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Energy absorption in structural frames

Amir Masoud Horr

University of Wollongong

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by

AMIR MASOUD HORR, B.Sc. (Hons), M.Eng.Sc. (Hons)

DEPARTMENT OF CIVIL AND MINING ENGINEERING

June, 1995
DECLARATION

This is to certify that the work presented in this thesis was carried out by the author in the Department of Civil and Mining Engineering, the University of Wollongong, and has not been submitted to any other university or institute for a degree except where specifically indicated.

..............................................

Amir Masoud Horr
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ABSTRACT

Based on the theory of fractional calculus and the spectral theory of vibration, a new spectrally-formulated finite-element method of analysis is developed which is capable of making accurate predictions of the dynamic response of structures with added dampers. The frequency-dependent and temperature-dependent damping characteristics of structural materials can be modelled accurately using the fractional derivative model. It is shown that the proposed method can be extended to develop a non-linear damping element which can be used to model structural dampers. The approach has an advantage over the usual viscous treatment, which appears to lack a physical basis.

The main features of the complex-spectral finite-element method of analysis have been presented in this thesis. This method is capable of making accurate predictions of the dynamic response of structural systems. Most structural systems can be analysed and designed by using the conventional finite element method. However, in order to guarantee stability and accuracy of the solution, the number of elements used to model the structure may be very large. Hence, it appears that, for large structures, it may be more effective to use the spectral approach presented in this thesis.

A set of the fractional derivative damping models, capable of representing different damping mechanisms, have been derived for solving the dissipation problem in damped systems. The modelling of elastomeric and viscoelastic components in damped structures often requires complex viscoelastic representations. While traditional differential operators are typically employed in such a formulation, fractional operators give rise to a richer variety of functional
families, and hence lead to an improved integro-differential type curve fitting of constitutive representations.

The fractional derivative viscoelastic damping model enables a single formulation of the complex dynamic stiffness matrix of the damped system to be developed. This leads to the successful formation of the frequency domain equations of motion for a structure containing both elastic and viscoelastic components. However, although the development of the fractional-spectral method focuses primarily on damped structural frames, the results can readily be extended to damped mechanical systems. In addition, although the analysis of damped systems is the primary object of this study, the spectral finite element method presented in Chapter Four can be used in the case of undamped dynamical systems.
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# NOTATION

