1994

Stability and reliability assessments of earth structures (under static and dynamic loading conditions)

Dawei Xu
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STABILITY AND RELIABILITY ASSESSMENTS OF EARTH STRUCTURES (under static and dynamic loading conditions)

A thesis submitted in fulfilment of the requirements for the award of the degree of

DOCTOR OF PHILOSOPHY

from

THE UNIVERSITY OF WOLLONGONG
DEPARTMENT OF CIVIL AND MINING ENGINEERING
AUSTRALIA

By

DAWEI XU, B.E., M.E.(Honours.)

January, 1994
DECLARATION

I hereby declare that the research work described in this thesis is my own work and has not been submitted for a degree to any university or institute except where specifically indicated.

Dawei Xu
January, 1994
ACKNOWLEDGMENTS

I wish to express my sincere gratitude to Professor Robin Chowdhury who directed my interests in probabilistic geomechanics and geotechnical earthquake engineering and supervised the whole research work of my Ph.D. thesis. His invaluable guidance, useful suggestions and discussions and constant encouragements are gratefully acknowledged.

Special thanks are also extended to Professor Chuanzhi Xiong who allowed me to undertake research in geotechnical engineering and supported my application to come to Australia for further study.

I am indebted to the University of Wollongong for a postgraduate award and the Water Engineering and Geomechanics Research Group in the Department of Civil and Mining Engineering for continual support.

I record my gratitude to all members of my family. Special acknowledgement is due to my parents, and my wife, Heng Wang, for their constant support and encouragements. My lovely son, Ferris Xu, who was born on January 15, 1993, gave me a lot of happiness during a period of my intense study.

Finally, I would like to thank the Department of Civil and Mining Engineering for providing all the necessary facilities and good conditions for my research work.
ABSTRACT

The basic concepts and methods for the stability and reliability assessment of a soil slope or an earth structure, under static and dynamic loading conditions, have been discussed in some detail in this thesis. A number of improvements and extensions to the current state-of-the-art approaches have been proposed and implemented with particular emphasis on both 'simplified' and 'rigorous' limit equilibrium models. The simplified Bishop method, the Generalised Procedure of Slices (with the Morgenstern and Price side force function) and the Sarma method have been used extensively in this thesis.

An optimisation procedure, based on the conjugate gradient algorithm, was developed for locating the critical slip surface with either the minimum factor of safety or the minimum critical seismic coefficient. This optimisation procedure can be used to search not only circular and non-circular slip surfaces in homogeneous or layered soil slopes but also including situations in which part of the potential slip surface is controlled by a weak soil zone or a weak surface. A very effective numerical technique, the rational polynomial technique (RPT), was introduced for solving non-linear equations and estimating the partial derivatives of inexplicit functions which are often encountered in geotechnical reliability assessments.

A comprehensive framework has been presented for improving or updating the current probabilistic methods of analysis for earth structures. This framework includes the availability of the three main method used for geotechnical reliability analysis, i.e., (a) First Order and Second Moment Method (FOSM), (b) Point Estimation Method (PEM) and (c) Monte Carlo Simulation Method (MCSM).

The performance function was expressed in terms of the factor of safety and may be defined on the basis of either the simplified or 'rigorous' limit equilibrium methods. The proposed probabilistic framework includes some new concepts and new approaches. An orthogonal transformation has been introduced in the Monte Carlo Simulation Method so
that correlated basic random variables can be considered. A comparison of the conventional definition of reliability index, $\beta$, with the so called 'invariant' reliability index, $\beta^*$, has been presented. The influence of spatial correlations of basic random variables on the reliability index has also been investigated. Comprehensive comparisons based on the three methods, i.e., FOSM, PEM and MCSM, have been carried out.

The evaluation of geotechnical system reliability is important for earth structures because of the spatial variability of soil properties. Comprehensive procedures have been developed for estimating the reliability bounds, 'upper' and 'lower' bounds of slope reliability taking into consideration the fact there are many potential slip surfaces in any slope. These evaluations of geotechnical system reliability can be carried out on the basis of either the simplified or the relatively 'rigorous' limit equilibrium methods. Therefore, reliability bounds have been evaluated by considering not only circular slip surfaces but also non-circular slip surfaces. Moreover, both independent and correlated basic random variables can be included in the proposed analysis procedures. The influence of spatial variation of basic random variables on the reliability bounds was also investigated.

On the basis of the limit equilibrium concept and the Newmark-type dynamic response approach, an innovative procedure was developed for the earthquake analysis of earth structures such as embankments and earth dams. The proposed analysis procedure can consider not only the critical seismic coefficient but also the dynamic properties of materials, such as damping ratio and natural frequency. More importantly the change in the critical seismic coefficient with time is included in the analysis and simulation process. The degradation of shear strength parameters during earthquake shaking may occur due to strain-softening characteristics of the earth materials. A method has been proposed and implemented to include this post-peak shear strength decrease in the earthquake analysis process. Shear strength may also decrease in some soils due to the development of dynamic excess pore water pressure during earthquake shaking. A different procedure has been used to include this type of shear strength decrease in the analysis procedure. The factor of safety and critical seismic coefficient are considered as
functions of time after the start of an earthquake. The permanent displacements of earth structures due to earthquake excitations can also be evaluated and illustrative examples are presented to show the influence of material properties on the estimated magnitudes of permanent deformations. Based on Gaussian non-stationary random process a procedure has been presented for simulating earthquake motion and, in particular the time acceleration histories for an earthquake of specified magnitude and duration.
The following symbols are used in this thesis:

\[ A = \text{Vector of constants in a linear performance function;} \]
\[ A_n = \text{A parameter corresponding to \( n \) cycles of loading;} \]
\[ \begin{bmatrix} C \end{bmatrix} = \text{Covariance matrix of basic random variables or parameters;} \]
\[ \begin{bmatrix} C' \end{bmatrix} = \text{Covariance matrix of the reduced or normalised random variables} \]
\[ X' = (x'_1, x'_2, \ldots, x'_n); \]
\[ \begin{bmatrix} C_Y \end{bmatrix} = \text{Covariance matrix of vector} \ Y; \]
\[ D = \text{Distance from a point} \ X' \text{on the failure surface to the origin of} \ X'; \]
\[ E = \text{Interslice normal force;} \]
\[ F = \text{Factor of safety} \]
\[ F_f = \text{Factor of safety based on force equilibrium equation;} \]
\[ F_m = \text{Factor of safety based on moment equilibrium equation;} \]
\[ f(X) = \text{A model for calculating the factor of safety of a slope;} \]
\[ F_X(x_1, x_2, \ldots, x_n) = \text{Cumulative probability density function;} \]
\[ g = \text{Gradient vector;} \]
\[ G(X) = \text{Performance function in} \ X \text{space (e.g., safety margin of a slope);} \]
\[ I(t) = \text{Intensity function;} \]
\[ I = \text{Identity matrix;} \]
\[ K = \text{Seismic coefficient;} \]
\[ K_c = \text{Critical seismic coefficient;} \]
\[ K(t) = \text{Time history of earthquake acceleration coefficient;} \]
\[ L_i \text{ or} \ l_j = \text{Length of the base of a slice;} \]
\[ p_k = \text{Conjugate direction} (k = 1, 2, \ldots, n); \]
\[ P_f = \text{Probability of failure of a slope;} \]
\[ P_+ \text{ and} \ P_- = \text{Weight coefficients in Point Estimate Method;} \]
\[ Q = \text{A symmetric and positive-defined matrix with constant components;} \]
R = Radius of a circular slip surface;
S(ω) = Power spectrum density function;
T = Interslice shear force;
T = Orthogonal transformation matrix;
x₀ = Abscissa of the centre of a circular slip surface;
X = Vector of basic random variables or parameters;
X = Vector of reduced or normalised random variables;
X* = Vector representing the most probable failure point on the failure surface;
ΔXᵢ = Width of a vertical slice;
y₀ = Ordinate of the centre of a circular slip surface;
Y = Vector of uncorrelated transformed variables;
Y* = Reduced or normalised vector corresponding to vector Y;
Z = Uncorrelated standard normal random vector;
ϕ(X) = A joint probability density distribution;
μₚ = Mean value of performance function;
μₓ = Vector of mean values of random parameters;
σₚ = Standard deviation of performance function;
[σₓ] = Matrix of standard deviations of basic random variables (a diagonal matrix);
[σᵧ] = Matrix of standard deviation of vector Y (a diagonal matrix);
β = Conventional definition of reliability index (based on Cornell’s definition);
β* = Reliability index based on Hasofer-Lind definition ('so called' invariant reliability index);
ρ = Correlation coefficient;
ω = Natural circular frequency of earth deposit or soil layer;
v = Damping ratio of earth deposit or soil layer;

\( \left( \frac{∂G}{∂X_i} \right)^T \) = \left( \frac{∂G}{∂x_1^*, \ldots, \frac{∂G}{∂x_n^*}} \right)^T
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INTRODUCTION AIMS AND SCOPE

1.1. GENERAL REMARKS

An important consideration in earth structure design is stability or reliability under static and dynamic loading conditions. The stability and reliability analysis of earth structures under static loading can usually be performed by deterministic and probabilistic approaches based on limit equilibrium concepts. In deterministic analysis, the 'factor of safety' is considered as an indicator of safety or reliability of a slope. It is usually defined as the ratio of available shear strength to shear stress at any point along a slip surface within a slope. The factor of safety is normally considered as a performance function for searching the critical slip surface in slopes. This implies that the critical slip surface is defined as that with the minimum factor of safety. Although the variabilities or uncertainties of soil and rock properties can not be considered in deterministic methods of analysis, these methods are still widely used to evaluate the stability of earth structures. Several deterministic calculation models based on limit equilibrium concepts have been developed by researchers and geotechnical engineers for assessing soil and rock slope
Chapter 1: Introduction


The stability of a slope is governed by many factors, e.g. the geometrical characteristics, shear strengths of different materials, pore water pressures and local stress fields. As mentioned above, most of these factors may be variable or uncertain. For instance, the strength of soil in a slope may vary spatially, the methods of measuring parameters are not perfect and the properties of samples are not fully representative of that of the overall material. Hence considerable uncertainty exists with regard to our knowledge of the input parameters. Based on these uncertain input parameters, the factor of safety, therefore, may not be a consistent measure of risk since slopes with the same factor of safety can have different levels of failure probability depending on the variability of those factors. Moreover, a deterministic slope design associated with the average values of input parameters does not take into account these uncertainties and may provide misleading results for slope reliability. In fact, designs based on the deterministic approach which are considered to be conservative may have a significant or unacceptable probability of failure associated with them (see, for example, Hoeg & Murarka, 1974). This cannot be detected unless assessment is made within a probabilistic framework. A slope reliability analysis associated with a probabilistic method can only be conducted if the input parameters are considered as random variables and have been statistically quantified and described. In the last two decades, probabilistic approaches for slope reliability analysis have developed continually and significant progress has been made in their application.

Although slope reliability analysis based on probabilistic models can take into account the uncertainties of input parameters, the computational effort required is usually more than that required in deterministic analysis. The magnitude of failure probability which may be regarded as acceptable cannot be determined by calculation. Experience and engineering judgement are required to establish an order of magnitude of the acceptable failure probability which depends significantly on the importance and service
time of a slope or earth structure as well as the consequences of failure, if and when failure occurs. Both deterministic and probabilistic methods should be performed and these alternative methods can be considered as complementary to each other.

**New procedures for searching critical slip surfaces in earth slopes are developed in this thesis. Extensions to procedures for probabilistic analysis have also been developed. In particular, correlations between geotechnical parameters have been taken into consideration. System reliability bounds have been considered and procedures developed to evaluate 'upper' and 'lower' bounds.**

The response of earth structures to earthquake excitations is a major concern of geotechnical engineers and researchers especially in countries and regions with significant seismic activity. Events of small and moderately sized failures caused by earthquakes have occurred in many parts of the world and these include the failures of natural slopes, embankments and earth dams. There are few examples of large landslides which can be attributed to earthquakes.

**The major tasks included in this thesis are summarised in Fig. 1-1 and Fig. 1-2.**

The design of earthquake-resistant earth structures requires a thorough understanding of their dynamic response. Large inertia forces are induced in an earth structure shaken by an earthquake. These forces alternate in direction many times and, in certain zones of an earth structure, their magnitude may be sufficiently large to reduce the factor of safety below unity several times during an earthquake. Decrease of shear strength of earth materials may occur due to strain-softening associated with significant relative deformation along a potential slip surface. Shear strength may also decrease due to the development of excess pore water pressure (EPWP) during an earthquake. Several popular calculation methods for the dynamic response analysis of earth structures are based on a sliding block model. The original model was developed by Newmark (1965) and there have been modifications and extensions by, among others, Goodman and Seed.
(1966); Sarma (1975) and Chugh (1982). However, the dynamic properties of soil materials (i.e. natural frequency and damping ratio) were not considered in the development of the basic equations for these models. Therefore, the energy loss mechanism which is part of earthquake response cannot be considered in these models. Again, the degradation of shear strength of soil materials due to earthquake shaking was not taken into consideration explicitly in these models. **In order to overcome these shortcomings, an innovative and comprehensive procedure for the dynamic response analysis of earth structures has been developed during the research work and is described in this thesis.**

### 1.2. AIMS AND SCOPE OF THIS RESEARCH

The principal objective of this research was to develop several calculation models and associated computer programs to assess the stability and reliability of slopes under static and dynamic loading conditions. To achieve this objective the following tasks were undertaken (see Figs. 1-1 and 1-2):

1. Select suitable limit equilibrium models for the stability analysis of slopes considering slip surfaces of circular and arbitrary shape. Based on these models develop suitable optimization techniques to search for critical slip surfaces based either on (a) the minimum value of the factor of safety or (b) the minimum value of the critical seismic coefficient.

2. Develop procedures for probabilistic analysis based on selected limit equilibrium models and compare three numerical procedures namely (a) first order second moment method, (b) point estimate method, and (c) Monte Carlo simulation method.

3. Demonstrate the application of these probabilistic models to multi-layer slopes considering slip surface of arbitrary shape.

4. Consider slope stability involving multiple potential slip surfaces as a ‘geotechnical system’ and develop a comprehensive procedure for system
reliability analysis considering slip surfaces of circular or arbitrary shape and
demonstrate the application of this procedure.

5. Develop a comprehensive and innovative procedure for the dynamic response
analysis of earth structures by significant extension of the Newmark (1965) model
in combination with comprehensive limit equilibrium approaches.

6. Demonstrate the application of the dynamic response model.

A brief outline of the contents of Chapters 2-8 is given below:

Previous research work concerning deterministic and probabilistic methods of
analysis as well as the dynamic response analysis of earth structures to earthquake
excitation is reviewed in Chapter 2.

Simplified and rigorous limit equilibrium methods, selected for use in this
research work, for calculating factor of safety and critical seismic coefficient, are
introduced in some detail in Chapter 3.

An optimization method based on the conjugate gradient algorithm is presented in
chapter 4. Moreover, a numerical technique, called the rational polynomial technique
(RPT), is proposed for estimating the values of non-linear functions and the magnitudes
of partial derivatives of such functions. On the basis of conjugate gradient algorithm and
rational polynomial technique, innovative procedures have been developed for searching
the location of critical circular or non-circular slip surfaces considering the alternative
criteria of (a) the minimum factor of safety and (b) the minimum critical seismic
coefficient. These procedures are also described in Chapter 4. Several example problems
are presented to illustrate the implementation of these new approaches.

Numerical methods for assessing failure probability of slopes are discussed in
Chapter 5. A geometrical interpretation of the conventional definition of reliability index \( \beta \)
is presented. An 'invariant' reliability index proposed by Hasofer and Lind (1974) is then
introduced and a simple and clear deductive procedure is presented for the derivation of a
general solution for this alternative reliability index.
In the Monte Carlo simulation approach, an orthogonal transformation associated with factorization method is used in order to handle correlated normal random variables. Detailed studies of inherent spatial variability of various geotechnical parameters in slopes are excluded from this research because many researchers have done a lot of remarkable work, e.g. Vanmarcke (1977a, 1977b, 1984), Asaoka and Grivas (1982), Chowdhury (1980, 1984), Chowdhury et al. (1987, 1988), Zhang (1990), and Chowdhury and Zhang (1993). However, judicious use is made of a new concept to deal with a statistically inhomogeneous medium. The real random field may be replaced by a multi-layer model in which each layer is considered to be statistically homogeneous.

Different probabilistic approaches for slope analysis are compared in chapter 5 and the influences of spatial correlations of basic input random variables on the reliability of slopes are systematically analysed.

The concept of reliability bounds and the formulations for upper and lower bounds of a system are considered in Chapter 6. For a geotechnical system with several potential failure modes, the overall reliability will depend on the reliability considering individual modes of failure as well as the correlations between the different failure modes. The correlation coefficients between basic random variables in the geotechnical system can be considered in the analysis procedure developed during this research. Moreover, alternative definitions of reliability index can be used in the computational process. An additional feature of great value is that slip surfaces of arbitrary shape can be considered for assessing the reliability of geotechnical systems. The influence of the spatial correlations of material properties on the reliability bounds of slope systems is also discussed in chapter 6.

In Chapter 7, a procedure for simulating earthquake motions is presented on the basis of Gaussian non-stationary random process. The response spectrum analyses of the simulated artificial earthquakes have also been performed.

Combining the limit equilibrium concept with Newmark-type response analyses, an innovative and comprehensive procedure for the dynamic response analysis for earth
structures is proposed in Chapter 8. The dynamic properties of the earth material or soil layer (i.e. natural frequency and damping ratio) can be considered in the developed method. The degradation of shear strength of such materials due to earthquakes can also be simulated whether decrease of shear strength is a consequence of (a) strain-softening or (b) the development of excess pore water pressure. The permanent deformations of earth structures caused by an earthquake can be estimated. More importantly, the factor of safety and critical seismic coefficient are no longer constants but are modelled as variables with time after the start of an earthquake. Each of these quantities, i.e., the factor of safety $F$ and the critical seismic coefficient $K_c$ has, in fact, been simulated as a function of time during the period of earthquake shaking. The application of the developed comprehensive method for analysing the dynamic response of an earth structure is demonstrated for a real dam. Several analyses have been carried out and the results are discussed in Chapter 8. Probabilistic assessment of reliability during earthquakes was outside the scope of the present thesis.

The conclusions of the research are summarised in the final chapter, Chapter 9.
Chapter 1: Introduction

Use three limit equilibrium methods (Bishop simplified, Morgenstern & Price, Sarma) as the basis for objective and performance functions of both optimisation and reliability analyses of slopes.

Develop conjugate gradient method for optimisation to locate critical slip surfaces based on minimum factor of safety or minimum critical seismic coefficient.

Develop rational polynomial technique to estimate the partial derivatives for optimization and reliability analyses.

Develop and compare results based on three numerical probabilistic methods, FOSM, PEM, and MCSM.

Develop system reliability approach considering a slope as a geotechnical system with several potential failure modes.

Develop dynamic response analysis of earth structures subject to earthquake shaking based on an extension of Newmark's approach (See Fig. 1-2 for full details).

Fig. 1-1 Flow Chart Showing Tasks and Methods Included in this Thesis for Stability and Reliability Analysis
Chapter 1: Introduction

Develop a procedure for generating earthquakes acceleration-time histories based on Gaussian random process and an assumed power spectrum function.

Develop a dynamic response model based on combining the limit equilibrium concept with an extended Newmark-type approach.

Develop a new integral procedure to estimate the permanent deformation of earth structures caused by earthquake shaking.

Develop a model to simulate post-peak decrease in shear strength of earth materials along a slip surface as a function of the permanent relative deformation along that surface.

Develop a computer model to simulate the development of excess pore water pressure in earth materials as a result of earthquake shaking.

Use each of these models of shear strength decrease in illustrative examples of embankment behaviour during earthquakes.

Fig. 1-2 Flow Chart Showing Tasks and Methods Developed in this Thesis for Earthquake Response Analysis of Earth Structures
DETERMINISTIC AND PROBABILISTIC APPROACHES FOR SLOPE RELIABILITY

A BRIEF REVIEW

(Part-I: Static Loading Condition)

2.1. GENERAL REMARKS

Currently available approaches for slope stability analysis include limit equilibrium methods, limit analysis methods, finite element methods and boundary element methods. Only the limit equilibrium model is considered in this thesis in order to discuss and further develop both deterministic and probabilistic methods of slope analysis. Plasticity and stress-deformation approaches are, of course, useful. However, these approaches are outside the scope of this thesis. Although limit equilibrium is the oldest numerical approach for the analysis of slope stability, it is still widely used by geotechnical engineers due to its simplicity and the fact that the realistic numerical modelling can be
facilitated on the basis of geotechnical parameters which are relatively easy to determine. Numerous calculation models, based on limit equilibrium concepts, have been developed and most of them require consideration of the potential sliding mass as an assemblage of elements or slices which are usually vertical. Recognition that significant uncertainties are associated with factors governing slope stability has been growing among geotechnical engineers. Probability approaches based on various limit equilibrium calculation models have been used to consider and analyse uncertainties in a systematic way.

Previous research work is reviewed in this chapter. Deterministic methods of slope reliability analysis are considered first followed by probabilistic methods.

### 2.2. DETERMINISTIC ESTIMATION METHODS OF SLOPE STABILITY

The degree or index of slope stability is often expressed by a factor of safety, F, which may be defined as the ratio of shear strength to shear stress at any point along a potential slip surface. It may be written in the following form:

\[
F = \frac{\tau_f}{\tau}
\]

where,  
F = factor of safety;  
\(\tau_f\) = shear strength at a point along a potential slip surface;  
\(\tau\) = shear stress along the same slip surface and at the same point.

The methods of limit equilibrium which consider subdivision into a number of imaginary slices (usually vertical) are very popular because of their ability to deal with non-homogeneous slopes, variable ground water conditions and irregular slope surfaces. These methods are often classified as a group called the Generalised Procedure of Slices (GPS) and statical equilibrium in terms of forces or moments or both may be considered for the slope as a whole. A brief description of some simplified as well as relatively rigorous methods is given below.
2.2.1. **ORDINARY OR FELLENIUS METHOD** (FELLENIUS, 1936)

The ordinary method of slices is the simplest one and is the only procedure that results in a linear factor of safety equation. The interslice forces are neglected in this method and only the overall moment equilibrium is satisfied provided the slip surface has a circular shape. The calculated factor of safety is conservative and may underestimate the real equilibrium value by as much as 60 percent (Whitman and Bialey, 1967).

2.2.2. **SIMPLIFIED BISHOP METHOD**

The simplified Bishop method neglects the vertical interslice shear forces and satisfies the vertical force equilibrium of each vertical slice on this assumption. The slip surface is considered to be of circular shape and only the overall moment equilibrium is satisfied. No consideration is given to horizontal force equilibrium either of any individual slice or of the potential sliding mass as a whole. The factor of safety equation derived from the summation of moments about the centre of the slip surface, is not an explicit equation. Therefore, an iterative procedure is required for calculating the factor of safety. The simplified Bishop method leads to relatively accurate results which compare favourably with results from most of the so-called 'rigorous' limit equilibrium methods. This method is discussed in some detail in the next chapter.

2.2.3. **JANBU’S SIMPLIFIED AND GENERALISED METHODS**

Janbu's simplified and generalised method may be used for slip surfaces of arbitrary shape and, while these methods may not be regarded as 'rigorous', they have been widely used. In general, method which satisfy both force and moment equilibrium are classified as 'rigorous'.

Janbu’s simplified method uses a correction factor, $f_0$, to account for the effect of the interslice shear forces. The correction factor is related to the cohesion, the angle of internal friction and the shape of the failure surface (Janbu, 1954; Janbu *et al.*, 1956).
The normal force on the base of a slice is derived from summation of vertical forces without the interslice shear forces. The horizontal force equilibrium equation for the whole of the potential sliding mass is used to derive the factor of safety $F_0$. The corrected factor of safety is the product of $F_0$ and $f_0$. No effort is made to satisfy moment equilibrium for the slope mass as a whole.

In Janbu's generalised method, (n-1) assumptions are made regarding the locations of the points of action of the interslice forces, i.e. a 'line of thrust' is specified. The principle of determining the 'line of thrust' is that for the case of $c' = 0$ the line of thrust should be selected at or very near the lower third point and for $c' > 0$ the line of thrust should be located somewhat above this point in a compression zone (passive condition) and somewhat below it in an expansion zone (active condition) (Janbu, 1973). Janbu's rigorous method differs from the simplified method in that the interslice shear forces are included in the derivation of the normal force on the base of a vertical slice. The factor of safety equation is again derived from the summation of horizontal forces for the whole mass. In order to solve the factor of safety equation, an iterative procedure is required. As for the simplified method moment equilibrium is not satisfied for the slope mass as a whole. Problems of convergence of the numerical solution arise for some slip surfaces.

As inferred by Janbu himself (Janbu, 1973), Lumsdaine and Tang (1982), Li (1986) and Li and White (1987a), this method often results in an approximate solution because no additional parameter is introduced into the analysis to balance the numbers of unknowns and equations. (See also Ching and Fredlund, 1983 as well as Li, 1991).

Zhang (1990) developed a modified calculation procedure based on the Janbu method in which tension cracks are included and sections of the slip surface with large curvature can be eliminated. Thus there are no numerical problems of convergence created by these sections of the slip surface. Moreover, this modified procedure is a powerful tool of slope stability analysis where tension cracks can be included with or
without water pressure. This leads to unique procedures for search of critical slip surfaces.

2.2.4. SPENCER’S METHOD

Spencer’s method assumes parallel interslice force. Thus there is a constant relationship between the magnitude of the interslice shear and normal forces (Spencer, 1967), i.e.

\[
\tan \theta = \frac{T_L}{E_L} = \frac{T_R}{E_R} \tag{2-2}
\]

where, \( \theta \) = angle of the resultant interslice force from the horizontal;

\( T \) and \( E \) = interslice shear and normal forces, subscript \( L \) denotes the left boundary of a vertical slice and subscript \( R \), the right boundary.

Spencer analysed force components in the direction perpendicular to the interslice forces to derive the expression for normal forces on the base of a slice. Two factor of safety equations, one based on the summation of moments about a common point and another on the summation of forces in a direction parallel to the interslice forces, were derived by Spencer (1967). Spencer’s method yields two factors of safety for each angle of the side or interslice forces. However, for a particular value of the angle of the interslice forces, the two factors of safety are equal and both moment and force equilibrium are satisfied. A more complex iterative procedure than that for Janbu’s and Bishop’s methods is, therefore, required. Spencer extended his method for slip surfaces of arbitrary shape and also considered the thrust line criterion in the same detail (Spencer, 1973).

2.2.5. MORGENSTERN AND PRICE METHOD

The Morgenstern and Price (M&P) method assumes an arbitrary mathematical function to describe the direction of the interslice forces, i.e.
where, $\lambda$ is a parameter to be evaluated when solving for the factor of safety;

$f(x)$ is a prescribed or assumed function of $x$, the horizontal co-ordinate.

Morgenstern and Price (1965) based their solution on the summation of forces tangential and normal to each slice. The force equilibrium equations were combined and then the Newton-Raphson numerical technique was used to solve the moment and force equilibrium equations for the factor of safety, $F$, and the parameter, $\lambda$. When the function $f(x)$ is a constant, this method is the same as the Spencer’s method. The main advantages of this method are that a non-circular slip surface can be used and a rigorous solution may be obtained. Because the original formulation and method of solution proposed for this method is very complex, Fredlund and Krahn (1976) developed a simpler alternative formulation for this method. As in Spencer’s method, two factor of safety equations are computed; one with respect to moment equilibrium and another with respect to force equilibrium. An iterative procedure is required for calculating the factor of safety and $\lambda$ based on Fredlund and Krahn’s method. The alternative formulation of the Morgenstern and Price method will be used in the following chapters of this thesis.

2.2.6. SARMA METHOD

In Sarma’s method (1979), the degree of mobilisation of interslice shear strength is considered explicitly. The magnitude of the factor of safety on interslice shear planes is assumed to be the same as on the slip surface. The interslice shear and normal forces on the slice boundary can be related by Mohr-Coulomb failure criterion, i.e.,

$$T = \frac{(E - P_w) \tan \phi}{F} + c'd$$

where, $\phi$ = average internal friction angle on the interslice plane;

c' = average cohesion on the same plane;
\( d \) = length of the interslice plane; 
\( P_w \) = force due to water pressure on the interslice boundary; 
\( T \) and \( E \) = interslice shear and normal forces respectively; 
\( F \) = the factor of safety.

In this method, the equilibrium equations are set up after including the pseudo-static forces due to horizontal acceleration, \( K_g \) (where \( g \) is gravitational acceleration), in addition to gravitational forces due to self weight. The magnitude of the acceleration factor \( K \) which brings the potential sliding mass to a condition of limiting equilibrium is then calculated and is denoted as the critical acceleration factor, \( K_c \). For the static limit equilibrium of slopes when the critical acceleration \( K_c \) is not equal to zero, the static factor of safety is calculated by reducing the shear strength simultaneously on all sliding surfaces by the factor of safety until the acceleration \( K_c \) reduces to zero (Hoek, 1986). Inclined boundaries of slices may be considered in this method and it is, therefore, very useful for analysis of jointed rock slopes.

2.2.7. CHARACTERISTICS OF THE ABOVE ANALYSIS METHODS

The essential characteristics of analysis methods introduced in the above subsections are summarised in Table 2-1.

Fredlund and Krahn (1976) indicated that the factor of safety with respect to moment equilibrium is relatively insensitive to the interslice force assumption and the factor of safety based on overall force equilibrium is far more sensitive to the side force assumption. Therefore, the magnitudes of factor of safety obtained by the Spencer method as well as Morgenstern and Price method are generally in agreement to those obtained by the simplified Bishop method. However, it should be noted that the Bishop simplified method is limited to slip surfaces of circular shape.
2.2.8. OTHER ‘GPS’ METHODS

In the last two decades a number of other variations of the limit equilibrium approach have been developed by researchers and geotechnical engineers. Each method involves some assumptions concerning the relationship between interslice shear and normal forces.

Chen and Morgenstern (1983) assumed that the relationship between interslice shear and normal forces can be expressed by the following expression:

\[ T = [f_0(x) + \lambda f(x)] E \]  \hspace{1cm} (2-5)

where, \( f(x) \) and \( f_0(x) \) are prescribed functions subject to the constraints that \( f_0(x) \) satisfies the given boundary conditions and that \( f(x) = 0 \) applied at the boundaries. By choosing

<table>
<thead>
<tr>
<th>Method</th>
<th>Interslice Force</th>
<th>Equilibrium Equation</th>
<th>Calculation of Factor of safety</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fellenius</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Simplified Bishop</td>
<td>x</td>
<td>o</td>
<td>x</td>
</tr>
<tr>
<td>Simplified Janbu</td>
<td>x</td>
<td>o</td>
<td>0</td>
</tr>
<tr>
<td>Rigorous Janbu</td>
<td>o</td>
<td>o</td>
<td>0</td>
</tr>
<tr>
<td>Spencer</td>
<td>o</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>M. &amp; P.</td>
<td>o</td>
<td>o</td>
<td>o</td>
</tr>
<tr>
<td>Sarma</td>
<td>o</td>
<td>o</td>
<td>o</td>
</tr>
</tbody>
</table>

- Consideration  × - No consideration
If \( f_0(x) = 0 \), the method reduces to the Morgenstern and Price method. If \( [f_0(x) + \lambda f(x)] \) is equal to a constant or zero, Spencer’s method or Bishop’s method can be derived.

Hardin and Hardin (1984) proposed the following assumption concerning the effective interslice forces:

\[
T = \lambda E' = \lambda (E - \Delta U^s)
\]  

(2-6)

where, \( U^s \) is the total thrust due to water pressure acting on an interslice boundary.

Madej (1984) represented the vertical interslice force by the following equation

\[
T = \lambda g(x)
\]  

(2-7)

where, \( g(x) \) is a function describing the variation of \( T \) from slice to slice. Some suggestions for the function \( g(x) \) are also given by Madej (1984). A similar method has been proposed by Correia (1988).

Lowe and Karafiath (1960), Bell (1966), Whitman and Bailey (1967), Sarma (1973), Baker and Garber (1978), Chugh (1981), Morrison and Greenwood (1984), Miller and Hamilton (1989), and Leshichinsky (1990) also proposed different assumptions for the relationship between interslice shear and normal forces as well as different analysis procedures. Li (1991) presented a so-called unified solution in which several existing slope stability methods are covered. Although numerous limit equilibrium methods of slope stability analysis have been proposed, the major differences between them relate only to different assumptions concerning the relationship between interslice shear and normal forces. Numerical procedures for calculating the factor of safety based on either force equilibrium or moment equilibrium or both can also be different for different methods.
2.3. DETERMINATION OF CRITICAL FAILURE SURFACES IN SLOPES

The conventional deterministic analysis of slope stability involves two important aspects. The first concerns the definition and calculation of the factor of safety for any specified or trial slip surface. The second aspect is the search for the critical slip surface, i.e., the slip surface associated with the minimum value of the factor of safety. Different methods for finding the critical slip surface have been developed and each method has its advantages and shortcomings. The available methods can be classified into four categories as follows:

(i) theoretical calculation methods based on the calculus of variations
(ii) conventional methods associated with repeated trials, i.e., grid search methods
(iii) direct search optimisation methods-unconstrained optimisation
(iv) gradient methods of unconstrained optimisation.

The basic concepts and main characteristics of these methods are briefly presented below.

2.3.1. THE METHODS OF CALCULUS OF VARIATIONS

The calculus of variations provides mathematical procedures to study maxima or minima of functionals instead of functions. In the case of stability of slopes, if the function \( y(x) \) represents the slip surface then the factor of safety corresponding to this function \( y(x) \) can be considered as a functional of \( y(x) \). Revilla and Castillo (1977) proposed a functional expression of the factor of safety as follows in terms of the ratio of integrals of functions S and G:
Euler's equation associated with Eq. (2 - 8) was presented as follows:

\[
F = \int_{x_0}^{x_1} \frac{S(x, y, y') \, dx}{G(x, y, y') \, dx} \quad (2 - 8)
\]

Garber (1973) and Baker and Garber (1977a, 1977b and 1978) presented a series of papers concerning the application of calculus of variations for locating the critical slip surface. The functionals, derived from equation of force and moment equilibrium, were minimised analytically with respect to the location \( y(x) \) of the slip surface and the normal stress distribution \( \sigma(x) \) along it.

Variational approaches have been the subject of controversy and it has been suggested by some critics that the associated solutions do not deal with statical equilibrium correctly. Others have questioned the mathematical formulation. Continued research and investigation is necessary in order to establish these methods formally.

2.3.2. CONVENTIONAL METHODS BASED ON REPEATED TRIALS

The grid-search method was the most commonly used method for locating critical slip surfaces before optimisation methods were introduced. Such methods involve an interpolation algorithm and are suitable only for circular slip surfaces. A critical circular slip surface with minimum factor of safety can be located easily but convergence is slow. While the grid search method is not suitable for slip surfaces of arbitrary (non-circular)
shape, repeated trial approach may be used in some other way for these general slip surfaces.

Celestino and Duncan (1981) used a method for searching critical circular slip surfaces and arbitrary slip surfaces which is known as an alternating-variable approach. Each location parameters is chosen in turn while all the other location parameters are kept constant. The relative minimum of the objective function (in this case, a function with a single variable) can then be calculated. The location parameter is then held constant at this 'conditioned' minimum point and the procedure is repeated for the next variable. Once all the location parameters have been searched, the process is repeated until the required accuracy is achieved. Li and White (1987a) adopted a modified alternating-variable method to locate critical slip surfaces based either on the minimum factor of safety or on the maximum failure probability. The alternating-variable method sometimes yields false minima since convergence depends on the shape of the optimal surface and the initial choice of parameters even though the optimal surface may be unimodal with a single optimum region (Beightler et al., 1974).

Other conventional methods are the random search scheme which was employed by Siegel (1975), Boutrop (1977), Boutrop and Lovell (1980) and Oboni and Bourdeau (1983) as well as the dynamic programming method which was used by Baker (1980).

2.3.3. DIRECT SEARCH METHODS

Direct search methods are based on unconstrained optimisation techniques. The major advantage of the direct search methods is that the partial derivatives of objective function are not required. In 'rigorous' slope stability methods, the objective function, i.e. factor of safety, cannot be expressed explicitly in terms of the coordinates of the nodal points defining a slip surface. Thus the partial derivatives of the objective function cannot be determined analytically.
The simplex reflection algorithm is the most popular direct search method. It was originally devised for specialised application by Spendley et al. (1962) and was modified by Nelder and Mead (1965) as well as Parkinson and Hutchinson (1972) to solve general unconstrained optimisation problems. The general framework and methodology for locating the critical slip surface has been presented by Nguyen (1984 and 1985), Hachich et al. (1985), Denatale and Awad (1987), Chen and Shao (1988), Chowdhury and Zhang (1988), Zhang (1990) and DeNatale (1991).

In general, the simplex reflection algorithm for searching the critical failure slip surface with \( n \) unknown parameters involves the following steps:

1. Set up an initial simplex with \((n+1)\) vertices equidistant to each other. Each vertex \( i \) of the simplex is a point defined by \( \mathbf{X}_i = (x_1, x_2, \ldots, x_n) \), in which \( \mathbf{X}_i = \) vector or coordinate set comprising the geometrical variables defining the slip surface;

2. At each of the \((n+1)\) vertices in turn, by using the relevant vector at each vertex evaluate optimal functions \( F = f(\mathbf{X}) \), in which \( F = \) factor of safety;

3. Check reflection constraints: (a) If the reflection in step 2 results in a new vertex with function \( F \) still the largest in the new simplex, further reflection will return to the exact position of the old vertex, and the reflection algorithm is locked in a vacillation loop. (b) Alternatively, if the reflection results in a new vertex with infeasible coordinates, it is then not possible to evaluate function \( F \) at this vertex. (c) If a single vertex remains stationary at one point after many successive reflections, simplex reflection merely circle the stationary vertex;

4. Repeat the simplex reflections in the manner described in the steps 2 and 3 until convergence.

The other direct search algorithm is Powell's (1964) conjugate direction approach which was first applied by Xu (1988) for locating the critical slip surfaces based on the minimum factor of safety. In this powerful method, the conjugate directions are progressively constructed by an iterative procedure and partial derivatives of the objective
function are not required. The conjugate direction method has the quadratic convergence for quadratic objective functions with $n$ independent variables. Therefore Powell's conjugate direction approach has rapid convergence.

2.3.4. GRADIENT METHODS OF UNCONSTRAINED OPTIMISATION

Gradient methods of unconstrained optimisation involve the steepest descent methods, quasi-Newton or variable matrix methods and conjugate gradient methods. The main characteristic of gradient methods is that partial derivatives of objective function $f(X)$ are required for performing the optimum calculations. The convergence rate usually is higher than that of other optimisation algorithms mentioned above.

The steepest descent method is usually called the negative gradient algorithm and the iterative procedure is the most simple among the gradient methods. Its descent direction points directly toward the minimum only when the contours of objective functions are circular and therefore the negative gradient is not a good global direction for nonlinear objective functions. The convergence rate of the negative gradient is linear and, therefore, slow.

Newton's method can overcome the shortcoming in the negative gradient method and Magasarian (1971) has shown, under rather mild regularity condition on objective functions $f(X)$, that Newton's method exhibits a quadratic rate of convergence. However, Newton's method involves the inverse of the Hessian matrix, $H^{-1}$, which consist of the second order partial derivatives of $f(X)$. Accordingly, the convergence rate is still slow when the dimension of $f(X)$ is large.

The conjugate gradient method was originally devised by Hestenes and Steifel (1952) for solving a system of linear algebraic equations. Unlike the method of steepest descent, the search vectors are not equal to the negative gradient vectors. A sequence of search vectors is determined in such a manner that each search vector is a function of both the current gradient vector and the previous search vector. When $f(X)$ is a quadratic
function with \( n \) independent variables, the convergence rate of this method is quadratic, i.e., the number of required iterations is no larger than \( n \) (the number of arguments of \( f(X) \)). It is, therefore, more effective than either the negative gradient method or Newton's method. Various conjugate gradient algorithms have been developed but the one proposed by Fletcher and Reeves (1964) is the most effective conjugate gradient algorithm. The details of F-R algorithm and its application to slope stability problems will be introduced in Chapter 4.

The principle advantages of quasi-Newton methods (or modified Newton methods) are that the convergence rate is fast and the inverse of the Hessian matrix is not required. The basic idea of such a method is to construct a sequence of matrices \( \{H_k\} \) to approach the inverse of the Hessian matrix, \( H^{-1} \). Several quasi-Newton methods have been proposed. The most effective algorithm, referred to as 'DFP', was first developed by Davidon (1959) and modified by Fletcher and Powell (1963).

Numerical procedures based on the steepest descent and 'DFP' methods were reported by Chen and Shao (1987) for determining critical slip surface with minimum factor of safety. In these calculation procedures, the method of slices presented by Chen and Morgenstern (1983) was employed to calculate the factor of safety which is defined as the objective function. The partial derivatives of objective function \( f(X) \) were approximated by the finite difference technique.

The F-R algorithm was first used by Xiong and Xu (1985) for locating critical slip surfaces associated with slope stability. However, the objective function was based on simple limit equilibrium models. Relatively 'rigorous' methods of limit equilibrium were not considered in the original paper. The partial derivatives of objective function \( f(X) \) were approximated by the finite difference technique. Considerable development was required so that the F-R method can be used for realistic slope stability assessment including slip surfaces of arbitrary shape and non-homogeneous or layered slopes. These
developments, which also include to search critical slip surface based on minimum critical seismic coefficients, are discussed fully in Chapter 4.

2.3.5. DETERMINATION OF MAXIMUM FAILURE PROBABILITY

As point out by Tobbutt and Richards (1979), a critical slip surface based on the minimum factor of safety may not be the surface with the maximum failure probability. Li (1987d) and Chowdhury and Zhang (1988) used optimisation techniques to search the critical slip surface with the minimum reliability index or the maximum failure probability. Their studies showed that the critical slip surfaces with the minimum factor of safety which is associated with mean values of geotechnical parameters may be quite different in location to that based on the minimum reliability index or the maximum failure probability. The location of a critical slip surface associated with the minimum reliability index is, in general, different from the one associated with the maximum failure probability. However, the highest failure probability does correspond to the lowest reliability index for normal or Gaussian probability distribution of the factor of safety. Procedures for calculating failure probability, based on arbitrary probability distribution functions, can be very involved and complex. Therefore, reliability index is often used as the objective function instead of the failure probability.

2.4. PROBABILISTIC ANALYSIS AND SYSTEM RELIABILITY ASSESSMENT OF SLOPES

There are many significant uncertainties associated with slope stability analysis. Moreover, there are measurement and testing errors related to estimation of geotechnical parameters. There are also uncertainties associated with slope stability models. Consequently, probabilistic methods of analysis of slopes have been developed to complement deterministic approaches.
2.4.1. INHERENT VARIABILITY OF GEOTECHNICAL PARAMETERS

Inherent variability of shear strength parameters is an important factor leading to uncertainty in slope analysis. Uncertainties related to pore water pressure can also be significant.

The inherent variability of geotechnical parameters (e.g. shear strength parameters) can be described by following three models, i.e.

(i) statistically homogeneous model with lumped parameters (e.g. Matsuo and Kuroda, 1974; Tobutt and Richards, 1979; Chowdhury, 1980). In this model, only the mean and coefficient of variation of each geotechnical parameter are used to describe its variability. Therefore, spatial variability is not accounted for.

(ii) random field model based on statistical treatment of spatial correlations (e.g. Alonso, 1976; Yong et al., 1977; Baecher and Einstein 1979). In this model, any variable \( \varepsilon(x, y) \) is referred to as a random variable in a stationary random field with a zero mean, \( E[\varepsilon(x, y)] = 0 \). The joint distribution properties between \( \varepsilon_i(x_i, y_i) \) and \( \varepsilon_j(x_j, y_j) \) are constant. The auto-correlation function can be defined as \( R(r) = \text{Cov}(\varepsilon_i, \varepsilon_j)/[\text{Var}^{1/2}(\varepsilon_i) \text{Var}^{1/2}(\varepsilon_j)] \) and \( r \) is the vector separation distance between points \( (x_i, y_i) \) and \( (x_j, y_j) \);

(iii) random field model based on the consideration of the local average process. This model was proposed by Vanmarcke (1977a and 1984). The variance function (or reduction factor) and the scale of fluctuation (or correlation distance parameter) are used to describe correlation structure of random field. Some probabilistic studies of slope stability based on this model have been reported (e.g. Vanmarcke, 1977b and 1980; Chowdhury, 1980 and 1984; Asaoko and Grivas, 1982; Tang, 1984; Chowdhury et al., 1987; Li and White, 1987c; Chowdhury, 1992; Chowdhury and Zhang, 1993).
2.4.2. DEFINITION OF FAILURE PROBABILITY

The factor of safety (FOS) is considered as an index for the assessment of slope stability within a deterministic framework. Similarly the probability of failure is considered as an index of instability within a probabilistic framework. The most commonly used definition of the probability of failure is 'the probability that the factor of safety is less than unity'. Denoting factor of safety by FOS, the probability of failure for a slope can be defined as follows:

\[ P_f = P\{FOS \leq 1\} \quad (2-10a) \]

or,

\[ P_f = P\{(FOS - 1) \leq 0\} \quad (2-10b) \]

The factor of safety is a function of several random variables which include shear strength parameters, unit weight, surcharge load, pore water pressure, and etc. The factor of safety is often considered as the performance function in probabilistic analysis (e.g. Alonso, 1976; Vanmarcke, 1980; Baecher, 1983; Midler, 1984; Whitman, 1984; Nguyen, 1985; Zhang, 1990).

An alternative to the 'factor of safety' as performance function is the 'safety margin' (SM) which may be defined as the difference of resistance and load or as the term \((FOS - 1)\). If SM is less then zero, failure of sloping earth mass is indicated. Accordingly, the probability of failure may be defined as

\[ P_f = P\{SM \leq 0\} \quad (2-11) \]

Some researchers have considered the safety margin as the performance function for certain types of probabilistic analysis (e.g. Catalan and Cornell, 1976; Chowdhury, et al., 1987; Li, 1987d). The adoption of this definition may not lead to a consistent approach in using deterministic and probabilistic methods to complement each other for slope stability assessment.

An alternative to probability of failure as a measure of slope stability is the reliability index. Two definitions of reliability index have been proposed. The first one is
the conventional reliability index $\beta$ defined by Cornell (1969). The second one is the so-called 'invariant' reliability index defined by Hasofer and Lind (1974). The definitions and formulations concerning these two alternatives are discussed in Chapter 5.

Probabilistic analysis of slopes requires to assume the probability distribution function of the basic random variables of the performance function and the commonly used distribution are normal (Gaussian), lognormal and beta distributions. The application of beta distribution in slope probability analysis has been discussed by Lumb (1970), Chowdhury and DeRooy (1985) and Li and White (1987c). Four parameters, which determine the scale and shape parameters of the beta function, are required for defining the beta distribution. Assumption of a normal or a lognormal distribution is often preferred because only two statistical parameters of the performance function are needed to describe the distribution and to evaluate the reliability index.

2.4.3. ANALYSIS OF PROBABILISTIC METHODS

pioneered by Chowdhury and others (e.g. Chowdhury, 1980; Chowdhury and Grivas, 1982; Grivas and Asaoka, 1982; Tang et al., 1985; Chowdhury and DeRooy, 1985; Chowdhury et al., 1987; Grivas and Chowdhury, 1988; Chowdhury and Zhang, 1990; Chowdhury, 1992; Chowdhury and Zhang 1993). Probabilistic studies of rock slopes along rock discontinuities as well as of mining spoil piles were reported by Young (1977), Herget (1978), Kim et al. (1978), Major et al. (1978), Marek and Sevely (1978), Pentz (1981), Einstein et al. (1983), Miller (1984), Nguyen and Chowdhury (1984,1985), Call (1985), Piteau et al. (1985), Savely (1985), and Chowdhury (1986, 1987).

Probabilistic approaches in slope stability analysis are customarily based on deterministic limit equilibrium models with certain input parameters being treated as random variables. A limit state function or performance function is first derived on the basis of a valid limit equilibrium model and then, by means of analytical or numerical approaches, the probability of failure is evaluated. The performance functions associated with relatively ‘rigorous’ limit equilibrium methods are inexplicit and it is, therefore, necessary to use approximate numerical methods for calculation. Three types of numerical approaches are used for geotechnical analysis within a probabilistic framework, namely:

(i) The first order second moment method (FOSM);
(ii) Point estimate method (PEM);
(iii) Monte Carlo simulation method (MCSM).

In the first order second moment method, the performance function is expanded by a Taylor series at mean values of the basic random parameters and is then approximated by first two terms of the series. The methodology has been developed and used for the analysis of slope reliability only during the last two decades. Calculation of partial derivatives of the performance function with respect to the basic random variables is necessary in this method. Based on the FOSM method, several variations to the
calculation procedures have been proposed. For example, Yucement et al. (1973), Lee et al. (1983), and Bao and Yu (1985) proposed the analysis procedures on the basis of ordinary method of slices; Alonso (1976), Tobutt and Richards (1979), and Felio et al. (1984) formulated the solution procedures based on the simplified Bishop method. Based on a relatively 'rigorous' procedure of slices, Li (1987d) proposed a model of the performance function formulated in terms of safety margin (SM) so that the derivatives of the performance function can be evaluated analytically. A probabilistic analysis procedure, in which the performance function is formulated in terms of the factor of safety based on relatively 'rigorous' limit equilibrium methods, is developed in Chapter 5 of this thesis.

A point estimate method was proposed by Rosenblueth (1975). The statistical moments of the factor of safety of slopes can be approximated by the sum of several weighted 'point' estimates. Rosenblueth (1975) derived expressions for the point estimates and weighting factors consistent with probability theory. Rosenblueth's method involves a total of $2^n$ evaluations of the factor of safety for a performance function with $n$ random variables. The number of evaluations increases exponentially with $n$. Accordingly Rosenblueth's method becomes impracticable when $n$ is large. In slope stability analysis, the number of random variables may be usually large, especially if a soil profile is modelled as a random field. Applications of the Rosenblueth method to probabilistic analysis of slopes have been reported by many researchers (e.g. DeRooy, 1980; Matsuo and Asaoka, 1983; Nguyen and Chowdhury, 1984, 1985; Wolff, 1985; Wolff and Harr, 1987; Chen and Usmen, 1987; Chowdhury and Zhang, 1988; Zhang, 1990; Wolff, 1991). For a performance function with small number of random variables, the point estimate method is a very effective approximate technique to estimate the failure probability. For large number of variables modified point estimates approaches have been suggested by Li (1991).

Monte Carlo simulation is one of the most commonly used simulation techniques in the probabilistic analysis of slopes. (e.g. Kraft and Mukohay, 1977; Major et al, 1977
and 1978; Kim et al, 1978; Marek and Savely, 1978; Tobutt, 1982; Einstein et al, 1983; Miller, 1984; Nguyen and Chowdhury, 1984). In this simulation technique, the probability distributions of the contributing random variables must be known or assumed. There are no limitations to the shape of these probability distributions. Therefore, Monte Carlo simulation is applicable to any problem of probabilistic analysis whatever the adopted limit equilibrium method. However, Monte Carlo simulation is inefficient and cumbersome when calculating probabilities associated with rare events as the number of trials must substantially exceed the inverse of the accuracy of the calculation required (Shooman, 1968; Ang and Tang, 1984). However, in general, it is a very useful technique of simulation and has been used successfully for slope stability problems. However, enough attention has not been given to correlated variables. A Monte Carlo simulation procedure considering the correlations between basic random variables is presented in Chapter 5.

2.4.4. SYSTEM RELIABILITY OF SLOPES

For a structural or geotechnical system with several components, the overall reliability will depend not only on the reliabilities of individual components but also on other factors. These factors include the number of components and the correlations between them or between parameters defining them. It has been pointed out by Cornell (1971) that the reliability of a slope should consider all the slip surfaces and not just the critical slip surface (i.e. the one with the lowest factor of safety or reliability). If the safety factors for different slip surfaces are highly correlated, the failure probability should approach that of the critical slip surface. However, if the safety factors are poorly correlated the upper and lower bounds of the failure probability may be quite different. A recent paper proposed by Oka and Wu (1990) presented system reliability analysis for a particular example in which the factor of safety of several slip surfaces are poorly correlated. The selected example involves only a linear and explicit performance function
based on the "$\phi = 0$" assumption (applicable to saturated clays under undrained conditions). The results indicate that the upper bound of system failure probability can be significantly higher than the failure probability associated with a critical slip surface.

It is important to note that the shear strength of soil layers was considered to be independent and therefore, the results are considered to be conservative. In fact soil layers belonging to the same geological formation will have some degree of correlation in the strength properties. With such a correlation the upper bound will reduce. Moreover, as Oka and Wu (1990) pointed out the results also depend on the choice of layers. The division between clay layers in the example cited was somewhat arbitrary. If the two clay layers were considered as one single layer whose strength increases with depth, the upper bound failure probability would again be lower than that calculated on the basis of two layers with independent shear strength parameters.

The development of a more comprehensive reliability analysis procedure for slope systems is required and such a procedure is introduced in Chapter 6. In such a procedure, both circular or non-circular slip surfaces are included and correlations between basic random variables are considered systematically.
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2.5. GENERAL REMARKS

Many parts of the world have been subjected to earthquake disturbances at one time or another and earthquakes often cause damage to structures and facilities. Moreover, some earthquakes cause loss of life, disruption of communications and the movement of natural slope masses resulting in landslides. For example, the most recent earthquake in Australia occurred on 28th December 1989 near Newcastle, NSW and resulted in the death of 12 people and hundreds of injuries. Thousands of homes and buildings were destroyed or seriously damaged. This earthquake had a magnitude of 5.6 on the Richter Scale (Poulos, 1991). Embankments and earth dams may also fail during an earthquake. A famous example is the San Fernando earthquake of February 9, 1971 which caused a slide in the upstream slope of the Lower San Fernando Dam. The slide movement resulting from the earthquake shaking involved both the upstream slope and the upper part of the downstream slope, leaving about 1.5 m of freeboard in a very precarious position (Seed et al., 1975).

The design and analysis of earthquake-resistant earth structures (e.g. earth structures capable of resisting a specified earthquake loading without complete failure) have been a major concern of geotechnical engineers for many years. Research related to earthquake effects and their simulation or prediction has been developing rapidly during the last few decades. The design of earth structures to safely withstand the destructive effects of earthquakes can be improved if their response to earthquake shaking is fully understood. However, the analysis of earth structures for earthquakes is quite a complex task. Sudden ground displacements occur in embankments or earth dams due to the inertia forces resulting from earthquakes. A slope of an embankment or earth dam is subjected to forces which alternate in direction many times during an earthquake and it is
necessary to determine the effects of these pulsating stresses, superimposed on the initial, static stresses.

Various ways in which an earthquake may cause failure of an earth dam or embankment were summarised by Seed (1979) as follows:

(a) Disruption of a dam by major fault movement in the foundation
(b) Loss of freeboard due to differential tectonic ground movement
(c) Slope failures induced by ground motions
(d) Loss of freeboard due to slope failure or soil compaction
(e) Sliding of a dam on weak foundation materials
(f) Piping failure through cracks induced by ground motions
(g) Overtopping of a dam due to seiches in the reservoir
(h) Overtopping of a dam due to slides or rockfalls into the reservoir
(i) Failure of spillway or outlet works

During an earthquake, the inertia forces in certain zones of an embankment or earth dam may be sufficiently large to reduce the factor of safety below unity several times, but only for brief periods of time. During such periods, permanent displacements occur but the movement may be arrested as the magnitude of deformation reduces to zero after it reaches a temporary peak. The overall effect of a series of large but brief inertia forces may well be cumulative displacements of an earth structure which are spatially variable. However, once the ground motions generating the inertia forces have ceased, no further deformation may occur unless the soil strength has been decreased significantly. The magnitude of deformations that develop will depend on the time history of the inertia forces. A logical method of design requires

(a) a determination of the variation of inertia forces with time
(b) an assessment of the embankment or earth dam deformation induced by these forces (Seed and Martin, 1966).
As stated by Marcuson et al. (1992), an estimate of deformation of an earth structure caused by earthquake shaking is not a solved problem and will be the focus of a great deal of research effort during the current decade, the 1990's.

2.6. DYNAMIC RESPONSE ANALYSIS METHOD

Many researchers have devoted considerable effort to developing methods for the analysis of the stability of earth structures during earthquakes. Only three decades back, the standard method of evaluating the performance of earth dams during earthquakes was based on the pseudo-static approach. In fact, this type of analysis is still performed by geotechnical engineers although it is now recognised that the results are often unreliable and sometimes misleading. In this method the effects of an earthquake on a potential slide mass are represented by an equivalent static horizontal force determined by the product of a seismic coefficient, K, and the weight of the potential slide mass. Good experience and engineering judgement is required to select the magnitude of K which should, of course, be related to the intensity and duration of the relevant design earthquake. Independent estimates have led to the following approximate values (Seed, 1979):

- Severe earthquakes, Rossi-Forel scale IX  \( K = 0.1 \)
- Violent, destructive earthquakes, Rossi-Forel scale X  \( K = 0.25 \)
- Catastrophic earthquakes  \( K = 0.5 \)

Seed and Martin (1966) used the shear beam analysis to study the dynamic response of embankments to seismic loads and presented a rational method to predict seismic coefficients and their variation with time for the design of earth dams. In order to study the failure of the Sheffield Dam in Santa Barbara, California, during the earthquake of June 29, 1925, Seed et al. (1969) used not only the conventional pseudo-static approach, but also dynamic analysis in which the changes in stress caused by the earthquake were estimated on the basis of the limit equilibrium concept, combined with data from cyclic loading triaxial compression soil tests. An estimate was then made of
the required values of seismic coefficients leading to a factor of safety of unity. The range
of seismic coefficients thus estimated was 0.15 to 0.21 based on the conventional pseudo-
static method and 0.17 to 0.19 based on dynamic analysis with soil test data from cyclic
loading triaxial compression tests. Comparison of values of the seismic coefficients,
which would predict failure along slip surfaces approximating those on which failure
actually occurred, was considered a convenient way of comparing different types of
analyses. Ambraseys and Sarma (1967) adopted a similar procedure to study the response
of embankments to a variety of earthquake motions.

Newmark (1965) pointed out that the conventional pseudo-static analysis based on
the peak acceleration may not be sufficient for determining the response of earth dams or
embankments to earthquakes. The study of velocities, deformations and differential
displacements of the ground, leading to fissures in the ground surface, may be of equal or
even greater importance than the computation of a factor of safety. Therefore, new
methods were suggested by Newmark (1965) and Seed (1966) for estimating the
permanent displacements of dams subjected to earthquake shaking. Obviously
displacements during an earthquake should be among the criteria of performance of an
earth structure. Unfortunately, the pseudo-static approach gives no idea of expected
deformations. Newmark's approach retains the limit equilibrium concept for the
calculation of the minimum factor of safety. However, it goes beyond that and enables
estimation of permanent deformations based on simple concepts of dynamics. Following
Newmark's approach based on a sliding block model, a number of researchers developed
procedures for estimating earthquake-induced permanent displacements of earth dams or
embankments, e.g., Goodman and Seed (1966), Sarma (1975), Makdisi and Seed
(1991a,b).

