forward calculation, and equations (2.3.16) to (2.3.18) are called the inverse calculations.
The following properties of the solution of nonlinear equations are important whilst solving the least square problem (2.3.15). Firstly, although existence of the solution for equation (2.3.15) has been guaranteed by the physical process itself, uniqueness of solution is not guaranteed unless the nonlinear function is convex in the domain of definition. In general, the convexness is not available for the reconstruction problem but it can be shown that the conductivity distribution we are searching for is the global minimum of the equation (2.3.15), which is a useful \textit{a priori} information to the reconstruction. Secondly, for a non-convex function, it is essential to select a suitable starting point in order for an iterative algorithm to reach a desired solution. This is possible only for limited situations and is one of the active research topics in applications of low frequency current tomography. Thirdly, the number of solutions of a non-linear equation is intimately related to the number of variables in the equation. As the number of variables increases, the number of solutions increases. This makes it more difficult, sometimes impossible, to select a suitable initial point for the iterative algorithm to arrive at the desired solution. Generally speaking, it is always preferable to solve an equation with less variables.

2.3.5 Evaluating Jacobian Matrix Using Adjoint Network Method

In the LSVM iterative algorithm, the Jacobian matrix $\mathbf{J}_F$ plays an important role in the iteration. Evaluation of $\mathbf{J}_F$ normally consumes a significant part of the computing time in each iteration step. In addition, programming is often a difficult task because of the complexity of the formula for evaluating terms in $\mathbf{J}_F$. However, in circuit analysis, $\mathbf{J}_F$ can be calculated effectively by using an adjoint network method.

The adjoint network method can be used for calculating the derivatives of a node voltage $v_i$ with respect to the $j$th component value $y_j$. The method requires the construction of an adjoint network (see appendix A) from a network under analysis, called the original network, based on the following rules:
The adjoint network is obtained by setting all the current sources to zero in the original network. Topology and component values in the adjoint network remain the same as that in original network.

A unit current source is applied between ground and the node \( i \), whose voltage derivatives with respect to component values in the network are to be found.

The derivative of the node voltage \( v_{ni} \) with respect to the component conductance (or admittance) \( y_j \) is then expressed in terms of branch voltages in both original and adjoint network as

\[
\frac{\partial v_{ni}}{\partial y_j} = - v_j \hat{v}_j
\]

where \( v_j \) and \( \hat{v}_j \) are the \( j \)'th branch voltage of the original network and the \( i \)'th adjoint network (i.e. the adjoint network with a unit current source applied between the \( i \)'th node and ground of the original network), respectively. Thus the Jacobian matrix in (2.3.17) can be expressed as

\[
J_F = \begin{bmatrix}
v_1 & \hat{v}_1 & \cdots & v_{NB} & \hat{v}_{NB} \\
\vdots & \ddots & \cdots & \vdots & \vdots \\
v_1 & \hat{v}_1 & \cdots & v_{NA} & \hat{v}_{NB}
\end{bmatrix}
\]

So far our attention is focused on the reconstruction of component values of a network from measured node voltages. An iterative algorithm for reconstruction of conductivity distribution can be established by replacing the vector \( y \) with a vector \( \sigma = [\sigma_1, \ldots, \sigma_{NE}]^T \) in the algorithm (2.3.15) - (2.3.19) and modifying entries in the Jacobian matrix accordingly. The modified Jacobian matrix consists of derivatives of node voltages with respect to conductivity in each element. Based on the relation
between the finite element and its equivalent network, as shown in Fig.2.2.5 and Fig.2.2.6, the entries in the modified Jacobian matrix are given by (Appendix A)

\[
(J_F)_{ij} = \frac{\partial v_{ni}}{\partial \sigma_j} = - \sum_{s=1}^{NS} g_s \left( v_s^i v_s^j \right) (j),
\]

\[i = 1, \ldots, NP, \quad j = 1, \ldots, NE\]  \hspace{1cm} (2.3.22)

where NS is the number of sides in the j'th element, \( g_s \) is the coefficient associated with component value on side s which is determined by the finite element method. For triangular elements, \( g_s \) is calculated from equation (2.2.15). Since the number of elements is always less than the number of components in an equivalent network, it is advantageous to perform the reconstruction with respect to conductivity vector \( \sigma \) so as to reduce computational complexity. In the following context, reconstructions are all carried out with respect to conductivity.

Iteration algorithm (2.3.16) requires inversion of a \( NE \times NE \) matrix \( J_F^T J_F \) but \( J_F \) in equation (2.3.22) is a \( NA \times NE \) matrix, where \( NA \) is smaller than \( NE \) for equivalent networks. From matrix theory, the rank of \( J_F^T J_F \) is at most \( NA \). Therefore \( J_F^T J_F \) is rank deficient and its inverse does not exist. This problem can be overcome again by applying a multi-excitation procedure so as to make the reconstruction problem solvable. Combining all the \( J_F \) for each boundary current pattern into one matrix \( J_F \), and combining each set of measured voltage vector \([\vec{v}_\xi - v_\xi(c)]\) into one vector \([\vec{v}_\xi - v_\xi(c)]\), we have

\[
J_F = \begin{bmatrix}
J_F^1 \\
\vdots \\
J_F^p
\end{bmatrix}, \quad \vec{v}_\xi - v_\xi(c) = \begin{bmatrix}
(\vec{v}_\xi - v_\xi(c))^1 \\
\vdots \\
(\vec{v}_\xi - v_\xi(c))^p
\end{bmatrix}
\]  \hspace{1cm} (2.3.23)
where the superscripts relate to the order of boundary current patterns. Accordingly, equation (2.3.16) is modified to be
\[ \Delta c = - \{ J^T F J \}^{-1} J^T F \{ \tilde{v}_\xi - v_{\xi(c)} \}. \] (2.3.24)

Equation (2.3.24) requires the existence of the inverse matrix \( \{ J^T F J \}^{-1} \).

From NP current patterns and NA accessible nodes we can have NP x NA node voltage data (equations) for reconstruction. Due to reciprocity of a linear network, there may be some redundancy in the node voltages. After the dependent (redundant) node voltages are removed, the dimension of vector \( [\tilde{v}_\xi - v_{\xi(c)}] \) becomes less than NP x NA. The necessary condition for the inverse matrix to exist requires that the total independent voltages be equal to or greater than the number of unknowns in the equation. However, the sufficient condition for the inverse matrix to exist is not available because even when all the node voltage data are independent, it is possible for different functions to have the same gradient values at certain points. In practice, existence of the inverse matrix is often examined numerically at each iteration step. Experience has shown that when more independent voltages are used in the nonlinear equations, there is a better chance that the inverse matrix exists.

The major advantage of evaluating the Jacobian matrix \( J_F \) using the adjoint network method is the saving of computational time. From equation (2.3.21) and (2.3.22), it is seen that entries of \( J_F \) are composed of branch voltages of both original and adjoint networks. Since the two networks have the same topology and component values, they have the same nodal admittance matrix \( Y_n \). Moreover, when locations and magnitudes of current sources in the networks are changed so that new distributions of node voltage are generated, the nodal admittance matrix \( Y_n \) remains unchanged. Therefore, at each iteration step of the reconstruction, only one matrix inversion \( Y_n^{-1} \) is required in order to calculate the Jacobian matrix and vectors in equation (2.3.23).
2.4 Relations Between Some Iterative Reconstruction Algorithms

Many iterative algorithms for impedance imaging have been reported. A comparison of the numerical performance of various algorithms were carried out by Yorkey et al [1987]. By using computer-simulated data they found that, in terms of convergence rate and residual error, the modified Newton-Raphson reconstruction algorithm was superior to other algorithms such as perturbation, double constraint, and simple backprojection.

In this section, we shall investigate theoretical connections between some iterative reconstruction algorithms, which helps the basic understanding of various iterative algorithms.

The LSVM iterative algorithm is very similar to the algorithm developed by Yorkey et al [1987] except in two respects. First, the algorithm is derived based on network theory with emphasis on the connection between network inverse problems and the continuous inverse problems. The derivation process reveals the relationship between the number of accessible nodes, the number of components in a network, and the nonlinearity of the objective functional. The nonlinear reconstruction algorithm becomes linear in its limiting case, i.e. when all the nodes are accessible. Secondly, the iterative algorithm includes a parameter $\lambda_s$ which speeds up the convergence by using optimal step length along each search direction.

In the imaging system of Dines and Lytle [1981], the continuous distribution of an object's conductivity was directly modelled by a resistive network instead of finite elements, and current between boundary electrode pairs was measured whilst voltage sources were applied to boundary electrodes. The reconstruction was nonlinear and an iterative solution procedure similar to the LSVM iterative algorithm was used. Success of their algorithm and some error analysis with simulated data were reported but no reconstruction from practical data was presented. The Dines-Lytle algorithm and the LSVM algorithm can be regarded as a dual algorithm as far as algorithm structure is
concerned but it is important to note that there is no rigorous mathematical connection between a continuous object and Dines-Lytle's network models and, therefore, convergence of the node voltages of a network to that of a continuous object is not guaranteed. The LSVM algorithm, however, does not have this problem based on the finite element analysis theory.

Murai and Kagawa [1985], Sakamoto et al [1983], and Nakayama et al [1981] have all used a reconstruction method based on Geselowitz's sensitivity theorem [1971]. The sensitivity theorem relates a change in boundary transfer impedance $Z$ to a change of interior conductivity in a continuous medium and takes a form of

$$
\Delta Z = Z(\sigma) - Z(\sigma+\Delta\sigma) = -\int_D \Delta\sigma \frac{\nabla \phi(\sigma)}{I_\phi} \frac{\nabla \psi(\sigma+\Delta\sigma)}{I_\psi} \, ds
$$

(2.4.1)

where $\phi$ is the potential distribution over the field $D$ when the current $I_\phi$ is applied to electrode pair (A, B), $\sigma$ is the conductivity distribution while $\phi$ is established by the $I_\phi$, $\psi$ is the potential distribution when current $I_\psi$ is applied to electrode pair (C, D), and $\sigma+\Delta\sigma$ is the conductivity distribution while $\psi$ is produced by the $I_\psi$. Above terminology is depicted in Fig.2.4.1.

![An illustration of the Geselowitz's sensitivity theorem](image)
The transfer impedance $Z(\sigma)$ is defined as

$$Z(\sigma) = \frac{\Phi_{CD}(\sigma)}{I_{\phi}} = \frac{\Psi_{AB}(\sigma)}{I_{\psi}}$$  \hspace{1cm} (2.4.2)$$

where $\Phi_{CD}$ and $\Psi_{AB}$ is the potential difference between electrode pair (C, D) and (A, B), respectively. Murai [1985] expanded $\nabla \psi(\sigma+\Delta \sigma)$ with respect to $\Delta \sigma$ and used the first order approximation to establish an iterative algorithm

$$\Delta Z^{(k)} = \tilde{Z} - Z^{(\sigma^{(k)})} = - \int_{D} \Delta \sigma^{(k)} \frac{\nabla \phi^{(k)}}{I_{\phi}} \frac{\nabla \psi^{(k)}}{I_{\psi}} \, ds$$  \hspace{1cm} (2.4.3)$$

$$\sigma^{(k+1)}(x, y) = \sigma^{(k)}(x, y) + \Delta \sigma^{(k)}(x, y)$$  \hspace{1cm} (2.4.4)$$

where $\tilde{Z}$ is the measured transfer impedance, and quantities with superscript $k$ are calculated at the $k$'th iteration. The integral in equation (2.4.3) is evaluated numerically with the help of the finite element method using triangular dividing and a linear interpolation function. Conductivity inside each element has been assumed to be a constant. The resulting discrete iteration algorithm is given as

$$\tilde{Z}_i - Z_i^{(\sigma^{(k)})} = \sum_{e=1}^{NE} S_{ie} \Delta \sigma_e^{(k)}, \quad i = 1, \ldots, NA$$  \hspace{1cm} (2.4.5)$$

where $NA$ is the number of electrodes (accessible nodes), $NE$ the number of finite elements. The quantity $S_{ie}$ is given as

$$S_{ie} = - \int_{D_e} \frac{\nabla \phi^{(k)}}{I_{\phi}} \frac{\nabla \psi^{(k)}}{I_{\psi}} \, ds_e$$

$$= - \frac{1}{I_{\phi} I_{\psi}} \Phi_e^t \Phi_e \left( b^t b + c^t c \right) \frac{1}{4 \Delta_e} \Psi_e$$  \hspace{1cm} (2.4.6)$$
where

\[ \Phi^e = [\phi_1, \phi_2, \phi_3], \quad \Psi^e = [\psi_1, \psi_2, \psi_3], \]

\[ b^e = [b_1, b_2, b_3], \quad c^e = [c_1, c_2, c_3], \]

\[ b_1 = y_2 - y_3, \quad b_2 = y_3 - y_1, \quad b_3 = y_1 - y_2, \]

\[ c_1 = x_3 - x_2, \quad c_2 = x_1 - x_3, \quad c_3 = x_2 - x_1, \quad (2.4.7) \]

\( \Phi^e \) and \( \Psi^e \) are potentials at three vertices of the \( e \)th element, and \( \Delta_e \) is the area of the element. It has been noticed that Murai's algorithm is equivalent to the LSVM algorithm under certain conditions, as explained below:

When a current source with constant magnitude is applied to a pair of boundary electrodes, equation (2.4.5) can be rewritten as

\[ \tilde{v}_i - v_i (\sigma^{(k)}) = I_\phi \sum_{e=1}^{\text{NE}} S_{ie} \Delta \sigma_{e}^{(k)}, \quad i = 1, \ldots, \text{NA}. \quad (2.4.8) \]

Equation (2.4.8) has the same structure as equation (2.3.16) provided entries of the Jacobian matrix \( J_F \) in (2.3.22) can be replaced by \( I_\phi S_{ie} \).

Denoting the 3 x 3 matrix \( \frac{b^e c^e + c^e b^e}{4 \Delta_e} \) in equation (2.4.6) by \( G = \{g_{ij}\} \) (i, j = 1, 2, 3) and noticing that

\[ g_{ij} = g_{ji} = \frac{b_i b_j + c_i c_j}{4 \Delta_e}, \quad i, j = 1, 2, 3 \]

and

\[ g_{11} = -(g_{12} + g_{13}), \quad g_{22} = -(g_{21} + g_{23}), \quad g_{33} = -(g_{31} + g_{32}) \]

we then have
\[ I_{\phi} S_{le} = \frac{-1}{I_{\psi}} \left\{ (\phi_1 - \phi_2) (\psi_1 - \psi_2) g_{12} + (\phi_1 + \phi_3) (\psi_1 - \psi_3) g_{13} + (\phi_2 - \phi_3) (\psi_2 - \psi_3) g_{23} \right\} \]

If \( I_{\psi} \) is chosen to be a unit current source and triangular division is used in equation (2.3.16), then equation (2.4.9) is identical to equation (2.3.16) with \( g_{ij} \) in equation (2.4.9) being replaced by \( g_s \) (\( s = 1, 2, 3 \)) in equation (2.3.16). Similar results can be obtained for rectangular dividing in the finite element model.

Relations between the LSVM reconstruction algorithm and above mentioned iterative algorithms reveal that all the studies on the LSVM apply to other algorithms, and vice versa. Furthermore, connection between the finite element modelling of continuous inverse problems and network inverse problems, such as circuit diagnosis, system parameters identification, etc., implies that techniques for network inverse problems may be applied to the continuous inverse problem.

## 2.5 Tests on the LSVM Reconstruction Algorithm

The LSVM algorithm, as well as the equivalent algorithms, solve a system of nonlinear equations iteratively. Amongst the minima of the nonlinear equations, the desired conductivity distribution of an object is the global minimum point of the objective function (2.3.15). Since the solution to the nonlinear equation is not unique, convergence of the iteration is strongly dependent on the starting distributions. In practice, the iteration usually starts from a uniform distribution unless \textit{a priori} knowledge of the solution is known. Thus, although the algorithm converges in most
cases, it may not converge to the desired solution. This results in the question of the robustness of the algorithm when uniform starting distribution is used.

The convergence of the algorithm is intimately related with the number of unknowns in the nonlinear equations. As the number of unknowns increase, the number of solutions to the equation grow drastically. This increases the possibility that the iterative algorithm converges to unwanted solutions when it begins with a given starting distribution. Therefore, the algorithm is expected to have a better performance for a system containing less unknowns than for a system with more unknowns.

The LSVM algorithm consists of forward and inverse calculations. The forward calculation is accurate for nodal analysis of a lumped electrical network, and approximate for potential distribution analysis of a continuous object. Based on the finite element method, a calculated potential distribution approaches a real potential distribution only when the size of all elements approaches zero. Thus, the algorithm will be less successful for continuous conductivity reconstruction than network component values identification.

In this section, simulations and real reconstruction experiments are carried out to investigate the following aspects of the LSVM algorithm:

(1) convergent behaviour of the algorithm with respect to the number of unknowns, the accessible nodes to inaccessible node ratio (AIR) and signal to noise ratio (SNR), under the condition of uniform starting distributions.

(2) performance of the algorithm using measured data from electrical networks.

(3) performance of the algorithm using measured data from real continuous objects.

2.5.1 Reconstructions Using Simulated Data and Uniform Starting Distributions

Yorkey et al [1987] performed simulations on their algorithm as well as some other algorithms. They compared the various algorithms in terms of convergence rates
and residual errors, and concluded that the iterative algorithm using the Newton-
Raphson method is superior to other algorithms.

Murai et al [1985] reported simulation results using their own algorithms. The
following were observed from the simulations:

(1) The boundary voltages are more sensitive to changes in the conductivity near the
surface rather than deep inside the object.

(2) The reconstruction based on a finite element mesh with more elements (i.e.
more unknowns) produces inferior results when compared to one with less
elements.

(3) The more independent boundary voltages used for reconstruction, the better the
results.

(4) Reconstructions are sensitive to the noise contained in the measured data.

Due to equivalence between Yorkey's, Murai's and the LSVM algorithms, the
above conclusions also apply to the LSVM algorithm. However, the influence of the
starting distributions to the convergence of the iterative algorithms is not clear.

Although in certain special cases, a priori information about an object is
available so that one can use a suitable starting distribution to speed up the convergence,
uniform starting distribution is still most widely used in iterative algorithms. It is
required, therefore, to investigate the convergence behaviour under the condition of
uniform starting distributions. The simulations in this subsection examine the
performance of the LSVM algorithm using uniform starting distributions in relation to
the number of unknowns, the accessible nodes to inaccessible node ratio (AIR) and
signal to noise ratio (SNR) in a system.

The simulations were performed on three networks, as shown in Fig.2.5.1 (a),
(b), and (c). All the networks were of square grid structure and each branch (i.e. line
between any two node pair) was assigned a random number as component value. The
random number had a uniform distribution within a range of 0.01 - 100 Ω. In each
network, one boundary node was chosen as datum and the remaining were treated as
accessible nodes. Thus the number of accessible nodes, denoted by $NA$, was equal to 11, 15 and 19 for the three networks, respectively.

\[ N_{\text{max}} = \frac{NA (NA + 1)}{2} \] \hspace{1cm} (2.5.1)

For each current pattern, boundary node voltages are calculated by nodal analysis and Gaussian random numbers are added to the calculated voltages to simulate
measurement noise. The Gaussian random numbers are obtained using the following formula [Morgan 1984]

\[ y_i = \left( \sum_{i=1}^{12} x_i \right) - 6 \]

where \( x_i \) is a random number of uniform distribution within (0, 1), and \( y_i \) is a random number of Gaussian distribution with zero mean and unit variance. Each simulation data is then written as

\[ v_{nj}^i + \text{Scale} \times \text{Vave} \times \text{Ran} \]  \hspace{1cm} (2.5.2)

where \( v_{nj}^i \) is the calculated node voltage on the j'th node for the i'th current pattern, "Ran" is a Gaussian random number, "Scale" is a parameter for controlling the amplitude of the added noise, and \( \text{Vave} \) is the arithmetic mean over the calculated noiseless boundary node voltages.

<table>
<thead>
<tr>
<th>Current Injection Port</th>
<th>Voltage Measurement</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 - 0</td>
<td>1 - 0, 2 - 0, 3 - 0, ... , (NA - 1) - 0, NA - 0</td>
</tr>
<tr>
<td>2 - 0</td>
<td>2 - 0, 3 - 0, ... , (NA - 1) - 0, NA - 0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(NA - 1) - 0</td>
<td>(NA - 1) - 0, NA - 0</td>
</tr>
<tr>
<td>NA - 0</td>
<td>NA - 0</td>
</tr>
</tbody>
</table>

**Table 2.5.1** Current source injection and node voltages measurement arrangement in the simulations, where NA is the number of accessible nodes, and 0 indicates the datum node.
Quality of a reconstruction in the simulation is measured by a root mean square parameter \( Q \), which is defined as

\[
Q = \sqrt{\frac{1}{\text{NB}} \sum_{b=1}^{\text{NB}} \left( \frac{y_b - \tilde{y}_b}{y_b} \right)^2}
\]  

where \( \text{NB} \) is the number of component in the network, \( y_b \) is the given resistance value of \( b' \)th component, and \( \tilde{y}_b \) is reconstructed component value.

<table>
<thead>
<tr>
<th>No. of component</th>
<th>network (a)</th>
<th>network (b)</th>
<th>network (c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NB</td>
<td>NB = 24</td>
<td>NB = 40</td>
<td>NB = 60</td>
</tr>
<tr>
<td>NA</td>
<td>NA = 11</td>
<td>NA = 15</td>
<td>NA = 19</td>
</tr>
<tr>
<td>AIR</td>
<td>(\frac{11}{14})</td>
<td>(\frac{15}{24})</td>
<td>(\frac{19}{35})</td>
</tr>
<tr>
<td>Max No. of</td>
<td>66</td>
<td>120</td>
<td>190</td>
</tr>
<tr>
<td>Independent data</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No. of</td>
<td>66</td>
<td>120</td>
<td>190</td>
</tr>
<tr>
<td>Equations solved</td>
<td>&lt; 56</td>
<td>&lt; 118</td>
<td></td>
</tr>
<tr>
<td>Noiseless</td>
<td>1.518E-12</td>
<td>6.004E-5</td>
<td>**</td>
</tr>
<tr>
<td>- 80 dB</td>
<td>6.071E-6</td>
<td>1.984E-3</td>
<td>**</td>
</tr>
<tr>
<td>- 60 dB</td>
<td>5.892E-4</td>
<td>1.776E-1</td>
<td>**</td>
</tr>
<tr>
<td>- 40 dB</td>
<td>4.256</td>
<td>**</td>
<td>**</td>
</tr>
</tbody>
</table>

**Table 2.5.2**  Mean errors with respect to size of networks, Accessible to Inaccessible node Ratio (AIR), number of equations, and noise scales. The "**" indicates that the mean error is greater than 10 so that the iteration is regarded having converged to an unwanted solutions.

The simulation results are presented in Table 2.5.2, with respect to a uniform starting distribution within a range of 0.1 to 500 \( \Omega \). Each network is simulated ten times and the ten values of \( Q \) are averaged and shown in the table with respect to the
size of networks, the AIR value, the scale of noise, and the number of equations used in the reconstruction.

The following conclusions are drawn from the simulations:

1) For the network containing less than 40 components, iteration is able to converge to the correct solution. As the number of components increases, the algorithm fails to converge to the desired solution.

2) The mean error $Q$ increases as the noise level increases. For network (a), reconstructions become unacceptable when the noise level is over -40 dB, and for network (b), reconstructions become meaningless when the noise level is over -60dB.

3) More accurate results are generated when more equations (i.e. more voltage data) are used.

Since the distribution of component values in the simulations consisted of random numbers, the results can then be regarded as a guide to the convergent robustness of the algorithm under the condition of uniform starting distributions.

### 2.5.2 Reconstructions Using Measured Data from Networks

The LSVM algorithm is mathematically rigorous for network component's values identification but approximate for conductivity reconstruction of continuous objects. It is, therefore, interesting to know the performance of the algorithm when applied to identify the values of components in a practical network.

The resistor network under test in this subsection consists of 18 independent nodes and 42 components of 5% tolerance, as shown in Fig.2.5.2. A 20 mA dc current source was applied to the circuit in a sequence shown in Table 2.5.1. Node 1 through to node 11 in the circuit were assumed accessible ($NA = 11, AIR = \frac{11}{18} = 0.611$) and node 0 was used as ground. The maximum number of independent node
voltages is 66 according to equation (2.5.1). The boundary node voltages were
measured with a high precision digital voltmeter.

Fig. 2.5.2 Connection of the resistor network.

Table 2.5.3 presents the reconstructed component values using measured data
of four and five digit precision, respectively. A uniform distribution of the network
component values was used to start the iterations. The relative error is calculated using
the formula given by

\[
\text{Relative error} = \left| \frac{\text{Nominal value} - \text{Calculated value}}{\text{Nominal value}} \right| \times 100 \%
\]
<table>
<thead>
<tr>
<th>Nominal values of resistors</th>
<th>Reconstructed values using five digit precision data</th>
<th>Reconstructed values using four digit precision data</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.</td>
<td>Value ±5%</td>
<td>No.</td>
</tr>
<tr>
<td>1</td>
<td>220</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>56</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>56</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>220</td>
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**Table 2.5.3** Reconstructed component values using measured voltage data of four and five digit precision, respectively. Iterations started from uniform initial distribution ranging from 0.2 to 200 Ω, and converged within 10 iterations.
It has been observed that when the amplitude of the uniform initial distribution is within the range of 0.2 - 200 Ω, the algorithm produced a satisfactory reconstruction, and that the higher precision data results in a better reconstruction. All the results are within the 5% tolerance range of the resistors in the network. When data of three precision was employed in the algorithm, no convergence was achieved for any uniform starting distribution of component values. In addition, if less than 65 equations were used, no convergence was obtained even if the data with five digit precision was used. It has been found that for all successful iterations, the convergence was completed within 10 iteration steps. The results agree with the simulation results in Table 2.5.2 concerning the size of the network, the noise level of measurement error, and the number of the equations used in reconstruction.