\[
\begin{align*}
A & \quad = \quad \text{cross sectional area} \\
A(x) & \quad = \quad \text{variation function for cross sectional area} \\
a & \quad = \quad \text{first modulus parameter} \\
b & \quad = \quad \text{second modulus parameter} \\
[C] & \quad = \quad \text{viscous damping matrix} \\
c & \quad = \quad \text{fifth modulus parameter} \\
D[ ] & \quad = \quad \text{fractional operator} \\
E & \quad = \quad \text{Young's modulus} \\
E_v & \quad = \quad \text{complex moduli} \\
E' & \quad = \quad \text{real part of complex moduli} \\
E'' & \quad = \quad \text{imaginary part of complex moduli} \\
\hat{E} & \quad = \quad \text{fractional complex moduli} \\
e & \quad = \quad \text{third modulus parameter} \\
f & \quad = \quad \text{fourth modulus parameter} \\
f(t) & \quad = \quad \text{time-dependent force function} \\
[F] & \quad = \quad \text{force vector} \\
\hat{F}(\omega) & \quad = \quad \text{complex force function} \\
F< > & \quad = \quad \text{Fourier transform function} \\
G & \quad = \quad \text{shear modulus} \\
G_v & \quad = \quad \text{complex shear moduli} \\
G' & \quad = \quad \text{real part of complex shear moduli} \\
G'' & \quad = \quad \text{imaginary part of complex shear moduli} \\
\hat{G} & \quad = \quad \text{fractional complex shear moduli} \\
H & \quad = \quad \text{element thickness} \\
I & \quad = \quad \text{moment of inertia}
\end{align*}
\]
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I(x)$</td>
<td>variation function for moment of inertia</td>
</tr>
<tr>
<td>$K$</td>
<td>shape factor in Timoshenko theory</td>
</tr>
<tr>
<td>$[K]$</td>
<td>stiffness matrix</td>
</tr>
<tr>
<td>$[\hat{K}]$</td>
<td>complex dynamic stiffness matrix</td>
</tr>
<tr>
<td>$k'$</td>
<td>real part of complex stiffness representation</td>
</tr>
<tr>
<td>$k''$</td>
<td>imaginary part of complex stiffness representation</td>
</tr>
<tr>
<td>$\hat{k}$</td>
<td>spectral dynamic stiffness of a member</td>
</tr>
<tr>
<td>$k_{FE}$</td>
<td>finite element stiffness term</td>
</tr>
<tr>
<td>$k_{spec}$</td>
<td>spectral stiffness term</td>
</tr>
<tr>
<td>$L$</td>
<td>length of an element</td>
</tr>
<tr>
<td>$M_n$</td>
<td>normalised mass</td>
</tr>
<tr>
<td>$[M]$</td>
<td>mass matrix</td>
</tr>
<tr>
<td>$N$</td>
<td>number of spectral representations</td>
</tr>
<tr>
<td>$Q$</td>
<td>quality factor</td>
</tr>
<tr>
<td>$R_n$</td>
<td>normalised rotary inertia</td>
</tr>
<tr>
<td>$[T]$</td>
<td>transformation matrix</td>
</tr>
<tr>
<td>$u(x)$</td>
<td>displacement function</td>
</tr>
<tr>
<td>$u(x,t)$</td>
<td>time-dependent displacement function</td>
</tr>
<tr>
<td>$u(\omega)$</td>
<td>frequency-dependent displacement function</td>
</tr>
<tr>
<td>$\hat{u}(\omega)$</td>
<td>complex displacement function</td>
</tr>
<tr>
<td>$\dot{u}$</td>
<td>velocity</td>
</tr>
<tr>
<td>$\ddot{u}$</td>
<td>acceleration</td>
</tr>
<tr>
<td>$u'$</td>
<td>partial derivative of $u$ with respect to coordinate</td>
</tr>
<tr>
<td>$u''$</td>
<td>second partial derivative of $u$ with respect to coordinate</td>
</tr>
<tr>
<td>$\hat{u}$</td>
<td>displacement vector in transform coordinates</td>
</tr>
<tr>
<td>$w_s(t)$</td>
<td>stored energy</td>
</tr>
<tr>
<td>$w_d(t)$</td>
<td>dissipated energy</td>
</tr>
<tr>
<td>$\dot{\psi}(t)$</td>
<td>time-dependent stress power function</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
</tr>
</tbody>
</table>
\[ \begin{align*}
\delta & = \text{log decrement} \\
\omega & = \text{natural frequency} \\
\omega_c & = \text{complex representation of frequency} \\
\omega' & = \text{real part of complex representation of frequency} \\
\omega'' & = \text{imaginary part of complex representation of frequency} \\
\zeta & = \text{damping ratio} \\
\lambda & = \text{Lame constant} \\
\kappa & = \text{wave number} \\
\kappa & = \text{complex wave number} \\
\eta & = \text{coefficient of viscosity} \\
\mu & = \text{Lame constant} \\
\psi & = \text{damping capacity} \\
\psi_s & = \text{structural damping capacity} \\
\psi_w & = \text{structural specific loss} \\
\sigma & = \text{stress} \\
\sigma(t) & = \text{time-dependent stress function} \\
\sigma(\omega) & = \text{frequency-dependent stress function} \\
\sigma(s) & = \text{Laplace transform of stress function} \\
\varepsilon & = \text{strain} \\
\varepsilon(t) & = \text{time-dependent strain function} \\
\varepsilon(\omega) & = \text{frequency-dependent strain function} \\
\varepsilon(s) & = \text{Laplace transform of strain function} \\
\theta & = \text{loss angle} \\
\theta_E & = \text{loss angle for complex moduli } E_v \\
\theta_G & = \text{loss angle for complex shear moduli } G_v \\
\theta_k & = \text{loss angle for stiffness complex representation} \\
\nu & = \text{Poisson's ratio}
\end{align*} \]
\( v \) = lateral degree of freedom

**Superscripts**

\( T \) = transpose of matrix

\( n \) = time step

\( (\cdot) \) = partial differential with respect to time

\( (\cdot') \) = partial differential with respect to coordinates

**Subscripts**

\( d \) = damped

\( FE \) = finite element term

\( e, p \) = elastic and plastic component respectively

\( o \) = initial

\( ud \) = undamped

\( spec \) = spectral term

\( t \) = time-dependent term

\( x \) = \( x \) component in global coordinate

\( y \) = \( y \) component in global coordinate

\( z \) = \( z \) component in global coordinate

\( ()_{ij} \) = matrix indices
LIST OF PUBLICATIONS DURING PhD STUDY


Chapter One

Introduction

1.1 GENERAL

The dynamic analyses of structures are often very complicated and require the use of some form of numerical analysis. Although vibration damping is an important aspect of many structural systems, current damping models are inaccurate and inadequate. These models are founded on mathematical convenience rather than on physical understanding and do not reflect the non-linear characteristics of material and structural damping.