All these developments are based on the sliding block model originally proposed
by Newmark (1965) and use the concept of critical or yield acceleration calculated within
the conventional limit equilibrium framework. Simultaneously however, stress-deformation approaches based on the finite element method have also been developed. A great deal of research effort has led to sophisticated dynamic finite element models. For instance, static as well as dynamic finite element methods were used to study the distribution of dynamic shear stresses in different parts of an earth structure using the 'Seed-Idriss-Lee' approach (Seed, 1979). In this approach, laboratory tests on embankment soils are also performed (e.g. cyclic triaxial or cyclic simple shear tests) to simulate the stress conditions predicted by the stress-deformation studies already motioned. Pore pressure development is monitored during these tests which also enable determination of the number of cycles required for liquefaction of the soil samples under different stress conditions. Based on a series of these tests, the potential strength loss in different sections of an embankment can be determined and, in particular, zones which are expected to liquefy can be identified. Adopting appropriately reduced strength values in the identified zones, static limit equilibrium studies are performed to determine the post-earthquake factor of safety. If these analyses show that the embankment is unlikely to fail (post-earthquake factor of safety greater than unity), Newmark-type analyses are also carried out to determine the expected embankment deformations.

Two dimensional finite element methods have been developed to evaluate the strain potentials and permanent deformations within earth dams and embankments subjected to seismic excitations, e.g., Clough and Chopra (1966), Idriss and Seed (1967), Idriss et al. (1973), Lee (1974), Seed et al. (1975), Serff et al. (1976), Chaney (1979), Taniguchi et al. (1983), Paskalov (1984), Prevost et al. (1985), Elgamal et al. (1987), Ishihara et al. (1990).

On the basis of the Newmark-type approach, alternative calculation procedures for earthquake-induced permanent displacements of earth dams or embankments are briefly reviewed below.
2.6.1. CALCULATION PROCEDURES BASED ON THE NEWMARK APPROACH

The basic elements of a procedure for evaluating the potential deformations of an embankment or earth dam due to earthquake shaking were presented by Newmark (1965). Permanent movement of a slope is considered to occur if the inertia forces on a potential slide mass are large enough to overcome the yield resistance and permanent movements are assumed to cease when the inertia forces are reversed, i.e., change direction.

Newmark recognised that the character of the dynamic response of different types of materials would be somewhat different. In general, for slopes of both cohesionless and cohesive soils, movements were considered to occur along well-defined planes or curved surfaces. However, in highly cohesive materials, the dynamic response may be nearly elastic in character, and a well-defined sliding surface may not develop. The main steps in Newmark's approach are summarised below:

Determination of Critical Seismic Coefficient

In order to determine the yield resistance of a potential sliding mass, three cases are considered by Newmark as follows:

(a) circular sliding surface
(b) plane sliding surface
(c) block sliding along a horizontal interface with a weak soil layer (See Fig. 2-1)

a. Circular Cylindrical Sliding Surface

For a potential sliding mass shown in Fig. 2-1(a), the following equation or the

*Foot note: $K_c$ will denote critical or yield acceleration coefficient in later chapters of this thesis. However Newmark used $N$ or $N'$ Goodman and Seed as well as Sarma used $K_y$. 
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(a) Circular Sliding Surface

(b) Plane Sliding Surface

(c) Block Sliding

Fig. 2-1 Forces Acting on A Potential Sliding Mass - Three Cases
(after Newmark, 1965)
critical seismic coefficient, $N$, was derived by Newmark from consideration of moment equilibrium:

$$ N = \frac{b}{h} \left( \frac{s_q}{\tau} - 1 \right) $$  \hspace{1cm} (2-12)

in which $s_q$ is the undrained shear strength and $\tau$ is the shear stress. This equation applies to cohesive soil under fully undrained conditions.

b. **Plane Sliding Surface**

For cohesionless and free-draining materials, with an inclined sliding surface, as in Fig. 2-1(b), the critical seismic coefficient was derived by Newmark in the following form

$$ N = (\text{FOS} - 1) \sin \theta $$  \hspace{1cm} (2-13)

where FOS is the static factor of safety and $\theta$ is the inclination of the planar slip surface.

c. **Block Sliding on Horizontal Surface through Weak Soil Layer**

For block sliding along a plane surface such as z-z in Fig. 2-1(c), the critical seismic coefficient was derived in the following form by Newmark from consideration of force equilibrium:

$$ N = \frac{s_q}{p'} (1 - r_u) $$  \hspace{1cm} (2-14)

where $r_u$ is the pore pressure ratio and $p'$ is the normal effective stress. The parameter $r_u$ may be determined as an average from the following:

$$ r_u = \frac{\sum u_p ds}{\sum \gamma h ds} $$  \hspace{1cm} (2-15)

in which $u_p$ is the pore water pressure at any point along the potential sliding surface and $ds$ represents small length of this surface. The effective overburden pressure is denoted
in Eq. (2-14) by \( p' = (\gamma h - u_p) \) and \( s_q \) denoted the drained shear strength of the creak layer.

**Sliding of A Rigid-Plastic Mass**

A simple procedure for estimating earthquake-induced permanent displacements of a rigid-plastic mass was developed by Newmark. The calculation is based on the assumption that the whole moving mass moves as a single rigid body with resistance mobilised along the sliding surface.

The acceleration considered was assumed to be a single rectangular pulse of magnitude \( A_g \), lasting for a time interval \( t_0 \). The maximum velocity corresponding to such an acceleration has a magnitude \( V \) given by the expression

\[
V = A_g t_0
\]  

(2-16)

The velocity response is shown as a function of time in Fig. 2-2.

![Fig. 2-2 Velocity Response to Rectangular Block Acceleration](after Newmark, 1965)

After the \( t_0 \) is reached, the velocity remains constant. The velocity corresponding to the resisting acceleration has the magnitude \( Ngt \) shown by the dashed line. At a time \( t_m \), the two velocities are equal and the net velocity becomes zero, or the body comes to rest relative to the ground. The value for \( t_m \) is obtained as follows by equating the velocity \( V \) to the quantity \( Ngt \),
\[ t_m = \frac{V}{Ng} \] (2-17)

The maximum displacement of the mass relative to the ground, \( u_m \), is obtained by computing the shaded triangular area in Fig. 2-2. Thus the expression for maximum deformation can be written in the form

\[ u_m = \frac{V^2}{2gN} \left(1 - \frac{N}{A}\right) \] (2-18)

The result given in equation (2-18) generally overestimates the relative displacement for an earthquake because it does not take into account the pulses in opposite directions. It also does not take into consideration the energy loss due to viscous damping. However, it should give a reasonable order of magnitude for the relative displacement. It does indicate that the displacement is proportional to the square of the maximum ground velocity.

For fairly stable slopes, Newmark’s sliding block analysis can be used to estimate deformations during earthquake shaking. Once the deformations have been estimated it is possible to determine whether they exceed acceptable levels (Marcuson et al., 1992).

2.6.2. METHOD PROPOSED BY GOODMAN AND SEED

The validity of the fundamental principles of Newmark’s approach was demonstrated in model tests (shaking table test) by Goodman and Seed (1966). Their procedure for evaluating slope displacements during an earthquake involved two steps

(a) the determination of the threshold or yield acceleration, i.e., the acceleration at which sliding will begin to occur and (b) the evaluation of the displacements developed in time intervals when the yield acceleration is exceeded. The slope material was considered to be a dense and dry cohesionless soil.
Determination of the Yield Acceleration

For a cohesionless soil with a tendency to slide along an inclined plane, the yield acceleration, $K_yg$, may be expressed as follows:

$$K_yg = \frac{\sin(\phi - \alpha)}{\cos(\phi - \alpha - \theta)} g \quad (2-19)$$

in which, $\alpha$ is the slope inclination of slope, $\theta$ is the inclination of ground acceleration towards the slope below the horizontal and $\phi$ is the internal friction angle of soil.

For dense, cohesionless soils, the strength can best be expressed by the equation:

$$s = s_i + p \tan \phi \quad (2-20)$$

in which $s_i$ = shear strength at zero normal pressure which was considered to vary from 0.009 to 0.144 kN/m$^2$. Considering the effect of the intercept, $s_i$, the horizontal yield acceleration may be expressed by

$$K_yg = \tan(\phi - \alpha + \phi_{s1})g \quad (2-21)$$

or $$K_yg = \tan(\phi_{eq} - \alpha) g \quad (2-22)$$

in which $\phi_{eq} = \phi + \phi_{s1}$, and $\phi_{s1}$ = a correction factor which is a function of the shear strength intercept, $s_i$, and the slope length.

When actual acceleration exceeds the value $K_yg$, the extent of the ensuing displacements will depend on the nature of the ground motions and the stress-deformation relationship of the slope material.

**Stress-Deformation Characteristics of Cohesionless Soils**

Based on the results of the sliding block direct shear test, the variation of strength parameters, $s_i$ and $\phi$, with the magnitude of deformation may be expressed as a function of deformation as shown in Fig. 2-3. It may be seen from Fig. 2-3 that the shear strength intercept, $s_i$, decreases to zero after a relatively small deformation; the rate of decrease of $\phi$ is somewhat slower, but a much reduced value corresponding to the
residual strength is attained after a displacement of several grain diameters. Beyond this point the strength parameters are essentially constant.

**Procedure for Calculating Permanent Displacements**

The down-slope displacement, $x$, as a function of time during an earthquake inducing horizontal accelerations expressed by $K(t)g$ can be determined by approximating the sliding mass as a block of soil with an angle of friction, $\phi_{eq}$, subjected to the force system shown in Fig. 2-4. With reference to Fig. 2-4, the equilibrium equations may be obtained as follows:

\[
W \sin \alpha + K(t)W \cos \alpha - N \tan \phi_{eq} = \frac{W}{g} \frac{d^2x}{dt^2} \tag{2-23}
\]

and

\[
N = W \cos \alpha - K(t) W \sin \alpha \tag{2-24}
\]
Combining Eqs. (2-22), (2-23) and (2-24), one may get a differential equation of motion of the sliding mass, i.e.,

\[
\frac{d^2x}{dt^2} = B(x)[K(t) - K_y]
\]  

(2-25)

in which, \( B(x) = g \frac{\cos(\alpha - \phi_{eq})}{\cos \phi_{eq}} \) is a function of displacement \( x \) because it depends on mobilised shear strength parameters \( \phi_m \) and \( s_m \), both of which vary with displacement (\( \phi_{eq} \) has already been defined by Eq. 2-22).

To calculate the displacements of the sliding mass for this type of motion, numerical integration of Eq.(2-25) may be used. Designating \( t = 0 \) when the acceleration attains a magnitude equal to the yield acceleration, and noting that initial displacement and velocity are zero, the velocity at time \( t_j \) may be calculated, using the trapezoidal rule, as follows:

\[
V(t_j) = \sum_{i=0}^{j-1} B(x) \left( \frac{K(t_i) + K(t_{i+1})}{2} - K_y \right) \Delta t
\]

(2-26a)

and the displacement at time \( t_n \) is

\[
x(t_n) = \sum_{j=0}^{n-1} \frac{V(t_j) + V(t_{j+1})}{2} \Delta t
\]

(2-26b)
Eq. (2-26b) is valid from time $t = 0$ to time $t = t_n$ when the velocity is again zero. In Eq. (2-26a), $B(x)$ may reasonably be considered to have a constant value for any one acceleration cycle and thus Eq. (2-26) can readily be programmed and $x$ evaluated by computer. In such an analysis, the yield acceleration may be considered to vary in magnitude as displacements increase.

2.6.3. METHOD PROPOSED BY SARMA

Sarma (1975) also applied Newmark's approach to earthquake analysis and included the effect of the developed excess pore-pressure on the factor of safety, the critical acceleration and the displacement during an earthquake. The analysis model was based on limit equilibrium principles and the sliding mass was assumed to obey the Mohr-Coulomb failure criterion in terms of effective stresses.

Determination of Factor of Safety

The stability of a rigid block resting on a plane surface, as shown in Fig. 2-5, was examined by Sarma (1975). It was assumed that the surface is inclined at an angle $\beta$ to the horizontal and the block and the plane are separated by a thin layer of soil with an effective friction angle $\phi'$.

Due to the earthquake, an excess pore-pressure $\Delta u$ may develop increasing the initial pore water pressure $u_0$ to the total pore water pressure $u_d$:

$$u_d = u_0 + \Delta u$$  \hspace{1cm} (2-27)

A factor of safety is calculated in the following form:
\[ F_d = [\cos \beta - K \sin(\beta - \theta) - u_0a/W - \Delta u_0a/W] - \tan \phi'[\sin \beta + K\cos(\beta - \theta)] \]  

(2-28)

in which, \( a \) = the length of the plane. From Eqs. (2-28) and (2-29), it can be seen that if \( \Delta u \) can be evaluated, then \( F_d \) can be calculated.

The following hypothesis was introduced for determining the excess pore-pressure \( \Delta u \) in the Sarma method. If \( \sigma' \) and \( \tau' \) are respectively the effective normal and shear stresses on a potential failure plane and \( F \) is the factor of safety, then the state of stress at any point along the failure surface will be the same as if the friction angle of the material were \( \psi = \tan^{-1}(\tan\phi'/F) \). With this hypothesis, it is possible to draw the Mohr’s circles of stress. From this circle, the total principal stresses for the static and the dynamic condition can be obtained. By introducing the increments of the total principal stresses between the static and dynamic state, the dynamic pore-pressure \( \Delta u \) may also be related to \( \Delta \sigma_1 \) and \( \Delta \sigma_3 \) in the manner suggested by Skempton (1954) for static loading under undrained conditions. This assumes that the rotation of principal axes will not have any significant effect on the pore-pressure parameters \( A_n \) and \( B \). These parameters are evaluated at failure and are assumed to be constant during shearing. The relationship between \( \Delta u, \Delta \sigma_1, \Delta \sigma_3, A_n \) and \( B \) may be written as:

\[ \Delta u = B[\Delta \sigma_3 + A_n (\Delta \sigma_1 - \Delta \sigma_3)] \]  

(2-29)

where \( A_n \) is a parameter corresponding to \( n \) cycles of loading. The parameter \( A_n \) will have a different value for different stress paths. An empirical formula for estimating \( A_n \) was proposed by Sarma (1988) based on the results of cyclic loading tests performed by Martin et al., 1975. More details concerning this procedure will be considered in Chapter 8 of this thesis.

Sarma (1988) obtained an equation for \( F_d \) in quadratic form as follows:

\[ \frac{F_d}{\tan\phi'} + \frac{B\tan\phi'}{F_d} - B(1-2A_n)\sqrt{1+\tan^2\phi'/F_d^2} \]
\[
\cos \beta - K(1-B)\sin(\beta-\theta) + B\sin\beta[\tan\psi_0 - (1-2A_n)\sec\psi_0] - U_0a/w
\]
\[
\sin\beta + K\cos(\beta-\theta)
\]

(2-30)

The value of \( F_d \) may be obtained by solving the above equation.

**Calculation of Critical Acceleration**

Critical acceleration \( K_y \) is defined as that acceleration which, when applied to the block will produce a state of critical or limiting equilibrium (It is also called the yield acceleration). This implies that \( F_d = 1 \) when \( K = K_y \). By equation (2-29), therefore, critical acceleration may be expressed as follows:

\[
K_y = \frac{\cos \beta + B\sin \beta[\tan \psi_0 - (1-2A_n)\sec \psi_0] - P\sin \beta - U_0a/w}{(1-B)\sin (\beta - \theta) + P\cos (\beta - \theta)}
\]

(2-31)

where \( P = \cot \phi' + B\tan \phi' - B(1-2A_n)\sec \phi' \)

**Calculation of Displacement**

If earthquake acceleration is larger than \( K_yg \), the factor of safety will be less than one and the block will slide along the surface. It will come to rest some time after the earthquake acceleration has become less than \( K_yg \). The displacement depends on the magnitude and duration of the earthquake acceleration.

The driving force along in the direction of the sliding surface is:

\[
D = W[\sin \beta + K\cos(\beta - \theta)]
\]

(2-32)

The resisting force along the same surface but in the opposite direction is:

\[
R = W[\cos \beta - K\sin(\beta - \theta) - u_la/W]\tan \phi'
\]

(2-33)

where \( u_l = W[(\cos \beta - K_y \sin (\beta - \theta)) - [\sin \beta + K_y \cos (\beta - \theta)]\cot \phi'/a]
\]

(2-34)
The acceleration of the block relative to the plane surface is then given by

\[ \frac{W}{g} \ddot{x} = D - R \]  

(2-35)

Substituting for \( D \) and \( R \) from Eqs. (2-32) and (2-33), and for \( u_f \) from Eq. (2-35), a differential equation for the motion of the block may be obtained as follows:

\[ \ddot{x} = g \frac{\cos (\beta - \theta - \phi')}{\cos \phi'} \left[ K(t) - K_y \right] \]  

(2-36)

Eq. (2-36) is the same as that derived by Goodman and Seed (1966) when \( \theta = 0 \).

For solving Eq. (2-36) three simplified earthquake pulses were assumed in Sarma’s method and the formulations for calculating the permanent displacements corresponding to these three pulses were obtained. The shapes of these three pulses are shown in Fig. 2-6.

![Fig. 2-6 Three Simplified Earthquake Pulses (after Sarma 1975)](image)

In Fig. 2-6, \( T \) is the predominant period obtained from the acceleration spectrum of the earthquake record and \( K_m \) is the maximum acceleration of the record. Sarma pointed out that for \( K_y/K_m \) greater than 0.5, the triangular pulse and for \( K_y/K_m \) less than 0.5 the rectangular pulse would effectively give the displacement of a sliding block subjected to an earthquake.
2.6.4. CALCULATION PROCEDURE PROPOSED BY CHUGH (1982)

Considering Spencer's limit equilibrium method and the concept of rigid-plastic block motion proposed by Newmark (1965), Chugh developed another procedure for evaluating the earthquake-induced permanent displacements of an earth structure. The main steps in this procedure are as follows:

1) Use Spencer's method to calculate the yield or critical acceleration (considering a potential slip surface of circular or arbitrary)
2) Develop the equation of motion to estimate the displacement along the base of a vertical slice
3) Use a numerical integration method for calculating the relative displacement along the base of the slice.

Slope Stability Equation

The forces acting on a typical vertical slice are shown in Fig. 2-7. \( H_L \) and \( H_R \) are the hydrostatic forces and \( F_e = K(t)W \) is the earthquake force, which corresponds to a constant acceleration coefficient, \( K \), times that of gravity, acting at an inclination \( \theta \) to the horizontal and through the centre of mass of the slice. Considering static equilibrium of forces, the recursive formulation may be obtained from Spencer's method.

With known or assumed values of \( K \) and \( \theta \) as well as the boundary conditions for the first slice 1 (i.e. known \( Z_L \) and \( h_1 \)) one can determine the solution to the slope stability problem by suitably adjusting the pair of values \( (F, \delta) \) until the calculated values of \( Z_R \) and \( h_2 \) for the last slice agree with the known boundary conditions on that slice. This is an iterative procedure which converges quickly. By changing the value of \( K \) and repeating the calculation, one can determine the value of \( K \) for which value of factor of safety equals unity. This value of \( K \), corresponding to inclination \( \theta_{\text{yield}} \), is defined as the yield acceleration and denoted \( K_{\text{yield}} \). For different \( \theta \) values different values of yield
acceleration $K_y$ can be calculated. Thus a minimum yield acceleration corresponding to the critical inclination $\theta$ can be calculated.

\[
\text{Mobilized shear strength } = \frac{1}{F} [c' + \frac{N - U}{b \sec \alpha} \tan \phi' \cos (\theta - \alpha - \phi)] \sec \alpha
\]

$Mx$ is an additional imaginary force, the inertia force required for equilibrium

Fig. 2-7 Forces Acting on A Typical Slice (after Chugh, 1982)

**Equation of Motion**

The equation of plane motion for a typical slice along its base is derived using D’Alembert’s principle of dynamic equilibrium (Biggs, 1964). According to this method, an additional imaginary force, the inertia force acting at an inclination $\alpha_1$ to the horizontal, equal to the product of the mass of the slice, $M$, and acceleration, $d^2x/dt^2$, is applied in the direction opposite to that of positive displacement, as shown in Fig. 2-7. Having added this force, the dynamic equilibrium is treated exactly as a static equilibrium problem. Substituting $F = 1$ and $K = K_{\text{yield}}$ in the equation of static equilibrium and using the condition of dynamic equilibrium, a differential equation of motion for a typical slice can be written in the form

\[
\ddot{x} = g \frac{\cos (\theta - \alpha + \phi)}{\cos (\alpha - \alpha_1 - \phi)} [K(t) - K_{\text{yield}}]
\] (2-37)
Comparing Eqs. (2-23), (2-36) and (2-37), it is clear that even though the calculation methods for critical acceleration suggested by Goodman & Seed (1966), Sarma (1975) and Chugh (1982) are different, the differential equations of motion obtained by them are identical (The notations used are, of course, different and $\alpha = \alpha_1$ is assumed in the models of Goodman and Seed (1966) and Sarma (1975)).

Displacement Calculations

In general, the movement of the slide mass occurs along a slip surface of arbitrary shape. Therefore, the movement of slice may be considered to occur along planes of different inclination. The displaced configuration of any two points in the slide mass is related through the rotational displacement of the slide mass. Thus, for calculating displacements of a slide mass during an earthquake, it is possible to work with any one wedge or vertical slice. Since one of the items of interest in predicting the behaviour of an embankment for a design earthquake is to calculate the vertical displacement of the crest of the dam, it may be desirable to select the wedge, in a shear surface geometry, next to the crest of the dam.

The calculation procedure for estimating displacements of a potential slide mass may involve the following steps:

(a) Calculate the yield acceleration, $K_y$, within the framework of limit equilibrium

(b) Calculate $(K_e - K_y)$ for each time step and let it be designated by $K_{e-y}$ (where $K_e$ is the earthquake acceleration)

(c) For the rigid plastic system assumed in the Newmark method, the relative displacement of the slide mass may occur only when $K_{e-y}$ is positive. Let $K_{e-y}$ change sign in the time interval $t_{j-1} < t < t_j$. The time at which $K_{e-y} = 0$, by linear interpolation, is

$$t_{K_{e-y}=0} = t_{j-1} + \frac{K_{e-y\downarrow\downarrow j}(t_j - t_{j-1})}{K_{e-y\downarrow\downarrow j-1} - K_{e-y\downarrow j}}$$  \hspace{1cm} (2-38)
At the instant of time when the earthquake acceleration exceeds the yield acceleration, the relative velocity of the slide mass begins to increase and the relative displacement of the slide mass starts.

(4) Knowing the relative velocity $\dot{X}_s$, at time $t_s$, the relative velocity $\dot{X}_{s+1}$ at time $t_{s+1}$ can be calculated, by the linear interpolation method, as

$$\dot{X}_{s+1} = \dot{X}_s + \frac{\dot{X}_s + \dot{X}_{s+1}}{2} (t_{s+1} - t_s)$$

(2-39)

The value $X_s$ corresponding to the time given by equation (2-40) must be zero. Starting from the time when $K_{e,y}$ equals zero, calculate the relative velocity of the slide mass using equation (2-40) until the time step when $X_s \leq 0$. Displacement continues until the relative velocity becomes zero. The time at which $\dot{X}_s$ becomes zero can be calculated by quadratic interpolation approach.

(5) The displacement of the slide mass, between the time when $K_e$ exceeds $K_y$ to the time when $X_s$ drops to zero, can be calculated from the formula, based on the linear interpolation method

$$X_{s+1} = X_s + \dot{X}_s (t_{s+1} - t_s) + \frac{2}{6} \frac{\dot{X}_s + \dot{X}_{s+1}}{(t_{s+1} - t_s)^2}$$

(2-40)

2.6.5. ANALYSIS PROCEDURE PROPOSED BY MAKDISI AND SEED

Makdisi and Seed (1978) proposed a simple yet rational approach to the design of small embankments under earthquake loading. The method was based on the concept of permanent deformations as proposed by Newmark (1965) but modified to allow for the dynamic response of the embankment as proposed by Seed and Martin (1966) and restricted in application to compacted clay embankments and to dry and dense cohesionless soils that experience very little reduction in strength due to cycling loading. The method was an approximate one and involved a number of simplifying assumptions that may lead to somewhat conservative results.
To examine the dynamic response of embankments, Makdisi and Seed established a relationship for the variation of induced maximum acceleration ratio with embankment depth, as shown in Fig. 2-8 in which \( \bar{U}_{\text{max}} \) is the maximum crest acceleration and \( K_{\text{max}} \) is the maximum average acceleration for a potential sliding mass extending to a specified depth, \( y \) (below crest). The depth is expressed as a ratio \( y/h \) where \( h \) is the total height of the embankment. The difference between the envelope of all data and the average relationship range from \( \pm10\% \) to \( \pm20\% \) for the upper portion of the embankment and from \( \pm20\% \) to \( \pm30\% \) for the lower portion of the embankment.

Design curves to estimate permanent deformations of embankments, in the height range of 30m - 60m, have been established based on equivalent linear dynamic finite element analyses for different magnitude earthquakes by Makdisi and Seed. The displacements normalised with respect to the maximum average acceleration \( K_{\text{max}} \) and the natural period of vibration of the embankment \( T_0 \) are shown in Fig. 2-9.

To calculate the permanent deformations of an embankment that does not change in strength significantly during an earthquake, the first step is to determine its maximum crest acceleration, \( \bar{U}_{\text{max}} \), and the first natural period, \( T_0 \), due to a specified earthquake.
Chapter 2: A Brief Review

Fig. 2-8 Variation of Maximum Acceleration Ratio with Depth of Sliding Mass
(after Makdisi & Seed, 1978)

Fig. 2-9 Average Normalised Displacement (after Makdisi & Seed, 1978)
From the relationship for the maximum acceleration ratio ($K_{max} / \bar{U}_{max}$) with depth of sliding mass, as shown in Fig. 2-8, the maximum value of average acceleration history, $K_{max}$, for a sliding mass extending to a specified depth, $y$, may be determined from Fig. 2-8. With the appropriate values of $K_{max}$ and $T_0$, the permanent displacements can be determined for any value of yield acceleration associated with that particular sliding surface based on Fig. 2-9.

2.7. SIMULATION OF EARTHQUAKE ACCELERATION

2.7.1. INTRODUCTION

In an earthquake, the earth moves in a nearly random fashion in all directions, both horizontally and vertically. Therefore, the earthquake ground motion (EGM) may be simulated by a random process. Many available simulation techniques for EGM have been developed during the past decades. Earthquake acceleration was first modelled as a random process by G.W. Housner (1947) in a study in which the acceleration was idealised as a series of pulses of a certain magnitude located randomly in time. Since that time, Hudson (1956), Bycroft (1960), Rosenblueth (1956), Rosenblueth and Bustamante (1962), Goodman et al. (1955), Ward (1965) and Tajimi (1960) applied the process of white noise (its power spectral density function is a constant, $S_0$, and auto-correlation function is $R(\tau) = 2\pi S_0 \delta(\tau)$) with or without frequency characteristics which were reflected in the filter to simulate stationary and nonstationary artificial earthquake motion. The use of white-noise and filtered white-noise processes has been justified on the basis that, for limited ranges of structural periods or for some values of damping, the member functions generated from these processes yield spectra similar to those of earthquakes.

For generating the earthquake acceleration, Barstein (1960) proposed a non-white but stationary process and Bogdanoff et al. (1961) and Goldber et al. (1964) developed a non-stationary process consisting of a finite sum of time-modulated harmonics. Cornell
(1964) and Shinozuka and Sato (1967) used filtered Poisson processes to simulate artificial earthquake motion.

Shinozuka and Sato (1967) developed a procedure for generating a stochastic process by subjecting a linear system to a Gaussian, white noise excitation. Levy et al. (1971) used the auto-covariance, maximum ground acceleration, response spectrum, and non-stationary characteristics as a qualitative basis for judging the acceptability of a simulation process.

Katukura et al. (1978), Izumi et al. (1980) and Kimura and Izumi (1989) have pointed out the importance of considering Fourier phase inclinations in the analysis and simulation of EGM. Ohsaki (1979) showed an analogy between the envelope shape of EGM in the time domain and the Fourier phase difference distribution in the frequency domain. Kubo (1987) investigated the correlation of Fourier phase angles with occurrence time of peak value based on time histories. Watable et al. (1987) have discussed the method of synthesising EGM by combining phase information with amplitude information in generating EGM.

### 2.7.2. Model of Non-Stationary Shot Noise

Lin (1963), Shinozuka and Sato (1967), Amin and Ang (1968) and Ruiz and Penzien (1969, 1971) considered earthquake excitation as a non-stationary Gaussian shot noise process and developed the simulation models of artificial earthquake motion associated with shot noise process.

A schematic diagram of a non-stationary filtered shot noise process is shown in Fig. 2-10. A zero-mean random process, \( x(t) \), is said to be nonstationary shot noise if its covariance function is given by

\[
\text{cov}_x(t_1, t_2) = I(t_1) \delta(t_2 - t_1) \quad t_2 \geq t_1
\]  \hspace{1cm} (2-41)
in which \( t_1 \) and \( t_2 \) are any two time instants, \( \delta \) is the Dirac delta function, and \( I(t) \) is a positive function, called the "intensity function", of \( x(t) \); \( I(t) \) is also assumed to be a continuous function. The non-stationary shot noise process may be defined as

\[
x(t) = \sum_{k=1}^{n_t} Y_k \delta(t - t_k)
\]

(2-42)

in which \( Y_k \) are mutually independent random variables with zero mean, and with variance equal to \( I(t_k) \Delta t \) and \( n_t \) is a specified number of impulses in the interval \((0, t]\).

\[
x(t): \text{shot noise process} \quad \xrightarrow{\text{Filter}} \quad \dot{y}(t) \quad \xrightarrow{\text{Structure}} \quad \text{Response}
\]

Ground earthquake acceleration

Fig. 2-10 Simulation Procedure of Non-Stationary Random Process Based on Shot Noise Process

If \( h(t) \) denotes the impulse response function of a linear filter and \( \dot{y}(t) \) denotes the output of the filter subjected to the excitation \( x(t) \) given by Eq.(2-43), the result of the filter output is a nonstationary filtered shot noise process and can be written as follows

\[
\dot{y}(t) = \sum_{k=1}^{n_t} X_k h(t - t_k)
\]

(2-43)

Three types of filters may be considered and the impulse response functions corresponding to these filters are

FL1 \( h_1(t) = e^{-\gamma t} \) (2-44)

FL2 \( h_2(t) = \frac{1}{\omega_{2d}} e^{-\gamma \omega_{2d} t} \sin \omega_{2d} t \) (2-45)

FL3 \( h_3(t) = e^{-\gamma \omega_{3d} t} (A \cos \omega_{3d} t + B \sin \omega_{3d} t) \) (2-46)

in which \( \omega_d = \omega \sqrt{1 - \gamma^2} \); \( A = 2\gamma_3 \omega_3 \); and \( B = \frac{\omega_3^2(1 - 2\gamma_3^2)}{\omega_3^2} \).
The filtered nonstationary shot noise can be used to simulate many physical processes with known variance and covariance functions. The function \( I(t) \) is selected to give the desired variance function, while the filter is selected to yield the desired covariance function. The intensity function \( I(t) \) was assumed to consist of three parts (Amin and Ang, 1968), i.e.

\[
I(t) = \begin{cases} 
I_0 \left( \frac{t}{x_1} \right)^2 & 0 \leq t \leq x_1 \\
I_0 & x_1 \leq t \leq x_2 \\
I_0 e^{-c(t-x_2)} & t \leq x_2 
\end{cases} \tag{2-47}
\]

The quantity \( I_0 \) is a measure of earthquake intensity and its value depends on the desired intensity intended for consideration. It was selected such that the average response spectra of a single-degree-of-freedom system subjected to the simulated ground motions are comparable to the corresponding average spectra resulting from real earthquakes (Amin and Ang, 1968).

To simulate artificial earthquake motions it is convenient to write Eq.(2-44) in a form to obtain a recursive relation for the filter response. The procedure is demonstrated for filter FL2, as an example. Using a trigonometric expansion,

\[
y_2(t) = \frac{1}{\omega_{2d}^2} \left[ R_c(t) \sin \omega_{2d} t - R_s(t) \cos \omega_{2d} t \right] \tag{2-48}
\]

in which

\[
\begin{align*}
\left\{ \begin{array}{c} R_c(t) \\ R_s(t) \end{array} \right\} &= \sum_{k=1}^{n_1} X_k e^{-\gamma_2 \omega_2 (t-t_k)} \begin{bmatrix} \cos \omega_{2d} t_k \\ \sin \omega_{2d} t_k \end{bmatrix} \\
\end{align*}
\tag{2-49}
\]

The functions \( R_c(t) \) and \( R_s(t) \) may then be computed recursively as follows:

\[
\begin{align*}
\left\{ \begin{array}{c} R_c(t+s) \\ R_s(t+s) \end{array} \right\} &= e^{-\gamma_2 \omega_2 s} \left\{ \begin{array}{c} R_c(t) \\ R_s(t) \end{array} \right\} + \sum_{k=n_1}^{n_2} X_k e^{-\gamma_2 \omega_2 (t+s-t_k)} \begin{bmatrix} \cos \omega_{2d} t_k \\ \sin \omega_{2d} t_k \end{bmatrix} \tag{2-50}
\end{align*}
\]

The value of \( y_2(t+s) \) is then obtained by using Eq.(2-50) and Eq.(2-48).
Amin and Ang (1968) used this simulation procedure to generate eight artificial earthquakes for each of the filters FL1, FL2 and FL3. They found that the ground motions generated with filters FL2 and FL3 were very much like those obtained from strong-motion accelerograms.

2.7.3. NON-STATIONARY GAUSSIAN MODELS

As a result of the analysis of many past earthquake motions, Kanai suggested in 1957 that the power spectrum of earthquakes observed at bedrock can be characterised by a constant pattern. Following that suggestion in 1960, Tajimi derived an approximate power spectrum density function of earthquake acceleration. Housner and Jennings (1964), Franklin (1965), Iyenger et al. (1970) and Lou (1975) used the product of a stationary Gaussian random process $x(t)$ with a prescribed power spectral density function and a specified function $g(t)$ to approximate the ground earthquake acceleration, i.e.

$$
\ddot{y}(t) = g(t) x(t) \quad (2-51)
$$

After analysing many real earthquake records, Iyengar et al. (1970) proposed the form of the specified function, $g(t)$, to be

$$
g(t) = (a_1 + a_2 t) e^{-pt} \quad (2-52)
$$

and the power spectrum density function to be defined as

$$
G(\omega) = A_1 e^{-\omega^2 c^2} + A_2 \omega^2 e^{-4\omega^2 c^2} \quad (2-53)
$$

The values of parameters in Eqs. (2-52) and (2-53) are shown in Table 2-2.

In this thesis, a numerical technique for generating a stationary Gaussian process developed by Franklin (1965) has been applied and the detailed procedures for this simulation method are introduced in Chapter 7.
Table 2-2 The Values of Parameters in Iyenger's Model (after Iyenger, 1970)

<table>
<thead>
<tr>
<th>Earthquake Model</th>
<th>$a_1/0.1g$</th>
<th>$a_2/0.1g$</th>
<th>$p$</th>
<th>$A_1$ (1/Sec)$^{-1}$</th>
<th>$A_2$ (1/Sec)$^{-2}$</th>
<th>$c^2$ (1/Sec)$^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0</td>
<td>7.6</td>
<td>2.44</td>
<td>0.017836</td>
<td>0.000285</td>
<td>0.000999</td>
</tr>
<tr>
<td>II</td>
<td>0.315</td>
<td>0.73</td>
<td>0.62</td>
<td>0.010799</td>
<td>0.000063</td>
<td>0.000366</td>
</tr>
<tr>
<td>III</td>
<td>0.0525</td>
<td>0.328</td>
<td>0.15</td>
<td>0.016969</td>
<td>0.000246</td>
<td>0.000905</td>
</tr>
<tr>
<td>IV</td>
<td>0</td>
<td>0.025</td>
<td>0.08</td>
<td>0.097365</td>
<td>0.046396</td>
<td>0.029782</td>
</tr>
</tbody>
</table>

2.7.4. OTHER SIMULATION METHODS

Vanmarcke and Lai (1980) proposed the concept of equivalent stationary motion. They used a portion of a stationary Gaussian random process with zero mean to replace a non-stationary true recorded motion and applied an energy criterion which makes it possible to replace an envelope function by the root-mean-square acceleration $\sigma$ and equivalent stationary duration $s$. Only four parameters were needed in the equivalent motion model. Several basic equations associated with the equivalent stationary motion have been derived by Vanmarcke and Lai (1980), Lin (1982) and Lin et al. (1983). Lin & Tyan (1986) attempted to justify an equivalent stationary motion model and endeavoured to illustrate its potential as an attractive analysis tool. Using 38 ground accelerations recorded at rock sites as a data base, they demonstrated that the equivalent stationary motions do give average response spectra compatible with real earthquake records. A study of their results also showed that, through the equivalent stationary model, a set of motions may be defined by specifying the joint distributions of the model parameters.

Some researchers (e.g. Polhemus and Cakmak, 1981; Chang et al., 1982) used autoregressive moving average (ARMA) models for the analysis of earthquake acceleration time series. The ARMA models can be represented as stochastic linear difference equations of finite order. Using a sequence of white noise generated and
passing through the estimated ARMA filter, a zero-mean, stationary process, with frequency characteristics which depend upon the parameters of the transfer function, may be obtained. The parameters of the transfer function can be estimated using non-linear regression for minimising the sum of squared residuals according to the observed earthquake record. Since earthquake motion is non-stationary, the stationary process of output is multiplied by some derived transformation to generate time series with non-stationary features.
CHAPTER 3

DETERMINISTIC ANALYSIS OF SLOPES

3.1. GENERAL REMARKS

In the deterministic stability analysis of embankments, earth dams and slopes, limit equilibrium analysis is one of the most commonly used methods. Stress-deformation analyses, which are the basis of the development of sophisticated numerical techniques (e.g. finite element and boundary element), require comprehensive and accurate input data which is generally not available to geotechnical engineers. Although approaches based on the concept of limit equilibrium cannot give accurate stress distributions in geotechnical structures, very useful conclusions can be drawn concerning overall stability or reliability on the basis of these concepts. Furthermore, shear strength parameters may be computed from back analysis of slope failures on the basis of equilibrium concepts.

In any limit equilibrium method of analysis it is usual to assume a potential slip surface so that the static equilibrium of the free-body above this slip surface can be analysed. This computed state of stress on the potential slip surface, which is known as
the mobilized stress, is not necessarily the true or accurate state of stress along this surface but it gives an estimate of average shear stress (the mobilized shear strength) necessary to maintain equilibrium in the slope. This state of stress is then compared with the available shear strength and the factor of safety, \( F \), is defined as that factor by which the available shear strength can be decreased to give the mobilized shear strength. The available shear strength is generally computed on the basis of the Mohr-Coulomb failure criterion. On the basis of the limit equilibrium concept, a pseudo-static analysis may also be performed considering earthquake inertial forces in addition to gravitative force. An equivalent static horizontal or inclined force determined as the product of seismic (or acceleration) coefficient, \( K \), and the weight of the potential slide mass is applied. A critical (yield) acceleration coefficient or seismic coefficient, \( K_c \), is defined as one which corresponds to critical equilibrium, i.e. a factor of safety of one. The critical seismic coefficient is considered to be an indicator of the capability of earth structure during an earthquake. However, there are significant limitations to this approach and better methods have been developed by combining limit equilibrium approaches with the Newmark-type sliding block model.

There are usually two kinds of solution to problems of limit equilibrium based on the generalised procedure of slices (GPS). The first is a 'simplified' solution where the conditions of static equilibrium are not rigorously satisfied. Significant assumptions have to be made to obtain the solution in a simple form. The second is a 'rigorous' solution where the equilibrium conditions are completely satisfied based on assumptions which are not restrictive. These solutions have to satisfy the conditions of acceptability, e.g., (a) the forces obtained from the solution do not violate the Mohr-Coulomb failure criterion anywhere within the sliding body, (b) no tension is implied, and (c) the directions of forces are kinematically admissible. An acceptable 'rigorous' solution does not necessarily provide the actual stresses in the slope and it is 'rigorous' only within the context of the method of solution and its stated assumptions(Sarma, 1979). It has also to be
emphasised that the satisfaction of overall moment equilibrium does not necessarily result in the satisfaction of moment equilibrium of individual slices.

In this chapter, three limit equilibrium methods are used which are quite well known. These are (a) the simplified Bishop method, (b) the generalised procedure of slices using the Morgenstern and Price method and (c) the Sarma method. The pseudo-static models are derived for the first two of these methods. All these methods are two-dimensional methods.

Some comparisons of the computation results based on those two dimensional methods are presented. The relationships among $F$, $K_c$ and the direction of critical acceleration, $\theta$, are discussed.

3.2. BASIC EQUATIONS AND CONDITIONS OF EQUILIBRIUM

3.2.1. BASIC EQUATIONS

For deriving the basic equations used in the GPS methods, symbols and notations used in these equations are shown in Fig. 3-1. A typical slice is shown in the upper part of the figure with boundary forces $E_a$ and $E_b$ and distance of various points from an arbitrary point $O$. The forces on a typical slice including interslice forces are shown in the lower part of the figure. The subscript $i$ in those equations represents the $i$-th slice and the superscript $'$ represents the effective stress parameters of shear strength. The slices are numbered from 1 to $n$ in the positive $x$ direction. The following derivations are similar to those derived by Li and White (1987b) and no original work is claimed in this section except the calculation formulas of pseudo-static analysis.

i) Mohr-Coulomb Failure Criterion

The equilibrium along the base of a slice on a potential failure surface in slope can be mathematically expressed by the Mohr-Coulomb failure criterion and the factor of safety as follows:
Fig. 3-1 Definitions and Notation Including Inertial Force for Earthquake

Used in the GPS Method (after Li and White, 1987a)
\[ \tau_i = \frac{c'_i + (\sigma_i - u_i) \tan \phi'_i}{F} \quad i = 1, 2, \ldots, n \] (3-1)

where, 
- \( \tau_i \) = mobilized shear strength;
- \( \sigma_i \) = total normal stress acting at the base of slice;
- \( u_i \) = pore water pressure at the base;
- \( c'_i \) = effective cohesion of soil;
- \( \phi'_i \) = effective internal friction angle;
- \( F \) = factor of safety.

ii) Vertical Force Equilibrium of A Slice

From the Fig. 3-1, the force equilibrium equation of i-th slice in a vertical direction may be obtained as follows:

\[ \sigma_i = p_i + t_i - \tau_i \tan \alpha_i \] (3-2)

where,
- \( p_i = \frac{(1 + K \sin \theta) W_i}{\Delta X_i} \);
- \( t_i = \frac{\Delta T_i}{\Delta X_i} \);
- \( W_i \) = weight of i-th slice;
- \( \Delta T_i = T_i - T_{i-1} \) - net shear force across a vertical slice;
- \( \alpha_i \) = inclination to the horizontal of i-th slice;
- \( K \) = acceleration coefficient;
- \( \theta_i \) = inclination of acceleration force to the horizontal.

iii) Horizontal Force Equilibrium of A Slice

The horizontal force equilibrium equation of i-th slice can be written as follows:

\[ \Delta E_i = (-\sigma_i \tan \alpha_i + \tau_i) \Delta X_i - K W_i \cos \theta \] (3-3a)

in which \( \Delta E_i = E_i - E_{i-1} \) - net normal force across a vertical slice.
By substituting Eqs. (3-1), (3-2) into (3-3a), an alternative expression for $\Delta E_i$ is obtained as follows:

$$
\Delta E_i = [c_i' + (p_i + t_i - u_i) \tan \phi_i'] m_i \Delta X_i - [K W_i \cos \theta + (p_i + t_i) \Delta X_i \tan \alpha_i]
$$

(3-3b)

where,

$$
m_i = \frac{1 + \tan^2 \alpha_i}{F + \tan \phi_i \tan \alpha_i}
$$

The above expression can also be written in a different form as follows:

$$
\Delta E_i = [c_i L_i + [(W_i + \Delta T_i) \sec \alpha_i - U_i] \tan \phi_i'] m_i'
\quad - [K W_i \cos \theta + (W_i + \Delta T_i) \tan \alpha_i], \quad i = 1, 2, ..., n
$$

(3-3c)

where,

$$
m_i' = \frac{\sec \alpha_i}{F + \tan \phi_i \tan \alpha_i}, \quad W_i = (1 + K \sin \theta) W_i
$$

iv) Moment Equilibrium of A Slice

By taking the moments about the centre of the base of each slice, a expression for $\Delta T_i$ can be derived as follows:

$$
\Delta T_i = 2 (-T_{i-1} + E_{i-1} \tan \alpha_i + \Delta E_i n_i + \frac{\Delta Q_i}{\Delta X_i} Z_{Qi})
$$

(3-4)

where,

$$
n_i = \frac{\tan \alpha_i + \frac{h_i}{\Delta X_i}}{2}
$$

Assuming different relationships between interslice shear and normal forces and considering either force equilibrium equations or moment equilibrium as both, the iterative procedure for a GPS method can be developed as follows.

3.2.2. THE NUMBER OF EQUATIONS FOR GPS METHODS

In the GPS methods, the body of mass contained within the assumed slip surface and free ground surface is divided into $n$ slices as shown in Fig. 3-1. Therefore, there are
following \((6n - 2)\) unknowns: \(n\) values of total normal stress \((\sigma_i)\) or total normal force \((N_i)\); \(n\) values of shear stress \((\tau_i)\) along the slip surface or shear force \((S_i)\); \((n-1)\) values of the interslice normal force \((E_i)\); \((n-1)\) values of the interslice shear force \((T_i)\); \((n-1)\) values of the points of application of \(E_i\) given by \(h_i\); \(n\) numbers of the points of application of the \(N_i\) forces given by assumption; and one value of the factor of safety \(F\).

From the static equilibrium conditions, we have the following three equations for each slice, i.e. vertical force equilibrium equation, horizontal force equilibrium equation, and moment equilibrium equation. The Mohr-Coulomb failure criterion for each slice gives \(\tau_i = f(\sigma_i)\), a function of \(\sigma_i\). Therefore, for \(n\) slices, we have \(4n\) equations. This implies that the solution is statically indeterminate. To obtain a solution, it is, therefore, necessary to make \((2n-2)\) independent assumptions.

### 3.3. THREE LIMIT EQUILIBRIUM METHODS SELECTED FOR THIS THESIS

Three well known limit equilibrium methods are introduced in the following subsections. A different type of assumption of the relationship between interslice shear and normal forces is made or implied in each of these methods.

#### 3.3.1. SIMPLIFIED BISHOP METHOD

The simplified Bishop method (1955) is one of the most popular among simplified limit equilibrium methods and is applicable only to slip surface of circular shape. The tangential or shear forces on the boundaries of slices are assumed to be zero \((T_i = 0, i = 1,2, ..., n-1)\) and the moment equilibrium of each slice is ignored in this method. The point of application of the normal stress is assumed at the middle of the base of the slice. The horizontal equilibrium of each slice cannot be satisfied with the computed factor of safety which is derived on the basis of overall moment equilibrium. For slip surfaces of circular shape this error is small and the accuracy of the method is generally acceptable.
After neglecting the interslice shear forces, by means of the normal force equilibrium equation (3-2), the total normal force acting on the base of a slice $N_i$ can be rewritten as,

$$N_i = (1 + K \sin \theta) W_i \sec \alpha_i - S_i \tan \alpha_i \tag{3-5}$$

where,

$$N_i = \sigma_i L_i;$$

$$S_i = \tau_i L_i = \frac{c_i L_i + (N_i - u_i L_i) \tan \phi_i}{F_i}.$$

The disturbing force can be considered in two parts. The first part is the component of slice weight along the slip base, i.e.,

$$D_i = W_i \sin \alpha_i \tag{3-6}$$

and the second part of the disturbing is extra load $K W_i \cos \theta$.

From Fig. 3-1 and assuming the $E_a = E_b = T_a = T_b = 0$ (boundary conditions), the overall moment equilibrium equation about the centre of the slip circle may be written as,

$$R \sum_{i=1}^{n} S_i = R \sum_{i=1}^{n} D_i + \sum_{i=1}^{n} K W_i \cos \theta y_i Q_i \tag{3-7a}$$

or

$$\sum_{i=1}^{n} S_i = \sum_{i=1}^{n} D_i + \frac{1}{R} \sum_{i=1}^{n} K W_i \cos \theta y_i Q_i \tag{3-7b}$$

Using Eqs. (3-5), (3-6) and (3-7), the factor of safety based on Bishop's assumption can be represented as follows:

$$F = \left( \frac{\sum_{i=1}^{n} \left[ c_i \Delta X_i + [(1 + K \sin \theta) W_i - u_i \Delta X_i] \tan \phi_i \right] / m_{\alpha_i}}{K \cos \theta \sum_{i=1}^{n} W_i y_i Q_i + \sum_{i=1}^{n} (1 + K \sin \theta) W_i \sin \alpha_i} \right) \tag{3-8a}$$

where,

$$m_{\alpha_i} = \frac{\tan \alpha_i \tan \phi_i}{F} \cos \alpha_i.$$

The expression for the factor of safety, $F$, based on the Bishop assumption contains $F$ in the term $m_{\alpha_i}$. Therefore, the solution of Eq.(3-8a) must be obtained by an
iterative procedure. The Bishop method is, in fact, derived on the basis of the overall moment equilibrium.

Let $F$ in Eq. (3-8a) equal to unity; the critical or yield acceleration coefficient can then be explicitly expressed as follows,

$$K_c = \frac{\sum_{i=1}^{n} \left( c_i \Delta X_i + [W_i - u_i \Delta X_i] \tan \phi_i \right) / m_{\alpha_i}}{\frac{\text{cos} \theta}{R} \sum_{i=1}^{n} W_i y_i Q_i + \sum_{i=1}^{n} (\sin \alpha_i - \sin \theta \tan \phi / m_{\alpha_i}) W_i}$$

(3-8b)

where, $m_{\alpha_i} = (1 + \tan \alpha_i \tan \phi) \cos \alpha_i$

### 3.3.2. MORGENSTERN AND PRICE METHOD (M&P METHOD)

Morgenstern and Price method (1967) is one of the most popular among relatively 'rigorous' GPS methods. The method is applicable to slip surfaces of both circular and arbitrary shape. The original formulation by Morgenstern and Price (M&P) is complex and involves the solution of simultaneous differential equations. Fredlund and Krahn (1976) developed a simpler formulation for the M&P method which is consistent with the GPS approach. In the M&P method, $(n-1)$ values of the relationship between interslice normal and shear forces $E$ and $T$ are assumed, i.e.,

$$T_i = \lambda \ f(x_i) \ E_i, \quad i = 1, 2, ..., n-1$$

(3-9)

where, $\lambda = \text{an unknown coefficient}$;

$f(x_i)$ = a prescribed function with respect to coordinate $x_i$.

The $n$ values of points of application of $N$ forces are assumed to be at the middle of the base of each slice. Thus the total assumptions made are $(2n-1)$. Because an unknown $\lambda$ is included in the simultaneous equation, the number of the total unknowns is increased by one to $(6n-1)$. This implies that the solution of the M&P method is rigorous from the point of view of statics. However, it is difficult to accurately assess the relationship
between the interslice normal and shear forces $E$ and $T$ because this relationship depends on a number of factors including the stress-deformation characteristics of the material of a slope. Therefore, various arbitrary assumptions are made and the particular assumption, which is consistent with the criteria of acceptability, is adopted.

Considering $T_i = \Delta T_i + T_{i-1}$ and $E_i = \Delta E_i + E_{i-1}$, Eq. (3-9) can be rewritten as,

$$T_{i-1} + \Delta T_i = \lambda f(x_i) (E_{i-1} + \Delta E_i), \quad i=1, 2, ..., n-1 \quad (3-10)$$

Combing Eqs. (3-3) and (3-10), one may obtain,

$$\Delta T_i = \left( E_{i-1} - \frac{T_{i-1}}{\lambda f(x_i)} \right) + \left( c_i' \Delta X_i + [(1 + K \sin \theta) W_i - u_i \Delta X_i \tan \phi_i] \right) m_i - W_i [K \sin \theta + (1 + K \sin \theta) \tan \alpha_i] \Omega \quad (3-11)$$

where,

$$\Omega = \frac{1}{\lambda f(x_i)} - m_i \tan \phi_i' + \tan \alpha_i';$$

$m_i$ is same as that in Eq.(3-3b).

Assuming the boundary conditions to be $E_a = E_b = T_a = T_b = 0$ and considering overall horizontal forces equilibrium ($\sum_{i=1}^{n} \Delta E_i = 0$), the following equation is obtained:

$$\sum_{i=1}^{n} \left\{ \left( c_i' \Delta X_i + [(1 + K \sin \theta) W_i + \Delta T_i - u_i \Delta X_i \tan \phi_i] \right) m_i - [K \cos \theta W_i + [(1 + K \sin \theta) W_i + \Delta T_i] \tan \alpha_i] \right\} = 0 \quad (3-12)$$

From Eq. (3-12), the factor of safety based on force equilibrium can be derived as follows:

$$F_f = \frac{\sum_{i=1}^{n} \left\{ c_i' \Delta X_i + [(1 + K \sin \theta) W_i + \Delta T_i - u_i \Delta X_i \tan \phi_i] \sec \alpha_i / m_{\alpha_i} \right\}} {\sum_{i=1}^{n} \left\{ K \cos \theta W_i + [(1 + K \sin \theta) W_i + \Delta T_i] \tan \alpha_i \right\}} \quad (3-13a)$$

By considering overall moment equilibrium about an arbitrary point 'O' shown in Fig. 3-1, the factor of safety can be expressed as follows:
Chapter 3: Deterministic Analysis of Slopes

3-11

\[
F_m = \sum_{i=1}^{n} \left( c_i' \Delta X_i + [(1 + K \sin \theta) W_i + \Delta T_i - u_i \Delta X_i] \tan \phi_i \right) y_{m_i} \sec \alpha_i / m_{\alpha_i}
\]

(3-14a)

\[
K \cos \theta \sum_{i=1}^{n} W_i y_{Q_i} + \sum_{i=1}^{n} \left( [(1 + K \sin \theta) W_i + \Delta T_i] y_{m_i} \tan \alpha_i - \Delta T_i x_{m_i} \right)
\]

In particular, if the slip surface is a circle and the moments are taken about the centre of the circle, Eq. (3-14a) simplifies to Eq. (3-8a).

On the basis of Eqs. (3-13a) and (3-14a), let \( F = F_m = F_f = 1 \), the critical acceleration coefficient, \( K_c \), can then be calculated by both force and moment equilibrium equations, i.e.,

\[
K_c^{(f)} = \frac{\sum_{i=1}^{n} \left( c_i' \Delta X_i + [(1 + K \sin \theta) W_i + \Delta T_i - u_i \Delta X_i] \tan \phi_i \right) \sec \alpha_i / m_{\alpha_i} - \sum_{i=1}^{n} (W_i + \Delta T_i) \tan \alpha_i}{\sum_{i=1}^{n} W_i \{ \cos \theta + \sin \theta \tan \alpha_i - \sin \theta \tan \phi_i \sec \alpha_i / m_{\alpha_i} \}}
\]

(3-13b)

and

\[
K_c^{(m)} = \left( \sum_{i=1}^{n} \left( c_i' \Delta X_i + [(1 + K \sin \theta) W_i + \Delta T_i - u_i \Delta X_i] \tan \phi_i \right) y_{m_i} \sec \alpha_i / m_{\alpha_i} \right. \\
- \left. \sum_{i=1}^{n} [(W_i + \Delta T_i) y_{m_i} \tan \alpha_i - \Delta T_i x_{m_i}] \right) / \Theta
\]

(3-14b)

where,

\[
\Theta = \sum_{i=1}^{n} W_i \{ \cos \theta \ y_{Q_i} + \sin \theta \ y_{m_i} \tan \alpha_i - \sin \theta \ y_{m_i} \tan \phi_i \sec \alpha_i / m_{\alpha_i} \}
\]

\[
m_{\alpha_i} = (1 + \tan \alpha_i \tan \phi_i) \cos \alpha_i
\]

From Eq. (3-11), one may find that \( \Delta T_1 \) only involves the interslice forces \( E_{1,1} \) and \( T_{1,1} \) on the left hand side of equation. Thereby the calculation of \( \Delta T_1 \) is explicit once the values of factor of safety, \( F \), or acceleration coefficient, \( K \), and \( \lambda \) as well as the boundary conditions are given. Based on the M&P method, two factors of safety or critical acceleration coefficients are contained, i.e. one based on force equilibrium \( F_f \) (or \( K_c^{(f)} \)) and another based on moment equilibrium \( F_m \) (or \( K_c^{(m)} \)). A relatively ‘rigorous’
solution requires that \( F_f = F_m \) and \( K_c^{(f)} = K_c^{(m)} \). For a prescribed function \( f(x) \) with respect to \( x \) given, these two factors of safety or critical acceleration coefficients may be made identical by adjusting the value of the coefficient \( \lambda \). Thus, one may use a function \( q(\lambda) \) with respect to \( \lambda \) as follows:

\[
q(\lambda) = F_m - F_f \quad (3-15a)
\]

or,

\[
q(\lambda) = K_c^{(m)} - K_c^{(f)} \quad (3-15b)
\]

The function \( q(\lambda) \) varies monotonically with \( \lambda \). Obviously, the rigorous solution of \( F_{mf} \) (or \( K_{cmf} \)) can be obtained by the root \( \lambda_{mf} \) of the equation \( q(\lambda) = 0 \). A two level iterative procedure is required for calculating the relatively 'rigorous' solution. The first level iterative procedure is used to calculate, for a given value of \( \lambda \), a pseudo factor of safety which can be either \( F_m \) or \( F_f \). For example, let \( F_{mP} \) be the pseudo FOS based on moment equilibrium equation, then Eq. (3-15a) can be rewritten as,

\[
q(\lambda) = F_{mP} - F_f \quad (3-16)
\]

The second level iterative procedure is then used to calculate the root of Eq. (3-16) where \( q(\lambda) = 0 \). Obviously, the first level iterative procedure is nested in the second level iterative procedure, i.e., for a different value of \( \lambda \), the first level iterative procedure has to be used again to calculate the pseudo factor of safety. The two iterative procedures are successively used until the difference between \( F_m \) and \( F_f \) is less than a given tolerance for some value of \( \lambda \). The iterative procedures involved in the M&P method are shown in Fig. 3-2 (a flowchart). A similar calculation procedure can also be used to compute the critical acceleration coefficient.

A compound iterative procedure is utilised for calculating the pseudo factor of safety in this section. Assuming the right hand side of Eq. (3-14a) as \( H(F) \), the value of \( F \) at the \((i+1)\)th iteration can then be obtained from the value \( F_i \) at the previous iteration:
\[ F_{i+1} = H(F_i) \]  \hspace{1cm} (3-17)

The compound iterative procedure involves the following steps:

1. Give an initial factor of safety \( F_0 \) and use the Eq. (3-17) to calculate the function value \( H(F_0) \);

2. Assign the value of \( H(F_0) \) to \( F_1 \) and use the Eq. (3-17) to calculate the function value \( H(F_1) \);

3. Calculate the difference between \( F_0 \) and \( F_1 \). If this difference is less than a given tolerance (for instance \( 10^{-5} \)), then the calculation stops; otherwise the following calculation steps are carried out;

4. Calculate the correction coefficient \( \kappa \) and the new factor of safety as follows:

![Flowchart of Iterative Procedure for M&P Method](image)
\[
\kappa = \frac{H(F_1) - H(F_0)}{F_1 - F_0}
\]

\[
F_2 = \frac{H(F_1) - \kappa F_1}{1 - \kappa}
\]

(5) Assign the values of \( F_1, H(F_1) \) and \( F_2 \) to \( F_0, H(F_0) \) and \( F_1 \) respectively;

(6) Calculate the \( H(F_1) \) using the Eq. (3-16) and return to step (3).