The experiments have demonstrated that, under the condition of uniform starting distribution, the LSVM algorithm is effective only for networks of small scale (e.g. network with less than 40 - 50 components). However, if the starting distribution is sufficiently close to the real distribution, the algorithm has been able to converge to the correct solution for networks of larger scale.

2.5.3 Reconstruction Using Measured Data from Real Objects

The experiments in this subsection were designed to investigate the performance of the LSVM algorithm when applied to practical continuous objects. The objects under test consisted of:

1. a piece of square conducting rubber, as shown in Fig.2.5.4 (a), and
2. a piece of circular conducting rubber, as shown in Fig.2.5.5 (a).

The conducting rubber was of 1.5 mm thickness and nominal uniform volume conductivity 8 ohm/cm³. 16 electrodes were placed on the boundary of the square object and 12 electrodes placed on the boundary of the circular object, both with No.1 electrode used as ground. A 10kHz ac current source of 20mA was applied to the
objects in the experiments. The current injection and boundary voltage measurement arrangement was similar to that in Table 2.5.1, except for the different number of boundary electrodes. On the shadow areas of the object in Fig.2.5.4 (a) and Fig.2.5.5 (a), silver paint was placed to provide a perturbation to the uniform conductivity.

A data collection system was constructed for current injection and boundary potential measurement in the experiments. The system consisted of ac current sources (10kHz - 60kHz), 16-channel multiplexers, differential amplifiers, peak-value detector, sample and hold, ADC of 12-bit precision, and a IBM/AT computer. The system block diagram is shown in Fig.2.5.3.

![Fig.2.5.3 A block diagram for the data collection system.](image)

The objects were divided into a number of finite element models, as shown in Fig.2.5.4 (b), (c), and Fig.2.5.5 (b), (c). The square meshes of Fig.2.5.4 (b) and (c)
matched silvered area in the object of Fig.2.5.4 (a) whilst the silvered area in Fig.2.5.5 can only be covered approximately by the mesh elements in Fig.2.5.5 (b) and (c).

It has been observed that, with uniform starting distributions, no acceptable reconstruction could be obtained when measured data was used, irrespective of the finite element meshes. Furthermore, even when the correct distributions of conductivity were used to start the iterations, the algorithm still failed to produce desired solutions. The poor performance of the algorithm was due to the errors caused by unknown contact impedance of the driving electrodes and the poor convergence of the calculated boundary voltages using the finite element method, which forced the LSVM iteration to converge to a unwanted solution.

Fig.2.5.4 (a) Square conducting rubber object with silver paint on the shadow area. (b) and (c) The finite element meshes for the object.
Fig. 2.5.5  (a) Circular conducting rubber object with silver paint on the shadow area.  (b) and (c) The finite element models for the object.

In the following section, the errors will be investigated and an improved algorithm will then be developed.

2.6 Error Identification and a Modified LSVM Algorithm

In this section, data error due to electrode contact impedance and the error due to slow convergence of calculated potential will be identified. Based on the error investigations, a modified LSVM algorithm will be developed and tested.
2.6.1 Data Error from Electrode Contact Impedance

It is well known that the contact impedance of electrodes may introduce error to measured potential data. In the paired current injection scheme, this error exists in the measured potentials on driving electrodes but not in the remaining source-free electrodes because of the extremely high input impedance of the measuring equipment which draws no appreciable current. The data error due to the unknown contact impedance on the driving electrodes in the experiment of subsection 2.5.3 was evident when comparing the measured and theoretical boundary voltage distributions of a circular uniform object with diagonal driving electrodes, as shown in Fig.2.6.1(a).

The measured data was obtained from the same experimental set up as that used in subsection 2.5.3 except that no silver paint was applied on the object. The centre of the object was chosen as potential reference during measurement.

The theoretical solution of the boundary potential distribution is obtained from the following model: The object is a uniform circle with unit radius whose centre is chosen as potential reference. A low frequency ac current source of unit amplitude is applied to a pair of diagonal boundary electrodes which generates a boundary current distribution consisting of a pair of spatial pulses with opposite polarity. The boundary electrodes are assumed small compared with dimension of the object, and therefore the magnitude of the current intensity pulses can be well approximated by 1/d, where d is the width of the electrodes in degree. The geometry of the model is shown in Fig.2.6.1 (a).

Theoretical boundary potential solution for the model of Fig.2.6.1 (a) is given by (see appendix B)

\[
\phi(\theta) = \sum_{i=1}^{\infty} \frac{1}{n} \left( C_n \cos n \theta + S_n \sin n \theta \right)
\]

(2.6.1)

where
\[ C_n = \frac{1}{\pi} \int_{0}^{2\pi} J(\theta) \cos n \theta \, d\theta, \quad S_n = \frac{1}{\pi} \int_{0}^{2\pi} J(\theta) \sin n \theta \, d\theta, \quad (2.6.2) \]

and the current density distribution \( J(\theta) \) is given as

\[
J(\theta) = \begin{cases} 
1/d & |\theta| < 0.5d \\
0 & \text{elsewhere.} 
\end{cases} \quad (2.6.3)
\]

To facilitate the comparison, the measured data was normalized in such a way that the theoretical and measured data on the electrode of position \( \theta = 15^\circ \) (i.e. the electrode neighbouring to the driving electrode) are equalized. The final results are shown in Fig.2.6.1 (b) where the electrode size in the theoretical potential has been chosen to be \( d=1^\circ \) to agree with the physical electrode size in the experimental set up. It can be seen that the measured and calculated voltages have satisfactory agreements on all source-free electrodes. The potential difference on the driving electrode indicates the existence of the contact impedance whose value is generally unknown. Due to the symmetrical geometry and data, only a quarter of the potentials on the object periphery have been presented.
Fig. 2.6.1  Comparison of theoretical and measured boundary potentials of a circular uniform object.

(a) Geometry of the object, where “d” refers to angle in the second fig.
(b) Comparison of the potentials.
2.6.2 Finite Element Solution of Electrical Potentials Under Pulse-like Neumann Boundary Condition

In addition to the contact impedance error, there is error between measured and calculated potentials. According to the finite element method, the calculated voltage distribution converges to a real distribution when the size of all the elements approaches zero. The convergence rate mainly depend on the element sizes in a finite element mesh and the boundary conditions of the problem.

Extensive literature is available on the convergence behaviour of finite element solutions of quasi-static potentials. Usually, interior potential distributions are of concern and boundary potential or current conditions are smoothly distributed. In EIT systems using paired electrodes injection, however, it is boundary potential distribution, not the interior potential, that is of concern. Moreover, the boundary current is of a pulse-like distribution (i.e. the boundary current is zero except on the driving electrodes). Thus the validity of the general conclusions on the convergence of finite element solutions of quasi-potentials needs to be examined.

The convergence property of calculated boundary potentials under the pulse-like current boundary condition (the Neumann condition) is investigated in the following numerical experiments. A circular object of unit radius is divided into three finite element meshes, as shown in Fig.2.6.2.
Fig. 2.6.2  Finite element meshes used in the calculation.
(a) Three layer mesh. (b) Four layer mesh. (c) Six layer mesh.

Potential distributions based on different meshes are calculated using the finite element method under the boundary condition given by equation (2.6.3). An exact theoretical potential is obtained from equation (2.6.1). Comparison of the results are shown in Fig.2.6.3, where the boundary current pulse has a width of $d = 1^\circ$.

Convergence of Finite Element Solutions

![Convergence of Finite Element Solutions](image)

**Fig. 2.6.3**  Convergence of the finite element solutions to the exact theoretical potential.
It is observed that the convergence of the finite element solution depends on the electrode position. Whilst the potentials at the source free electrodes are in good agreement with the theoretical one, there is still significant difference between the theoretical and calculated potentials at the driving electrode. The error is reducing when the object is divided into finer and finer meshes but the convergent rate is very slow.

The width of the boundary current pulses also affects the convergence of the calculated potentials. To see this, a comparison of the theoretical potentials with respect to different width of current pulses are performed and the results are shown in Fig.2.6.4.

![Potentials for Various Current Pulse Width](image)

**Fig.2.6.4** Comparison of potentials for different width of current pulses.

It is seen that the major differences between potentials appear at the driving electrode and its near neighbours. The potential on boundary points about 15 degrees away from the driving electrode is almost independent of the width of current pulse. Therefore the width of current electrode has only a local effect on the resulting boundary potentials.
Notice that in the Fig.2.6.3, the calculated potentials using the finite element method have distributions which resemble the theoretical potentials of Fig.2.6.4 that are induced by boundary currents with larger pulse width. This indicates that the finite element method may have a better convergence under smooth boundary current conditions.

2.6.3 A Modified Algorithm -- Partial LSVM Reconstruction

Algorithm

The calculated potential by the finite element method converges to the true potential at different rates for different locations on an object's boundary. When an object is divided into a relatively coarse mesh and paired boundary current injection is used, the error contained in the finite element solution can be significant on the part of the boundary around the driving electrodes. Additionally, the unpredictable contact impedance on driving electrodes may introduce considerable error to the measured voltage data. These errors can mislead the LSVM iteration and unacceptable reconstructions occur.

According to the convergence study of finite element solutions, the pulse-like boundary current intensity results in fairly slow convergences of potentials on and close to the driving electrodes, whilst potentials on electrodes more than 15° away from the driving electrodes produced satisfactorily fast convergence for all the three meshes used.

The above observations suggest that if the potentials on and close to the driving electrodes are eliminated, two major sources of error will be removed from the measured data and the LSVM algorithm may produce meaningful results. A modified algorithm has been developed based on this idea. The algorithm is called Partial Least Square Voltage Matching (PLSVM) reconstruction because the voltage matching is performed on part of boundary potentials.
The PLSVM algorithm has the same current injection and potential measurement arrangement as shown in Table 2.5.1. The objective function for PLSVM is given by

$$F(c) = \frac{1}{2} \| \tilde{v} - v(c) \|^2 = \min.$$  \hspace{1cm} (2.6.1)

where the measured boundary potential vector \( \tilde{v} \) is given as

$$\tilde{v} = [ \tilde{v}_2, \tilde{v}_3, \ldots, \tilde{v}_{NA-1}; \tilde{v}_{NA-2} ],$$

and the calculated potential vector \( v(c) \) is equal to

$$v(c) = [ (v_{12}, v_{13}, \ldots, v_{1(NA-1)}) \ast C_1; (v_{23}, v_{24}, \ldots, v_{2(NA-2)}) \ast C_2; \ldots; (v_{NA-1(NA-2)} \ast C_{NA-2}) ](c).$$

The superscript is the number of electrode where a current source is applied, and the subscript is the number of electrode on which potential is measured. The constant \( C_i = v^i_{i+1}, i = 1, 2, \ldots, NA-2 \), are used to normalize the calculated boundary potentials with respect to the measured one for each boundary current pattern.

Equation (2.6.1) is solved iteratively by the Newton-Raphson method given in equations (2.3.16) - (2.3.19), resulting in a similar formula to that of LSVM.

Eliminating the potentials on and close to the driving electrodes avoids the major errors from contact impedance and finite element solution. However, potentials on remaining electrodes still contain errors caused by the contact impedance on the datum electrode, which adds a unknown constant level to the measured potentials and results in waveform shift. To reduce the waveform shift error, a local normalization process was introduced to the objective function (2.6.1). The process performs normalization to the data for each current injection.

In addition to reducing the waveform shift error, the local normalization has the merit of simplifying the selection of starting uniform distribution for the iteration. After the local normalization, the scale of the conductivity to be reconstructed does not affect the calculated potentials and, as a result, the range of starting distributions for the
iteration is extended. The absolute value of the reconstructed conductivity distribution can be restored from the known values of current source and measured potentials.

2.6.4 Reconstructions Using PLSVM and Measured Data

The PLSVM algorithm is tested using measured data from the same experimental set up as shown in Fig.2.5.3. The object is divided into 16 square elements with 15 boundary electrodes, as shown in Fig.2.5.3 (b). Two situations have been tested, as shown in Fig.2.6.5 (a) and (b), where shadow areas are painted with silver. The boundary current injection and potential measurement arrangement are the same as that of Table 2.5.1. The measured data is of four digit precision. For each current injection, only the potential on the driving electrode is eliminated in the PLSVM algorithm. Thus the maximum number of equations is 90. All the reconstructions are obtained from uniform starting distributions of unit amplitude.

Fig.2.6.5 (c) and (d) present the reconstructed images for corresponding objects in Fig.2.6.5 (a) and (b). The results are meaningful which proved the usefulness of the PLSVM algorithm.
Fig. 2.6.5 Testing results of the PLSVM algorithm on conducting rubber objects. (a) and (b) are objects with silver paint on the shadow area. (c) and (d) are reconstructed images.

2.7 Conclusions

A Least Square Voltage Matching (LSVM) reconstruction algorithm was developed using linear network analysis theory as well as finite element analysis. The derivation exhibited the mathematical equivalence between a finite element modelling and a linear network modelling of the continuous object. Relationship between the LSVM algorithm and a number of existing algorithms were discussed which revealed the equivalences of the algorithms in many respects.

Two distinctions exist between identification of network component values and reconstruction of a continuous conductivity distribution using the LSVM algorithm. For network component values identification or circuit diagnosis problems, information about network topology and nominal component values distribution are usually available. This provides a powerful constraint on the resulting voltage and current distribution of the network, and the nominal component values distribution of the network provide a suitable starting point for the iterative reconstruction procedure. The LSVM algorithm can, therefore, work successfully for networks of relative small scale.
(e.g. network with less than 40 - 50 components) when starting from a uniform distribution, or work effectively for larger network containing only a few fault components when starting from a nominal distribution of component values.

When a continuous conductivity distribution is to be reconstructed using the iterative algorithm, the continuum can be modelled by different finite element meshes. Therefore the unique topology constraint for network problems is no longer valid for the continuous distribution reconstruction. In addition, \textit{a priori} information for the starting distribution of the iteration is generally not available so that the iteration often begins with a uniform distribution. Thus, the performance of the LSVM algorithm may degrade when applied to practical reconstruction problems.

Tests on the LSVM algorithm has indicated that, in order to obtain a correct result under the condition of a uniform distribution, the number of unknowns in an equivalent network should be relatively small. This limitation is due to the non-unique minima of the set of nonlinear equations. As the number of unknown in the set of equations increases, the number of minima increase drastically so that the iteration has more chance to arrive at an unwanted solution. However, if a suitable starting distribution (i.e. a distribution very close to the wanted solution) is used, then the iteration may produce a satisfactory result for a system with larger number of unknowns.

The convergence of the finite element method used in forward calculations of many iterative algorithms is important to the performance of the algorithms. The study on the convergence behaviour of finite element analysis under pulse-like boundary current distribution has indicated that the data error on the driving electrodes is great enough to lead the LSVM to unwanted images.

An improved LSVM algorithm, i.e. PLSVM algorithm, was developed by eliminating the measured potentials on the driving electrodes from the algorithm. The PLSVM was able to effectively remove the influence of the data error on the reconstruction and meaningful images had been produced.
Chapter 3

Linear Reconstruction Algorithms
3.1 Introduction

Electrical impedance tomography is intrinsically a nonlinear problem and the nonlinearity arises from Poisson's equation. Under certain conditions, Poisson's equation can be linearized, leading to some useful linear reconstruction methods.

Linear reconstruction algorithms are one step reconstruction procedures which allow fast processing. However, linear algorithms only produce approximate reconstructions because of the linearization of Poisson's equation.

One of the most useful linear reconstructions is the filtered back-projection method developed by Barber and Brown in 1983. The method assumes that Radon's back-projection theory (appendix E) can be applied to electrical impedance tomography by back-projecting measured boundary potentials along the equipotential lines of an uniform object. Both \textit{in vivo} and \textit{in vitro} images in clinical applications have since been reported. Nevertheless, the method does not have rigorous mathematical support and further improvement of the imaging method appears to be difficult.

Based on the same linearized Poisson's equation and measurement arrangement used by Barber and Brown, a linear reconstruction method is developed in this chapter. The method reconstructs the conductivity distribution of objects by solving a set of integral equations. Apart from the linearization of Poisson's equation, the integral equation is rigorously derived and thus provides sound mathematical ground for the study of linear reconstruction algorithms.

Also in the chapter, properties of the integral equation, the relation between the integral equation and Radon's back-projection theory, and the relation between the integral equation method and the filtered back-projection method are investigated. The study on the integral equation establishes the mathematical relation between the filtered back-projection method and Radon's back-projection theory. Merits and disadvantages of the integral equation method are discussed as well.
3.2 Filtered Back-projection Method

Barber and Brown [1983, 1986] developed a linear reconstruction algorithm based on a linearized Poisson's equation and Radon's back-projection method used in X-ray Computerized Tomography. The algorithm assumes that a conductivity distribution can be reconstructed by back-projecting the boundary potential differences between electrode pairs into the object domain along the equipotential lines of an uniform object. Useful images from both in vitro and in vivo data have been produced.

In this section, Barber-Brown's filtered back-projection algorithm is briefly reviewed and discussed, which is important to the development of a novel integral equation reconstruction method.

3.2.1 Linearized Poisson's Equation and Conformal Transformation

*Linear approximation of Poisson's equation:*

The potential distribution \( \phi = \phi(x,y) \) within a source free, isotropic two-dimensional conductive medium at low frequency is governed by Poisson's equation

\[
\nabla(\sigma \nabla \phi) = \sigma \nabla^2 \phi + \nabla \sigma \cdot \nabla \phi = 0,
\]

(3.2.1)

where \( \sigma = \sigma(x,y) \) is the distribution of conductivity within the region. An alternative expression for equation (3.2.1) can be given as

\[
\nabla^2 \phi = \nabla r \cdot \nabla \phi,
\]

(3.2.2)
where \( r = - \ln \sigma \) is commonly called log-resistance. Equations (3.2.1) and (3.2.2) represent a nonlinear relationship between potential and log-resistance because potential itself is a function of conductivity.

In many cases, it is useful to separate the total potential into two parts such that

\[
\phi = \phi_0 + \phi_p , \tag{3.2.3}
\]

where \( \phi_0 \) is the potential distribution of an object with uniform conductivity distribution and \( \phi_p \) is the perturbed part of the total potential caused by the non-uniform distribution of conductivity. The \( \phi_0 \) satisfies the Laplace equation \( \nabla^2 \phi_0 = 0 \). Substituting equation (3.2.3) into equation (3.2.2) yields

\[
\nabla^2 \phi = \nabla^2 \phi_0 = \nabla r \cdot \nabla \phi_0 + \nabla r \cdot \nabla \phi_0 . \tag{3.2.4}
\]

If, for any reason, the term \( \nabla r \cdot \nabla \phi_0 \) can be eliminated from equation (3.2.4), then equation (3.2.4) can be approximated as:

\[
\nabla^2 \phi = \nabla^2 \phi_0 = \nabla r \cdot \nabla \phi_0 . \tag{3.2.5}
\]

A common assumption allowing the term \( \nabla r \cdot \nabla \phi_0 \) to be ignored is that the perturbation is small, i.e.

\[
\| \nabla r \cdot \nabla \phi_0 \| \ll \| \nabla r \cdot \nabla \phi_0 \| , \tag{3.2.6}
\]

where \( \| \cdot \| \) stands for the norm in \( L^2 \) space. The validity of the assumption in equation (3.2.5) and (3.2.6) will be discussed in chapter four and here we assume that equation (3.2.5) is acceptable.

Equation (3.2.5) represents a linear relation between \( r \) and \( \phi_p \) because \( \phi_0 \) can be known either theoretically or measured experimentally. The linear equation
provides a suitable ground for developing a fast reconstruction algorithm in electrical impedance tomography. It should be appreciated, as well, that the equation (3.2.5) is different from a normal boundary problem because both \( \phi \) and \( \tau \) are unknown inside the region and only the information about \( \phi \) on the boundary is measurable. Thus techniques for solving boundary problems are not applicable to equation (3.2.5).

*Conformal coordinates transformation:*

The Barber-Brown back-projection algorithm was designed for a circular object with adjacent electrode pair current driving configuration. In the system, electrodes are evenly placed around the boundary of the circular object. For each current pattern, only one pair of adjacent electrodes are connected to the current source, and potentials on the remaining electrodes are measured. The adjacent current injection allows the use of conformal transformation to simplify the mathematical manipulations in developing the algorithm. In addition, the current driving configuration has the least number of driving electrodes and the maximum number of source free electrodes, which avoids the error contained in the measured boundary voltages due to uncertain contact resistances. The current configuration may also provide the best resolution [Barber and Seagar, 1987].

It is known that a circular domain may be transformed into a more convenient rectangular coordinate system using a bipolar transformation [Margenau and Murphy, 1964]. When the distance between electrodes is small, the driving pair can be approximated by a current dipole. For such geometry, an appropriate transformation is given by

\[
w = u + jv = \frac{1}{x + jy} = \frac{1}{z}, \quad z = x + jy, \quad (3.2.7)
\]

or, equivalently
\[ u = \frac{x}{x^2 + y^2}, \quad v = -\frac{y}{x^2 + y^2}, \quad (3.2.8) \]

\[ x = \frac{u}{u^2 + v^2}, \quad y = -\frac{v}{u^2 + v^2}. \quad (3.2.9) \]

The relation between the two coordinate systems is shown in Fig.3.2.1. After the transformation, a current dipole at the origin of the x-y plane becomes two line current sources at \( u = \pm \infty \) in u-v plane. The boundary of the circle is mapped to \( v_b = -0.5 \) and the centre mapped to \( v_0 = -1 \) in u-v plane, respectively. The conformal transformation (3.2.7) has been used in developing the back-projection algorithm.

**Fig.3.2.1** Relation between x-y plane and u-v plane under transformation (3.2.7).

Equipotential lines and current density lines on a circular uniform object with a current dipole on the boundary are shown in Fig.3.2.2 (a). The solution procedure is given in appendix E. It is important to note that, after the transformation (3.2.7), the curved equipotential lines and current density lines in x-y plane have been converted to straight lines parallel to u-axis and v-axis, respectively, and the current density has
uniform magnitude with constant polarity in the u-v plane. Fig.3.2.2 shows the comparison between the two planes.

Another useful property of the transform is that both Laplace's equation and Poisson's equation are invariant in expressions under the conformal transformation [Morse and Feshbach, 1957]. Thus, in u-v plane, equation (3.2.5) has a simpler form:

\[ \nabla^2 \phi_p (u,v) = \frac{\partial \phi_0}{\partial u} (u,v) \frac{\partial r}{\partial u} (u,v) = E_u \frac{\partial r}{\partial u} (u,v), \]  

(3.2.10)

where \( E_u = \frac{\partial \phi_0}{\partial u} \) is a constant. It is worth noting that \( E_u \) may not remain a constant when transformed into other planes.

![Fig. 3.2.2](image-url)  

**Fig. 3.2.2** Equipotential lines and current density lines in x-y plane and u-v plane.
3.2.2 Back-projection along Equipotential Lines

Fourier analysis:

Baber and Brown used Fourier transform in developing their back-projection method. Taking the Fourier transform of the equation (3.2.10) with respect to \( u \) and \( v \) yields

\[
-(\omega_u^2 + \omega_v^2) \Phi_p(\omega_u, \omega_v) = E_u \cdot [j \omega_u R(\omega_u, \omega_v)] ,
\]  

(3.2.11)

where \( \omega_u \) and \( \omega_v \) are frequency variables with respect to spatial variables \( u \) and \( v \), respectively; \( \Phi_p(\omega_u, \omega_v) \), \( R(\omega_u, \omega_v) \) are the Fourier transforms of \( \phi_p(u,v) \) and \( r(u,v) \), respectively. Multiplying equation (3.2.11) by the function

\[
\frac{-j\omega_u}{\omega_u^2 + \omega_v^2}
\]

results in

\[
G(\omega_u, \omega_v) = K(\omega_u, \omega_v) E_u R(\omega_u, \omega_v)
\]  

(3.2.12)

where

\[
G(\omega_u, \omega_v) = j\omega_u \Phi_p(\omega_u, \omega_v) ,
\]  

(3.2.13)

\[
K(\omega_u, \omega_v) = \frac{\omega_u^2}{\omega_u^2 + \omega_v^2} .
\]  

(3.2.14)

Thus, differential equation (3.2.10) has been converted into an algebraic equation (3.2.12) which relates the log-resistance and the partial potential gradient in the frequency domain.