The finite element method is a popular and frequently used method of dynamic analysis because of its great ability to model complex geometries by piecing together different small elements. The method is capable of solving field equations of motion using piecewise-continuous trial displacement functions. These functions are assumed over a local region of the structural system, and individual element matrices are calculated using the principle of variations or weighted residuals. The global system matrices, then, can be assembled using element matrices. The result of this procedure is a set of discretised equations of motion that are easily solved using standard numerical techniques on computers.
In recent years, both researchers and practising engineers have tried to bring dissipation analysis up to the same level of accuracy as stiffness and mass analysis for structural systems. In fact, a highly accurate damping model is required to design damped systems. Most structures can be analysed and designed by using the conventional finite element method. However, current popular treatments of dissipation in this approach are unable to reproduce the non-linear damping characteristics in structures. Furthermore, in order to guarantee stability and accuracy of the solution, the number of elements used to model the structure may be very large indeed; more precisely, an accurate result can be obtained only after a substantial computational effort. As a consequence, it appears that for large structures it may be more effective to use alternative mathematical modelling approaches. In this respect, attention is paid herein to the alternative spectral approach which works in the frequency domain and draws its robustness from the speed and switching capabilities of the Fast Fourier Transform.

This thesis develops a spectral finite element method designed for solving damped three-dimensional structures. There has been considerable effort to apply the method of spectral analysis to the vibration of structures. The spectral formulation starts at the same starting point as the conventional finite element formulation. However, it is formulated in the frequency domain. Using a fractional derivative model, the non-linear damping characteristics of structural materials and also the characteristics of additional external dampers can be adapted into the spectral method. The assembled system of equations can then be solved in the frequency domain, and is then transformed to the time domain using a Fast Fourier Transform.

1.2 OBJECTIVE

The resistance of structures against dynamic loads depends in part on their ability to dissipate or propagate energy away from the impact site. In recent years,
it has been recognised that energy dissipation dampers can provide an efficient means for controlling the structure response induced by dynamic forces. Most structures can be designed to withstand severe dynamic forces by providing ductility and energy absorption capacity to the structural elements, but at the expense of substantial damage in the structural elements, and also in non-structural elements and services.

On the other hand, by dissipating the vibratory energy in structural dampers, the risk of the structure experiencing excessive deformations or accelerations can be reduced. As a result, less ductility or inelastic energy demand is required in the structural frame. In particular, structural isolation systems can be designed essentially to limit the non-linear behaviour to the isolation devices, thereby imposing very small or no ductility demand on the structure itself.

Passive and active damping of vibration in structures can be very important for several reasons. In terms of performance, higher damping can reduce steady-state vibration time and levels, and can also reduce the time needed for transient vibrations to settle. Generally, passive damping can reduce the complexity of the active control needed. However, it couples vibration modes (natural frequencies) which are closely spaced in the frequency spectrum. Hence, for an accurate design of a damped system, an accurate mathematical damping model, able to predict realistic results for damping mechanisms, is needed.

Structural damping mostly arises as a result of many energy dissipation mechanisms acting in a system. These sources of damping might be classified by considering a structure as an assemblage of elements which interact at interfaces (nodes). Material damping occurs within the elements, and can be added to joint impact and friction damping coming from the interaction of one element with another at a common interface, or from the interaction of the structure with a non-
structural internal or external environment. Material damping is generally a complex function of frequency, temperature, type of deformation, amplitude and structural geometry. It is probably true to say that current popular treatments of damping in structural dynamics are not physically motivated, and are unable to reproduce this fundamental physical behaviour. It can also be proved that no simple, element-based damping matrix can accurately represent the behaviour of real engineering materials over a range of frequencies.

This thesis primarily concentrates on the dynamic analysis of damped structures. The objective is to develop an efficient numerical method and corresponding computer implementation for the prediction of the dynamic response of damped three-dimensional structures.

1.3 SCOPE

This research addresses the need for a full three-dimensional numerical method of dynamic analysis for damped structures. The numerical method developed in this thesis is mainly focussed on the following areas:

(I) establishing a connection between the intrinsic material damping and structural damping.

(II) establishing a frequency domain spectral finite element method for the prediction of dynamic response of structures.

(III) establishing a fractional derivative formulation for the constitutive relation between the stresses and strains in engineering materials.
(IV) extending the fractional derivative formulation to establish a flexible damping model for the prediction of damping in structural materials.