It was found from experience that the convergence of the compound iterative procedure is very fast. Generally 4 or 5 iterations are enough. The iterative procedure for solving the simplified Bishop method is similar.

For obtaining the root \( \lambda_{mf} \) of the equation \( q(\lambda) = 0 \), a rapid convergence, iterative algorithm based on the rational polynomial technique is used. The rational polynomial technique was employed by Li and White (1987b) for solving some nonlinear equations of geotechnical structure stability. The use of this technique has now been extended to probabilistic analysis and the details of this technique are presented in the next chapter.

3.3.3. SARMA METHOD

A limit equilibrium approach based on subdivision of the sloping mass into non-vertical slices was originally presented by Sarma (1979). This method may be regarded as an extended 2-D wedge method of solution. In this method, the geometry of the slices is considered in accordance with the notation shown in Fig 3-3.

In contrast to other 'rigorous' methods (especially GPS methods discussed in previous section, such as the M&P method), Sarma method considers only the force equilibrium of each slice and overall force equilibrium of all the slices. It does not consider the overall moment equilibrium.

This method invokes the possibility of shear failure inside the sliding mass and assumes the degree of mobilisation of shear strength inside the soil or rock mass to agree with that on the slip surface, i.e. the factor of safety is the same on the slip surface and the internal shear planes. Therefore, Mohr-Coulomb failure criterion can be used on internal
shear planes and thus (n-1) relationships between the $E_i$ and $T_i$ forces are available. Points of application of normal forces (N) are assumed so that the line of thrust of interslice forces can be determined. Thus, a total of (2n-2) assumptions are made.

In this section, Sarma method as modified by Hoek (1986), is presented. The static factor of safety is calculated by reducing the shear strength simultaneously on all sliding surfaces and internal shear planes until the acceleration coefficient $K$ reduces to zero.

The forces acting on the $i$-th slice are as shown in Fig. 3-3. From the vertical and horizontal equilibrium of slice, one can obtain

Fig. 3-3 Geometry of 2-D Wedge and Forces Acting on Individual Inclined Slice (after Sarma, 1979).
\[ N_i \cos \alpha_i + S_i \sin \alpha_i = W_i + T_{i+1} \cos \delta_{i+1} - T_i \cos \delta_i - E_{i+1} \sin \delta_{i+1} + E_i \sin \delta_i \]  
(3-17)

\[ S_i \cos \alpha_i - N_i \sin \alpha_i = K W_i + T_{i+1} \sin \delta_{i+1} - T_i \sin \delta_i + E_{i+1} \cos \delta_{i+1} - E_i \cos \delta_i \]  
(3-18)

where, \( S_i \) can be expressed by Mohr-Coulomb failure criterion, i.e.

\[ S_i = \frac{c'_b \sec \alpha_i + (N_i - U_i) \tan \phi'_i}{F} \]  
(3-19)

According to previous assumptions, the interslice shear and normal force can be written as follows:

\[
\begin{align*}
T_i &= \frac{c'_i d_i + (E_i - P W_i) \tan \phi'_i}{F} \\
T_{i+1} &= \frac{c'_i d_{i+1} + (E_{i+1} - P W_{i+1}) \tan \phi'_{i+1}}{F}
\end{align*}
\]  
(3-20)

in which \( \phi'_i \) is the average friction angle on the inclined plane; \( c'_i \) is the average cohesion on the same plane; \( d_i \) is the length of the inclined plane; and \( P W_i \) is the force due to water pressure on the plane. Putting Eqs. (3-19) and (3-20) in Eqs. (3-18) and (3-17) thereby eliminating \( S_i, T_i \) and \( T_{i+1} \) and then eliminating \( N_i \) from the resulting two equations, one can get write as follows (the notation is defined after Eq. 3-22):

\[ E_{i+1} = a_i - p_i K + E_i e_i \]  
(3-21)

Eq. (3-21) is a recurrence relation and from this and boundary conditions \( E_{n+1} = E_1 = 0 \), the acceleration coefficient \( K \) can be derived in terms of the factor of safety, \( F \), on the basis of the concept of limit equilibrium as follows:

\[ K = \frac{a_n + \sum_{j=1}^{n-1} a_j \prod_{i=j+1}^{n} e_i}{p_n + \sum_{j=1}^{n-1} p_j \prod_{i=j+1}^{n} e_i} \]  
(3-22)
where,

\[ a_i = q_i \{ W_i \sin(\phi_i' - \alpha_i) + r_i \cos \phi_i' + S_{i+1} \sin (\phi_i' - \alpha_i - \delta_{i+1}) \} \]

- \( S_i \sin(\phi_i' - \alpha_i - \delta_i) \};

\[ p_i = q_i W_i \cos (\phi_i' - \alpha_i); \]

\[ e_i = q_i \cos (\phi_i' - \alpha_i + \phi_i' - \delta_i) \sec \phi_i; \]

\[ q_i = \frac{\cos \phi_{i+1}}{\cos (\phi_i' - \alpha_i + \phi_i' - \delta_i)}; \]

\[ s_i = \frac{c_i d_i - PW_i \tan \phi_i}{F}; \]

\[ r_i = \frac{c_i b_i \sec \alpha_i - U_i \tan \phi_i}{F}. \]

An iterative procedure is required for calculating the static factor of safety, \( F \), from Eq. (3-22) for the value of \( K = 0 \). It may be noted that, if \( K = 0 \), the solution of Eq. (3-22) with respect to \( F \) can be obtained by the rational polynomial technique. Since inclined slices can be used and also the shear strength along internal shear planes can be considered the method is suitable for the stability analysis of jointed rock slopes. In Eq.(3-22), the point application of total normal forces (\( N \)) are not used, i.e. the moment equilibrium is not included in Eq.(3-22). Therefore the static factor of safety (\( F \)) obtained by Eq.(3-22) is not a 'rigorous' solution. The number of assumptions exceeds by one the number actually required. The effective normal stresses acting on the base and the sides of a slice calculated by this method may not be positive in certain cases. On the basis of Eq. (3-22), it is obvious that the critical acceleration coefficient, \( K_c \), can be explicitly calculated when the factor of safety, \( F \), in Eq.(3-22) equals to unity.
### Chapter 3: Deterministic Analysis of Slopes

#### 3.4. ILLUSTRATIVE EXAMPLES-ANALYSES WITH ALTERNATIVE METHODS

In order to illustrate the methods of analysis presented in the previous sections a simple and homogeneous slope has been assumed and is shown in Fig. 3-4. A specified circular slip surface is considered in these analyses. This slip surface along with the data are shown in Fig 3-4.

![Graph showing potential circular slip surface](image)

**Fig. 3-4 The Geometry and Shear Strengths of the Example Problem**

#### 3.4.1 THE EFFECTS OF THE ASSUMPTION BETWEEN INTERSLICE FORCES ON THE FACTOR OF SAFETY (FOS) USING M&P METHOD

As previously mentioned in Section 3.3.2., the Morgenstern and Price method assumes an arbitrary mathematical function to describe the relationship between interslice normal and shear forces. Several types of functions $f(x)$ are usually used to represent the changes of the relationship between interslice forces from one end of a sliding mass to the
other. The functions generally assumed are the following (the shapes of these functions are shown in Fig. 3-5):

(i) \( f(x) = \text{constant} \) (this implies parallel interslice forces);
(ii) \( f(x) = \text{half sine curve} \);
(iii) \( f(x) = \text{half sine curve of amplitude 2 clipped} \);
(iv) \( f(x) = \text{half sine wave added to a trapezoid} \). Here \( a \) and \( b \) are specified by the use of a computer programme;
(v) \( f(x) = \text{full sine wave adjusted as shown} \);
(vi) \( f(x) = \text{full sine wave of amplitude 2 but clipped at values greater than 1} \);
(vii) \( f(x) = \text{full sine wave added to trapezoid} \);
(viii) \( f(x) = \text{arbitrary with convex characteristics} \);
(ix) \( f(x) = \text{arbitrary with wave characteristics} \);
(x) \( f(x) = \text{arbitrary with concave characteristics} \).

Factors of safety corresponding to these alternative assumptions are presented in Table 3-1. The results indicate that the maximum FOS is associated with the assumption in which the relationship between interslice forces is an arbitrary wave function and the minimum FOS is associated with the assumption in which the relationship between interslice forces is a full sine wave function. It is, however, interesting to note that the differences of FOS based on the different interslice force assumptions are not significant. The difference between the two extreme values of FOS is 0.004 (0.23%). The average factor of safety based on these assumptions is 1.7203. The results show that the coefficient \( \lambda \) changes significantly with the different interslice force assumptions.

An admissible or acceptable limit equilibrium solution must give a reasonable distribution of normal stresses on the assumed failure surface. The general limit equilibrium slope stability problem is statically indeterminate and there is no unique normal stress distribution for a given failure surface. In the Morgenstern and Price method a set of assumptions is made to render the problem statically determinate and each
Fig. 3-5. Some Side Force Assumptions for M&P Method (after Hamel, 1968)
set of assumptions leads to a corresponding normal stress distribution. For the above ten assumptions of the relationship between the interslice forces, the normal stress distributions on the assumed failure surface are shown in the Fig.3-6. These distribution curves indicate that the shapes of distribution are essentially the same except for the assumption of (a) arbitrary wave function and (b) arbitrary concave function.

Table 3-1 Factors of Safety Based on the Different Assumptions of Interslice Forces and Associated $\lambda$ Values

<table>
<thead>
<tr>
<th>Types of Function</th>
<th>Factor of Safety</th>
<th>Coefficient $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>i</td>
<td>1.7210</td>
<td>0.2867</td>
</tr>
<tr>
<td>ii</td>
<td>1.7196</td>
<td>0.3750</td>
</tr>
<tr>
<td>iii</td>
<td>1.7202</td>
<td>0.3119</td>
</tr>
<tr>
<td>iv</td>
<td>1.7201</td>
<td>0.3482</td>
</tr>
<tr>
<td>v</td>
<td>1.7191</td>
<td>0.4371</td>
</tr>
<tr>
<td>vi</td>
<td>1.7195</td>
<td>0.3448</td>
</tr>
<tr>
<td>vii</td>
<td>1.7192</td>
<td>0.3904</td>
</tr>
<tr>
<td>viii</td>
<td>1.7196</td>
<td>0.4010</td>
</tr>
<tr>
<td>ix</td>
<td>1.7231</td>
<td>0.6231</td>
</tr>
<tr>
<td>x</td>
<td>1.7220</td>
<td>0.4158</td>
</tr>
</tbody>
</table>

Fig. 3-6 Distribution of Normal Stresses on the Base of Slice with Different Assumptions between Interslice Forces
3.4.2. The Relationships Between $\lambda$, $F_m$ and $F_f$

The variation of the factor of safety based either on moment equilibrium ($F_m$) or on force equilibrium ($F_f$) with the value of $\lambda$ are shown in Fig. 3-7. The factors of safety based on simplified Bishop and simplified Janbu methods can be obtained from this figure when $\lambda$ equals to zero. The relationship curves show that the factor of safety based on moment equilibrium is relatively insensitive to $\lambda$ whereas the factor of safety based on the force equilibrium is very sensitive to $\lambda$. Therefore, the moment equilibrium equation is usually chosen to calculate the pseudo factor of safety in first level iterative procedure for calculating the real factor of safety based on Morgenstern and Price method. The relationship curves also show that the factors of safety based on moment and force equilibrium equations will be equal at a particular value of $\lambda$.

![Fig. 3-7 Comparison of Factors of Safety with Different $\lambda$](image-url)
3.4.3. **THE COMPARISON OF DIFFERENT ANALYSIS MODELS**

All methods introduced in Section 3.3 were used to analyse the example problem of Fig. 3-4 considering two different pore water pressure conditions. The first condition corresponds to no pore pressure and hence the value of the homogeneous pore pressure ratio \( r_u = 0 \). The second condition corresponds to a value of \( r_u = 0.25 \). The results are presented in Table 3-2. It is clear from those that the factors of safety obtained by the Morgenstern and Price method are very close to those computed by the simplified Bishop method. This is not surprising since the assumption of side force function has an insignificant influence on factor of safety based on the moment equilibrium (Fredlund and Krehn, 1976). On the other hand the values of factor of safety based on the Sarma method are significantly higher than those based on the M&P method. The reason is that the shear strength between the slices is considered to be mobilized to the same extent as that along the base of each slice.

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Conditions</th>
<th>Factor of Safety</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Bishop</td>
</tr>
<tr>
<td>1</td>
<td>( r_u = 0.25 )</td>
<td>1.7231</td>
</tr>
<tr>
<td>2</td>
<td>( r_u = 0.0 )</td>
<td>2.0217</td>
</tr>
</tbody>
</table>

From a practical point of view, the analysis results associated with the Bishop method and with the Morgenstern and Price method are more conservative than those associated with Sarma method. It should be noted that overall moment equilibrium is not considered in the Sarma method.

3.5. **RELATIONSHIPS AMONG F, \( \theta \) AND \( K_c \)**

It was considered of interest to evaluate the value of the factor of safety for a wide range of values of \( \theta \) which defines the direction of the pseudo-static earthquake force. The value of \( \theta \) was varied between the limit -90° to + 90°. For each direction the
horizontal and vertical components of the seismic or acceleration coefficient, $K_h$ and $K_v$, can also be computed. In order to demonstrate the relationship between factors of safety and acceleration directions, a simple slope with a potential slip surface was analysed. The geometry of the slope and slip surface as well as other data are shown in Fig. 3-8.

![Fig. 3-8 Geometry and Strength Parameters for A Homogeneous Slope Case](image)

The pseudo-static seismic or acceleration coefficient is assumed as $K = 0.1$ and the homogeneous pore water pressure ratio $r_u = 0.4$. The factors of safety, based on Bishop simplified and Morgenstern & Price method are shown in Table 3-3.

The calculation results shown in Table 3-3 indicate that there is an acceleration direction which gives the minimum factor of safety. This may be called the critical value of $\theta = \theta_c$. These calculation results are also presented as curves in Fig. 3-9. On the basis of Table 3-3 or Fig. 3-9, the value of critical direction is approximately $\theta_c = -9^\circ$ for this particular example.

On the basis of Eqs. (3-8b), (3-13b) and (3-14b), it is possible to find the relationship between factor of safety and seismic or acceleration coefficient $K$ for a acceleration direction. The factors of safety corresponding to the various acceleration coefficients can be calculated using both the simplified Bishop method and the Morgenstern & Price method. The curves in Fig. 3-10 show how the factor of safety decreases with the increase in the acceleration coefficient.
coefficient. The critical acceleration coefficient \( K = K_c \) corresponds to a value of \( F = 1 \).

Based on the Bishop method \( K_c = 0.195 \) for this particular example and the corresponding value based on the Morgenstern and Price method is 0.210.

Table 3-3 Numerical Analysis Results Associated with Fig. 3-8

<table>
<thead>
<tr>
<th>( \theta^\circ )</th>
<th>( K_h )</th>
<th>( K_v )</th>
<th>Bishop Method ( F = 1.5664 )</th>
<th>M&amp;P Method ( F = 1.5793 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-90</td>
<td>0</td>
<td>-0.1</td>
<td>1.5444</td>
<td>1.5793</td>
</tr>
<tr>
<td>-70</td>
<td>0.0342</td>
<td>-0.0910</td>
<td>1.4037</td>
<td>1.4188</td>
</tr>
<tr>
<td>-50</td>
<td>0.0643</td>
<td>-0.0766</td>
<td>1.3040</td>
<td>1.3203</td>
</tr>
<tr>
<td>-30</td>
<td>0.0866</td>
<td>-0.0500</td>
<td>1.2461</td>
<td>1.2632</td>
</tr>
<tr>
<td>-15</td>
<td>0.0966</td>
<td>-0.0259</td>
<td>1.2281</td>
<td>1.2454</td>
</tr>
<tr>
<td>-12</td>
<td>0.0978</td>
<td>-0.0208</td>
<td>1.2270</td>
<td>1.2442</td>
</tr>
<tr>
<td>-10</td>
<td>0.0985</td>
<td>-0.0174</td>
<td>1.2266</td>
<td>1.2439</td>
</tr>
<tr>
<td>-8</td>
<td>0.0990</td>
<td>-0.0139</td>
<td>1.2266</td>
<td>1.2439</td>
</tr>
<tr>
<td>-5</td>
<td>0.0996</td>
<td>-0.0087</td>
<td>1.2273</td>
<td>1.2445</td>
</tr>
<tr>
<td>0</td>
<td>0.1</td>
<td>0</td>
<td>1.2300</td>
<td>1.2471</td>
</tr>
<tr>
<td>5</td>
<td>0.0996</td>
<td>0.0087</td>
<td>1.2348</td>
<td>1.2517</td>
</tr>
<tr>
<td>8</td>
<td>0.0990</td>
<td>0.0139</td>
<td>1.2386</td>
<td>1.2554</td>
</tr>
<tr>
<td>10</td>
<td>0.0985</td>
<td>0.0174</td>
<td>1.2415</td>
<td>1.2582</td>
</tr>
<tr>
<td>15</td>
<td>0.0966</td>
<td>0.0259</td>
<td>1.2501</td>
<td>1.2667</td>
</tr>
<tr>
<td>30</td>
<td>0.0866</td>
<td>0.0500</td>
<td>1.2872</td>
<td>1.3029</td>
</tr>
<tr>
<td>50</td>
<td>0.0643</td>
<td>0.0766</td>
<td>1.3612</td>
<td>1.3756</td>
</tr>
<tr>
<td>70</td>
<td>0.0342</td>
<td>0.0940</td>
<td>1.4613</td>
<td>1.4744</td>
</tr>
<tr>
<td>90</td>
<td>0</td>
<td>0.1</td>
<td>1.5844</td>
<td>1.5964</td>
</tr>
</tbody>
</table>

The value of the critical acceleration coefficient, \( K_c \), depends not only on the geometry and strength parameters of slopes but also on the acceleration direction. For simplified Bishop method, the critical direction \( \theta_c \) can be derived analytically by differentiating \( K_c \) with respect to \( \theta \) based on Eq. (3-8a) and equating to zero. The value of \( \tan \theta_c \) can be expressed in the following form:
Figure 3-9: Relation between Factors of Safety and Directions of Acceleration (K = 0.1)

Figure 3-10: Relationship between Factor of Safety and Acceleration Coefficient

\[
\tan \theta_c = \frac{(\overline{C} - \overline{D}) R}{B} \quad (3-23)
\]

or,

\[
\theta_c = \tan^{-1}\left(\frac{(\overline{C} - \overline{D}) R}{B}\right) \quad (3-24)
\]

where,

\[
\overline{C} = \sum_{i=1}^{n} W_i \sin \alpha_i ;
\]

\[
\overline{D} = \sum_{i=1}^{n} W_i \tan \phi_i / m \alpha_i ;
\]
Based on the given geometrical and shear strength parameters values, for the minimum critical seismic coefficient and critical acceleration direction, $\theta_c$, can be obtained by Eq. (3-24). The values are $\theta_c = -15^\circ$ and the minimum $K_c = 0.194$. It is obvious that the critical acceleration coefficient with respect to $\theta = -15^\circ$ is less than that associated with $\theta = -9^\circ$.

For Morgenstern and Price method, the value of $K_c$ cannot be determined analytically because Eqs. (3-13b) and (3-14b) are not explicit with respect to the acceleration direction. A numerical approach may, however, be used to evaluate the critical acceleration direction.

3.6. SUMMARY

1. The basic characteristics of a generalised procedure of slices approach (GPS) to limit equilibrium analysis have been presented in this chapter. Three methods of limit equilibrium analysis have been presented in some detail. These are (a) simplified Bishop method, (b) Morgenstern and Price method and (c) Sarma's method with non-vertical slices. Moreover, the pseudo-static seismic forces have been included in the formulations.

2. The analyses of a typical example indicate that the choice of the function $f(x)$, which is used to describe the relationship between interslice forces in the Morgenstern and Price method, does not have a significant influence on the calculated value of the factor of safety. Ten different forms of the function $f(x)$ were chosen for these examinations. The relative difference of the factors of safety was less than 1%. However, the distributions of the normal stresses on the base of slice may not be reasonable for some assumed interslice force functions.
3. The factor of safety based on moment equilibrium is much less sensitive to the coefficient \((\lambda)\) than that based on force equilibrium. Comparisons of results, based on different analysis models, show that the values of the factors of safety associated with simplified Bishop and M&P methods are very close. On the other hand the Sarma method gives consistently higher value of the factor of safety.

4. On the basis of both simplified Bishop method and Morgenstern & Price method, the expressions for the critical seismic or acceleration coefficient have been derived. The influence of acceleration direction, \(\theta\), on the factor of safety was investigated. The analysis results reveal that there is a critical acceleration direction, \(\theta_c\), which results in a minimum value of the factor of safety. The relationship between factor of safety and acceleration coefficient is also discussed by means of the developed formulas. Associated with simplified Bishop method, an analytical formula has been derived for determining the minimum \(K_c\) with respect to the acceleration direction \(\theta\).
CHAPTER 4
IDENTIFICATION OF SLIP SURFACES USING OPTIMISATION TECHNIQUE

4.1. GENERAL REMARKS

Studies concerned with the stability of slopes are usually carried out within the framework of limit equilibrium methods. Whatever the chosen method of limit equilibrium, a deterministic analysis of the slope stability generally involves two procedures: one for the calculation of the factor of safety (FOS) based on a specified potential slip surface and the other for locating the critical slip surface.

Several methods of analysis have been introduced in the previous chapter. This chapter is mainly concerned with the procedures for the location of critical slip surfaces. Many approaches which have been developed in the last several decades for searching the critical slip surface use a line linked by a set of nodes to define a potential non-circular slip surface. By changing the nodal coordinates iteratively, a critical slip surface with the minimum FOS may be located. Different search techniques correspond to the different
ways of changing the location of the nodes. A method of unconstrained optimisation, the conjugate gradient algorithm, is presented in this chapter and used for searching the critical slip surface with the minimum factor of safety. **Based on the calculation formulas developed in Chapter 3, however, a critical slip surface with the minimum pseudo-static acceleration or seismic coefficient may also be found by this optimisation approach.**

For a real slope, there may be some geometrical constraints to the development or location of a slip surface such as discontinuities with low shear resistance or bed rock at shallow depth. Therefore the critical slip surface will follow or be tangent to these known surfaces. The nodes defining these special surfaces may be fixed during the search for a critical slip surface. In such problems the optimisation approach to be used is called 'unconstrained' optimisation approach although the surface itself has constraints.

The application of the conjugate gradient algorithm for searching the critical slip surface requires the evaluation of the derivatives of an objective function \( f(X) \) with respect to its variables. The factor of safety is considered as an objective function with respect to the coordinates of the nodes defining the potential slip surface. This objective function \( f(X) \) in slope stability, the expression for the factor of safety, is not only non-linear but also inexplicit with respect to those coordinates. The finite difference technique has been used to evaluate the partial derivatives of the objective function \( f(X) \) (Xiong & Xu, 1985 and Chen & Shao, 1987) when the gradient methods are used to search for the critical slip surfaces. The accuracy of the finite difference technique depends on the type of objective function and the magnitudes of the argument increment used. An alternative technique is presented in this chapter to evaluate the partial derivatives of the objective function. This technique is called the rational polynomial technique (RPT) and was first used by Li and White (1987a) in connection with slope stability problems. In this thesis the application is extended to estimate the partial derivatives of inexplicit functions.
4.2. METHOD OF CONJUGATE GRADIENTS

Consider a function of n variables whose value \( f(X) \) and gradient vector \( g(X) \) can be calculated at any point \( X \). In the neighbourhood of the required minimum \( X^* \), the function may be expanded in the following form:

\[
f(X) = f(X^*) + \frac{1}{2} (X - X^*)^T A (X - X^*) + \text{higher terms} \quad (4-1)
\]

where, \( A \) is the matrix of second-order partial derivatives, i.e., a Hessian matrix which is a square matrix each term of which is a second-order partial derivative of the same function.

For a slope stability problem, \( f(X) \) represents factor of safety \( F \) which is a function of the geometrical parameters of the slip surface represented by vector of \( X \). For example, if the slip surface is of circular shape, \( X = (x_0, y_0, R) \) where \((x_0, y_0)\) are the coordinates of the centre and \( R \) is the radius.

The above equation, Eq. 4-1, follows the Taylor series procedure for the approximation of a function at the location \( X^* \).

4.2.1. BASIC CONCEPT OF CONJUGATE DIRECTIONS

Virtually all iterative minimisation procedures locate \( X^* \) as the limit of a sequence \( X_0, X_1, X_2, \ldots \) where \( X_0 \) is an initial approximation to the position of the minimum, and where for each \( i \geq 0 \), \( X_{i+1} \) is the position of the minimum of \( f(X) \) with respect to variations along the line through \( X_i \) in some specified direction \( p_i \). Thus, for example, the method of steepest descent uses the direction of the negative gradient of \( f(X) \) at \( X_i \), and the method of alternating directions uses cyclically the directions of the \( n \) coordinate axes. Methods which calculate each new direction of search as part of the iteration cycle are inherently more powerful than those in which the directions are assigned in advance, in
that any accumulated knowledge of the local behaviour of the function can be taken into account. The conjugate gradient method is one such powerful method.

As is well known, any positive-definite quadratic function $f(X)$ may be written as

$$f(X) = \frac{1}{2} X^T Q X + b^T X + c$$  \hspace{1cm} (4-2)

where, $Q$ is a constant coefficient, symmetric and positive definite matrix; $b$ is a constant vector and $c$ is a scalar constant.

In order to objectively describe the search procedure based on the method of conjugate directions, a two-dimensional quadratic function is examined. Setting an initial point $X_0$, then the new point $X_1$ can be obtained by the linear search technique along the direction $p_0$. The direction $p_0$ can be determined by the following relation:

$$g^T_1 p_0 = 0$$  \hspace{1cm} (4-3)

where, $g_1 = g(X_1)$ is a gradient vector of $f(X)$ at the point $X_1$.

Eq.(4-3) means that the line $L_0$ in the direction $p_0$ is tangential to a contour at point $X_1$ as shown in Fig. 4-1. If the negative gradient of $f(X)$ at $X_1$ is chosen as the search direction such as the method of steepest descent, then an oscillatory phenomenon will occur. In order to avoid this disadvantage, let the search direction of the next iterative step directly point to the minimum point $X^*$. If this direction can be found then the minimum point of the two dimensional quadratic $f(X)$ can be obtained after two sequential linear searches. Therefore the new search direction $p_1$ may be related with the minimum point $X^*$ and point $X_1$, i.e.

$$X^* = X_1 + t_1 p_1$$  \hspace{1cm} (4-4)

where, $t_1$ is a scalar parameter.

For a two-dimensional quadratic $f(X)$ (the function has two independent coordinates), the derivative vector $g(X)$ is,

$$g(X) = Q X + b$$  \hspace{1cm} (4-5)
Because $X^*$ is the minimum point of the $f(X)$, the following equation can be derived from optimisation theory, i.e.

$$g(X^*) = Q X^* + b = 0$$  \hfill (4-6)
Multiplying each term of Eq.(4-7) on the left hand side by the vector $p_0^T$, the following condition making $p_1$ directly point to the minimum point $X^*$ can be obtained (since the first term become zero, $g(X)$, being orthogonal to $p_0^T$).

$$p_0^T Q p_1 = 0 \quad (4-8)$$

If the two directions $p_0$ and $p_1$ satisfy this condition, i.e. Eq. (4-8), then these two directions are called as the mutually conjugate with respect to the matrix $Q$.

Assuming the direction $p_1$ can be expressed by a linear combination of the two direction vectors $g(X_1)$ and $p_0$,

$$p_1 = -g(X_1) + \alpha_0 p_0 \quad (4-9)$$

Multiplying both sides of Eq.(4-9) by $p_0^T Q$ at the left side and using Eq. (4-8), we have

$$p_0^T Q p_1 = -p_0^T Q g(X_1) + \alpha_0 p_0^T Q p_0 = 0 \quad (4-10)$$

From Eq. (4-10), the scalar parameter $\alpha_0$ can be expressed as follows:

$$\alpha_0 = \frac{p_0^T Q g(X_1)}{p_0^T Q p_0} \quad (4-11)$$

Hence, the expression for direction $p_1$ may be rewritten as,

$$p_1 = -g(X_1) + \frac{p_0^T Q g(X_1)}{p_0^T Q p_0} p_0 \quad (4-12)$$

Therefore, the minimum point $X^*$ can be obtained by substituting Eq. (4-12) into Eq. (4-4). This implies that the minimum point $X^*$ of the two dimensional quadratic functions $f(X)$ may be found after two linear searches along the two conjugate directions $p_0$ and $p_1$ are performed by starting at an arbitrary initial point $X_0$. 
This idea can be extended to an n-dimensional case. First, let \( p_0, p_1, \ldots, p_{k-1} \) be linearly independent (or orthogonal) vectors, these vector can constitute a \( K \)-plane, \( \pi_k \). The minimum point, \( X_k \), of \( f(X) \) on this \( K \)-plane, \( \pi_k \), can be obtained through a given initial point \( X_0 \) if these nonnull vectors \( p_0, p_1, \ldots, p_{k-1} \), which are a basis of \( \pi_k \), are mutually conjugate with respect to matrix \( Q \), i.e.,

\[
p_j^T Q p_j = 0 \quad (i \neq j, i = 0, 1, \ldots, k-1)
\]  

(4-13)

In order to obtain a formula for the minimum point \( X_k \) of \( f(X) \) on the plane \( \pi_k \) through an inceptive point \( X_0 \), an important theorem of the method in conjugate direction methods must be introduced as follows.

**Theorem 4.1.** Let \( Q \) be a \( n \times n \)-dimensional, symmetric and positive definite matrix and let \( p_0, p_1, \ldots, p_{k-1} \) be a conjugate system with respect to \( Q \). For a given initial point \( X_0 \), the \( k \) linear searches are sequentially performed along those conjugate vectors for the \( n \)-dimensional quadratic objective function \( f(X) \), i.e.

\[
X_{i+1} = l s (X_i, p_j) \quad i = 0, 1, \ldots, k-1
\]  

(4-14)

Then the following two conclusions can be drawn,

(i) \( p_j^T Q g(X_k) = 0 \quad 0 \leq j \leq k \)  

(4-15)

(ii) \( X_k \) is a minimum point of \( f(X) \) on the plane \( \pi_k \). This minimum point can be expressed by a linear combination of the conjugate system

\[
X_k = X_0 + \sum_{i=0}^{k-1} \alpha_i p_i
\]  

(4-16)

This theorem can be objectively demonstrated by means of a geometrical graph as shown in Fig. 4-2. Let \( n = 3 \) and \( k = 2 \); then a two-dimensional plane \( \mathbb{R}^2 \), which is constituted by the two conjugate vectors \( p_0 \) and \( p_1 \) with respect to \( Q \), through the original point of the coordinate system, may be obtained. After a linear search along \( p_0 \) through
the initial point $X_0$, the point $X_1$ can be located. Then a linear search along $p_1$, through the point $X_1$, is again performed for obtaining point $X_2$. It is the plane $\pi_2$ that passes through the point $X_0$ and is constituted by the vectors $p_0$ and $p_1$. This plane is parallel to the plane $R^2$. The gradient of $f(X)$ at the point $X_2$ must be perpendicular to $p_0$ as well as $p_1$ and $X_2$ is a minimum point of $f(X)$ on the plane $\pi_2$.

![Geometrical Graph for Interpreting Theorem 4.1](after Xie, 1981)

Two important corollaries of theorem 4.1. are:

**Corollary 4.1.** When $k = n$ in theorem 4.1., $X_n$ is a minimum point of the positive definite quadratic function $f(X)$ on the plane $R^n$.

**Corollary 4.2.** Any linear combination of $p_0, p_1, ..., p_{k-1}$ in theorem 4.1., i.e. $\sum_{i=0}^{k-1} \beta_i p_i$, is perpendicular to the $g(X_k)$ (where, $\beta_0, \beta_1, ..., \beta_{k-1}$ are arbitrary real constants).
4.2.2 METHODS OF CONJUGATE DIRECTION

According to the theorem 4.1., as well as the two corollaries 4.1. and 4.2., a general algorithm of conjugate directions for searching the minimum point of a positive definite quadratic function \( f(X) \) can be summarised as follows:

(i) Select an initial point \( X_0 \) as well as a descent direction vector \( p_0 \) and let \( k = 0 \);

(ii) Perform a linear search along the direction \( p_k \) for determining the point \( X_{k+1} \), i.e., \( X_{k+1} = ls(X_k, p_k) \) and \( X_{k+1} \) can be explicitly calculated by a formula,

\[
X_{k+1} = X_k - \frac{p_k^\top g(X_k)}{p_k^\top Q p_k} p_k
\]

(iii) If \( ||g(X_{k+1})|| < \varepsilon \) (\( \varepsilon \) is a given tolerance) is valid then \( X_{k+1} \) is a minimum point of \( f(X) \) otherwise the following steps are carried out;

(iv) Determine a conjugate direction \( p_{k+1} \) so that the following formula is satisfied, i.e.,

\[
p_j^\top Q p_{k+1} = 0 \quad j = 0, 1, ..., k
\]

(v) Let \( k = (k+1) \) and go back to step (ii).

In this algorithm the conjugate vector system can be determined by various methods. For the general function, this algorithm can still be applied to search the minimum point \( X^\star \) of \( f(X) \) because the value of the function at a neighbourhood of the \( X^\star \) may be approximately expressed by a quadratic function as mentioned above. In this case the matrix \( Q \) is the matrix \( A \) in Eq.(4-1) and it is called a Hessian matrix.

4.2.3. THE METHOD OF CONJUGATE GRADIENT

After selecting a point \( X_0 \), let the conjugate direction \( p_0 \) equal the steepest descent vector at the point \( X_0 \), i.e., \( p_0 = -g(X_0) \). Determine the sequence conjugate direction \( p_k \).
by means of the linear combination of the negative gradient \(-g(X_k)\) at the point \(X_k\) as well as the previously obtained conjugate direction \(p_{k-1}\). This leads to a specific conjugate direction algorithm which is called as conjugate gradient algorithm. In this algorithm each conjugate direction is constructed by the negative gradients at iterative point.

For a quadratic function, therefore, the direction \(p_k\) can be written as:

\[
p_k = -g(X_k) + \alpha_{k-1} p_{k-1} \tag{4-17}
\]

where,

\[
\alpha_{k-1} = \frac{g_k^\top Q p_{k-1}}{p_{k-1}^\top Q p_{k-1}} \tag{4-18}
\]

Performing a linear search along the direction \(p_k\), the new point \(X_{k+1}\) can be located thus:

\[
X_{k+1} = X_k + t_k p_k \tag{4-19}
\]

where,

\[
t_k = \frac{-p_k^\top g_k}{p_k^\top Q p_k} = \frac{g_k^\top g_k}{p_k^\top Q p_k} \tag{4-20}
\]

In order to apply the conjugate gradient approach to a general function, the matrix \(Q\) in Eqs. (4-18) and (4-20) has to be eliminated because the matrix \(Q\) is a Hessian matrix and it is, therefore, impossible to calculate the matrix \(Q\) (or \(A\)) when the function \(f(X)\) is inexplicit. Fletcher and Reeves (1964) proposed an alternative formula for calculating the parameter \(\alpha_k\) as follows:

\[
\alpha_k = \frac{||g_{k+1}||^2}{||g_k||^2} \tag{4-21}
\]

In the algorithm of Fletcher and Reeves a numerical approach was used for the linear search. Thus the calculation of Hessian matrix for the general objective function is not required in their algorithm.
The Fletcher and Reeves algorithm (F-R algorithm) is a most effective conjugate gradient method. The algorithm for the n-dimensional non-quadratic function is introduced as follows:

(i) Select an arbitrary initial point \( X_0 \) and calculate the values of the function and gradient at \( X_0 \), i.e., \( f_0 = f(X_0) \) and \( g_0 = g(X_0) \). Let \( p_0 = -g_0 \) and \( k = 0 \);

(ii) Perform the linearly search, i.e. \( X_{k+1} = ls (X_k, p_k) \), based on a numerical approach. Let \( f_{k+1} = f(X_k) \) and \( g_{k+1} = g(X_k) \);

(iii) If the convergence condition is valid then \( X_{k+1} \) is the minimum point of \( f(X) \); otherwise the following steps will be carried out;

(iv) If \( k = n \) then let \( X_0 = X_{k+1}, f_0 = f_{k+1}, g_0 = g_{k+1}, p_0 = -g_0 \) and \( k = 0 \). Go back to step (ii) otherwise carry out the next step;

(v) Use Fletcher and Reeves algorithm to calculate the conjugate direction \( p_{k+1} \), i.e.,

\[
\alpha_k = \frac{||g_{k+1}||^2}{||g_k||^2}, \quad p_{k+1} = -g_{k+1} + \alpha_k p_k
\]

(vi) If \( p_{k+1}^T g_{k+1} < 0 \) is valid then let \( k = 0 \) and \( p_0 = -g_{k+1} \) and go back to step (ii) otherwise let \( k = (k+1) \) and go back to step (ii).

This process is assured, apart from rounding errors, to locate the minimum of any quadratic function with \( n \) arguments within at most \( n \) iterations. For non-quadratic objective function more than \( n \)-step iterations may be required, but the convergence rate is over-linear. The golden section method is used as the linear search method in the calculation program. The flowchart of the F-R Algorithm is shown in Fig. 4-3.

4.2.4. GOLDEN SECTION METHOD

According to Eq. (4-19), when the search direction \( p_k \) and point \( X_k \) are known the function \( f(X_{k+1}) \) becomes a one dimensional function and the minimum point \( X_{k+1} \) of the function \( f(X) \) can be obtained by a linear search, i.e.,

\[
f(X_k + t_k p_k) = \min_t f(X_k + t p_k) = \min_t \phi(t) \quad (4-22)
\]
Chapter 4: Identification of Slip Surfaces Using Optimisation Technique

Begin

Give the initial point, i.e., \( X_0 = (x_{01}, x_{02}, \ldots, x_{0n}) \)

Calculate: \( f_0 = f(X_0) \) and \( g_0 = g(X_0) \)

where, \( g_0 = \left( \frac{\partial f(X)}{\partial x_1}, \frac{\partial f(X)}{\partial x_2}, \ldots, \frac{\partial f(X)}{\partial x_n} \right)^T \)

Let \( k = 0 \) and \( p_k = -g_k \)

Perform a linear search along the direction \( p_k \) i.e., \( X_{k+1} = l_s (X_k, p_k) \)

or, \( X_{k+1} = X_k + t_k p_k \)

where, \( t_k \) can be determined by the golden section method.

Calculate: \( f_{k+1} = f(X_{k+1}) \) and \( g_{k+1} = g(X_{k+1}) \)

where, \( g_{k+1} = \left( \frac{\partial f(X)}{\partial x_1}, \frac{\partial f(X)}{\partial x_2}, \ldots, \frac{\partial f(X)}{\partial x_n} \right)^T \)

Use Fletcher and Reeves formula to calculate a new conjugate direction, i.e.,

\[
\alpha_k = \frac{||g_{k+1}||^2}{||g_k||^2} \quad \text{and} \quad p_{k+1} = -g_{k+1} + \alpha_k p_k
\]

Yes

Yes

Convergence criterion is satisfied

No

Print the minimum point \( X_{k+1} \)

Stop

Yes

No

\( k = k + 1 \)

Fig. 4-3. Calculation Flowchart of Fletcher and Reeves (F-R) Algorithm
Although the calculation effort for the one dimensional or linear search is large, it is necessary in order to avoid the determination of the Hessian matrix of the general function $f(X)$.

There have been several algorithms for the linear search, such as Newton-Raphson search, Cubic-convergent search without second derivatives, Quadratic-convergent search without derivatives, Fibonacci search and Golden section search, etc. In this research the golden section algorithm has been applied because the derivative of the function $\phi(t)$ is not required. Thus the algorithm of the golden section is relative simple.

Consider the symmetric placement of two trial points as shown in Fig. 4-4. Starting with a unit interval (purely for convenience), two trials are located a fraction $\tau$ from either end. With this symmetric placement, regardless of which of the corresponding function values is smaller, the length of the remaining interval is always $\tau$. Suppose that the right-hand subinterval is eliminated. It is apparent from Fig. 4-5 that the remaining subinterval of length $\tau$ has the one old trial located interior to it at a distance $(1 - \tau)$ from the left end point.

![Fig. 4-4 Golden Section Search](image1.png)

![Fig. 4-5 Golden Section Interval](image2.png)

In order to retain the symmetry of the search pattern, the distance $(1 - \tau)$ should correspond to a fraction $\tau$ of the length of interval (which itself is of length $\tau$). With this
choice of \( \tau \), the next evaluation can be located at a fraction \( \tau \) of the length of the interval from the right-hand end point (see Fig 4-6).

\[
\begin{array}{c}
\text{0} \quad \text{1-} \frac{1}{\tau} \\
\hline
\text{\( \tau \)} \quad \text{1-} \frac{1}{\tau} \\
\end{array}
\]

Fig. 4-6 Golden Section Symmetry

Hence, with the choice of \( \tau \) satisfying \((1 - \tau) = \tau^2\), the symmetric search pattern of Fig. 4-4 is retained in the reduced interval of Fig. 4-5. The solution of the quadratic equation is

\[
\tau = \frac{-1 \pm \sqrt{5}}{2}
\]

The positive root of this equation is \( \tau = 0.61803 \ldots \) The search scheme for locating trial points based on this ratio is known as the golden section search. After the first two evaluations, each succeeding evaluation will eliminate \((1 - \tau)\) of the remaining interval. Hence, the interval remaining after \( N \) evaluations, assuming the initial was of unit length, will be of length \( \tau^{N-1} \). It can be shown that this is the asymptotic form of the optimum minimum search.

For an arbitrary interval \((a,b)\) the golden section algorithm may be given as follows:

(i) adopt an initial searching interval \((a,b)\) for the function \( \phi(t) \) and let \( \tau = 0.618 \);
(ii) Calculate \( t_2 = a + \tau (b - a) \) and let \( \phi_2 = \phi(t_2) \);
(iii) Calculate \( t_1 = a + b - t_2 \) and let \( \phi_1 = \phi(t_1) \);
(iv) If \( |t_1 - t_2| < \varepsilon \) (a given tolerance), let \( t^* = \frac{t_1 + t_2}{2} \) and stop calculation; if \( |t_1 - t_2| > \varepsilon \)
perform the next step;
(v) If $\phi_1 \leq \phi_2$, let $b = t_2$, $t_2 = t_1$ and $\phi_2 = \phi_1$ and go back to step (iii); if $\phi_1 > \phi_2$ let $a = t_1$, $t_1 = t_2$, $\phi_1 = \phi_2$, $t_2 = a + \beta (b - a)$ and $\phi_2 = \phi(t_2)$ and go back to step (iv).

4.2.5. SIMPLE ILLUSTRATIVE EXAMPLE

A three dimensional quadratic function is examined based on the numerical optimum algorithm proposed by Fletcher and Reeves. Consider the following:

$$f(x_1, x_2, x_3) = 3x_1^2 - 10x_1 + 2x_1x_2 - 3x_1x_3 + 4x_2^2 - 4x_2 + 10x_3^2 - 8x_2x_3 + 3x_3 + 10 \quad (4-24)$$

The components of the gradient direction of the function can be explicitly obtained as follows:

$$g_1 = \frac{\partial f(x_1, x_2, x_3)}{\partial x_1} = 6x_1 + 2x_2 - 3x_3 - 10$$
$$g_2 = \frac{\partial f(x_1, x_2, x_3)}{\partial x_2} = 2x_1 + 8x_2 - 8x_3 - 4$$
$$g_3 = \frac{\partial f(x_1, x_2, x_3)}{\partial x_3} = -3x_1 - 8x_2 + 20x_3 + 3 \quad (4-25)$$

The results of the optimisation process for this function are shown in Table 4-1. The inception point was $(x_1, x_2, x_3) = (2, 4, 6)$ and the initial function value was 216. From Table 4-1 one can find that the minimum value is reached after four iterations and this value is almost identical with the minimum value after five iterations. According to conjugate direction theory, the minimum value of this function should be found in three iterative steps because Eq. (4-24) is a quadratic function with three arguments. However, a numerical linear search procedure, i.e. golden section method, is used to calculate the scalar $t_k$ instead of Eq. (4-20). Therefore, the errors of the numerical process will result in calculation errors associated with the new points.

Table 4-2 shows the calculation results based on the steepest descent algorithm for the same function (Eq. 4-24). The value of the function is close to the minimum value of
Table 4.1: Calculating Results Based on Conjugate Gradient Method

<table>
<thead>
<tr>
<th>Iteration No.</th>
<th>Point Coordinates</th>
<th>Components of gradients of f(X)</th>
<th>Components of conjugate direction</th>
<th>Scalar t</th>
<th>Scalar t</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x₁</td>
<td>x₂</td>
<td>x₃</td>
<td>g₁</td>
<td>g₂</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>216.0</td>
<td>-8.0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.6797</td>
<td>0.3051</td>
<td>6</td>
<td>0.2267</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1.6797</td>
<td>0.3051</td>
<td>6</td>
<td>0.2267</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>1.6797</td>
<td>0.3051</td>
<td>6</td>
<td>0.2267</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>1.6797</td>
<td>0.3051</td>
<td>6</td>
<td>0.2267</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>1.6797</td>
<td>0.3051</td>
<td>6</td>
<td>0.2267</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>1.6797</td>
<td>0.3051</td>
<td>6</td>
<td>0.2267</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>1.6797</td>
<td>0.3051</td>
<td>6</td>
<td>0.2267</td>
</tr>
</tbody>
</table>

Objective function f(X)
the function after twelve iterative steps are performed. It is obvious that the calculation effectiveness of Fletcher and Reeves Algorithm (F-R Algorithm) is much higher than the steepest descent method and is generally twice faster than the latter. Although the calculation of conjugate directions is required, the main calculation effort of gradient methods is concentrated on the computation of the function gradients when the function is general and inexplicit.

Table 4-2 Calculation Results During Steepest Descent Iterative Process

<table>
<thead>
<tr>
<th>Iteration No.</th>
<th>Point Coordinates</th>
<th>f(X)</th>
<th>Negative Gradient of f(X)</th>
<th>Scalar t</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x₁</td>
<td>x₂</td>
<td>x₃</td>
<td>-g₁</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>216</td>
</tr>
<tr>
<td>1</td>
<td>2.3483</td>
<td>4.6967</td>
<td>2.2990</td>
<td>51.7407</td>
</tr>
<tr>
<td>4</td>
<td>1.5145</td>
<td>0.6161</td>
<td>0.5970</td>
<td>2.3564</td>
</tr>
<tr>
<td>8</td>
<td>1.6628</td>
<td>0.3289</td>
<td>0.2533</td>
<td>1.3491</td>
</tr>
<tr>
<td>12</td>
<td>1.6759</td>
<td>0.3053</td>
<td>0.2253</td>
<td>1.3423</td>
</tr>
<tr>
<td>14</td>
<td>1.6765</td>
<td>0.3048</td>
<td>0.2237</td>
<td>1.3423</td>
</tr>
</tbody>
</table>
4.3. THE APPLICATION OF RATIONAL POLYNOMIAL TECHNIQUE

The rational polynomial technique is an interpolation method of approximation which can be used in engineering analyses. It is a useful tool for estimating the approximate values of non-linear functions and their derivatives as well as their integrations. The method of reciprocal difference (also known as Thiele's method) for interpolation of univariate functions was extended by Zhou (1982) to multivariate functions and to optimum design involving such functions. Li and White (1987a) used this technique to solve numerical problems in slope stability analysis, such as calculating the factor of safety based on the M&P method and searching the critical slip surfaces. During the research reported in this thesis the application of the method was extended to numerical calculations involved in probabilistic analyses of slope stability (Chowdhury and Xu, 1993 or 1994).

4.3.1. INTERPOLATION OF FUNCTIONS AND A RATIONAL POLYNOMIAL

Consider a set of points \((x_i, y_i)\), \(i = 0, 1, 2, ..., m\) and let \(y\) be a function \(f(x)\) of \(x\) which may be defined explicitly by a mathematical expression or implicitly by a calculation process. Thus,

\[ y_i = f(x_i) \quad i = 0, 1, 2, ..., m \quad (4-26) \]

The points \((x_i, y_i)\) will be called interpolation points. By means of these interpolation points, a rational polynomial \(R_{p,q}(x)\), which closely approximates the given function \(f(x)\) within the domain of interest, may be constructed. In particular the interpolation rational polynomial passes through those interpolation points, i.e.,

\[ R_{p,q}(x) = y_i \quad i = 0, 1, 2, ..., m \quad (4-27) \]

A rational polynomial \(R_{p,q}(x)\) is defined as a ratio of two polynomials \(N_p(x)\) and \(D_q(x)\). The degree of each of these polynomials is not greater than \(p\) and \(q\) respectively.
and they do not have a common factor. Therefore, the rational polynomial $R_{p,q}(x)$ may be expressed as follows:

$$R_{p,q}(x) = \frac{N_p(x)}{D_q(x)} = \frac{c_0 + c_1 + c_2x + \cdots + c_px^p}{b_0 + b_1 + b_2x + \cdots + b_qx^q} \quad (4-28)$$

This equation, Eq.(4-28), defines a rational polynomial with degree $plq$. It is obvious that multiplying $N_p(x)$ and $D_q(x)$ by a non-zero constant, say $1/b_0$, does not affect the value of $R_{p,q}(x)$. Therefore, only $(p+q+1)$ coefficients of $R_{p,q}(x)$ are independent. By means of given points $(x_i, y_i)$ $(i = 0, 1, 2, ..., m)$ and the interpolation conditions Eq. (4-27), the coefficients $c_i$ $(i = 0, p)$ and $b_i$ $(i = 0, q)$ may be determined. Thus, a given function $y = f(x)$ can be approximated by Eq. (4-28).

There are many options in the choice of $R_{p,q}(x)$ for a given set of interpolation points. For instance, if $m = 5$, the possible choices of $R_{p,q}(x)$ are shown in Fig. 4-7. In particular, $R_{5,0}$ represents the polynomial interpolation of the function. However, the rational polynomial located in the shaded region in Fig. 4-7 usually give more accurate approximation for estimating a given function (Stoer and Bulirsch, 1980). That implies $lp - q1 \leq 1$. Therefore, for $m = 5$, it is preferable to use either $R_{3,2}$ or $R_{2,3}$.

Several methods are available for calculating the coefficients of the rational polynomial, for example the matrix method, Stoer's method, and Thiele's method. Thiele's method is introduced in this thesis. This method is also commonly known as the method of reciprocal difference. The interpolation rational polynomial is expressed in the form of a continued fraction as follows:

$$R_{p,q}(x) = a_0 + \frac{x - x_0}{a_1 + \frac{x - x_1}{a_2 + \cdots + \frac{x - x_{m-1}}{a_m}}} \quad (4-29)$$
Eq. (4-29) may also be written in a more compact form as follows

\[
\begin{align*}
R_{p,q}(x) &= \phi_0(x) \\
\vdots \\
\phi_i(x) &= a_i + \frac{x - x_i}{\phi_{i+1}(x)} \\
\vdots \\
\phi_m(x) &= a_m
\end{align*}
\] (4-30)

The coefficient \(a_i\) in Eq. (4-29) can be determined from the interpolation points using the following recurrence relations:
\[
\begin{align*}
\hat{a}_{i,0} &= y_i \\
\hat{a}_{i,j} &= \frac{x_i - x_{j-1}}{a_{i,j-1} - a_{j-1}}, \quad j = 1, 2, \ldots, i \\
\hat{a}_i &= a_{i,i}
\end{align*}
\] (4-31)

The calculation procedure is illustrated in Table 4-3. It can be observed from Eq.(4-31) or Table 4-3 that the coefficient \(a_i\) only depends on \(x_j\) (\(j = 0, 1, 2, \ldots, i\)). Therefore, the addition of an extra point to the continued fractions does not affect the coefficients already calculated. Once the coefficients \(a_i\) are determined, the rational polynomial \(R_{p,q}(x)\) can be recovered by using either forward or backward algorithm. Although Thiele's method is simpler than the matrix or Stoer's method, ill-conditioning may occur at sometime for Thiele's method. However, it can usually be remedied by replacing the interpolation point inducing the ill-condition with another point in its vicinity.

Table 4-3 Coefficient of Rational Polynomial

<table>
<thead>
<tr>
<th>(a_0)</th>
<th>(a_1)</th>
<th>(a_2)</th>
<th>(a_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_0 = y_0)</td>
<td>(a_1 = \frac{x_1 - x_0}{a_{1,0} - a_0})</td>
<td>(a_2 = \frac{x_2 - x_1}{a_{2,1} - a_1})</td>
<td>(a_3 = \frac{x_3 - x_2}{a_{3,2} - a_2})</td>
</tr>
<tr>
<td>(a_{1,0} = y_1)</td>
<td>(a_{2,0} = y_2)</td>
<td>(a_{3,0} = y_3)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(a_{1,1} = \frac{x_1}{a_{1,0} - a_0})</td>
<td>(a_{2,1} = \frac{x_2}{a_{2,0} - a_0})</td>
<td>(a_{3,1} = \frac{x_3}{a_{3,0} - a_0})</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(a_{1,2} = \frac{x_1}{a_{1,1} - a_1})</td>
<td>(a_{2,2} = \frac{x_2}{a_{2,1} - a_1})</td>
<td>(a_{3,2} = \frac{x_3}{a_{3,1} - a_1})</td>
<td>(\ldots)</td>
</tr>
</tbody>
</table>

4.3.2. APPLICATION OF THE RATIONAL POLYNOMIAL METHOD

The rational polynomial technique based on Thiele's method can be used to evaluate the values and the derivatives of linear and non-linear functions and to solve linear or non-linear equations. In this section, the solution of non-linear equations and the estimation of the partial derivatives of multivariate functions will be introduced.
4.3.2.1. **Solution of Non-linear Equations**

A nonlinear equation may be written in the following simple form:

\[ f(x) = 0 \]  
(4-32)

For some engineering problems these may not be an explicit equation but only a calculation procedure. Let \( a \) be the solution of Eq.(4-32) (i.e. \( f(a) = 0 \)) and assume that \( g(y) \), the inverse function of \( f(x) \), exists in the neighbourhood of \( a \). Thus, we have

\[
\begin{align*}
  x &= g(y) \\
  y &= f(x)
\end{align*}
\]

where, \( a \) is given by

\[ a = g(0) \]  
(4-34)

Consider now a set of points \((x_i, y_i)\) \(i = 0, 1, 2, ..., m\). The solution of \( f(x) = 0 \) can be estimated by means of the rational polynomial and these points. Because the inverse function of \( f(x) \) is assumed to be in existence, this inverse function \( g(y) \) can be approximated by the following rational polynomial:

\[
x = g(y) \approx a_0 + \frac{y - y_0}{y - y_1} a_1 + \frac{y - y_2}{y - y_{m-1}} a_2 + \cdots + \frac{y - y_m}{a_m}
\]

(4-35)

The coefficients \( a_i \) \((i = 0, 1, 2, ..., m)\) can be calculated by Table 4-3 in which \( x_i \) and \( y_i \) \((i = 0, 1, 2, ... m)\) are alternated. A estimate of \( a \) is obtained by letting \( y = 0 \). As mentioned above, extra interpolation points can be continuously introduced and do not affect the coefficients already calculated. Therefore only a few initial interpolation points are required. In realistic calculation, three or four initial interpolation points usually are
enough and each new interpolation point based on the previous calculation results will be
used until \( f(x) \) approximates to zero or is less than a given tolerance (very small value).

Two illustrative examples are presented so that the suitability of the rational
calculating of the factor of safety of a slope based on the M&P method. The
graph and properties of the slope are shown in Fig. 3-7 in Chapter 3. Two factors of
in the M&P method, (a) \( F_f \), which is based on the overall equilibrium of
forces and (b) \( F_m \), which is based on the overall moment equilibrium. Both \( F_f \) and \( F_m \) are
functions of parameter \( \lambda \) which is unknown and determined as part of the calculation
This means that the following equation has to be solved:

\[
q(\lambda) = F_f(\lambda) - F_m(\lambda) = 0
\]  

(4-36)

In fact, Eq.(4-36) is only expressed by a calculation procedure rather than an explicit
mathematical formula. The numerical solution can be achieved by using the rational
d. The results of numerical calculation are shown in Table 4-4. Three
chosen tolerance for \( q(\lambda) \) is less than or equals to \( 10^{-7} \).

Table 4-4 Calculation Results of FOS Based on M&P method
   with Rational Polynomial Technique

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( F_f )</th>
<th>( F_m )</th>
<th>( q(\lambda) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08*</td>
<td>1.645832</td>
<td>1.722383</td>
<td>-0.076552</td>
</tr>
<tr>
<td>0.2*</td>
<td>1.682551</td>
<td>1.721338</td>
<td>-0.038787</td>
</tr>
<tr>
<td>0.4*</td>
<td>1.756901</td>
<td>1.719600</td>
<td>0.037301</td>
</tr>
<tr>
<td>0.308071</td>
<td>1.720482</td>
<td>1.720390</td>
<td>0.000092</td>
</tr>
<tr>
<td>0.307831</td>
<td>1.720392</td>
<td>1.720392</td>
<td>( 10^{-7} )</td>
</tr>
</tbody>
</table>

Note: '* indicates each of the three initial interpolation points.'
The second example is the following simple nonlinear equation:

\[ f(x) = e^{-5x} - 1.0^{-5} x = 0 \]

Newton’s method (a well known numerical approximation method) was also used to solve this equation in order to compare the results to those obtained by the rational polynomial method. The calculation results in the iterative procedure are shown in Table 4-5. The initial value is \( x_0 = 3 \) for Newton’s method. The two interpolation points for the rational polynomial method are \( x_0 = 3 \) and \( x_1 = 0.6 \). The number of iterative steps required in the rational polynomial technique is less than that required in Newton’s method to achieve the same accuracy in solving this equation. Also the calculations of the derivative of \( f(x) \) during the iterative procedure are required for Newton’s method.

| Table 4-5 Results of Iterative Procedure Based on Newton’s and Rational polynomial Methods |
|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| Newton’s Method                 | Rational Polynomial Method      |
| \( x \)                        | \( f(x) \)                      | \( x \)                        | \( f(x) \)                      |
| 3                              | \(-2.96941 \times 10^{-5}\)     | 3*                             | \(-2.96941 \times 10^{-5}\)     |
| 0.4245139                      | 0.1197183                       | 0.6*                           | 0.0497811                       |
| 0.6245035                      | 0.0440399                       | 2.998569                       | \(-2.96778 \times 10^{-5}\)     |
| 0.8244660                      | \(1.619846 \times 10^{-2}\)    | 1.757246                       | \(1.352504 \times 10^{-4}\)     |
| 1.024340                       | \(5.955638 \times 10^{-3}\)    | 2.110629                       | \(5.004898 \times 10^{-6}\)     |
| 1.223929                       | \(2.186993 \times 10^{-3}\)    | 2.154675                       | \(-5.96842 \times 10^{-7}\)     |
| 1.422636                       | \(8.000763 \times 10^{-4}\)    | 2.149311                       | \(2.65405 \times 10^{-8}\)      |
| 1.618660                       | \(2.893929 \times 10^{-4}\)    | 2.149535                       | \(1.24145 \times 10^{-11}\)     |
| 1.806834                       | \(1.011955 \times 10^{-4}\)    | 2.149535                       | \(1.24145 \times 10^{-11}\)     |
| 1.973736                       | \(3.203371 \times 10^{-5}\)    |                               |                                 |
| 2.092884                       | \(7.604949 \times 10^{-6}\)    |                               |                                 |
| 2.142698                       | \(8.159212 \times 10^{-7}\)    |                               |                                 |
| 2.149429                       | \(1.245127 \times 10^{-8}\)    |                               |                                 |
| 2.149535                       | \(1.24145 \times 10^{-11}\)    |                               |                                 |
| 2.149535                       | \(1.24145 \times 10^{-11}\)    |                               |                                 |
4.3.2.2 ESTIMATION OF PARTIAL DERIVATIVES FOR MULTIVARIATE FUNCTION

Usually, the factor of safety of a slope cannot be explicitly expressed by a mathematical formula based on limit equilibrium concepts. Moreover, location of a critical slip surface with the conjugate gradient method requires estimation of the partial derivatives of the factor of safety with respect to geometrical parameters. The evaluation of these partial derivatives cannot be carried out by an analytical approach and numerical methods have, therefore, to be used. The forward or backward finite difference method has been used to evaluate the partial derivatives of factor of safety with respect to the geometrical parameters (Xiong and Xu, 1985; Chen and Shao, 1987). However, for the research reported in this thesis the rational polynomial technique is used.