Now consider a point object positioned at \((u_p, v_p)\) in a uniform region with conductivity distribution given by
\[ \sigma(u,v) = e^{\delta(u-u_p)} \delta(v-v_p). \quad (3.2.15a) \]

The log-resistance is equal to

\[ r(u,v) = \delta(u-u_p) \delta(v-v_p). \quad (3.2.15b) \]

For such an object equation (3.2.12) becomes

\[ G(\omega_u, \omega_v) = K(\omega_u, \omega_v) e^{j(\omega_u u_p + \omega_v v_p)}. \quad (3.2.16) \]

Performing inverse Fourier transform of equation (3.2.16) with respect to \( \omega_v \) yields

\[ \tilde{g}(\omega_u, v) = \tilde{K}(\omega_u, v-v_p) e^{-j\omega_u u_p}, \quad (3.2.17) \]

where \( \tilde{g}(\omega_u, v) \) and \( \tilde{K}(\omega_u, v) \) are inverse Fourier transforms of \( G(\omega_u, \omega_v) \) and \( K(\omega_u, \omega_v) \) with respect to \( \omega_v \), respectively, and

\[ \tilde{K}(\omega_u, v-v_p) = |\omega_u| e^{-|\omega_u||v-v_p|}. \quad (3.2.18) \]

The Fourier transform of function \( |\omega_u| e^{-a|\omega_u|} \) (a > 0) is given by

\[ \frac{1}{\pi} \frac{a^2 - u^2}{(a^2 + u^2)^2}, \quad (3.2.19) \]

which indicates that the inverse Fourier transform of equation (3.2.18) does not exist for \( v = v_p \). Nevertheless, the generalized Fourier transform of the equation (3.2.18) exist and has the same expression as equation (3.2.19). Thus the inverse Fourier transform of the equation (3.3.18) is found to be
On the boundary of the object \( v = v_b \), the partial potential gradient is

\[
\frac{\partial \phi_p}{\partial u}(u, v) = \frac{(v-v_p)^2 - (u-u_p)^2}{(u-u_p)^2 + (v-v_p)^2}.
\]  

(3.2.20)

where \( q = (v_b-v_p) \) represents the depth of the point object from the boundary \( v_b \).

Equation (3.2.21) is the basis for the filtered back-projection method.

Equation (3.3.21) can also be obtained with the help of Green's function, as seen in appendix D. This provides an alternative approach in the mathematical analysis.

It is important to note that the point object response (3.2.21) is obtained from the linearized Poisson's equation and is therefore only an approximate result. The accurate point object response may not be solved analytically due to the non-linearity of Poisson's equation.

The boundary potential gradient in equation (3.2.21) has a pulse-like waveform with the centre of the pulse located at \( u = u_p \), as shown in Fig.3.2.3. Since the boundary potential gradient for a single point object is similar to the projection waveform of a point object in X-ray tomography, Barber and Brown have assumed that the back-projection procedure used in X-ray CT (see appendix E) can be applied to the electrical impedance tomography. Note that in the u-v plane, the u-coordinate lines are the equipotential lines for the uniform circular object, the approach is therefore named as the back-projection method along equipotential lines.
Fig. 3.2.3 Wave form of boundary potential gradient (3.2.21)

In practice, the domain of the object is divided into strips in parallel with \( v \)-axis and the boundary potential is back-projected along the strips. After boundary potential gradients for all the current injections are processed, an approximate point object can be reconstructed. The filtered back-projection procedure in \( u \)-\( v \) plane and \( x \)-\( y \) plane is illustrated in Fig.3.2.4 (a) and (b), respectively.

The filtered back-projection method has been applied to general cases other than single point objects and useful images were produced using both \textit{in vitro} and \textit{in vivo} data. Nevertheless, mathematical support for the method is incomplete.
3.2.3 Physical Interpretations, Filtering, and Weighting Functions

Barber and Brown interpreted the potential gradient response of a point object in two ways. The first is based on an interesting observation that equation (3.2.21) has the same expression as the potential gradient of a current dipole located at point \((u_p, v_p)\) (see appendix D). However, the potential gradient response of the point object was obtained from the linearized Poisson's equation (3.2.5) and it is therefore only an approximate point object response for Poisson's equation. Secondly, equation (3.2.18) was divided into two terms and the variable \(v\) was located on the boundary, i.e. \(v = v_b\). It was found that the exponential term can be thought of as the Fourier transform of a point response function measured at the boundary for a point object at depth \(v_b-v_p\) (see appendix E). The remaining term in equation (3.2.18), i.e. \(|\omega_u|\), is a ramp filter. Thus Barber and Brown concluded that back-projection of the
boundary data can be processed without further ramp filtering which is necessary in X-ray computerized tomography.

![Diagram of point objects at different depth](image)

**Fig. 3.2.5** Potential gradient for point objects at different depth inside a circle with unit radius, where \( \rho_1 = 0.75, \rho_2 = 0.25 \). In the graph, the taller pulse is for object (a), which is closer to the boundary than object (b); and the shorter and flatter pulse is for object (b).
The potential gradient of a point object has been found to be dependent on the depth from the boundary. Fig.3.2.5 shows the boundary potential gradient for two point objects at different depths. Significant difference exists in the peak amplitudes of the two wave forms, as can be seen in Fig.3.2.5. The boundary potential gradient for object (a) has narrower pulse width and much higher magnitude than that of object (b). Therefore the back-projection approach may produce a better reconstruction in case (a) than in case (b).

In order to compensate for reconstruction artefacts due to the depth dependence of the point object response, Barber and Brown processed the measured data using some filtering techniques before the back-projection. Therefore the method is also called the filtered back-projection method.

In practice, the back-projection is performed in the x-y plane along curved equipotential lines. Care must be taken to keep angular uniformity of back-projection through each image point in order to obtain acceptable reconstructions and a weighting function has been proposed [Barber and Brown, 1986].

3.3 Linear Reconstruction Algorithm by Solving Integral Equations

In this section, a linear integral equation description for electrical impedance tomography is derived and a numerical reconstruction algorithm is developed [Chen and Paoloni, 1990]. The derivation of the integral equation is mathematically rigorous apart from the linearization of Poisson's equation. Relations between the integral equation method, the Radon back-projection theory, and the Barber-Brown method are investigated. Reconstruction experiments are carried out and effects of the method are discussed.
3.3.1 The Integral Equation

In section 3.2, by linearization of Poisson's equation, conformal transformation, and Fourier transform, a linear algebraic equation (3.2.12) has been obtained. Taking the inverse Fourier transform of equation (3.2.12) with respect to \( \omega_v \) yields

\[
\tilde{g}(\omega_u, v) = \tilde{K}(\omega_u, v) \ast \{ E_u \tilde{r}(\omega_u, v) \}
\]

\[
= E_u \int_{-\infty}^{\infty} \tilde{K}(\omega_u, v-\zeta) \tilde{r}(\omega_u, \zeta) \, d\zeta ,
\]

(3.3.1)

where \( \tilde{g}(\omega_u, v) \), \( \tilde{K}(\omega_u, v) \) and \( \tilde{r}(\omega_u, v) \) are inverse Fourier transforms of \( G(\omega_u, \omega_v) \), \( K(\omega_u, \omega_v) \) and \( R(\omega_u, \omega_v) \) with respect to \( \omega_v \), respectively, and " \( \ast \) " stands for convolution with respect to \( v \). Substituting equation (3.2.18) into equation (3.3.1) and choosing coordinate \( v \) to lie along the boundary of the object (\( v = v_b \)) yields

\[
\tilde{g}(\omega_u, v_b) = E_u \int_{-\infty}^{v_b} |\omega_u| e^{-|\omega_u| \zeta} \tilde{r}(\omega_u, \zeta) \, d\zeta .
\]

(3.3.2)

The inverse Fourier transform of equation (3.3.2) with respect to \( \omega_u \) cannot be carried out directly because when \( \zeta \) is equal to \( v_b \), the transform does not exist. By multiplying both sides of equation (3.3.2) with a function:

\[
e^{-\alpha |\omega_u|} , \quad \alpha > 0 ,
\]

(3.3.3)

the inverse Fourier transform with respect to \( \omega_u \) can then be performed and the integral equation is obtained:
where the kernel $k(u, \xi, \zeta)$ is given by

$$k(u, \xi, \zeta) = \frac{(v_b - \zeta + \alpha)^2 - (u - \xi)^2}{[(v_b - \zeta + \alpha)^2 + (u - \xi)^2]^2} \mathcal{E}_u,$$

and

$$h(u,v_b) = \int_{-\infty}^{\infty} \frac{\alpha}{(u - \xi)^2 + \alpha^2} \frac{d\phi(\xi, v_b)}{d\xi} d\xi \tag{3.3.6a}$$

$$= -\int_{-\infty}^{\infty} \frac{2\alpha(u - \xi)}{[(u - \xi)^2 + \alpha^2]^2} \phi(\xi, v_b) d\xi. \tag{3.3.6b}$$

The integral equation (3.3.4) provides a mathematical connection between the log-resistance and the boundary partial potential gradient or boundary potential. From this equation a linear reconstruction algorithm will be developed.

In the derivation of both Barber-Brown's algorithm and the integral equation algorithms, there is a requirement on the boundary current injections. The requirement is that the equipotential lines on a uniform object produced by the current injections must align with the coordinate lines. In this case Poisson's equation can be reduced into a simpler form, such as equation (3.2.10), which is essential to the linear reconstruction algorithms.

### 3.3.2 Properties of the Integral Equation

Equation (3.3.4) is a special two dimensional Fredholm integral equation of the first kind. It is special because the variable $v$ has been constrained to the boundary
\( v_b \), whereas a normal integral equation allows \( v \) to vary inside the whole object. Equation (3.3.4) is also a singular equation in the sense that the range of integration extends to infinity.

The major properties of the integral equation are discussed as follows:

1. Measured boundary data for a single boundary current pattern is insufficient to solve equation (3.3.4). Therefore multi-current-injections are required in order to generate adequate data for the reconstruction. Consequently, a set of integral equations must be solved instead of a single equation. For each individual current injection there is an integral equation defined on a \( u-v \) plane associated with that current injection geometry. Different \( u-v \) planes are related by the common \( x-y \) plane, or the common polar system.

In numerical solution of the set of integral equations, each integral equation is discretized into a under determined matrix equation. The resulting matrix equations are solved simultaneously to obtain a reconstruction. According to the theory of linear algebra, all the matrix equations must be consistent in order for the set of matrix equations to have a correct solution [Silov, 1971, Mirsky, 1963].

2. The solution of equation (3.3.4) is not unique. This is because

\[
\frac{\partial}{\partial \xi} \left[ \frac{(u-\zeta)}{(v_b-\zeta+\alpha)^2 + (\xi-u)^2} \right] \\
= \frac{(u-\zeta)^2 - (v_b-\zeta+\alpha)^2}{[ (v_b-\zeta+\alpha)^2 + (u-\zeta)^2 ]^2},
\]

(3.3.7)

and therefore
Equation (3.3.7) reveals that for any log-resistance satisfying \( r_0(u,v) = r_0(v) \), the right hand side of equation (3.3.4) vanishes. In other words, if \( r(u,v) \) is a solution of equation (3.3.4) then \( r(u,v) + r_0(v) \) also satisfies the equation. The non-uniqueness can be effectively reduced when a set of integral equations is solved simultaneously. This is because if \( r_0(u,v) \) has the representation of \( r_0(v) \) in one \( u-v \) plane, it cannot depend only on \( v \) in all the remaining \( u-v \) planes. Consequently the set of integral equations will contain a single non-unique solution \( r_0(u,v) = \text{constant} \), which simply affects the background level of reconstructed images.

(3) It is almost impossible to solve the equation (3.3.4) analytically, let alone a set of integral equations. Numerical solution is therefore inevitable. In numerical treatment of the integral equations, the continuous object is discretized by a mesh and the integral equation is approximated by quadrature formulas. Studies on the numerical solution of the integral equation of the first kind indicated that the set of discrete equations obtained from the numerical integration almost always have a system matrix with very poor condition. The poor conditioned matrix equation often leads to an unstable solution which contains high frequency noise. Therefore, care must be taken in the numerical solution of the integral equations.

(4) The kernel \( k(u,\xi,\zeta) \) plays an important part in the behaviour of the integral equation. The overall shapes of the kernel function are shown in Fig.3.3.1, where the function is seen to have a two-dimensional single pulse-like distribution. The centre of the pulse is located at \( \xi = u \) and \( \zeta = v_b \), and the pulse width varies with the values of parameter \( \alpha \).
Fig. 3.3.1 Overall shapes of the kernel function (3.4.5a) with different values of $\alpha$: (a) $\alpha = 0.25$; (b) $\alpha = 0.5$; (c) $\alpha = 1.0$; and (d) $\alpha = 2.0$. In all the cases, $u \in (-5, 5)$ and $v \in (-10.5, -0.5)$.

Mathematical analysis of the kernel function reveals the following:

The zeros of the kernel function are located at

$$u - \xi = \pm (v_b - \zeta + \alpha), \quad (3.3.9)$$
and the partial derivatives of the kernel function are given by

\[
\frac{\partial}{\partial \xi} k(u, \xi, \zeta) = \frac{-2(u-\xi) \left[ 3(v_b \cdot \zeta + \alpha)^2 - (u-\xi)^2 \right]}{\left[ (u-\xi)^2 + (v_b \cdot \zeta + \alpha)^2 \right]^3}, \quad (3.3.10)
\]

\[
\frac{\partial}{\partial \zeta} k(u, \xi, \zeta) = \frac{2(v_b \cdot \zeta + \alpha) \left[ 3(u-\xi)^2 - (v_b \cdot \zeta + \alpha)^2 \right]}{\left[ (u-\xi)^2 + (v_b \cdot \zeta + \alpha)^2 \right]^3}. \quad (3.3.11)
\]

Critical points of the kernel are then summarized below:

- zeros of \( k(u, \xi, \zeta) \): \( u-\xi = \pm (v_b \cdot \zeta + \alpha) \), \( (3.3.12a) \)

- Max. of \( k(u, \xi, \zeta) \) for fixed \( \zeta \): \( \xi = u \), \( (3.3.12b) \)

- Min. of \( k(u, \xi, \zeta) \) for fixed \( \zeta \): \( \xi = u \pm \sqrt{3(v_b \cdot \zeta + \alpha)} \), \( (3.3.12c) \)

- Max. of \( k(u, \xi, \zeta) \) for fixed \( \zeta \): \( \zeta = v_b + \alpha \), \( (3.3.12d) \)

- Min. of \( k(u, \xi, \zeta) \) for fixed \( \zeta \): \( \zeta = v_b + \alpha \pm \sqrt{3(\zeta-u)} \), \( (3.3.12e) \)

Of particular interest is the fact that along the line \( \zeta = u \) the kernel reaches its maximum with respect to \( \xi \). Along this line, the kernel function reduces to

\[
k(u, \xi, \zeta) = \frac{1}{(v_b \cdot \zeta + \alpha)^2}, \quad (3.3.13)
\]

which monotonically decreases as \( \zeta \) varies within \((-\infty, v_b)\). On the boundary where \( \zeta = v_b \), equation (3.3.13) reaches its peak value \( \frac{1}{\alpha^2} \). The waveforms of the function (3.3.13) are depicted in Fig.3.3.2 with different values of \( \alpha \).
Another useful feature is the distribution of the kernel function along the boundary, where the equation (3.3.5) reduces to

\[
\frac{\alpha^2 - (u-\xi)^2}{[\alpha^2 + (u-\xi)^2]^2}.
\]

(3.3.14)

The waveform of function (3.3.14) is shown in Fig.3.3.3.
(5) The parameter $\alpha$ was introduced into the integral equation (3.3.4) by multiplying equation (3.3.2) with the function $e^{-\alpha |\omega_u|}$ to ensure the existence of the inverse Fourier transformation. The value of the parameter $\alpha$ is essential to the dynamic range and smoothness of the kernel $k(u, \xi, \zeta)$, as shown in Fig.3.3.1. The maximum value of the kernel is $\frac{1}{\alpha^2}$ according to equation (3.3.13). In addition, the function $e^{-\alpha |\omega_u|}$ is a low-pass filter with respect to $\omega_u$, which attenuates the high frequency component of the boundary potential gradient. Transfer characteristics of the low-pass filter for different values of $\alpha$ is illustrated in Fig.3.3.4. As can be seen, larger value of $\alpha$ leads to narrower bandwidth. As $\alpha \to 0$, the bandwidth approaches infinity and the filter becomes all-pass.
The kernel function $k(u, \xi, \zeta)$ serves as a weight function to log-resistance $r(\xi, \zeta)$ in the integral equation (3.3.4). Because of the pulse-like distribution of $k(u, \xi, \zeta)$, significant contribution to the equation comes from the log-resistance close to the point $\xi = u$ and $\zeta = v_b$. Thus, the integral equation is much more sensitive to the value of $r(\xi, \zeta)$ close to the boundary point $(u, v_b)$ than that in the central area of the object.

The sensitivity of the integral equation to the log-resistance at the inner parts of the object depends on the kernel function. If the parameter $\alpha$ is too small, the kernel becomes a very narrow pulse so that the integral equation becomes less sensitive to the log-resistance deep inside. To ensure that the set of integral equations contains contributions of the log-resistance from every part of the object, the parameter $\alpha$ should be relatively large so that the kernel is smooth. In practice, adjacent current injection is applied to every pair of boundary electrodes. If, in each current injection, the kernel is relatively smooth within the shaded area shown in Fig.3.3.5, then the...
whole object can be covered piece by piece by the kernel after all the current injection be completed.

Fig. 3.3.5 Piece-wise covering of the object domain by smooth parts of the kernel function (shadowed areas) for adjacent current injection at different locations.

Suppose we define the smoothness of the kernel to be the ratio of the kernel value at two points along the line $\xi=u$. One of the points is the boundary point ($\xi=u$, $\zeta=v_b$) and the other is the centre of the object ($\xi=u$, $\zeta=v_0$). By substituting the coordinates into equation (3.3.5), we obtain the following equation for a specified smoothness value $S$:

$$\left[ \frac{v_b - v_0 + \alpha}{\alpha} \right]^2 = S .$$

(3.3.15)
Consider a unit circular object where \( v_b = -0.5 \) and \( v_0 = -1 \). If the smoothness value \( S \) is chosen to lie in the range 4 to 9, then \( \alpha \) is found to lie between 0.5 and 0.25. The kernel amplitude distribution along the line \( \xi = u \) for the value \( \alpha = 0.5 \) is quite smooth within the range of \( \xi \in (v_0, v_b) \).

The \( \alpha \) cannot be too large because the low pass filter \( e^{-\alpha |\omega_u|} \) will otherwise suppress too much high frequency information contained in the measured data, which may result in deficiency of information for reconstruction. It is seen, from Fig.3.3.4, that the low pass filter has a moderately decreasing rate for \( \alpha \) between 0.5 and 0.25. As a trade-off between smoothness of the kernel and bandwidth of the filter, \( \alpha \) has been chosen to be 0.5 (i.e. \( \alpha = -v_b \)).

(6) The kernel in the convolution (3.3.6b) has the property that

\[
\int_{-\infty}^{\infty} \frac{2\alpha(u-\xi)}{[(u-\xi)^2 + \alpha^2]^2} \, d\xi = 0. \tag{3.3.16}
\]

This reveals that the integral equation is independent of the reference potential of the system.

(7) The point response (3.2.21) used by Barber and Brown to develop the filtered back-projection method can be regarded as a special case of the integral equation. Substituting the point log-resistance representation (3.2.15b) into integral equation (3.3.4) yields

\[
h(u, v_b) = \frac{(v_b-v_p+\alpha)^2 - (u-u_p)^2}{[(v_b-v_p+\alpha)^2 + (u-u_p)^2]^2} \, E_u, \tag{3.3.17}
\]
where $h(u, v_b)$ is given by (3.4.6a). As $a \to 0$, the kernel in the convolution (3.4.6a) approaches a delta function $\delta (\xi - u)$, and thus $h(u, v_b)$ reduces to $\frac{d\phi_p}{du}(u, v_b)$.

Equation (3.3.17) then becomes

$$\frac{d\phi_p}{du}(u, v_b) = \frac{(v_b - v_p)^2 - (u - u_p)^2}{[(v_b - v_p)^2 + (u - u_p)^2]^2} E_u,$$

which has the same expression as equation (3.2.21) except for the constant $E_u$.

(8) The kernel of the integral equation is independent of the distribution of unknown conductivity and measured data. When the discrete mesh and the quadrature formula are determined, the system matrix and its inverse can be found and stored in computer memory before a reconstruction takes place. The reconstruction is, therefore, a simple matrix multiplication process and can be performed in a short time.

### 3.4 The Integral Equation Method and the Filtered Back-Projection Method

The filtered back-projection method assumed that Radon's back-projection theory can be applied to the impedance tomography because the single point object response given by equation (3.2.21) had a pulse waveform similar to the point response of X-ray CT. The method back-projuncts the normalized variations of the measured boundary voltages along the equipotential lines of an uniform object with a simple matrix multiplication.

The filtered back-projection method is related to the integral equation method and this relation will be discussed in the following section.
3.4.1 Reconstructions of Point Objects

In the case of a single point object, the filtered back-projection method offers a blurred image of the original point object. The blurring also depends on the depth of the point object [Barber and Brown, 1986].

When there are several point objects to be imaged, the method may have difficulty in distinguishing them. For example, consider two separate point objects located at (i): $u_p = 0.2$, $v_p = -0.77$, and (ii): $u_p = -0.1$, $v_p = -0.61$, as shown in Fig.3.4.1(a). The total boundary potential gradient is the sum of potential gradients of individual point objects, as shown in Fig.3.4.1(b). It is seen that the total boundary response remains a single peaked pulse and, if this boundary response is back-projected into the object domain, a single blurred object will be reconstructed instead of two distinct points.

(a) Two point objects at different locations.
Fig.3.4.2 Geometry of two point objects and boundary responses.

The integral equation method, on the other hand, is rigorously derived from linearized Poisson's equation for arbitrary distribution of log-resistance. In the case of isolated point objects, the integral equation may even be inverted analytically to obtain the images of the point objects.

Consider $n$ isolated points in a uniform circular domain. The boundary potential gradient is given by (see equation (3.2.20))

$$
\frac{\partial \phi}{\partial u}(u, v) = E_u \sum_{i=1}^{n} \frac{(u-u_p^i)^2 - (v-v_p^i)^2}{[(u-u_p^i)^2 + (v-v_p^i)^2]^2},
$$

where $n$ is the number of the point objects, $(u_p^i, v_p^i)$ is the position of the $i$th point object. Substituting equation (3.4.1) into equation (3.3.4) and taking Fourier transform with respect to $u$ yields
\[ \sum_{i=1}^{n} e^{-j\omega_{ul} (v_{b}^i - v_{p}^i)} e^{-j\omega_{up}^i} = \int_{-\infty}^{v_b} e^{-j\omega_{ul} (v_{b} - \zeta)} \tilde{r}(\omega_{ul}, \zeta) \, d\zeta, \quad (3.4.2) \]

where \( j = \sqrt{-1} \). Since the term \( e^{-j\omega_{up}^i} \) is a complex number, then the function \( \tilde{r}(\omega_{ul}, \zeta) \) must have the following form

\[ \tilde{r}(\omega_{ul}, \zeta) = \sum_{i=1}^{n} e^{-j\omega_{up}^i \tilde{r}_i(\omega_{ul}, \zeta)} \quad (3.4.3) \]

or

\[ r(\xi, \zeta) = \sum_{i=1}^{n} \delta(\xi - u_{p}^i) r_i(\xi, \zeta) \quad , \quad (3.4.4) \]

where \( r(\xi, \zeta) \) and \( r_i(\xi, \zeta) \) are the Fourier transform of \( \tilde{r}(\omega_{ul}, \zeta) \) and \( \tilde{r}_i(\omega_{ul}, \zeta) \) with respect to \( \omega_{ul} \), respectively. Substitute equation (3.4.3) into equation (3.4.2), then both side of the equation contains the term \( e^{-j\omega_{up}^i} \), \( i = 1, \ldots, n \). Notice that the variable \( \omega_{ul} \) in equation (3.4.2) is defined on \((0, \infty)\) and thus the coefficients of each term \( e^{-j\omega_{ups}^i} \) on both sides of the equation must be equal. We then obtain the following equation:

\[ e^{-j\omega_{ul} (v_{b}^i - v_{p}^i)} = \int_{-\infty}^{v_b} e^{-j\omega_{ul} (v_{b} - \zeta)} \tilde{r}_i(\omega_{ul}, \zeta) \, d\zeta , \quad i = 1, 2, \ldots, n \quad . \quad (3.4.5) \]
Making use of the sampling property of the delta function, i.e.

\[ f(x) = \int_{-\infty}^{\infty} f(t) \delta(x-t) \, dt \quad (3.4.6) \]

we arrive at

\[ r_\pi(\omega_\mu, \zeta) = \delta(\zeta - \nu_p^i), \quad i = 1, 2, \ldots, n \quad (3.4.7) \]

Substituting equation (3.4.7) into (3.4.3) and performing inverse Fourier transformation yields

\[ r(u, v) = \sum_{i=1}^{n} \delta(\zeta - u_p^i) \delta(\xi - \nu_p^i) \quad (3.4.8) \]

This example implies that, theoretically, the integral equation method may have better resolution than the filtered back-projection method. In practice, imaging quality of the integral equation method will be affected by many factors including data error, numerical error, modeling error, etc.

3.4.2 Filtered Back-projection as a Pseudo-inversion of the Integral Equations

The filtered back-projection method is based on the assumption that Radon's back-projection theory can be adopted for the impedance tomography. Two conditions are essential to Radon's back-projection theory. One is that the measured boundary data should be in a form of one-dimensional projection (i.e., linear integral) of the distribution function of an object and the other is that the boundary data at different view angles can be related by coordinate rotation. When the two conditions are
satisfied, the distribution function can be expressed explicitly as a two-dimensional linear integral in terms of measured boundary data using Radon's transformation. The Radon's back-projection method is therefore a single pass process and is able to offer robust reconstructions.