(V) establishing a fractional derivative damping element for structures with added external damping devices.

(VI) adapting the fractional derivative damping model into the frequency domain spectral method.

(VII) comparison of results of frequency domain fractional-spectral method with existing time domain methods (i.e., finite element method).

(VIII) extending the developed numerical method to non-linear dynamic analysis in order to account for non-linear damping effects.

(IX) applying the developed numerical method to study some fundamental problems related to structural dynamics.

1.4 OUTLINE OF THE THESIS

The thesis takes a developmental approach. It is devoted to the development of a fresh numerical method for dynamic analysis of damped structures. The thesis is divided into eight chapters and four appendices.
Chapter One introduces the research project. It states the problem, objectives and scope of the research.

Chapter Two reviews previous research work and highlights the need for further research in the area of structural dynamics.

Chapter Three establishes the mathematical bases for the dissipation of energy in damped structures. The fundamental connection between the intrinsic material damping and structural damping ratio is derived and is shown graphically. Comparative analyses for damping models in different vibration theories are performed, and special emphasis is placed on the Timoshenko beam theory.

Chapter Four is devoted to the development of the novel frequency domain spectral finite element method. The mathematical basis of the spectral solution to the governing differential equation of structures is discussed. It is then extended to establish the numerical method for the prediction of the dynamic response of structures in a matrix format. The computer implementation is presented, and it is shown that the method is accurate and efficient compared with conventional finite element methods.

Chapter Five describes the mathematical basis of fractional calculus. The whole derivation procedure to obtain the constitutive relation for viscoelastic materials is presented, and is compared with existing conventional constitutive relations.

Chapter Six presents the fractional-spectral method. The fractional derivative damping model is adapted into the frequency domain spectral finite
element method. Emphasis is placed on the derivation of a global dynamic stiffness matrix which is assembled using the local spectral-dynamic stiffness matrices and damping elements.

Chapter Seven is devoted to the application of the developed numerical method. Several linear and non-linear problems related to structural dynamics are analysed in this Chapter.

Chapter Eight presents the summary, conclusions and recommendations for future research work.
Chapter Two

REVIEW OF PAST WORK

2.1 GENERAL

A vibrating system has a component that stores potential energy and releases it as kinetic energy in the form of motion. Vibration analysis is concerned with the oscillatory motions of masses and forces associated with them. As all engineering structural systems possess mass and elasticity, they are capable of vibration to some degree, and their design generally requires consideration of their oscillatory behaviour. Typical examples of vibration are the swaying of a large building due to wind or an earthquake, the motion of an airplane's wing and vibration of automobiles or motorcycles on the road. Oscillatory systems can generally be characterised as linear or non-linear systems in which the governing equations of motion have to be solved with the help of mathematical techniques.

The mechanisms of damping by which energy is dissipated in materials and structural systems are great in number and also complexity. In many systems the vibrational motions are assumed to be undesirable and have to be suppressed. In fact, one of the important design considerations for dynamical systems focuses on the skills needed to determine ways of adjusting the physical parameters of the system in such a way that the vibration response meets some specified shape or performance criterion.
Passive and active damping of vibration in structural systems may depend on many parameters such as: type of material, geometry, chemical composition, temperature, pre-stress, initial strain and amplitude, and frequency of the excitation forces. These damping parameters can be very important in the design of dynamical systems for several reasons. In terms of dynamic performance, higher damping can reduce steady-state vibration time and levels, and can also reduce the time needed for transient vibration to settle.

### 2.2 CURRENT DAMPING MODELS

Many different methods for incorporating damping into structural systems have been proposed by researchers over recent decades. Classical dynamic analysis treats the damping characteristic in materials with appropriate assumptions concerning the proportionality of damping with the velocity of motion. Generally, there is no known procedure for determining the damping matrix elements in numerical procedures, except in cases involving discrete viscous damping. It is also well known that no simple, element-based damping matrix can accurately represent the behaviour of real engineering materials over a range of frequencies.

#### 2.2.1 Viscous Damping Model

In this model of damping, all damping forces are assumed to be proportional to the rate of change of displacement or strain. This simplified assumption leads to the simplest mathematical treatment of damping in structures. The model is perhaps appropriate to model resisting forces in viscous fluids, or, in deformation, perfectly-plastic materials. However, as the model has the property that damping increases monotonically with frequency, it cannot accurately characterise most engineering materials over a broad frequency range.
As the energy dissipated per cycle at a single frequency is matched to experimental measurement in forced vibration to obtain the viscous damping coefficient, the viscous model poorly approximates frequency-dependent material behaviour (Hobbs, 1971).