In this chapter, the rational polynomial technique is introduced for estimating the partial derivatives of a general multivariate function.

Consider a multivariate function of the following form:

\[ f(\mathbf{X}) = f(x_1, x_2, \ldots, x_n) \]  

(4-37)

The derivatives of \( f(\mathbf{X}) \) at the point \((\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)\) (a considered point) can be estimated by the rational polynomial technique. First, keep the values of \( x_2 \) to \( x_n \) equal respectively to \( \bar{x}_2, \ldots, \bar{x}_n \) and consider \( m \) sets of values of \( x_1^{(i)} \) as given \((i = 0, 1, \ldots, m)\). In other words, \( m \) discrete values are assigned to variable \( x_1 \) and these values must lie between the lower and upper bound values of this variable (Generally, the lower and upper bound values are close to \( x_1 \)). Thus there are \( m \) partial function values with respect to \( x_1 \), which may be written as follows:

\[
\begin{align*}
F_{10} &= f(x_1^{(0)}, \bar{x}_2, \ldots, \bar{x}_n) \\
F_{11} &= f(x_1^{(1)}, \bar{x}_2, \ldots, \bar{x}_n) \\
&\vdots \\
F_{1m} &= f(x_1^{(m)}, \bar{x}_2, \ldots, \bar{x}_n)
\end{align*}
\]  

(4-38)
Chapter 4: Identification of Slip Surfaces Using Optimisation Technique

4.26

The number of sets $m$ need not be large and in fact a value of $m=3$ or $m=5$ is often sufficient. As shown by the following examples, accurate results can be obtained with 3 or 5 sets of values of each variable, i.e. $m=3$ or $m=5$. More complex functions involving realistic problems are to be solved by computer and the calculation time does not increase significantly if higher values of $m$ are used.

The partial function on argument $x_1$ can be approximated by means of a rational polynomial of the type shown in Eq. (4.29). The expression of the rational polynomial about the partial function can be written as,

$$f_1(x_1, x_2, \ldots, x_n) \approx a_0 + \frac{x_1 - x_1^{(0)}}{a_1 + a_2 + \ldots + \frac{x_1 - x_1^{(m-1)}}{a_m}}$$

(4.39)

The coefficients $a_0, a_1, \ldots, a_m$ may be calculated by replacing $y_i$ and $(x_i - x_j)$ with $F_i$ and $x_1^{(i)} - x_1^{(j)}$ (i.e., $i=0, 1, \ldots, m$ and $j < i$) respectively in Table 4.3.

Eq.(4.39) can be written in a more compact form like Eq. (4.30), i.e.,

$$f(x_1, x_2, \ldots, x_n) = \phi_0 (x_1)$$

(4.40)

The relationship between $\phi_i (x_1)$ and $\phi_{i+1} (x_1)$ can be written in the form:

$$\phi_i (x_1) = a_i + \frac{x_1^{(i)} - x_1^{(0)}}{\phi_{i+1} (x_1)}$$

(4.41)

The last value of $\phi(x_1)$ i.e. $f_m(x_1)$ is given by

$$\phi_m(x_1) = a_m$$

(4.42)
The partial derivative of $f(X)$ with respect to argument $x_1$ can now be estimated with the rational polynomial technique. Using equations (4-39), (4-40) and (4-41), the partial derivative of $f(X)$ can be approximated as follows:

$$\frac{\partial f(X)}{\partial x_1} = \phi_0(x_1)$$  \hspace{1cm} (4-43)

By applying the quotient rule of differentiation repeatedly to Eq.(4-41) with $i = 0$, Eq.(4-43) can be written as follows:

$$\frac{\partial f(X)}{\partial x_1} = \phi'_0(x_1) = \frac{\phi_1(x_1) - (x_1 - x_1^0) \phi_1'(x_1)}{[\phi_1(x_1)]^2}$$ \hspace{1cm} (4-44a)

In general, the differential of Eq.(4-41) will give:

$$\phi'_i(x_1) = \frac{\phi_{i+1}(x_1) - (x_1 - x_1^0) \phi_{i+1}'(x_1)}{[\phi_{i+1}(x_1)]^2}$$ \hspace{1cm} (4-44b)

When $i = m$, the differential of Eq.(4-42) will be

$$\phi'_m(x_1) = 0$$ \hspace{1cm} (4-44c)

Substituting $x_1$ by $\bar{x}_1$ in Eq.(4-44), we can get the partial derivative of $f(X)$ with respect to $x_1$ at point $\bar{X} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, i.e.,

$$\frac{\partial f(\bar{X})}{\partial x_1} = \phi'_0(\bar{x}_1)$$ \hspace{1cm} (4-45)

By the same method, we can obtain the partial derivatives of $f(X)$ with respect to the other arguments at point $\bar{X} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$. The calculation illustrations have been shown in Appendix-1.
4.4 DETERMINATION OF CRITICAL SLIP SURFACES

The critical slip surface with the minimum factor of safety or the minimum acceleration coefficient can be found by the methods introduced in the previous sections. Slip surfaces in two-dimensional slope stability analyses are considered to be cylindrical with a cross-section of either circular shape or of arbitrary non-circular shape. Both of types of slip surfaces will be discussed in this section.

4.4.1. CIRCULAR SLIP SURFACE

The search for critical slip surface of circular shape is simpler than the search for non-circular critical slip surface. For a circular slip surface, there are only three location parameters, i.e. the abscissa $x_0$ and the ordinate $y_0$ of the centre of the slip circle as well as the radius $R$ of this circle. Therefore, the factor of safety $F$ can be defined as a function with respect to these three variables in the form:

$$F = f(x_0, y_0, R)$$  \hspace{1cm} (4-46)

Eq. (4-46) may be considered as an objective function and thus the minimum point $X^* = (x_0^*, y_0^*, R^*)$ which is used to define the critical circular slip surface can be found by the conjugate gradient optimisation technique introduced earlier in this chapter. Of course, for different limit equilibrium methods, the objective function $G(X)$ is different. Calculations are performed in this section based on the M&P method. The geometrical parameters and soil properties of the slope are the same as shown in Fig. 3-4 in Chapter 3. The initial and critical circular slip surfaces are shown in Fig 4-8. The calculation results during the search procedure are demonstrated in Table 4-6.
For this homogeneous slope the relative changes of geometrical parameter of the slip surfaces at the initial and final situations are very large, i.e., \( \frac{|x_0^{(0)} - x^*_0|}{x^*_0} \approx 42\%, \frac{|y_0^{(0)} - y^*_0|}{y^*_0} \approx 36\%, \frac{|R^{(0)} - R^*|}{R^*} \approx 31\%, \text{ and } \frac{|S^{(0)} - S^*|}{S^*} \approx 15\% \) (S means the section area of sliding mass) but the relative change of factor of safety is only 4.5%.

However, the factor of safety may not be so insensitive to slip surface geometry in layered or non-homogeneous slopes especially when pore water pressure are considered in a realistic manner. A false minimum of factor of safety may be reached as in search number 2 of Table 4-6. A repeat search based on the previous calculation results is usually required.
Table 4-6 Results in Search Procedure

<table>
<thead>
<tr>
<th>No. of linear search</th>
<th>xo</th>
<th>yo</th>
<th>R</th>
<th>FOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
<td>30</td>
<td>31</td>
<td>1.7730</td>
</tr>
<tr>
<td>1</td>
<td>5.393</td>
<td>29.917</td>
<td>30.966</td>
<td>1.7613</td>
</tr>
<tr>
<td>2</td>
<td>5.667</td>
<td>31.405</td>
<td>32.511</td>
<td>1.7752</td>
</tr>
<tr>
<td>3</td>
<td>7.417</td>
<td>30.598</td>
<td>31.889</td>
<td>1.7593</td>
</tr>
<tr>
<td>4</td>
<td>8.429</td>
<td>21.991</td>
<td>24.094</td>
<td>1.7039</td>
</tr>
<tr>
<td>5</td>
<td>8.080</td>
<td>21.565</td>
<td>23.507</td>
<td>1.7024</td>
</tr>
<tr>
<td>6</td>
<td>8.587</td>
<td>21.977</td>
<td>23.595</td>
<td>1.6972</td>
</tr>
<tr>
<td>7</td>
<td>8.587</td>
<td>21.977</td>
<td>23.595</td>
<td>1.6972</td>
</tr>
</tbody>
</table>

4.4.2. NON-CIRCULAR SLIP SURFACE

In realistic slope stability analyses, slip surface of non-circular or arbitrary shape must often be considered. A non-circular slip surface is defined by connecting several discrete nodes. Several approaches for searching critical non-circular slip surfaces have been published (e.g. Nguyen, 1984 & 1985; Li and White, 1987; Chen and Shao, 1988; Zhang, 1990). In this section, a comprehensive and effective optimisation procedure, based on the conjugate gradient algorithm and Golden section method, is proposed for locating an admissible critical failure surface with the minimum factor of safety.

There are a variety of ways to describe a non-circular slip surface based on a multi-dimensional space. For example, the nodal coordinates \((x_i, y_i)\) along an arbitrary slip surface, may be used as arguments of the objective function in terms of the factor of safety. In this research work, limit equilibrium models with both vertical and inclined slices have been considered. For vertical slices, the slip surface is defined by the inclinations \((\alpha_1, \alpha_2, ..., \alpha_n)\) of the slice bases, the abscissa \((x_0)\) of the inception point at the lowest end of the slip surface and the width of the slice \((\Delta X)\), as shown in Fig. 4-9.

For inclined slices, the outcrop positions of the slice boundaries on the surfaces of the slope are known. Therefore, an arbitrary slip surface, as shown in Fig. 4-10, may be
defined by the ordinates \((y_i)\) \((i = 1, 2, \ldots, n-1)\) and an abscissa \((x_n)\) of the last node. These two cases are now discussed detail in the following sub-sections.

![Arbitrary Slip Surface Associated with Vertical Slices](image)

**Fig. 4-9** Arbitrary Slip Surface Associated with Vertical Slices

The coordinates \((x_i, y_i)\) and inclinations \(\delta_i\) \((i = 1, \ldots, n)\) are assumed to be known

![Arbitrary Slip Surface Associated with Inclined Slices](image)

**Fig. 4-10** Arbitrary Slip Surface Associated with Inclined Slices
4.4.2.1. CASE OF VERTICAL SLICES

Herein, the objective function-FOS is based on M&P method. As mentioned above, the inclination of the slice base and width of the slice $\Delta X$ as well as $x_0$ are considered as the arguments of the objective function, i.e.,

$$F = f(\alpha_1, \alpha_2, \ldots, \alpha_n, \Delta X, x_0)$$  \hspace{1cm} (4-47)

However, considering the different dimensions of the variables, they may not be simultaneously involved in a multi-dimensional space for optimisation. Thus, the slice width $\Delta X$ and abscissa of inception point $x_0$ are first considered as constant in Eq. (4-47). The conjugate gradient method is used to search a pseudo-minimum FOS, denoted as $F_1$, based on the objective function $F_1 = f_1(\alpha_1, \alpha_2, \ldots, \alpha_n)$. After the pseudo-critical slip surface is found, the all inclinations ($\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$) of the slice base are kept as constant and another optimisation procedure associated with golden section is separately used to search optimum values of $\Delta X$ and $x_0$ based on the objective functions $F_2 = f_2(\Delta X)$ and $F_3 = f_3(x_0)$. Once the optimum values $\Delta X^*$ and $x_0^*$ are obtained, the conjugate gradient method is again used to calculate $\min\{F_1\}$. These calculation procedures are repeated sequentially until the three pseudo-minimum FOS, i.e. $F_1$, $F_2$ and $F_3$ are identical or nearly so.

When there are some known weak layers or rock boundaries which locate part of the slip surface, the inclinations of the slice bases located on these known surfaces are considered as constants and therefore, we can set the partial derivatives of objective function with respect to those known inclinations equal to zero.

The calculation procedure described above was used to search an critical arbitrary slip surface for a homogeneous slope. All the parameters of the geometry and soil property of the slope are illustrated in Fig. 4-11. The initial values of the slice width and $x_0$ are 9m and -95m, respectively. An assumed initial arbitrary slip surface is shown in Fig. 4-11. The initial factor of safety is 1.3171. The search of the critical arbitrary slip
surface is then carried out based on the optimisation process proposed above. The minimum factor of safety found by this optimisation process is 1.0630 and the slice width corresponding to this minimum FOS is 8.92m corresponding to 15 slices. The critical slip surface of arbitrary shape shown in Fig. 4-11, and its location is significantly different from that of the initial surface of arbitrary shape.

![Critical Slip Surface and Initial Slip Surface](image)

Fig. 4-11 Initial and Critical Slip Surfaces of Arbitrary Shape Based on M&P Method

4.4.2.2 EXAMPLE WITH INCLINED SLICES

In realistic rock slope stability problems, the structure of the rock mass, described by the geometry of discontinuities, is of critical importance. The shear strengths along discontinuities are generally lower than that of the rock material or the rock mass as a whole. On the basis of engineering geological investigations, the outcrop positions \((x_i, y_i)\) and inclinations \(\delta_i\) of the discontinuity planes on the slope surface may be determined. The geometrical parameters of the discontinuity planes, considered as boundaries of slices are considered to be known in the following optimisation process. The arguments of the objective function associated with the Sarma method are only the ordinates of the nodes which form the intersections of the slip surface with the
discontinuity planes. The abscissa of the last nodal point, which is the intersection of slip surface and slope surface, is also treated as an argument of the function. Hence, the objective function may be defined as follows:

\[ F = f(y_1, y_2, \ldots, y_{n-1}, x_n) \]  

Referring to Fig. 4-10, the abscissa \( x_i \) (\( i = 1, 2, \ldots, n-1 \)) of nodes used to define the potential slip surface can be calculated by following formula:

\[ x_i = x_{i+1} + \tan \delta_i (y_{i+1} - y_i) \]  

and \( x_n \) is an abscissa of the last node which is located on the ground surface of slope.

Since the arguments of the objective function are geometrically similar, the conjugate gradient method can directly be used in Eq. (4-48) for searching the critical arbitrary slip surface with the minimum factor of safety.

A non-homogeneous rock slope which is divided by several inclined discontinuity planes is shown in Fig. 4-12. The shear strength parameters on the discontinuity planes and slip surface are shown in Table 4-7. For an assumed (initial) slip surface, the factor of safety associated with the Sarma method is 1.8201 and the slip surface shape is shown in Fig. 4-12.

The objective function based on Eq. (4-48) is optimised on the basis of the conjugate gradient approach. The critical slip surface is associated with a minimum factor of safety of 1.5012 and is shown in Fig. 4-12.
Table 4-7 Shear Strength Parameters on Discontinuity Planes and on the Slip Surface

<table>
<thead>
<tr>
<th>No. of Rock Block</th>
<th>Discontinuous Planes</th>
<th>Slip Surface</th>
<th>Rock Unit Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>c' (kN/m²)</td>
<td>φ' (°)</td>
<td>c' (kN/m²)</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>25°</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>25°</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>25°</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>20°</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>20°</td>
<td>20</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>20°</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>20°</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>20°</td>
<td>15</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td>12</td>
</tr>
</tbody>
</table>

Fig. 4-12 Initial and Critical Slip Surfaces of Arbitrary Shape Based on Sarma Method
4.4.3. ADDITIONAL ILLUSTRATIVE EXAMPLES

The first illustrative example in this section was taken from Baker (1980). Two kinds of soil profiles were considered, i.e. homogeneous and non-homogeneous (with a weak layer). Comparisons of minimum factor of safety with the corresponding results determined by Baker are presented. The second illustrative example is concerned with the search of the critical slip surface based on the critical pseudo-static acceleration coefficient. The critical slip surface is defined as one which gives a minimum value of the critical seismic coefficient. The Lower San Fernando Dam which suffered a significant slip during the San Fernando, (California), earthquake of February 9, 1971, was selected for this analysis.

4.4.3.1. DETERMINATION OF CRITICAL SLIP SURFACE BASED ON THE MINIMUM FACTOR OF SAFETY

An example used by Baker (1980) for searching the critical slip surface is investigated with or without the inclusion of a thin, weak layer within the soil mass. Three different pore water pressure conditions are considered as follows:

(a) no pore water pressure

(b) homogeneous pore water pressure ratio, $r_u$ value of 0.25, and

(c) pore water pressure defined by a linear piezometric surface.

The strength properties of the soil and weak layer are tabulated in Table 4-8 along with the soil unit weight.

<table>
<thead>
<tr>
<th>Table 4-8 Material Properties of The Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>Properties of Material</td>
</tr>
<tr>
<td>-----------------------------</td>
</tr>
<tr>
<td>Soil</td>
</tr>
<tr>
<td>Weak Layer</td>
</tr>
</tbody>
</table>

Critical slip surfaces of arbitrary shape are located by the proposed search procedure combined with the Morgenstern and Price method. For the homogeneous case, the critical
slip surfaces are shown in Fig. 4-13 (a), (b) and (c) respectively for $r_u = 0$, $r_u = 0.25$ and a specified piezometric surface. The calculated minimum factors of safety by the proposed approach are very close to those obtained by Baker who used an effective minimisation procedure based on dynamic programming and the Spencer method was used to calculate the factor of safety of slopes.

For the case of slope with a weak base, a non-circular slip surface will pass the range of the weak layer and therefore, the shape is significantly controlled by this layer. Therefore, the inclinations of the bases of some slices are known. The critical slip surfaces are shown in Fig. 4-14 (a) (b) and (c) for the three pore water pressure conditions. The calculated minimum factors of safety are again very close to the corresponding values published by Baker.
Fig. 4-13 Critical Slip Surfaces for Homogeneous Slope Located by Proposed Analysis Procedure and Comparison of Factors of Safety with Those Obtained by Baker
Fig. 4-15 Critical Slip Surfaces for Slope with Weak Layer Located by Proposed Analysis Procedure and Comparison of Minimum Factor of Safety with Those Obtained by Baker
4.4.3.2. DETERMINATION OF THE CRITICAL SLIP SURFACE BASED ON THE MINIMUM CRITICAL SEISMIC (PSEUDO-STATIC) ACCELERATION COEFFICIENT

To investigate the applicability of the approaches developed in this chapter for determining the minimum critical acceleration coefficient, a pseudo-static analysis of the Lower San Fernando dam was performed. On the basis of the test results presented by Lee et al. (1974), the shear strength parameters and unit weights of the dam materials are summarised in Table 4-9. The slip surface is assumed to be of arbitrary shape and the pseudo-static analysis is based on Morgenstern and Price method. The initial and minimum value of the factor of safety without the influence of earthquake are 1.93 and 1.87, respectively. The initial and minimum critical acceleration coefficient are 0.457 and 0.358, respectively. All the slip surfaces, i.e. the initial slip surface, the critical slip surfaces with the minimum factor of safety and the critical slip surface with the minimum critical acceleration coefficient, are shown in Fig. 4-15.

The critical slip surface corresponding to the minimum factor of safety is significantly different from that based on the minimum critical acceleration coefficient. The critical failure surfaces do not extend into the foundation soil and this corresponds with the observed failure surface as interpreted after the failure of this dam.

Table 4-16 Soil Parameters of Lower San Fernando Dam (after Lee et al., 1975)

<table>
<thead>
<tr>
<th>Material (No.)</th>
<th>Cohesion - c'</th>
<th>Friction Angle - φ'</th>
<th>Unit Weight - y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alluvium (1)</td>
<td>0</td>
<td>38°</td>
<td>17.6 (kN/m³)</td>
</tr>
<tr>
<td>Hydraulic Fill Sand (2)</td>
<td>0</td>
<td>37°</td>
<td>17 (kN/m³)</td>
</tr>
<tr>
<td>Clay Core (3)</td>
<td>81.4 (kN/m²)</td>
<td>0</td>
<td>19 (kN/m³)</td>
</tr>
<tr>
<td>Ground Shale-Hydraulic Fill (4)</td>
<td>0</td>
<td>33°</td>
<td>20 (kN/m³)</td>
</tr>
<tr>
<td>Rolled Fill (5)</td>
<td>81.4 (kN/m²)</td>
<td>0</td>
<td>19 (kN/m³)</td>
</tr>
</tbody>
</table>
4.5. SUMMARY

1. For location of critical slip surfaces a method within the class of conjugate direction methods, the conjugate gradient approach, was introduced as an alternative to other methods such as (a) grid search with the repeated trails and (b) direct search techniques. The F-R algorithm is a very efficient optimisation approach and does not require calculation of the Hessian matrix of the objective function when the function is non-quadratic. The oscillatory behaviour, which is often characteristic of the steepest descent method during iteration, can be avoided. Another important advantage of the F-R algorithm is that the development of a computer program for the F-R algorithm is easy. However, the partial derivatives of the objective function with respect to the variables are required for using the F-R algorithm. The objective function in terms of the factor of safety is usually non-linear and inexplicit. As a consequence, suitable numerical methods for estimating these partial derivative have to be used.
2. In comparison with other popular unconstrained gradient optimisation techniques, such as the steepest descent algorithm, fewer iterations are required in the conjugate gradient method. Although the calculation of conjugate directions is required for the conjugate gradient algorithm in each iterative step, the number of such calculations is fewer than that for other gradient methods.

3. A very effective numerical approximation approach, the rational polynomial technique, is presented for solving non-linear equations and estimating the derivatives of functions. A modified calculation procedure for estimating the partial derivatives of a multivariate function is proposed. The rational polynomial technique has proved to be more effective than Newton's tangential method for solving non-linear equations. Calculation of derivatives of a non-linear equation is required during each iteration step for Newton's method. This is not required for the rational polynomial technique. Five or six iterations normally are enough for calculating the FOS based on Morgenstern and Price method when the rational polynomial technique is used. The accuracy of the partial derivatives of a multivariate function, associated with rational polynomial technique, is higher than that associated with a finite difference approach. Moreover, the accuracy of the partial derivatives of a non-linear function with the finite difference approach is dependent on the increment argument (ΔX).

4. Specific strategies have been developed for locating critical circular and non-circular slip surfaces in slope stability problems and the sliding mass may be considered as an assemblage of vertical slices or of non-vertical slices. A comprehensive optimisation procedure which combines conjugate gradient algorithm with the golden section method is presented for locating critical slip surfaces of arbitrary shape based on Morgenstern and Price method. All calculation results show that the proposed optimisation procedures are effective for locating both circular and non-circular critical slip surfaces with different limit equilibrium methods. It is, however, necessary to note that false minimum FOS may sometimes be reached in the iterations because of the behaviour
of the objective function and the errors of the linear search. However, this can easily be detected by a repeat search after an apparent minimum has been reached.

5. The results presented in this chapter are reasonable and, for a selected case history, in close agreement with those obtained previously by other methods. The computation results indicate that the proposed optimisation procedure can be used to locate a critical slip surface of arbitrary shape which is controlled by a weak layer. On the basis of the optimisation method presented in this chapter and a pseudo-static analysis model, a critical slip surface with the minimum critical seismic coefficient can also be located. A realistic failure case induced by a strong earthquake is selected to demonstrate the applicability of the method.
5.1. GENERAL REMARKS

The two previous chapters were concerned mainly with analysis of slope stability based on conventional limit equilibrium concepts. The material properties and pore water pressure are considered as deterministic quantities or constants for any homogeneous soil layer. However, significant randomness or variability is often associated with various parameters involved in limit equilibrium analyses of slope stability. Therefore, there is a significant uncertainty associated with the calculated value of the factor of safety.

Over the past two decades, probabilistic methods have been developed to supplement deterministic approaches for assessing the reliability of slopes. A number of researchers have contributed to the development of probabilistic approaches for geotechnical application. For example, elementary analyses based on a probabilistic approach have already been introduced in some textbooks (e.g. Harr, 1977; He and Wei, 1979; Lee et al, 1983; Ang and Tang, 1984; Madsen et al, 1986). There is little doubt that probabilistic methods could become powerful tools for the reliability assessment of geotechnical structures in engineering practice.
Chapter 5: Probabilistic Analysis of Slopes and Applications

The calculation model for slope stability analyses based on the concept of limit equilibrium cannot usually be expressed by an explicit mathematical formula. The probability distributions of significant random variables are usually not known or known only approximately. Moreover, the joint probability density function (the probability distribution) of the factor of safety or the safety margin is generally unknown. It is not possible, in any case, to determine the accurate mean value and variance of the performance function when the performance function is defined in terms of a factor of safety associated with relatively 'rigorous' limit equilibrium methods of slope stability analysis. The two statistical moments (mean and variance) of the performance function must be evaluated by approximate methods. The following three numerical approaches are generally used for geotechnical probabilistic analyses:

(i) The first order second moment method, FOSM;
(ii) Point estimate method, PEM;
(iii) Monte Carlo simulation method, MCSM.

The application of these three numerical methods for the assessment of failure probabilities of slopes is discussed in this chapter. The main aim is to develop a more efficient framework for assessing slope reliability. Another aim is the comparisons of these three numerical methods considering the random variables describing shear strength parameters to be either uncorrelated or correlated.

For the probabilistic analyses discussed in this thesis, the factor of safety, F, is always considered as the performance function in preference to the safety margin. The factor of safety has a wide acceptance as an indicator of performance whereas the safety margin is rarely used in geotechnical practice. In order that probabilistic and deterministic analyses are complementary to each other it is desirable to use the factor of safety as an indicator of performance in both types of analysis.
5.2. DEFINITION OF PERFORMANCE FUNCTION AND DESCRIPTION OF BASIC PARAMETERS

5.2.1. PERFORMANCE FUNCTION

The performance or response of an engineering system may be depicted explicitly by means of a mathematical expression or inexplicitly by a calculation procedure such as a computer program. Such a function is called the performance function or, in probabilistic terminology, as the limit state function $G(X)$ which may be written in the following form:

$$G(X) = G(x_1, x_2, ..., x_n)$$  \hspace{1cm} (5-1)

where, $X = (x_1, x_2, ..., x_n)$ is a vector of the input random variables. For slope stability assessment, $G(X) > 0$ implies that the slope is in a ‘safe state’, whereas a ‘failure state’ is implied by $G(X) \leq 0$. The boundary separating a safe and a failure states is the ‘limit state’ of a slope, defined by the equation:

$$G(X) = 0$$  \hspace{1cm} (5-2)

Therefore, the failure probability of the slope can be obtained by

$$P_f = \Pr \{G(X) \leq 0\}$$  \hspace{1cm} (5-3)

In slope stability analyses, the factor of safety, $F$, is a function of several variables such as material strength parameters, pore water pressures, density of materials and geometrical parameters of the slope.

The performance function can be defined as follows:

$$G(X) = F - 1 = f(X) - 1$$  \hspace{1cm} (5-4)

where, $f(X)$ is a mathematical formula or a calculation procedure associated with any one of the available limit equilibrium methods. Based on Eqs. (5-3) and (5-4) the failure probability of a slope can be expressed in the following form:
\[ P_f = \Pr\{f(X) \leq 1\} \]  \hfill (5-5)

or,

\[ P_f = \int \cdots \int \phi(X) \, dx_1 \cdots dx_n \]  \hfill (5-6)

where \( \phi(X) = \phi(x_1, x_2, \ldots, x_n) \) is a joint probabilistic density distribution of all the basic input random variables and \( G(X) \leq 0 \) is a region of n-dimensional integration. If \( \phi(X) \) and \( G(X) \) are known exactly then the failure probability of slopes can be calculated exactly by Eq. (5-6) provided that the integration can be carried out.

5.2.2. DESCRIPTION OF BASIC RANDOM VARIABLES

In this research the shear strength parameters \( (c' \text{ and } \phi') \), unit weight \( \gamma \) of slope materials and pore water pressure ratio \( r_u \) are considered as the basic input random variables. For the Monte Carlo simulation method all random variables are assumed to follow normal (Gaussian) distribution.

The correlations of shear strength parameters are often important for the realistic assessment of failure probability of slopes. It is difficult to determine these coefficients. In particular, the analysis of spatial variability of shear strength is a vast subject in its own right. Some researchers have proposed mathematical models such as the random field model (Vanmarcke, 1977a and 1984). Although the material properties in slope randomly vary from a point to another point it is, in fact, very difficult to describe this variation and the spatial correlation characteristics of each variable through a mathematical model. It may be more effective in practice to use a multi-layer model (with each layer statistically homogeneous) to approximate a non-homogeneous random field. In such a model the mean and standard deviation of basic input random variables are considered as constant within each layer. The spatial correlation characteristics of these random variables must be obtained on the basis of an appropriate methodology supplemented by engineering judgement. The larger the number of layers or regions the better is the accuracy that can be expected. Random field models based on the random process theory are not easily understood and accepted by geotechnical engineers. Therefore, the
proposed alternative approach could be very valuable in encouraging adoption of probabilistic models in geotechnical practice.

5.3. FIRST ORDER SECOND MOMENT METHOD (FOSM)

5.3.1. DEFINITION OF FOSM

The joint probability density distribution $\phi(X)$ in Eq. (5-6) is usually unknown. Even the probability distribution of the basic random variables is often unavailable or difficult to obtain for reasons of insufficient data. Furthermore, even when the required distributions can be specified, the exact evaluation of the probabilities may be impractical and numerical approaches are required to calculate the probability distribution of the performance function or, at least, its first few statistical moments. Practical measures of safety or reliability of slopes are often limited to functions of the first two statistical moments (mean and variance). The implementation of reliability concepts usually involves the second-moment formulation based on expansion of the performance function $G(X)$ as a Taylor series (Cornell, 1969; Ang and Cornell, 1974). Thereby, the mean and variance of the performance function $G(X)$ may be approximated respectively as follows,

$$
\mu_G = E\{G(X)\} = G(\mu_X)
$$

$$
\sigma_G^2 = \text{Var}\{G(X)\} \approx \nabla G^\top [C] \nabla G
$$

in which, $\mu_X = (\mu_{x_1}, \mu_{x_2}, ..., \mu_{x_n})$ is a vector of the mean values of the basic random variables, $[C]$ is a matrix of covariance of the basic input random variables, and $\Delta G$ is a vector of the partial derivatives of performance function $G(X)$ with respect to the mean values of the basic random variables.

An index associated with these two statistical moments of the performance function can be defined and is called a reliability index. Two definitions of reliability index are introduced in following sections.
5.3.2. RELIABILITY INDEX - $\beta$

The reliability index $\beta$ is most commonly used in geotechnical reliability studies. As an alternative reliability format to the conventional factor of safety, the reliability index $\beta$ was first defined by Cornell (1969) in the following form:

$$\beta = \frac{\mu_G}{\sigma_G}$$  \hspace{1cm} (5-9)

or,

$$\beta = \frac{\mu_F - 1}{\sigma_F}$$  \hspace{1cm} (5-10)

where, $\mu_G$ and $\sigma_G$ can be estimated by an appropriate numerical approach such as first order second moment method. The reason for using $\beta$ as a measure of safety or reliability may be explained by considering again the definition of failure probability. Define a variable $Z$ as follow:

$$Z = \frac{G(X) - \mu_G}{\sigma_G}$$  \hspace{1cm} (5-11)

the probability of failure may now be written as follows:

$$P_f = Pr\{G(X) \leq 0\}$$

$$= Pr\left\{ \frac{G(X) - \mu_G}{\sigma_G} \leq -\frac{\mu_G}{\sigma_G} \right\}$$

$$= Pr\{Z \leq -\beta\}$$

$$= \int_{-\infty}^{-\beta} \psi(z) \, dz$$

$$= \Psi(-\beta)$$  \hspace{1cm} (5-12)

where $\psi(z)$ and $\Psi(z)$ are respectively the probability density function (PDF) and cumulative distribution function (CDF) of $Z$. Since CDF is always a non-decreasing function, a one-to-one correspondence exists between the failure probability and the reliability index. The uncertainties associated with individual random variables are thus condensed into the single reliability index $\beta$. Provided that the reliability index for two different slopes are equal, they are considered to have a similar reliability although the variability of individual random variables may be different for the two slopes. Fig. 5-1
shows a schematic representation of the PDF of $G(X)$. If the value of $\beta$ is large, the mean value of $G(X)$ will be further away from the cut-off point $G(X) = 0$ and the failure probability will therefore be smaller.

![Fig. 5-1 Probability Density Function of $G(X)$](image)

The value of the reliability index obtained from Eq. (5-9) or Eq. (5-10) depends on the definition of the limit state. For example, for the equivalent limit-state events $[R(X) - Q(X)]$ and $[R(X)/Q(X) - 1 < 0]$, respective reliability index values $\beta_1$ and $\beta_2$ based on Eq. (5-9) will be different (Ang and Tang, 1984), i.e.,

$$\beta_1 = \frac{\mu_R - \mu_Q}{\sqrt{\sigma_R^2 - 2\rho\sigma_R\sigma_Q + \sigma_Q^2}} \tag{5-13}$$

$$\beta_2 = \frac{\mu_R}{\mu_Q} \sqrt{\frac{\sigma_R^2}{\mu_R^2} - 2\rho \frac{\sigma_R}{\mu_R} \frac{\sigma_Q}{\mu_Q} + \frac{\sigma_Q^2}{\mu_Q^2}} \tag{5-14}$$

Therefore, it has been suggested that an invariant definition of the index, such as that proposed by Hasofer and Lind (1974), should be preferred.

5.3.3. ALTERNATIVE RELIABILITY INDEX - $\beta^*$

Hasofer and Lind (1974) proposed an alternative definition of reliability index which is designated here as $\beta^*$ and is defined as the minimum distance from the surface $G(X) = 0$ to the origin of the uncorrelated reduced random variates. This definition may
be illustrated in two dimensional space as shown in Fig. 5-2. The tangent plane to the failure surface at the point \( (x_1^*, x_2^*, \cdots, x_n^*) \) may then be used to approximate the actual failure surface, and the required reliability index or probability of safety may be evaluated from the first two terms of the Taylor series expansion of the performance functions.

![Fig. 5-2 Definition of \( \beta^* \) Based on Space of Reduced Variates \( x_1' \) and \( x_2' \) (after Ang and Tang, 1984)](image)

The use of the invariant reliability index \( \beta^* \) as a reliability measure has gained popularity in structural reliability analyses, but it has not been used in slope stability analysis until recently (e.g. Ramachandran and Hosking, 1985; Gussman, 1985; Li and Lumb, 1987; Li, 1987(d); Luckman, 1987 and Chowdhury and Xu, 1992). The use of invariant reliability index has also been demonstrated by Nguyen (1985) but, in a later publication (Nguyen, 1990), he questioned whether Hasofer and Lind definition really implied an invariant index of reliability. The chief advantage of this definition of reliability index is that it does not depend, in any way, on the precise analytical form of the performance function \( G(X) \). Indeed, Shinozuka (1983) has shown that the point on the failure surface with minimum distance to the origin of the reduced variates is the maximum likelihood failure point provided that \( X' \) is a vector of uncorrelated random variables. The methods for calculation of \( \beta^* \) have been derived by Hasofer and Lind (1974), Parkinson (1979) as well as Ang and Tang (1984). One main aim in following
section is to present an interpretation which requires only a knowledge of linear algebra and is, therefore, simple, direct and easier to understand than any of the available procedures. Moreover, an alternative technique is presented for the iterative numerical solution when the performance function is non-linear (Chowdhury and Xu, 1992).

5.3.4. DEVELOPMENT OF CALCULATION FORMULAE FOR $\beta^*$

Consider again the vector of the basic random variables $X = (x_1, x_2, \ldots, x_n)$, the variables $x_i$ having mean values $\mu_{x_i}$ and standard deviations $\sigma_{x_i}$. It is useful to consider separately the cases of uncorrelated and correlated variables or variates. The case of uncorrelated variates $X$ is considered first below.

5.3.4.1. UNCORRELATED VARIATES

Consider a single variable $x_i$ with mean $\mu_{x_i}$ and standard deviation $\sigma_{x_i}$. A reduced or normalised variable or variate is defined as follows:

$$x_i = \frac{x_i - \mu_{x_i}}{\sigma_{x_i}} \quad (5-15)$$

All variates may be normalised or reduced in this way and the expression for a set of reduced variates can be written in compact form as follows:

$$X' = [\sigma_X]^{-1}(X - \mu_X) \quad (5-16)$$

in which the diagonal matrix $[\sigma_X]$ and row vector $\mu_X$ are:

$$\sigma_X = \begin{bmatrix} \sigma_{x_1} & 0 \\ 0 & \sigma_{x_2} & \cdots & \sigma_{x_n} \end{bmatrix}$$

$$\mu_X = (\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_n})^T$$

The distance from a point $X' = (x'_1, x'_2, \ldots, x'_n)$ on the failure surface $G(X) = 0$ to the origin of $X'$ is given by the following:
\[ D = (X^T X')^{1/2} \]  
\[ (5-17) \]

On the failure surface, a point denoted by, \( X'^* = (x_1'^*, x_2'^*, \ldots, x_n'^*) \), and having the minimum distance to the origin may be determined by minimising the function \( D \), subject to the constraint \( G(X) = 0 \).

By means of the method of Lagrange's multiplier, the minimum distance, or \( \beta^* \) can be obtained as follows (See, for instance, Ang and Tang(1984)):

\[ \beta^* = - \frac{\left( \frac{\partial G}{\partial X'} \right)^T X'^*}{\left( \left( \frac{\partial G}{\partial X'} \right)^T \frac{\partial G}{\partial X'} \right)^{1/2}} \]  
\[ (5-18) \]

in which,

\[ \left( \frac{\partial G}{\partial X'} \right)^T = \begin{pmatrix} \frac{\partial G}{\partial x_1'^*} & \frac{\partial G}{\partial x_2'^*} & \cdots & \frac{\partial G}{\partial x_n'^*} \end{pmatrix} \]

The corresponding point on the failure surface is given by the following expression:

\[ X'^* = - \frac{\left( \frac{\partial G}{\partial X'} \right)^T \beta^*}{\left( \left( \frac{\partial G}{\partial X'} \right)^T \frac{\partial G}{\partial X'} \right)^{1/2}} \]  
\[ (5-19) \]

Eq. (5-18) can be rewritten in terms of original variables in the following form:

\[ \beta^* = - \frac{\left( \frac{\partial G}{\partial X} \right)^T (X^* - \mu_X)}{\left( \left( \frac{\partial G}{\partial X} \right)^T \frac{\partial G}{\partial X} \right)^{1/2}} \]  
\[ (5-20) \]

For correlated variables it is necessary to obtain a set of transformed variates which are uncorrelated and the procedure for doing this is discussed in the following section.
5.3.4.2. CORRELATED VARIATES

The procedure described above for evaluating the reliability index $\beta^*$ is based on the assumption that the random variables $X = (x_1, x_2, \ldots, x_n)$ are uncorrelated. In general, these variables are correlated and the coefficients of correlation between any two variables may vary between the limits -1 and +1.

The covariance matrix of the original variates of performance function $G(X)$ may be written in the form:

$$[C] = \begin{pmatrix}
\sigma_1^2 & \text{cov}(x_1, x_2) & \cdots & \text{cov}(x_1, x_n) \\
\text{cov}(x_2, x_1) & \sigma_2^2 & \cdots & \text{cov}(x_2, x_n) \\
\vdots & \vdots & \ddots & \vdots \\
\text{cov}(x_n, x_1) & \text{cov}(x_n, x_2) & \cdots & \sigma_n^2
\end{pmatrix} \tag{5-21}$$

where the elements, $\text{cov}(x_i, x_j)$ ($i \neq j$), are the respective covariances between the pairs of variables $x_i$ and $x_j$. The corresponding covariance between a pair of reduced or dimensionless variates, $x'_i$ and $x'_j$, can be derived based on the definition of a reduced variate given above in Eq. (5-15). This covariance can be written in the following form as a consequence of that definition:

$$\text{cov} (x'_i, x'_j) = \rho_{x'_ix'_j} \tag{5-22}$$

According to Eq. (5-22), the covariance between a pair of reduced variates, $x'_i$ and $x'_j$, is equal to the correlation coefficient between the corresponding pair of original variates $x_i$ and $x_j$. The complete original matrix of correlation coefficients may be written in the following form and it should be noted that it is also the covariance matrix for the reduced variates:

$$[C'] = \begin{pmatrix}
1 & \rho_{12} & \cdots & \rho_{1n} \\
\rho_{21} & 1 & \cdots & \rho_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{n1} & \rho_{n2} & \cdots & 1
\end{pmatrix} \tag{5-23}$$
The required set of (uncorrelated) transformed variates can be obtained from $X'$ through the following orthogonal transformation.

$$Y = T^T X'$$  \hspace{1cm} (5-24)

in which, $X' = \begin{pmatrix} x'_1 & x'_2 & \cdots & x'_n \end{pmatrix}^T$ is a vector of reduced variates

$Y = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \end{pmatrix}^T$ is the required vector of uncorrelated transformed variates and

$T$ is an orthogonal transformation matrix.

According to the approach used in generalised error propagation theory, the covariance matrix of $Y$ is

$$[C_Y] = E\{YY^T\} = T^T [C'] T = \sigma_y^2$$  \hspace{1cm} (5-25)

Since $T$ is an orthogonal matrix, $T^{-1} = T^T$, inversion of Eq.(5-24) yields

$$X' = T Y$$  \hspace{1cm} (5-26)

and,

$$X = [\sigma_x] TY + \mu_x$$  \hspace{1cm} (5-27)

The uncorrelated reduced variates $Y' = \begin{pmatrix} y'_1 & y'_2 & \cdots & y'_n \end{pmatrix}^T$ can be written as follows:

$$Y' = [\sigma_y]^{-1} Y$$

and,

$$Y = [\sigma_y] Y'$$  \hspace{1cm} (5-28)

putting Eq.(5-28) into Eq.(5-27), the relation between $Y'$ and $X$ can be obtained as follows:

$$Y' = [\sigma_y]^{-1} T^T [\sigma_y]^{-1} (X - \mu_x)$$  \hspace{1cm} (5-29)

where $[\sigma_y]$ is a diagonal matrix.

Because orthogonal transformation is an equivalent transformation, the minimum distance ($D_{\text{min}}$) in $Y'$ space is equal to that in $X$ space. From Eq.(5-29), the partial derivatives ($\partial G/\partial Y'$) can be obtained through the rule of matrix differentiation as follows:
\[
\left( \frac{\partial G}{\partial Y} \right)^T = \left( [\sigma_x] [\sigma_y]^T \right)^T \left( \frac{\partial G}{\partial X} \right) \quad (5-30)
\]

From Eq.(5-25), one can write the following:

\[
\left( \frac{\partial G}{\partial Y} \right)^T = \left( \frac{\partial G}{\partial X} \right)^T \left( [\sigma_x] [\sigma_y]^T \right) \left( \frac{\partial G}{\partial X} \right) \\
= \left( \frac{\partial G}{\partial X} \right)^T \left( [\sigma_x] [\sigma_y]^T \right) \left( \frac{\partial G}{\partial X} \right) = \left[ C \right] \left( \frac{\partial G}{\partial X} \right) \quad (5-31)
\]

Similarly using Eqs.(5-19) and (5-30), the following expression can be obtained:

\[
\left( \frac{\partial G}{\partial Y} \right)^T (X - \mu_x) = \left( \frac{\partial G}{\partial X} \right)^T (X - \mu_x) \quad (5-32)
\]

By means of Eqs. (5-31) and (5-32) as well as the formulation, Eq.(5-18), described above for calculating reliability index based on uncorrelated random variables the formulation of reliability index based on the original, untransformed design parameters can now be written as follows:

\[
\beta^* = \frac{\left( \frac{\partial G}{\partial X} \right)^T (X^* - \mu_x)}{\left( \left( \frac{\partial G}{\partial X} \right)^T [C] \left( \frac{\partial G}{\partial X} \right) \right)^{1/2}} \quad (5-33)
\]

in which, \( \left( \frac{\partial G}{\partial X} \right)_* \) is the gradient vector at the point \( X^* = (x_1^*, x_2^*, \ldots, x_n^*) \) on failure surface \( (G(X^*) = 0) \).

It is now easy to show that Eq.(5-33) becomes Eq.(5-20) if all correlation coefficients are zero since the full matrix \([C]\) becomes the diagonal matrix \([\sigma_x^2]\).

5.3.4.3. NUMERICAL SOLUTION OF \( \beta^* \)

The reliability index, \( \beta^* \), may be determined from Eq.(5-33) for given \( \mu_x \) and \([C]\), after the point \( X^* \) on failure surface has been determined. In order to do this, two
types of performance function have to be considered separately, i.e., linear and non-linear.

5.3.4.3.1. LINEAR PERFORMANCE FUNCTION

In this case, the performance function can be written as:

\[ G(X) = a_0 + A^\top X \]  \hspace{1cm} (5-34)

where, \( A = (a_1 \ a_2 \ldots \ a_n)^\top \) is a vector of constants.

The partial derivatives in Eq.(5-33) are independent of the variables, i.e.,

\[ \left( \frac{\partial G}{\partial X} \right) = A \]  \hspace{1cm} (5-35)

Therefore, one can write as follows:

\[ \left( \frac{\partial G}{\partial X} \right)_* (X^* - \mu_X) = A^\top X^* - A^\top \mu_X \]

\[ = -(a_0 + A^\top \mu_X) \]  \hspace{1cm} (5-36)

Substituting the Eq.(5-36) into Eq.(5-33), the reliability index, for a linear performance function with correlated variates, can be determined directly as follows:

\[ \beta^* = \frac{a_0 + A^\top \mu_X}{\left[ A^\top [C] A \right]^{1/2}} \]  \hspace{1cm} (5-37)

where \([C]\) is the covariance matrix defined earlier.

It is of interest to evaluate \( \beta \) based on Eq.(5-9) for this case of linear performance function. Firstly, the expressions for \( \mu_G \) and \( \sigma_G \) are:

\[ \mu_G = a_0 + A^\top \mu_X \]

\[ \sigma_G = \left\{ A^\top [C] A \right\}^{1/2} \]

and hence,
\[
\beta = \frac{a_0 + \mu_x}{\left[ A^T [C] A \right]^{1/2}}
\]

Thus alternative definitions of the reliability index, \((\beta \text{ or } \beta^*)\), lead to the same result, if the performance function is linear.

5.3.4.3.2. NON-LINEAR PERFORMANCE FUNCTION

When the performance function is non-linear, the reliability index \(\beta^*\) has to be obtained by an iterative procedure because the true failure point on the failure surface is initially unknown. An iterative solution procedure for uncorrelated variables was given by Hasofer and Lind (1974). A different solution procedure based on space analytical geometry was proposed by Parkinson (1979) considering correlated variables. A alternative development of Parkinson's formulation is presented in this section based on linear algebra theory. The procedure for deriving this iterative expression is given below.

Assume that \(X_{(i)}^*\) is the value of \(X^*\) in the i-th iteration and \(G(X_{(i)}^*) = 0\). According to Eq.(5-19), the new values in \(Y^*\)-space can be obtained as:

\[
Y_{(i+1)}^* = -\frac{\left[ \frac{\partial G}{\partial Y_{(i)}^*} \right]_{\beta_{(i)}}^*}{\left\{ \left( \frac{\partial G}{\partial Y_{(i)}^*} \right)^T \left( \frac{\partial G}{\partial Y_{(i)}^*} \right) \right\}^{1/2}}
\]

From Eq.(5-29) and Eq.(5-30), Eq.(5-38a) can be rewritten as:

\[
\left[ [\sigma_x] T [\sigma_y] \right] (X_{(i+1)}^* - \mu_x) = -\frac{\left( [\sigma_x] T [\sigma_y] \right) \left( \frac{\partial G}{\partial X_{(i)}^*} \right)_{\beta_{(i)}}^*}{\left( \left( \frac{\partial G}{\partial X_{(i)}^*} \right)^T [C] \left( \frac{\partial G}{\partial X_{(i)}^*} \right) \right)^{1/2}}
\]

Substituting Eq.(5-33) into Eq.(5-38b), an iterative equation can be obtained as follows
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5.3.5. COMPARISON OF $\beta$ AND $\beta^*$ BASED ON THE GEOMETRICAL REPRESENTATION

Different definition of the two reliability indices $\beta$ and $\beta^*$ will usually mean that the calculated values of $\beta$ and $\beta^*$ are different since the performance function $G(X)$ is a nonlinear function for slope stability problems. It is necessary to discuss in some detail why the computed values of reliability index are different with the alternative definition. This discussion is facilitated by considering a graphical representation of the performance function.

For simplicity of graphical representation, consider the case in which only two uncorrelated basic random variables are involved in the performance function. The performance function may be expanded as a Taylor series at the mean point $\mu_x = (\mu_{x_1}, \mu_{x_2})$, i.e.,

$$G(X) = G(\mu_x) + \sum_{i=1}^{2} (x_i - \mu_{x_i}) \left[ \frac{\partial G}{\partial x_i} \right]_{x = \mu_x} +$$
\[ \frac{1}{2!} \sum_{j=1}^{2} \sum_{i=1}^{2} (x_i - \mu_x)(x_j - \mu_x) \left( \frac{\partial^2 G}{\partial x_i \partial x_j} \right)_{x=\mu_x} + \cdots \]  

(5-40)

in which, \( G(\mu_x) = F(\mu_x) - 1 \). Truncating the above series at the second-order term, the performance function may be approximately expressed as follows:

\[
G(\mathbf{X}) = G(\mu_x) + \sum_{i=1}^{2} (x_i - \mu_x) \left( \frac{\partial G}{\partial x_i} \right)_{x=\mu_x}
\]  

(5-41)

Eq. (5-41) is the equation of a plane in \( \mathbf{X} \)-space and the format of this equation in \( \mathbf{X}' \) space is as follows:

\[
G(\mathbf{X}') = G(\mu_x) + \sum_{i=1}^{2} \sigma_{x_i} x'_i \left( \frac{\partial G}{\partial x_i} \right)_{x=\mu_x}
\]  

(5-42)

in which, \( x'_i = \frac{x_i - \mu_x}{\sigma_{x_i}} \) (\( i = 1, 2 \))

Eq.(5-42) is an equation of the tangent plane, passing through the origin point in \( \mathbf{X}' \) space or the mean point in \( \mathbf{X} \) space, on the surface expressed by the performance function. Thus, the equation of the intersecting line between the tangent plane in \( \mathbf{X}' \) space and the plane consisting of the reduced random variables is:

\[
\sigma_{x_1} x'_1 \left( \frac{\partial G}{\partial x_1} \right)_{x=\mu_x} + \sigma_{x_2} x'_2 \left( \frac{\partial G}{\partial x_2} \right)_{x=\mu_x} + G(\mu_x) = 0
\]  

(5-43)

The distance from this straight line to the origin of the reduced random variable \( \mathbf{X}' \) is given (from analytic geometry) as follows:

\[
d = \frac{G(\mu_x)}{\sqrt{\sigma_{x_1}^2 \left( \frac{\partial G}{\partial x_1} \right)_{x=\mu_x}^2 + \sigma_{x_2}^2 \left( \frac{\partial G}{\partial x_2} \right)_{x=\mu_x}^2}} = \frac{\mu_G}{\sigma_G}
\]  

(5-44)

Eq.(5-44) is the same as the conventional definition of the reliability index(See, Eqs. (5-9), (5-7) and (5-8)). A geometrical interpretation of the conventional definition of the reliability index \( \beta \) can now be given. This has not previously been attempted by other researchers and, therefore, it is of considerable interest so that \( \beta^* \) and \( \beta \) can be seen together in one graphical representation.
Referring to Fig. 5-3, the origin of reduced random variables is $0'$ and the curve shows the trace or projection of the surface which represents the performance function. The trace represents the failure surface. The performance function is considered here to be plotted in the direction perpendicular to the plane of the paper and if the origin $0'$ is projected vertically upwards, it will meet the surface at some point. If a tangent is drawn to the surface at that point, then the line PP' (also designated as $G'$) is a projection of that tangent plane on the co-ordinate plane (the plane of the paper). Now from Eq.(5-44) it is obvious that the reliability index $\beta$ represents the distance from origin $0'$ to the intersection line PP' (or $G'$). In summary, the geometrical interpretation is as follows:

"The distance from the origin of reduced variables to the trace in $X'$-space of a plane which is tangent to the performance function surface at the point where the vertical projection of $0'$ meets that surface".

Fig. 5-3 The Geometrical Representation of the Definitions of the Reliability Index.

The geometrical interpretation of the reliability index $\beta^*$ is the minimum distance from origin of reduced variates to the most probable failure point on the failure surface ($G(X) = 0$). This definition is also shown in Fig. 5-3 where $X^*$ is the most probable
failure point on \( G(X) = 0 \) and \( \overline{X} \) is the base of the perpendicular from the mean point in X-space (or origin in X'-space) to the straight line-G' (or PP').

If the contour \( G(X) = 0 \) passes through the mean point, the points \( X^* \) and \( \overline{X} \) will overlap, i.e., \( \beta^* = \beta = 0 \); If the contour \( G(X) = 0 \) does not pass through the mean point, the positions of the points \( X^* \) and \( \overline{X} \) are not identical. In this case (i.e., \( G(X) \) is convex corresponding to the reduced space), the point \( \overline{X} \) usually falls in the area of \( G(X) > 0 \). Therefore, the value of \( \beta^* \) is generally larger than that of \( \beta \). This means that the reliability index \( \beta \) based on Cornell's definition is conservative because it underestimates reliability. The difference between \( \beta^* \) and \( \beta \), shown in Fig. 5-3, will increase with the increment of \( \beta \) and \( \beta^* \) or with the decrease of the coefficient of variation of basic input random variables.

5.3.6. ACCURACY OF LINEAR APPROXIMATION

The 'linear' approximation of a nonlinear performance function implies the replacement of a n-dimensional failure surface (a hyper-surface) with a hyper-plane tangent to the failure surface at the 'most probable failure point'. This changes the boundary between the safe state, \( G(X) > 0 \), and the failure state, \( G(X) < 0 \), from a general curvilinear surface to a plane surface; the failure probability, \( p_F \), is then the generalised volume integral of the joint PDF over the failure region \( G(X) < 0 \). The accuracy of linear approximation has been discussed in some detail by Ang and Tang (1984). For a specific slope, a sensitivity study has been made during this research to explore the accuracy of the linear approximation and the results are discussed below.

In probabilistic analyses of slope stability, the performance function \( G(X) \) in terms of the factor of safety is a function of the random variables \( c', \phi', \gamma \). Consider an arbitrary homogeneous slope with a circular slip surface and assume that one of the three random variables is varied independently by increasing or decreasing its value by \( i \) standard deviations from the respective mean value. The coefficient of variation (\( \delta \)) of each variable is assumed to be 0.15 or 15%. Thus calculations of the performance
function based on the M&P and Sarma methods can be made for the three following cases:

(i) $\delta_c = 15\%, \ \delta_{\phi} = \delta_\gamma = 0$

(ii) $\delta_{\phi} = 15\%, \ \delta_c = \delta_\gamma = 0$

(iii) $\delta_\gamma = 15\%, \ \delta_c = \delta_{\phi} = 0$

where $c'$, $\phi'$ and $\gamma$ are considered as the variables with values in each case between the limits of (a) the mean value plus three standard deviations and (b) the mean value minus three standard deviations.

Based on an assumed homogeneous slope which is shown in Fig. 5-4(a) the values of the calculated factor of safety for this slope are exhibited in Fig. 5-4(b) and (c). It is interesting to note from Fig. 5-4(b) and (c) that the partial performance function (this means that only one random variable is varied and others are considered as constant) is almost linear with respect to both $c'$ and $\phi'$ whether the M&P method or the Sarma method of slope stability analysis is used. On the other hand, there is significant non-linearity of the partial performance function with respect to the unit weight considering a wide range of $\gamma$ values. However, in the neighbourhood of the mean of unit weight $\gamma$, the partial performance function with respect to $\gamma$ has a relatively higher degree of linearity. The smaller the variation coefficients of basic random variables, the higher the degree of linearity of the performance functions. These conclusions are valid only for a slope formed within a single soil layer. Further studies are required for a multi-layer soil medium and for slopes with non-circular slip surfaces.
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Slope Geometry

(a)

![Slope Geometry](image)

- $\phi' = 18^\circ$
- $c' = 15 \text{ kN/m}^2$
- $\gamma = 20 \text{ kN/m}^3$

(b) M&P method

- Cohesion $c' = \mu_c + i \delta_c \mu_c$, $\delta_c = 15\%$, $\delta_\phi = \delta_\gamma = 0$
- Friction angle $\phi' = \mu_\phi + i \delta_\phi \mu_\phi$, $\delta_\phi = 15\%$, $\delta_c = \delta_\gamma = 0$
- Unit weight $\gamma = \mu_\gamma + i \delta_\gamma \mu_\gamma$, $\delta_\gamma = 15\%$, $\delta_c = \delta_\phi = 0$

(c) Sarma method

- Cohesion $c' = \mu_c + i \delta_c \mu_c$, $\delta_c = 15\%$, $\delta_\phi = \delta_\gamma = 0$
- Friction angle $\phi' = \mu_\phi + i \delta_\phi \mu_\phi$, $\delta_\phi = 15\%$, $\delta_c = \delta_\gamma = 0$
- Unit weight $\gamma = \mu_\gamma + i \delta_\gamma \mu_\gamma$, $\delta_\gamma = 15\%$, $\delta_c = \delta_\phi = 0$

Fig. 5-4 Variation of the Performance Function (the Factor of Safety) with Increment or Decrement of Basic Random Variables
5.4. POINT ESTIMATION METHOD (PEM)

5.4.1. BRIEF INTRODUCTION

The point estimation method was first proposed by Rosenblueth (1975) for approximately evaluating the statistical moments of a function of two or more random variables by means of the sum of several weighted 'point estimates'. The values of the point estimates and the weight factors can be obtained from elementary rules of probability theory. The main advantage of the PEM is that the derivatives of a performance function are not required for estimating the first two or three moments of the performance function. The applications of the PEM to geotechnical engineering have been presented by several researchers (e.g. DeRooy, 1980; Nguyen & Chowdhury, 1984, 1985; Wolff, 1985; Lau & Kuroda, 1986; Zhang, 1990).