The integral equation (3.3.4) represents a relation between the boundary data and the weighted two-dimensional integral of a log-resistance distribution. The weight function, i.e. the kernel of the integral equation, has a two-dimensional pulse-like distribution centred at $\xi = u$ and $\zeta = v_b$, as shown in Fig.3.3.1. On the boundary of the object, kernels of the integrals on the both sides of equation (3.3.4) approach $\delta$-functions as the parameter $\alpha$ approaches zero. Therefore, if we replace the weight function by a function $k_a(u, \xi, \zeta)$ of the form

$$k_a(u, \xi, \zeta) = \delta(\xi - u) \quad (3.4.9)$$

and replace the kernel in the convolution (3.3.6a) by a function $\delta(\xi-u)$, we then obtain a line integral equation, or a projection integral, given by

$$\frac{\partial \phi}{\partial u}(u, v_b) = \int_{-\infty}^{v_b} r(u, \zeta) d\zeta. \quad (3.4.10)$$

Equation (3.4.10) implies that it may be the possible to apply the Radon's back-projection method to electrical impedance tomography.

There are, however, two major differences between Radon's back-projection theory and impedance tomography. Firstly, difference exists between the kernel $k(u, \xi, \zeta)$ in equation (3.3.6a) and the kernel $k_a(u, \xi, \zeta)$ of equation (3.4.9) and therefore equation (3.4.10) is only valid approximately. Secondly, for each current injection pattern, there is a $u$-$v$ plane being generated and the different $u$-$v$ planes cannot be related by coordinate rotation transformation. Thus, directly applying Radon's back-projection method to impedance tomography may only produce
approximate reconstructions. Additional signal processing is necessary to compensate for reconstruction errors.

The integral equation reduces to equation (3.2.21) when a point object is concerned. The point object response (3.2.21) forms the basis of the filtered back-projection method. Thus, the integral equation provides a mathematical link between Radon's back-projection theory and the filtered back-projection method.

Radon's back-projection method reconstructs an image by back-projecting the measured data along the straight ray path of the transmitting signals. The algorithm consists of a matrix multiplication which is an accurate inversion of an integral equation for projections. Barber-Brown's method is also a matrix multiplication process which mimics Radon's back-projection method. In this sense, the filtered back-projection method can be regarded as a pseudo-inversion of the integral equation (3.3.4).

3.5 Numerical solution of the integral equation

Equation (3.3.4) is not suitable for numerical solution because the range of integration is infinite. Coordinate transformation is required to limit the range of integration before numerical treatment of the integral equation can proceed.

The coordinate transform used here is as follows: the u-v plane is transformed back into x-y plane through transformation (3.2.7), with the origin of the x-y coordinate system being on the boundary of the circular object. The Jacobian for the transformation is given by

\[ \text{Jacobian}_1 = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{(x^2 + y^2)^2} \cdot (3.5.1) \]
In the x-y plane, the term $E_u$ in equation (3.3.4) is no longer a constant but a function of the variables $x$ and $y$ given by

$$E_u = v = \frac{-y}{x^2 + y^2} E,$$  \hspace{1cm} (3.5.2)

where $E$ is a constant. Equation (3.5.2) can also be obtained using the physical constraint that between the two terminals of the current source, the integration of the electric field along different current lines produces the same voltage.

The x-y plane is then transformed into a polar system, with its origin at the centre of the object, so that

$$x = r \sin \theta, \quad y = 1 - r \cos \theta, \quad (0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi). \hspace{1cm} (3.5.3)$$

and the Jacobian of the transformation is given by

$$\text{Jacobian}_2 = \rho.$$  \hspace{1cm} (3.5.4)

In polar coordinates, equation (3.3.4) becomes

$$\ln(h, v_b) = \int_0^{2\pi} \int_0^1 k(\Theta, \rho, \theta) r(\rho, \theta) \, d\theta \, d\rho$$  \hspace{1cm} (3.5.5)

where

$$k(\Theta, \rho, \theta) = \frac{\zeta^2 - (u_\Theta - \xi)^2}{[\zeta^2 + (u_\Theta - \xi)^2]^2} \frac{\zeta}{(\rho^2 - 2\rho \cos \theta + 1)^2}, \hspace{1cm} (3.5.6)$$
\[ h(\theta, v_b) = \frac{2\pi}{\int_0^\infty \frac{v_b}{(u\theta - \xi(\theta))^2 + v_b^2} \frac{d\theta}{\phi_p(\theta, v_b)} d\theta \quad (3.5.7) \]

\[ = \int_0^\frac{2\pi}{\frac{v_b}{\left[ (u\theta - \xi(\theta))^2 + v_b^2 \right]^{\frac{3}{2}}} \frac{\sin \theta}{1 - \cos \theta} \phi_p(\theta, v_b) d\theta, \quad (3.5.8) \]

and

\[ \xi = \frac{\rho \sin \theta}{(\rho^2 - 2\rho \cos \theta + 1)}, \quad \zeta = \frac{\rho \cos \theta - 1}{(\rho^2 - 2\rho \cos \theta + 1)}. \quad (3.5.9) \]

\[ u\theta = 0.5 \frac{\sin \theta}{1 - \cos \theta}. \quad (3.5.10) \]

In the above derivations, \( \alpha = -0.5 = -v_b \).

The Jacobian has a singular point at \( x = y = 0 \), or \( \rho = 1, \theta = 0 \), which needs careful treatment before numerical solution of equation (3.5.5). Assume that the conductivity within a sufficiently small area around the singularity is a constant, denoted by \( r_s \). The integral value on the small area can then be evaluated analytically.

It is convenient to treat the singular point in the \( u-v \) plane and, according to the coordinate transformation (3.2.7), the point \( x = y = 0 \) is corresponding to \( u = \pm \infty, v = -\infty \). Let an area \( D_s \) on \( u-v \) plane be defined by

\[ -\infty \leq v \leq -B, \quad A \leq |u| \leq \infty \quad (3.5.11) \]

where \( A \) and \( B \) are positive constants, and denote the integral on the \( D_s \) by \( I_s \). The integral equation (3.3.4) then becomes

\[ h(u, v_b) = \int_{-B}^{A} \int_{-\infty}^{-B} k(u, \xi, \zeta) \cdot \frac{r(\xi, \zeta)}{\mathcal{R}} d\xi d\zeta + I_s \quad (3.5.12) \]
where

$$I_S = r_S \left[ \tan^{-1}\left(\frac{v_b}{u-A}\right) - \tan^{-1}\left(\frac{v_b}{u+A}\right) \right] + \tan^{-1}\left(\frac{B}{u-A}\right) - \tan^{-1}\left(\frac{B}{u+A}\right) \right].$$

(3.5.13)

Thus, the singularity of the integral equation (3.4.4) has been removed from the equation (3.5.12).

In the polar system, equation (3.5.12) becomes

$$h(\theta, v_b) = \int \int_{D - \Delta D} k(\theta, \rho, \theta) r(\rho, \theta) \, d\theta \, d\rho + I_S,$$

(3.5.14)

where $D$ is the domain of the whole object, $\Delta D$ is a small area corresponding to the area in the $u$-$v$ plane defined by equation (3.5.11). The $r_S$ in equation (3.5.13) is still an unknown and will be solved together with other unknown log-resistance elements during numerical solution of the integral equation (3.5.14).

The singularity $I_S$ is evaluated in the $u$-$v$ plane whilst equation (3.5.14) is solved numerically using a polar coordinate grid and thus it is necessary to relate areas in the different coordinate systems. Equation (3.5.11) defines an area in the $u$-$v$ plane, as shown in Fig.3.5.1 (b), that does not coincide with polar coordinate lines. This area is now approximated by the region:

$$\rho_B \leq \rho \leq 1, \quad 0 \leq |\theta| \leq \theta_A,$$

(3.5.15)

which is shown in Fig.3.5.1(a). $\rho_B$ and $\theta_A$ are found by equating the two areas.
Studies on numerical solutions of Fredholm integral equation of the first kind have indicated that the system matrix of the integral equation is almost always ill-conditioned. This ill-conditioned system matrix normally leads to a highly oscillated solution when conventional matrix inversion techniques are used. Singular Values Decomposition (SVD) of the ill-conditioned matrix, together with truncation of certain small singular values, has been used with some success in solving one-dimensional integral equations of the first kind [Lawson, 1974, Hanson, 1971]. The technique is adopted here to solve the set of two-dimensional integral equations (3.5.14) in order to generate acceptable reconstructions.

Fig. 3.5.1 Shadow areas defined by (3.5.15) and (3.5.11) on the same x-y plane. (a) Shadow area defined by polar system coordinates $\pm \theta_A, \rho_B$, and 1. (b) Shadow area defined by $u = \pm A, v = -B$, and $v = v_b$.

3.6. Simulations and Experiment Results

Numerical performance of the integral equation method has been tested by simulations and real experiments. Results are demonstrated and discussed in this section.
3.6.1 Condition of the System Matrices

Numerical solution of the integral equation approximates the continuous equation by a set of simultaneous equations which are usually ill-condition. The condition of a matrix can be measured in a number of ways and one useful measure is the ratio of the maximum to minimum singular values of the matrix. The condition of the matrix deteriorates when the ratio becomes large.

To investigate the singular value distributions of the integral equation, the uniform unit circle on which the equation is defined is discretized into three different meshes, as shown in Fig.6.1 (a)-(c). Numerical integration of integral equation (3.5.14) is carried out for each of the three meshes using rectangular quadrature. The resulting system matrices are then decomposed by SVD method and the normalized singular value distributions are illustrated in Fig.6.2.

Fig.3.6.1 Mesbes used in SVD analysis of the integral equation.
Fig.3.6.2 Normalized singular value distributions of the three meshes, where mesh No.1, 2 and 3 refer to Fig.3.6.1 (a), (b) and (c), respectively.

The maximum and minimum singular values for all the three meshes range between approximately 10 and $10^{-9}$. The very large difference of the singular values reveals the serious ill-condition of the system matrix.

3.6.2 Point Objects Reconstruction Simulations

The point object response obtained from the linearized Poisson’s equation is given by equation (3.2.21). Reconstructions of single point objects using the filtered back-projection method has been reported [Barber and Brown, 1986].

In the case of several point objects located within a uniform circle, the back-projection method may have difficulty in distinguishing individual point objects but the integral equation is capable of doing so even analytically, as discussed in section 3.4. However, when numerical reconstructions are performed, results may be affected by
round-off error and modeling error during numerical processing. The following simulations demonstrate the reconstruction results of point objects using the integral equation method.

The geometry of the two separate point objects is shown in Fig.3.4.2 (a). The total boundary potential gradients for current injections at different angle were obtained using equation (3.4.1). The reconstruction was performed on mesh No.3 in order to have good spatial resolution of the image. In the reconstruction, singular values less than $10^{-7}$ were truncated in order to suppress unwanted oscillations in the result.

The normalized reconstruction result is shown in Fig.3.6.3, where object (i) is represented by the small pulse and object (ii) is represented by the outer tall pulse. The location of each reconstructed pulse is correct but the amplitudes of the two pulses are not equal which indicates that the integral equation method is more sensitive to anomalies close to boundary.

![Reconstructed image for two point objects.](image)

The size of the reconstructed point objects depends on the element size of the mesh. The reconstructed image in Fig.3.6.3 appears to be considerably narrower than the reconstruction of single point presented by Barber and Brown [1986]. Further
reduction of element size may increase the spatial resolution, but the penalty is that the resulting larger dimension system matrix is more sensitive to numerical errors.

3.6.3 Reconstructions of Concentric Objects

Simulations

The objects under test had a concentric log-resistance distribution given by:

\[ r(\rho, \theta) = \begin{cases} \rho < \rho_c & r_c \\ 0 & \rho_c \leq \rho \leq 1 \end{cases} \] (3.6.1)

For such a geometry, theoretical solution to boundary potential was available (appendix B) and was used as simulated data for reconstructions.

In simulations, various combinations of the parameters \( \rho_c \) and \( r_c (= -\ln\sigma) \) had been tested. Simulations were performed on all the three meshes, respectively, and acceptable results were obtained for mesh No.1 and No.2. Reconstructions using mesh No.3 produced highly oscillated images that masked the wanted object. In all cases, truncation thresholds were chosen to be between \( 10^{-6} \) and \( 10^{-7} \). Normalized reconstruction results using mesh No.1 for different values of \( \rho_c \) are shown in Fig.3.6.4 - Fig.3.6.6, respectively.
Simulation Result for $\rho_c = 0.25$, $\sigma = 2.0$

Fig. 3.6.4  Reconstruction of object with $\rho_c = 0.25$. Mesh No.1 is used.

Simulation Result for $\rho_c = 0.5$, $\sigma = 2.0$

Fig. 3.6.5  Reconstruction of object with $\rho_c = 0.5$. Mesh No.1 is used.
Features of the results are summarized as follows:

1. All the solutions oscillate around the required results and there is variation amongst elements on different layers. The oscillations can be easily suppressed by a smoothing process among elements of the same layer.

2. It is necessary to truncate the singular values below a threshold in order to suppress unwanted oscillations in the solutions. Raising threshold contributes little to the oscillation suppression and will gradually degrade authenticity of the reconstructed image. For the result in Fig.3.6.6, the relatively large oscillations can be reduced but at a cost of increasing image distortion.

3. The value the threshold is related with the mesh used, the quadrature formula, and the level of measurement noise. In practice, the threshold is determined by a trial-and-error approach.

4. In the three reconstructions, \( \rho_c \) were chosen to be 0.25, 0.5, and 0.75, which coincide with the layer positions of the mesh. However, when \( \rho_c \) deviated slightly
from above values, similar results were obtained. This indicates that the method is less sensitive to the form of meshes used.

(5) The integral method only offers a relative reconstruction of conductivity. When the value of $r_c$ varies, the values in the reconstructed images do not vary accordingly. This is because the integral equation was derived from the linearized Poisson’s equation and is therefore an approximate algorithm.

(6) The reconstruction is sensitive to the number of elements in a mesh. For a mesh consists of large number of elements, such as mesh No.3, the integral equation method may not be able to produce reasonable reconstructions. This occurs because as dimension of system matrices becomes larger, the solution turns out to be more sensitive to the errors in the integral equation as well as numerical inversion of a system matrix.

**Reconstructions using measured data**

The reconstruction algorithm was also tested on real objects which consisted of discs of conducting rubber with 100 mm diameter, 2 mm thickness and $8\Omega/\text{cm}^2$ conductivity. On the central circle ($\rho < \rho_c$), silver paint was applied as a perturbation to the uniform conductivity. Two objects were tested with $\rho_c = 0.25$ and $\rho_c = 0.5$, respectively. 16 thin cylindrical electrodes were placed around the boundary of the object with uniform spacing. A 10 kHz ac current source of 15 mA p-p was applied to neighbouring pairs of electrodes as required by the algorithm. Electrode voltage was measured to an accuracy of three decimal places. Boundary potentials between adjacent electrodes were obtained by interpolation and were used to calculate the $h(\Theta, v_B)$ in equation (3.5.14). The perturbed boundary potential $\phi_p$ was obtained by subtracting the theoretical potential distribution for an uniform object from the measured total boundary potentials.

The reconstructed results using mesh No.1 are displayed in Fig.3.6.7 and Fig.3.6.8. Singular values were truncated at a threshold of $10^{-4}$, which was larger than that used in previous simulations. The increased threshold was due to the noise
contained in the measured data and this agreed with the reported results for numerical solution of Fredholm integral equation of the first kind [Lawson, 1974, Hanson, 1972]. Oscillations are visible in the graphs but the images are clear reconstructions of the original objects.

![Graph showing normalized conductivity vs element number](image)

**Fig. 3.6.7** Reconstruction using measured data. Mesh No. 1 was used and $\rho_c = 0.25$. (a) Element values. (b) Three-dimensional plot.
Fig. 3.6.8  Reconstruction using measured data. Mesh No.1 was used and \( p_c = 0.5 \). (a) Element values.  (b) Three-dimensional plot.

It has been observed that the contact impedance of driving electrodes introduced error into the measured potentials. This was confirmed by comparing the
measured and theoretical potentials of a uniform object. However, the error had little effect on the reconstructions of concentric objects, as illustrated by the results in Fig.3.3.7 and Fig.3.6.8.

### 3.6.4 Reconstruction of Offset Objects

The offset objects that were tested consist of uniform circle with a circular anomaly defined by offset distance $d$, radius $\rho_o$, and log-resistance $r_c$, as shown in Fig.3.6.9. Theoretical solutions of the boundary potentials for the objects can be obtained by a conformal transformation approach. The resulting exact boundary potentials were used in the reconstruction simulations.

![Geometry of an offset object.](image)

For each current injection, the theoretical boundary potential distribution (appendix C) was calculated and used to evaluate the convolution function $h(\Theta, v_b)$ in equation (3.5.14). This $h(\Theta, v_b)$ was then used in solution of the integral equation in the reconstructions of concentric objects. It has been found that, in the case of offset objects, the individual convolution function $h(\Theta, v_b)$ needs to be re-scaled before being used to solve the integral equations.
The re-scaling operation is as follows: For a given boundary current injection, the known conductivity distribution is substituted into the integral on the right-hand-side of the integral equation. The values of the integral with respect to variable $\Theta$ are calculated and the maximum value is denoted by $F_m$. For the same current injection, a convolution function $h(\Theta, v_b)$ is also calculated with respect to $\Theta$. The $h(\Theta, v_b)$ is then re-scaled such that its maximum is equal to $F_m$. This procedure is applied to every current injection before the numerical solution of the set of integral equations takes place.

In Fig.3.6.10, a reconstructed image for an offset object with $d = 0.625$, $\rho_0 = 0.125$, and $\sigma = 2.0$ is displayed. In the reconstruction, mesh No.1 was used and the offset object was located in elements 19 and 42. The truncation threshold of the singular values was chosen to be $10^{-6}$. The reconstructed result exhibits a clear contrast between conductivity values of element 19, 42 and that of the remaining elements.
Fig. 3.6.10 Reconstruction using simulated data for an offset object with $d = 0.625$, $\rho_0 = 0.125$ and $\sigma = 2.0$.

(a) Element values. (b) Three-dimensional plot.

3.6.5 Discussions

The integral equation and the filtered methods were both derived from the linearized Poisson's equation. An error study on the linearized Poisson's equation, which will be described in chapter 4, indicates that the small perturbation assumption fails in most cases and for large perturbations, the dominant error caused by the linearization of Poisson's equation is a re-scaling error, which allows reconstruction of a relative conductivity image.

In the integral equation method, a set of integral equations given in equation (3.3.4) is obtained using multiple current injection. The equations must be consistent in order to acquire a correct solution. For small perturbations, the consistency requirement is satisfied and the integral equation method appears to be a suitable
approach to the impedance imaging. In the case of large perturbations, however, the equations may become inconsistent due to the re-scaling errors.

In the appearance of re-scaling errors, the set of discretized integral equations derived from equation (3.3.4) can be written as

\[
\begin{align*}
\lambda_1 A_1 \sigma &= h_1 \\
\vdots \\
\lambda_{NP} A_{NP} \sigma &= h_{NP}
\end{align*}
\]  

(3.6.2)

or

\[
\begin{align*}
A_1 \sigma &= \lambda_1 h_1 \\
\vdots \\
A_{NP} \sigma &= \lambda_{NP} h_{NP}
\end{align*}
\]  

(3.6.3)

where \( \sigma \) is a vector consisting of conductivity values of each elements, \( A_i \) \( (i = 1, NP) \) is the matrix obtained from the i'\text{th} integral equation, \( h_i \) \( (i = 1, NP) \) is a vector consisting of the i'\text{th} convolution values, the subscript \( i \) stands for the i'\text{th} current injection. The re-scaling factors \( \lambda_i \) and \( \lambda_i' \), \( i = 1, NP \), are required to compensate for the error caused by linear approximation of Poisson's equation.

For concentric objects, all the re-scaling factors are of the same value due to symmetry and thus the set of equations (3.6.2) and (3.6.3) are consistent. Moreover, reconstructions can be performed without knowledge of the actual value of the re-scaling factor because the value only affect the dynamic range of the images.

In the case of asymmetrical objects, the values of the re-scaling factors are essential to the consistency of the equations (3.6.2) and (3.6.3). The offset object reconstruction experiments in subsection 3.6.4 have found that values of the re-scaling factors for different current injections vary considerably. For example, the re-scaling factors for the experimental set-up in Fig.3.6.11 (a) and (b) are 0.633 and 0.106, respectively.
The reconstruction tests on the offset objects have proven that, when proper re-scaling factors are used, the integral equation method is able to produce meaningful images. This confirms the dominance of the re-scaling error in the case of asymmetrical objects.

**Fig. 3.6.11** Measurement arrangement for an offset object with two different boundary current injections.
In practice, the values of the re-scaling factors depend on the unknown conductivity distribution and are not known before reconstruction. A natural practice is to treat the re-scaling factors as variables. By doing so, however, equation (3.6.2) becomes a set of nonlinear equation and all the advantages of a linear equation will be lost. For equation (3.6.3), the reconstruction becomes a problem of finding the non-trivial solutions of the homogeneous equation, which is more complicated than solving a set of linear equation.

Other than to solve a set of linear equations, the filtered method back-projects the measured boundary data for individual current injection. This method avoids the problem of consistency between the data for different current injections and maintains the merits of a linear algorithm. The problem of different values of the re-scaling factors is treated by normalization within the set of measured data for each current injection.

3.7 Conclusions

A linear integral equation approach to electrical impedance tomography has been developed. In the case of both small and large perturbations, the integral equation is a suitable and useful mathematical description for the study of linear reconstruction algorithms in electrical impedance tomography.

Properties of the integral equation are investigated. Relations between the integral equation method, the X-ray back-projection method, and Barber-Brown's filtered back-projection approach are discussed.

A numerical reconstruction algorithm is established. The algorithm is one step reconstruction approach and by discretizing the kernel of integral and storing the inverted system matrix beforehand, fast reconstruction can be achieved.

As a Fredholm integral equation of the first kind, the integral equation is sensitive to data error and numerical error. Singular value decomposition technique
has been found necessary during numerical solution of the integral equation to acquire acceptable results.

The integral equation reconstruction method needs to solve a set of integral equations and for small perturbations or symmetrical objects with large perturbation, the method is able to produce useful results. In the case of asymmetrical objects with large perturbation, the method can generate meaningful images provided the values of the re-scaling factors are known. In practice, however, the values of the re-scaling factors are not available and thus the requirement for the integral equations to be consistent cannot be satisfied. The problem can be avoided by using pseudo-inversions of the set of integral equations, such as the filtered back-projection method. Other pseudo-inversions needs to be developed and investigated.
Chapter 4

Error Study of Linearized Poisson's Equation
4.1 Introduction

The Barber-Brown [1986] filtered back-projection method and the integral equation method are both based on the linearized Poisson's equation. The linearized Poisson's equation was obtained by discarding the term $\nabla r \cdot \nabla \phi_p$ from the equation (3.2.4). A common procedure that allows the dropping of the term $\nabla r \cdot \nabla \phi_p$ is the assumption that perturbation is small, i.e.

$$\| \nabla \sigma \cdot \nabla \phi_p \| \ll \| \nabla \sigma \cdot \nabla \phi_0 \|,$$  \hspace{1cm} (4.1.1a)

or

$$\| \nabla r \cdot \nabla \phi_p \| \ll \| \nabla r \cdot \nabla \phi_0 \|. \hspace{1cm} (4.1.1b)$$

Equation (4.1.1) requires that the projection of the perturbed electric field $\nabla \phi_p$ on the vector $\nabla r$ (or $\nabla \sigma$) be ignorable compared to the projection of the reference electric field $\nabla \phi_0$ on the $\nabla r$ (or $\nabla \sigma$).

When the conductivity distribution of an object is only slightly different from a uniform distribution, the term $\nabla r \cdot \nabla \phi_p$ will have small value and, therefore, the small perturbation assumption (4.1.1) can be valid. However, studies on the distinguishability of electrical impedance tomography [Issacson, 1986] has shown that when the perturbation of conductivity is small, the resulting $\phi_p$ will also be small so that it may become impossible to detect the inhomogeneity due to the finite accuracy of measuring systems and digital computers. Thus, a dilemma exists between the small perturbation condition (4.1.1) for linearization of the Poisson's equation and the distinguishability of the electrical impedance tomography.

In practice, electrical conductivity of various biological tissues are known to cover a wide range of values [Barber and Brown, 1984]. Even within soft tissues, differences are quite substantial. Moreover, discontinuities of conductivity often occur
at the boundary of different organs. Similar situations occur in mineral compositions in geophysical engineering. The wide range of conductivity variation and the presence of discontinuities do help to distinguish the conductivity anomalies but, at the same time, conflict with the small perturbation condition for the linearized Poisson's equation to be valid.

Applications of the filtered back-projection method and the integral equation method have demonstrated that even for objects with large perturbations and discontinuities where the small perturbation (4.1.1) fails, the reconstruction algorithms are still able to produce useful images. Mechanism of this phenomenon needs to be investigated in order to improve the existing reconstruction algorithms as well as to develop new reconstruction methods.

Two types of errors are associated with linear reconstruction algorithms. One is concerned with the error caused by elimination of the term $V_r \nabla \phi_p$ from Poisson's equation. The other deals with the effect of noise in the measured boundary data. The first type of error has been found to be dominant in linear reconstruction algorithms and it will be studied in this chapter. The study includes error mechanism analysis, case studies, quantitative evaluation of the error, as well as filtering effect on the errors.

A piecewise finite element modeling of a continuous conductivity distribution is established to facilitate the error analysis. Properties of Poisson's equation at the border of discontinuity of the conductivity are examined, which is important to the error analysis.