To correct a perceived deficiency of the viscous damping model, a frequency-dependent viscous damping model was proposed by Myklestad (1952). As some experimental results indicated that damping forces may be proportional to displacement amplitudes instead of velocity, the proposed frequency-dependent viscous model replaced the constant damping coefficient by a constant divided by frequency. The advantage of this approach is that the damping can be represented as an experimentally-determined function of frequency, temperature and material (Myklestad, 1952).

As the damped equation of motion in the case of a frequency-dependent damping model mixes time derivatives and frequency, Myklestad (1952), proposed a complex modulus method of solution. His solution is based on a complex stiffness term in which damping in an entire structure, substructure or an element can be included in the form of a loss factor into the stiffness terms. This method is sometimes called the hysteretic or structural damping model, and it is a very popular treatment of damping in structures.

2.2.2 Raleigh Damping Model

The finite element method is undoubtedly the most popular numerical technique in structural dynamics. It uses variational and interpolation methods for modelling and solving boundary value problems. A popular method for the damping
treatment in the finite element analysis of dynamic problems is that of Rayleigh damping. In this approach the damping matrix is assumed to be proportional to the mass and stiffness matrix, and hence it can be represented as a linear combination of the mass and stiffness matrices (Clough et. al., 1975).

The Rayleigh damping model leads to a damping matrix in which the damping in each individual element is modelled as concentrated properties at nodes. The advantage of this approach is that the global damping matrix for an entire structure is in a form by which the dynamic equations of motion can be uncoupled.

As the damping matrix is a linear combination of the mass and the stiffness matrices \([C]=\alpha[M]+\beta[K]\), the coefficients \(\alpha\) and \(\beta\) can be calculated using experimental measurements (Horr and Schmidt, 1995). If the damping behaviour of the material is plotted versus the frequency, the two coefficients may be calculated using the damping values at two specific frequencies.

### 2.2.3 Modal Damping

Another popular method of incorporating damping in the finite element analysis is the modal damping method. It gives the ability to specify different damping ratios for different modes of vibration. In this way, damping can be incorporated in the usual mode superposition method of dynamic analysis. All damping can initially be ignored, and the natural frequencies and mode shapes can be found by one of the conventional eigenvalue methods. After deriving the modal stiffness and modal mass matrices, the damping matrix, which is proportional to the mass or stiffness matrices or both, may be introduced into the nominally uncoupled second-order modal equations.
However, the major drawback in this method is that agreement at natural frequencies of the system in no way guarantees agreement at frequencies far from these natural frequencies. Another drawback is that, as damping can couple natural modes which are closely spaced in frequency, the resulting magnitude and phase angle of response can be incorrect in the forced response analysis.

2.2.4 Viscoelastic Damping Treatment

Damping can also manifest itself through a time dependent constitutive relation. A material is called viscoelastic because it exhibits both elastic and viscous behaviour. The stress-strain relationship for viscoelastic material can be summarised by extending the modulus of a material to a complex modulus. Materials that exhibit viscoelastic behaviour are rubber and rubberlike plexiglass, vinyl and nylon. A common use of these viscoelastic materials in design is as an additive damper device (i.e., viscoelastic damper, rubber bearing) to increase the combined structure’s damping, or as an isolator.

2.3 RECENT ADVANCED DAMPING FORMULATIONS

Conventional damping models currently used in structural dynamics do not accurately represent the behaviour of real materials over a broad range of frequency. Although the formulation of accurate models like the viscoelastic damping model are already in place, the practical use of these models have not been widely accepted.

One of the obstacles is the compatibilities of these models with existing numerical methods like the conventional finite element method. Another problem is that some of these accurate models do not have conditions which allow the equations of motion to become uncoupled. Hence, an approximate method capable
of decoupling the motion equations is needed to solve the damped dynamical problem.

2.3.1 \textit{ATF Damping Model}

The ATF (Augmented Thermodynamic Fields) method was proposed by Lesieutre (1989), in which he had developed a time-domain continuum model of material damping. His model exhibits the frequency-dependent characteristic of real materials. In addition, the model is compatible with current conventional finite element structural analysis methods.

Thermoelastic damping, a result of the well-understood coupling of temperature and strain, provided the initial basis for the ATF damping model. The model uses the techniques of nonequilibrium thermodynamics to derive coupled material constitutive equations and to develop coupled partial differential equations of evolution.