5.4.2. FUNCTION OF A SINGLE VARIABLE

Let \( x \) and \( y \) be random variables and \( y = G(x) \) be a well-behaved performance function. Given the mean value \( (\mu_x) \), standard deviation \( \sigma_x \), and skewness coefficient \( \nu_x \) of \( x \), the \( m \)th-order statistical moment of \( y \) can be approximately expressed as follows:

\[
E[y^m] \approx P_+ y_+^m + P_- y_-^m
\]  
(5-45)

where, \( y_+ = G(x_+) \) and \( m \) is the order number of the moment.

As Rosenblueth (1974) pointed out, \( P_+ \) and \( x_+ \) must satisfy the following simultaneous equations:

\[
\begin{align*}
P_+ + P_- & = 1 \\
P_+ x_+ + P_- x_- & = \mu_x \\
P_+ (x_+ - \mu_x)^2 + P_- (x_- - \mu_x)^2 & = \sigma_x^2 \\
P_+ (x_+ - \mu_x)^3 + P_- (x_- - \mu_x)^3 & = \nu_x \sigma_x^3
\end{align*}
\]  
(5-46)

The solutions of Eq. (5-46) are:
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\[ P_+ = \frac{1}{2} \left[ 1 \pm \sqrt{1 - \frac{1}{1 - (\nu_x / 2)^2}} \right] \]  
(5-47)

\[ P_- = 1 - P_+ \]  
(5-48)

\[ x_+ = \mu_x + \sigma_x \sqrt{P_+ / P_-} \]  
(5-49)

\[ x_- = \mu_x - \sigma_x \sqrt{P_- / P_+} \]  
(5-50)

Very small values of the skewness coefficient are approximated by the value \( \nu_x = 0 \). Therefore, \( P_+ \) and \( P_- \) in Eqs. (5-47) and (5-48) equal 1/2. Thus, the first and second order moments of \( y \) can be simply expressed as follows:

\[ \mu_y = \frac{y_+ + y_-}{2} \]

\[ \sigma_y = \frac{|y_+ - y_-|}{2} \]  
(5-51)

5.4.3. MULTIVARIABLE PERFORMANCE FUNCTIONS

The procedure was extended by Rosenblueth (1975) to multivariable performance functions in which the basic random variables may be correlated. The joint probability density of the performance function is concentrated at \( 2^n \) points in \( n \)-dimensional space defined by \( n \) random variables. Then the \( m \)-th order statistical moments of the performance function can be approximated by a formula similar to that for the one dimensional case. Consider cases with two variables (\( n = 2 \)) and three variables respectively:

Let \( y = G(x_1, x_2) \) and \( y = G(x_1, x_2, x_3) \) respectively for the two cases. The \( m \)-th order statistical moments can be expressed respectively by Eqs. (5-52) and (5-53) below:

\[ E[y^m] \approx P_{++}y_{++}^m + P_{+-}y_{+-}^m + P_{-+}y_{-+}^m + P_{--}y_{--}^m \]  
(5-52)

\[ E[y^m] \approx P_{++}y_{++}^m + P_{+-}y_{+-}^m + P_{-+}y_{-+}^m + P_{--}y_{--}^m + P_{+++}y_{+++}^m + P_{++}y_{++}^m + P_{++}y_{++}^m + P_{+++}y_{+++}^m + P_{+++}y_{+++}^m + P_{+++}y_{+++}^m \]  
(5-53)

in which,

\[ y_{++} = G(\mu_{x_1} + \sigma_{x_1}, \mu_{x_2} + \sigma_{x_2}) \]

\[ y_{+-} = G(\mu_{x_1} + \sigma_{x_1}, \mu_{x_2} - \sigma_{x_2}) \]

\[ \vdots \]
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\[ y \ldots = G(\mu_{x_1} - \sigma_{x_1}, \mu_{x_2} - \sigma_{x_2}, \mu_{x_3} - \sigma_{x_3}) \]

Let \( \rho_{ij} \) be the correlation coefficient of \( x_i \) and \( x_j \), the weight factors in Eqs.(5-52) and (5-53) can be calculated by the following equations:

(For Two Variable Case)

\[
\begin{align*}
P_{++} &= P_{--} = \frac{1}{4} (1 + \rho_{12}) \\
P_{+-} &= P_{-+} = \frac{1}{4} (1 - \rho_{12})
\end{align*}
\]

(5-54)

(For Three Variable Case)

\[
\begin{align*}
P_{+++} &= P_{---} = \frac{1}{8} (1 + \rho_{12} + \rho_{13} + \rho_{23}) \\
P_{++-} &= P_{--+} = \frac{1}{8} (1 + \rho_{12} - \rho_{13} - \rho_{23}) \\
P_{+--} &= P_{-+-} = \frac{1}{8} (1 - \rho_{12} + \rho_{13} - \rho_{23}) \\
P_{++-} &= P_{--+} = \frac{1}{8} (1 - \rho_{12} - \rho_{13} + \rho_{23})
\end{align*}
\]

(5-55)

The definitions of weight factors in Eqs. (5-54) and (5-55) meet the requirements of elementary probabilistic theory because the sum of all weight factors is equal to unity.

The coordinates of the \( 2^n \) points and the relevant weight factors are shown in Figs. 5-5 and 5-6 for \( n = 2 \) and 3 respectively.

Fig. 5-5 The Coordinates of Points and The Relevant Weight Factors in Two Dimensional Case.
(after Rosenblueth, 1975)
The first moment (mean value) and second central moment (variance) of a multivariable performance function $G(X)$ can be calculated by the statistical moments of the function. For a two-dimensional performance function $y = G(x_1, x_2)$, the mean value of $y$ can be estimated by Eq.(5-52) with $m = 1$ and the variance of $y$ is given, by definition, as follows:

$$\sigma_y^2 = E\left\{ [y - E(y)]^2 \right\} = E[y^2] - [E(y)]^2 = E[y^2] - \mu_y^2 \quad (5-56)$$

In Eq.(5-56), the first term can be estimated by Eq.(5-52) with $m = 2$. Using a similar procedure, the third and fourth central moments of $y$ can also be estimated.

5.4.4. APPLICATION OF PEM TO PROBABILISTIC ANALYSIS OF SLOPE STABILITY

The factor of safety associated with any homogeneous or multi-layer slope can be treated as a multivariable performance function with respect to basic random variables such as the shear strength parameters ($c_t$ and $\phi_t$) and unit weight ($\gamma_{lt}$) of the slope materials and pore water pressure ratio ($r_u$). The calculation processes for using PEM to perform the reliability analysis of slopes may be summarised as follows:
(1) Determine the mean value and the standard deviation of each basic random variable as well as the correlation coefficient between any two random variables.

(2) Determine the location of $2^n$ points in n-dimensional space and the relevant weight factor on these points (as shown in Figs. 5-5 and 5-6).

(3) Calculate the factor of safety of the slope for each combination of these points sequentially based on any of the limit equilibrium methods introduced in Chapter 3.

(4) Sum these weighted factors of safety with $m = 1$ and $m = 2$ based on Eq.(5-52) and estimate the first and the second statistical moments of factor of safety then the variance of the factor of safety can be estimated by Eq.(5-56).

(5) Calculate the slope reliability index based on the Cornell's definition, i.e., Eq. (5-10).

The calculation effort required for the PEM method increases exponentially with the number of basic random variables in the performance function. For $n = 10$ the number of calculations for the factor of safety is 1024 and $n = 12$ the number is 4096. Thus the computational efficiency of the PEM method decreases significantly as the number of basic random variables increases. However, advanced PEM methods have recently been proposed by Li (1991).

### 5.5 MONTE CARLO SIMULATION METHOD (MCSM)

#### 5.5.1. INTRODUCTION

Simulation is the process of replicating the real world based on various kinds of experiments and mathematical models. The simulation associated with mathematical models is usually based on the use of numerical methods. In this computer age, numerical simulation is a powerful tool for studying the performance or response of engineering systems. Through repeated numerical simulations, the sensitivity of system performance to variation in the values of system parameters may be assessed.
Monte Carlo simulation technique is a very important tool for simulating the probability distribution of a system when the variables in the system are random variables. During the simulation process, a sample of the system solutions or responses, each corresponding to a different set of values of the basic random variables, can be obtained. A sample from Monte Carlo simulation is similar to a sample of experimental observations. Therefore, the results of Monte Carlo simulations may be treated statistically and such results may be presented in the form of histograms or used to calculate the statistical moments of the performance function which describes the system. Moreover, some known probability distributions may be used to model the results by appropriate choice of the values of the significant parameters of that probability distribution. The results based on MCSM method are subject to, of course, sampling errors. Although the generation of the probability distribution based on Monte Carlo simulation technique requires large numbers of calculations using randomly generated data from the known or assumed distributions of the basic variables, it is widely used to calculate the reliability of complex engineering systems. The reason for this fact is that there are few available alternatives to simulate complex systems.

The Monte Carlo simulation approach has been widely use by researchers concerned with slope stability (e.g. Kraft and Mukhohay, 1977; Major et al, 1977; Marek and Savely. 1978; Miller, 1984; Nguyen and Chowdhury, 1984, etc).

5.5.2. ELEMENTARY THEORY OF THE MONTE CARLO SIMULATION METHOD (MCSM)

A key task of the Monte Carlo simulation is the generation of appropriate values of each random variable in accordance with its prescribed probability distribution. The automatic generation of the requisite random numbers with specified distributions is, therefore, necessary. Uniformly distributed random number between 0 and 1.0 is first generated for each variable. Through appropriate transformations the corresponding random variable with the specified probability distribution is then obtained.

Consider a random variable \( X \) with the cumulative distribution function (CDF) \( F_X(x) \). Then, at a given cumulative probability \( F_X(x) = u \), the value of \( X \) is
\[ x = F_X^{-1}(u) \]  

(5-57)

Now suppose that \( u \) is a value of the standard uniform variate, \( U \), with a uniform probabilistic density function (PDF) between 0 and 1.0, then CDF of \( U \) is,

\[ F_U(u) = u \]  

(5-58)

that is, the cumulative probability of \( U \leq u \) is equal to \( u \). The PDF and CDF of standard uniform variate \( U \) are shown in Fig (5-7) after Ang and Tang (1984).

Therefore, if \( u \) is a value of \( U \), the corresponding value of the variate \( X \) obtained through Eq. (5-57) will have a cumulative probability,

\[
\begin{align*}
P(X \leq x) &= P[F_X^{-1}(U) \leq x] = P[U \leq F_X(x)] \\
&= F_U[F_X(x)] = F_X(x)
\end{align*}
\]  

(5-59)

Fig. 5-7 PDF and CDF of Standard Uniform Variate \( U \)

(after Ang and Tang, 1984)
Accordingly, if \((u_1, u_2, \ldots, u_n)\) is a set of values from \(U\), the corresponding set of values obtained through Eq.\((5-57)\) are:

\[ x_i = F_X^{-1}(u_i) \quad i = 1, 2, \ldots, n \quad (5-60) \]

These values (Eq. 5-60) will have the desired CDF - \(F_X(x)\). The relationship between \(u\) and \(x\) is shown in Fig. 5-8 after Ang and Tang (1984).

Random numbers based on a standard uniform distribution can be generated by various mathematical and physical methods. However, the random numbers generated by these approaches are not really random and may be called 'pseudo random numbers'. In this research, the 'pseudo random numbers' with standard uniform distribution are generated by a special function in the computer system.

5.5.3. RANDOM VARIATE SAMPLING

According to the elementary concept of MCSM introduced above, any random variable with a prescribed probability distribution may be generated from random numbers with standard uniform distribution if the inverse function of the prescribed CDF exists and can be expressed explicitly. The formulae of random variable sampling associated with some important probability distributions are presented in the following sections.
5.5.3.1. STANDARD NORMAL RANDOM VARIABLE

Box and Muller (1958) have shown that if \( u_1 \) and \( u_2 \) are two independent standard uniform random variates, then the independent random variables \( z_1 \) and \( z_2 \) from a standard normal distribution (with zero mean and unit variance) can be obtained from the following equations:

\[
\begin{align*}
    z_1 &= \sqrt{-2 \ln u_1 \cos 2\pi u_2} \\
    z_2 &= \sqrt{-2 \ln u_1 \sin 2\pi u_2}
\end{align*}
\]  

(5-61)

Therefore, a pair of independent random variables from a normal distribution \( N(\mu, \sigma) \) may be generated by:

\[
\begin{align*}
    x_1 &= \mu + \sigma \sqrt{-2 \ln u_1 \cos 2\pi u_2} \\
    x_2 &= \mu + \sigma \sqrt{-2 \ln u_1 \sin 2\pi u_2}
\end{align*}
\]  

(5-62)

However, if the 'pseudo random numbers' are used then the normal random variables obtained from Eqs. (5-61) and (5-62) are pseudo normal random variables. The proof of Eq. (5-61) is excluded from this section and the reader may refer the paper published by Box and Muller (1958).

For a lognormal variate \( y \) with parameters \( \lambda \) and \( \zeta \), it has been shown that \( \ln y \) is a normal variable with mean \( \lambda \) and standard deviation \( \zeta \). Therefore, if \( x \) is a value from a normal distribution \( N(\lambda, \zeta) \), then the lognormal random variable \( y \) can be calculated by

\[
y = e^x
\]  

(5-63)

5.5.3.2. BETA-DISTRIBUTED RANDOM VARIABLES

The PDF of the standard Beta distribution with shape parameters \( q \) and \( r \) is given by

\[
f_z(z) = \frac{1}{B(q,r)} z^{q-1}(1 - z)^{r-1} \quad 0 \leq z \leq 1
\]  

(5-64)

in which \( B(q,r) \) is the Beta function which is defined by
\[ B(q,r) = \int_0^1 t^{q-1} (1 - t)^{r-1} \, dt \]  

(5-65)

A random variable from the Beta distribution may be generated by following formula

\[ z = \frac{u_1^{1/q}}{u_1^{1/q} + u_2^{1/r}} \]  

(5-66)

where \( u_1 \) and \( u_2 \) are two random values from the standard uniform distribution which have to satisfy the condition: \( u_1^{1/q} + u_2^{1/r} \leq 1 \). The generation procedure of Beta-distributed random variable was proposed by Jöhnk (1964). The random variable \( x \) from a general Beta distribution, that is, with lower and upper bounds \( a \) and \( b \), may then be obtained through the following:

\[ x = \frac{z - a}{b - a} \]  

(5-67)

where \( z \) is the corresponding random variable from the standard Beta distribution.

5.5.3.3. EXponentially-DISTRIBUTED RANDOM VARIABLE

The PDF of an exponential distribution is

\[ f_x(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \]  

(5-68)

where the parameter \( \lambda \) is larger than zero \( (\lambda > 0) \).

According to the elementary theory of the Monte Carlo simulation, a random variable from the exponential distribution may be generated as

\[ x = -\frac{1}{\lambda} \ln u \]  

(5-69)

where \( u \) is random variable from the standard uniform distribution.
5.5.4. GENERATION OF RANDOM VECTORS-CORRELATED VARIABLES

Let \( \mathbf{X} = (X_1, X_2, \ldots, X_n)^T \) be a vector of random variables and the joint distribution of all these random variables be \( f_{\mathbf{X}}(X_1, X_2, \ldots, X_n) \). If the components of the vector \( \mathbf{X} \) are statistically independent, each random component can be generated separately and independently by using the method presented above.

In realistic problems, the components of \( \mathbf{X} \) are often correlated. In this case, the joint PDF of the components of random vector \( \mathbf{X} \) may be expressed as

\[
f_{\mathbf{X}}(X_1, X_2, \ldots, X_n) = f_{X_1}(x_1)f_{X_2}(x_2|x_1) \cdots f_{X_n}(x_n|x_1, \ldots, x_{n-1})
\]

where \( f_{X_1}(x_1) \) is the marginal PDF of \( X_1 \), and \( f_{X_k}(x_k|x_1, \ldots, x_{k-1}) \) is the conditional PDF of \( X_k \). The corresponding joint CDF is

\[
F_{\mathbf{X}}(x_1, x_2, \ldots, x_n) = F_{X_1}(x_1)F_{X_2}(x_2|x_1) \cdots F_{X_n}(x_n|x_1, \ldots, x_{n-1})
\]

where \( F_{X_1}(x_1) \) and \( F_{X_k}(x_k|x_1, \ldots, x_{k-1}) \) are the marginal and conditional CDF of \( X_1 \) and \( X_k \), respectively.

When components of \( \mathbf{X} \) are correlated, the required random variables cannot be generated independently one by one. The following procedure may be used for generating the required set of correlated random variables.

Consider a set of uniformly distributed and independent random variables \( (u_1, u_2, \ldots, u_n) \) which have been generated. Then, a value \( x_1 \) may be determined independently as

\[
x_1 = F_{X_1}^{-1}(u_1)
\]

With this value of \( x_1 \), the conditional CDF \( F_{X_2}(x_2|x_1) \) is a function only of \( x_2 \), and hence a value \( x_2 \) may be determined from

\[
x_2 = F_{X_2}^{-1}(u_2|x_1)
\]

Similar, using the values \( x_1, x_2, \ldots, x_{n-1} \) already obtained, the value of \( x_n \) can be determined as
\[ x_n = F^{-1}_{X_n}(u_n|x_1, x_2, \ldots, x_{n-1}) \] (5-72c)

Therefore, recursively the required set of dependent random variables \((x_1, x_2, \ldots, x_n)\) can be determined, provided that the inverse functions of the marginal and conditional CDF, i.e. \(x_1 = F_{X_1}^{-1}(x_1)\) and \(x_k = F_{X_k}^{-1}(u_k|x_1, x_2, \ldots, x_{k-1})\), are known.

The joint probability density function of random vector \(X\) and the conditional distributions are generally unknown and cannot be determined for realistic problems. However, the density function \(f_{X_i}(x_i)\) of the component \(x_i\) and the correlation coefficient \(\rho_{ij}\) between components \((i, j = 1, 2, \ldots, n)\) may be estimated from experimental data combined with experience and judgement. If the random vector \(X\) follows a normal distribution its joint density function can then be determined uniquely. For other cases, this uniqueness does not exist. Thus the random variable vector which has a given marginal density function \(f_{X_i}(x_i)\) and correlation coefficient \(\rho_{ij}\) can be generated from alternative joint probability density functions. This implies that the generated random vector \(X\), with a given distribution function and correlation coefficients, is not unique when \(X\) is not a normal random vector. The generation of correlated bivariate normal random vector on the basis of linear transformation was discussed by Nguyen and Chowdhury (1985). However, a solution process of linear simultaneous equations is required for this linear transformation. A more general and comprehensive procedure is proposed below for generating correlated multivariate normal random vector.

Suppose that the random vector \(X = (x_1, x_2, \ldots, x_n)\) is subjected to the multivariate Gaussian distribution. A joint-normal density distribution function of several variables may be written in the following form:

\[
 f_X(x_1, x_2, \ldots, x_n) = (2\pi)^{-n/2}|C|^{1/2}\exp\left\{-\frac{1}{2}(X - \mu_X)^TC^{-1}(X - \mu_X)\right\} 
\]

(5-73)

in which, \(\mu_X = (\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_n})^T\) is mean of random vector \(X\),

\[
 C = \begin{pmatrix}
  c_{11} & c_{12} & \cdots & c_{1n} \\
  c_{21} & c_{22} & \cdots & c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n1} & c_{n2} & \cdots & c_{nn}
 \end{pmatrix}
\]

is the covariance matrix of random vector \(X\);
The components of \( C \) can be calculated from the definition of covariance, i.e.,

\[
c_{ij} = E[(x_i - \mu_x_i)(x_j - \mu_x_j)] \quad (i, j = 1, 2, \ldots, n)
\]

When coefficients of correlation and \( \mu_X \) equal to zero as well as the variance equals unity, the covariance matrix \( C \) becomes a \( I \)-matrix and the components of \( X \) are independent of each other and follow a standard normal distribution, i.e. \( N(0, 1) \). In this case, the uncorrelated standard normal random vector, \( Z \), may be generated by the method introduced in the previous section. There is a general linear transformation between the random vectors \( X \) and \( Z \) and this linear transformation can be expressed as follows:

\[
X = T \cdot Z + \mu_X \quad (5-74)
\]

The matrix \( T \) may be obtained by various methods. A most popular method is factorization. The matrix \( C \) is a positive definite and real symmetrical matrix and can be calculated by,

\[
C = E[(X - \mu_X)(X - \mu_X)^T] \quad (5-75)
\]

substituting Eq.(5-74) into Eq.(5-75) and considering \( E[ZZ^T] \) to equal \( I \), an identity matrix, the matrix \( C \) can be expressed as follows:

\[
C = T \cdot T^T \quad (5-76)
\]

According to the Crout factorization (Franklin, 1965), the matrix \( T \) in Eq.(5-74) or Eq.(5-76) can be constituted by a lower-triangular matrix with positive elements along its main diagonal, i.e.,

\[
T = \begin{pmatrix}
t_{11} & 0 & \ldots & 0 \\
t_{21} & t_{22} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
t_{n1} & t_{n2} & \ldots & t_{nn}
\end{pmatrix}
\]
The components $t_{ij}$ of $T$ can be computed from the following equation:

$$
t_{ij} = \left\{ \begin{array}{l}
\frac{c_{ij} - \sum_{k=1}^{j-1} t_{ik} t_{jk}}{\sqrt{c_{jj} - \sum_{k=1}^{j-1} t_{jk}^2}} \\
\sum_{k=1}^{0} t_{ik} t_{jk} = 0 \quad 0 \leq j \leq i \leq n
\end{array} \right. \quad (5-77)
$$

Therefore, the required random vector $X$ from a multivariate Gaussian distribution with mean vector $\mu_X$ and covariance matrix $C$ may be generated from Eq.(5-74) once the linear transformation matrix $T$ is determined.

5.5.5. APPLICATION OF MCSM TO PROBABILISTIC ANALYSIS OF SLOPE STABILITY

The probability distribution of the factor of safety of a homogeneous or multi-layer slope may be obtained once the probability density functions of all basic input random variables and the correlation coefficients between them are specified. In this research, the PDF of basic input random variables is assumed to follow a normal (Gaussian) distribution and the correlation coefficients between these random variables can be considered in the manner discussed above. The steps of the Monte Carlo simulation method for the probabilistic analysis of slopes are the following:

1. Determine mean value and standard deviation of each basic random variable as well as correlation coefficient of all pairs of random variables.
2. Assume the probability density function for all basic input random variables and specify the number of simulations to be used.
3. Calculate the lower-triangular matrix $T$ by means of Eq. (5-77) based on the covariance matrix $C$ of those basic input random variables which are assumed as normal random variables.
4. Generate the basic input values of variables.
5. Use these generated input values of simulation variables to calculate the factor of safety based on any one of the limit equilibrium methods introduced in Chapter 3.
(6) Go back to Step (4) until the given number of simulations is reached.

(7) Use a histogram to approximately describe the distribution of the factor of safety and evaluate the mean and variance of factor of safety from basic formulas given below:

\[
\mu_F \approx \frac{1}{N} \sum_{i=1}^{N} F_i \tag{5-78}
\]

\[
\sigma_F^2 \approx \frac{1}{N-1} \sum_{i=1}^{N} (F_i - \mu_F)^2 \tag{5-79}
\]

where, N is the number of simulations.

(8) Calculate the reliability index of slope based on Eq.(5-10).

5.6. COMPARISONS OF THE THREE NUMERICAL METHODS

5.6.1. INTRODUCTION

Several numerical methods have been developed for use in probabilistic analysis of slopes. However, few systematic comparisons of these methods have been performed by geotechnical researchers. Comparisons among the three numerical methods (i.e. FOSM, PEM and MCSM) in connection with slope of reliability are presented in this section. The performance functions are based on alternative limit equilibrium methods proposed in Chapter 3. The shear strength parameters, the soil unit weight and the pore water pressure ratio are considered as random variables. The correlation coefficients between the basic input random variables are considered. However, the influence of the auto-correlation and cross-correlation on the slope of reliability is discussed in subsequent sections. The maximum number of random variables has been limited to 13 in these studies. For PEM, therefore, the maximum number of FOS calculations is 8192 (= 2^{13}). The Monte Carlo simulation studies have been performed with 3000 simulations of the performance function. All the basic input random variables are considered as Gaussian (normal) random variables.
To make the comparisons among these three numerical methods, reliability indices based on three alternative limit equilibrium methods have been considered separately.

5.6.2. RELIABILITY INDICES BASED ON SIMPLIFIED BISHOP METHOD-TWO EXAMPLES

The first example concerns the stability of a homogeneous slope and the second that of a non-homogeneous slope. In each case, three numerical methods are used to estimate the reliability indices.

5.6.2.1. HOMOGENEOUS SLOPE

The geometry of a slope along with a specified circular slip surface is shown in Fig. 5-9. The coordinates of centre defining the slip surface and its radius are respectively $X_0 = -61.74 \text{m}$, $Y_0 = 393.74 \text{m}$, and $R = 83.66 \text{m}$. Shear strength parameters $c'$ and $\phi'$, unit weight $\gamma$ and the pore water pressure ratio, $r_u$, are assumed to be the only random variables. The mean values of these random variables are taken as follows:

<table>
<thead>
<tr>
<th>$\mu_\gamma$</th>
<th>$\mu_{c'}$</th>
<th>$\mu_{\phi'}$</th>
<th>$\mu_{r_u}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.3 (kN/m$^3$)</td>
<td>37 (kN/m$^2$)</td>
<td>15*</td>
<td>0.2</td>
</tr>
</tbody>
</table>

The factors of safety, under this conditions with and without pore water pressure, corresponding to the specified circular slip surface are 1.23 and 1.43, respectively.

Coefficients of variation of all the random variables are assumed to vary from 5% to 30%. The three random variables $c'$, $\phi'$, and $\gamma$ are considered either as independent or correlated. The coefficients of correlation are assumed as follows:

<table>
<thead>
<tr>
<th>$\rho_{c\gamma}$</th>
<th>$\rho_{c\phi}$</th>
<th>$\rho_{\gamma\phi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.3</td>
<td>-0.25</td>
</tr>
</tbody>
</table>
The reliability indices estimated by these three numerical methods are shown in Figs. 5-10 and 5-11 for the case when pore water pressure is ignored. FOSM-1 in the figures means that reliability index is calculated by Cornell's definition and FOSM-2 by

Fig. 5-9 Geometry of A Homogeneous Slope with Circular Slip Surface

Fig. 5-10 Comparisons of Reliability Index Calculated by Three Numerical Methods with Independent Random Variables
Hasofer and Lind’s definition. The differences between reliability indices, based on Cornell’s and Hasofer and Lind’s definitions, reduce with increasing coefficients of variation of basic random variables. This result is in agreement with the conclusion in Section 5.3.5. The reliability indices based on three numerical methods show close agreement with each other.

For the case when the pore water pressure is included in the analyses, the reliability indices of the slope, based on the three numerical methods, are shown in Figs. 5-12 and 5-13. Again, there is close agreement among the three methods. Comparing Figs. 5-10 and 5-11 with Figs. 5-12 and 5-13 respectively, agreement is not as close as in the case with no water pressure.
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Fig. 5-12 Comparisons of Reliability Index Calculated by Three Numerical Methods with Independent Random Variables ($r_u$ included as a random variable)

Fig. 5-13 Comparisons of Reliability Index Calculated by Three Numerical Methods with Correlated Random Variables ($r_u$ included as a random variable)
5.6.2.2. NON-HOMOGENEOUS SLOPE

The geometry of a non-homogeneous slope with four different soil layers is shown in Fig. 5-14. The geometrical parameters defining the circular critical slip surface are \( X_0 = 2.59 \text{m}, \) \( Y_0 = 32.19 \text{m}, \) and \( R = 32.29 \text{m}, \) respectively. The mean values of shear strength parameters and unit weight of the soil layers are shown in Table 5-1. The standard deviations of all these random variables and the correlation coefficients between pairs of random variables in the same soil layer are also tabulated (in Table 5-1). The correlations between the random variables in different soil layers are excluded from the calculations. In other words, each layer is considered to have properties independent of other layers.

Table 5-1 Basic Input Random Variables and Correlation Coefficients

<table>
<thead>
<tr>
<th>No. Layers</th>
<th>Mean Values</th>
<th>Standard Deviations</th>
<th>Correlation Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \mu_\gamma ) (kN/m(^2))</td>
<td>( \mu_c ) (kN/m(^2))</td>
<td>( \mu_{\phi} ) (degree)</td>
</tr>
<tr>
<td>1</td>
<td>16</td>
<td>25</td>
<td>15°</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>26</td>
<td>18°</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
<td>28</td>
<td>20°</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>30</td>
<td>22°</td>
</tr>
</tbody>
</table>

Fig. 5-14 The Geometry of a Non-homogeneous Slope with Circular Slip Surface
The factors of safety with and without pore water pressure, corresponding to the specified circular slip surface shown in Fig. 5-14, are 1.43 and 1.63, respectively.

The slope reliability indices estimated by the three numerical methods, for the slope with uncorrelated variable, are shown in Table 5-2. The results presented in Table 5-2 indicate that the magnitude of the reliability index $\beta^*$ (FOSM) is the highest. Reliability indices $\beta$ (FOSM), $\beta$ (PEM), and $\beta$ (MCSM) are relatively close to each other whether pore water pressure is excluded or included.

The main reason for differences in calculated values is the different definitions of reliability index.

Table 5-2. Reliability Indices of FOS without Any Correlation between Variables

<table>
<thead>
<tr>
<th>PWPR</th>
<th>Reliability Indices</th>
<th>$\beta$ (FOSM)</th>
<th>$\beta^*$ (FOSM)</th>
<th>$\beta$ (PEM)</th>
<th>$\beta$ (MCSM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_u = 0$</td>
<td></td>
<td>3.028</td>
<td>3.395</td>
<td>3.036</td>
<td>2.992</td>
</tr>
<tr>
<td>$r_u = 0.2$</td>
<td></td>
<td>2.285</td>
<td>2.549</td>
<td>2.306</td>
<td>2.328</td>
</tr>
</tbody>
</table>

Coefficient of Variation of PWPR: $\delta_{r_u} = 10\%$
Correlation Coefficients: $\rho_{yc} = \rho_{yc} = \rho_{c\phi} = 0$

Slope reliability indices estimated by the three numerical methods, for the case with correlated variables, are shown in Table 5-3. Once again $\beta^*$ (FOSM) has the largest magnitude whereas the other three values are in close agreement.

Table 5-3. Reliability Indexes of FOS with Correlated Variables

<table>
<thead>
<tr>
<th>PWPR</th>
<th>Reliability Indices</th>
<th>$\beta$ (FOSM)</th>
<th>$\beta^*$ (FOSM)</th>
<th>$\beta$ (PEM)</th>
<th>$\beta$ (MCSM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_u = 0$</td>
<td></td>
<td>3.360</td>
<td>3.773</td>
<td>3.372</td>
<td>3.354</td>
</tr>
<tr>
<td>$r_u = 0.2$</td>
<td></td>
<td>2.576</td>
<td>2.855</td>
<td>2.597</td>
<td>2.504</td>
</tr>
</tbody>
</table>

Coefficient of Variation of PWPR: $\delta_{r_u} = 10\%$
Correlation Coefficients: $\rho_{yc} \neq 0$, $\rho_{c\phi} \neq 0$, $\rho_{c\phi} \neq 0$
5.6.3. RELIABILITY INDICES BASED ON THE M&P METHOD-TWO EXAMPLES

The first example concerns the stability of a homogeneous slope and second one that of a non-homogeneous slope. In each case, three numerical methods are used to estimate the reliability indices. The performance functions in the reliability analysis are defined by the M&P method.

5.6.3.1. HOMOGENEOUS SLOPE

The geometry of a slope is shown in Fig. 5-15. The calculations are performed for a specified non-circular slip surface shown in the figure. Shear strength parameters \( c', \phi' \), unit weight \( \gamma \) and pore water pressure ratio \( r_u \) are assumed to have the same mean values as in the previous example (Subsection 5.6.2.1). The factors of safety with and without pore water pressure, corresponding to the specified non-circular slip surface shown in Fig. 5-15, are 1.22 and 1.41, respectively.

Coefficients of variation of all the random variables are assumed to vary from 5% to 30%. The three random variables \( c', \phi', \) and \( \gamma \) are considered either as independent or correlated. The coefficients of correlation between pairs of basic random variables are also assumed to be the same as in the previous example (Subsection 5.6.2.1).

![Fig. 5-15 The Geometry of A Slope with A Specified Arbitrary Slip Surface](image-url)

The reliability indices estimated by the three numerical methods are shown in Figs. 5-16 and 5-17 for the case when the pore water pressure is ignored. The FOSM-1
in the figures means that the reliability index is calculated by Cornell's definition and the FOSM-2 by the Hasofer and Lind's definition. Reliability indices based on these three numerical methods are in close agreement with each other. Differences between reliability indexes estimated by FOSM, based on Cornell's and Hasofer & Lind's definitions, are reduced with increasing coefficients of variation of basic random variables. This result is agreement with that obtained in Subsection 5.6.2.1.

Fig. 5-16 Comparisons of Reliability Index Calculated by Three Numerical Methods with Independent Random Variables

Fig 5-17 Comparisons of Reliability Index Calculated by Three Numerical Methods with Correlated Random Variables
The slope reliability indices based on the three numerical methods are shown in Figs. 5-18 and 5-19 for the case with pore water pressure. Once again there is close agreement between the three methods whether variables are independent or correlated.

**Fig. 5-18** Comparisons of Reliability Index Calculated by Three Numerical Methods with Independent Random Variables ($r_u$ included as a random variable)

**Fig. 5-19** Comparisons of Reliability Index Calculated by Three Numerical Methods with Correlated Random Variables ($r_u$ included as a random variable).
5.6.3.2. NON-HOMOGENEOUS SLOPE

The geometry of a non-homogeneous slope is shown in Fig. 5-20. The calculations are performed for a specified non-circular slip surface shown in the figure. The shear strength parameters and unit weights of these soil layers are the same as shown in Table 5-1. The standard deviations of these random variables and the correlations between the random variables within the same soil layer are also shown in Table 5-1. The correlations between random variables in different soil layers are excluded from the calculations. The factors of safety with and without pore water pressure ratio, corresponding to the critical non-circular slip surface shown in Fig. 5-20, are 1.38 and 1.57, respectively.

![Morgenstern and Price Method](image)

**Fig. 5-20** The Geometry of A Non-homogeneous Slope With Non-circular Slip Surface

The slope reliability indexes estimated by the three numerical methods, considering independent variables, are shown in Table 5-4. The magnitude of the reliability index-\( \beta^* \) (FOSM) is the highest. Reliability indices-\( \beta \) (FOSM) and \( \beta \) (PEM) are in close agreement with each other when pore water pressure is not included in the analyses. The reliability indices-\( \beta \) (FOSM), \( \beta \) (PEM), and \( \beta \) (MCSM) show less agreement when the pore water pressure is included as random variable.
Table 5-4. Reliability Indices of FOS without any correlation between Variables

<table>
<thead>
<tr>
<th>PWPR</th>
<th>Reliability Indices</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>β (FOSM)</td>
<td>β* (FOSM)</td>
<td>β (PEM)</td>
<td>β (MCSM)</td>
</tr>
<tr>
<td>r_u = 0</td>
<td>2.873</td>
<td>3.261</td>
<td>2.896</td>
<td>2.756</td>
</tr>
<tr>
<td>r_u = 0.2</td>
<td>2.096</td>
<td>2.367</td>
<td>2.148</td>
<td>2.283</td>
</tr>
</tbody>
</table>

Coefficient of Variation of PWPR: \( \delta_r = 10\% \)
Correlation Coefficients: \( \rho_{yc} = \rho_{yc} = \rho_{cp} = 0 \)

The slope reliability indices estimated by the three numerical methods, considering the case with correlated variables, are shown in Table 5-5. Once again \( \beta^*(\text{FOSM}) \) has the largest magnitude.

Table 5-5. Reliability Indices of FOS with Correlated variables

<table>
<thead>
<tr>
<th>PWPR</th>
<th>Reliability Indices</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>β (FOSM)</td>
<td>β* (FOSM)</td>
<td>β (PEM)</td>
<td>β (MCSM)</td>
</tr>
<tr>
<td>r_u = 0</td>
<td>3.282</td>
<td>3.745</td>
<td>3.314</td>
<td>3.362</td>
</tr>
<tr>
<td>r_u = 0.2</td>
<td>2.390</td>
<td>2.722</td>
<td>2.464</td>
<td>2.548</td>
</tr>
</tbody>
</table>

Coefficient of Variation of PWPR: \( \delta_r = 10\% \)
Correlation Coefficients: \( \rho_{yc} \neq 0, \rho_{\gamma \psi} \neq 0, \rho_{cp} \neq 0 \)

5.6.4. RELIABILITY INDICES BASED ON SARMA METHOD-TWO EXAMPLES

The first example concerns a homogeneous slope and the second concerns a non-homogeneous slope. The slope reliability indices are calculated on the basis of the Sarma method with non-vertical slices. In each example, independent and correlated variables are considered separately as in the previous sections.
5.6.4.1. HOMOGENEOUS SLOPE

The geometry of a slope with inclined slices is shown in Fig. 5-21. Reliability analyses were performed for a specified non-circular slip surface shown in the figure. Shear strength parameters on internal shear planes are considered in stability analyses using the Sarma method. The values of these parameters are assumed to be the same as those along the slip surface. The shear strength parameters $c'$, $\phi'$ and unit weight $\gamma$ as well as the pore water pressure $r_u$ are considered as random variables. The mean values of these random variables are taken as follows:

<table>
<thead>
<tr>
<th>$\mu_\gamma$</th>
<th>$\mu_{c'}$</th>
<th>$\mu_{\phi'}$</th>
<th>$\mu_{r_u}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 (kN/m$^3$)</td>
<td>20 (kN/m$^3$)</td>
<td>25°</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Fig. 5-21 The Geometry of A Slope with A Specified Non-circular Slip Surface

The factors of safety under conditions without and with pore water pressure, corresponding to the critical non-circular slip surface shown in Fig. 5-21, are 1.66 and 1.44, respectively.

Coefficients of variation of all the random variables are assumed to vary from 5% to 30%. The three random variables $c'$, $\phi'$, and $\gamma$ are considered either as independent or
correlated. The coefficients of correlation between the three pairs are assumed as follows:

<table>
<thead>
<tr>
<th></th>
<th>$\rho_{\gamma_c}$</th>
<th>$\rho_{\gamma_{\phi}}$</th>
<th>$\rho_{c_{\phi}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.2</td>
<td>0.3</td>
<td>-0.25</td>
</tr>
</tbody>
</table>

Reliability indices estimated by the three numerical methods are shown in Figs. 5-22 and 5-23 for the case when the pore water pressure is ignored. The notation FOSM-1 in the figures means that the reliability index is calculated by Cornell’s definition and

![Graph](image1)

**Fig. 5-22** Comparisons of Reliability Index Calculated by Three Numerical Methods with Independent Random Variables ($r_u = 0$)

![Graph](image2)

**Fig. 5-23** Comparisons of Reliability Index Calculated by Three Numerical Methods with Correlated Random Variables ($r_u = 0$)
FOSM-2 by the Hasofer and Lind's definition. In general, there is close agreement between the three methods. The differences between reliability indices estimated by FOSM, based on Cornell's and Hasofer & Lind's definitions, are reduced with increase of the coefficients of variation of basic random variables. Differences of reliability indices between MCSM and others methods increase with increase of the coefficient of variation of the basic random variables.

For the case when the pore water pressure is included as a random variable in the performance function, the reliability indices of the slope based on the three numerical methods are shown in Figs. 5-24 and 5-25. Reliability indices based on the Monte Carlo simulation are significantly lower than estimates based on other methods if coefficient of variation of the variables are greater than about 15%. However, the reliability indexes of other two methods (FOSM and PEM) are almost identical.

![Fig. 5-24 Comparisons of Reliability Index Calculated by Three Numerical Methods with Independent Random Variables (r_u included as a random variable)](image-url)
5.6.4.2. NON-HOMOGENEOUS SLOPE

The geometry of a non-homogeneous slope with two material types is shown in Fig. 5-26. The shear strength parameters and unit weights of these two materials are shown in Table 5-6. The standard deviations of all random variables and the correlation coefficients between pairs of random variables in the same type of material are tabulated (Table 5-6). The correlation coefficients between any pair of random variables in different soil layers are excluded from these calculations. In each material, the shear parameters on internal shear planes are assumed to be identical with those on slip surface. The factors of safety without and with pore water pressure, corresponding to the specified non-circular slip surface shown in Fig. 5-27, are 1.69 and 1.88, respectively.
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Fig. 5-26 The Geometry of A Non-homogeneous Slope with Non-circular Slip Surface and Inclined Slices

Table 5-6 Basic Input Random Variables and Correlated Coefficients

<table>
<thead>
<tr>
<th>No. Type</th>
<th>Mean Values</th>
<th>Standard Deviations</th>
<th>Correlated Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mu_\gamma$ (kN/m$^3$)</td>
<td>$\mu_c$ (kN/m$^3$)</td>
<td>$\mu_\psi$ (degree)</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>20</td>
<td>15°</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>15</td>
<td>12°</td>
</tr>
</tbody>
</table>

The slope reliability indices estimated by the three numerical methods, considering independent variables, are shown in Table 5-7. The magnitudes of the reliability index-$\beta^*$ (FOSM) is the largest and the reliability index-$\beta$ (FOSM), $\beta$ (PEM), and $\beta$ (MCSM) are close to each other whether pore water pressure is included in the computations or not.
Table 5-7: Reliability Indexes of FOS without Any Correlation between Variables

<table>
<thead>
<tr>
<th>PWPR</th>
<th>Reliability Indexes</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>β (FOSM)</td>
<td>β* (FOSM)</td>
<td>β (PEM)</td>
<td>β (MCSM)</td>
</tr>
<tr>
<td>r_u = 0</td>
<td>2.837</td>
<td>3.908</td>
<td>2.864</td>
<td>2.892</td>
</tr>
<tr>
<td>r_u = 0.2</td>
<td>2.448</td>
<td>3.242</td>
<td>2.496</td>
<td>2.420</td>
</tr>
</tbody>
</table>

Coefficient of Variation of PWPR: $\delta_{r_u} = 10\%$
Correlation Coefficient: $\rho_{\gamma_c} = \rho_{\gamma_c} = \rho_{\phi_c} = 0$

Slope reliability indices estimated by the three numerical methods, considering correlation coefficients, are shown in Table 5-8. The maximum relative errors among reliability indices based on $\beta^*$, with and without pore water pressure, are 25% and 29%, respectively. The differences of reliability indexes between Hasofer & Lind’s method and others methods are again quite significant.

Table 5-8: Reliability Indexes of FOS with Correlated Variables

<table>
<thead>
<tr>
<th>PWPR</th>
<th>Reliability Indexes</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>β (FOSM)</td>
<td>β* (FOSM)</td>
<td>β (PEM)</td>
<td>β (MCSM)</td>
</tr>
<tr>
<td>r_u = 0</td>
<td>2.976</td>
<td>4.151</td>
<td>3.006</td>
<td>2.937</td>
</tr>
<tr>
<td>r_u = 0.2</td>
<td>2.580</td>
<td>3.450</td>
<td>2.632</td>
<td>2.651</td>
</tr>
</tbody>
</table>

Coefficient of Variation of PWPR: $\delta_{r_u} = 10\%$
Correlation Coefficient: $\rho_{\gamma_c} \neq 0$, $\rho_{\gamma_c} \neq 0$, $\rho_{\phi_c} \neq 0$
5.6.5. DISCUSSION

The comparative studies presented in this chapter, based on the three numerical methods, showed that the slope reliability indexes - $\beta$ (FOSM), $\beta^*$ (FOSM), $\beta$ (PEM), and $\beta$ (MCSM) are usually in close agreement with each other. For a slope in which pore water pressure is included as a basic random variable the differences among these reliability indices are larger than that for the case with no pore water pressure. The slope reliability index based on Hasofer and Lind's definition - $\beta^*$ (FOSM) was found to be greater than alternative values whether a slope was homogeneous or non-homogeneous, whether pore water pressure was included or not and whether circular or non-circular slip surface was considered. In general, the different definition of reliability index used has a significant influence on the reliability index. Differences between numerical results by the three methods are greater for non-circular slip surfaces compared to those for circular slip surfaces. Again differences are greater for non-homogeneous slopes compared to homogeneous slopes. The reliability index - $\beta$ (FOSM) is always conservative compared with $\beta$ (PEM) and $\beta^*$ (FOSM). The differences between $\beta$ (FOSM) and $\beta$ (MCSM) are not consistent. The reliability indices - $\beta$ (FOSM) and $\beta$ (PEM) are usually in close agreement. The differences between $\beta$ (FOSM) and $\beta$ (PEM) as well as between $\beta$ (FOSM) and $\beta$ (MCSM) are increased with the increase in the variation coefficients of basic random variables. Yet, the variation coefficients are small (normally less than 10%).

In conclusion, all three numerical methods can be used to provide a choice for estimating reliability indices of slopes. However, choice of slope stability model may have a significant influence on the calculated magnitudes.
5.7. INFLUENCE OF CROSS-CORRELATION COEFFICIENTS OF VARIABLES ON RELIABILITY INDEX OF SLOPES

In this section, the influence of cross-correlation coefficients of variables on reliability index of slopes will be investigated in some detail based on the probabilistic analysis methods presented in previous sections and on slope stability analysis approaches introduced in Chapter 3. The influence of these correlation coefficients on the probability distribution of factor of safety will also be discussed. Relatively little has been published in this regard. However, the importance of correlation has been recognised. For example, significant negative correlation between shear strength parameters $c'$ and $\phi'$ has been reported by Matsuo and Kuroda (1974), Forster and Weber (1981) and Grivas (1981). Therefore, the correlation characteristics of $c'$ and $\phi'$, i.e. cross-correlation of $c'$ and $\phi'$, should be considered in probabilistic designs of slopes.

The Green Creek Slide was first presented by Crawford and Eden (1967). The actual case is concerned with the stability of slopes in sensitive clays. A probabilistic study was performed by Alonso (1976). The mean values and coefficients of variation of soil properties are shown in Table 5-11. The influences of cross-correlation coefficients on the calculated reliability indices will be discussed in following subsections based on three limit equilibrium methods.

<table>
<thead>
<tr>
<th>Table 5-11 Parameters of Soil Property</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Unit Weight</td>
</tr>
<tr>
<td><strong>Mean Value</strong></td>
</tr>
<tr>
<td>16 (kN/m$^3$)</td>
</tr>
<tr>
<td><strong>Coefficient of Variation</strong></td>
</tr>
<tr>
<td>0.3</td>
</tr>
<tr>
<td>Cohesion</td>
</tr>
<tr>
<td><strong>Mean Value</strong></td>
</tr>
<tr>
<td>29.43 (kPa)</td>
</tr>
<tr>
<td><strong>Coefficient of Variation</strong></td>
</tr>
<tr>
<td>0.15</td>
</tr>
<tr>
<td>Internal Friction Angle</td>
</tr>
<tr>
<td><strong>Mean Value</strong></td>
</tr>
<tr>
<td>19°</td>
</tr>
<tr>
<td><strong>Coefficient of Variation</strong></td>
</tr>
<tr>
<td>0.05</td>
</tr>
<tr>
<td>Pore Pressure Ratio</td>
</tr>
<tr>
<td><strong>Mean Value</strong></td>
</tr>
<tr>
<td>0.3</td>
</tr>
<tr>
<td><strong>Coefficient of Variation</strong></td>
</tr>
<tr>
<td>0.67</td>
</tr>
</tbody>
</table>
5.7.1. SIMPLIFIED BISHOP METHOD

On the basis of soil properties tabulated in Table 5-11, a circular critical slip surface based on the Bishop method may be located. The geometry of this critical slip surface is shown in Fig. 5-27. The minimum factor of safety corresponding to the critical slip surface is 1.60.

![Fig. 5-27 The Geometry of the Slope and Critical Circular Slip Surface](Green Creek Slide)

The cross-correlation coefficients among $c'$, $\phi'$ and $\gamma$ are assumed to vary individually from -0.5 to +0.5 with 0.1 as a step. The relationships between reliability indices and $\rho_{c\phi}$, $\rho_{yc}$ as well as $\rho_{\gamma\phi}$ are shown Figs. 5-28, 5-29 and 5-30 respectively. Referring Fig. 5-28, the reliability index decreases with the increase of $\rho_{c\phi}$ from negative to positive for FOSM-1, FOSM-2 and PEM. This means that reliability of slopes is enhanced with negative correlation whereas positive correlation will result in an underestimation of reliability. The reliability indexes based on FOSM-1 and PEM are in close agreement. Reliability indices based on FOSM-1 and PEM are more sensitive to the change of $\rho_{c\phi}$ than those based on FOSM-2.

Referring now to Figs. 5-29 and 5-30, reliability index of the slope increases with the increase of $\rho_{yc}$ or $\rho_{\gamma\phi}$ from negative to positive. The calculated values of
reliability index associated with FOSM-1 and PEM are in close agreement to each other. There are some differences between FOSM-1 and FOSM-2 as well as FOSM-2 and PEM which are basically constant with the variation of $\rho_{yc}$ and $\rho_{\gamma\phi}$ from negative to positive. The rates of increase FOSM-1, FOSM-2 and PEM for the change of $\rho_{yc}$ are 6%, 5% and 4%, respectively. The rates of increase FOSM-1, FOSM-2 and PEM for the change of $\rho_{\gamma\phi}$ are 3%, 2% and 3%, respectively. Reliability index is more sensitive to the change of $\rho_{c\phi}$ than that of $\rho_{yc}$ or $\rho_{\gamma\phi}$. An important finding is that the relationship between the reliability index and the cross-correlation coefficient is nearly linear in each case.

![Graph](image1)

**Fig. 5-28** The Relationship between Reliability Index and $\rho_{c\phi}$

![Graph](image2)

**Fig. 5-29** The Relationship between Reliability index and $\rho_{yc}$
The influences of cross-correlation on the probability distribution of the factor of safety may now be considered. These investigations are performed based on the Monte Carlo simulation method presented in Section 5.5. All the random variables are assumed to be normal variates in the analyses. Five cross-correlation values, i.e. $\rho = -0.5, -0.2, 0, 0.2, 0.5$, were used for these studies. The histograms of the factor of safety (FOS) for $\rho_{c\phi}$, $\rho_{yc}$ and $\rho_{y\psi}$ are shown in Figs. 5-31, 5-32 and 5-33 respectively (in each, the other two correlation coefficients are kept as zero). The main finding is that provided all the basic input random variables are normal variates, the distribution of FOS approximates to a normal distribution whether the cross-correlation coefficients equal to zero or not. However, the distributions of FOS are somewhat asymmetrical for certain values of $\rho_{c\phi}$, $\rho_{yc}$ and $\rho_{y\psi}$. This may be caused by sampling errors associated with the numerical simulation method itself.
Fig. 5-31 The Distributions of FOS Associated with $\rho_{c\theta}$
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Fig. 5-32 The Distribution of FOS Associated with $\rho_{yc}$
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Fig. 5-33 The Distribution of FOS Associated with $\rho_{\gamma k}$
Calculated reliability indices corresponding to the different assumptions are tabulated in Table 5-12. The reliability index decreases with the increase of $\rho_{c\phi}$ from negative to positive. The increase of $\rho_{\gamma c}$ or $\rho_{\gamma \phi}$ from negative to positive does not lead to a consistent increase of the corresponding calculated reliability index values. This trend may indicate lack of sensitivity or it may indicate numerical sampling errors. However, the general tendency of the reliability index is an increase with the variation of $\rho_{\gamma c}$ and $\rho_{\gamma \phi}$ from negative to positive. Comparing Figs. 5-28, 5-29, 5-30 and Table 5-12, the differences among reliability indexes based on FOSM-1, FOSM-2, PEM and MCSM are not very significant for a given cross-correlation coefficient.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>-0.5</th>
<th>-0.2</th>
<th>0</th>
<th>0.2</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>R.I. for $\rho_{c\phi}$ ($\rho_{\gamma c} = \rho_{\gamma \phi} = 0$)</td>
<td>1.788</td>
<td>1.699</td>
<td>1.628</td>
<td>1.579</td>
<td>1.491</td>
</tr>
<tr>
<td>R.I. for $\rho_{\gamma c}$ ($\rho_{c\phi} = \rho_{\gamma \phi} = 0$)</td>
<td>1.579</td>
<td>1.552</td>
<td>1.652</td>
<td>1.656</td>
<td>1.723</td>
</tr>
<tr>
<td>R.I. for $\rho_{\gamma \phi}$ ($\rho_{\gamma c} = \rho_{c\phi} = 0$)</td>
<td>1.567</td>
<td>1.591</td>
<td>1.664</td>
<td>1.634</td>
<td>1.636</td>
</tr>
</tbody>
</table>

5.7.2. MORGENSTERN AND PRICE METHOD

A non-circular critical slip surface, associated with Morgenstern and Price method, is shown in Fig. 5-34 and is based on the soil properties presented in Table 5-11. The minimum FOS corresponding to this critical slip surface is 1.59.

In order to investigate the influence of the cross-correlation coefficients on the reliability index, the cross-correlation coefficients among $c'$, $\phi'$ and $\gamma$ were assumed to vary from -0.5 to +0.5. The relationships between reliability indices and each of the parameters $\rho_{c\phi}$, $\rho_{\gamma c}$ and $\rho_{\gamma \phi}$ are shown respectively in Figs. 5-35, 5-36 and 5-37. The trends in the relationships based on different numerical methods, are similar to those obtained on the basis of the simplified Bishop method.
Mogenstern and Price's Method

Critical Non-circular Slip Surface

Fig. 5-34 The Geometry of the Critical Slip Surface Based on M&P Method

Fig. 5-35 The Relationships between Reliability Index and \( \rho_{c\phi} \)

Fig. 5-36 The Relationships between Reliability Index and \( \rho_{c\gamma} \)
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Fig. 5-37 The Relationships between Reliability Index and $\rho_{\gamma\phi}$

On the basis of the Monte Carlo Simulation method, the influences of cross-correlation coefficients on the probability distribution of FOS were examined. Five cross-correlation values, i.e. $\rho = -0.5, -0.2, 0, 0.2, 0.5$, were used in the analyses. The histograms of FOS for $\rho_{c\phi}$, $\rho_{\gamma c}$ and $\rho_{\gamma\phi}$ are shown in Figs. 5-38, 5-39 and 5-40 respectively. The influences of cross-correlation coefficients on the distribution of FOS are not significant. The figures also indicate that the probability distributions are close to the normal (Gaussian).
Fig. 5-38 The Distributions of FOS Associated with $p_{c\phi}$
Fig. 5-39 The Distribution of FOS Associated with $\rho_{yc}$
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5-67

Morgenstern and Price's Method
\( \rho_{\gamma\phi} = -0.5 \)

Morgenstern and Price's Method
\( \rho_{\gamma\phi} = -0.2 \)

Morgenstern and Price's Method
\( \rho_{\gamma\phi} = 0 \)

Morgenstern and Price's Method
\( \rho_{\gamma\phi} = 0.2 \)

Morgenstern and Price's Method
\( \rho_{\gamma\phi} = 0.5 \)

Fig. 5-40 The Distribution of FOS Associated with \( \rho_{\gamma\phi} \)
The reliability indices based on the Monte Carlo simulation method (MCSM) are tabulated in Table 5-13 considering different cross-correlation coefficients. The results shown in Table 5-13 reveal that the reliability index of the slope decreases with the variation of $\rho_{c\phi}$ from negative to positive but the changes of the reliability index with changes in $\rho_{yc}$ and $\rho_{y\phi}$ are not consistent.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>-0.5</th>
<th>-0.2</th>
<th>0</th>
<th>0.2</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>R.I. for $\rho_{c\phi}$ ($\rho_{yc} = \rho_{y\phi} = 0$)</td>
<td>1.907</td>
<td>1.707</td>
<td>1.659</td>
<td>1.583</td>
<td>1.520</td>
</tr>
<tr>
<td>R.I. for $\rho_{yc}$ ($\rho_{c\phi} = \rho_{y\phi} = 0$)</td>
<td>1.641</td>
<td>1.677</td>
<td>1.668</td>
<td>1.657</td>
<td>1.701</td>
</tr>
<tr>
<td>R.I. for $\rho_{y\phi}$ ($\rho_{yc} = \rho_{c\phi} = 0$)</td>
<td>1.615</td>
<td>1.610</td>
<td>1.667</td>
<td>1.630</td>
<td>1.677</td>
</tr>
</tbody>
</table>

5.7.3. SARMA (1979) METHOD

In this method, the values of shear strength parameters inside the soil mass (along interslice boundaries) are assumed to be same as those on the base of any slice or segment within a homogeneous slope. In other words the magnitude of the safety factor is assumed to be the same along the interslice boundaries as along the slip surface. A non-circular critical slip surface based on the Sarma method is shown in Fig. 5-41. The minimum FOS corresponding to this critical slip surface is 1.76.

The cross-correlation coefficients among $c'$, $\phi'$ and $\gamma$ are assumed to vary from -0.5 to +0.5 for examining the influences of these cross-correlation coefficients on the reliability index of the slope. The relationships between the reliability index and each of $\rho_{c\phi}$, $\rho_{yc}$ and $\rho_{y\phi}$ are shown respectively in Figs. 5-42, 5-43 and 5-44. Fig. 5-42 shows that the reliability index, based on each of the three numerical methods, decreases with the change of $\rho_{c\phi}$ from negative to positive. By comparing Figs. 5-28, 5-35 and 5-42, it is clear that the results based on the Sarma method are somewhat different than those based on simplified Bishop method and Morgenstern and Price method. The differences between FOSM-1 and FOSM-2 as well as FOSM-2 and PEM are nearly constant with the change of $\rho_{c\phi}$ from negative to positive. These values are higher than those associated with Simplified Bishop method and M&P method. This implies that the influences of
\( \rho_{c\phi} \) on the reliability index of the slope, based on Sarma method, are larger than those based on the simplified Bishop method and on the M&P method. Figs. 5-43 and 5-44 show that the reliability index increases with the change of either \( \rho_{\gamma c} \) or \( \rho_{\psi \phi} \) from negative to positive. With varying \( \rho_{\gamma c} \), the changing rates of reliability index associated with FOSM-1, FOSM-2, and PEM are 8\%, 5\% and 6\%, respectively while the changing rates of reliability index, with varying \( \rho_{\psi \phi} \), are only 4\%, 2\% and 4\%, respectively. It is obvious that the influence of change in \( \rho_{c\phi} \) on the reliability index is more significant than that of either \( \rho_{\gamma c} \) or \( \rho_{\psi \phi} \).