The error study indicates that the commonly used small perturbation assumption fails in most practical situations. The linearization of Poisson's equation can cause larger error to linear reconstruction algorithms. However, it has been found that the dominant error is re-scaling error which allows the reconstruction of relative images by the linear algorithms.
4.2 Piecewise Finite Element Modelling and Boundary Properties of Poisson's Equation

4.2.1 A Piecewise Finite Element Modeling of Conductivity

In order to simplify mathematical manipulation in the error study, a Piecewise Finite Element Modeling (PFEM) of the continuous conductivity distribution $\sigma(x,y)$ of an object is introduced. The PFEM is constructed by dividing the domain of the object, denoted by $D$, into $m$ non-overlapping subdomains $D_i$, $i=1,2,\ldots,m$. The subdomains are called elements and the set of the subdomains covers the whole area of the object. The boundaries of $D$ and the $D_i$'s are denoted by $\overline{D}$ and $\overline{D_i}$, respectively. On each element, a constant conductivity $\sigma_i$ is assigned which is equal to the arithmetic mean of the continuous conductivity distribution within that element. The so defined PFEM can be written as

$$\sigma(x,y) = \sum_{i=1}^{m} \sigma_i \cdot c_i(x,y), \quad c_i(x,y) = \begin{cases} 1 & x, y \in D_i \\ \frac{1}{2} & x, y \in \overline{D_i} \cap \overline{D_j} \\ 0 & \text{otherwise} \end{cases} \quad (4.2.1)$$

Likewise, a PFEM for the log-resistance $r = -\ln \sigma$ is given by

$$r(x,y) = \sum_{i=1}^{m} r_i \cdot c_i(x,y), \quad c_i(x,y) = \begin{cases} 1 & x, y \in D_i \\ \frac{1}{2} & x, y \in \overline{D_i} \cap \overline{D_j} \\ 0 & \text{otherwise} \end{cases} \quad (4.2.2)$$

A typical PFEM is depicted in Fig.4.2.1.

On the border of elements, discontinuities of conductivity may appear. The value of PFEM at the discontinuity is defined as the mean value of the element.
conductivity on both sides of the border. For the given \( m \) and \( D \), different partition of the area \( D \) results in different PFEM. When \( m \) approaches infinity and the maximum area of elements approaches to zero, the function \( \sigma(x, y) \) and \( r(x, y) \) will approach the continuous distribution \( \sigma(x, y) \) and \( r(x, y) \), respectively. For uniform objects, the function \( \sigma(x, y) \) and \( r(x, y) \) are identical to \( \sigma(x, y) \) and \( r(x, y) \).

Fig. 4.2.1 A typical piecewise finite element modeling.

The use of the PFEM is also supported by several practical considerations. Firstly, in computerized electrical impedance tomography, the number of unknowns (or pixels) is always finite and within each element the conductivity is treated as a constant. Secondly, reconstructed images are usually displayed on equipment with finite display elements such as digital CRT, dot matrix printer, etc.. Thirdly, the practical objects to be imaged often have distributions similar to the PFEM. For instance, the cross section distribution of conductivity of the human thorax usually consists of finite sub-areas. Each sub-area may consist of one organ or the same tissue which has an approximately constant conductivity. On the interface between different organs, the conductivity may have a rapid change which can be treated as discontinuity of conductivity. In all the cases, the PFEM is obviously a suitable choice.

The PFEM function \( \sigma(x, y) \) has the following useful properties:
(1) Zero gradient inside elements:

\[ \nabla \sigma(x, y) = 0, \quad x, y \in D_i, \quad i = 1, 2, \ldots, m \]  

(4.2.3)

(2) Tangential continuity of gradient:

\[ \frac{\partial \sigma(x, y)}{\partial t} = 0, \]  

(4.2.4)

(3) Normal discontinuity of gradient:

\[ \frac{\partial \sigma(x, y)}{\partial n} = (\sigma_j - \sigma_i) \delta(\cdot), \quad x, y \in \overline{D_i \cap D_j}, \]  

(4.2.5)

where \( \frac{\partial \sigma(x, y)}{\partial t} \) and \( \frac{\partial \sigma(x, y)}{\partial n} \) represent the tangential and normal component of \( \nabla \sigma(x, y) \) on the boundary \( \overline{D_i \cap D_j} \), and \( \delta(\cdot) \) is the delta function along the normal direction on the boundary point \((x, y)\). The \( \overline{D_i \cap D_j} \) is the intersection between element boundary \( \overline{D_i} \) and \( \overline{D_j} \). The normal is defined as the direction from element \( D_i \) to element \( D_j \).

Similar results are available for the function \( r(x, y) \), as shown below:

\[ \nabla r(x, y) = 0, \quad x, y \in D_i, \quad i = 1, 2, \ldots, m \]  

(4.2.6)

\[ \frac{\partial r(x, y)}{\partial t} = 0, \]  

(4.2.7)

\[ \frac{\partial r(x, y)}{\partial n} = (r_j - r_i) \delta(\cdot), \quad x, y \in \overline{D_i \cap D_j}. \]  

(4.2.8)

Substituting equations (4.2.6) - (4.2.8) into Poisson's equation (3.2.4) yields

\[ \nabla^2 \phi = \nabla^2 \phi_p = \nabla \cdot (\nabla \phi_0 + \nabla \phi_p) = 0, \]
Equation (4.2.9) reveals the important fact that the perturbed potential $\phi_p$ for the PFEM functions (4.2.1) and (4.2.2) also satisfies Laplace's equation within each element, as does the potential $\phi_0$. Therefore discarding the term $\nabla \cdot \nabla \phi_p$ from equation (4.2.9) does not cause any error. In other words, the error caused by linearizing Poisson's equation for a PFEM distribution may only appear on the boundary of elements.

4.2.2 Boundary Properties of Poisson's Equation

Electrical potentials generated by low frequency current or voltage sources satisfy Poisson's equation inside every element. On the boundary of the elements, the potentials satisfy boundary conditions instead of Poisson's equation. The boundary conditions require that the total normal current density and the total tangential electric field be continuous across borders between any two neighbouring elements $i$ and $j$. The mathematical descriptions of the boundary conditions are given by

$$\frac{\partial \phi}{\partial t} (D_i) = \frac{\partial \phi}{\partial t} (D_j),$$  \hspace{1cm} (4.2.10)

$$\sigma_i \frac{\partial \phi}{\partial n} (D_i) = \sigma_j \frac{\partial \phi}{\partial n} (D_j).$$  \hspace{1cm} (4.2.11)

In particular, the boundary condition for the potential of a uniform object are given by

$$\frac{\partial \phi_0}{\partial t} (D_i) = \frac{\partial \phi_0}{\partial t} (D_j),$$  \hspace{1cm} (4.2.12)

$$\frac{\partial \phi_0}{\partial n} (D_i) = \frac{\partial \phi_0}{\partial n} (D_j).$$  \hspace{1cm} (4.2.13)
In the development of Barber-Brown's filtered back-projection algorithm and the integral equation algorithm, however, Fourier transformation was applied to the Poisson's equation not only inside every element, but also on the boundaries of elements, where Poisson's equation may not be valid. For a PFEM function $r(x, y)$, the Fourier transform of Poisson's equation has zero values inside every element. On the boundaries of elements, the Fourier transform may have non-zero values. It is necessary, therefore, to investigate the behaviour of Poisson's equation on the boundaries of elements.

At the boundary $\bar{D}_i \cap \bar{D}_j$, terms on both sides of Poisson's equation (3.2.4) become

$$\nabla^2 \phi = \nabla^2 \phi_p = \frac{\partial^2 \phi_p}{\partial n^2} + \frac{\partial^2 \phi_p}{\partial t^2}$$  \hspace{1cm} (4.2.14)

$$- \frac{1}{\sigma} \nabla \sigma \cdot \nabla \phi = - \frac{1}{\sigma} \frac{\partial \sigma}{\partial n} \frac{\partial \phi}{\partial n} .$$  \hspace{1cm} (4.2.15)

If $\sigma_i \neq \sigma_j$ then, according to equation (4.2.11), $\frac{\partial \phi}{\partial n}$ has a step discontinuity across the boundary. Thus term $\frac{\partial^2 \phi_p}{\partial n^2}$ in equation (4.2.14) can be expressed as

$$\frac{\partial^2 \phi_p}{\partial n^2} = \left[ \frac{\partial \phi_p}{\partial n} (\bar{D}_j) - \frac{\partial \phi_p}{\partial n} (\bar{D}_i) \right] \delta \left( \cdot \right) = A_p \delta \left( \cdot \right) ,$$  \hspace{1cm} (4.2.16)

where $A_p = \frac{\partial \phi_p}{\partial n} (\bar{D}_j) - \frac{\partial \phi_p}{\partial n} (\bar{D}_i) = \frac{\partial \phi}{\partial n} (\bar{D}_j) - \frac{\partial \phi}{\partial n} (\bar{D}_i)$. Substituting equation (4.2.11) into equation (4.2.16) yields

$$\frac{\partial^2 \phi_p}{\partial n^2} = \frac{\sigma_i - \sigma_j}{\sigma_i} \frac{\partial \phi}{\partial n} (\bar{D}_j) \delta \left( \cdot \right) .$$  \hspace{1cm} (4.2.17)
Using the definition (4.2.1) and equations (4.2.4) - (4.2.5), and noticing that

\[ \frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial n} (\bar{\Omega}_j) + \frac{\partial \phi}{\partial n} (\bar{\Omega}_i) \]

(4.2.18)

equation (4.2.15) can then be written as

\[ \frac{1}{\sigma} \frac{\partial \sigma}{\partial n} \frac{\partial \phi}{\partial n} = -\frac{\sigma_i - \sigma_j}{\sigma_i} \frac{\partial \phi}{\partial n} (\bar{\Omega}_j) \delta (.) \]

(4.2.19)

Comparing equation (4.2.17) with equation (4.2.19) yields

\[ \frac{\partial^2 \phi_p}{\partial n^2} = -\frac{1}{\sigma} \frac{\partial \sigma}{\partial n} \frac{\partial \phi}{\partial n} , \quad x, y \in \bar{\Omega}_i \cap \bar{\Omega}_j . \]

(4.2.20)

Equation (4.2.20) reveals that unless

\[ \frac{\partial^2 \phi_p}{\partial t^2} = 0 , \quad x, y \in \bar{\Omega}_i \cap \bar{\Omega}_j , \]

(4.2.21)

Poisson's equation will not be valid on boundaries of the elements. In general, tangential electric field continuity does not automatically lead to condition (4.2.21).

While the term \( \frac{\partial^2 \phi_p}{\partial n^2} \) is a delta function on boundaries, the term \( \frac{\partial^2 \phi_p}{\partial t^2} \) is continuous and has finite values according to the boundary conditions (4.2.4) and (4.2.5). Since Poisson's equation has non-zero values only on the finite number of boundary lines, the two-dimensional Fourier transform of the function \( \frac{\partial^2 \phi_p}{\partial t^2} \) has, therefore, infinitely small value due to the finite values of the function and the infinitely small areas of integration on the lines. On the other hand, the Fourier transform of the
function $\frac{\partial^2 \phi_p}{\partial n^2}$ has non-zero values because of the delta functions. Therefore we arrive at the following conclusions:

Poisson's equation is valid inside elements of a PFEM but may not be so on the element boundaries. However, the Fourier transform of Poisson's equation is valid within elements as well as on the element boundaries. The Fourier transform has zero values within elements, where Poisson's equation reduces to Laplace's equation, and non-zero value on boundary lines.

4.3 Error analysis of linearized Poisson's equation

In the previous section, it has been found that the Fourier transform of Poisson's equation is valid for both continuous object and the object with discontinuous distribution. The linearized Poisson's equation, however, contains error due to dropping of the term $\nabla \sigma \nabla \phi_p$ or $\nabla r \nabla \phi_p$. This section is then devoted to the error analysis of the linearized Poisson's equation. In addition, reduction of the error by filtering during development of linear reconstruction algorithms is also discussed.

4.3.1 Classification of Errors

In section 4.2, it is found that the Fourier transform of Poisson's equation (3.2.4) is equivalent to the Fourier transform of equation (4.2.20). The error analysis of linearized Poisson's equation will, therefore, be based on equation (4.2.20) which is re-written as

$$\frac{\partial^2 \phi_p}{\partial n^2} = -\frac{1}{\sigma} \frac{\partial \sigma}{\partial n} \left[ \frac{\partial \phi_p}{\partial n} + \frac{\partial \phi_o}{\partial n} \right], \quad x,y \in \bar{D}_i \cap \bar{D}_j. \quad (4.3.1)$$

or, in terms of the log-resistance, as
\[ \frac{\partial^2 \phi_p}{\partial n^2} = \frac{\partial r}{\partial n} \left[ -\frac{\partial \phi_p}{\partial n} + \frac{\partial \phi_o}{\partial n} \right], \quad x, y \in \bar{D}_i \cap \bar{D}_j. \quad (4.3.2) \]

Dropping the term \( \nabla \sigma \cdot \nabla \phi_p \) or \( \nabla r \cdot \nabla \phi_p \) in Poisson's equation (3.2.4) is equivalent to dropping the term \( \frac{\partial \phi_p}{\partial n} \) from equation (4.3.1). Three types of error associated with eliminating the term \( \frac{\partial \phi_p}{\partial n} \) will be discussed below. The error analysis focuses on the difference between terms \( \frac{\partial^2 \phi_p}{\partial n^2} \) and \( -\frac{1}{\sigma} \frac{\partial \sigma}{\partial n} \frac{\partial \phi_o}{\partial n} \), as seen from equation (4.3.1).

A. Re-scaling error:

Substituting equation (4.2.1), (4.2.4), (4.2.5) and (4.2.16) into equation (4.3.2) yields

\[ A_p \delta(\cdot) = (r_j - r_i) \left[ \frac{\partial \phi_p}{\partial n} + \frac{\partial \phi_o}{\partial n} \right] \delta(\cdot). \quad (4.3.3) \]

Equation (4.3.3) indicates that discarding the term \( \nabla r \cdot \nabla \phi_p \) (i.e., the term \( \frac{\partial \phi_p}{\partial n} \) in the equation) will cause amplitude distortion to the delta function \( \delta(\cdot) \) on the right hand side of equation (4.3.3). Nevertheless, information about the location of the conductivity discontinuity remains in the equation since the function \( \delta(\cdot) \) still exists in the remaining parts of the equation after the term \( \frac{\partial \phi_p}{\partial n} \) is removed. Moreover, if we introduce a scaling factor \( \lambda \) to the remaining parts of the equation, we then have

\[ A_p \delta(\cdot) = \lambda \frac{\partial \phi_o}{\partial n} \delta(\cdot). \quad (4.3.4) \]
Equation (4.3.4) implies that if an impedance image is reconstructed by an algorithm derived from the linearized Poisson's equation (3.2.5), then a re-scaled log-resistance distribution will be obtained with accurate location of discontinuities. The scaling factor \( \lambda \) is found to be

\[
\lambda = \frac{\frac{\partial \phi_p}{\partial n} + \frac{\partial \phi_o}{\partial n}}{\frac{\partial \phi_o}{\partial n}}, \quad x,y \in D_1 \cap D_j \quad (4.3.5)
\]

and, because \( \lambda \) takes different values on different boundaries, the reconstructed image will not be uniformly re-scaled.

From the relationship between conductivity and log-resistance, it is found that the scaling factor \( \lambda \) serves as a power factor for the conductivity \( \sigma(x,y) \), i.e.,

\[
\lambda \sigma = \ln \left( \frac{1}{\sigma} \right)^{\lambda}. \quad (4.3.6)
\]

Based on equation (4.3.5), if \( \frac{\partial \phi_p}{\partial n} \) has the same sign as \( \frac{\partial \phi_o}{\partial n} \) on a boundary, then \( \lambda > 1 \) so that the reconstructed image will have enhanced contrast compared with the true image, otherwise the contrast of the image will be weakened.

**B. boundary distribution error:**

The re-scaling distortion analysis is only concerned with the error at individual points on a boundary. There is, however, another type of error associated with the distribution of normal gradient \( \frac{\partial \phi_p}{\partial n} \) along the boundary, which we call the boundary distribution error.

Consider a conductivity distribution, as shown in Fig.4.3.1. On the boundary \( x = x_1 \) the derivative of the conductivity with respect to \( x \) is given by
\[
\frac{\partial \sigma}{\partial x} = \begin{cases} 
(\sigma_1 - \sigma_0) \delta (x-x_1) & y \in (y_1 \ y_2) \\
0 & \text{otherwise}
\end{cases} \quad (4.3.7)
\]

Note that the amplitude of the delta function in equation (4.3.8) has a rectangular window distribution with respect to \(y\), as sketched in Fig. 4.3.2 (a).

The field distribution \(\frac{\partial \phi}{\partial n} = E\) can be achieved in a number of ways. For example, the field produced by a current dipole on the boundary of a circular uniform
object can be mapped into another plane (a u-v plane) through the conformal transformation (3.2.7). The current density becomes a constant with uniform polarity in the new plane (i.e., the u-v plane).

The term \( \frac{\partial \phi_p}{\partial x} \) is the normal component of the electric field and contains a step discontinuity on the boundary lines \( x = x_1 \) for \( y \in (y_1, y_2) \). On the boundary lines \( y = y_1 \) and \( y = y_2 \) with \( x \in (x_1, x_2) \), the term becomes a tangential component of the electric field which is continuous across the boundaries, or in other words, \( \frac{\partial^2 \phi_p}{\partial x^2} \) vanishes on the border. Thus the term \( \frac{\partial^2 \phi_p}{\partial x^2} \) is expected to have an expression of the form

\[
\frac{\partial^2 \phi_p}{\partial x^2} = A_p(y) \delta(x-x_1), \quad y \in (y_1, y_2)
\]

(4.3.9)

where \( A_p(y) \) is a smooth function with respect to \( y \). As \( y \) approaches \( y_1 \) or \( y_2 \), the amplitude of \( A_p(y) \) approaches zero because at \( y = y_1 \) or \( y = y_2 \) conductivity discontinuity with respect to \( x \) disappears so that the normal component of current density becomes continuous. Function (4.3.9) is sketched in Fig.4.3.2 (b).

Comparing the normalized amplitude distributions of \( \delta(.) \) in equation (4.3.8) and (4.3.9), it is found that, after dropping the term \( \frac{\partial \phi_p}{\partial n} \), there is distortion between the amplitude distribution of the delta functions on both sides of the linearized Poisson's equation. The largest disagreement occurs at the two ends of the interval \( (y_1, y_2) \), as seen from Fig.4.3.2. The amplitude difference of the delta functions between equation (4.3.8) and (4.3.9) can be restored by including the missing part, i.e., the \( \frac{\partial \phi_p}{\partial n} \), into Poisson's equation.
Fig. 4.3.2 (a) Function (4.3.8) where $\Pi(y)$ stands for a window function defined on $(y_1, y_2)$.
(b) Sketch of function (4.3.9)

C. The third kind of error:

In the case that the normal gradient $\frac{\partial \phi_o}{\partial n}$ vanishes on a boundary, discarding the term $\frac{\partial \phi_p}{\partial n}$ from equation (4.3.1) will result in an additional error.

When $\frac{\partial \phi_o}{\partial n}$ is zero, the right hand side of the linearized Poisson's equation (4.3.1) becomes zero but the left hand side may not be zero. Thus the linearized Poisson's equation becomes unbalanced. This error is different from either re-scaling or boundary distribution error and is therefore named as the third kind of error.

The third kind of error appears when the total electric field of a uniform object, i.e. $\nabla \phi_o$, is aligning along the tangential direction of the boundary. In this case, the normal component of the total electric field $\nabla \phi$ contains the perturbed electric field $\nabla \phi_p$ only. Since the perturbed electric field has been dropped from the linearized Poisson's equation, error is therefore introduced to the equation.
4.3.2 Filtering effect and error reduction

In developing Barber-Brown's back-projection algorithm and the integral equation method, Fourier transformation has been applied to the linearized Poisson's equation. The result is given by

\[(\omega_1^2 + \omega_2^2) \phi_p = F(\nabla r \cdot \nabla \phi_0)\]  \hspace{1cm} (4.3.10)

where \(F(.)\) stands for the Fourier transform of the function ",", and \(\omega_1\) and \(\omega_2\) are frequency variables with respect to spatial coordinates \(x_1\) and \(x_2\). The function

\[\frac{1}{\omega_1^2 + \omega_2^2}\]  \hspace{1cm} (4.3.11)

is then multiplied to equation (4.3.10) in order to cancel the term \((\omega_1^2 + \omega_2^2)\) on the left hand side of the equation. It is interesting to note that the function (4.3.11) also serves as a two-dimensional low pass filter which, when applied to equation (4.3.10), effectively reduces the boundary distribution error.

Consider function \(A_p(y)\) which is defined to be

\[A_p(y) = \begin{cases} \cos[\frac{\pi}{2a} (y - b)] & y \in (y_1 \ y_2) \\ 0 & \text{otherwise} \end{cases}\]  \hspace{1cm} (4.3.12)

where \(a = \frac{y_2 - y_1}{2}\), and \(b = \frac{y_2 + y_1}{2}\). The function has its maximum at the centre of the interval \((y_1, y_2)\) and zero at the two ends of the interval. Let us compare \(A_p(y)\) and the rectangular window function (4.3.8). The difference between the two functions is obvious.
The Fourier transforms of the window function and the function $A_p(y)$, denoted by $F_1(\omega_2)$ and $F_2(\omega_2)$, respectively, are found to be

$$|F_1(\omega_2)| = \frac{\sin(a\omega_2)}{\omega_2}$$  \hspace{1cm} (4.3.13)

and

$$|F_2(\omega_2)| = \frac{\pi}{a} \frac{\cos(a\omega_2)}{\left(\frac{\pi}{2a}\right)^2 - \omega_2^2}.$$  \hspace{1cm} (4.3.14)

Graphs of the two spectrums are shown in Fig.4.3.3(a) and there is clear difference between the two waveforms.

Applying the filter (4.3.11) to spectrums (4.3.13) and (4.3.14), we obtain two new waveforms which are shown in Fig.4.3.3(b). The difference between the two spectrums has been reduced substantially after the filtering, especially when the half width of the interval, 'a', is small.

Being symmetrical to variables $\omega_1$ and $\omega_2$, the filter (4.3.11) is equally effective on boundary distribution errors along both x and y coordinates.

Errors in the linearized Poisson's equation can be reduced by several ways. Re-scaling error can be removed by data normalization processing and tangential distribution error can be relieved by the filtering processing. Although the third kind error cannot be reduced by the above methods, chances for it to appear are few and the value of it is usually small. Thus, the overall errors in the linearized Poisson's equation are normally small and, even in the case of large perturbations, successful reconstructions using linear algorithms have been achieved.
Fig. 4.3.3 Filtering effect of function (4.3.11). $a = 1.0$, and unit of frequency is in radian. 
(a) Waveforms before filtering. 
(b) Waveforms after filtering.
In summary, following conclusions are obtained from the error analysis:

(1) For small perturbations, linearized Poisson's equation is valid but, according to
the distinguishability study of impedance imaging [Issacson, 1986], it is very likely
that the inhomogeneities become indistinguishable.

In the case of large perturbations, conditions (4.1.1) are normally violated but,
based on the error analysis of this chapter, the overall errors may still be acceptably
small with the help of re-scaling and filtering processing.

(2) Although there are amplitude distortions, the linearized Poisson's equation
retains information about location of the discontinuities, which are represented by the
delta functions in the boundary analysis of the equation. The linear algorithms are,
therefore, able to reconstruct discontinuities of a conductivity distribution.

(3) In general, it is unfeasible to obtain absolute value distributions of conductivity
by the linear reconstruction algorithms because of errors in the linearized Poisson's
approximation. In practice, however, relative distributions of conductivity are often
useful enough and, in many cases, information about conductivity discontinuities are
most desirable. In such circumstances, linear reconstruction methods are of particular
value.

4.4 Case studies

Case study 1:
Consider an object shown in Fig.4.4.1 with a boundary current distribution

\[ J(\theta) = \cos \theta \]  \hspace{1cm} (4.4.1)
Fig. 4.4.1 A concentric conductivity distribution.