Although, the ATF model can comparatively be more accurate than conventional damping models, an additional set of state coordinates must be employed in order to develop a finite element analysis. In addition, the system matrix dimensions in the finite element method developed by Lesieutre (which uses the ATF damping model) are very large.

2.3.2 \textit{GHM Damping Model}

The GHM method was initially proposed by Golla and Hughes (1985), and then by McTavish and Hughes (1987). Their method is essentially based on the same general perceived need for additional dissipation coordinates as the ATF
method. The developers derived a time domain finite element formulation of viscoelastic material damping.

Although their technique was successfully used to fit a portion of an experimentally determined curve of damping versus frequency, their model was restricted to microstructural damping mechanisms. However, the GHM model can easily be adapted into existing numerical tools like the finite element method.

2.3.3 Perturbation Damping Model

For a slightly viscoelastic damped structure, Segalman (1987), proposed a new finite element method. His method is essentially a perturbation technique in which a perturbation solution is used to develop the stiffness and damping matrices of the structure.

The proposed method works in the time domain but, unlike the ATF or GHM method, it avoids introducing additional dissipation coordinates. However, the applicability of the perturbation method is limited to materials with small viscoelastic behaviour. In addition, the resulting stiffness and damping matrices are unsymmetric and they are not compatible with the standard finite element method.

2.3.4 Fractional Derivative Damping Model

The generalisation of derivatives and integrals to fractional order can be dated back to the last century. Gemant (1936), was the first to use this concept in the constitutive equations of viscoelastic materials. Scott-Blair and Caffyn (1949), were among others who used it to develop a damping model for engineering materials.
In late 60’s and 70’s, Caputo (1966, 1976), developed a fractional derivative damping model in which he found good agreement with experimental results when applying the model to some metals and also to soil.

Among other recent researchers, working on the fractional derivative damping model, were Bagley and Torvik (1979, 1983). The core of their concept is to use fractional time derivatives in material constitutive relations. Their development was motivated by the observation that the fractional derivative of order 1/2 arises naturally in the shear stress-strain relation of polymer solids with no crosslinking and under certain restrictions. Using a five parameter damping model, they have successfully modelled the elastic and dissipative behaviour of over 100 materials.

Zimm (1956) has also proposed the fractional derivative molecular theory taking into account the intermolecular hydrodynamic forces. He showed that the fractional derivative of order 2/3 can accurately represent some material behaviour. His theoretical findings provide a link between the microscopic theories of ideal viscoelastic media and the macroscopic behaviour of real materials.

The concept of using fractional calculus in the formulation of constitutive equations for materials with memory has also been proposed by Koeller (1984). He has shown that the theory of hereditary solid mechanics (spring-dashpot damping model) is equivalent to the requirement that the stress in the dashpot be proportional to the fractional derivative of the strain in the dashpot.
Expressions for creep and relaxation functions, in terms of the fractional derivative parameter, were obtained for the fractional spring-dashpot damping element in Koeller's model. It is also shown that the fractional calculus constitutive equation in the damping element allows for a continuous transition from the solid state to the fluid state when the memory parameter varies from zero to one.

Koh and Kelly (1990), have also used the concept of fractional derivatives in the formulation of a stress-strain relationship for elastomers. In their model, an oscillator consisting of a mass and a fractional spring-dashpot element is used to model elastomeric bearings in base isolation systems. They have also developed an efficient numerical multi-step scheme for the dynamic analysis of a single-degree-of-freedom fractional oscillator in the time domain.

The fractional derivative model is found to agree well with experimental results when it is applied to shaking table tests of a base isolated bridge deck. Koh and Kelly (1990), have also shown that their multi-step numerical schemes are in good agreement with the Laplace and Fourier solutions.

Recently, Kasai et. al. (1993) proposed a fractional derivative damping element for the modelling of viscoelastic dampers. The model accounts for the effects of excitation frequency, ambient temperature and temperature rise due to energy dissipation. Performance of their proposed model was demonstrated through comparison with experimental results using different excitation histories as well as temperatures.

In addition, they have shown that viscoelastic damper behaviour is sensitive to temperature and frequency, and these effects cannot be neglected in the seismic design of viscoelastically-damped buildings. However, in the case of wind loading,
their thermomechanic rule used to predict the temperature rise should be modified due to the much longer duration of loading.

Although the application of the fractional derivative damping model in damped systems is quite successful and it is an area of continuing research, the general fractional derivative approach to time domain analysis is cumbersome. In the study herein, the author attempted to use the concept of generalised derivatives in the frequency domain dynamic analysis.