Fig. 5-41 The Critical Slip Surface Associated with Sarma Method

![Critical Non-circular Slip Surface](image)

Fig. 5-42 The Relationship between Reliability Index and \( \rho_{c\phi} \)
Using the Monte Carlo Simulation method, the influence of cross-correlation coefficients on the probability distribution of FOS was again investigated for five different values of each cross-correlation coefficient, i.e. $\rho = -0.5, -0.2, 0, 0.2, 0.5$. The histograms of FOS for various values of $\rho_{c\phi}, \rho_{\gamma c}$ and $\rho_{\phi\gamma}$ are shown respectively in Figs. 5-45, 5-46 and 5-47. From these figures, it can be noted that the shape of the distribution of FOS does not change significantly as each of the cross-correlation coefficients is changed from negative to positive. The shape of the distribution in each case approximates to normal (Gaussian).
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Fig. 5-45 The Distribution of FOS Associated with $\rho_{c\phi}$
Fig. 5-46 The Distribution of FOS Associated with $\rho_{\gamma_e}$
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(a) Sarma's Method
\( \rho_{\gamma_b} = -0.5 \)

(b) Sarma's Method
\( \rho_{\gamma_b} = -0.2 \)

(c) Sarma's Method
\( \rho_{\gamma_b} = 0 \)

(d) Sarma's Method
\( \rho_{\gamma_b} = 0.2 \)

(e) Sarma's Method
\( \rho_{\gamma_b} = 0.5 \)

Fig. 5-47 The Distribution of FOS Associated with \( \rho_{\gamma_b} \)
The reliability indices based on the Monte Carlo simulation method (MCSM) are tabulated in Table 5-14 for various values of cross-correlation coefficients. The results in Table 5-14 reveal that the reliability index of the slope decreases with variation of \( \rho_{c\phi} \) from negative to positive. However the changes in reliability index are not consistent for either varying \( \rho_{yc} \) or varying \( \rho_{y\theta} \) although there is a general increase in each case from values at \( \rho = -0.5 \) to corresponding values at \( \rho = +0.5 \).

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( -0.5 )</th>
<th>( -0.2 )</th>
<th>( 0 )</th>
<th>( 0.2 )</th>
<th>( 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>R.I. for ( \rho_{c\phi} (\rho_{yc} = \rho_{y\theta} = 0) )</td>
<td>2.503</td>
<td>2.168</td>
<td>1.956</td>
<td>1.906</td>
<td>1.631</td>
</tr>
<tr>
<td>R.I. for ( \rho_{yc} (\rho_{c\phi} = \rho_{y\theta} = 0) )</td>
<td>1.931</td>
<td>1.948</td>
<td>2.078</td>
<td>2.041</td>
<td>2.141</td>
</tr>
<tr>
<td>R.I. for ( \rho_{y\theta} (\rho_{yc} = \rho_{c\phi} = 0) )</td>
<td>1.982</td>
<td>1.975</td>
<td>2.001</td>
<td>2.021</td>
<td>2.100</td>
</tr>
</tbody>
</table>

5.7.4. DISCUSSIONS

The influences of cross-correlation coefficients between soil properties (\( c', \phi' \) and \( \gamma \)) on the reliability index of slopes were investigated on the basis of limit equilibrium approaches introduced in Chapter 3. The value of the reliability index decreases with increase of \( \rho_{c\phi} \) from negative value to positive. On the other hand, the value of the reliability index increases with increase of either \( \rho_{yc} \) or \( \rho_{y\theta} \). It was interesting to note that the influence of \( \rho_{c\phi} \) on the reliability index of a slope is larger than that of either \( \rho_{yc} \) or \( \rho_{y\theta} \), i.e., the reliability index is more sensitive to \( \rho_{c\phi} \) than to \( \rho_{yc} \) or \( \rho_{y\theta} \). A conservative estimate may be obtained by ignoring \( \rho_{c\phi} \) provided the actual value of \( \rho_{c\phi} \) is negative as has been suggested in the literature. Based on either the simplified Bishop or the Morgenstern & Price method, the differences of the reliability index computed from FOSM-1 and FOSM-2 or the differences in estimates computed from PEM and FOSM-2 increase with increases of \( \rho_{c\phi} \) from negative to positive based on the Sarma method, however, these differences are constant.

Monte Carlo Simulation method was used to investigate the influences of each cross-correlation coefficient on the probability distribution of FOS. The investigations show that these influences are not very significant for the shape of the probability
distribution. The investigations also reveal that the computed probability distributions approximate to normal (Gaussian) provided that all basic input random variables (the material properties) are assumed to be normal variates.

5.8. INFLUENCES OF AUTO-CORRELATION COEFFICIENT ON RELIABILITY INDEX OF SLOPES

It is now well known that spatial variability of soil parameters can be important even within a seemingly homogeneous soil profile. To appropriately model or simulate the stochastic nature of soil properties, the random field model has often been used. Calculations based on the random field model can be tedious, the selection of appropriate mathematical formulae is not easy and the required random field parameters are difficult to measure. Moreover, the results of probabilistic analysis are very sensitive to choice of those parameters and the choice of the mathematical model of a random field. It may be more convenient to use a multi-layer model (each layer considered as statistically homogeneous) instead of the random field model. Of course, the auto-correlation coefficients of soil properties have to be included in such a multi-layer model. Usually, the auto-correlation coefficients of soil profile are significant with regard to the shear strength parameters, i.e. $c'$ and $\phi'$. In the following analyses, two limit equilibrium methods, i.e. Simplified Bishop method and Morgenstern and Price method, are used. For investigating the influences of the auto-correlation coefficient on the reliability index of slopes, a five-layered model is used for a "homogeneous" soil slope and the auto-correlation coefficient of any soil parameter between any pair of layers is assumed to vary from 0 to 1.0.

5.8.1. INFLUENCES OF AUTO-CORRELATION COEFFICIENT OF COHESION ON RELIABILITY INDEX

In this investigation, the undrained cohesion of a soil is considered as the only random variable. The shear strength parameters and unit weights of a five-layered soil slope are tabulated in Table 5-15. Two limit equilibrium methods, Simplified Bishop
method and Morgenstern and Price method, are separately used for investigating these influences as follows.

<table>
<thead>
<tr>
<th>Soil No.</th>
<th>$\gamma$ (kN/m$^3$)</th>
<th>$\mu_{cu}$ (kPa)</th>
<th>$\sigma_{cu}$ (kPa)</th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>18.0</td>
<td>36.5</td>
<td>8.3</td>
<td>0°</td>
</tr>
<tr>
<td>II</td>
<td>18.5</td>
<td>42.6</td>
<td>12.2</td>
<td>0°</td>
</tr>
<tr>
<td>III</td>
<td>18.8</td>
<td>55.7</td>
<td>15.2</td>
<td>0°</td>
</tr>
<tr>
<td>IV</td>
<td>19.0</td>
<td>84.7</td>
<td>20.5</td>
<td>0°</td>
</tr>
<tr>
<td>V</td>
<td>19.5</td>
<td>125.1</td>
<td>26.7</td>
<td>0°</td>
</tr>
</tbody>
</table>

5.8.1.1. SIMPLIFIED BISHOP METHOD

A five-layered soil slope and a specified circular slip surface are shown in Fig. 5-48. The factor of safety based on mean values of shear strength corresponding to the slip surface is 1.53.

On the basis of the numerical methods, FOSM-1, FOSM-2 and PEM, the relationship between the reliability indices and the auto-correlation coefficient is presented in Fig. 5-49. The figure shows that reliability index decreases with increase of the auto-correlation coefficient of cohesion. The figure also indicates that the value of the reliability index based on any of the three numerical methods is exactly the same. This is
a consequence of the fact that the performance function associated with the simplified Bishop method is linear if only the cohesion of soils is considered as the random variate and all other soil parameters are assumed to be constants. The influence of auto-correlation coefficient of cohesion on the reliability index of the slope is significant. Reliability is underestimated by assuming a highly correlated variable and overestimated by ignoring auto-correlation.

Based on the Monte Carlo Simulation method, the influences of auto-correlation coefficients on the shape of the distribution of FOS was investigated. Three auto-correlation coefficient values, i.e. $\rho_{cc} = 0.1, 0.5, \text{ and } 1$, were assumed to examine these influences. In the analyses, cohesion was assumed to be the only random variate (Gaussian). The histograms of FOS are shown in Fig. 5-50. The influence of auto-correlation coefficient on the type of the distribution is not very significant whereas the influence is significant for the shape of the distribution. As expected, the reliability index decreases with increase of the auto-correlation coefficient. The differences of reliability index between MCSM and either FOSM or PEM are not significant. Fig. 5-50 also shows that the distribution type of FOS approximates to normal (Gaussian).
5.8.1.2. MORGENSTERN AND PRICE METHOD

A specified non-circular slip surface is shown in Fig. 5-51. The factor of safety corresponding to the slip surface based on mean values of the random variables is 1.52.

Based on three numerical methods (FOSM-1, FOSM-2 and PEM) the relationship between the reliability index and the auto-correlation coefficient is presented in Fig. 5-52. The reliability index decreases with an increase of the auto-correlation coefficient of cohesion. The figure also indicates that the reliability index values based on FOSM-1 are
slightly different from those associated with FOSM-2 and PEM. Moreover, the reliability index values calculated on the basis of FOSM-2 and PEM are exactly the same. This is a consequence of the fact that the performance function associated with M&P method is not linear even if only cohesions of different layers are considered as random variates. The rates of decrease of the reliability index are less than those based on the simplified Bishop method.

Based on the Monte Carlo Simulation method, an investigation was carried out concerning with influences of auto-correlation coefficients on the probability distribution
of the factor of safety. Three auto-correlation coefficient values ($\rho_{cc} = 0.1, 0.5, \text{ and } 1$) were selected. The histograms of FOS are shown in Fig. 5-53 for the assumed values of auto-correlation coefficients. Again it is obvious that the distributions approximate to normal (Gaussian) and that the shape of distribution is sensitive to the value of the auto-correlation coefficient.

![Histogram of FOS for different auto-correlation coefficients](image)

Fig. 5-53 The Distributions of FOS Associated with $\rho_{cc}$
5.8.2. **INFLUENCE OF THE AUTO-CORRELATION COEFFICIENT OF FRICTION ANGLE ON RELIABILITY INDEX**

In this sub-section, the friction angle of soils is considered as the only random variable. The shear strength parameters and unit weights of various soil layers are tabulated in Table 5-16. Two limit equilibrium methods, simplified Bishop method and Morgenstern and Price method, have been used for this investigation.

<table>
<thead>
<tr>
<th>Soil No.</th>
<th>( \gamma ) (kN/m(^3))</th>
<th>( c' ) (kPa)</th>
<th>( \mu_{\phi} )</th>
<th>( \sigma_{\phi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>18.0</td>
<td>10.5</td>
<td>8°</td>
<td>2°</td>
</tr>
<tr>
<td>II</td>
<td>18.5</td>
<td>16.5</td>
<td>16°</td>
<td>4°</td>
</tr>
<tr>
<td>III</td>
<td>18.8</td>
<td>18.5</td>
<td>20°</td>
<td>5°</td>
</tr>
<tr>
<td>IV</td>
<td>19.0</td>
<td>20.5</td>
<td>22°</td>
<td>6°</td>
</tr>
<tr>
<td>V</td>
<td>19.5</td>
<td>22.5</td>
<td>25°</td>
<td>8°</td>
</tr>
</tbody>
</table>

### 5.8.2.1. SIMPLIFIED BISHOP METHOD

A specified circular slip surface, as shown in Fig. 5-48, has been considered as the basis for these analyses. The factor of safety corresponding to this specified slip surface is 1.97.

On the basis of the simplified Bishop method, the influence of auto-correlation coefficient of friction angle on the reliability index of the slope is shown in Fig. 5-54. The figure shows that the reliability index decreases with increase of auto-correlation coefficient of friction angle. There are some differences between reliability indices which are calculated on the basis of FOSM-1, FOSM-2 and PEM when \( \rho_{\phi} \) is changed from 0 to 1. Reliability index values associated with FOSM-1 and PEM are very close to each other. Reliability index values associated with FOSM-2 are always larger than the corresponding values based on FOSM-1 and PEM when \( \rho_{\phi} \) varies from 0 to 1. This can be attributed to the fact that the performance function associated with the simplified Bishop method is not linear with respect to the friction angle. The figure also indicates that reliability index decreases by 16%, 12% and 11% respectively for the FOSM-1,
FOSM-2 and PEM if $\rho_{\phi\phi}$ is increased from 0 to 1. Comparing Figs. 5-49 and 5-54, the $\rho_{cc}$ is more sensitive to the reliability index of the slope than $\rho_{\phi\phi}$.

![Fig. 5-54 The Relationship between Reliability Index and Auto-correlation Coefficient](image)

The histograms of FOS, based on MCSM, are shown in Fig. 5-55 for $\rho_{\phi\phi} = 0.1$, 0.5 and 1. The results based on MCSM again indicate that the reliability index decreases with increase of $\rho_{\phi\phi}$ from 0 to 1. All the calculation results reveal that an overestimation of reliability index may be obtained for $\rho_{\phi\phi} = 0$ while an underestimation of reliability index gotten for $\rho_{\phi\phi} = 1$.

![Histograms](image)
5.8.2.2. MORGENSTERN AND PRICE METHOD

A specified non-circular slip surface, as shown in Fig. 5-51, has been considered as the basis for these analyses. The factor of safety corresponding to this specified slip surface is 2.08.

The influence of auto-correlation coefficient of friction angle on the reliability index of the slope is shown in Fig. 5-56. Based on MCSM, the histograms of FOS is shown in Fig. 5-57.
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Morgenstern and Price Method
(only friction angle is considered as random variable)
Auto-correlation Coefficient $\rho_\psi = 0.1$

$\beta = 1.936$

Fig. 5-57 The Distributions of FOS Associated with $\rho_\psi$

Morgenstern and Price Method
(only friction angle is considered as random variable)
Auto-correlation Coefficient $\rho_\psi = 0.5$

$\beta = 1.878$

Morgenstern and Price Method
(only friction angle is considered as random variable)
Auto-correlation Coefficient $\rho_\psi = 1.0$

$\beta = 1.774$
5.8.3. DISCUSSIONS

A multi-layer model is presented to examine the influences of auto-correlation coefficient on the reliability index of a slope. Both the simplified Bishop method and the Morgenstern & Price method have been used as the basis of the analysis. The results from each type of analysis show that reliability index values are underestimated if perfect autocorrelation is assumed either with respect to cohesion or with respect to internal friction angle. Similarly, reliability index values are overestimated if zero autocorrelation is assumed either with respect to cohesion or with respect to internal friction angle. The analysis results also indicate that the influence of \( \rho_{cc} \) on the reliability index is more significant than that of \( \rho_{\phi\phi} \). The influence of auto-correlation coefficients of both shear strength parameters, \( c' \) and \( \phi' \), on the difference of reliability indexes among FOSM-1, FOSM-2 and PEM is not significant, i.e. these differences are not changed with the increase of \( \rho_{cc} \) or \( \rho_{\phi\phi} \). On the basis of the Monte Carlo Simulation method and both limit equilibrium approaches, the simplified Bishop method and the Morgenstern & Price method, the influences of \( \rho_{cc} \) and \( \rho_{\phi\phi} \) on the distributions of FOS were investigated. The investigations have shown that these influences on the distribution type of FOS are very slight. The distribution of FOS are approximately normal (Gaussian) provided that all the basic input random variables are the normal variates whatever the magnitudes of \( \rho_{cc} \) and \( \rho_{\phi\phi} \). However, the influence of either \( \rho_{cc} \) or \( \rho_{\phi\phi} \) on the shape of distribution is significant.

5.9. RELIABILITY ANALYSIS OF EARTH STRUCTURES FOR EARTHQUAKES-PSEUDO-STATIC MODEL

The critical acceleration or seismic coefficient, based on pseudo-static analysis, can be considered as a random function with respect to the material properties of an earth structure because of the uncertainty of these parameters. It is useful to investigate the influence of the basic input random variables on the reliability when an earthquake force is included.
A realistic case, the Lower San Fernando dam which failed as a consequence of a strong earthquake, was used for analysis. The geometry of that dam has been shown in Fig. 4-15. For performing a probabilistic or reliability analysis based on pseudo static approach, a slip surface associated with minimum critical acceleration coefficient, \( \min \{ K_c \} \) or a specified slip surface, has to be used. A slip surface with minimum \( K_c \), which has been shown in Fig. 4-15, may be considered in this section. The mean values of the material parameters were assumed to be the same as tabulated in Table 4-16. All the material properties, i.e. \( \gamma, c', \phi' \), were considered as random variables with coefficients of variation ranging from 5% to 30%. The Morgenstern and Price method was used for determining the critical acceleration or seismic coefficient. On the basis of first order second moment method (FOSM) and point estimate method (PEM), the reliability indices and the standard deviations of \( K_c \) were obtained for different coefficients of variation of the soil parameters. All the calculation results are shown in Figs. 5-58(a) and 5-58(b). On the basis of the Monte Carlo simulation method, probability distributions were generated for different values of the coefficient of variation (see Figs. 5-59(a), 5-59(b) and 5-59(c)).
Fig. 5-58(a) shows that the reliability indices calculated by the FOSM-1 are very close to those calculated on the basis of PEM; the differences between them are slightly increased with an increase of the coefficient of variation of the basic input random variables. However, the differences between FOSM-1 and FOSM-2 or PEM and FOSM-2 are significant especially for small magnitudes of the coefficient of variation. The standard deviations of critical acceleration coefficient calculated by FOSM-1, FOSM-2 and PEM, are shown in Fig. 5-58(b). Differences between values calculated by different approaches increase with an increase in the coefficients of variation of soil parameters. The calculation results also show that the reliability of earth structures resisting earthquakes will decrease with the increase of the variation coefficient of basic input random variables. Figs. 5-59(a), 5-59(b) and 5-59(c) show that the influence of the variation coefficient of random variables on the distribution type of $K_c$ is significant. When the variation coefficient is small the distribution of $K_c$ approximates to a normal (Gaussian). The distribution of $K_c$ is no longer normal (Gaussian) for relatively high values of the coefficients of variation of random variables ($\delta = 15\%$ and $\delta = 20\%$). The degree of non-linearity of the performance function in terms of $K_c$ is increased when the magnitude domain of the basic input random variables becomes wider. This may also be the reason for the asymmetrical (non-Gaussian) distributions obtained.
Fig. 5-59 The Histogram Distribution of $K_c$ with Different Variation Coefficient

5.10. SUMMARY

1. Basic input random variables affecting slope reliability have been identified and comparative studies have been made using different numerical methods. To account for spatial variability a simplified soil model, the multi-layer model, is proposed instead of the random field model. On the basis of this probabilistic framework, the correlations between any pair of basic random variables may be included in analyses.
2. Using the rational polynomial technique, the first order and second moment approximation was successfully implemented to estimate slope reliability. The performance function was defined in terms of the factor of safety. A simple and clear deductive procedure has been proposed for the derivation of a general solution of the so called 'invariant' second moment reliability index considering any number of basic random variables. Moreover, an independent approach has been presented for an iterative numerical solution when considering a non-linear performance function. The geometrical interpretation of the conventional definition of the reliability index is given and compared with the geometrical definition of the so called 'invariant' reliability index.

3. The application of the point estimate method (PEM) has also been demonstrated. For multi-layer slopes, this method is not efficient because of the exponential increase in the number of computations as the number of variables increase.

4. The application of the Monte Carlo simulation method has also been demonstrated. In particular, a general procedure for generating n-dimensional correlated normal random vector has been proposed and its application to slope reliability analyses has been presented. As a consequence, the MCSM can be used in the reliability assessment of a complex slope such as a multi-layer slope in which the various basic input random variables in the different layers may be correlated. However, the correlated random variable cannot be uniquely generated when the probabilistic distributions of these random variables are not normal distributions. It has to be noted that the influence of sampling errors on calculation results may also be significant.

5. Comparative studies of slope reliability based on FOSM, PEM, and MCSM have been performed systematically. This is the first time that such a comprehensive comparison has been made. The significance of the results has been explored and discussed.

6. The influence of spatial correlation of basic input random variables on the reliability index of slopes has been investigated in a systematic way. The influences of spatial correlation coefficients on the distribution type of FOS have been examined. The calculation results showed that the influences of $\rho_{c\phi}$, $\rho_{cc}$ and $\rho_{\phi\phi}$ on the reliability index
of slopes are significant. However, the influence of spatial correlation coefficients of basic random variable on the distribution type of FOS is not significant.

7. The influences of the coefficients of variation of basic input random variables on the reliability of earth structures resisting earthquakes has been examined on the basis of the pseudo-static approach. The calculation results indicated that the reliability of earth structures is reduced with increase in the coefficient of variation of the basic random variables such as c and φ. The distribution shape of the critical acceleration coefficient is also influenced by the coefficient of variation of basic random variables.
6.1. GENERAL REMARKS

An engineering system generally consists of a number of elements and the performance function of these elements may be independent or correlated to each other. The performance of the system as a whole will depend not only on the relevant parameters of individual elements but also on correlations between them. In an engineering system, it is useful to identify all the potential failure modes before system failure can be assessed.

For a geotechnical system with several potential failure modes, overall reliability depends on the reliability considering each individual failure mode as well as on the correlations between the different failure modes. The calculation of system reliability in such problems is not easy. The correlations between the performance of different pairs of elements must be known in order to analyse the behaviour of the system. Usually the relationships between different parts or elements in a system are complex. If appropriate, one may simplify the problem by considering it as a simple series system (failure occurs
if any element of the system fails) or a parallel system (the failure of one element of the system leads to further loading of other elements and consequent decrease in reliability but the system does not fail unless all elements fail). As Harr (1977) showed, the combinatorial probabilities of failure can vary by many orders of magnitude. The assumption of a series system often implies significantly high failure probabilities and the assumption of a parallel system implies extremely low failure probabilities.

In geotechnical engineering systems, as in structural systems, the situation is usually far more complex. Exact mathematical solutions to the probability of failure or reliability of a system are difficult to obtain. The problem has, therefore, been considered in a different way so that the 'upper bound' and the 'lower bound' to the reliability of a system can be estimated. Solutions for such 'bounds' for structural systems were originally proposed by Cornell (1967) and Moses & Kinser (1967). The usefulness of these bounds depends on how close they are to each other and according to Ditlevsen (1979) the original solutions gave rather wide bounds in many cases. Therefore, a solution for narrow reliability bounds was presented by Ditlevsen (1979). He showed that the proposed bounds could be generalised to structural systems with a random number of potential failure modes. The general solution is now well known and has, for instance, been incorporated in the text book by Ang and Tang (1984).

6.2. SLOPE SYSTEM RELIABILITY

This chapter is concerned with the estimation of the reliability of a slope considered as a system with different failure models. It is customary in soil slope stability problems to 'search' for the critical slip surface on the basis of the conventional factor of safety. Considering a given limit equilibrium model as the basis of calculation, the slip surface with the lowest factor of safety is characterised as the critical slip surface. Within a probabilistic framework, a similar 'search' procedure may be adopted but the critical slip surface could be defined as that which is associated with the highest probability of failure or the lowest reliability.
The calculation of the minimum factor of safety or the minimum reliability means that only one mode of potential failure has been considered, as was the case with the formulation and results presented in previous chapters of this thesis. There may be an infinitely number of admissible slip surfaces and the failure probability of each of them may differ. Therefore, the slope stability problem can be considered as a system with many possible failure modes. The overall probability of failure of a slope with many potential slip surface is greater than the probability of failure along any individual slip surface. The difference between the system failure probability on the one hand and the probability of failure of the critical slip surface on the other hand will depend on the correlation between the failure models associated with various slip surfaces. System reliability analysis for a particular example in which the factors of safety of several slip surfaces are poorly correlated was considered by Oka and Wu (1990). The example involves only a linear and explicit performance function based on the "$\phi = 0$" assumption applicable to saturated clays under undrained conditions. The results show that the upper bound of system failure probability can be significantly higher than the failure probability associated with a critical slip surface.

It is important to develop methods for dealing with the general slope stability problem in which several parameters can be regarded as random variables. These may include both components of shear strength (cohesion and internal friction angle), the unit weight, as well as the pore water pressure. Such a general procedure has been developed by Chowdhury and Xu (1994a) so that linear as well as non-linear performance functions can be used. Moreover, correlations between basic random variables of a system can also be considered in the proposed procedure (Chowdhury and Xu 1994b). For performing the reliability analysis of a slope system, the procedure developed herein must be able to handle slopes involving several soil layers with different values of cohesion and internal friction angle. More importantly, any general slope stability problem may involve a non-linear or an inexplicit performance function and this aspect must be addressed in the reliability analysis procedure.
6.3. ELEMENTARY THEORY OF SYSTEM RELIABILITY

Some basic concepts of system reliability analyses are presented below.

6.3.1. RELIABILITY OF MULTIPLE FAILURE MODES

The reliability of a multi-component system is essentially a problem involving multiple modes of failure. The failures of different components, or different sets of components constitute distinct and different failure modes of a given system. The consideration of multiple modes of failure, therefore, is fundamental to the problem of system reliability. Consider a slope system with M potential slip surfaces or failure modes, the different failure modes would then have different performance functions. Suppose that the respective performance functions may be represented as

\[ G_j(X) = G_j(x_1, x_2, ..., x_n) \quad j = 1, 2, ..., M \] (6-1)

such that the individual failure events are

\[ E_j = [G_j(X) < 0] \] (6-2)

Then the complements of \( E_j \) are the safe events, i.e.,

\[ \bar{E}_j = [G_j(X) > 0] \] (6-3)

In the case of the performance function with two variables, the above events may be sketched in Fig. 6-1, in which three failure modes represented by the limit-state equations \( G_j(X) = 0, \ j = 1, 2, 3 \), are shown.

For a series system, the system reliability or safety is the event in which none of the M potential failure modes occur, i.e.,

\[ \bar{E} = \bar{E}_1 \cap \bar{E}_2 \cap \cdots \cap \bar{E}_M \] (6-4)

Theoretically, therefore, the probability of the safety of the system may be expressed as the following the volume integral:
whereas the probability of failure of the system would be expressed as:

\[ p_f = 1 - p_s \]  \hspace{1cm} (6-6)

In contrast to the series system, however, one may consider a system in which the elements are in parallel. Therefore, the complete failure of a parallel system means that all the individual elements have been in failure, i.e.,

\[ E = E_1 \cap E_2 \cap \cdots \cap E_M \]  \hspace{1cm} (6-7)

The probability of the failure of the system may be expressed as follows:

\[ p_f = \int_{E_1 \cap \cdots \cap E_M} f_X(x_1, x_2, \ldots, x_n) \, dx_1 \cdots dx_n \]  \hspace{1cm} (6-8)

whereas the probability of safety of the system would be as follows:

\[ p_s = 1 - p_f \]  \hspace{1cm} (6-9)
When all the failure modes $G_i$ ($i = 1, 2, \ldots, M$) are independent, the probability of safety of a series system can be calculated as follows:

$$p_s = p_{s_1}p_{s_2} \cdots p_{s_M} \quad (6-10)$$

in which, $p_{s_i}$ is the probability of safety of the $i$-th potential individual failure mode. If all the elements have the same reliability or the probability of failure, then Eq. (6-10) may be rewritten as follows:

$$p_s = (1-p)^M \quad (6-11)$$

where, $p$ is the probability of failure of the elements. The probability of failure of the system is:

$$p_f = 1 - p_s = 1 - (1-p)^M \quad (6-12)$$

The probability of failure of a parallel system with $M$ independent elements may be expressed as follows:

$$p_f = p_{f_1}p_{f_2} \cdots p_{f_M} \quad (6-13)$$

in which, $p_{f_i}$ is the probability of failure of the $i$-th individual element. If all the elements have the same reliability or the probability of failure, the Eq. (6-13) can be rewritten as follows:

$$p_f = (1-p')^M \quad (6-14)$$

where, $p'$ is the probability of safety of the elements. The reliability of the system is,

$$p_s = 1 - p_f = 1 - (1-p')^M \quad (6-15)$$

Obviously, it is very difficult to perform the integrations in Eqs. (6-5) and (6-8) when all the individual elements in either a series or a parallel system are correlated. This is because the joint probability density function of all the basic random variables in the performance functions and the integration region composed by $\left( E_1 \cap \cdots \cap E_M \right)$ or
(E_1 \cap \cdots \cap E_M) are usually unknown. As a consequence, therefore, mathematically approximate estimates have to be made based on the so-called 'upper bound' and 'lower bound' solutions.

6.3.2. PROBABILITY BOUNDS OF A SYSTEM

As mentioned in the previous subsection, the calculation of the probability of safety or failure of a system is generally difficult. The approximate estimation of system reliability associated with 'upper' and 'lower' bounds is discussed below. However, these two approximate approaches are associated with the series system. This presentation follows very closely the approach outlined by Ang and Tang (1984).

6.3.2.1. UNI-MODEL BOUNDS

First consider the correlation coefficient between any two failure modes to be positive, i.e. \( \rho_{ij} > 0 \). This means, that for the two events \( E_i \) and \( E_j \),

\[
p(E_j | E_i) \geq p(E_j)
\]

which also means

\[
p(\overline{E}_i | \overline{E}_j) \geq p(\overline{E}_j)
\]

Therefore,

\[
p(\overline{E}_i \overline{E}_j) \geq p(\overline{E}_i) p(\overline{E}_j)
\]

For a series system with \( M \) elements, the probability of safety of the system can be generalised to yield,

\[
p(\overline{E}) = p(\overline{E}_1 \overline{E}_2 \cdots \overline{E}_M) \geq \prod_{i=1}^{M} p(\overline{E}_i) \tag{6-16}
\]

Conversely, the event of \( \overline{E}_1 \cap \overline{E}_2 \cap \cdots \cap \overline{E}_M \) is comprised by the event \( \overline{E}_j \), i.e.

\[
\overline{E}_1 \cap \overline{E}_2 \cap \cdots \cap \overline{E}_M \subseteq \overline{E}_j \quad \text{for any } j
\]
and, in particular,

\[ \bar{E}_1 \cap \bar{E}_2 \cap \cdots \cap \bar{E}_M \subset \min_j \bar{E}_j \quad j = 1, 2, \ldots, M \]

therefore,

\[ p(\bar{E}) \leq \min_j p(\bar{E}_j) \quad (6-17) \]

If the probability of safety of the i-th failure mode is denoted as \( p_s = p(\bar{E}_i) \) and the probability of safety of the system with M failure modes as \( p_s = p(\bar{E}) \), \( p_s \) can then be bounded according to Eqs. (6-16) and (6-17) as follows (Ang and Amin, 1968),

\[ \prod_{i=1}^{M} p_{s_i} \leq p_s \leq \min_i p_{s_i} \quad (6-18) \]

Conversely, the corresponding bounds for the failure probability \( p_f = P(E) \) would be,

\[ \max_i p_{f_i} \leq p_f \leq 1 - \prod_{i=1}^{M} (1 - p_{f_i}) \quad (6-19) \]

where \( p_{f_i} = p(E_i) \) is the failure probability of the i-th failure mode.

When elements in a system are negatively dependent, i.e. \( \rho_{ij} < 0 \), for two elements \( E_i \) and \( E_j \) one has,

\[ p(E_j | E_i) \leq p(E_j) \]

and

\[ p(\bar{E}_j | \bar{E}_i) \leq p(\bar{E}_j) \]

therefore,

\[ p(\bar{E}_i, \bar{E}_j) \leq p(\bar{E}_i) p(\bar{E}_j) \]

and same as above,

\[ p(\bar{E}) = p(\bar{E}_1 \bar{E}_2 \cdots \bar{E}_M) \leq \prod_{i=1}^{M} p(\bar{E}_i) \]
Trivially, of course, $p(E) \geq 0$. Thus,

$$0 \leq p(E) \leq \prod_{i=1}^{M} p(E_i) \quad (6-20a)$$

Conversely, the failure probability of the system is

$$p_f \geq 1 - \prod_{i=1}^{M} p(E_i) \quad (6-20b)$$

As mentioned by Ang and Tang (1984), the separation between the lower and upper bounds will depend on the number of potential failure modes, and on the relative magnitudes of the individual mode probabilities. If there is a dominant mode, the probability of safety or failure will be dominated by this mode. The bounds may be widely separated especially if the number of potential failure modes is large. The bounds estimated by Eqs. (6-18), (6-19) and (6-20) may be called as the 'first-order' or 'uni-mode' bounds on $p_s$ and $p_f$ since the lower and upper bound probabilities involve single mode probabilities.

### 6.3.2.2. BI-MODAL BOUNDS

The bounds described above may be improved by taking into account the correlation between pairs of potential failure modes, i.e. the resulting improved bounds will require the probabilities of joint events, such as $E_i E_j$ or $E_i E_j$, and thus may be called as 'bi-modal' or 'second-order' bounds.

The failure event may be decomposed as follows with (see Ang and Tang, 1984) $E_1$ assumed to be the largest set and sketched in Fig. 6-2 for $M = 3$. As explained by Ang and Tang (1984) the derivation of bounds involves following step. The even $E$ may be expressed thus:

$$E = E_1 \cup E_2 \cup E_3 \cup (E_1 \cap E_2) \cup \ldots \cup E_M \cup (E_1 \cap \cdots \cap E_{M-1}) \quad (6-21)$$
According to deMorgan's rule, one can write the following:

\[ \bar{E}_1 \bar{E}_2 \cdots \bar{E}_{i-1} = \bar{E}_1 \cup \bar{E}_2 \cdots \cup \bar{E}_{i-1} \quad i = 2, 3, \ldots, M \]

Therefore, one can write:

\[ E_i (\bar{E}_1 \bar{E}_2 \cdots \bar{E}_{i-1}) = E_i (E_1 \cup E_2 \cdots \cup E_{i-1}) \]

It may be noted that:

\[ E_i (E_1 \cup E_2 \cdots \cup E_{i-1}) \cup E_i (E_1 \cup \cdots \cup E_{i-1}) = E_i \]

Accordingly, one may write as follows:

\[ p[E_i (\bar{E}_1 \bar{E}_2 \cdots \bar{E}_{i-1})] = p(E_i) - p(E_iE_1) + p(E_iE_2) + \cdots + p(E_iE_{i-1}) \]

However, it is to be noted that:

\[ p(E_iE_1 \cup E_iE_2 \cdots \cup E_iE_{i-1}) \leq p(E_iE_1) + p(E_iE_2) + \cdots + p(E_iE_{i-1}) \]

Therefore,
Hence, with Eq.(6-21) one has

\[
p(E) \geq p(E_i) + \max \left[ \sum_{i=2}^{M} \left( p(E_j) - \sum_{j=1}^{i-1} p(E_i E_j) \right); 0 \right]
\]  

(6-22)

On the other hand, one can write the following:

\[E_1 E_2 \cdots E_{i-1} \subseteq E_j; \text{ for any } j\]

and, in particular, one can write

\[E_1 E_2 \cdots E_{i-1} \subseteq \min_{j<i} E_j\]

Accordingly

\[(E_1 E_2 \cdots E_{i-1}) E_i \subseteq \left( \min_{j<i} E_j \right) E_i\]

It may be noted that

\[
\left( \min_{j<i} E_j \right) E_i \cup \left( \max_{j<i} E_j \right) E_i = E_i \left( \min_{j<i} E_j \cup \max_{j<i} E_j \right) = E_i
\]

Accordingly,

\[p(E_1 E_2 \cdots E_{i-1} E_i) \leq p(E_i) - \max_{j<i} p(E_i E_j)\]

Therefore, using Eq.(6-21), one can write:

\[p(E) \leq p(E_i) + \sum_{i=2}^{M} \left[ p(E_i) - \max_{j<i} p(E_i E_j) \right]\]

or

\[p(E) \leq \sum_{i=1}^{M} p(E_i) - \sum_{i=2}^{M} \max_{j<i} p(E_i E_j)
\]

(6-23)

By combining Eqs. (6-22) and (6-23), the failure probabilities of 'bi-model' lower and upper bound failure probabilities for M potential failure modes may be represented as follows (Kounias, 1968; Hunter, 1976)
p_{f_1} + \max \left[ \sum_{i=2}^{M} \left( p_{f_i} - \sum_{j=1}^{i-1} p(E_i E_j) \right) \right] \leq p_f \leq \sum_{i=1}^{M} p_{f_i} - \sum_{i=2}^{M} \max p(E_i E_j) \quad (6-24)

Some of the failure modes may be correlated and this should be taken into consideration for the calculation of the joint probabilities $p(E_i E_j)$ in Eq. (6-24).

6.3.3. ESTIMATION OF JOINT PROBABILITY OF CORRELATED EVENTS

6.3.3.1. LINEAR PERFORMANCE FUNCTIONS

A weakened estimation of the above 'second-order' bounds has been proposed by Ditlevsen (1979) for Gaussian random variables. Suppose the basic random variables $X = (x_1, x_2, \ldots, x_n)$ are normal variates. Consider two potential failure modes, $E_i$ and $E_j$ defined by the limit-state equations $G_i(X) = 0$ and $G_j(X) = 0$, respectively, with a positive correlation $\rho_{ij} > 0$. In the space of the reduced (or normalised) variates $x'_1, x'_2, \ldots, x'_n$, these limit state equations represent two intersecting hyper-surfaces, which are approximated by the respective tangent planes, with corresponding distances $\beta^*_i$ and $\beta^*_j$ from the origin of the reduced variates. The geometrical representation for the two-variable case is shown in Fig. 6-3. reproduced from Ang and Tang (1984).

In Fig. 6-3, the cosine of the angle $\theta$ between $G_i(X) = 0$ and $G_j(X) = 0$ is the correlation coefficient $\rho_{ij}$, i.e. $\cos \theta = \rho_{ij}$. Let the performance functions $G_i(X)$ and $G_j(X)$ be linear, i.e.,

$$G_i(X) = a_0 + a_1 x_1 + a_2 x_2$$
$$G_j(X) = b_0 + b_1 x_1 + b_2 x_2$$

where $x_1$ and $x_2$ are uncorrelated, i.e. $\text{Cov}(x_1, x_2) = 0$. Then the covariance of these two performance functions is

$$\text{Cov}(G_i, G_j) = a_1 b_1 \sigma^2_{x_1} + a_2 b_2 \sigma^2_{x_2}$$

and the coefficient between $G_i(X)$ and $G_j(X)$ is
Fig. 6-3 Two Intersecting Tangent Planes (positively correlated failure events) (after Ang and Tang, 1984)

\[
\rho_{ij} = \frac{\text{Cov}(G_i, G_j)}{\sigma_{G_i} \sigma_{G_j}} = \frac{a_1 b_1 \sigma_{x_i}^2 + a_2 b_2 \sigma_{x_2}^2}{\sqrt{(a_1^2 \sigma_{x_i}^2 + a_2^2 \sigma_{x_2}^2)(b_1^2 \sigma_{x_i}^2 + b_2^2 \sigma_{x_2}^2)}}
\]

For the direction cosines associated with \( G_i(X) = 0 \) and \( G_j(X) = 0 \), one has

\[
\cos \theta_i = \frac{a_2 \sigma_{x_2}}{\sqrt{a_1^2 \sigma_{x_1}^2 + a_2^2 \sigma_{x_2}^2}}
\]

\[
\cos \theta_j = \frac{b_2 \sigma_{x_2}}{\sqrt{b_1^2 \sigma_{x_1}^2 + b_2^2 \sigma_{x_2}^2}}
\]

Also from Fig. 6-3,

\[
\cos \theta = \cos (\theta - \theta_i) = \cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j
\]

\[
= \frac{a_1 b_1 \sigma_{x_1}^2 + a_2 b_2 \sigma_{x_2}^2}{\sqrt{(a_1^2 \sigma_{x_1}^2 + a_2^2 \sigma_{x_2}^2)(b_1^2 \sigma_{x_1}^2 + b_2^2 \sigma_{x_2}^2)}}
\]

Therefore, \( \rho_{ij} = \cos \theta \).

The joint failure event, \( E_iE_j \), is the shaded region shown in Fig. 6.3. Obviously,
also,
\[ \mathbf{E}_i \mathbf{E}_j \supseteq B \]
where, A and B are defined as Fig. 6-3.

Thus, observing Fig. 6-3 again, one may obtain the 'upper' and 'lower' probabilistic bounds of the event, \( \mathbf{E}_i \mathbf{E}_j \), as follows

\[
\max[p(A), p(B)] \leq p(\mathbf{E}_i \mathbf{E}_j) \leq p(A) + p(B) \quad (6-25)
\]

The magnitudes of the distances, a and b, shown in Fig. 6-3 may be calculated as following equations

\[
a = \frac{\beta_i - \rho_{ij} \beta_j^*}{\sqrt{1 - \rho_{ij}^2}}
\]

\[
b = \frac{\beta_i^* - \rho_{ij} \beta_j}{\sqrt{1 - \rho_{ij}^2}}
\]

Therefore, by reason of orthogonality (see Fig. 6-3) and Eq. (6-26), the probabilities of A and B may be calculated as follows

\[
p(A) = \Phi(-\beta_i^*) \Phi(-a) = \Phi(-\beta_i^*) \Phi\left(\frac{\beta_i^* - \rho_{ij} \beta_j^*}{\sqrt{1 - \rho_{ij}^2}}\right)
\]

\[
p(B) = \Phi(-\beta_j^*) \Phi(-b) = \Phi(-\beta_j^*) \Phi\left(\frac{\beta_i^* - \rho_{ij} \beta_j^*}{\sqrt{1 - \rho_{ij}^2}}\right)
\]

The probabilities of the joint events, \( p(\mathbf{E}_i \mathbf{E}_j) \), in Eq.(6-24) may then be approximated with the appropriate sides of Eq. (6-25), i.e. for the lower bound, Eq. (6-22), the joint probabilities \( p(\mathbf{E}_i \mathbf{E}_j) \) may be replaced by \([p(A) + p(B)]\) whereas for the upper bound, Eq. (6-23), by \( \max[p(A), p(B)] \). Of course, the reliability indices used in Eq.(6-27) may also be obtained on the basis of a different definition of the reliability index, i.e. Cornell's definition.
Actually, Eq. (6-25) can be applied to failure modes involving \( n \) variables. However, the estimation of the 'upper' and 'lower' probability bounds of the joint event, \( E_1E_p \), in Eq. (6-25) is based on the assumption that any pair of basic random variables of the performance function is uncorrelated. In realistic geotechnical engineering, however, the basic random variables are usually correlated. Therefore, an orthogonal transformation introduced in Chapter 5 has to be used for transforming the correlated variables to uncorrelated variables. Consider a linear system with \( M \) elements and each element has \( n \) random variables, i.e.,

\[
G = A X + A_0
\]  
(6-28)

where,

\[
G = (G_1, G_2, \ldots, G_M)^T;
\]

\[
X = (x_1, x_2, \ldots, x_n)^T;
\]

\[
A_0 = (a_{10}, a_{20}, \ldots, a_{M0})^T;
\]

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots  & \vdots  & \ddots & \vdots  \\
    a_{M1} & a_{M2} & \cdots & a_{Mn}
\end{pmatrix}
\]

The components of the normal random vector \( X \) are correlated to each other and the elements of the covariance matrix, \( C \), of these components are assumed to be known. According to Eq. (5-74), the vector \( X \) can be represented by a uncorrelated standard normal vector \( Z \), i.e.,

\[
X = T Z + \mu_X
\]  
(6-29)

in which, \( T \) is a lower-triangular matrix with positive elements on its main diagonal and may be calculated by Eq. (5-77).

Substituting Eq. (6-29) into Eq. (6-28), the linear system can be rewritten as,

\[
G = A(T Z + \mu_X) + A_0 = B Z + B_0
\]  
(6-30)

where,

\[
B = A T = \begin{pmatrix}
    b_{11} & b_{12} & \cdots & b_{1n} \\
    b_{21} & b_{22} & \cdots & b_{2n} \\
    \vdots  & \vdots  & \ddots & \vdots  \\
    b_{M1} & b_{M2} & \cdots & b_{Mn}
\end{pmatrix}
\]

\[
B_0 = A \mu_X + A_0 = (b_{10}, b_{20}, \ldots, b_{M0})^T - \text{a constant vector}
\]
\( Z = (z_1, z_2, \ldots, z_n)^T \) - an independent standard normal vector.

Therefore, the required correlation coefficient between \( G_i(X) \) and \( G_j(X) \) can be directly obtained by the previous derivations, i.e.,

\[
\rho_{ij} = \frac{\text{Cov}(G_i, G_j)}{\sigma_{G_i} \sigma_{G_j}} = \frac{\sum_{k=1}^{n} b_{ik} b_{jk}}{\sqrt{\left(\sum_{k=1}^{n} b_{ik}^2\right)\left(\sum_{k=1}^{n} b_{jk}^2\right)}}
\]  

(6-31)

where, \( \sigma_{z_k} = 1 \) (\( k = 1, 2, \ldots, n \)).

The 'second-order' bounds, Eq. (6-24), will depend on the ordering of the individual failure modes, i.e., different orderings of the individual failure modes may yield different values for Eq. (6-22), and thus the bounds corresponding to different ordering may have to be evaluated to determine the sharpest bounds. However, if the mode which has the maximum failure probability in all the single-modes is denoted as \( E_1 \) in Eq.(6-22) the range of the reliability bounds may then be the most narrow.

6.3.3.2. NON-LINEAR PERFORMANCE FUNCTIONS

In general, the performance functions in geotechnical engineering are not linear. It is, therefore, necessary to replace a n-dimensional failure surface defined by the non-linear performance function with a hyper-plane tangent to the failure surface at some points, i.e., a non-linear system may be linearized by the Taylor series expansion without second-order and higher order partial derivatives. Thus, a non-linear system may be approximately expressed as follows:

\[
G = A X + A_0
\]

(6-32)

where, \( A_0 \) is a constant vector (not required in system analysis)

\[
A = \begin{pmatrix}
\frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \cdots & \frac{\partial G_1}{\partial x_n} \\
\frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2} & \cdots & \frac{\partial G_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial G_m}{\partial x_1} & \frac{\partial G_m}{\partial x_2} & \cdots & \frac{\partial G_m}{\partial x_n}
\end{pmatrix}
\]
The matrix $A$ in the above equation may be different for different definitions of the reliability index. For Cornell's definition, the partial derivatives in the matrix $A$ are calculated at the mean points of the basic random variables whereas for the Hasofer and Lind' definition the partial derivatives are computed at the 'most probable failure points'. Once Eq. (6-32) is obtained, the mutual correlations between the non-linear performance functions may be estimated by Eqs. (6-30) and (6-31) provided that the covariance matrix, $C$, of the basic random variables in these performance functions is known.

If $E_i$ and $E_j$ are negatively dependent, i.e., $\rho_{ij} < 0$, the limit-state equations would be as depicted in Fig. 6-4, where the joint failure event $E_iE_j$ is the shaded region. In such case, it can be seen from Fig. 6-4) that the upper bound of $p(E_iE_j)$ is

$$p(E_iE_j) \leq \min[p(A), p(B)]$$  \hspace{1cm} (6-33)

where $p(A)$ and $p(B)$ are given by Eq.(6-27). The lower bound is trivially zero.

Fig. 6-4 Two Tangent Planes for Negatively Dependent Failure Events
(after Ang and Tang, 1984)
Chapter 6: System Reliability of Slopes and Applications

6.4. SYSTEM RELIABILITY ANALYSES OF A SLOPE

6.4.1. INTRODUCTION

The basis of the proposed reliability analysis of slope systems is the concept of limit equilibrium. Any popular method of slices, simplified or rigorous, may be used. The simplified Bishop method and the relatively 'rigorous' Morgenstern and Price method are adopted in this chapter. However, the procedure can also be used in conjunction with any other limit equilibrium method. The 'reliability index' is used as a basis for the calculation of the probability of failure assuming the performance function, the factor of safety, to be a random variable which follows a normal or Gaussian distribution. Two alternative definitions of the 'reliability index', i.e. Cornell's definition and Hasofer & Lind's definition, have been used for alternative evaluations of system reliability. In the slope system reliability analyses, the basic input random variables may include the shear strength parameters (c' and φ'), unit weight of material (γ), and pore water pressure ratio (r_u). An optimisation or search procedure is required for identifying the critical slip surface so that reliability associated with such a surface can be compared to the reliability bounds for the system. The conjugate gradient method presented in Chapter 4 is used for this purpose and the critical slip surface is identified on the basis of the minimum factor of safety.

The formulation for linear and non-linear slope systems is considered separately below.

6.4.2. LINEAR AND EXPLICIT PERFORMANCE FUNCTION "$ \phi = 0$" CASE

Consider a slope above a slip surface of circular shape to be subdivided into n vertical slices. The i-th slice has a weight W_i and inclination of its base to the horizontal is $\alpha_i$. For the "$\phi = 0$" case, the factor of safety of a slope may be defined as the ratio of resisting and overturning moments as follows:
where, \( c_i \) is the undrained cohesion along the base of the i-th slice and \( l_i \) is the length of slip surface in the i-th slice.

If there are \( m \) soil layers with different undrained shear strengths \( c_1, c_2, ..., c_m \), Eq. (6-34) can be rewritten as:

\[
F = \frac{\sum_{i=1}^{n} c_i l_i}{\sum_{i=1}^{n} W_i \sin \alpha_i} \tag{6-35}
\]

where, \( L_j \) (\( j = 1, 2, ..., m \)) is the length of the slip surface in layer \( j \).

Eq. (6-35) is a linear function with respect to the undrained shear strengths \( c_i \) (\( i = 1, 2, ..., m \)), and, therefore, the two alternative definitions of reliability index give identical results as mentioned in Chapter 5. The performance function considering an individual slip surface \( k \) may be written as follows:

\[
G_k = a_{0k} + \sum_{j=1}^{m} a_{kj} c_j \tag{6-36}
\]

in which, \( a_{0k} = -1 \);

\[
a_{kj} = \frac{W_{ki} \sin \alpha_{ki}}{\sum_{i=1}^{n} \sum_{k} W_{ki} \sin \alpha_{ki}}
\]

Thus the reliability index associated with slip surface or failure mode \( k \) is:

\[
\beta_k = \frac{b_{0k}}{\sqrt{\sum_{j=1}^{m} b_{kj}^2}} \tag{6-37}
\]
where, \[ b_{k0} = \frac{\sum_{j=1}^{m} L_{kj} \mu_{cj}}{\sum_{i=1}^{n} W_{ki} \sin \alpha_{ki}} - 1 \]
\[ b_{kj} = \frac{L_{kj} \sigma_{cj}}{\sum_{i=1}^{n} W_{ki} \sin \alpha_{ki}} \]
\[ \mu_{cj} = \text{the mean value of cohesion in layer } j \]
\[ \sigma_{cj} = \text{the standard deviation of cohesion in layer } j \]

For system reliability formulation it is necessary to calculate the correlation coefficient between different failure modes based on the different slip surfaces. Considering two slip surfaces k and l with m layers of soil the correlation coefficient between two failure modes associated with the slip surfaces k and l can be written as follows (see previous section)

\[ \rho_{kl} = \frac{\sum_{j=1}^{m} b_{kj} b_{lj}}{\left[ \sum_{j=1}^{m} b_{kj} \sum_{j=1}^{m} b_{lj}^2 \right]^{1/2}} \quad (6-38) \]

Of course, an orthogonal transformation, proposed in previous section, must be used when any of the cohesion soils in pair of layers is correlated.

6.4.3. NON-LINEAR AND INEXPLICIT PERFORMANCE FUNCTION

In reliability analysis of a slope, a performance function may be defined in terms of safety margin which is related to the factor of safety as follows

\[ G(\mathbf{X}) = F(\mathbf{X}) - 1 \quad (6-39) \]

in which, \( F(\mathbf{X}) \) is the factor of safety of a slope and is itself a function of the basic random variables and \( \mathbf{X} \) is a vector of basic random variables. The factor of safety, \( F(\mathbf{X}) \), can be calculated by various simplified or relatively 'rigorous' limit equilibrium
methods. In general, however, the performance function defined by Eq. (6-39) is nonlinear and inexplicit for most limit equilibrium methods (except for the case presented in previous subsection). It is, therefore, difficult to calculate accurately the correlation coefficients between any pair of failure modes. To overcome this problem in slope system reliability analysis, a linearizing procedure proposed in the previous section must be used. A performance function may expanded as a Taylor series at certain points and linearly approximated by truncating the series at the second order partial derivatives. Based on the two alternative definitions of reliability index, the expansion of a performance function as a Taylor series can be carried out at mean value points of basic random variables for Cornell's definition and at the 'most probable failure points' for the Hasofer-Lind definition.

The formulations based on two alternative definitions are given below.

6.3.3.1. CONVENTIONAL DEFINITION OF RELIABILITY INDEX

The definition of reliability index has been presented in Chapter 5, i.e., the reliability index of a slope is equal to a ratio of the mean safety margin and the standard deviation of factor of safety. Now expanding the performance function as a Taylor series at the mean value points of the basic random variables, one may write:

\[
G(X) = F(\mu_X) - 1 + \sum_{i=1}^{m} (x_i - \mu_{x_i}) \left( \frac{\partial G}{\partial x_i} \right)_{x=\mu_X} + \sum_{j=1}^{m} \sum_{i=1}^{m} (x_i - \mu_{x_i})(x_j - \mu_{x_j}) \left( \frac{\partial^2 G}{\partial x_i \partial x_j} \right)_{x=\mu_X} + \ldots
\]

where, \(x_i\) and \(\mu_{x_i}\) \((i = 1, 2, ..., m)\) are respectively the random variables and their mean values. In linear approximation, that is, truncating Eq.(6-40) at the second order partial derivative, the performance function for slip surface \(k\) may be written as:

\[
G_k = a_{k0} + \sum_{i=1}^{m} a_{ki} x_i
\]
Therefore, if all the basic random variables are independent, the correlation
coefficient between failure modes $k$ and $l$ may then be approximately estimated as
follows,

$$\rho_{kl} = \frac{\sum_{j=1}^{m} a_{jk}^2 \sigma_{x_j}^2}{\left(\sum_{j=1}^{m} a_{jk}^2 \sigma_{x_j}^2\right)^{1/2}}$$  \hspace{1cm} (6-42)

where, $m$ is the number of basic random variables in a slope system, $\sigma_{x_i}$ is the standard
deviation of the $i$-th random variable and $a_{kj}$ or $a_{lj}$ are the partial derivatives of
performance function $k$ or $l$ with respect to of the $j$-th random variable at the mean value
point.

An orthogonal transformation associated with Eq.(6-30), however, is required
for calculating the correlation coefficients based on Eq.(6-31) when the basic random
variables in performance functions are correlated and the covariance matrix, $C$, of the
basic random variables is known. In this case, the matrix $A$ in Eq.(6-30) is constituted
by the partial derivatives of all the performance functions with respect to basic input
random variables at the mean values.

6.4.3.2. HASOER-LIND RELIABILITY INDEX

Hasofer and Lind (1974) proposed a definition of reliability index as the minimum
distance from the failure surface ($G(X) = 0$) to the origin of the reduce random variables.
According to this definition, the reliability index can be calculated as follows:

$$\beta^* = -\frac{(\partial G/\partial X)^T_{*} (X^* - \mu_X)}{\left[(\partial G/\partial X)^T_{*} [C] (\partial G/\partial X)^T_{*}\right]^{1/2}}$$  \hspace{1cm} (6-43)

in which, $(\partial G/\partial X)^T_{*}$ is the gradient vector of the performance function at the most
probable failure point $X^* = (x_{1*}, x_{2*}, \ldots, x_{m*})$ on failure surface ($G(X^*) = 0$); $\mu_X$ and $C$
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are respectively the vector of the mean values and the covariance matrix of the basic random variables in a slope system.

An iterative procedure proposed in Chapter 5 must be carried out for calculating the reliability index $\beta^*$ and $X^*$ because the performance function based on either the Bishop method or the M&P method is non-linear and inexplicit.

After obtaining the most probable failure point $X^*$, the performance function can be expanded by Taylor series at the point $X^*$, which is on the failure surface $G(X^*) = 0$. The performance function for slip surface $k$ may be approximately represented as:

$$G_k = a_{k0}^* + \sum_{i=1}^{m} a_{ki}^* x_i$$

(6-44)

The correlation coefficient between the slip surfaces (or potential failure modes) $k$ and $l$ would be approximated as the following equation if all the random variables in the slope system are independent:

$$\rho_{kl}^* = \frac{\sum_{j=1}^{m} a_{kj}^* a_{l,j}^* \sigma_{x_j}^*}{\left[\left(\sum_{j=1}^{m} a_{kj}^* \sigma_{x_j}^* \right) \left(\sum_{j=1}^{m} a_{l,j}^* \sigma_{x_j}^* \right)\right]^{1/2}}$$

(645)

where, $a_{kj}^*$ is the partial derivative of the performance function $(k)$ with respect to the $j$-th random variables at the most probable failure point $X^*$. An orthogonal transformation proposed in previous Section, of course, is also needed for calculating the required correlation coefficient between the performance functions if the random variables in the performance functions are mutually correlated.

The partial derivatives of the performance function can be evaluated by the rational polynomial technique presented in Chapter 4.
6.5. ILLUSTRATIVE EXAMPLES

Three limit equilibrium methods of the slope stability analysis are used in following subsections to evaluate the system reliability of specified slopes. Any pair of basic random variables in the same material layer may be considered either as correlated or as independent variables. However the first example involves only one basic random variable in each layer.

6.4.1. LINEAR PERFORMANCE FUNCTION - "\( \phi = 0 \)" CASE

The geometry of the slope is shown in Fig. 6-5 and this example is the well known Congress street cut (Ireland, 1954). It is also the case history analysed by Oka and Wu (1990). The clay deposit is divided into three layers following Ireland (1954). All the slip surfaces are assumed to be tangent to the bases of two layers. The set of slip surfaces tangential to the base of clay 2 is denoted as set 1 and the set of slip surfaces tangential to the base of clay 3 is denoted as set 2. The top layer of sand has negligible influence on stability because of zero cohesion and low normal stress and has, therefore, been neglected as in the analysis of Ireland (1954) and Oka & Wu (1990). The unit weight of soils is assumed to be a constant and equals to 20 (kN/m³). Because of the assumptions of "\( \phi = 0 \)" and a constant unit weight, the performance functions in terms of the factor of safety will be linear and explicit with respect to all the basic random variables, namely, the undrained cohesion of each of the three clay layers. Three sets of mean values and standard deviations of the undrained shear strength (cohesion) are considered in following analyses. The first set is presented in Table 6-1.

| Table 6-1 Shear Strength Parameters for First Example |
|---------------------------------------------|-----------------|-----------------|-----------------|
|                | Clay 1 \( c_1 \) | Clay 2 \( c_2 \) | Clay 3 \( c_3 \) |
| Means (kPa)    | 55              | 43              | 56              |
| Stan. dev. (kPa)| 20.4            | 8.2             | 13.2            |
| Coefficients of variation | 37%             | 19%             | 24%             |
The results for the upper and lower bounds of the system failure probability are shown in Table 6-2 along with failure probability associated with the critical slip surface.

Table 6-2 Computed Failure Probabilities for The First Example

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Slip Surface Set 1-Tangent to Layer 2 (8 slip surfaces)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.268222</td>
<td>1.1167</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.315570</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.268222</td>
<td>-</td>
</tr>
<tr>
<td><strong>Slip Surface Set 2- Tangent to Layer 3 (8 slip surfaces)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.277294</td>
<td>1.1072</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.305404</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.277292</td>
<td>-</td>
</tr>
<tr>
<td><strong>All Slip Surface-Tangent to Layer 2 and 3 (16 slip surfaces)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.277294</td>
<td>1.1072</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.586016</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.277294</td>
<td>-</td>
</tr>
</tbody>
</table>
considering either slip surface set 1 or slip surface set 2 or both together, i.e. the whole system. In this example, the results for set 1 and set 2, considered separately reveal that the lower bound of failure probability is identical with that based on the corresponding critical slip surface. There are, however, some differences between the system failure probability bounds. Even when the whole system is considered together, the lower bound is identical to the critical failure surface probability but the difference between the reliability bounds of the whole system is significantly increased. The failure probability representing the upper bound is almost twice as large as that representing the lower bound. In other words the reliability bounds become wide when the whole system is considered together. A total of 16 slip surfaces were considered in these calculations. From experience, it was found that there would be little gain in accuracy by further increasing the number of slip surfaces. This has also been confirmed in additional work carried out after original completion of this thesis.