This is the optimal current distribution used by Issacson [1986] to increase the distinguishibility of conductivity. The exact solution of Poisson's equation is found as

\[
\phi(\rho, \theta) = \begin{cases} 
\gamma \rho \cos \theta & 0 \leq \rho < \rho_1 \\
(\alpha \rho + \frac{\beta}{\rho}) \cos \theta & \rho_1 < \rho \leq 1 
\end{cases} \tag{4.4.2}
\]

\[
\phi_\rho(\rho, \theta) = \begin{cases} 
(\gamma - 1) \rho \cos \theta & 0 \leq \rho < \rho_1 \\
[(\alpha - 1)\rho + \frac{\beta}{\rho}] \cos \theta & \rho_1 < \rho \leq 1 
\end{cases} \tag{4.4.3}
\]

\[
\phi_\theta(\rho, \theta) = \rho \cos \theta \tag{4.4.4}
\]

where

\[
\gamma = \frac{1}{\sigma_1 + 1} \frac{2}{1 + \mu \rho_1^2} \tag{4.4.5a}
\]

\[
\beta = -\frac{\mu \rho_1^2}{1 + \mu \rho_1^2} \tag{4.4.5b}
\]
\[ \alpha = \frac{1}{1 + \mu \rho_1^2} . \]  
\hspace{4cm} (4.4.5c)

and

\[ \mu = \frac{\sigma_1 - 1}{\sigma_1 + 1} . \]  
\hspace{4cm} (4.4.6)

In a polar coordinates, Poisson's equation takes the form:

\[ \frac{\partial^2 \phi_p}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \phi_p}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \phi_p}{\partial \theta^2} = -\frac{1}{\sigma} \left[ \frac{\partial \sigma}{\partial \rho} \frac{\partial \phi}{\partial \rho} + \frac{\partial \sigma}{\partial \theta} \frac{\partial \phi}{\partial \theta} \right] . \]  
\hspace{4cm} (4.4.7)

On the boundary of the conductivity discontinuity \( \rho = \rho_1 \), it is found that

\[ \frac{\partial \phi}{\partial \rho} = \frac{1}{2} \left[ \gamma + \alpha - \frac{\beta}{\rho_1^2} \right] \cos \theta = \alpha \cos \theta , \]  
\hspace{4cm} (4.4.8)

\[ \frac{\partial \phi_p}{\partial \rho} = \frac{1}{2} \left[ \gamma + \alpha - 2 - \frac{\beta}{\rho_1^2} \right] \cos \theta = \beta \cos \theta , \]  
\hspace{4cm} (4.4.9)

\[ \frac{\partial^2 \phi_p}{\partial \theta^2} = \frac{1}{2} \left[ \gamma + \alpha - 2 + \frac{\beta}{\rho_1^2} \right] \rho_1 \cos \theta = (1 - \gamma) \rho_1 \cos \theta , \]  
\hspace{4cm} (4.4.10)

\[ \frac{\partial^2 \phi_p}{\partial \rho^2} = \left[ \alpha - \gamma - \frac{\beta}{\rho_1^2} \right] \cos \theta \delta(\rho - \rho_1) = 2 \alpha \mu \cos \theta \delta(\rho - \rho_1) , \]  
\hspace{4cm} (4.4.11)

\[ -\frac{1}{\sigma} \frac{\partial \sigma}{\partial \rho} = 2 \mu \delta(\rho - \rho_1) , \]  
\hspace{9cm} \frac{\partial \sigma}{\partial \theta} = 0 . \]  
\hspace{4cm} (4.4.12)
Substituting equation (4.4.8) - (4.4.12) into equation (4.4.7) and comparing coefficients for the delta function yield

\[ \frac{\partial^2 \Phi_p}{\partial \rho^2} = - \frac{1}{\sigma} \frac{\partial \sigma}{\partial \rho} \left( \frac{\partial \phi_o}{\partial \rho} + \frac{\partial \phi_p}{\partial \rho} \right). \]  

(4.4.13)

The remaining part other than the term \( \frac{\partial^2 \phi_p}{\partial \rho^2} \) on the left hand side of the equation (4.4.7) is equal to

\[ \frac{1}{\rho_1} \frac{\partial \phi_p}{\partial \rho} + \frac{1}{\rho_1^2} \frac{\partial^2 \phi_p}{\partial \theta^2} = - \frac{\beta}{\rho_1^3} \cos \theta \neq 0. \]  

(4.4.14)

Thus Poisson's equation is not valid on the boundary \( \rho = \rho_1 \), which confirms the discussions in the previous section. Furthermore, we find that

\[ \nabla \sigma \cdot \nabla \phi_o = (1 - \sigma_1) \cos \theta \delta (\rho - \rho_1), \]  

(4.4.15)

\[ \nabla \sigma \cdot \nabla \phi_p = (1 - \sigma_1) \beta \cos \theta \delta (\rho - \rho_1), \]  

(4.4.16)

and

\[ \frac{\ll \nabla \sigma \cdot \nabla \phi_p \ll}{\ll \nabla \sigma \cdot \nabla \phi_o \ll} = |\beta| = \frac{\mu \rho_1^2}{1 + \mu \rho_1^2}. \]  

(4.4.17)
Equation (4.4.17) shows that the small perturbation condition (3.3.6) fails in most cases. For example, let \( p_1 = 0.75, \sigma_1 = 5 \), the value of the ratio (4.4.17) is found to be \( \frac{3}{11} \), which is not small enough to be ignored.

Nevertheless, if we still remove the term \( \nabla \sigma \nabla \phi_p \) from Poisson's equation and then take the Fourier transform of the remaining part, only re-scaling error is incurred in this case. This can be seen because the two-dimensional Fourier transform of equation (4.4.14) is zero and discarding \( \frac{\partial \phi_p}{\partial \rho} \) in equation (4.4.13) only leads to amplitude unbalance of delta functions which can be restored by a re-scaling procedure.

**Case study 2:**

In Barber-Brown's back-projection method and the integral equation method, a current dipole was applied to the boundary of a circular object. The circular domain is then transformed into a half u-v plane via a conformal transformation. After the transformation, the current dipole becomes two line current sources uniformly located along \( v \in (v_b, -\infty), u = \pm \infty \).

For a uniform circular object, the transformed potential field is such that equipotential lines coincide with the \( v \) coordinate and current density lines uniformly align along the \( u \) coordinate, as shown in Fig.3.2.2. Mathematical expressions for the field distribution are given by

\[
\frac{\partial \phi_0}{\partial v} = 0, \tag{4.4.18a}
\]

\[
\frac{\partial \phi_0}{\partial u} = E = \text{const.}. \tag{4.4.18b}
\]

Assume that on the half plane \( v < 0, u \in (-\infty, \infty), \) a conductivity distribution is given by
\[ \sigma(u, v) = \begin{cases} \sigma_i & u \in (u_{2i-1}, u_{2i}) \\ \frac{\sigma_0 + \sigma_i}{2} & u = u_{2i-1} \text{ and } u_{2i} \\ \sigma_0 & \text{otherwise} \end{cases} \quad i = 1, 2, \ldots, n \] (4.4.19)

A plot of the distribution is shown in Fig.4.4.2.

Based on the conductivity distribution (4.4.19), the following equations are obtained

\[ \frac{\partial \sigma}{\partial v} = 0, \quad (4.4.20a) \]

\[ \frac{\partial \sigma}{\partial u} = \sum (\sigma_i - \sigma_0) [\delta(u-u_{2i-1}) - \delta(u-u_i)]. \quad (4.4.20b) \]

**Fig.4.4.2** Conductivity distribution of function (4.4.19).
Using boundary conditions (4.2.7), (4.2.11), (4.2.13) and (4.4.19) we obtain

\[
\frac{\partial \phi_p}{\partial v} = 0, \quad \text{(4.4.21a)}
\]

\[
\frac{\partial \phi_p}{\partial u} = \begin{cases} 
\frac{\sigma_o - \sigma_i}{E} & u \in (u_{2i-1}, u_{2i}) \\
\frac{\sigma_o - \sigma_i}{\sigma_i} & u = u_{2i-1} \text{ and } u_{2i} \\
0 & \text{otherwise}
\end{cases} \quad \text{(4.4.21b)}
\]

\[
i = 1, 2, \ldots, n,
\]

\[
\frac{\partial^2 \phi_p}{\partial u^2} = E \sum_{i=1}^{n} \frac{\sigma_o - \sigma_i}{\sigma_i} [\delta(u - u_{2i-1}) - \delta(u - u_i)]. \quad \text{(4.4.22)}
\]

Under the conformal transformation, Poisson's equation is invariant and thus we obtain the following equation

\[
\frac{\partial^2 \phi_p}{\partial u^2} + \frac{\partial^2 \phi_p}{\partial v^2} = -\frac{1}{\sigma} \left( \frac{\partial \sigma}{\partial u} \frac{\partial \phi}{\partial u} + \frac{\partial \sigma}{\partial v} \frac{\partial \phi}{\partial v} \right). \quad \text{(4.4.23)}
\]

Substituting equations (4.4.18a) and (4.4.21a) into equation (4.4.23) yields

\[
\frac{\partial^2 \phi_p}{\partial u^2} = -\frac{1}{\sigma} \frac{\partial \sigma}{\partial u} \frac{\partial \phi}{\partial u}
\]

\[
= -\frac{1}{\sigma} \frac{\partial \sigma}{\partial u} \left( -\frac{\partial \phi_0}{\partial u} + \frac{\partial \phi_p}{\partial u} \right). \quad \text{(4.4.24)}
\]
Then substituting equations (4.4.18b) - (4.4.22) into equation (4.4.24) and comparing the amplitude of the delta functions, we find that Poisson's equation (4.4.24) is satisfied both inside the uniform area and on the boundary lines, \( u = u_{2i-1}, u_{2i}, i = 1, 2, \ldots, n \) (remember that in case study 1, Poisson's equation is not valid on boundary \( \rho = \rho_1 \)).

Furthermore, on i'th border we obtain

\[
\frac{\| \nabla \sigma \cdot \nabla \phi_p \|}{\| \nabla \sigma \cdot \nabla \phi_0 \|} = \frac{|\sigma_o - \sigma_i|}{2 \sigma_i}. \tag{4.4.25}
\]

For \( \sigma_i \approx \sigma_o \), equation (4.4.25) can be of large value (approaches 0.5) and thus the small perturbation assumption is not valid. As well, discarding the term \( \nabla \sigma \cdot \nabla \phi_p \) in Poisson's equation only causes re-scaling error in this case.

**Case study 3:**

Assume in case study 2 that the conductivity has a finite range of inhomogeneity, as shown in Fig.4.3.1. In such circumstance, both the boundary distribution error and the third kind of error exist in addition to the re-scaling error. The appearance of the boundary distribution error is due to the finite length of the object along the \( v \) coordinate, as discussed in section 4.3. The third kind of error is caused by the zero \( v \)-component of the electric field for a uniform object, given by equation (4.4.18a).

The boundary distribution error can be reduced by filter processing, but not the third kind of error. However, since the current sources are uniformly distributed along \( v \in (v_b, \infty) \), \( u = \pm \infty \), the total electric field is mainly parallel with \( u \)-axis. This implies that the \( v \) component of the electric field is much smaller than the \( u \) component. Thus, the third kind of error will be small and ignorable even if it exists. The above analysis will be tested numerically in the following section.
4.5 Numerical test of the errors

In linear reconstruction algorithms, re-scaling error, boundary distribution error and the third kind error have been integrated by mathematical manipulations such as Fourier transform, coordinate conversion, etc.. Therefore what we can observe is overall error. In this section, we will calculated the overall errors using the integral equation (3.5.14).

Procedures in the error test are as follows: A conductivity distribution is assumed on which exact analytical solution of potential is available. The resulting boundary potential distributions with respect to different boundary current injections are evaluated using theoretical formula and used to obtain the convolution $h(\Theta, v_b)$ numerically from equations (3.5.7) or (3.5.8). The given conductivity is then substituted into the right hand side of the integral equation (3.5.14) to obtain numerical integration with respect to variable $\theta$. The differences between the normalized numerical integration and the normalized convolution yield the overall errors.

The error test has been performed on symmetric and centre offset objects. The results are exhibited and discussed below.

4.5.1 Error Test of Concentric Objects

The concentric objects under test had a geometry shown in Fig.4.4.1 and were characterized by two parameters, $\sigma_1$ and $\rho_1$. In the error calculations, the circular domain of the objects was discretized into different meshes as shown in Fig.3.6.1. The exact theoretical boundary potentials distribution were calculated (Appendix B) and used to evaluate the numerical convolution $h(\Theta, v_b)$. The known conductivity distribution was then substituted into the right hand side of equation (3.5.14) and the numerical integration was obtained. The numerical integration was normalized and compared with the normalized convolution. The error in the calculation was defined as the difference between the normalized integration values and the normalized
convolution. Due to angular symmetry of the concentric objects, error calculations were performed for only one current injection position for each object.

Test results for two objects are illustrated in Fig. 4.5.1 and Fig. 4.5.2. Object 1 has a structure of $p_1 = 0.25, \sigma_1 = 2$ and for object 2, $p_1 = 0.75, \sigma_1 = 2$. Mesh No. 1 (66 elements) and No. 3 (296 elements) are used in the calculations.

It is seen from Fig. 4.5.1 that excellent agreement between normalized convolution and integration has been achieved. Moreover, in terms of calculated errors, the result using mesh No. 3 is better than that using mesh No. 1.

Test results on object 2 are displayed in Fig. 4.5.2. Again, we obtain good agreement between the convolution and integration. In addition, test result using mesh No. 1 turns out to be superior to that using mesh No. 3.

The error calculations indicate that re-scaling error is dominant in the case of concentric objects and the error can be reduced by the re-scaling process. The boundary distribution error, as well as the third kind of error, are ignorable. This has been confirmed by the reconstruction experiments in section 3.6.

Because of the symmetrical geometry of the objects, the same value of re-scaling factor $\lambda$ applies to various of current injections.
**Fig. 4.5.1** Error calculations for object 1 ($\rho_1 = 0.25$, $\sigma_1 = 2$).

(a) Result using mesh No.1. (b) Result using mesh No.3.
Fig. 4.5.2 Error calculations for object 2 (\( \rho_1 = 0.75, \sigma_1 = 2. \)).

(a) Result using mesh No.1. (b) Result using mesh No.3.
4.5.2 Error Test of Offset Objects

The offset objects under test are shown in Fig.3.6.9. Exact theoretical solution for boundary potentials were obtained using the approach shown in appendix C and the error calculations followed the same procedure as that for concentric objects. Results for the offset objects are illustrated in Fig.4.5.2 and features are discussed as follows.

1) Unlike concentric objects, calculated errors for the offset objects vary with the injection position of current dipoles. In Fig.4.5.3, the current dipole is injected along the symmetrical axis of the offset object. The resulting errors are within a similar range to that for concentric objects. When the current dipole is applied at an asymmetrical position, as shown in Fig.4.5.4, the errors become larger. In both cases, however, acceptable agreement between the integration and convolution are obtained, which indicates that re-scaling error is dominant in comparison with boundary distribution error and the third kind of error.

2) For different current injections, re-scaling factors are different. This has been observed during the error calculations.

The re-scaling factor \( \lambda \) in the calculations is obtained as the ratio of the peak values of the integration and convolution before normalization. For the experimental set-up of Fig.4.5.3 and Fig.4.5.4, \( \lambda \) is found to be 0.633 and 0.106, respectively. The different re-scaling factors reflect that the errors caused by the linearization of Poisson's equation are associated with the relative positions of current injection and anomalies, which will be discussed in the following subsection.

3) By normalizing the convolution data with respect to the peak value of the integration data for each current injection, and using the normalized convolution data in the integral equation method, meaningful images have been reconstructed, as shown in section 3.6. This indicates that the re-scaling error is the major error source to the reconstruction algorithms derived from the linearized Poisson's equation.
Fig. 4.5.3  Error calculation for offset object with current dipole injected at symmetrical position.  (a) Object and current injection geometry.  (b) Error calculation results.
Fig. 4.5.4  Error calculation for offset object with current dipole injected at asymmetrical position.  (a) Object and current injection geometry.  
(b) Error calculation results.
4.5.3 Discussions

The integral equation was derived in the u-v plane where the electric field is a constant and in parallel to the u-axis. The objects used in error calculations are in a x-y plane and they are to be mapped into a common u-v plane for the purpose of the error discussions. Using the conformal transformation (3.3.7), the unit circle is transformed into half u-v plane and perturbed areas of conductivity in both concentric and offset objects are mapped into finite areas.

In Fig.4.5.5, a mapping image of the concentric object 1 (\( \rho_1 = 0.25 \)) via the conformal transformation is illustrated. The new object remains a circular domain but with different radius, and the image is symmetry with respect to the v - axis.

In Fig.4.5.6, two offset objects are transformed into u-v plane. The offset object 1 is corresponds to that in Fig.4.5.3 and the offset object 2 to that in Fig.4.5.4. Whilst the two offset objects have the same size in x-y plane, the mapped images exhibit a considerable difference in object's size.

The following has been observed from the conformal transformation results:

1. All the mapped objects are of finite range and, according to case study 3, all the three errors exist. However, both the boundary distribution error and the third kind of error are expected to be small because the smooth boundary of the conductivity discontinuity avoids a significant boundary distribution error, and there is little place along the boundary where \( \frac{\partial \phi_0}{\partial n} \) is equal to zero. Thus, re-scaling error becomes the dominating error. This conclusion applies to the situations where anomalies have smooth boundaries.

2. Due to the symmetrical geometry of a concentric object, the conformal transformation results in the same images irrespective of current injection positions. Therefore a uniform re-scaling factor can be used in reconstruction.
For offset objects, current injection at different positions results in different re-scaling factors. This is because the same object with different boundary current injections has totally different images after the conformal mapping, which then produce different perturbed potentials. Therefore, the numerical convolutions $h(\Theta, \nu_b)$ calculated from the perturbed potentials will vary with current injection substantially, but the magnitude of the integrations on the right hand side of equation (3.5.14) is independent of the current injection positions. Consequently, the re-scaling factor varies with current injections in the case of offset anomalies.

**Fig. 4.5.5** A mapped image of the boundary of the central circle in the concentric object 1. The central circle has a radius of 0.25, and the mapped image remains a circle of different size in the u-v plane.
Offset objects and its transformed images. The circle on the y-axis in (a) is corresponding to the larger circle of Fig.(b), and the circle off the y-axis is mapped as the smaller circle.
4.6 Conclusions

Poisson's equation is correct only within the domain of continuous conductivity. At discontinuities, Poisson's equation may not be valid. However, the Fourier transform of Poisson's equation is valid both inside the domains of continuous conductivity and on the boundaries of discontinuities.

Three types of error associated with the linearized Poisson's equation have been discussed. The error analysis is based on the condition that the electric field in a uniform object has a uniform distribution. The condition was used in derivation of Barber-Brown's back-projection algorithm and the integral equation algorithm.

Case studies showed that the small perturbation assumption used in many existing algorithms may not be correct for conductivity distributions consisting of discontinuities. Both case studies and the numerical tests have indicated that errors existing in the linearized Poisson's equation can be considerably large, and amongst the three errors, the re-scaling error is dominant in most cases.

The re-scaling error only causes error to the absolute value of reconstructions. Relative reconstruction can still be obtained from the data containing re-scaling errors because information about locations of discontinuities remains in the linearized Poisson's equation.

The error analysis provides a useful method of understanding the errors generated in the linearized Poisson's equation and reveals the effectiveness of the linear reconstruction algorithms under large perturbations.
Chapter 5

Conclusions and suggestions
5.1 Conclusions

A least square voltage matching (LSVM) iterative reconstruction algorithm was developed based on linear network theory and the equivalence between finite element modelling and a linear electrical network. In each iteration, an improved Newton-Raphson method was used to search for the minimum of the objective function and an adjoint network technique was adopted for efficient calculation of the iteration matrix. The algorithm reconstructs the conductivity distribution of an object by matching the calculated and measured boundary voltages.

The relationship between the LSVM algorithm and Murai's and Yorkey's algorithm was investigated. It was found that the algorithms were equivalent although the derivation were different.

Performance of iterative algorithms depends on the starting distribution and the number of unknowns (number of elements or components) to be reconstructed. Simulations of arbitrary networks under the condition of uniform starting distribution were carried out and it was shown that the LSVM algorithm was able to produce robust reconstructions when the networks consisted of a relatively small number of components (e.g. less than 50 components). For large networks, meaningful reconstructions were possible only if proper a priori information was applied.

Using the measured data, it was found that the LSVM algorithm has better performance for a network than a continuous problem. This was because, in the reconstruction of network component values, the topology of a network forms an effective constraint during the reconstruction and a suitable starting distribution was normally available. For a continuous object, different network modellings can be used for the same object and the topology constraint no longer exists. Thus, reconstruction of a continuous object becomes less robust.
The finite element method has been used in the LSVM algorithm to calculate boundary voltage distributions for a given boundary current distribution. It was found that in the case of paired current injections, the calculated boundary voltage had a much slower convergence rate on the driving electrodes than on the remaining electrodes, resulting in serious modelling errors in the measured data. The error can be reduced by refining the finite element model but the number of elements in the equivalent network will increase significantly. As a result, the refined network may violate the requirement of 'a small network' and lead to unwanted images. Therefore, there is a contradiction between the requirement of the convergence of the calculated boundary voltage and the requirement of the size of the network in order to obtain a reliable reconstruction.

An improved LSVM algorithm, called partial least square voltage matching (PLSVM) algorithm, was developed by eliminating the measured voltages of the driving electrodes from the original algorithm. The PLSVM algorithm reduced the influence of modelling error and the error caused by contact impedance of driving electrodes. Successful reconstructions using measured data had been obtained which indicated the effectiveness of the algorithm.

A linear integral equation was derived from a linearized Poisson's equation using conformal and Fourier transformations. The integral equation provides a useful mathematical description for the study of the linear reconstruction approach. Properties of the integral equation were discussed and the relationship between the integral equation method, the Radon back-projection method and the Barber-Brown's filtered back-projection method was investigated. It was found that the filtered back-projection method can be regarded as a pseudo-inverse of the integral equation.

A numerical reconstruction algorithm was developed from the integral equation. The algorithm consisted of solution of a set of matrix equations obtained from discretization of the integral equations. The system matrix had a poor condition and a
singular value decomposition technique was found to be necessary for achieving useful reconstructions. The algorithm can be performed in a short time because the system matrix can be pre-inversed and stored in computer memory and thus reconstruction is simply a matrix multiplication processing.

In applications of the integral equation method, consistency between equations related to different boundary current injections is essential. Reconstruction using both simulated and measured data indicated that the method was able to produce useful images in the case of concentric objects. For offset objects, the method was able to generate reasonable results when the set of integral equations were consistent. In practice, the consistency condition is often violated due to the large perturbations and this has created difficulty for the method to produce desired reconstructions, particularly for measured data.

An error analysis of a linearized Poisson's equation was carried out in order to investigate the error behaviour of linear reconstruction algorithms. Three types of error have been discussed based on a piecewise finite element model of continuous objects. It was discovered that the commonly used small perturbation assumption fails in most cases and amongst three types of error, the re-scaling error is usually dominant. Case studies and numerical tests confirmed the above findings.

Based on the error analysis, it was found that many reported reconstructions contained large perturbations which conflicted with the small perturbation required by linear algorithms. It was also found that the dominance of the re-scaling error in the appearance of large perturbations still allowed reconstruction of a useful relative conductivity image. Thus, the error analysis has provided a new explanation to the mechanism of linear algorithms, as well as a useful mathematical support to the application of algorithms in cases of both small and large perturbations.
5.2 Suggestions

The linear approach has fast reconstruction speed and avoids non-unique solution and starting distribution problems. The error study indicates that even under large perturbation, it is still possible to reconstruct an image of relative conductivity distribution by the linear approach. Thus, future research interests will be in the direction of the linear reconstruction approach development.

The linear integral equation has certainly provided a useful mathematical ground for the further development of the linear reconstruction approach. In the case of large perturbations, however, the method suffers from inconsistency due to the solution of a set of simultaneous equations. The problem can be solved by introducing a set of re-scaling variables to the set of integral equations and this will turn the original linear algorithm into either a nonlinear problem or a problem of finding the non-trivial solution of a homogeneous equation. Research effort is required to investigate the advantages and disadvantages of such treatment.

Pseudo-inverse of the integral equation method is a promising direction of research in the development of the linear reconstruction approach. A successful pseudo-inverse should be as close to the accurate inverse of the equation as possible, and should be able to process the measured boundary data for each current injection individually in order to avoid the inconsistency problem.

As a pseudo-inverse of the integral equation, the filtered back-projection method has many properties to be explored in relation to the integral equation. A study focusing on this respect may bring out useful results for the further improvement of the method.
References
Alan, A. and Pritsker, B., 1986,
Introduction to simulation and SLAM II,

Allaire, P. E., 1985,
Basics of the Finite Element Method,
Wm. C. Brown Publishers, Bubuque, Iowa.

Baker, C. T. H., 1977,
The numerical treatment of integral equations,
Oxford University Press.

Baker, L. E., 1971,
"Biomedical applications of electrical impedance measurements,"
Medical electronics monographs, pp. 1 - 6,

Balabanian, N., 1969,
Electrical network theory,
John Wiley & sons, Inc..

Barber, D. C., Brown, B. H. Brown and Freeston, I. L., 1983,
"Imaging spatial distributions of resistivity using applied potential tomography,"

Barber, D. C. and Brown, B. H., 1984,
"Applied potential tomography,"

Barber, D.C. and Brown, B.H., 1986,
"Recent developments in applied potential tomography - APT,"
Information Processing in Medical Imaging, Ed. S.L. Bacharach, Martinus Nijhoff,

Barber, D. C. and Seagar, A. D., 1987,
"Fast reconstruction of resistance images,"
Bates, R. H. T., McKinnon, G. C. and Seagar, A. D., 1980,
"A limitation on systems for imaging electrical conductivity distributions,"

Bates, R. H. T., 1984,
"Full-wave computed tomography Part 1: Fundamental theory,"

Bracewell, R. N., 1956,
"Strip integration in radioastronomy,"

Blum, H., Lin, Q., and Rannacher, R., 1986,
"Asymptotic error expansion and Richardson extrapolation for linear
finite elements,"

Brown, B. H., 1983,
"Tissue impedance measurement,"
Imaging with non-ionising radiations, ed. D. F. Jackson, Surrey University Press,
Guildford, pp. 85 - 110.

Brown, B. H. and Seagar, A. D., 1987,
"The Sheffield data collection system,"

Burger, H. C. and van Dongen, R., 1961,
"Specific electric resistance of body tissues,"

Chen, Chi - Tsong, 1984,
Linear system theory and design,
Holt, Rinehat and Winston.