2.4 DYNAMIC ANALYSIS TECHNIQUES

Determining the dynamic response of a damped structure is one of the most demanding challenges for structural dynamicists. The successful determination of the dynamic response of a structural system to prescribed loading time histories depends on the successful solution of two problems. The first problem is that of describing the mechanical properties of the structural materials (stiffness, mass, damping) in a mathematically rational manner. The second problem is solving the resulting equations of motion for the structure. This section introduces the numerical techniques which can be used to solve the dynamic equation of motion.

2.4.1 Time Domain Dynamic Analysis

The evaluation of the vibration response for a dynamic system may proceed following two basic methodological approaches: the time domain integration method or the frequency domain integration method. The superiority of one approach compared with the other strongly depends on the problem. The conventional time domain method is based on satisfaction of dynamic equilibrium at selected time intervals. Hence the extension to non-linear problems can be dealt with by introducing iterative time-stepping techniques.
Direct Integration Method: Direct integration in the time domain for determining the dynamic response of a damped system may be conceived as a sequence of pseudo-static problems at each time step. The term “direct” means that prior to the numerical integration, no coordinate transformation of the dynamic equations is carried out. The equations for direct integration are either explicit or implicit. In the explicit equations, neither the displacement nor the velocity at the current time step is a function of the acceleration at the next time step. On the other hand, in the implicit equations, the acceleration is a function of the displacement and velocity in last time step.

As the set of dynamic equilibrium equations are regarded as a system of ordinary differential equations with constant coefficients, any convenient finite difference expressions to approximate the accelerations and velocities in terms of displacements can be used. However, as far as the efficiency is concerned, only a small number of different finite difference schemes need be considered.

The central difference method (Collatz 1966) is an efficient explicit integration method in which the algorithm for the step-by-step solution can be easily operated. The computation starts with describing the displacement and velocity initial conditions for the dynamic system. From these two conditions, the acceleration can be found using the equations of motion.

To construct the central difference algorithm, difference expressions for the nodal velocities and acceleration at the current time are first written. The difference equations and the equation of motion can now be used repeatedly; the equation of motion gives the displacements at each time step while the difference equations gives accelerations and velocities, and then the process is repeated at each time step.
The Houbolt (Houbolt 1950) and Wilson θ (Wilson et. al., 1973) methods are also considered efficient numerical schemes in dealing with dynamical systems. However, they differ from the central difference method since they use an implicit integration scheme. Although the acceleration and velocity are approximated in terms of the displacement components as for the central difference method, a linear variation of acceleration from the current time step to the next time step is assumed.

The Newmark method (Newmark 1959) is also an implicit type finite difference scheme which can be considered to be an extension of the Houbolt and Wilson methods. The difference expressions are more complicated than the previous linear acceleration methods, and two individual parameters are used to obtain integration accuracy and stability. The method is an unconditionally stable scheme which is based on the constant-average-acceleration method. One of the advantages of the Newmark scheme is that it reduces to the static approach when the inertia and damping effects are neglected. Another advantage of the method is that a larger time step can be used in the integration scheme compared with other finite difference methods.

*Modal Analysis:* As the number of operations required in the direct integration are directly proportional to the number of time steps used in the analysis, the use of the direct integration technique can be expected to be efficient when the response for a relatively short duration of time is required. If the integration needs be carried out for a long time window, it seems that it would be more effective to use an alternative form of equilibrium equations. The power of modal analysis is that it shows the way for replacing a large dynamic system by one of much smaller size.
The objective of the transformation is to obtain new system stiffness, mass and damping matrices which have a smaller bandwidth than the original system. An alternative model of the system which describes the dynamic properties as a function of frequency, can be formed using a partial eigenvalue analysis. The structure’s modal matrices can be used to transform the original equations of motion so that the resulting uncoupled equations of motion can be solved.

2.4.2 Frequency Domain Dynamic Analysis

The use of numerical methods in the frequency domain to obtain the dynamical response of a system can be efficient when it is noted that this frequency-dependent behaviour of structural materials can be treated linearly in the transform frequency domain instead of in the direct time domain. In fact, the frequency domain strategy to solve dynamic problems is an alternative approach to the classical time domain method.

The dynamic analysis in the frequency domain has some advantages such as spectral decomposition of the forcing function, which helps to set bounds on the dynamical problem. Also, the stability and efficiency of the Fast Fourier Transform, which allows the use of a larger time step compared with the direct integration time step which is an advantage. Moreover, for systems with frequency dependent parameters, and also in those cases with long time duration or high stiffness coupling, frequency domain analysis may be the only effective means of determining the dynamic response.