The second analysis was performed with different shear strength data for the clay layers as shown in Table 6-3. The slope geometry is the same as shown in Fig. 6-5 and the analysis is again for the \( \phi = 0 \) assumption.

The results are tabulated in Table 6-4. The results for slip surface set 1 show that the upper bound of system failure probability is only slightly different from the lower bound. The result for slip surface set 2 show that the corresponding bounds are relatively further apart than those for slip surface set 1. There is an increased difference between the reliability bounds for the whole system but the difference is much lower than that noted from previous data set (Tables 6-1 and 6-2). This could be explained by the relatively very low coefficients of variation assumed in Table 6-3 compared to those assumed in Table 6-1.

<table>
<thead>
<tr>
<th>Table 6-3 Shear Strength Parameters for Second Example</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Means (kPa)</td>
</tr>
<tr>
<td>Stan. Dev. (kPa)</td>
</tr>
<tr>
<td>Coefficient of Variation</td>
</tr>
</tbody>
</table>
Table 6-4 Computed Failure Probabilities for Second Example

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 2 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.005077</td>
<td>1.1090</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.005078</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.005077</td>
<td>-</td>
</tr>
<tr>
<td>Layer 3 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.019438</td>
<td>1.0571</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.022181</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.019438</td>
<td>-</td>
</tr>
<tr>
<td>Layer 2 and 3 (16 slip surface)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.019438</td>
<td>1.0571</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.026339</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.019438</td>
<td>-</td>
</tr>
</tbody>
</table>

The third analysis was made for the data set shown in Table 6-5. It may be noted that the mean values of cohesion for the three layer are assumed to be significantly higher than those assumed in Table 6-1 and 6-3. However, the coefficients of variation assumed in Table 6-5 are quite close to those assumed in Table 6-1. The computed failure probabilities are tabulated in Table 6-6. In this case the upper and lower bounds of failure probability are very close to each other for the two sub-systems. There is, however, a significant difference between the upper and lower bounds of the failure probability of the whole system. The upper bound is almost twice as large as the lower bound, a result similar to the one for the first data set. This result confirms the conclusion that the higher the coefficients of variation of the basic random variables, the greater the difference between lower and upper bounds of system failure probability.

Table 6-5 Shear Strength Parameters for Third Example

<table>
<thead>
<tr>
<th></th>
<th>Clay 1</th>
<th>Clay 2</th>
<th>Clay 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$c_3$</td>
</tr>
<tr>
<td>Means (kPa)</td>
<td>136</td>
<td>80</td>
<td>102</td>
</tr>
<tr>
<td>Std. Dev. (kPa)</td>
<td>50</td>
<td>15</td>
<td>24</td>
</tr>
<tr>
<td>Coefficients of Variation</td>
<td>36.8%</td>
<td>18.8%</td>
<td>23.5%</td>
</tr>
</tbody>
</table>
Table 6-6 Computed Failure Probabilities for Third Example

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Layer 2 (8 slip surfaces)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000616</td>
<td>2.3145</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000646</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000616</td>
<td>-</td>
</tr>
<tr>
<td><strong>Layer 3 (8 slip surfaces)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000515</td>
<td>2.1310</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000565</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000515</td>
<td>-</td>
</tr>
<tr>
<td><strong>Layer 2 and 3 (16 slip surface)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000616</td>
<td>2.3145</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.001193</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000616</td>
<td>-</td>
</tr>
</tbody>
</table>

By comparing two sub-systems, it is interesting to note that the minimum factor of safety for set 1 is larger than that for set 2 but the failure probability for set 1 is also larger than that for set 2. This implies that the influence of the standard deviation of cohesion of clay-1 on the results for the critical slip surface among slip surface set 1 is larger than that on the results for the critical slip surface among slip surface set 2. Note that the critical slip surface is defined on the basis of minimum F.

6.5.2. NON-LINEAR PERFORMANCE FUNCTION

In slope reliability analyses, most performance functions based on simplified and relatively 'rigorous' GPS methods are non-linear and inexplicit. Two popular limit equilibrium methods, the simplified Bishop method and the Morgenstern & Price method, were used for system reliability analyses presented in this subsection.

6.5.2.1. ANALYSES BASED ON THE SIMPLIFIED BISHOP METHOD
The geometry of the slope is the same as shown in Fig. 6-5. All potential slip surfaces are assumed to be circular in shape and tangent to the bases of either layer 2 or layer 3 as shown in Fig. 6-6. All the shear strength parameters and unit weights of soils are considered as the basic random variables. The mean values and the standard deviations of the random variables are shown in Table 6-7. Two alternative definitions of the reliability index, the Cornell definition and the Hasofer - Lind definition, are used for comparison.

![Diagram of Circular Slip Surfaces Based on the Simplified Bishop Method](image)

**Fig. 6-6 Circular Slip Surfaces Based on the Simplified Bishop Method**

<table>
<thead>
<tr>
<th>Table 6-7 Shear Strength Parameters and Unit Weights of Soils</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Clay 1</strong></td>
</tr>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>Mean</td>
</tr>
<tr>
<td>$\sigma$</td>
</tr>
<tr>
<td>$\delta$</td>
</tr>
</tbody>
</table>

**NOTE:** $\sigma$ - standard deviation of basic random variables; $\delta$ - coefficient of variation of basic random variables; The units of $c$ and $\gamma$ are 'kPa' and 'kN/m$^3$', respectively.

The correlation coefficients, $\rho$, between any pair of basic random variables in the same soil layer may be included in the system analyses and they are assumed as follows...
Soil - 1

<table>
<thead>
<tr>
<th></th>
<th>( \gamma )</th>
<th>( c' )</th>
<th>( \phi' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td>1</td>
<td>0.15</td>
<td>0.25</td>
</tr>
<tr>
<td>( c' )</td>
<td>0.15</td>
<td>1</td>
<td>-0.1</td>
</tr>
<tr>
<td>( \phi' )</td>
<td>0.25</td>
<td>-0.1</td>
<td>1</td>
</tr>
</tbody>
</table>

Soil - 2

<table>
<thead>
<tr>
<th></th>
<th>( \gamma )</th>
<th>( c' )</th>
<th>( \phi' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td>1</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>( c' )</td>
<td>0.2</td>
<td>1</td>
<td>-0.15</td>
</tr>
<tr>
<td>( \phi' )</td>
<td>0.3</td>
<td>-0.15</td>
<td>1</td>
</tr>
</tbody>
</table>

Soil - 3

<table>
<thead>
<tr>
<th></th>
<th>( \gamma )</th>
<th>( c' )</th>
<th>( \phi' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma )</td>
<td>1</td>
<td>0.1</td>
<td>0.35</td>
</tr>
<tr>
<td>( c' )</td>
<td>0.1</td>
<td>1</td>
<td>-0.05</td>
</tr>
<tr>
<td>( \phi' )</td>
<td>0.35</td>
<td>-0.05</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: There is an assumption that there is no correlation of any parameter between layers.

For conventional (Cornell's) definition of reliability index, the reliability bounds of slope systems are shown in Table 6-8 when the basic random variables are considered as independent. In this case the upper and lower bounds of failure probability are somewhat different for each of the two sub-systems. Moreover, the upper bound of failure probability for the first sub-system is very close to that for the whole system. This implies that the reliability bounds of the whole system is controlled by the first sub-system for the given conditions. When the correlation coefficients of the random variables are included in the analyses, the reliability bounds of slope systems are tabulated in Table 6-9. The absolute differences between reliability bounds of the whole
### Table 6-8 Failure Probability Bounds for Independent Random Variables
(based on conventional definition)

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 2 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.017926</td>
<td>1.4699</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.024713</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.017926</td>
<td>-</td>
</tr>
<tr>
<td>Layer 3 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000512</td>
<td>1.7778</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000790</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000512</td>
<td>-</td>
</tr>
<tr>
<td>Layer 2 and 3 (16 slip surface)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.017926</td>
<td>1.4699</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.025363</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.017926</td>
<td>-</td>
</tr>
</tbody>
</table>

### Table 6-9 Failure Probability Bounds for Correlated Random Variables
(based on conventional definition)

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 2 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.010869</td>
<td>1.4699</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.016257</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.010869</td>
<td>-</td>
</tr>
<tr>
<td>Layer 3 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000345</td>
<td>1.7778</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000571</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000345</td>
<td>-</td>
</tr>
<tr>
<td>Layer 2 and 3 (16 slip surface)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.010869</td>
<td>1.4699</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.016776</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.010869</td>
<td>-</td>
</tr>
</tbody>
</table>
system for the case with independent variables are larger than that for case with correlated variables. In other words, the reliability bounds of slope system become narrower or the reliability of the slope system is increased or improved when the assumed correlations between basic random variables are considered. In fact, in this instance, the upper bound in Table 6-9 is smaller than the lower bound in Table 6-8. No general conclusion should, however, be drawn because the results could be quite different for another set of correlation coefficients.

For Hasofer and Lind's definition of reliability index, the reliability bounds of slope systems are shown in Tables 6-10 and 6-11 when the basic random variables are considered as independent or correlated respectively. It is interesting to note that the absolute differences between the reliability bounds based on the Hasofer-Lind definition are less than that based on the conventional definition, i.e. the Hasofer-Lind definition gives narrower bounds for system reliability than does the conventional definition.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 2 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.014219</td>
<td>1.4699</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.019248</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.014219</td>
<td>-</td>
</tr>
<tr>
<td>Layer 3 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000281</td>
<td>1.7778</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000412</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000281</td>
<td>-</td>
</tr>
<tr>
<td>Layer 2 and 3 (16 slip surface)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.014219</td>
<td>1.4699</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.019611</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.014219</td>
<td>-</td>
</tr>
</tbody>
</table>
Table 6-11 Failure Probability Bounds for Correlated Random Variables
(based on Hasofer-Lind's definition)

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Layer 2 (8 slip surfaces)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.009442</td>
<td>1.4699</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.013329</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.009442</td>
<td>-</td>
</tr>
<tr>
<td><strong>Layer 3 (8 slip surfaces)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000213</td>
<td>1.7778</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000316</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000213</td>
<td>-</td>
</tr>
<tr>
<td><strong>Layer 2 and 3 (16 slip surface)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.009442</td>
<td>1.4699</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.013620</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.009442</td>
<td>-</td>
</tr>
</tbody>
</table>

6.5.2.2. ANALYSES BASED ON THE MORGENSTERN AND PRICE METHOD

In this illustration, all potential slip surfaces are assumed to be of arbitrary shape (non-circular). The geometry of the slope is the same as shown in Fig. 6-5. The shear strength parameters and unit weights of soil are the same as presented in Table 6-7. The correlation coefficients of the basic random variables are assumed to the same as in the previous example. Two sub-systems are again considered in following analyses (slip surface tangent to layer 2 being one sub-system and slip surface tangent to layer 3 being the second sub-system). The slip surfaces are shown in Fig 6-7.

For convention definition of reliability index, the reliability bounds of the slope system are shown in Table 6-12 when the basic random variables are considered as independent. There are some differences between the reliability bounds for the two sub-systems. The reliability bounds of the whole system are very close to the corresponding bounds for the first sub-system (set 1) in this case. This conclusion is similar to the conclusion reached in the previous sub-section when analyses were performed on the basis of potential slip surfaces of circular shape. When the basic
random variables are considered to be correlated the reliability bounds of the slope system are tabulated in Table 6-13. Comparing Tables 6-12 and 6-13, the absolute values of reliability bounds are smaller for correlated variables than for independent variables. Moreover, for correlated variables, the bounds are narrower. These results cannot be generalised and are valid for the specific data on correlation coefficient. However, the conclusions are similar to those for analyses based on slip surfaces of circular shape.

Based on the Hasofer-Lind definition of reliability index, the reliability bounds of the slope system are shown in Tables 6-14 and 6-15 when the basic random variables are considered as independent and correlated respectively. It is interesting to note that the absolute values of the reliability bounds associated with Hasofer-Lind's definition are smaller than those based on the conventional definition whether the basic random variables are independent or correlated. It is also obvious that Hasofer-Lind's definition gives narrower bounds of reliability than those based on the convention definition. The conclusion is similar to the one for analyses based on potential slip surface of circular shape.

Fig. 6-7 Cross Section of Congress Street Cut with Non-circular Slip Surface
Table 6-12 Failure Probability Bounds for Independent Random Variables
(based on conventional definition)

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 2 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.017024</td>
<td>1.4740</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.029286</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.017024</td>
<td>-</td>
</tr>
<tr>
<td>Layer 3 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000499</td>
<td>1.7804</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000707</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000499</td>
<td>-</td>
</tr>
<tr>
<td>Layer 2 and 3 (16 slip surface)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.017024</td>
<td>1.4740</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.029933</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.017024</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6-13 Failure Probability Bounds for Correlated Random Variables
(based on conventional definition)

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 2 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.010345</td>
<td>1.4740</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.019397</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.010345</td>
<td>-</td>
</tr>
<tr>
<td>Layer 3 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000361</td>
<td>1.7804</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000511</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000361</td>
<td>-</td>
</tr>
<tr>
<td>Layer 2 and 3 (16 slip surface)</td>
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<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.010345</td>
<td>1.4740</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.019909</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.010345</td>
<td>-</td>
</tr>
</tbody>
</table>
### Table 6-14 Failure Probabilistic Bounds for Independent Random Variables
(based on Hasofer-Lind's definition)

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 2 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.013829</td>
<td>1.4740</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.026623</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.013829</td>
<td>-</td>
</tr>
<tr>
<td>Layer 3 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000273</td>
<td>1.7804</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000379</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000273</td>
<td>-</td>
</tr>
<tr>
<td>Layer 2 and 3 (16 slip surface)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.013829</td>
<td>1.4740</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.026999</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.013829</td>
<td>-</td>
</tr>
</tbody>
</table>

### Table 6-15 Failure Probabilistic Bounds for Correlated Random Variables
(based on Hasofer-Lind's definition)

<table>
<thead>
<tr>
<th>Condition</th>
<th>Failure Probability</th>
<th>Safety Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Layer 2 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.009165</td>
<td>1.4740</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.021222</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.009165</td>
<td>-</td>
</tr>
<tr>
<td>Layer 3 (8 slip surfaces)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.000210</td>
<td>1.7804</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.000342</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.000210</td>
<td>-</td>
</tr>
<tr>
<td>Layer 2 and 3 (16 slip surface)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Critical slip surface</td>
<td>0.009165</td>
<td>1.4740</td>
</tr>
<tr>
<td>Upper bound of system</td>
<td>0.021576</td>
<td>-</td>
</tr>
<tr>
<td>Lower bound of system</td>
<td>0.009165</td>
<td>-</td>
</tr>
</tbody>
</table>
6.5.3. DISCUSSIONS

The stability of a slope has traditionally been considered in relation to the critical slip surface, i.e., a slip surface which leads to the minimum factor of safety. Within a probabilistic framework, a slip surface with the maximum probability of failure can be identified and such a slip surface may be regarded as an alternative to the conventional critical slip surface. However, neither of these may lead to a realistic estimate of the slope reliability. This is because there are many potential slip surfaces and each has a probability of failure associated with it. Therefore, the slope stability problem must be considered as a system reliability problem. In such a problem, the correlations between the various 'elements' in the system are, of course, very important. The determination of the exact system reliability is not possible except for very simple systems.

For 'homogeneous' slope systems, the correlation between the various elements of the system are very high. In other words, the probability of failure along different potential slip surfaces is highly correlated and therefore the reliability bounds of the systems are close or narrow. The correlation between any pair of failure modes would be low when some basic random variables involved in that pair of elements are different. Consequently, the reliability bounds of the system will become wider. In general, the reliability bounds of the whole system are always wider than that of any sub-system. On the one hand, the lower the correlation the greater the difference between the reliability bounds of the slope system and on the other hand, the upper bound failure probability of the system is found to be higher than the maximum failure probability associated with a critical slip surface.

The results shown in the above section indicate that the range of reliability bounds will become narrow if the single-mode failure probabilities decrease, i.e. the less the failure probability of the single-mode the narrower is the reliability range for the slope system. This conclusion is similar to those reached by Ditlevsen (1979) and Ma & Ang (1981).

For slope systems with independent basic random variables, the absolute differences of the reliability bounds are larger than those for slope systems with
correlated basic random variables. However, this result corresponds to the chosen sets of correlation coefficients. A different trend might be noted with a different set of correlation coefficients. Comparing both alternative definitions of the reliability index, it is interesting to note that generally the reliability bounds based on the Hasofer and Lind's definition are somewhat narrower than those associated with the conventional definition.

6.6. INFLUENCES OF AUTO-CORRELATION COEFFICIENTS ON PROBABILITY BOUNDS OF SLOPE SYSTEMS

In general, the basic input random variables in soil profiles for a slope system are spatially correlated. The influences of cross-correlation coefficients of basic random variables within each soil layer on the probability bounds of a slope system, based on the simplified Bishop method as well as on the Morgenstern and Price method, have already been discussed in Section 6.4. It may also be useful to consider the influences of auto-correlation coefficients on the probability bounds of a slope system consisting of several soil layers which are statistically homogenous with respect to each random variable. The influences of auto-correlation coefficients of shear strength parameters on the probabilistic bounds of a slope system are examined in this section by varying these coefficients from 0 to 1. In the first case, only the auto-correlation coefficient of cohesion, $\rho_{cc}$, is considered. In the second case, only the auto-correlation coefficient of friction angle, $\rho_{\phi\phi}$, is considered. In the third case both auto-correlation coefficients, i.e. $\rho_{cc}$ and $\rho_{\phi\phi}$, are varied together by the same amount. The simplified Bishop method was used as the basis for the performance function for these studies. Of course, other limit equilibrium methods, such as Morgenstern and Price method, can also be used to define the performance function. The reliability index was based on Cornell's definition in these investigations. The slope geometry and slip surfaces are the same as shown in Fig. 6-6. The influences of $\rho_{cc}$ and $\rho_{\phi\phi}$ on the probability bounds, based on the three cases, were separately investigated below.

**CASE-1:** In this case, only cohesion of the slope soil is considered as the random variable and the shear strength parameters and unit weight of soil are assumed to
be same as those presented in Table 6-7 except that standard deviations are zero for all variables other than cohesion. All the auto-correlation coefficients of the cohesion are simultaneously varied from 0 to 1. The probability bounds of the whole slope system corresponding to different values of $\rho_{cc}$ are shown in Fig 6-8. The probability bounds become narrow as $\rho_{cc}$ increases from 0 to 1. It is also interesting to note that the upper and lower bounds of failure probability of a slope system are exactly equal to that associated with the critical slip surface when $\rho_{cc} = 1$.

**CASE-2:** The internal friction angle of the slope soil is considered as the only random variable and the shear strength parameters and unit weight of soil are assumed to be same as in Table 6-7 except that standard deviations are zero for all parameters other than internal friction angle. All the auto-correlation coefficients of the friction angles are simultaneously varied from 0 to 1. The probability bounds of the slope system corresponding to different values of $\rho_{\phi\phi}$ are shown in Fig 6-9. From Fig. 6-9, it can be seen that the failure probability of upper and lower bounds increases with the increase of $\rho_{\phi\phi}$ from 0 to 1. It is also interesting to note that the upper and lower bounds of failure probability of a slope system are exactly equal to that associated with the critical slip surface when $\rho_{\phi\phi} = 1$. However, the interval between upper and lower bounds of the

![Image](https://via.placeholder.com/150)
whole slope system first increases and then decreases which is quite different from the previous case, **CASE-1**, when cohesions are the only random variables.

![Fig. 6-9 Relationship between Probabilistic Bounds And $\rho_{\phi\phi}$](image)

**CASE-3:** Both soil parameters, cohesion and internal friction angle, are considered as random variables for each soil layer. The shear strength parameters and unit weight of various soil layers are assumed to be same as in Table 6-7. All the auto-correlation coefficients of the cohesion and friction angle, $\rho_{cc}$ and $\rho_{\phi\phi}$, are simultaneously varied from 0 to 1. The probability bounds of the whole slope system corresponding to these magnitudes of $\rho_{cc}$ and $\rho_{\phi\phi}$ are shown in Fig 6-10. The failure probability of upper and lower bounds increases with the increase of $\rho_{cc}$ and $\rho_{\phi\phi}$ from 0 to 1. However, the upper and lower bounds of failure probability of the slope system are not equal to each other when $\rho_{cc}$ and $\rho_{\phi\phi}$ are both equal 1. The interval between upper and lower bounds of the reliability whole slope system fluctuates slightly with change of $\rho_{cc}$ and $\rho_{\phi\phi}$ from 0 to 1.
Fig. 6-10 Relationship between Probabilistic Bounds And $\rho_{cc}$ as well as $\rho_{\phi\phi}$

All the computation results presented above indicate that the failure probability bounds (upper and lower bounds) of the whole slope system increase with the increase of $\rho_{cc}$ or $\rho_{\phi\phi}$ or both together. However, the type of increase and the interval between upper and lower bounds differs in each case. Considering cohesion as the only random variable in each layer the probability bounds of the slope system become narrow with the increase of $\rho_{cc}$ and are equal to each other when $\rho_{cc} = 1$. Considering internal friction angle as the only random variable in each layer, however, the change of the interval between the probabilistic bounds fluctuates with the increase of $\rho_{\phi\phi}$. Considering both cohesion and internal friction angle as random variable in each layer, the interval between the probability bounds of the slope system fluctuates only slightly with the increase of $\rho_{cc}$ and $\rho_{\phi\phi}$ but the upper and lower probability bounds are not equal to each other when $\rho_{cc}$ and $\rho_{\phi\phi}$ are both equal to 1.

The results obtained are not conclusive and further work is required to establish general trends concerning the effect of autocorrelation coefficients on reliability bounds.
6.7. SUMMARY

1. The problem of slope stability may be formulated as one of system reliability since there are many potential slip surfaces in any soil slope. However, procedures have so far not been developed for exact evaluation of system reliability. Therefore, the 'upper' and 'lower' bounds of the system reliability may be evaluated.

2. The calculation procedures of system reliability are presented based on linear and non-linear performance functions. A simple orthogonal transformation procedure is proposed so that correlated basic random variables of performance functions can be considered for system reliability analyses.

3. The analysis procedures of slope system reliability, associated with explicit and linear as well as inexplicit and non-linear performance functions, are developed in this chapter. Both circular and non-circular slip surfaces can be included in the analysis procedure. Some comprehensive GPS methods (Bishop and Morgenstern & Price) are used as the basis for performance functions in terms of FOS for the analyses of slope system reliability. The shear strength parameters, $c$ and $\phi$, as well as the unit weight $\gamma$ of soil can be considered as the basic random variables of performance functions. Two alternative definitions of the reliability index are used for comparison.

4. The calculation results show that the lower bound of the failure probability of a slope system is always equal to the maximum failure probability of associated with the critical slip surface. Conversely, the upper bound of the failure probability is always larger than the maximum failure probability associated with the critical slip surface. The range of the reliability bounds of any system depends on the degree of correlations between the elements of the system.

5. In general, the range of reliability bounds, based on Hasofer and Lind' definition of reliability index, is narrower than that associated with the conventional definition. The range of the reliability bounds, for cross-correlated basic random variables, is more narrow than that for independent variables. However, the influence of auto-correlations on the range of the reliability bounds is not easy to predict. The range of the reliability bounds based on the sub-system is usually narrower than that of the whole system. The
range of the reliability bounds is also dependent on the failure probabilities of the single-mode. The larger the failure probabilities associated with single modes the wider system range of the reliability bounds.
7.1. GENERAL REMARKS

Accelerograms of strong earthquake motions recorded on firm ground are extremely irregular, and exhibit certain features of random processes. Because the information obtained from earth structural response studies based on only a single real earthquake record is insufficient for earthquake-resistant designs, it would be desirable to have a large sample of accelerograms of strong earthquake motion for various class of intensity and duration. Alternatively, simulation procedures may be used to generate many samples of artificial earthquake records. The simulations can be devised to be statistically consistent with past earthquakes, or varied to emphasise particular adverse aspects in anticipation of future earthquakes. However, the simulations of earthquakes, based on random processes, have to satisfy certain conditions, such as magnitude of earthquakes, duration time, frequency features, geological conditions, etc. In order to
simulate an earthquake accelerogram associated with random process theory, some basic definitions related to random process simulation are given below:

**Random Process** - A random process, also called a time series or stochastic process, is an ensemble of time function $X_n(t), \ n = 1, 2, \ldots$ (could be uncountable), such that the ensemble can be characterised through statistical properties. This definition can be used to describe the records of strong motion induced by earthquakes.

**Gaussian Process** - The random process is called Gaussian if, for every finite collection of times, $t_1 < \cdots < t_r$, the random variables $x(t_1), \ldots, x(t_r)$ have a multivariate Gaussian distribution.

**Auto-correlation Function** - It is defined as

$$R(t, t_{i+1}) = E[x(t_i) x(t_{i+1})]$$  \hfill (7-1)

**Stationary Process** - The process is called stationary if and when, for any fixed increment $\Delta t$, the random variables $x(t_i + \Delta t)$ have the same joint distribution as random variables $x(t_i)$. Therefore, the auto-correlation function of a stationary process can be written as,

$$R(\tau) = E[x(t) x(t+\tau)]$$ \hfill (7-2)

**Power Spectral Density** - For stationary random processes, the power spectral density, PSD, is defined as,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega \tau} d\tau$$ \hfill (7-3)

On the basis of the stationary Gaussian process, an algorithm of generating pseudo non-stationary processes is presented for simulating artificial earthquakes in this chapter. In conjunction with those generated artificial earthquake time-histories, the response spectra, which can be used to predict the maximum response of a linear structure, will be investigated as well.
7.2 SIMULATION OF NON-STATIONARY RANDOM PROCESSES

Observations have shown that the recorded earthquake motions behave in a non-stationary manner, i.e. the process of an earthquake motion has an uprising portion at their inception and decaying portion at their tail. However, such a non-stationary process may approximately be represented by the product of a stationary random process, \( x(t) \), and a deterministic time function, \( I(t) \). This is a non-stationary random process which can expressed as,

\[
K(t) = I(t) x(t)
\]  
(7-4)

If an earthquake motion, \( K(t) \), is considered as a non-stationary Gaussian process, then \( x(t) \) in Eq.(7-4) is a stationary Gaussian process with zero mean value. It is, therefore, obvious that a stationary Gaussian process can be used for simulating earthquake motions involving a non-stationary Gaussian process.

7.2.1. SIMULATION OF STATIONARY GAUSSIAN PROCESS

A numerical simulation algorithm of stationary Gaussian random processes has been proposed by Franklin (1965). This numerical procedure can simulate a Gaussian process which has a given power spectral density function. On the basis of Franklin's method and a specific power spectral density function proposed by Kanai-Tajimi (1960), the procedure for generating a stationary Gaussian random process can be presented as follows.

7.2.1.1. ELEMENTARY CONCEPT

A linear process with time-invariant elements may be described mathematically as a linear transformation, i.e.,

\[
x(t) = \int_{-\infty}^{t} g(t - \tau) w(\tau)d\tau
\]  
(7-5)
This transformation produces an output \( x(t) \) depending on the input process \( w(\tau) \) for times \( \tau \leq t \). The power spectral densities \( S_w(\omega) \) and \( S_x(\omega) \) of input and output are related by the equation:

\[
S_x(\omega) = |G(\omega)|^2 S_w(\omega) \tag{7-6}
\]

where \( e^{j\omega t}G(\omega) \) is the response of the filter to the input \( e^{j\omega t} \); equivalently, \( G(\omega) \) is the Fourier transform of \( g(t) \).

The condition \( S_w(\omega) = 1 \) defines an idealised random process \( w(t) \) known as white noise. Although white noise is convenient for theoretical discussion, it is not suitable for direct input in digital computation. If one assumes that \( S(\omega) \) can be represented with sufficient accuracy by a rational function, i.e., a quotient of two polynomials in \( \omega \), then it is possible to find a transform which satisfies the condition

\[
|G(\omega)|^2 = S(\omega). \tag{7-7}
\]

where \( P(z) \) and \( Q(z) \) are polynomial in \( z \) with real coefficients and the degree of \( P \) is less than that of \( Q \). Thus one may choose,

\[
G(\omega) = \frac{P(i\omega)}{Q(i\omega)} \tag{7-8}
\]

According to Kanai-Tajimi model, however, the power spectral density function of a linear, single degree-of-freedom (SDOF) system may be expressed as the following rational polynomial function:

\[
S(\omega) = \frac{1 + 4 \xi^2 \left(\frac{\omega}{\omega_0}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + 4 \xi^2 \left(\frac{\omega}{\omega_0}\right)^2} S_0 \tag{7-9}
\]

The three parameters in Eq.(7-9), namely \( S_0, \omega_0 \) and \( \xi \), represent the spectrum level (normalised to unit mass) of the broad-band excitation at the base, the natural frequency
in bed-rock, and the ratio of damping to the critical damping in bed-rock, respectively. These parameters may be adjusted according to the earthquake magnitude, ground resonance frequency, and attenuation of seismic waves in the ground. The most attractive feature of this model is the ability to simulate ground response in a very simple way.

Based on Eqs. (7-7) and (7-8), Better form of Eq. (7-9) can be presented as follows:

\[
S(\omega) = \frac{\left(\omega_0 \sqrt{S_0}\right)^2 + \frac{2}{2} \xi \omega_0 \sqrt{S_0} \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + (2 \xi \omega_0)^2 \omega^2}
\]

or,

\[
G(\omega) = \frac{b_0 (i\omega) + b_1}{(i\omega)^2 + a_1 (i\omega) + a_2} = \frac{P(i\omega)}{Q(i\omega)}
\]

in which, \( b_0 = 2 \xi \omega_0 \sqrt{S_0} \), \( b_1 = \omega_0 \sqrt{S_0} \), \( a_1 = 2 \xi \omega_0 \), \( a_2 = \omega_0^2 \).

Denoting \( D = d / dt \) as the differential operator, the random process, \( x(t) \), may be written as,

\[
x(t) = \frac{P(D)}{Q(D)} w(t)
\]

or,

\[
x(t) = P(D) \phi(t)
\]

where, \( w(t) \) is the white noise and \( \phi(t) \) is the steady-state solution.

The simulation procedure, based on Eqs. (7-10) and (7-11), is described in the next sub-section.
7.2.1.2. SIMULATION STEPS OF STATIONARY GAUSSIAN PROCESS

On the basis of the derivations proposed by Franklin (1965), the major simulation steps of the stationary Gaussian random process may be summarised as follows:

**STEP-1** Compute the moment matrix $M$ when $t = 0$

In this case, according to Franklin's method, the matrix, $M$, can be defined as,

$$M = \begin{bmatrix} m_0 & 0 \\ 0 & m_1 \end{bmatrix}$$  \hspace{1cm} (7-12)

The $m_0$ and $m_1$ in Eq.(7-12) may be calculated by following equations, i.e.,

$$\begin{bmatrix} a_2 & -1 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$  \hspace{1cm} (7-13)

thus,

$$M = \begin{bmatrix} m_0 & 0 \\ 0 & m_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 \xi \omega_0^3 & 0 \\ 0 & 4 \xi \omega_0 \end{bmatrix}$$  \hspace{1cm} (7-14)

The matrix $M$ is a positive definite, real, symmetric matrix. Therefore, $M$ may be factored by the Crout factorisation which was introduced in Chapter 5, i.e.,

$$M = TT^T$$  \hspace{1cm} (7-15a)

or,

$$M = \begin{bmatrix} t_{11} & 0 \\ t_{21} t_{22} & 0 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{bmatrix}$$  \hspace{1cm} (7-15b)

From Eqs. (7-14) and (7-15b), the components of matrix, $T$, may be obtained by following equations,

$$t_{11} = \frac{1}{2 \omega_0 \sqrt{\xi \omega_0}}, \quad t_{21} = t_{12} = 0, \quad t_{22} = \frac{1}{2 \sqrt{\xi \omega_0}}$$

**STEP-2** Compute the initial vector $Z(0)$

On the basis of the calculation procedure proposed by Franklin, the initial vector, $Z(0)$, can be computed as
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\[ Z(0) = T W^0 = \begin{bmatrix} \frac{1}{2 \omega_0 \sqrt{\xi \omega_0}} & 0 \\ 0 & \frac{1}{2 \omega_0 \sqrt{\xi \omega_0}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \] (7-16)

in which, \( w_1, w_2, \ldots \) simulate independent samples from the Gaussian distribution with mean 0 and variance 1 as discussed in Chapter 5.

**STEP-3 Compute the matrix \( \exp(At) \)**

According to the definition proposed by Franklin, the matrix, \( \exp(At) \), may be written as follows,

\[ \exp(At) = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2'(t) \end{bmatrix} = X(t) \] (7-17)

in which, the components of the matrix, \( \exp(At) \), have to satisfy the following differential equations, i.e.,

\[ \phi_{j''} + a_1 \phi_j' + a_2 \phi_j = 0 \quad j = 1, 2 \] (7-18a)

or,

\[ \phi_{j''} + 2 \xi \omega_0 \phi_j' + \omega_0^2 \phi_j = 0 \quad j = 1, 2 \] (7-18b)

Because \( X(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \), the initial conditions for Eq. (7-18a) or Eq. (7-18b) are,

\[ \phi_1(0) = 1, \quad \phi_1'(0) = 0, \quad \phi_2(0) = 0, \quad \phi_2'(0) = 1 \]

The eigen-equation of the differential equation (7-18) is

\[ r^2 + a_1 r + a_2 = 0 \] (7-19)

The solutions of the quadratic equation (7-19) are

\[ r_1 = -\xi \omega_0 + \sqrt{1 - \xi^2} \quad r_2 = -\xi \omega_0 - \sqrt{1 - \xi^2} \] (7-20)

Therefore, the solution of the initial-value problem for the differential equation (7-15) is

\[ \phi(t) = e^{-\xi \omega_0 t} \left[ C_1 \cos \omega_0 t + C_2 \sin \omega_0 t \right] \] (7-21)
According to the initial conditions, the expressions for \( \phi_1(t) \), \( \phi_1'(t) \), \( \phi_2(t) \) and \( \phi_2'(t) \) may be obtained as follows:

\[
\phi_1(t) = e^{-\xi \omega_0 t} \left[ \cos \left( \omega_0 \sqrt{1 - \xi^2} t \right) + \frac{\xi}{\sqrt{1 - \xi^2}} \sin \left( \omega_0 \sqrt{1 - \xi^2} t \right) \right]
\]

\[
\phi_1'(t) = -\frac{\omega_0}{\sqrt{1 - \xi^2}} e^{-\xi \omega_0 t} \sin \left( \omega_0 \sqrt{1 - \xi^2} t \right)
\]

\[
\phi_2(t) = \frac{e^{-\xi \omega_0 t}}{\omega_0 \sqrt{1 - \xi^2}} \sin \left( \omega_0 \sqrt{1 - \xi^2} t \right)
\]

\[
\phi_2'(t) = e^{-\xi \omega_0 t} \left[ \cos \left( \omega_0 \sqrt{1 - \xi^2} t \right) - \frac{\xi}{\sqrt{1 - \xi^2}} \sin \left( \omega_0 \sqrt{1 - \xi^2} t \right) \right]
\]

By means of Eqs.(7-22) and (7-23), matrix \( e^A_t \) may be rewritten as follows:

\[
e^A_t = e^{-\xi \omega_0 t} \begin{bmatrix} e_{11}(t) & e_{12}(t) \\ e_{21}(t) & e_{22}(t) \end{bmatrix}
\]

in which,

\[
e_{11}(t) = \cos \left( \omega_0 \sqrt{1 - \xi^2} t \right) + \frac{\xi}{\sqrt{1 - \xi^2}} \sin \left( \omega_0 \sqrt{1 - \xi^2} t \right)
\]

\[
e_{21}(t) = -\frac{\omega_0}{\sqrt{1 - \xi^2}} \sin \left( \omega_0 \sqrt{1 - \xi^2} t \right)
\]

\[
e_{12}(t) = \frac{1}{\omega_0 \sqrt{1 - \xi^2}} \sin \left( \omega_0 \sqrt{1 - \xi^2} t \right)
\]

\[
e_{22}(t) = \cos \left( \omega_0 \sqrt{1 - \xi^2} t \right) - \frac{\xi}{\sqrt{1 - \xi^2}} \sin \left( \omega_0 \sqrt{1 - \xi^2} t \right)
\]

**STEP - 4 Compute the moment matrix \( M_r \) for \( t > 0 \)**

According to the calculation procedure proposed by Franklin and based on the given power spectral density function, the positive semi-definite matrix, \( C \), can be written as:

\[
C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]
and the matrix, \( \mathbf{A} \), may be equal to
\[
\mathbf{A} = \begin{bmatrix}
0 & 1 \\
-a_1 & -a_2
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-\omega_0^2 & -2\xi\omega_0
\end{bmatrix}
\]

The moment matrix, \( \mathbf{M}_r \), is symmetrical and depends on \( \Delta t \) but not on \( t \). The following equation may, therefore, be obtained:
\[
e^{A\Delta t} \mathbf{C} [e^{A\Delta t}]^T - \mathbf{C} = \mathbf{A} \mathbf{M}_r + \mathbf{M}_r \mathbf{A}^T
\]
where, the matrix, \( \mathbf{M}_r \), can be defined as,
\[
\mathbf{M}_r = \begin{bmatrix}
a & b \\
b & c
\end{bmatrix}
\]
On the basis of Eqs. (7-24) and (7-25) as well as the definitions of the matrices, \( \mathbf{C} \) and \( \mathbf{M}_r \), the components of \( \mathbf{M}_r \) can be written as following equations:
\[
b = \frac{1}{2} \bar{e}_{11}(\Delta t) \\
c = -[\bar{e}_{22}(\Delta t) + 2\omega_0^2 b] / (4\xi\omega_0) \\
a = \frac{[c - 2\xi\omega_0 b - \bar{e}_{21}(\Delta t)]}{\omega_0^2}
\]
\[
(7-26)
\]
in which,
\[
\bar{e}_{11}(\Delta t) = e^{-2\xi\omega_0\Delta t}e_{11}(\Delta t) \\
\bar{e}_{12}(\Delta t) = \bar{e}_{21}(\Delta t) = e^{-2\xi\omega_0\Delta t}e_{12}(\Delta t) e_{22}(\Delta t) \\
\bar{e}_{22}(\Delta t) = e^{-2\xi\omega_0\Delta t}e_{22}(\Delta t) - 1
\]

**STEP - 5** Compute the vector \( \mathbf{z}(t_k) \) and generate the random process

By means of the Crout factorisation, the matrix, \( \mathbf{M}_r \), can be rewritten as,
\[
\mathbf{M}_r = \begin{bmatrix}
a & b \\
b & c
\end{bmatrix} = \mathbf{T}_r \mathbf{T}_r^T = \begin{bmatrix}
t_{11} & 0 \\
t_{21} & t_{12}
\end{bmatrix} \begin{bmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{bmatrix}
\]
\[
(7-27)
\]
Therefore, the components of the matrix, \( \mathbf{T}_r \), are
\[
t_{11} = \sqrt{a}, \ t_{21} = b / \sqrt{a}, \ t_{22} = \sqrt{c a - b^2} / \sqrt{a}
\]
On the basis of the algorithm proposed by Franklin, the vector, \( \mathbf{z}(t_{k+1}) \), may be calculated as follows

\[
\mathbf{z}(t_{k+1}) = e^{A\Delta t} \mathbf{z}(t_k) + \mathbf{T}_r \mathbf{W}^{(k+1)}
\]

or,

\[
\begin{bmatrix}
Z_1(t_{k+1}) \\
Z_2(t_{k+1})
\end{bmatrix}
= e^{\xi \omega_0 \Delta t}
\begin{bmatrix}
e_{11}(\Delta t) & e_{12}(\Delta t) \\
e_{21}(\Delta t) & e_{22}(\Delta t)
\end{bmatrix}
\begin{bmatrix}
Z_1(t_k) \\
Z_2(t_k)
\end{bmatrix}
+ 
\begin{bmatrix}
t_{11} & 0 \\
t_{21} & t_{22}
\end{bmatrix}
\begin{bmatrix}
w_{2k+3} \\
w_{2k+4}
\end{bmatrix}
\]

in which, the vector \( \mathbf{W}^{(k+1)} \) is a standard normal random vector which may be generated by the approach presented in Chapter 5.

Therefore, the samples based on the stationary Gaussian random process can be simulated as the following equation,

\[
x(t_k) = b_0 Z_1(t_k) + b_1 Z_2(t_k)
\]

\[
= \sqrt{S_0} [2 \xi \omega_0 Z_1(t_k) + \omega_0 Z_2(t_k)] \quad k = 0, 1, 2, \ldots
\]

On the basis of the simulation procedure described above, an artificial earthquake time history, i.e. a non-stationary random process may be obtained.

### 7.2.2. Generating a Non-stationary Random Process

As mentioned previous section, a non-stationary random process may be approximately represented by the product of a stationary random process and a deterministic time function. Such a deterministic time function is usually called the intensity function. In accordance with many real earthquake records, the time history of an earthquake may have three distinct phases, i.e. rising, steady and decaying portions. Therefore, the intensity function, \( I(t) \), may be defined by a step function. A number of researchers have proposed alternative definitions for the intensity function (e.g. Lin, 1963; Amin and Ang, 1968; etc.). In this research work, an intensity function slightly different from Amin and Ang's definition is proposed as follows:
\[
I(t) = \begin{cases} 
I_0 \left( \frac{t}{T_1} \right) & 0 < t \leq T_1 \\
I_0 & T_1 < t < T_2 \\
I_0 e^{-p(t-T_2)} & T_2 < t 
\end{cases} 
\quad (7-30)
\]

in which \( I_0 \) and \( p \) are positive constants, \( T_1 \) is the lasting time of the earthquake acceleration rising and \( T_2 \) is the inception time of earthquake acceleration decaying. The quantity of \( I_0 \) is a measure of earthquake intensity and it depends on the peak acceleration of the design earthquake, \( p \) is a shape parameter of earthquake acceleration decaying and it depends not only on the design earthquake acceleration but also the duration of the design earthquake. The larger the magnitude of \( p \), the quicker the decaying of the earthquake acceleration. \( T_1 \) and \( T_2 \) may be obtained by the duration time of the design earthquake. Eq. (7-30) can be illustrated by a simple diagram as shown in Fig. 7-1.

Fig. 7-1 Shape of Intensity Function \( I(t) \)

One of the important parameters for simulating the random process, \( x(t) \), is the time increment or time step \( \Delta t \). If a large time step, \( \Delta t \), is chosen then the high frequency components in \( x(t) \) may be lost. Conversely, if a very small time step, \( \Delta t \), is used then the low frequency components of \( x(t) \) may be lost. It is, therefore, necessary to select an appropriate magnitude of \( \Delta t \) for generating an artificial earthquake time history. According to published research papers, the value of \( \Delta t \) is usually chosen in the domain of 0.001 to 0.005 sec.
7.3 GENERATING ARTIFICIAL EARTHQUAKES AND CARRYING OUT SPECTRAL ANALYSIS

The elementary theory and necessary mathematical derivations for simulating artificial earthquakes have been presented in the previous sections. The applications of this simulation procedure to the design of earthquakes and to response spectra analysis may now be discussed.

7.3.1. DESIGN OF ARTIFICIAL EARTHQUAKES

In order to simulate an artificial earthquake, the parameters mentioned in the preceding section have to be assumed. Five artificial earthquakes were generated and the duration time of all these artificial earthquakes was assumed to be 15 sec. The motion rising time, $T_1$, and the duration time before the motion begins to decay, $T_2$, were assumed to be 2.5 sec and 10 sec, respectively. The shape parameter of the motion decaying, $p$, was assumed to be 0.25. The time increment or step, $\Delta t$, was taken as 0.005 sec. The design maximum or peak acceleration was chosen as 0.6g. The power spectra intensity of the white noise, $S_0$, was assumed as 0.01. The five different damping ratios, $\xi$, and natural circular frequencies, $\omega_0$, of bed-rock were assumed as tabulated in Table 7-1.

<table>
<thead>
<tr>
<th>Case No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
<td>0.1</td>
<td>0.05</td>
<td>0.15</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>12</td>
<td>20</td>
<td>25</td>
<td>30</td>
<td>35</td>
</tr>
</tbody>
</table>

On the basis of the simulation procedure of non-stationary random process presented in the above section and the assumptions stated in this section, five artificial earthquake motions were obtained. These motion records, represented in terms of the acceleration coefficient, are shown in Figs. 7-2, 7-3, 7-4, 7-5 and 7-6.
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Fig. 7-2 Artificial Earthquake Time History - Case 1

Fig. 7-3 Artificial Earthquake Time History - Case 2

Fig. 7-4 Artificial Earthquake Time History - Case 3
7.3.2. RESPONSE SPECTRUM ANALYSIS OF DESIGNED ARTIFICIAL EARTHQUAKES

The response spectrum approach is based on dynamic analysis taking into account the characteristics of ground motions as well as the dynamic properties of structures. The response spectrum is defined as the maximum elastic response of a single degree of freedom (SDOF) system oscillator to a given earthquake excitation. The response of a structure is usually obtained for a wide range of vibration frequencies and damping ratios envisaged in practical applications. The equation of motion of a SDOF system can be expressed by a second-order differential equation, i.e.,
\[ \ddot{x}(t) + 2\nu \omega \dot{x}(t) + \omega^2 x(t) = -a(t) \quad (7-31) \]

in which \( x(t) \) is the displacement response of the SDOF system, \( \nu \) and \( \omega \) are respectively the damping ratio and circular frequency of the SDOF system and \( a(t) \) is the input earthquake acceleration.

On the basis of Eq. (7-31), the response of a SDOF system at any time \( t \), i.e. the displacement of the system, can be expressed by Duhamel integral,

\[ x(t,\omega,\nu) = -\frac{1}{\omega_v} \int_0^t a(\tau) e^{-\nu \omega_v (t-\tau)} \sin \omega_v (t - \tau) \, d\tau \quad (7-32) \]

where, \( \omega_v = \omega \sqrt{1 - \nu^2} \)

The maximum displacement response for all \( t \) is called the spectral displacement, denoted here by the symbol \( S_d \), which can be defined as follows:

\[ S_d = \left| x(t, \omega, \nu) \right|_{\text{max}} \quad (7-33) \]

The spectral velocity, denoted as \( S_v \), can then be approximately defined as,

\[ S_v = \omega S_d \quad (7-34) \]

Further, the spectral acceleration of a SDOF system, \( S_a \), may be expressed as,

\[ S_a = \omega^2 S_d \quad (7-35) \]

An approximate and simple numerical integration approach has been developed by Clough (1972) based on a recursive algorithm. However, the accuracy of this numerical integration is not very high especially for a relatively large time increment, \( \Delta t \). A more accurate numerical integration method, which is as simple as Clough’s approach, was proposed and developed during this research. The details of this new approach are presented in Chapter 8.

Based on an earthquake acceleration time history (real or simulated), the maximum response values of a SDOF system can be detected for a given circular frequency, \( \omega \), or the natural period, \( T = 2\pi/\omega \), and damping ratio, \( \nu \), of the system. The response spectra of five artificial earthquakes presented above can be obtained on the basis of Eqs. (7-33), (7-34) and (7-35). These response spectra correspond to six
different damping ratios, viz $\nu = 0\%, 1\%, 5\%, 10\%, 15\%$ and $20\%$ and are shown respectively in Figs. 7-7, 7-8, 7-9, 7-10 and 7-11. These figures reveal that the higher the circular frequency of earthquake excitation, $\omega_0$, the smaller is the magnitude of the response spectrum of a SDOF system.
Displacement Response Spectrum

(a)

Velocity Response Spectrum

(b)

Acceleration Response Spectrum

(c)

Fig. 7-7 Response Spectrums for The Artificial Earthquake - Case 1
Fig. 7-8 Response Spectrums of The Artificial Earthquake - Case 2
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Fig. 7-9 Response Spectrums of The Artificial Earthquake - Case 3
Fig. 7-10 Response Spectrums of The Artificial Earthquake - Case 4
Fig. 7-11 Response Spectrums of The Artificial Earthquake - Case 5
7.4. SUMMARY

On the basis of simulation method proposed by Franklin (1965), a procedure for generating non-stationary random processes associated with a given power spectral density function has been presented in detail. Five artificial earthquakes based on such a simulation procedure were generated under certain assumptions. The simulation results indicate that different earthquake design parameters lead to different artificial earthquakes. For each generated artificial earthquake time history, response spectral analyses were carried out. The results of these analyses showed that the influence of the damping ratio and natural period of a SDOF system on the maximum responses (displacement, velocity and acceleration) of the system is significant for a given earthquake. Moreover, the magnitudes of response spectra of a SDOF system decrease with increase of the natural circular frequency of bed-rock, $\omega_0$. 
CHAPTER 8

DYNAMIC RESPONSE ANALYSIS OF EARTH STRUCTURES AND APPLICATIONS

8.1. GENERAL REMARKS

The design of earthquake-resistant earth structures is of great interest to geotechnical engineers in many parts of the world. Considerable efforts have been made to develop methods for assessing the capability of earth structures to resist earthquake shaking. Under static loading conditions, it is often considered sufficient to achieve a specified factor of safety based on the limit equilibrium concept. Deformations can be minimised effectively with good design followed by effective construction control. However, under earthquake loading, excessive permanent deformations may occur even if overall stability is maintained. Obviously, the magnitude of the permanent displacements mainly depends not only on the earthquake time history but also on the capability of earth structure to resist sliding and on the dynamic properties of materials constituting the earth structure (e.g. natural frequency and damping ratio).

The use of the pseudo-static approach for the analysis of stability under earthquake conditions is common even today. However, the limitations of this approach
are now well understood. The main difficulty, based on pseudo-static approach, has always been associated with selecting a suitable seismic coefficient $K$ for the design of an earth structure. Moreover, the permanent deformations of an earth structure during an earthquake cannot be estimated by this approach. The approach proposed by Newmark (1965), based on a sliding block model, was a significant improvement in this regard. Newmark's approach involves the estimation of a yield acceleration coefficient $K_C$ (also called the critical acceleration coefficient) based on the pseudo-static method. Relative deformation of the potential sliding mass along a potential slip surface is considered to occur if and when the real acceleration coefficient exceeds the critical acceleration coefficient. An idealised time history of earthquake acceleration is selected for simple dynamic analysis, and the permanent deformation of a potential sliding mass can then be estimated by integration. There have been several improvements and extensions of the Newmark approach such as those by Seed and Martin (1966), Ambraseys and Sarma (1967), Sarma (1975), Makdisi and Seed (1978), Chugh (1982), Sarma (1988) as well as Elgamal et al. (1990). However, none of these modifications or improvements simulate the variation of the critical acceleration coefficient $K_C$ during an earthquake.

Embankments or earth dams involving sandy soils may be susceptible to development of high excess pore water pressures during earthquakes. For the analysis of such earth structures, an advanced analysis procedure called the 'Seed-Idriss-Lee approach' was outlined by Seed (1979). A very concise description of this approach has been given in Chapter 2.

A recent review of geotechnical practice and advances with regard to earthquake stability of embankments over the last 25 years (Marcuson et al., 1992) reveals that Newmark-type approaches (Newmark, 1965) and the Seed-Idriss-Lee approach (Seed, 1979) are still considered to be the best available for studying the response of earth structure to earthquakes. There has been further progress in the evaluation of soil properties for use in seismic stability analyses. For instance, attention has been focussed on the undrained 'residual' (post-earthquake, post-liquefaction) strength of sandy soils (Marcuson et al., 1992, Martin, 1992). Nevertheless there still are considerable gaps in
our knowledge of embankment performance during earthquakes. In particular, there is significant scope for development of more sophisticated, consistent and reliable methods of stability analysis.

The further development and extension of the Newmark-type approach have been implemented by Chowdhury and Xu (1993a, b; 1994). A procedure has been developed to simulate the change in the yield acceleration coefficient (or critical acceleration coefficient) $K_c$ as a function of time after the start of an earthquake. The formulation of the problem includes the natural frequency and damping ratio of a soil mass, both of which have been neglected in all previous Newmark-type methods. The new approach presented in this chapter can be used for both 'total stress' and 'effective stress' analyses involving, respectively, effective stress and total stress parameters of shear strength.

The critical acceleration coefficient and the static factor of safety may decrease during an earthquake because of strength loss in some zones of an embankment or an earth dam. The shear strength degradation may be due to either or both of the following:

(a) strain-softening associated with significant relative deformation along a potential slip surface, and

(b) development of excess pore water pressure during an earthquake.

Both mechanisms can be considered in the new approach outlined in this chapter.

8.2. STATE-OF-THE-ART APPROACH FOR SEISMIC DEFORMATION ASSESSMENT

According to Marcuson et. al. (1992) the assessment of deformation induced by earthquakes is not a solved problem and will be the focus of a great deal of research effort in the 1990s. As part of this research, a comprehensive development of the Newmark approach was carried out. This new approach has the following features:

(a) A comprehensive limit equilibrium procedure which can handle slip surfaces of circular and arbitrary shape enabling determination of factor of safety as well as critical acceleration coefficient. The method can be used to locate critical slip
surfaces based either on the minimum factor of safety, $F_{\text{min}}$, or on the minimum critical acceleration coefficient, $\min\{K_c\}$.

(b) A method for generating different time-acceleration histories to simulate earthquakes.

(c) A dynamic response analysis model of the Newmark-type but including natural frequency and damping ratio in the formulations.

(d) An approach for simulating the strength loss due to strain-softening if this mechanism is appropriate for any of the soils forming the earth structure.

(e) An approach for estimating pore pressure development under earthquake conditions based on generated dynamic shear stresses and laboratory test data on the embankment soils for which this mechanism of strength degradation is appropriate.

The approaches used for carrying out steps (a) and (b) have been described in Chapter 3, 4 and 7. The developments and procedures required for steps (c), (d) and (e) are presented in this chapter.

8.2.1. DYNAMIC RESPONSE ANALYSIS

In considering the influence of an earthquake on an earth dam or embankment, it is necessary to consider all aspects of the motion. In other words, the peak acceleration alone may not be significant in determining the response of the earth dam or embankment. Newmark (1963) first proposed that the effects of earthquakes on earth structure stability should be assessed in terms of the deformations caused by earthquake rather than the minimum factor of safety. Further, he presented a method of evaluating the potential deformations of an earth dam or embankment due to earthquake excitation based on this concept and the assumption of rigid plastic materials. Since that time, several refined and modified Newmark-type methods have been proposed for assessing the seismically-induced deformation of earth structures.
8.2.1.1. EQUATION OF MOTION

A typical vertical element from a soil mass with a potential slip surface is shown in Fig. 8-1 along with the internal and external forces. For any time interval when the factor of safety is less than unity (or earthquake acceleration is greater than $K_c g$), the element will slide along the slip surface and will come to rest some time after the earthquake acceleration has become less than $K_c g$. The net permanent displacement of the slice depends not only on the magnitude and duration of the earthquake acceleration but also the dynamic properties of the slice (i.e. natural frequency and damping properties). The equation of motion can be derived by using the principle of dynamic equilibrium.

$$\ddot{X}_i + C \dot{X}_i + K X_i = 0$$

![Fig. 8-1 Forces Involved in the Dynamic Equilibrium of a Typical Slice or Element](image)

Three additional imaginary forces, i.e. the inertial force which equals the product of the mass of the slice, $M_i$, and acceleration of the mass, $\ddot{X}_i$, the damping force which equals the product of the damping coefficient of the material, $C$, and the velocity of mass, $\dot{X}_i$, and the elastic force which equals the product of the stiffness coefficient of the material, $K_s$, and the displacement of the mass, $X_i$, are applied in a direction opposite to that of potential displacement. The directions of all these three imaginary forces are assumed to be parallel to the base of the soil element.
The equilibrium equation of the forces in a direction parallel to the base of the slice may be written as,

\[ K(t) \dot{W}_j \cos (\alpha_i - \theta) + \Delta E_i \cos \alpha_i (W_i + \Delta T_i) \sin \alpha_i - S_i = M_i \ddot{X}_i + C \dot{X}_i + K_s \dot{X}_i \]  
(8-1)

The equilibrium equation of forces in a direction normal to the base of slice can be expressed as,

\[ N_i = (W_i + \Delta T_i) \cos \alpha_i - K(t) W_i \sin (\alpha_i - \theta) - \Delta E_i \sin \alpha_i \]  
(8-2)

Based on the Mohr-Coulomb criterion, the resisting shear force \( S_i \) can be represented as follows:

\[ S_i = c' \Delta X_i \sec \alpha_i + (N_i - u_i \Delta X_i \sec \alpha_i) \tan \phi_i \]  
(8-3)

where, \( u_i \) = pore water pressure.

Eliminating \( N_i \) from Eqs. (8-2) and (8-3), \( S_i \) may be rewritten as,

\[ S_i = c' \Delta X_i \sec \alpha_i - u_i \Delta X_i \sec \alpha_i \tan \phi_i + [(W_i + \Delta T_i) \cos \alpha_i - K(t) \dot{W}_i \sin (\alpha_i - \theta) - \Delta E_i \sin \alpha_i] \tan \phi_i \]  
(8-4)

Substituting Eq. (8-4) into Eq. (8-1), the following equation can be obtained:

\[ M_i \ddot{X}_i + C \dot{X}_i + K_s \dot{X}_i = K(t) W_i \frac{\cos (\alpha_i - \theta - \phi_i)}{\cos \phi_i} + (W_i + \Delta T_i) \frac{\sin (\alpha_i - \phi_i)}{\cos \phi_i} + \Delta E_i \frac{\cos (\alpha_i - \phi_i)}{\cos \phi_i} - (c' \Delta X_i \sec \alpha_i - u_i \Delta X_i \sec \alpha_i \tan \phi_i) \]  
(8-5)

At the instant of limiting or critical equilibrium, \( F = 1 \) and \( K(t) = K_c \), and the initial, conditions of motion are \( \dot{X}_i = \ddot{X}_i = \dot{X}_i = 0 \). Therefore Eq. (8-5) can be rewritten as,

\[ K_c W_i \frac{\cos (\alpha_i - \theta - \phi_i)}{\cos \phi_i} = -(W_i + \Delta T_i) \frac{\sin (\alpha_i - \phi_i)}{\cos \phi_i} - \Delta E_i \frac{\cos (\alpha_i - \phi_i)}{\cos \phi_i} - (c' \Delta X_i \sec \alpha_i - u_i \Delta X_i \sec \alpha_i \tan \phi_i) \]  
(8-6)
By combining Eqs.(8-5) and (8-6) the critical acceleration coefficient $K_c$ may be incorporated in the governing equation of motion while the internal forces $\Delta E_i$ and $\Delta T_i$ are eliminated. Therefore the resulting expression is:

$$M_i \ddot{X}_i(t) + C_i \dot{X}_i(t) + K_i X_i(t) = W_i \frac{\cos (\alpha_i - \theta - \phi_i)}{\cos \phi_i} (K(t) - K_c)$$  \hspace{1cm} (8-7a)$$

or,

$$\ddot{X}_i(t) + 2v \omega \dot{X}_i(t) + \omega^2 X_i(t) = g \frac{\cos (\alpha_i - \theta - \phi_i)}{\cos \phi_i} (K(t) - K_c)$$  \hspace{1cm} (8-7b)$$

where,

$v = \text{damping ratio of the slice material};$

$\omega = \text{natural circular frequency of the slice material};$

$C = 2v \omega M_i;$

$\omega^2 = K_i / M_i.$

Obviously, Eq. (8-7) is a second order inhomogeneous differential equation with constant coefficients. The magnitude of the critical seismic coefficient, $K_c$, in Eq.(8-7) will depend on the particular limit equilibrium method adopted for slope stability analysis. For instance, either the Bishop simplified or the Morgenstern & Price methods may be used to calculate the critical seismic coefficient. If the damping ratio, $v$, and the natural circular frequency, $\omega$, in Eq. (8-7), are ignored, then the form of Eq. (8-7) is same as those derived by Goodman & Seed (1966), Sarma (1975) and Chugh (1978). It is interesting to note that Eq. (8-7) appears to be independent of cohesion, $c'$. However, this is not so since cohesion is included in the critical seismic coefficient, $K_c$. It should be noted the magnitude of $K_c$ will change during an earthquake. The change in $K_c$ during an earthquake is a consequence of decreased shear strength which is associated with mechanisms as mentioned in Section 8.1.