Cheng, Kuo-Sheng, Issacson, D., Newell, J. C. and Gisser, D. G., 1989,
"Electrode models for electric current computed tomography,"
Cheng, Kuo-Sheng et al, 1990,
"Errors due to measuring voltage on current-carrying electrodes in electric
current computed tomography,"

Chua, L. O. and Lin, P. M., 1975,
Computer-aided analysis of electronic circuits,
Prentice - Hall.

Cromwell, L., Weibell, F. J. and Pfeiffer, A. E., 1980,
Biomedical Instrumentation and Measurements,
2nd Ed., Prentice Hall, New Jersey.

Davis, A. J., 1980,
The finite element method,

Delves, L. M. Delves and Walsh, J., 1974,
Numerical solution of integral equations,

Dines, K. A. and Lytle, R. J., 1981,
"Analysis of electrical conductivity imaging,"

El-Turky, F. M., 1986,
"Efficient computation of network sensitivities,"

Frewer, R. A., 1974,
"The electrical conductivity of flowing blood,"
Biomedical Engineering, vol. 9, pp. 552 - 555.

Geddes, L. A. and Baker, L. E., 1967,
"The specific resistance of biological material - a compendium of data for the
biomedical engineer and physiologist,"
Geselowitz, D. B., 1971,
 "An application of electrocardiographic lead theory to impedance plethysmography"

Gisser, D. G., Issacson, D. and Newell, J. C., 1987,
 "Current topics in impedance imaging,"

Gisser, D. G., Issacson, D. and Newell, J. C., 1988,
 "Theory and performance of an adaptive current tomography system,"

Golub, G. H. and Kaham, W., 1965,
 "Calculating the singular values and pseudo-inverse of a matrix,"

Haley, S. B., 1986,
 "Pole sensitivity to network component change,"

Hanson, R. J, 1971,
 "A numerical method for solving Fredholm integral equation of the first kind
 using singular values,"

Hanson, R. J, 1972,
 "Integral equations of immunology,"

Henderson, R. P. and Webster, J. G., 1985,
 "An impedance camera for spatially specific measurements of the thorax,"

Herman, G. T. and Natterer, F., 1981,
 Mathematical aspects of computerized tomography,

Hua, P., Webster, J. G. and Tompkins, W. J., 1987,
"Effect of the measurement method on noise handling and image quality of EIT imaging."

Hua, P., Webster, J. G. and Tompkins, W. J., 1988,
"A regularised electrical impedance tomography reconstruction algorithm."

Hua, P., Woo, E. J., Tompkins, W. J. and Webster, J. G., 1989,
"An electrical impedance tomography using compound electrodes,"

Hua, P., 1990
Modeling and reconstruction methods for electrical impedance tomography,

Issacson, D., 1986,
"Distinguishability of conductivities by electric current computed tomography,"

Jackson, W. D., 1962,
Classic Electrodynamics,

Jossinet, J. and Kardous, G., 1987,
"Physical study of the sensitivity distribution in multi-electrode systems,"

Katz, B., 1966,
Nerve, Muscle and Synapse,

Kaufman, W. and Johnston, F. D., 1943,
"The electrical conductivity of tissue near the heart,"

Kim, Y., Webster, J. G., and Tompkins, W. J., 1983,
"Electrical impedance imaging of the thorax,"

J. Microwave Power, vol. 18, No. 3, pp. 245-257.

Klema, V. C., 1980,
"The singular value decomposition and some applications,"

Lawson, C. L., 1974,
Solving least squares problems,

Lehr, J., 1972,
"A vector derivation useful in impedance plethysmographic field calculation,"

Lin, Q., and Zhu, Q. D., 1986,
"Local asymptotic expansion and extrapolation for finite elements,"

Lytle, R. J. and Dines, K. A., 1978,
"An impedance camera: A system for determining the spatial variation of electrical conductivity,"
Lawrence Livermore Laboratory, Livermore, CA, Rep. UCRL-52413.

Mathews, J. H., 1977,
Complex variables for mathematics and engineering,
WM. C. Brown publishers, Dubuque, Iowa, USA.

Margenau, K. and Murphy, G. M., 1964,
The mathematics of physics and chemistry,

Mehra, R. K., 1976,
System identification advances and case studies,
Academic Press.

Minard, R. A., Robinson, B. S. and Bates, R. H. T., 1985,
"Full-wave compute tomography Part 3: Coherent shift-and add imaging,"
Mirsky, L., 1963,
An introduction to linear algebra,
Oxford at the Clarendon Press.

Morgan, B. J. T., 1984,
Elements of simulation,
Chapman and Hall Ltd.

Morse, P. M. and Feshbach, H., 1953,
Methods of theoretical physics,

Murai, T. and Kagawa, Y., 1985,
"Electrical impedance computed tomography based on a finite element model,"

Nakayama, K., Yagi, W. and Yagi, S., 1981,
"Fundamental study on electrical impedance CT algorithm utilizing sensitivity
theorem on impedance plethysmography,"
Proc. 5th Int. Conf. on Electrical Bio-impedance, Tokyo, pp. 99 -102.

Newell, J. C., Disser, D. G., and Issacson, D., 1988,
"An electric current tomography,"

Nowotny, R. and Nowotny, C., 1980,
"Determination of electrode impedance with an FFT spectrum analyser,"

Ortega, J. M., 1970,
Iteration solution of nonlinear equations in several variables,
Academic Press.

Phillips, D. L., 1962,
"A technique for the numerical solution of certain integral equations of the
first kind,"
J. ACM, vol. 9, pp. 84 - 97.
Plonsey, P., 1969,  
Bioelectric Phenomena,  

Plonsey, R. and Barr, R., 1982,  
"The four electrode technique applied to cardiac muscle,"  

Powell, H. M., Barber, D. C. and Freeston, I. L., 1987,  
"Impedance imaging using linear electrode arrays,"  

Pryce, L. R., 1979,  
"Electrical impedance computer tomography (ICT): a new CT imaging technique,"  

Ramachandran, G. N. and Lakshminarayanan, A. V., 1971,  
"Three dimensional reconstruction from radiographic and electron micrographic application of convolutions instead of Fourier transforms,"  

Rannacher, R., and Scott, R., 1982,  
"Some optimal error estimates for piecewise linear finite element approximations,"  

Rapisarda, L. and Decailo, R., 1983,  
"Analog multifrequency fault diagnosis,"  

Rapisarda, L. and Decailo, R., 1986,  
"Corrections to analog multifrequency fault diagnosis,"  

Reddy, G. N. and Subratam Saha, 1984,  
"Electrical and dielectric properties of wet bone as a function of frequency,"  
Roberts, A. W., 1973,
Convex functions,
Academic Press.

Rush, S. Abildskov J. A. and McFee, R., 1963,
"Resistivity of body tissues at low frequencies,"

Sakamoto, K. and Kanai, H., 1983,
"A fundamental study of an electrical impedance CT algorithm,"
Proc. 6th Int. Conf. on Electrical Bio-impedance, Yugoslavia, pp. 349 - 52.

Schwan, H. P. and Kay, C. F., 1956,
"Specific resistance of body tissues,"

Scudder, H. J., 1978,
"Introduction to computer aided tomography,"

Seagar, A. D., 1983,
Probing with low frequency electric current,

Seagar, A. D. and Yeo, T. S., 1984,
"Full-wave computed tomography Part 2: Resolution limits,"

Seagar, A. D. and Bates, R. H. T., 1985,
"Full-wave computed tomography Part 4: Low-frequency electric current CT,"

Seagar, A. D. and Brown, B. H. Brown, 1987a,
"Theoretical limits to sensitivity and resolution in impedance imaging,"

Seagar, A. D. and Brown, B. H., 1987b,
"Limitations in hardware design in impedance imaging,"
Shepp, L. A. and Logan, B. F., 1974,
"The Fourier reconstruction of a head section,"

Silvester, P. P. and Ferrari, R. L., 1986,
Finite elements for electrical engineers,
Cambridge University Press.

Shilov, G. E. and Silverman, R. A., 1971,
Linear algebra,
Prentice-Hall Inc., Englewood Cliffs, N.J.

Shimazu, H., Yamakoshi, K., Tagawa, T., Fukuoka, M. and Ito, H., 1981,
Non-invasive measurement of blood resistivity and haematicrit,
Proc. VI th Int. Conf. Electrical Bioimpedance (ICEBI), Tokyo, pp. 233 - 238.

Tompkins, W. J. and Webster, J. G., 1986,
Medical instruments and apparatus - design and construction,
Englewood cliffs, London.

Tozoni, O. V., 1968,
Mathematical models for the evaluation of electric and magnetic field,
Iliffe Books Ltd, London.

Tucker, R. D., Schmitt, O. H., Sievert, S. G., Ellsworth, S. G. and Silvis, S. E., 1982,
"Targets of convenience for computer automated measurements of electrical tissue impedance spectra,"

Witsoe, D. A. and Kinner, E., 1967,
"Electrical resistivity of lung at 100 kHz,"

Yorkey, T. J., Webster, J. G. and Tompkins, W. J., 1987,
"Comparing reconstruction algorithms for electrical impedance tomography,"
Appendices
Appendix A  Evaluation of Nodal Voltage Derivatives with respect to Component Values Using Adjoint Network Method

Evaluation of nodal voltage derivatives with respect to element conductivity is required in LSVM, PLSVM and many other iterative reconstruction algorithms. One efficient technique for doing so is the use of an adjoint network method.

The adjoint network method was developed for efficient network sensitivity analysis where voltage derivatives with respect to component values in a network are of concern. The method may readily be adopted in the impedance imaging because of the equivalence between finite element modelling and network analysis problems.

A.1 Basic Equations

For a network under consideration, called the original network, we first extract all the independent current sources and those nodes whose voltage derivatives are to be evaluated to form a multi-port network, as shown in Fig.A.1. We then construct an adjoint network from the original network by open-circuiting the current source ports, and applying an unit current source between ground and the node whose derivative is to be found.

Denote the port current and voltage vector by \( \mathbf{I}_p \) and \( \mathbf{V}_p \), respectively. Denote the non-port branch current and voltage vector by \( \mathbf{I}_b \) and \( \mathbf{V}_b \), respectively. We then have

\[
\mathbf{V}_b = \mathbf{Z}_b \mathbf{I}_b , \quad \dot{\mathbf{V}}_b = \dot{\mathbf{Z}}_b \dot{\mathbf{I}}_b , \tag{A.1.1}
\]
where all the terms with cap refer to the quantities for the adjoint network; $Z_b$ and $\hat{Z}_b$ are component impedance matrices which are diagonal if the original network has no dependent sources; $Z_{oc}$ and $\hat{Z}_{oc}$ are open-circuit impedance matrices for the two networks. Negative signs in equation (A.1.2) are due to the reference direction definition for port current and voltages. Impedance matrices in the two networks are related by the following equations

\[
Z_b^t = \hat{Z}_b, \quad Z_{oc}^t = \hat{Z}_{oc}.
\]  

Fig. A.1 An original network with current sources and concerned nodes being extracted to form a multi-port.

From Tellegen's theorem of electrical network analysis, we have

\[
\hat{\mathbf{I}}^t \Delta \mathbf{V} = \hat{\mathbf{V}}^t \Delta \mathbf{I}.
\]  

(A.1.4)
Expressing quantities of equation (A.1.4) in terms of $I_p$, $V_p$, $I_b$ and $V_b$, equation (A.1.4) becomes

$$\begin{align*}
\hat{I}_t^t & \Delta V - \hat{V}_b^t \Delta I = (\hat{I}_b^t \Delta V_b + \hat{I}_p^t \Delta V_p) - (\hat{V}_b^t \Delta I_b + \hat{V}_p^t \Delta I_p) = 0,
\end{align*}$$

and therefore

$$- (\hat{I}_b^t \Delta V_b - \hat{V}_b^t \Delta I_b) = (\hat{I}_p^t \Delta V_p - \hat{V}_p^t \Delta I_p). \quad (A.1.5)$$

Note that perturbed terms $\Delta V_p$ and $\Delta I_p$ need not be small changes for equation (A.1.5) to be valid.

Assume there are very small changes in the elements of $Z_b$. Then, to a first-order approximation, equation (A.1.1) yields the following relationships:

$$\begin{align*}
\Delta V_b &= \Delta(Z_b I_b) = \Delta Z_b I_b + Z_b \Delta I_b, \quad (A.1.6) \\
\Delta V_p &= -\Delta(Z_{oc} I_p) = -\Delta Z_{oc} I_p - Z_{oc} \Delta I_p. \quad (A.1.7)
\end{align*}$$

Substituting equation (A.1.6) into (A.1.5) yields

$$\hat{I}_b^t \Delta Z_b I_b = \hat{I}_b^t \Delta Z_b I_b, \quad (A.1.8)$$

and similarly we can have

$$\hat{I}_p^t \Delta Z_{oc} I_p = -\hat{I}_p^t \Delta Z_{oc} I_p. \quad (A.1.9)$$

Substituting equations (A.1.8) and (A.1.9) into equation (A.1.5) yields

$$\hat{I}_p^t \Delta Z_{oc} I_p = \hat{I}_b^t \Delta Z_b I_b. \quad (A.1.10)$$
Equation (A.1.10) is an useful result for calculating derivatives of open-circuit impedances with respect to component values in a network. By properly arranging the current sources at certain ports of the adjoint network, derivatives of any individual open-circuit impedance can be obtained.

A.2 Evaluation of Nodal Voltage Derivatives

Partial derivatives of node voltage with respect to component admittances, i.e. \( \frac{\partial v}{\partial y} \), can be found from the value \( \frac{\partial Z_{oc}}{\partial z} \) as follows. Since all the current sources connected to the ports of the network have constant values, then from equation (A.1.7) we have

\[
\Delta V_p = -\Delta Z_{oc} I_p .
\]  
(A.2.1)

Substituting equation (A.2.1) into equation (A.1.10) yields

\[
I_p ^t \Delta V_p = - I_b ^t \Delta Z_b I_b .
\]  
(A.2.2)

Assume that the partial derivative of the i'th node voltage with respect to the j'th component impedance is to be found, namely \( \frac{\partial v_{ni}}{\partial z_j} \). Then by applying a unit current source to the i'th node in the adjoint network with current flowing into the i'th node, the vector \( \hat{I}_p ^t \) is given by \( \hat{I}_p ^t = [ 0, \ldots, -1, \ldots, 0 ] \) where the -1 appears at the i'th position of the vector \( \hat{I}_p ^t \). Substituting \( \hat{I}_p ^t \) into equation (A.2.2) yields

\[
\frac{\partial v_{ni}}{\partial z_j} = \hat{i}_j .
\]  
(A.2.3)

Since admittance is the inverse of impedance, we obtain
The above equation reveals that \( \frac{\partial v_{ni}}{\partial y_j} \) is equal to the product of j'th branch voltages of the original and the adjoint networks.

### A.3 Example of Application of the Method

Consider a circuit as shown in Fig.A.2.

The nodal equation of the circuit is found to be

\[
\begin{bmatrix}
y_1 + y_2 & -y_2 & 0 \\
-y_2 & y_2 + y_3 + y_5 & -y_3 \\
0 & -y_3 & y_3 + y_4
\end{bmatrix}
\begin{bmatrix}
v_1 \\ v_2 \\ v_3
\end{bmatrix}
= \begin{bmatrix}
j_1 \\ j_2 \\ j_3
\end{bmatrix}.
\] (A.3.1)
Inversion of the nodal-admittance matrix gives

$$Y_n^{-1} = \frac{1}{|Y_n|} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$$ (A.3.2)

where

$$y_{11} = (y_3 + y_4)(y_2 + y_3 + y_5) - y_3^2,$$
$$y_{22} = (y_1 + y_2)(y_3 y_4),$$
$$y_{33} = (y_1 + y_2)(y_2 + y_3 + y_5) - y_2^2,$$
$$y_{12} = y_{21} = y_2 (y_3 + y_4),$$
$$y_{23} = y_{32} = y_3 (y_1 + y_2),$$
$$y_{13} = y_{31} = y_2 y_3,$$

and

$$|Y_n| = (y_1 + y_2)(y_3 + y_4)(y_2 + y_3 + y_5)$$

$$- y_3^2 (y_1 + y_2) - y_2^2 (y_3 + y_4).$$

By substituting values of components and current sources into equation (A.3.1), the node voltages of the original network are found to be

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 20 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 11 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 26 \\ 16 \\ 17 \end{bmatrix}.$$ (A.3.3)
Node voltages for the adjoint network with a unit current source connected to node No.1 is found as

\[
\begin{bmatrix}
\hat{v}_1 \\
\hat{v}_2 \\
\hat{v}_3 \\
\end{bmatrix} = \frac{1}{36} \begin{bmatrix}
20 & 4 & 2 \\
4 & 8 & 4 \\
2 & 4 & 11 \\
\end{bmatrix} \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} = \frac{1}{36} \begin{bmatrix}
20 \\
4 \\
2 \\
\end{bmatrix} .
\]  
(A.3.4)

We then calculate branch voltages for both original and adjoint networks and substitute them into equation (A.2.4). The derivatives for voltage of node No.1 are then obtained as

\[
\begin{bmatrix}
\frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} & \frac{\partial v_1}{\partial y_3} & \frac{\partial v_1}{\partial y_4} & \frac{\partial v_1}{\partial y_5} \\
\end{bmatrix} = \begin{bmatrix}
20 \times 26 & 16 \times 10 & 1 \times 2 & 17 \times 2 & 16 \times 4 \\
\frac{1296}{1296} & \frac{1296}{1296} & \frac{1296}{1296} & \frac{1296}{1296} & \frac{1296}{1296} \\
\end{bmatrix} .
\]  
(A.3.5)

Partial derivatives of other node voltages can be calculated likewise.

In order to confirm the correctness of the results from the adjoint network method, analytical solution of the node voltage derivatives is evaluated and given below. From equation (A.3.2) we obtain

\[
v_1 = \frac{1}{\|Y_n\|} (y_{11} j_1 + y_{12} j_2 + y_{13} j_3 ),
\]  
(A.3.6)

Evaluating derivatives of \(v_1\) with respect to component values using equation (A.3.6), we then arrive at equation (A.3.5).
Appendix B  Solutions of Laplace Equation in a Unit Circular Domain

B.1 Potential Distribution in a Uniform Circular Region

Potential distributions within a conductive medium is governed by Poisson's equation

\[ \nabla^2 \phi = -\frac{1}{\sigma} \nabla \sigma \cdot \nabla \phi \]  \hspace{1cm} (B.1.1)

where \( \sigma \) and \( \phi \) are conductivity and potential distributions, respectively. Consider a circular domain of unit radius within which the conductivity has a uniform distribution and thus equation (B.1.1) reduces to a Laplace equation. In polar coordinates Laplace's equation is

\[ \nabla^2 \phi = \frac{1}{\rho} \frac{\partial \phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0 \]  \hspace{1cm} (B.1.2)

Assume that on the boundary of the object the following current density distribution is applied:

\[ J(\theta) = \frac{\partial \phi}{\partial \rho} (\rho = 1, \theta) = \sum_{i=1}^{\infty} \{ C_n \cos n \theta + S_n \sin n \theta \} \]  \hspace{1cm} (B.1.3)

where
\[ C_n = \frac{1}{\pi} \int_0^{2\pi} J(\theta) \cos n \theta \, d\theta , \quad (B.1.4) \]

\[ S_n = \frac{1}{\pi} \int_0^{2\pi} J(\theta) \sin n \theta \, d\theta . \quad (B.1.5) \]

At the centre of the object we define the potential to be zero, i.e.

\[ \phi (\rho = 0, \theta) = 0 . \quad (B.1.6) \]

By separation of variables, we obtain

\[ \phi (\rho, \theta) = \sum_{i=1}^{\infty} \frac{\rho^n}{n} \{ C_n \cos n \theta + S_n \sin n \theta \}, \quad (B.1.7) \]

and on the boundary \( \rho = 1 \), we have

\[ \phi (\rho = 1, \theta) = \sum_{i=1}^{\infty} \frac{1}{n} \{ C_n \cos n \theta + S_n \sin n \theta \}. \quad (B.1.8) \]

**B.2 Potential Distribution of a Concentric Object**

For a conductivity distribution given by

\[ \sigma (\rho, \theta) = \begin{cases} \sigma_1 & 0 < \rho < \rho_1 \\ 1 & \rho_1 < \rho < 1 \end{cases} , \quad (B.2.1) \]
conditions in equation (B.1.3) and (B.1.6) still apply. In addition, the potential solution must satisfy the following conditions at the boundary \( \rho = \rho_1 \):

\[
\phi (\rho_1^-, \theta) = \phi (\rho_1^+, \theta), \quad (B.2.2)
\]

\[
\sigma_1 \frac{\partial \phi}{\partial \rho} (\rho_1^-, \theta) = \frac{\partial \phi}{\partial \rho} (\rho_1^+, \theta). \quad (B.2.3)
\]

Again, by separation of variables we obtain

\[
\phi(\rho, \theta) = \sum_{i=1}^{\infty} \Phi_n(\rho) \{C_n \cos n \theta + S_n \sin n \theta\} \quad (B.2.4)
\]

\[
\Phi_n(\rho) = \begin{cases} 
\gamma_n \rho^n & 0 < \rho < \rho_1 \\
\alpha_n \rho^n + \beta_n \rho^{-n} & \rho_1 < \rho < 1 
\end{cases} \quad (B.2.5)
\]

where the coefficients \( \alpha_n, \beta_n \) and \( \gamma_n \) are determined by the boundary conditions. Substituting equations (B.2.4) and (B.2.5) into equations (B.1.6), (B.2.2) and (B.2.3) yields

\[
\alpha_n - \beta_n = \frac{1}{n}, \quad (B.2.6)
\]

\[
\alpha_n + \beta_n \rho_1^{-2n} = \gamma_n, \quad (B.2.7)
\]

\[
\alpha_n - \beta_n \rho_1^{-2n} = \sigma_1 \gamma_n. \quad (B.2.8)
\]

Solving equation (B.2.6) - (B.2.8) gives
\[ \alpha_n = \frac{1}{n} \frac{\sigma_1 + 1}{(\sigma_1 - 1)\rho_1^{2n} + (\sigma_1 + 1)} , \quad \text{(B.2.9)} \]

\[ \beta_n = -\frac{1}{n} \frac{(\sigma_1 - 1)\rho_1^{2n}}{(\sigma_1 - 1)\rho_1^{2n} + (\sigma_1 + 1)} , \quad \text{(B.2.10)} \]

\[ \gamma_n = \frac{1}{n} \frac{2}{(\sigma_1 - 1)\rho_1^{2n} + (\sigma_1 + 1)} , \quad \text{(B.2.11)} \]

and

\[ \alpha_n + \beta_n = \frac{1}{n} \frac{1 - \mu\rho_1^{2n}}{1 + \mu\rho_1^{2n}} , \quad \text{(B.2.12)} \]

where

\[ \mu = \frac{\sigma_1 - 1}{\sigma_1 + 1} . \quad \text{(B.2.13)} \]

Thus, potential distribution of the concentric object is determined and, on the boundary \( \rho = 1 \), we have

\[ \phi (\rho = 1, \theta) = \sum_{i=1}^{\infty} \frac{1}{n} \frac{1 - \mu R^{2n}}{1 + \mu R^{2n}} \{ C_n \cos n \theta + S_n \sin n \theta \} . \quad \text{(B.2.14)} \]

### B.3 Fourier Expansion of Boundary Current Distributions

For a boundary current density distribution shown in Fig.B.1 where
\[ J(\theta) = \begin{cases} \frac{1}{\Delta_1} & \theta_0 < \theta < \theta_1 \\ -\frac{1}{\Delta_2} & \theta_2 < \theta < \theta_3 \\ 0 & \text{otherwise} \end{cases} \] (B.3.1)

and \( I \) is a constant, the Fourier coefficients \( C_n \) and \( S_n \) are found to be

\[
C_n = \frac{2I}{n\pi} \left\{ \frac{1}{\Delta_1} \sin \frac{n\Delta_1}{2} \cos n (\theta_0 + \frac{\Delta_1}{2}) \\
- \frac{1}{\Delta_2} \sin \frac{n\Delta_2}{2} \cos n (\theta_0 + \alpha + \frac{\Delta_2}{2}) \right\} \] (B.3.2)

and

\[
S_n = \frac{2I}{n\pi} \left\{ \frac{1}{\Delta_1} \sin \frac{n\Delta_1}{2} \sin n (\theta_0 + \frac{\Delta_1}{2}) \\
- \frac{1}{\Delta_2} \sin \frac{n\Delta_2}{2} \sin n (\theta_0 + \alpha + \frac{\Delta_2}{2}) \right\} \] (B.3.3)

where

\[
\Delta_1 = \theta_1 - \theta_0, \quad \Delta_2 = \theta_3 - \theta_2, \quad \alpha = \theta_2 - \theta_0. \] (B.3.4)
Using equations (B.3.2) and (B.3.3), we further obtain

\[ C_n \cos n\theta + S_n \sin n\theta = \frac{2I}{n\pi} \left\{ \frac{1}{\Delta_1} \sin \frac{n\Delta_1}{2} \cos n \left( \theta_o + \frac{\Delta_1}{2} - \theta \right) \\
- \frac{1}{\Delta_2} \sin \frac{n\Delta_2}{2} \cos n \left( \theta_o + \frac{\Delta_2}{2} - \theta \right) \right\}. \tag{B.3.5} \]

In the case of \( \Delta_1 = \Delta_2 = \Delta \), equation (B.3.5) is simplified as

\[ C_n \cos n\theta + S_n \sin n\theta = \frac{4I}{n\pi\Delta} \sin \frac{n\alpha}{2} \sin \frac{n\alpha}{2} \sin n \left( \theta_o + \frac{\Delta+\alpha}{2} - \theta \right). \tag{B.3.6} \]

If \( \theta_o = -\frac{\alpha}{2} \), i.e. the two current pulses with opposite polarity are placed symmetrically about axis \( \theta = 0 \), then equation (B.3.6) becomes

\[ C_n \cos n\theta + S_n \sin n\theta = \frac{4I}{n\pi\Delta} \sin \frac{n\alpha}{2} \sin \frac{n\alpha}{2} \sin n \left( \frac{\Delta}{2} - \theta \right). \tag{B.3.7} \]
B.4 Potential Distribution in a Uniform Circular Region with Trigonometry Boundary Current

For a uniform circular object with sinusoidal boundary current condition, potential solution can be obtained from equation (B.1.7) and is given by

\[ \phi(\rho, \theta) = \rho \sin \theta = y. \] (B.4.1)

From equation (B.4.1) it is found that equipotential lines are in parallel to x axis and current lines are in parallel to y axis, as shown in Fig. B.2 (a). In addition, the current has a uniform distribution inside the domain of the object.