In fact, Eq. (8-7) represents the vibration equation of a SDOF system. For studying a SDOF system, the damping ratio, $v$, and the circular frequency, $\omega$, are very important dynamic properties. The earthquake energy in an actual system may be continually lost due to damping of the system. However, the previous Newmark-type methods did not include these two dynamic parameters of the system. Therefore, they
include no energy loss mechanism. The lost earthquake energy has an important effect in reducing the response to earthquake excitation. Thus the influence of damping ratio, \( \nu \), and natural circular frequency, \( \omega \), should be considered and this can be done by including the relevant terms in the governing equation of motion as shown above.

8.2.1.2. NUMERICAL CALCULATION OF DUHAMAL INTEGRATION - NEW APPROACH

The response of the differential equation (8-7) to an input earthquake excitation can be given by the Duhamel integral, i.e.,

\[
X_i(t) = \frac{1}{\omega_v} \int_0^t A(\tau) e^{-\nu \omega t} \sin \omega_v (t - \tau) \, d\tau
\]  

where,

\[
A(\tau) = g \frac{\cos (\alpha_i - \theta - \phi_i)}{\cos \phi_i} [K(\tau) - K_c] ;
\]

\[
\omega_v = \omega \sqrt{1 - \nu^2}.
\]

The integral of Eq. (8-8) may be calculated analytically if and when \( A(t) \) can be expressed mathematically. Because \( A(t) \) is related to the any arbitrary loading history or earthquake excitation, i.e. \( K(t) \) cannot be expressed by a mathematical formulation, it is necessary to use a numerical integration process for evaluating the magnitude of the response. Many numerical procedures are available for this purpose, such as simple summation method, trapezoidal quadrature, Simpson’s method, etc. A simple and approximate procedure based on recursive algorithm has been proposed by Clough (1970). A procedure was developed which proved to be more accurate than Clough’s but as simple in its implementation. This procedure is now presented below:

Using the following trigonometric identity,

\[
\sin \omega_v (t - \tau) = \sin \omega_v t \cos \omega_v \tau - \cos \omega_v t \sin \omega_v \tau
\]

Eq. (8-8) can be rewritten in following form:

\[
X_i(t) = \frac{1}{\omega_v} \{ \sin \omega_v t \int_0^t A(\tau) e^{-\nu \omega (t - \tau) \cos \omega_v \tau} \, d\tau
\]
- \cos \omega_\tau \int_0^t A(\tau) e^{-\nu_\omega(t-\tau)} \sin \omega_\tau \, d\tau \tag{8-9a}

or,

\begin{align*}
X_i(t) &= \frac{1}{\omega_\nu} \{ \bar{A}(t) \sin \omega_\nu t - \bar{B}(t) \cos \omega_\nu t \} \\
\tag{8-9b}
\end{align*}

When the dynamic response is calculated at equal intervals \(\Delta \tau\), the modified recursive formulas, based on the closed Newton-Cotes numerical integral algorithm, can be obtained for computing \(\bar{A}(t)\) and \(\bar{B}(t)\), i.e.,

\begin{align*}
\bar{A}(t) &= \bar{A}(t-\Delta \tau) e^{-\nu_\omega \Delta \tau} + \frac{\Delta \tau}{2} \left[ A(t-\Delta \tau) e^{-\nu_\omega \Delta \tau} \cos \omega_\nu (t-\Delta \tau) + A(t) \cos \omega_\nu t \right] \\
\bar{B}(t) &= \bar{B}(t-\Delta \tau) e^{-\nu_\omega \Delta \tau} + \frac{\Delta \tau}{2} \left[ A(t-\Delta \tau) e^{-\nu_\omega \Delta \tau} \sin \omega_\nu (t-\Delta \tau) + A(t) \sin \omega_\nu t \right] \\
\tag{8-10}
\end{align*}

It can be seen that if the integral boundaries are from \((t - \Delta \tau)\) to \(t\), then the error of Clough's method is \(O[\Delta \tau^2 f'(t)]\) and that of the modified method is \(O[\Delta \tau^3 f''(t)]\) in which \(f(t)\) is an integrated function. This implies that the accuracy of the modified method is much higher than that of the method proposed by Clough. An illustrative example is shown below for calibrating the precision of the new recursive integral formula and of Clough's approach. Assume that a function \(F(t)\) can be expressed as follows:

\[ F(t) = \int_0^t e^{-\nu_\omega (t-\tau)} \cos \omega \tau \, d\tau \]

The function \(F(t)\) can be analytical formulated by integrating the above formulation in time interval [0, \(t\)], i.e.

\[ F(t) = \frac{1}{\omega (1 + \nu^2)} \left[ \sin \omega t + \nu \cos \omega t - \nu e^{-\nu_\omega t} \right] \]

After values of \(\omega\) and \(\nu\) are given, the function \(F(t)\) at different times \((t = t_1, t_2, \ldots, t_n)\) can be accurately calculated by the above formula. Assume the following values:

\[ \Delta \tau = 0.01 \text{sec}, \, \omega = 18, \, \nu = 0.15 \]

The relative errors between numerical solutions based on Clough recursive method and new recursive method as well as accurate solutions computed by the above formula can be obtained at different points of time (e.g. \(t = 0.5, 1.0, 1.5, \ldots, 10\text{sec}\)).
These relative errors are shown in Fig. 8-2. From Fig. 8-2 it is evident that the relative errors between new recursive method and accurate calculation are almost zero but that between Clough recursive method and accurate calculation are identical only at several points of time and the fluctuation of relative errors with time is significant.

![Comparison of Results from Clough’s Method and proposed New Recursive Method with the Accurate Solution.](image)

8.2.2. SIMULATIONS OF SHEAR STRENGTH DEGRADATION

The degradation of the material shear strength may be caused by either or both of the mechanisms mentioned in section 8.1. These mechanisms are (a) strain-softening associated with significant relative deformation and (b) development of excess pore water pressure associated with earthquake shaking. The cumulative permanent displacement of the slide mass may be significantly influenced by decrease in shear strength. It is, therefore, necessary to simulate this decrease as a function of time after the start of earthquake motions. The displacement of a potential slide mass depends on the earthquake time history, i.e. $K(t)$, and also on the critical seismic coefficient, $K_c$. Because $K_c$ is a function of the shear strength parameters, $c'$ and $\phi'$, as well as pore
water pressure, \( u \), its magnitude changes with any change in these parameters. Any decrease of \( K_c \) will lead, in turn, to an increase of the relative deformation of the soil mass. Such a decrease may result in further strength loss if the soil mass exhibits strain-softening behaviour.

The development of excess pore water pressure will occur in a saturated soil mass and will depend on the type of soil as well as the characteristics of the earthquake motion. Essentially, developed excess pore water pressure is a function of the cyclic shear stress as well as the equivalent number of cycles. The process of pore pressure development is quite complex under dynamic loads and one has to rely on data from appropriate laboratory studies before any assessment can be carried out.

8.2.2.1. SIMULATION OF STRAIN -SOFTENING

The post-peak shear strength of a soil depends on the level of shear strain. It is well known that significant relative deformation along a slip surface is required to reduce the shear strength to a residual value. In this chapter, the peak and residual values of cohesion and internal friction coefficient may be denoted by \((c_p, \tan \phi_p)\) and \((c_r, \tan \phi_r)\) respectively. Define the following ratios

\[
R_c = \frac{c_p - c_r}{c_p}, \quad R_{\tan \phi} = \frac{\tan \phi_p - \tan \phi_r}{\tan \phi_p}
\]

Let \( X_m \) represent the magnitude of relative deformation required to reduce the strength parameters to residual values and \( X \) be any deformation smaller than \( X_m \). An appropriate type of function can be selected so that the shapes of the curves describing post-peak strength degradation can be changed by changing a single parameter.

The proposed strength decrease equations including the chosen function of deformation ratio \((X/X_m)\) are presented for cohesion and friction coefficient as follows

\[
c = c_p \left[ 1 - f(X/X_m) \ R_c \right]
\]
\[
\tan \phi = \tan \phi_p \left[ 1 - f(X/X_m) \ R_{\tan \phi} \right]
\]

where,
\[ f(X/X_m) = 1 - \exp\left( (X/X_m)^\alpha \ln \bar{a} \right) \]

in which, \( \bar{a} \) is a small quantity (such as 0.001 or 0.0001); the smaller the value of \( \bar{a} \), the closer do \( c \) and \( \phi \) approach \( c_r \) and \( \phi_r \) at \( X = X_m \). Increasingly brittle, post-peak behaviour can be modelled by \( \alpha \) approaching zero whereas perfectly plastic behaviour can be modelled as \( \alpha \) approaches infinity.

As an example, assume the peak and residual values of cohesion and internal friction coefficient to be \( c'_p = 20 \text{ KN/m}^2 \), \( c'_r = 5 \text{ KN/m}^2 \), \( \tan \phi'_p = 0.6428 \) and \( \tan \phi'_r = 0.2679 \) respectively, the shapes of curves of post-peak cohesion and friction coefficient for different values of parameter \( \alpha \) and \( X_m = 200 \text{ cm} \) are then shown in Figs. 8-3 and 8-4. The figures show that \( \alpha = 0.5 \) represents a brittle soil whereas \( \alpha = 3 \) indicates a relatively non-brittle soil.

![Fig. 8-3 Post-Peak Cohesion with Different \( \alpha \)](image-url)
8.2.2.2. SIMULATION OF EXCESS PORE WATER PRESSURE

Much has been written about the generation of excess pore water pressures under static loading and Skempton's pore pressure coefficients A and B are widely used for estimation of transient pore pressure under static loading conditions. Under dynamic loading the generation of pore water pressures has been studied mainly from laboratory tests. The influences of initial stress conditions, the stress level of cyclic loading and the number of cycles of loading have also been explored by many researchers. No simple equation relating the increase of pore water pressure to the increase of shear stresses during earthquakes is available. However, Sarma (1988) has proposed the determination of a dynamic pore pressure parameter $A_n$ similar to the Skempton coefficient A for static loading.

The following relationships have been proposed relating excess pore water pressure $\Delta u$ to increments of either the major principal stress $\Delta \sigma_1$ (triaxial loading conditions) or the shear stress $\Delta \tau$ (simple shear conditions) during dynamic loading of saturated soil:

$$\frac{\Delta u}{\sigma_{3c}} = A_n \frac{\Delta \sigma_1}{\sigma_{3c}}$$  (8-12)
where $\sigma_{3c}$ is the minor principal effective consolidation stress and $\sigma'_{v0}$ is the effective vertical overburden stress.

By re-plotting published data concerning the results of cyclic loading tests on different sands, linear relationships were found to exist between $A^1/2_n$ and log$(n)$ (where $n$ is the number of cycles) for different values of the ratio $\frac{\Delta \tau}{\sigma'_{v0}}$ or $\frac{\Delta \sigma_{1}}{2 \sigma_{3c}}$. The following equation was proposed for $A_n$ by Sarma (1988)

$$A^1/2_n = A^1/2_1 + \beta \log n$$

(8-14)

where, $A_1$ is the parameter for one cycle of loading and $\beta$ is a constant for a particular soil.

By using Sarma's suggested definition of $A_n$ the variation of $A_n$ during the time history of an earthquake may be determined if a concept such as that of 'the number equivalent uniform stress cycles' is invoked. If results of pore pressure development from more sophisticated tests (such as shaking table tests) are available, the variation of $A_n$ during an earthquake may be determined even more realistically.

During an earthquake, the excess pore pressures are primarily generated as a consequence of the increments in shear stress due to dynamic loading. Therefore, at any time $t$, the cumulative excess pore water pressure may be related to the time-average of the maximum shear stress increment as follows:

$$\Delta u(t) = A_n(t) \{[\Delta \sigma_{1}(t) - \Delta \sigma_{3}(t)]_{\text{average}}\}$$

(8-15)

It is important to understand that $A_n$ is a parameter relevant to the cumulative excess pore water pressure and not to the development of pore water pressure for the very small time interval between time $t$ and $(t + \Delta t)$. The increments in principal stresses with reference to the initial principal stresses fluctuate dramatically with time due to the nature of earthquake motion. Therefore, the best approach for calculating the cumulative excess pore water pressure is to use the averaging process as proposed above in Eq. (8-
15) for the increment of shear stress at any time. The time-average of the maximum shear stress increment is, therefore, always calculated considering the reference stresses as those at the start of the earthquake.

For calculation of principal stress increments required in Eq. (8-15), the following hypothesis is introduced. If $\sigma'_{m}$ and $\tau'_{m}$ are the mobilized effective normal and shear stresses on a possible failure plane, then the state of stress at any point along the base of the slice will be the same as if the friction angle of the material is $\phi'_{m}$ and cohesion is $c'_{m}$ (when $c'_{m} = \frac{c}{F}$ and $\tan \phi'_{m} = \tan \phi'_{m}$). With this hypothesis, it is possible to draw the Mohr's circle of stresses. The circle will pass through the point $(\sigma'_{m}, \tau'_{m})$ and will be tangent to the strength envelope inclined at an angle $\phi'_{m}$ at the point. For a given failure surface the direction of the mobilized effective principal stresses will change during the application of the mobilized effective stresses $(\sigma'_{m}, \tau'_{m})$. The Mohr circle of stress under the condition of the static limit equilibrium can be drawn as shown in Fig. 8-5. From the geometry of the circle, one has following relationships,

\[
\begin{align*}
    r &= \tau'_{m} \sec \phi'_{m} \\
    BE &= \tau'_{m} \tan \phi'_{m}
\end{align*}
\]  

From Eq. (8-16) and Fig. 8-5, the mobilized effective principal stresses $(\sigma'_{1m}, \sigma'_{3m})$ can be expressed as follows:

\[
\begin{align*}
    \sigma'_{1m} &= \sigma'_{m} + \tau'_{m} \left( \tan \phi'_{m} + \sec \phi'_{m} \right) \\
    \sigma'_{3m} &= \sigma'_{m} + \tau'_{m} \left( \tan \phi'_{m} - \sec \phi'_{m} \right)
\end{align*}
\]  

Therefore, the initial mobilized effective principal stresses $(\sigma'_{1m}(0), \sigma'_{3m}(0))$ can be calculated from Eq. (8-17) after the factor of safety is computed under the static condition. When a value of acceleration coefficient based on the acceleration time-history, $K(t)$, is introduced to represent a transient earthquake force in the pseudo-static analysis, the factor of safety will certainly be changed. This change of the factor of safety will then result in a new state of the mobilized effective principal stresses along a potential failure
Chapter 8: Dynamic Response Analysis of Earth Structures And Applications

![Diagram of Mohr's Circle of Stresses for the Static limit](image)

The subscript 'm' represents the mobilized values:

\[ c'_m = \frac{c'}{F} \quad \text{and} \quad \phi'_m = \tan^{-1} \left( \frac{\tan \phi'}{F} \right) \]

\[ \frac{c'_m}{\tan \phi'_m} = \frac{c'}{\tan \left[ \tan^{-1} \left( \frac{\tan \phi'}{F} \right) \right] / \tan \phi'} \]

Fig. 8-5 Mohr's Circle of Stresses for the Static limit

Equilibrium Condition

If these new values of the mobilized effective principal stresses are designated as \( (\sigma'_{1m}(t), \sigma'_{3m}(t)) \), then the relative increment of the mobilized principal stresses with respect to initial principal stresses are:

\[
\begin{align*}
\Delta \sigma_{1m} &= \sigma'_{1m}(t) - \sigma'_{1m}(0) \\
\Delta \sigma_{3m} &= \sigma'_{3m}(t) - \sigma'_{3m}(0)
\end{align*}
\]

Excess pore water pressure \( \Delta u \) at time \( t_i \) can then be computed based on Eq. (8-15) by averaging all individual values of \( \Delta \sigma_{1m} \) and \( \Delta \sigma_{3m} \) between \( t = 0 \) and \( t = t_i \).

8.2.3. ASSESSMENT OF DISPLACEMENT

In order to simplify the calculation procedure for displacements, the changes of the geometric configuration of the slide mass are ignored. The movement of the slide mass is not along a single inclined plane; the movement takes place along planes of different inclination. The inclination would also vary somewhat with displacement and
the geometric shape and size of the slide mass will change after the displacements occur. These changes will influence the values of the critical seismic coefficient $K_c$ as well as the geometric parameters to be used in the equation of motion. If changes in geometry are included the modelling and analysis will be too complex. Therefore, it is sensible to ignore changes in geometry and to deal separately with each slice or segment with a single inclination of slice base.

For a time history of acceleration of the real or design earthquake, the relative displacement of the slide mass begins to increase when the earthquake acceleration exceeds the critical acceleration ($K_{cg}$). The relative displacement of the slide mass will continue to increase until the relative velocity become zero. At this time, the relative displacement reaches a maximum value. The movement of the slide mass will then stop. If and when the acceleration of the earthquake again exceeds the critical acceleration ($K_c'$) (the use of notation $K_c'$ means that its value is different from previous value of $K_c$) at another time instant, the relative displacement of slide mass again begins to increase. This process can be illustrated as in Fig. 8-6.

![Fig. 8-6 Illustrative Sketch Showing Different Increments of Permanent Displacement as a Function of Time](image-url)
Therefore, the total displacement of the slide mass can be calculated as follows

\[ D = \sum_{i=1}^{n} d_{\text{max}}^{i} \]  

(8-19)

The numerical simulation process is very complex because consideration has to be given to any decrease of shear strength which may occur with time. Iterative limit equilibrium analyses have to be carried out to calculate new values static factor of safety \( F \) and critical seismic coefficient \( K_c \). The process has to be repeated for successive time steps and each time step has to be very small in order to properly discretise the acceleration time-history of the earthquake. The process may be understood from a flowchart as shown in Fig. 8-7.
Input geometrical and shear strength parameters as well as initial pore water pressure

Input earthquake acceleration time history

Calculate static factor of safety and critical seismic coefficient

Time: \( t_j = t_{j-1} + \Delta t \), \( j = 1, m \)

Calculate the excess pore water pressure induced by earthquake based on the proposed approach.

Calculate the displacement of all slices associated with the response analysis procedure proposed by this research work

Displacement occur

Yes

Calculate new shear strength parameters based on the proposed model of post-peak deformation

Calculate new values of \( F \) and \( K_c \)

No

\( j = m \)

Yes

Stop

Fig. 8-7 Flowchart of Dynamic Response Analysis
8.3. APPLICATIONS OF THE PROPOSED APPROACHES

The analysis procedure proposed in the previous sections was applied to carry out a number of case studies. The influence the dynamic properties of soil and of various degrees of strain-softening on the permanent deformations was investigated. Moreover, for soil susceptible to development of excess pore water pressure during earthquake shaking separate simulation studies were made.

8.3.1. INFLUENCE OF STRAIN SOFTENING OF SOIL ON DISPLACEMENTS OF SLICE

The response of an earth structure depends on the geotechnical properties and, in particular, on the shear strength parameters. If soil shear strength decreases during an earthquake, the deformations will be higher than those experienced by an earth structure in which soil shear strength does not decrease. Therefore, strain-softening characteristics of embankment soils must be considered in assessing the earthquake response. It is well known that shear strength decreases from a peak to a residual value with increasing relative deformation along a slip surface. Therefore, the nature of post-peak decrease in shear strength is important in addition to the magnitude of decrease in shear strength. The decrease will be abrupt for brittle soils and gradual for non-brittle soils. A simple model has been proposed in the previous section to simulate the post-peak soil behaviour.

8.3.1.1. PROBLEM GEOMETRY, SOIL PARAMETERS AND OTHER DETAILS

A homogeneous embankment is considered (Fig.8-8) with undrained soil strength parameters and other data shown in Table 8-1. The value of brittleness parameters $\alpha = 0.5$ in Table 8-1 represents a relatively brittle soil and $\alpha = \infty$ a perfectly-plastic soil. The parameter $X_m$ in Table 8-1 is the magnitude of $X$ required to reduce shear strength to residual and $X_m = 2m$ is chosen in this analysis. The embankment is subjected to an earthquake which is similar to the time-history used by Seed and his co-workers for the analysis of the failure of the lower and upper San Fernando dams during the San Fernando earthquake of 1971 (Seed, 1979). Such a time-acceleration history may be
generated using the simulation procedure introduced in Chapter 7. The time-acceleration
history is shown in Fig.8-9 in terms of the acceleration coefficient.

Table 8-1. Soil Properties And Assumed Data

<table>
<thead>
<tr>
<th>Unit Weight $\gamma \text{(kN/m}^3\text{)}$</th>
<th>Cohesion $c_p$, $c_r \text{(kN/m}^2\text{)}$</th>
<th>Friction Angle $\phi_p$, $\phi_r \text{(°)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.5</td>
<td>57.5</td>
<td>27</td>
</tr>
</tbody>
</table>

$X_m = 2m$  $\alpha = 0.5, 0.8, 1.5, \infty$

Natural Period $T = \frac{2\pi}{\omega} = 1.5 \text{ sec.}$

Damping Ratio $\nu = 0.01$

Note: sensitivity analyses with different values of $T$ and $\nu$ are excluded from this section.

Fig.8-8 A Homogeneous Embankment with A Critical Slip Surface of Arbitrary Shape
It is assumed that the embankment material is such that excess pore pressures will not be generated during dynamic loading. As emphasised by Seed (1979), clays, clayey soils and some dense cohesionless materials are not susceptible to significant pore pressure increase during earthquakes. Two alternative cases were used for analysis, namely,

a) Embankment without reservoir (no water retained)

b) Embankment with reservoir (submerged slope)

8.3.1.2. RESULTS OF ANALYSIS

The shapes of the curves of post-peak cohesion for different values of the parameter $\alpha$ are shown in Fig.8-10, with $\alpha = 0.5$ indicating a brittle soil and $\alpha = 1.5$ representing a material with relatively low brittleness. Similar curves can be plotted for the post-peak values of the internal friction coefficient, $\tan \phi$.

The potential sliding mass was divided into 15 vertical slices numbered consecutively from 1 at the toe to 15 at the crest (1 to 13 for analyses with reservoir).
Considering an embankment with no water in the reservoir the variation of F and K_c with time is shown for an embankment of non-brittle soil ($\alpha = 1.5$) in Fig. 8-11. Even at the end of the earthquake the factor of safety remains significantly higher than one. The response of an embankment of brittle soil ($\alpha = 0.5$) is shown in Fig. 8-12. In this case the values $F = 1$, $K_c = 0$ are reached after only 4.8 seconds of earthquake motion (The corresponding time with $\alpha = 0.8$ was found to be 8.6 seconds and with $\alpha = 1.0$ to be 13.1 seconds).
The cumulative permanent deformation as a function of time for a typical slice (slice 5) is shown for different values of $\alpha$ in Fig. 8-13. The significant influence of the value of $\alpha$ is evident from this figure. For each value of $\alpha$ the position of the vertical arrow shown in Fig. 8-13 indicates the time when $K_c = 0$ which may be called the 'critical time' and catastrophic failure is thus predicted at that time based on limit equilibrium concepts. However, based on the dynamic response model, deformation can still be calculated for increasing time. In this paper $K_c$ is assumed to have a zero value beyond this 'critical time'. If, however, the implied negative values of $K_c$ were used, estimated deformations would be much higher than those shown.
Embankment With Reservoir

The influence of soil brittleness can again be considered with different values of $\alpha$. With $\alpha = 1.5$, the variation of $K_c$ and $F$ with time was found to be insignificant. However, a significant decrease of both parameters can be noted if the embankment soil is assumed to be relatively brittle with $\alpha = 0.5$ (Fig. 8-14). Yet, overall stability is maintained because $F > 1$, $K_c > 0$ at the end of earthquake shaking.

The cumulative permanent deformation is shown as a function of time in Fig. 8-15. It is interesting to note that there is negligible deformation for the first 7.5 seconds of the earthquake and even after that time the deformation is insignificant for all values of $\alpha$ except $\alpha = 0.5$ representing a relatively brittle material.
8.3.1.3. DISCUSSIONS

In this section, the applications of the proposed new, Newmark-type approach have been presented for assessing the response of an embankment to earthquakes. On the basis of the proposed strain softening model and the dynamic response analysis procedure, the shear strength of embankment materials during earthquake excitation was
first simulated. The time histories of critical acceleration coefficient, $K_c$, and the static factor of safety, $F$, can be obtained by combining such a simulation procedure with the limit equilibrium analysis. Soil brittleness has a significant influence on the permanent displacements of the embankment, critical acceleration coefficient and the static factor of safety. However, disregarding brittleness of embankment materials can lead to significant under-estimates of embankment deformations. Therefore, the proposed approach can be potentially very useful in realistic assessment of embankment response during earthquakes.

8.3.2. INFLUENCE OF EXCESS PORE WATER PRESSURE ON DISPLACEMENTS

Some types of soil such as dry sands and dense saturated sands do not lose significant resistance to deformation as a result of earthquake loading. However, saturated sands of low to medium density are susceptible to contraction during dynamic loading. Consequently, excess pore water pressures will generally develop in such soils during dynamic loading. Local excess pore pressure will, of course, redistribute within the earth structure during or after an earthquake. The simulation procedure of excess pore water pressure during an earthquake has been proposed in Section 8.2.2.2. The applications of such a procedure incorporated in a seismic response analysis is presented below.

8.3.2.1. BASIC INFORMATION OF CASE STUDIES

The method for simulation of excess pore water pressure, developed in the previous section, was used to carry out analyses of the Lower San Fernando Dam which has previously been studied by Seed and his co-workers (Seed, et al., 1975). The embankment cross-section is shown in Fig. 8-16 along with one of the observed slip surfaces. The simulated earthquake acceleration-time history used for the analyses was the same as shown in Fig. 8-9. The soil properties used for the determination of the values of factor of safety and critical seismic coefficient are shown in Table 8-2.
8.3.2.2. RESULTS OF CALCULATION

The variation of generated excess pore water pressures with time for three typical slices is shown in Fig. 8-17 based on the assumed values of dynamic pore water pressure coefficient, $A_1 = 0.67$ and $A_5 = 6$. This means that the number of equivalent uniform cycles is considered to be $n = 5$ and that $A_n$ has a value varying from 0.67 to 6 during the 15 second period of earthquake motion; the value is considered at be constant ($A_1 = 0.67$) over the first 3 seconds. Fig. 8-17 shows that the time of the excess pore water pressure exceeding the initial effective normal stress is different for the different slices. The variation of the factor of safety with time and the variation of the critical seismic

Table 8-2  Soil Properties Used in Analysis

<table>
<thead>
<tr>
<th>Material No.</th>
<th>$c_u$ (kN/m²)</th>
<th>$\phi_u$</th>
<th>$\gamma$ (kN/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>57.5</td>
<td>20°</td>
<td>17.3</td>
</tr>
<tr>
<td>2</td>
<td>57.5</td>
<td>20°</td>
<td>16.0</td>
</tr>
<tr>
<td>3</td>
<td>103</td>
<td>0</td>
<td>19.0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>33°</td>
<td>20.0</td>
</tr>
<tr>
<td>5</td>
<td>103</td>
<td>0</td>
<td>19.0</td>
</tr>
</tbody>
</table>

Note: These properties are taken from the paper by Lee et. al., 1975
coefficient with time are shown in Fig. 8-18 for $A_1 = 0.67$ and $A_5 = 6.0$. The shapes of the curves of $K_c$ and $F$ with time are different from the corresponding curves in Fig. 8-11 or 8-12. The decrease of $K_c$ and $F$ with time is discontinuous when the mechanism of strain-softening is invoked but the decrease is continuous in Fig. 8-18. This is because excess pore water pressure develops continuously during in the acceleration time history.

![Graph](image)

**Fig. 8-17** Variation of Excess Pore Water Pressure with Time for $A_1 = 0.67$ and $A_5 = 6.0$

For different values of $A_5$, the estimated maximum and minimum displacements are shown in Table 8-3 along with the number of the relevant slice (in brackets). The number of total slices is 20 and slices 19 and 20 are located near the top (crest) of the slope. The fact that adjacent slices show maximum and minimum displacement implies that a tension crack would have developed between them, i.e. near the crest of the slope.
Fig. 8-18 Variation of Factor of Safety and Critical Seismic Coefficient with Time for $A_1 = 0.67$ and $A_5 = 6.0$

<table>
<thead>
<tr>
<th>$A_n = A_5$</th>
<th>Ave. Disp. (cm)</th>
<th>Max. Disp. (cm) (slice No.)</th>
<th>Min. Disp. (cm) (slice No.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.67</td>
<td>5.28</td>
<td>6.64 (19)</td>
<td>4.35 (20)</td>
</tr>
<tr>
<td>3</td>
<td>9.04</td>
<td>11.07 (19)</td>
<td>7.45 (20)</td>
</tr>
<tr>
<td>5</td>
<td>19.82</td>
<td>24.25 (19)</td>
<td>16.32 (20)</td>
</tr>
<tr>
<td>6</td>
<td>37.01</td>
<td>45.45 (19)</td>
<td>30.59 (20)</td>
</tr>
</tbody>
</table>

8.3.2.3. DISCUSSIONS

Based on the new approach for earthquake response analysis outlined in the previous section, an example illustrating the simulation model of excess pore water pressure has been presented above. It has been shown that the magnitudes of critical seismic coefficient and the factor of safety of an embankment reduce significantly and continuously during an earthquake and may reach critical values if the soil shear strength decreases sufficiently. This can happen in soils such as saturated sands which develop significant pore pressures during dynamic loading. From the results presented in Table
8-3 it is clear that displacements are quite small at a value of $A_5 = 1.67$. The displacements do increase with an increase in the value of $A_5$ but, even with $A_5 = 6$, the maximum displacement is only about 46cm. With such a high value of $A_5$ the critical seismic coefficient does eventually reach zero almost towards the end of the earthquake. Therefore, a slide could be expected to occur only if material behaviour was known to correspond to this high value of $A_5$. However, values of $A_5$ in the range (1.35-2.58) have been interpreted by Sarma (1988) on the basis of available data concerning the upper and lower San Fernando dams. It is obvious that failure is not predicted on the basis of undrained shear strength parameters used for static and pseudo-static analyses of these dams before the earthquake.

These results confirm that failure of the Lower San Fernando dam during the San Fernando earthquake could not have been predicted on the basis of undrained shear strength parameters (for the sands) which were used for static and pseudo-static analysis during the design stages.

Further work has been carried out based on effective shear strength parameters but the details of this additional work are outside the scope of this thesis.

8.3.3. INFLUENCE OF DYNAMIC PROPERTIES ON DISPLACEMENTS OF SLICES

In considering the response of a system to dynamic loading, the damping ratio ($\nu$) and the natural circular frequency ($\omega$) of the system are important dynamic properties. The previous dynamic response analysis methods based on Newmark-type models do not involve these two dynamic parameters. Therefore, they include no energy loss mechanism. An actual system continually absorbs earthquake energy during motion, and the earthquake energy loss has an important effect in reducing the response to earthquake excitation. The influence of damping ratio ($\nu$) and natural frequency ($\omega$) on the permanent deformations of an earth structure during an earthquake will be examined in this section.
8.3.3.1. Illustrative Example

In order to show the significant influence of natural frequency and damping ratio on the permanent deformations of an embankment during an earthquake, a homogeneous embankment without reservoir is considered again. The soil properties are the same as shown in Table 8-1. A simulated acceleration-time history, presented in Fig. 8-9, was again considered as the input earthquake excitation.

The calculations of permanent embankment displacement were made for a range of values of the natural circular frequency \( \omega \) (the natural period \( T \) varying from 0 to 2 seconds where \( T = \frac{2\pi}{\omega} \)). The damping ratio \( \nu \) was assumed to vary in the range 0.01 to 0.6.

The variation of permanent deformation with natural period and damping ratio is shown for vertical slice No. 2 in Fig.8-19 for an assumed value of \( \alpha = 0.5 \) and in Fig.8-20 for an assumed value of \( \alpha = 0.8 \). Both values represent brittle soils, the latter representing a somewhat less brittle soil than the former.

For two damping ratio \( \nu = 0.05 \) and \( \nu = 0.3 \), the influence of soil brittleness on displacements of slice No. 15 is shown for a range of values of the natural period in Tables 8-4 and 8-5, respectively.

---

**Fig. 8-19** Permanent Deformations of Slice No.2 As A Function of Natural Period \( T \) And Damping Ratio \( \nu \) (\( \alpha = 0.5 \))
Fig. 8-20 Permanent Deformations of Slice No.2 As A Function of Natural Period T And Damping Ratio \( \nu \) (\( \alpha = 0.8 \)).

### Table 8-4 Displacement of Slice No. 15 with Embankment Soils of Different Brittleness And \( \nu = 0.05 \)

<table>
<thead>
<tr>
<th>Brittleness Parameter</th>
<th>Natural Period-T (sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>33.65 (cm)</td>
</tr>
<tr>
<td>0.8</td>
<td>15.97 (cm)</td>
</tr>
<tr>
<td>2.5</td>
<td>5.46 (cm)</td>
</tr>
</tbody>
</table>

### Table 8-5 Displacement of Slice No. 15 with Embankment Soils of Different Brittleness And \( \nu = 0.3 \)

<table>
<thead>
<tr>
<th>Brittleness Parameter</th>
<th>Natural Period-T (sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0.5</td>
</tr>
<tr>
<td>0.5</td>
<td>23.38 (cm)</td>
</tr>
<tr>
<td>0.8</td>
<td>8.86 (cm)</td>
</tr>
<tr>
<td>2.5</td>
<td>4.04 (cm)</td>
</tr>
</tbody>
</table>
8.3.3.2 Discussions

For a given value of $\alpha$, both the natural circular frequency, $\omega$, and damping ratio, $\nu$, of soil materials have a significant influence on embankment deformations. The higher the natural frequency the less the deformation. Meanwhile, the less the damping ratio the larger the deformation. This is because the less the damping ratio of an embankment the less the earthquake energy loss in the embankment.

From the studies reported in this section it is very clear that calculated displacements would be quite inaccurate and even misleading if natural frequency, damping ratio and soil brittleness are ignored. By assuming zero values of natural frequency and damping ratio, the displacements are overestimated. By assuming soil of zero brittleness ($\alpha = \infty$) the displacements are underestimated.

8.4. SUMMARY

1. A new comprehensive procedure for estimating the response of earth structures to earthquakes has been proposed by combining the limit equilibrium analysis with the proposed new approach, a Newmark-type dynamic analysis. The dynamic response equation based on such a procedure includes the critical seismic coefficient, $K_c$. It also includes the dynamic properties of the earth materials i.e., the natural frequency, $\omega$, and damping ratio, $\nu$. Any arbitrary acceleration-time history representing a real or simulated earthquake can be considered for use with this new approach.

2. The proposed analysis procedure has been developed to take into consideration the decrease of the shear strength of embankment materials during earthquake shaking. In order to simulate such a shear strength degradation, two mechanisms have been proposed. The first one is the strain-softening mechanism and the second one is the development of excess pore water pressure due to earthquake shaking. For the proposed model to simulate strain-softening, soils different brittleness can be considered by varying the shape parameter, $\alpha$, of the post-peak curve of shear stress versus shear deformation.
Based on the equation proposed by Sarma (1988) for determining a dynamic pore pressure parameter, $A_n$, a model for simulating the excess pore water pressure caused by earthquake shaking has also been proposed and a comprehensive procedure has been developed to consider the consequent changes in shear strength and factor of safety with time.

3. Considering either one of the mechanisms of shear strength decrease due to earthquake shaking, the critical seismic coefficient, $K_c$, is no longer a constant over the period of shaking. The proposed models can simulate this variation of both $K_c$ and FOS with time. This is the major advance over all previous Newmark-type models.

4. A new numerical procedure has been proposed for estimating the Duhamel integral and this new procedure has been shown to be more accurate than the numerical approach originally proposed by Clough. Because the new integral method is associated with a recursive algorithm, the calculation procedure is simpler than alternative numerical methods.

5. Examples of analysis have been presented to illustrate the models and methods proposed. The calculation results indicate that the brittleness of soil, i.e. the shape of the post-peak curve showing degradation of shear strength, has a significant influence on embankment deformations. Relatively brittle soils (low values of parameter $\alpha$) lead to large deformations whereas relatively non-brittle soils (high values of parameter $\alpha$) lead to small deformations. It is evident that the influence of the development of excess pore water pressure on deformations of an earth structure is significant. The influence on deformations of the magnitude of the dynamic pore pressure parameter, $A_n$, is shown to be dominant by using the proposed simulation model for development of excess pore water pressure. Therefore, accurate or reliable determination of this parameter, $A_n$, is important. The calculation results also show that the influence of the dynamic properties of earth structure namely, the natural frequency and the damping ratio, on permanent deformations is significant.
CONCLUSIONS

9.1. INTRODUCTION

The research work presented in this thesis is mainly concerned with the stability and reliability assessments of earth structures under static and dynamic loading conditions. On the basis of two-dimensional limit equilibrium approaches, which are associated with the simplified and 'rigorous' models, comprehensive analysis procedures and the relevant programs have been developed for the deterministic and probabilistic design of earth structures under static loading conditions. Moreover, an innovative and comprehensive analysis procedure based on combining the limit equilibrium concept with Newmark-type dynamic response analysis has been proposed for the stability assessment of earth structures. The proposed comprehensive approaches are suitable for the solution of realistic and complex geotechnical engineering problems including slopes, embankments and earth dams. The significant contributions associated with this research work are summarised below.

On the basis of the simplified and 'rigorous' limit equilibrium methods, such as the simplified Bishop method and the generalised procedure of slices (with the
Morgenstern and Price side force assumption), pseudo-static analysis procedures were presented for determining the critical seismic coefficient, $K_c$, which may be used to describe the capability of earth structures subjected to earthquake excitations.

A new optimisation procedure, based on conjugate gradient algorithm, was proposed for searching critical slip surfaces based either on the minimum factor of safety or on the minimum critical seismic coefficient. This comprehensive procedure enables the location of critical slip surfaces of circular and non-circular shape in homogeneous and layered media and also when part of the slip surface is controlled by the existence of a weak zone, layer or surface.

The objective function may be defined either in terms of the factor of safety or in terms of the critical seismic coefficient for both simplified and 'rigorous' limit equilibrium models.

On the basis of the three main probabilistic methods, (a) The First-Order-Second Moment (FOSM), (b) The Point Estimate Method (PEM) and (c) The Monte Carlo Simulation Method (MCSM), a comprehensive probabilistic framework has been developed to update the current state-of-the-art of geotechnical reliability assessment. The factor of safety, associated with simplified and rigorous limit equilibrium models, was used to define the performance function. It is very desirable to define the factor of safety as the performance function because, in geotechnical practice and, especially for slope stability analysis, the factor of safety has a wide acceptance as an indicator of performance whereas the safety margin is rarely used. In order that probabilistic and deterministic analyses are complementary to each other it is desirable to use the factor of safety as an indicator of performance in both types of analysis. All these three numerical probabilistic approaches are systematically described in this thesis. Some new ideas, such as using a multi-layer model (with each layer considered as statistically homogeneous) to approximate the random field model have also been explored. It is relatively difficult to communicate the abstract concept of random field or random process to geotechnical engineers. However, the proposed multi-layer model can be understood and is likely to be accepted.
On the basis of the proposed probabilistic framework as well as the simplified and 'rigorous' limit equilibrium models, a comprehensive procedure has been proposed for estimating the reliability of slope systems. This analysis procedure can incorporate slip surfaces of both circular and arbitrary shape. Moreover, the correlation coefficients between all pairs of basic random variates can be included in the analysis procedure. Because a slope system is very complex, the assessment of reliability has to be considered in an approximate way. This means the estimation of an 'upper bound' and a 'lower bound' of reliability for a given problem.

Another important effort of this research work was to develop a comprehensive method for assessing the response of an earth structure to earthquake shaking. On the basis of the limit equilibrium concept and the Newmark-type dynamic response analysis, a new and comprehensive analysis procedure has been proposed. The proposed new equation of motion includes not only the critical seismic coefficient but also the dynamic properties of earth materials or soil layers in the earth structure. All previously published models have ignored the natural frequency and damping ratio of the soil layers.

By simulating the process of strain-softening or, alternatively, the development of excess pore water pressure, the shear strength degradation can also be included in this analysis procedure. Therefore, the critical seismic coefficient is no longer a constant during the period of an earthquake. In order to examine the responses of an earth structure to different earthquake excitations, generated earthquake time-histories are usually required. Therefore, a procedure for generating artificial earthquake motions, associated with a special power spectrum density function, has been presented in this thesis based on Gaussian random process.
9.2. MAIN CONCLUSIONS

A - OPTIMISATION APPROACH AND NUMERICAL TECHNIQUE

1. Based on the limit equilibrium concept, several methods of slope stability analysis have been introduced. Based on the pseudo-static analysis model for earthquake analysis, equations for critical acceleration $K_c$ have been derived.

2. A new optimisation technique using conjugate gradients, was developed for locating the critical slip surface based on either the minimum factor of safety or on the minimum critical seismic coefficient. An appropriate numerical technique, the rational polynomial technique (RPT), was used for estimating the partial derivatives of objective functions. The accuracy of the partial derivatives, calculated by the RPT, was found to be better than that achieved by using the finite difference approach for a multivariate function.

3. The results obtained by using the proposed optimisation technique show close agreement with those based on other techniques which have been published previously. Moreover, the critical slip surface with the minimum critical seismic coefficient can also be located on the basis of the new procedures.

B - PROBABILISTIC FRAMEWORK

4. A comprehensive probabilistic framework for slope reliability assessment has been developed using three well known numerical approaches. The spatial correlation characteristics of basic random variables (i.e. cross-correlation coefficients and auto-correlation coefficients) can be incorporated in such a probabilistic framework.

5. The first order and second moment method (FOSM) has been used for the estimation of slope reliability using both simplified and relatively 'rigorous' limit equilibrium methods. A simple and clear deductive procedure has been proposed for the derivation of a general solution of the so called 'invariant' second moment reliability index, $\beta^*$, considering any number of basic random variables. Moreover, an
independent approach has been presented for an iterative solution which is required for obtaining the numerical solution when considering a non-linear performance function. The geometrical interpretation of the conventional definition of the reliability index, $\beta$, was given and compared with the geometrical definition of the so called 'invariant' reliability index.

6. A general procedure for generating an n-dimensional correlated normal random vector was proposed for use with the Monte Carlo Simulation method (MCSM). This enables the use of this method for probabilistic analysis of a multi-layer medium or one which is modelled as a multi-layer medium.

7. Comparative studies of slope reliability have been performed on the basis of the three alternative methods (FOSM, PEM, and MCSM). Separate comparisons have been made using simplified or rigorous limit equilibrium models. This is the first time such systematic comparisons have been made. The calculated magnitudes of reliability indices were generally close to each other no matter which limit equilibrium method was used. Some differences in the magnitudes of reliability indices were noted between alternative techniques of reliability analysis, i.e., FOSM, PEM, MCSM. Such differences tend to increase with increase in the number of soil layers.

PEM is not suited for the multi-layer model because, with increase of the number of basic random variables, computation effort increases exponentially.

8. The influence of spatial correlation of soil properties on the reliability index of slopes was investigated. Monte Carlo simulation method was used to investigate the influences of the cross-correlation coefficients on the form of probability distribution of the factor of safety (FOS). The investigations showed that the influence is not very significant. The investigations also revealed that the distributions of FOS are generally normal (Gaussian) provided that all the basic input random variables are normal variates.

9. The influence of auto-correlation coefficient on the reliability index of slopes was investigated based on the simplified Bishop method and the Morgenstern & Price
method. Generally, reliability index will be overestimated if variables are assumed to be spatially uncorrelated \((\rho_{cc} \text{ and } \rho_{\phi\phi} = 0)\). On the contrary reliability index is underestimated if the variables are perfectly correlated spatially \((\rho_{cc} \text{ and } \rho_{\phi\phi} = 1)\).

**C - System Reliability of Slopes**

10. A comprehensive approach has been developed for system reliability analysis of slopes considering slip surfaces of circular or arbitrary shape. A simple orthogonal transformation procedure was developed so that the correlated basic random variables in slope systems can be considered in the system reliability analysis.

11. The magnitude of the lower bound of system reliability is the same as that based on the critical slip surface. However the magnitude of the upper bound of system reliability is always smaller than the reliability based on the critical slip surface. Conversely, system failure probability has an upper bound which is greater than the failure probability based on the critical slip surface.

The range of reliability bounds also depends on the failure probabilities associated with single modes. The larger the failure probabilities associated with single modes the wider the range of bounds associated with the slope system.

12. It is important to investigate the influences of \(\rho_{cc}\) and \(\rho_{\phi\phi}\) on the behaviour of slope systems. The computation results indicated that the magnitudes of the bounds of the failure probability (upper and lower bounds) of the slope system increases with increase of either \(\rho_{cc}\) or \(\rho_{\phi\phi}\) or both together. The correlation between the geotechnical parameters in the different soil layers has a significant influence on the reliability bounds; the higher the correlation, the narrower the bounds.

**D - Earthquake Analysis of Earth Structures**

13. A procedure for generating artificial earthquakes has been presented based on Gaussian random process. A given power spectrum density function can be considered in this procedure. The response spectral analyses based on the simulated artificial earthquake time histories have been performed.
14. Combining the limit equilibrium concept with a Newmark-type dynamic response analysis, an innovative and comprehensive procedure for estimating the response of earth structures to earthquakes has been proposed. The dynamic response equation developed in this research work includes not only the critical seismic coefficient but also the dynamic properties of earth materials (soil layers) such as the natural frequency and damping ratio.

15. The decrease of the shear strength of earth materials can be taken into consideration by simulations of (a) the process of strain softening and (b) the development of excess pore water pressure depending on which mechanism is considered to be relevant. Both the strain softening model and the excess pore water pressure model were used to estimate the response of an earth structure to a given earthquake excitation. An important aspect of the new procedure is that the factor of safety and the critical seismic coefficient are regarded as functions of time after the start of an earthquake.

16. A new numerical algorithm has been proposed for estimating the Duhamel integral based on the recursive method. The method is not only as simple as that proposed by Clough but also has a higher accuracy than that of Clough.

17. The influences of various parameters on the permanent deformations of an earth structure caused by an earthquake have been investigated using the developed comprehensive procedures. The influence of the brittleness of soils on the permanent deformation of earth structures was found to be very significant. The influence of dynamic excess pore water pressure parameter, $A_n$, on the permanent deformation of earth structures was also found to be very significant.

The comprehensive approach developed for the dynamic response analysis was, therefore, found to be a very effective tool for assessing the safety of earth structures during earthquakes as well as for the estimation of permanent displacements.
REFERENCES


References

Seismological Society of America, Vol. 51.


Significance and Utilization of Coal Mining Wastes, pp. 95-109.


References


Hardin, B.O. and Hardin, K.O. (1984) *A New Statically Consistent Formulation*


K1, pp.49-54.


References


Parkinson, J.M. and Hutchinson, D. (1972) *An Investigation into the Efficiency*


References


References


CALCULATION ILLUSTRATIONS FOR ESTIMATING PARTIAL DERIVATIVES OF MULTIVARIATE FUNCTION BASED ON RPT

To demonstrate the calculation steps of the rational polynomial technique (RPT), two simple examples are considered for which the exact partial function values and partial derivatives are known. Both are explicit functions; one consisting of two variables is linear with respect to these two variables and the other, containing three variables, is non-linear. Inexplicit functions are included in the main illustrative examples of slope stability considered in the next section.

(i) Linear Function of Two Arguments

Consider a function of two arguments

\[ f(x_1, x_2) = x_1 + x_2 \]

Let the discrete values of each variable be three, i.e. \( m = 2 \) and given point be \( \bar{X} = (20,10) \). Assume the upper and lower bound with these two variables are \( 20 \pm 4 \) and \( 10 \pm 2 \). Therefore the discrete values are,
From Eqs. (4-39) to (4-42), considering partial function values with variable x₁:

\[ f(x_1, \bar{x}_2) = \phi_0(x_1) = a_0 + \frac{x_1 - 16}{a_1 + \frac{x_1 - 20}{a_2}} \]

\[ \phi_1(x_1) = a_1 + \frac{x_1 - 20}{a_2} \]

\[ \phi_2(x_1) = a_2 \]

From Table 4-3, the values of \( a_0 \), \( a_{1,0} \) and \( a_{2,0} \) are:

\[ a_0 = F_{10} = f(x^{(0)}_1, \bar{x}_2) = f(16, 10) = 16 + 10 = 26 \]

\[ a_{1,0} = F_{11} = f(x^{(1)}_1, \bar{x}_2) = f(20, 10) = 20 + 10 = 30 \]

\[ a_{2,0} = F_{12} = f(x^{(2)}_1, \bar{x}_2) = f(24, 10) = 24 + 10 = 34 \]

Substituting these values in the other expressions in Table 4-3, we have:

\[ a_1 = \frac{x^{(1)}_1 - x^{(0)}_1}{a_{1,0} - a_0} = \frac{24 - 16}{30 - 26} = 1 \]

\[ a_{2,1} = \frac{x^{(2)}_1 - x^{(1)}_1}{a_{2,0} - a_0} = \frac{24 - 16}{34 - 26} = 1 \]

\[ a_2 = \frac{x^{(2)}_1 - x^{(1)}_1}{a_2 - a_1} = \frac{24 - 20}{1 - 1} = \infty \text{ (constant)} \]

Accordingly considering Eqs.(4-42), (4-41) and (4-40) in that order:

\[ \phi_2(x_1) = \infty, \quad \phi_1(x_1) = 1 + \frac{x_1 - 20}{\infty} = 1, \]

and \[ \phi_0(x_1) = a_1 + \frac{x_1 - 16}{1} = 26 + x_1 - 16 = x_1 + 10 \]
Thus, the numerical partial function value with respect to the variable \( x_1 \) is given by Eq. (4-39) is found to be

\[
f(x_1, \bar{x}_2) = \phi_1(x_1) = x_1 + 10
\]

This is, of course, the accurate partial function value.

Next consider the numerical value of the derivative with respect to Eqs. (4-43) and (4-44). First, consider the partial derivative of \( \phi_2(x_1) \):

\[
\frac{\partial \phi_1(x_1)}{\partial x_1} = \frac{\phi_2(x_1) - (x_1 - x_1^{(i)}) \frac{\partial \phi_2(x_1)}{\partial x_1}}{[\phi_2(x_1)]^2} = 0, \text{ since } \phi_2(x_1) = \infty \text{ (constant)}
\]

Now consider the partial derivative of \( \phi_0(x_1) \), referring again to Eqs. (4-43) and (4-44),

\[
\frac{\partial \phi_0(x_1)}{\partial x_1} = \frac{\phi_1(x_1) - (x_1 - x_1^{(i)}) \frac{\partial \phi_1(x_1)}{\partial x_1}}{[\phi_1(x_1)]^2} = \frac{\phi_1(x_1) - 0}{[\phi_1(x_1)]^2} = \frac{1}{\phi_1(x_1)} = 1
\]

Obviously the numerical value of the partial derivative of the function with respect to \( x_1 \) is 1. This is the accurate value of partial derivative for this simple function, i.e.,

\[
\frac{\partial f(x_1, \bar{x}_2)}{\partial x_1} = 1
\]

Following exactly the same procedure for the partial function value with respect to \( x_2 \), the following results are obtained:

\[
a_0 = 28, \quad a_{1,0} = 30, \quad a_1 = 1, \quad a_{2,0} = 32, \quad a_{2,1} = 1, \quad a_2 = \infty
\]

\[
\phi_2(x_2) = a_2 = \infty, \quad \phi_1(x_2) = 1, \quad \phi_0(x_2) = 20 + x_2
\]

Thus the partial function value obtained numerically is again the exact partial function value. The derivative may be calculated numerically as above and it is found that

\[
\frac{\partial f(x_1, x_2)}{\partial x_2} = 1
\]
This is the accurate value of the partial derivative with respect to $x_2$.

The results could now be written in tabular form as below:

Table A-1 Coefficients of Rational Polynomial
(Each line is a diagonal of Table 4-3)

<table>
<thead>
<tr>
<th>Variables</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a_0$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>26</td>
</tr>
<tr>
<td>$x_2$</td>
<td>28</td>
</tr>
</tbody>
</table>

Table A-2 The Values of $\phi_i(*) i = 2, 1, 0$

<table>
<thead>
<tr>
<th>Variables</th>
<th>Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi_2(*)$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Table A-3 The Values of $\phi_i'(*) i = 2, 1, 0$

<table>
<thead>
<tr>
<th>Variables</th>
<th>Derivatives</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\phi_2'(*)$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

(ii) Non-Linear Function of Three Variables

Consider the following function:

$$f(x_1, x_2, x_3) = \frac{x_1^2 - 2x_2^2 + x_3^2}{x_1 x_2 x_3}$$

The partial derivative values of this function are to be evaluated at the point $\bar{x}_1 = 2$, $\bar{x}_2 = -3$ and $\bar{x}_3 = 4$.

The exact values of the partial derivatives at this point are,
In order to calculate these partial derivatives using the numerical approach based on the rational polynomial technique, discrete values of the three variables are to be assumed. Five discrete values are chosen for this purpose, one value is the chosen point and the others 10% and 15% on either side of it.

Thus the discrete values can be shown in the following table, i.e.

**Table A-4 The Discrete Values for These Three Variables**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Discrete Values (i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1^0)</td>
<td>0.0</td>
</tr>
<tr>
<td>(x_2^0)</td>
<td>-2.55</td>
</tr>
<tr>
<td>(x_3^0)</td>
<td>3.4</td>
</tr>
</tbody>
</table>

**Note:** The order in which the discrete values are taken is not important. Thus the numbers above are not taken in increasing or decreasing order.

The coefficients \(a_0\) to \(a_4\) of the rational polynomial may now be evaluated based on Table 4-3 for each variable. These values are written in tabular form below,

**Table A-5 Coefficients of Rational Polynomial**

(Each line shows diagonal values of Table 4-3 for each variable)

<table>
<thead>
<tr>
<th>Variables</th>
<th>Coefficients of Rational Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>(a_0)</td>
</tr>
<tr>
<td>(x_2)</td>
<td>-0.0436</td>
</tr>
<tr>
<td>(x_3)</td>
<td>-0.3429</td>
</tr>
</tbody>
</table>

Using the first line of the above table, the partial function with respect to \(x_1\) can be written as follows,
\[ f(x_1, x_2, x_3) = -0.0436 + \frac{x_1 - 1.7}{-7.2569 + \frac{x_1 - 1.8}{-0.7330 + \frac{x_1 - 2.3}{-4.7430 + \frac{x_1 - 2.2}{810.4218}}} \]

The partial function values at the chosen point are shown in the following table,

Table A-6 Function \( \phi_i(\cdot) \) \( (i = 4, 3, 2, 1, 0) \) for All Three Variables

<table>
<thead>
<tr>
<th>Variables</th>
<th>( \phi_4(\cdot) )</th>
<th>( \phi_3(\cdot) )</th>
<th>( \phi_2(\cdot) )</th>
<th>( \phi_1(\cdot) )</th>
<th>( \phi_0(\cdot) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 = 2 )</td>
<td>-0.0833</td>
<td>-7.555</td>
<td>-0.6697</td>
<td>-4.7430</td>
<td>810.4218</td>
</tr>
<tr>
<td>( x_2 = -3 )</td>
<td>-0.0833</td>
<td>-1.7337</td>
<td>2.9218</td>
<td>-2.3690</td>
<td>1.213\times10^5</td>
</tr>
<tr>
<td>( x_3 = 4 )</td>
<td>-0.0833</td>
<td>-2.9565</td>
<td>-2.5358</td>
<td>-3.2012</td>
<td>1.376\times10^4</td>
</tr>
</tbody>
</table>

The partial derivatives of the functions \( \phi_i(\cdot) \) at the chosen point are calculated by Eq.(4-44) and the results are shown in the table below,

Table A-7 Derivatives of \( \phi_i(\cdot) \) \( (i = 4, 3, 2, 1, 0) \) for All Three Variables

<table>
<thead>
<tr>
<th>Variables</th>
<th>( \phi_4'(\cdot) )</th>
<th>( \phi_3'(\cdot) )</th>
<th>( \phi_2'(\cdot) )</th>
<th>( \phi_1'(\cdot) )</th>
<th>( \phi_0'(\cdot) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 = 2 )</td>
<td>0</td>
<td>0.0012</td>
<td>-0.2108</td>
<td>-1.3992</td>
<td>-0.1250</td>
</tr>
<tr>
<td>( x_2 = -3 )</td>
<td>0</td>
<td>8.24\times10^{-6}</td>
<td>-0.4221</td>
<td>0.3274</td>
<td>-0.5278</td>
</tr>
<tr>
<td>( x_3 = 4 )</td>
<td>0</td>
<td>7.27\times10^{-5}</td>
<td>-0.3124</td>
<td>-0.3749</td>
<td>-0.3125</td>
</tr>
</tbody>
</table>

The results in the last column are the required partial derivatives with respect to \( x_1, x_2 \) and \( x_3 \) at the chosen point \( \mathbf{x} = (2, -3, -4) \) and they are identical to the exact solutions.

For this non-linear function, the estimations of the partial derivative of the function based on the forward finite difference method are performed at the same point, i.e. \( \mathbf{x} = (2, -3, 4) \). Let \( \Delta \mathbf{x} = \delta \mathbf{x} \) then the partial derivatives corresponding to different \( \delta \) can be evaluated. The approximated results are shown in the following table. From the table one can find that the estimation values of the partial derivative are close to the exact values only when \( \delta = 0.001 \) or \( \delta = 0.0001 \) in this case. It is, therefore, important to
adequately choose the \( \Delta X \) for gating the good approximate values of the partial derivative.

Table A-8 The Partial Derivatives Based on Forward Finite Difference Method

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( \frac{\partial f(X)}{\partial x_1} )</th>
<th>( \frac{\partial f(X)}{\partial x_2} )</th>
<th>( \frac{\partial f(X)}{\partial x_3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.1212</td>
<td>-0.5025</td>
<td>-0.2992</td>
</tr>
<tr>
<td>0.01</td>
<td>-0.1246</td>
<td>-0.5250</td>
<td>-0.3111</td>
</tr>
<tr>
<td>0.001</td>
<td>-0.1250</td>
<td>-0.5275</td>
<td>-0.3124</td>
</tr>
<tr>
<td>0.0001</td>
<td>-0.1250</td>
<td>-0.5277</td>
<td>-0.3125</td>
</tr>
<tr>
<td>0.00001</td>
<td>-0.1252</td>
<td>-0.5285</td>
<td>-0.3129</td>
</tr>
<tr>
<td>0.000001</td>
<td>-0.1192</td>
<td>-0.5464</td>
<td>-0.2980</td>
</tr>
</tbody>
</table>