Similarly, potential distribution in a circular uniform region with cosine boundary current can be obtained as

\[ \phi(\rho, \theta) = \rho \cos \theta = x. \] (B.4.2)

Therefore equipotential lines are in parallel to y axis and current lines are in parallel to x axis with constant amplitude, as shown in Fig. B.2 (b).
Fig. B.2  Equipotential lines and current lines of a uniform circular object with boundary current of:

(a) a sine distribution and (b) a cosine distribution.
Appendix C  Potential Distribution of an Offset Object Using a Conformal Transformation

Potential distribution of an offset object, as shown in Fig.C.1(b), can be obtained using a conformal transformation in terms of potential solution for a concentric object as shown in Fig.C.1(a). Both object domains are unit circles. The shadow area in the concentric object has a radius $\rho_O$, and the shadow area in the offset object is $\bar{\rho}$ with an offset distance 'd'.

**Fig. C.1** An offset object (b) can be mapped onto a concentric object (a) via a conformal transformation.

Under conformal transformations, Laplace's equation is invariant. This feature of conformal transformation allows the solutions of a Laplace's equation in simple geometries to be transformed and applied to complicated geometries.
The conformal transformation between the offset and concentric object are given by the following:

Concentric Object, $z$ plane

$z = x + jy$

$w = \frac{d + z}{zd + 1}$

$\bar{\rho} = \frac{\rho_o (1 - d^2)}{1 - \rho_o^2 d^2}$

$c = \frac{d (1 - \rho_o^2)}{1 - \rho_o^2 d^2}$

$z = \frac{d - w}{wd - 1}$

$\rho_o = \frac{1 - c^2 + \bar{\rho}^2 - \sqrt{(1 - c^2 + \bar{\rho}^2)^2 - 4\bar{\rho}^2}}{2\bar{\rho}}$

$d = \frac{1 + c^2 - \bar{\rho}^2 - \sqrt{(1 + c^2 + \bar{\rho}^2)^2 - 4c^2}}{2c}$

Offset Object, $w$ plane

$w = u + i v$

Potential distribution for the concentric object is given in Appendix B and can be used to determine the potential distribution of the offset object. The procedure is as follows:
For a given boundary point of the offset object in w-plane, the corresponding boundary point of the concentric object in z-plane is given by

\[
x = \frac{(d - u)(u - \frac{1}{d}) - v^2}{d[ (u - \frac{1}{d})^2 + v^2 ]}
\]

(C.1)

\[
y = \frac{v (-\frac{1}{d} - d)}{d[ (u - \frac{1}{d})^2 + v^2 ]}
\]

(C.2)

where d is determined by

\[
d = \frac{1+c^2\rho^2 - \sqrt{(1+c^2\rho^2)^2 - 4c^2}}{2c}
\]

(C.3)

From the x, y we can evaluate the values of \(\rho\) and \(\theta\), the radius and angle variables in polar coordinate, as

\[
\rho = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).
\]

(C.4)

We then calculate the radius of the central circle \(\rho_0\) of the concentric object from the formula

\[
\rho_0 = \frac{1-c^2+\rho^2 - \sqrt{(1-c^2+\rho^2)^2 - 4\rho^2}}{2\rho}
\]

(C.5)

Analytical solution of the boundary potential for the concentric object is given in Appendix D, and it is used to calculate the potential at point \((r, \theta)\) in z-plane. This potential value is equal to the value of the offset object at the point \((u,v)\) in the w-plane.
It should be appreciated that when a current boundary condition is imposed upon a boundary point of the offset object in the $z$-plane, it must be scaled by a factor $\frac{dw}{dz}$ before being imposed upon the mapped boundary point in the $w$-plane [Mathews, 1977].
Appendix D  Potential Distribution of a Current Dipole  
in a Uniform Conductive Medium

D.1 Basic Equations  

Assume that in a uniform, isotropic conductive medium, there is a low frequency current source distributed within an area $D_s$. The governing equation for the resulting potential $\phi(x, y)$ is given by

$$\nabla^2 \phi(x, y) = -\frac{1}{\sigma} J(x, y). \quad (D.1.1)$$

where $\sigma$ is the conductivity of the medium, $J(x, y)$ is the current of the current source. If the area $D_s$ approaches to zero and concentrated to a point $(x_s, y_s)$, whilst the magnitude of the current source remains a constant $I$, then $J(x, y)$ becomes a Delta function so that equation (D.1.1) becomes

$$\nabla^2 \phi_1(x, y) = -\frac{I}{\sigma} \delta(x - x_s) \delta(y - y_s). \quad (D.1.2)$$

When two of the Delta current densities of opposite sign are aligned in parallel with the $x$-axis with the distance between them approaches to zero, we obtain a current density dipole, or current dipole for short. The corresponding potential distribution can be obtained by solving the equation:

$$\nabla^2 \phi_2(x, y) = -\frac{I}{c} \frac{\partial}{\partial x} \delta(x - x_s) \delta(y - y_s). \quad (D.1.3)$$
D.2 Solutions of Potential

Poisson's equations (D.1.2) and (D.1.3) can be solved in term of a two-dimensional Green's function. The Green's function in a plane is given by (Morse and Feshbach, 1958)

\[ G(x, y) = -\frac{1}{4\pi} \ln \left[ (x-x_s)^2 + (y-y_s)^2 \right]. \tag{D.2.1} \]

The Laplace of the Green's function is found to be

\[ \nabla^2 G(x, y) = \begin{cases} 0 & x \neq x_s \quad y \neq y_s \\ -\delta(x-x_s) \delta(y-y_s) & x = x_s \quad y = y_s \end{cases}. \tag{D.2.2} \]

Thus the solution to equation (D.1.2) is

\[ \phi_1(x, y) = -\frac{1}{4\pi \sigma} \ln \left[ (x-x_s)^2 + (y-y_s)^2 \right], \tag{D.2.3} \]

and the solution to equation (D.1.3) is given by:

\[ \phi_2(x, y) = \frac{\partial}{\partial x} \phi_1(x, y) = -\frac{1}{2\pi \sigma} \frac{x - x_s}{(x-x_s)^2 + (y-y_s)^2}. \tag{D.2.4} \]

D.3 Applications

In the Barber-Brown's back projection method, partial gradient and partial Fourier transform of \( \phi_2(x, y) \) are used. The partial gradient of \( \phi_2(x, y) \) is obtained from equation (D.2.4) as

\[ \frac{\partial}{\partial x} \phi_2(x, y) = \frac{1}{2\pi \sigma} \frac{(x-x_s)^2 - (y-y_s)^2}{[ (x-x_s)^2 + (y-y_s)^2 ]^2}, \tag{D.3.1} \]
and the partial Fourier transform of $\phi_2(x, y)$ with respect to $(y - y_s)$ is found as:

$$\frac{-1}{2\sigma} e^{-\frac{1}{2}x|x_s - y||\omega_y|}, \quad x \neq x_s. \quad (D.3.2)$$

The equipotential and current lines of the potential distribution $\phi_2(x, y)$ is shown in Fig.D.1:

![Diagram showing equipotential and current lines](image)

**Fig.D.1** Equipotential lines and current density lines for a current dipole located at origin aligning with x-axis.

The results (D.1.3) and (D.2.4) - (D.3.2) can be immediately applied to the case of a circular uniform object with a current dipole on its boundary, as shown in Fig.D.2. The special pattern of the potential distribution for the experimental set-up of Fig.D.2 allows the simplification of mathematical analysis in the study of electrical impedance tomography. For example, Barber-Brown's algorithm and the integral
equation method were both based on the measurement set-up and used a coordinate to simplify Poisson's equation.

![Diagram of a circular uniform object with a current dipole on the boundary.](image)

**Fig. D.2** Geometry of a circular uniform object with a current dipole on the boundary.
Appendix E  Radon's Back-projection Theory

Back-projection theory was first discussed by the eminent mathematician J. Radon in 1917. The theory has been successfully applied to X-ray CT since 1972. Applications of the theory are also found in various of tomographic systems other than the X-ray CT.

Barber and Brown developed a linear algorithm for electrical impedance imaging which mimics Radon's back-projection method. The study on the integral equation method established a relationship between the Barber-Brown's algorithm and Radon's back-projection method. The Radon's back-projection theory is briefly reviewed here.

E.1 Radon's transformation

Consider the parallel geometry shown in Fig.E.1. An stationary object defined by the distribution of parameter $\mu$ is located in a coordinate system $(x, y)$ with a source-detector system rotating about the origin. The signal penetrates the object and travels in a straight line path between the source and detector. At the detector, an attenuated signal is received which contains information about the two-dimensional distribution function $\mu(x,y)$. The task of the tomographic imaging systems is to reconstruct the distribution function $\mu(x,y)$ using the measured data from receivers around the object.

The rotation angle is designated by $\Theta$, and a rotating coordinate system $(\hat{x}, \hat{y})$ is used which centred at the same origin as the $(x, y)$ system. The positions of the source and detector are expressed in terms of $\hat{x}$. Any point on the object can be represented by either $(x, y)$ or $(\hat{x}, \hat{y})$. The two coordinate systems are related by a rotational transformation:

$$\hat{x} = x \cos \Theta + y \sin \Theta$$

(E.1.1a)
\[ \hat{y} = -x \sin \Theta + y \cos \Theta \quad \text{(E.1.1b)} \]

or inversely

\[ x = \hat{x} \cos \Theta - \hat{y} \sin \Theta \quad \text{(E.1.2a)} \]

\[ y = \hat{x} \sin \Theta + \hat{y} \cos \Theta \quad \text{(E.1.2b)} \]

Fig. E.1 Geometry of the tomographic system.

Denoting the function \( \mu(x, y) \) in the rotational coordinate \( (\hat{x}, \hat{y}) \) by \( \mu_\Theta(\hat{x}, \hat{y}) \), then

\[ \mu_\Theta(\hat{x}, \hat{y}) = \mu(x, y) = \mu(\hat{x} \cos \Theta - \hat{y} \sin \Theta, \hat{x} \sin \Theta + \hat{y} \cos \Theta). \quad \text{(E.1.3)} \]
The two-dimensional Fourier transformations of the functions in (F1.3) are given by:

\[
U(X, Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mu(x, y) e^{-j2\pi(xX+yY)} \, dx \, dy
\]  
(E.1.4)

\[
U_\Theta(\hat{X}, \hat{Y}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mu_\Theta(\hat{x}, \hat{y}) e^{-j2\pi(\hat{x}\hat{X}+\hat{y}\hat{Y})} \, d\hat{x} \, d\hat{y}.
\]  
(E.1.5)

Substituting equation (E.1.3) into (E.1.4), recognizing that \(dx \, dy = d\hat{x} \, d\hat{y}\), i.e., the Jacobain is 1, and using equations (E.1.2) in (E.1.4), yield

\[
U(X, Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mu_\Theta(\hat{x}, \hat{y}) e^{-j2\pi[(\hat{x}\cos\Theta-\hat{y}\sin\Theta)X+(\hat{x}\sin\Theta+\hat{y}\cos\Theta)Y]} \, d\hat{x} \, d\hat{y}
\]  
(E.1.6)

Let

\[
\hat{X} = X \cos\Theta + Y \sin\Theta
\]  
(E.1.7a)

\[
\hat{Y} = -X \sin\Theta + Y \cos\Theta,
\]  
(E.1.7b)

then from equation (E.1.6) we have

\[
U(X, Y) = U_\Theta(\hat{X}, \hat{Y}).
\]  
(E.1.8)

Equation (E.1.8) reveals that the two Fourier transforms are the same if axes \((\hat{X}, \hat{Y})\) are a rotation of \((X, Y)\) in the frequency domain by a view angle \(\Theta\).
The projection of the object at a view angle $\Theta$ is a linear integral along a straight line, expressed as

$$p_{\Theta}(\hat{x}) = \int_{L} \mu_{\Theta}(\hat{x}, \hat{y}) \, d\hat{y},$$  \hspace{1cm} (E.1.9)

where $\hat{x}$ is the detector position and $\hat{y}$ is along a ray of signal penetrating the object. In some practical tomographic systems, such as X-ray CT, signals obtained at receivers can be represented by equation (E.1.9). Taking one-dimensional Fourier transform of the function $p_{\Theta}(\hat{x})$ at spatial frequency $f$ yields

$$P_{\Theta}(f) = \int_{-\infty}^{+\infty} p_{\Theta}(\hat{x}) \, e^{-j2\pi f \hat{x}} \, d\hat{x}. \hspace{1cm} (E.1.10)$$

Substituting equation (E.1.9) into (E.1.10) yields

$$P_{\Theta}(f) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mu_{\Theta}(\hat{x}, \hat{y}) \, e^{j2\pi \hat{x} f} \, d\hat{x} \, d\hat{y}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mu_{\Theta}(\hat{x}, \hat{y}) \, e^{j2\pi (\hat{x} f + 0\hat{y})} \, d\hat{x} \, d\hat{y}. \hspace{1cm} (E.1.11)$$

Comparing equation (E.1.11) with (E.1.5) and using equation (E.1.8), we obtain

$$P_{\Theta}(f) = U_{\Theta}(f, 0) = U(f \cos \Theta, f \sin \Theta). \hspace{1cm} (E.1.12)$$

Equation (E.1.12) is commonly known as Radon's transformation or the projection theorem. The theorem indicates that the Fourier transform of a projection is a center-cross-section of the Fourier transform of the image. Most modern tomographic systems are based on this fundamental theorem.
It is important to note that the back-projection theory requires the measured projections be in a form of straight line integral over a two-dimensional distribution to be reconstructed.

E.2 Frequency domain and spatial domain approaches

Two approaches are available for reconstructing an image from projections. One is frequency domain approach and the other is spatial domain approach.

The frequency domain approach calculates one-dimensional Fourier spectrums of each projection. Two-dimensional spectrum of the object distribution is then constructed from the projection spectrums by interpolation. The interpolation process is necessary because the usual display medium is on a TV screen which requires a rectangular raster whereas the projection (E.1.12) is in a polar system. The so constructed two-dimensional spectrum is then transformed back to spatial domain by the inverse Fourier transform to achieve reconstruction.

The frequency domain approach is elegant in mathematics but time consuming in computation. Most modern commercial machines works in the spatial domain, using what is called the convolution back-projection algorithm. The convolution back-projection algorithm is simply an extension of the frequency domain approach.

Consider the inverse Fourier transform of $U( X, Y )$

$$\mu (x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} U( X, Y ) e^{j2\pi (xX+yY)} \, dX \, dY , \quad (E.2.1)$$

and polar coordinates

$$x = \rho \cos \Phi, \quad y = \rho \sin \Phi \quad (E.2.2)$$

$$X = R \cos \Theta, \quad Y = R \sin \Theta, \quad (E.2.3)$$
\[ R = \text{sgn}(Y) \sqrt{X^2 + Y^2}, \]  
(E.2.4)

\[ \Theta = \arctan\left(\frac{Y}{X}\right) \mod \pi \]  
(E.2.5)

where

\[ 0 < \rho < \infty, \quad 0 < \Phi < 2\pi, \]  
(E.2.6)

\[ -\infty < R < \infty, \quad 0 < \Theta < \pi. \]  
(E.2.7)

Converting equation (E.2.1) into the polar coordinates yields

\[
\begin{aligned}
\mu(x, y) &= \int_0^\infty \int_{-\infty}^{\infty} P_\Theta(R) e^{j2\pi \rho R \cos \Theta \sin \Phi + \sin \Theta \sin \Phi} |R| dR d\Theta \\
&= \int_0^{\pi} \int_{-\infty}^{\infty} P_\Theta(R) |R| e^{j2\pi \rho R \cos (\Theta - \Phi)} dR d\Theta, \\
&= \int_0^{\pi} d\Theta \int_{-\infty}^{\infty} P_\Theta(R) |R| e^{j2\pi \rho R \cos (\Theta - \Phi)} dR d\Theta, 
\end{aligned}
\]  
(E.2.8)

where \( P_\Theta(R) = U(R \cos \Theta, R \sin \Theta) \) has been used. The inner integral of equation (E.2.8) is the inverse Fourier transform of a product of two terms. The first is the transform of the projection at view angle \( \Theta \), and the second is the function \(|R|\). Based on convolution theory, the inner integral of equation (E.2.8) can be written as

\[ g_\Theta(s) = \int_{-\infty}^{\infty} p_\Theta(s') k(s - s') ds', \]  
(E.2.9)

where \( p_\Theta(x) \) is the inverse transform of \( P_\Theta(R) \), and \( k(x) \) is the inverse transform of the function \(|R|\). Substituting equation (E.2.9) into (E.2.8) yields
\[ \mu(x, y) = \int_0^\pi g(\theta \cos(\Theta - \Phi)) \, d\Theta. \]  

(E.2.10)

The integral (E.2.10) is known as a back-projection and it reveals the relationship between the desired distribution \( \mu(x, y) \) and the measured boundary projections. Equations (E.2.9) and (E.2.10) can be easily approximated on a computer as one dimensional finite sums.

The back-projection procedure is illustrated in Fig. E.2. The projection data at various view angle \( \theta \) are convolved with the kernel function \( k \left( s - s' \right) \), the results, i.e. \( g_\theta(s) \), are then dumped back into the object domain to produce an image, as shown in the figure.

**Fig. E.2** Illustration of the back-projection procedure.

It is worth noting that the convolved detector readings \( g_\theta(x) \) is the derivative of the Hilbert transform of the projection functions \( p_\theta(x) \) [Scudder, 1978]. This provides
an alternative approach to evaluating some theoretically interesting phantoms, such as cylinder or ellipse [Shepp, 1974].

E.3 The kernel function

The kernel function \( k(x) \) in equation (E.2.9) is obtained from inverse Fourier transform of a ramp function \( |R| \). The ramp function \( |R| \) is not a normal function but a "generalised function," because the second derivative of \( |R| \) is a \( \delta \)-function. As a result, the classical Fourier transform of the function \( |R| \) does not exist because of the unbounded energy of \( |R| \). However, a generalised Fourier transform may be used to obtain a Fourier transform of \( |R| \). The generalised Fourier transform treats \( |R| \) as the limit of a "well behaved" function. A classical Fourier transform of the well behaved function is performed followed by a limit operation. For instance, the function \( |R| \) can be considered as the limit of a well-behaved function \( |R| e^{-\epsilon|R|} \) for \( \epsilon \to 0 \). A comparison of the two functions is sketched in Fig.E.3.

![Comparison of functions](image)

**Fig. E.3** Comparison of well-behaved function \( |R| e^{-\epsilon|R|} \) with the function \( |R| \). The top curve is for the function \( |R| \), and the remaining are for function \( |R| e^{-\epsilon|R|} \).
The inverse Fourier transform of the function $|R|e^{-|R|}$ with respect to $R$ is found to be

$$k_\varepsilon(x) = \frac{\varepsilon^2 - (2\pi x)^2}{(\varepsilon^2 + (2\pi x)^2)^2}.$$  \hspace{1cm} (E.3.1)

For large $x$, $k_\varepsilon(x) \approx k(x)$, and for any finite $\varepsilon$, $k_\varepsilon(x)$ approaches $\frac{1}{\varepsilon^2}$ as $x \to 0$. A sketch of the kernel function for small $\varepsilon$ is shown in Fig. E.4.

**Waveform of function $k_\varepsilon(x)$ ($\varepsilon = 0.5$)**

![Waveform of function](image)

Fig. E.4 Sketch of the function $k_\varepsilon(x)$ for $\varepsilon = 0.5$.

In practice, it is often assumed that the function $\mu(x,y)$ is band-limited by a spatial frequency $B$ cm$^{-1}$ and that the measured data is collected at the Nyquist frequency, with a spacing of $\frac{1}{2B}$ cm. This leads to a kernel of

$$k_L(x) = \int_{-B}^{B} |R| e^{j2\pi xR} dR$$ \hspace{1cm} (E.3.2)

which has an expression in the frequency domain as
where \( \text{rect}(x) = 1 \) for \( |x| < \frac{1}{2} \), \( \text{rect}(x) = 0 \) for \( |x| > \frac{1}{2} \), and " * " stands for convolution operation. The inverse Fourier transform of the function (E.3.3) is given by

\[
k_L(x) = 2B^2 \frac{\sin \pi x}{\pi x} - B^2 \sin^2(Bx)
\]

where \( \sin(x) = \frac{\sin \pi x}{\pi x} \). A sketch of the function (E.3.4) is shown in Fig.E.5.

The kernel function (E.3.3) was first discussed by Ramachandran and Lakshinarayanan [1971] and has been frequently used in the convolution back-projection approach. Other kernels are possible and are sometimes used, but all have properties similar to the Lakshinarayanan kernel.
Appendix F  Integral Equation

Most physical phenomena are governed by either differential equations or integral equations. Differential equations concern about the local behaviour of the physical phenomena and integral equations care for the integrated effect of the physical phenomena over certain regions.

Solution approaches to differential and integral equations are quite different. Techniques for differential equations are more developed than those for integral equations. An brief introduction to integral equation will be given in this Appendix.

F.1 Linear Integral Equations

General form of a one-dimensional linear integral equation may be written as

$$h(x) f(x) + \int_{a}^{b} K(x,y) f(y) \, dy = g(x), \quad x \in (a, b) \quad (F.1.1)$$

where functions $h(x)$, $g(x)$ and $K(x,y)$ are known and are assumed to be bounded and continuous. $K(x,y)$ is called the kernel of the integral equation and plays an important role in determining the properties of the equation.

When $h(x) = 0$, the equation is called Fredholm integral equation of the first kind and, in this case, $x$ is not necessarily bounded to $(a, b)$; if $h(x) \neq 0$ for $x \in (a, b)$, then the equation is called Fredholm integral equation of the second kind. If the region $(a, b)$ extends to infinity, or the kernel $K(x,y)$ or function $g(x)$ is unbounded, then the integral equation is called a singular equation. The classification of the one-dimensional integral equation also applies to multi-dimensional integral equations.
F.2 Numerical Solution of Integral Equations

Integral equations can rarely be solved analytically and therefore numerical treatment has to be employed. A simple but effective way to solve integral equation numerically is to replace the continuous integral by some quadrature formulas and match the whole equation at a set of discrete points of \( x \). This results in a set of linear equations which yields a discrete and approximate solution of the unknown function \( f(y) \).

A general formulation of the numerical solution of the integral equation (F.1.1) is given by

\[
 h(x_i) f(x_i) + \sum_{j=1}^{n} w_j K(x_i, y_j) f(y_j) = g(x_i) \quad (F.2.1)
\]

\[
i = 0, 1, 2, \ldots, n,
\]

where \( x_i \) and \( y_j \) are the discrete points within the region \((a, b)\), \( n \) is the number of the discrete points, and \( w_j \) are the weights that depends on the quadrature used.

In addition to the simple method, the Moment Method is also widely used in solving integral equations. In this method, an assumed solution containing a set of unknown parameters is substituted into an integral equation and the projections of the integral equation onto a set of known functions (often called basis) are evaluated. The unknown parameters in the assumed function are chosen in such a way that the errors between the projections of the both sides of the integral equation reach to minimum. The resulting set of parameters are then substituted into the assumed solution to obtain an approximate solution of the integral equation.

Integral equations of the second kind are much easier to solve than the first kind. Extensive literature on the solvability and stability of linear integral equations of the second kind are available. However, the literature on linear integral equations of the first kind is sparse. The major difficulty is that the equation does not have a bounded inverse which can be seen as follows (in fact it may not have an inverse, but we will assume here that it does).
Let $f(y)$ be a solution to equation (F.1.1) and add to it the function $f_m = \sin(my)$.

For any integrable kernel it is known that

$$g_m = \int_a^b K(x,y) \sin (my) \, dy \to 0, \text{ as } m \to \infty .$$  \text{(F.2.2)}

Equation (F.2.2) reveals that only an infinitesimal change $g_m$ in $g(x)$ may cause a finite change $f_m$ in $f(x)$. Additionally, as $m \to \infty$, $g_m$ vanishes faster for flat smooth kernels than for sharply peaked kernels. In the extreme of a $\delta$-function kernel, i.e. $K(x,y) = \delta(x-y)$, we obtain $g_m = f_m$, which indicates that $g_m$ may even not vanish.

The above discussion reveals that the success in solving integral equations of the first kind by any method depends primarily on the accuracy of $g(x)$ and the shape of $K(x,y)$. Moreover, although the existence of the solution for an integral equation of the first kind has been ensured by the physical phenomena itself in many situations, it is essential to use proper numerical method in order to obtain correct solution because solutions of the equations are inherently unstable.