Doyle (1989), in particular proposed the frequency domain spectral method of dynamic analysis which concerns the synthesis of waveforms from the superposition of many frequency components. The spectral approach is based on the spectral solution of the governing differential equations for the dynamic motion of
structures. However, the approach differs from the classical method because it uses the Fast Fourier Transform for conversion purposes. As the method uses discrete rather than continuous transforms, the frequency range is finite. Discrete points can be obtained by summing the components over frequency and wave numbers.

One of the advantages of the frequency domain spectral theory is that the frequency-dependent properties can be incorporated easily. In fact, to adapt the frequency-dependent damping model into the complex spectral method, the changes required are that the spectral relations should be modified.

2.4.3 Efficiency Study

The numerical methods described in the previous sections have their own advantages and disadvantages. Although the superiority of one numerical method compared with others is strongly problem-dependent, the following general advantages and shortcomings can be found by carefully reviewing these methods:

- The advantage of using the central difference method is that as the solution can essentially be carried out at the element level, there is no need for the global assembled stiffness and mass matrices.

- In so far as computer memory is concerned, the central difference algorithm needs little memory storage.

- One of the major shortcomings of the central difference method is that the system matrices should be diagonal. This means that a dynamical system with non-proportional damping cannot be solved with this method.
• Another major shortcoming is that the central difference method requires a smaller time step to have an unconditionally stable solution compared with the Houbolt and Wilson methods.

• A major advantage of the Houbolt and Wilson θ methods is that the dynamic solution scheme based on these methods reduces directly to a static analysis when the inertia and damping forces are neglected.

• Another advantage of the Houbolt and Wilson θ methods is that as the displacements, velocities and accelerations at the current time step are expressed in terms of the same quantities in the last time step, they need no special starting procedures.

• The Newmark integration scheme is unconditionally stable (Newmark 1959).

The use of a combination of numerical schemes for the integration of dynamic response is also advantageous when the stiffness and mass of the system are quite different (e.g., as in a fluid-structure system).

2.4.4 Finite Element Method

The finite element method is the most powerful and popular method for solving field equations of motion. The application of the finite element method to structural dynamics is popular and also successful. However, in order to guarantee stability and accuracy of the solution, the number of elements used to model a
structure may be very large indeed; more precisely, accurate results can be obtained only after a substantial computational effort.

The finite element method approximates structural systems in two distinct ways. Firstly, it divides the structural system into a number of small simple parts. The procedure of dividing up the system is called discretisation. The second approximation made in the finite element modelling is to solve the equation of vibration for each individual finite element by using a linear combination of low-order polynomials.

Based on the polynomial displacement functions, which are assumed over an element of the system, individual element matrices are calculated and then assembled into global system matrices. These matrix equations of motion are decoupled through different numerical schemes and the results are then obtained.

2.4.5 Recent Damped Finite Element Methods

It is well known that the solution of the forced vibration of a non-linear multi-degree-of-freedom system is very cumbersome when conditions which allow the equations to be uncoupled do not exist. In practice, as most real-life dynamic systems are comprised of different subcomponents, non-proportional damping is usually a common occurrence. Several numerical techniques have been proposed in recent decades to overcome the problem, but none of these methods has the generality to analyse the wide range of dynamical problems requiring solution.

Udwadia and Esfandiari (1990), have proposed a finite element scheme which uncouples the equations of motion of non-classically damped systems and computes the response iteratively. They show that convergence is guaranteed for
certain classes of matrices even when the undamped natural frequencies may be relatively small. However, their convergence results are limited to only those matrices which are either strictly diagonally dominant or which belong to a special class of symmetric and positive-definite matrices.

Recently Udwadia and Kumar (1994), have introduced two different sets of iterative schemes for determining the response of non-classically damped dynamic systems. Their schemes are superior to the previous proposed scheme (Udwadia and Esfandiari, 1990) in that they are applicable to a much wider class of matrices. They have also shown that the new iterative sets are computationally more efficient.

For the frequency domain dynamic analysis of multi-degree-of-freedom systems with non-proportional damping, Humar and Xia (1993) have proposed a new numerical procedure. They argue that, as the difference arises from the convolution in the frequency domain because of overlap between two subsequent periods of the time functions, a numerical procedure can be derived to compensate for the errors caused by such overlap. Their method is applicable to both single-and multi-degree-of-freedom systems as well as to analyses using the substructuring technique.

Among other recent researchers who developed damped finite elements, Golla and Hughes (1985) proposed a damped finite element scheme which assumes that the structural damping is due to material viscoelasticity. In their approach, appropriate material properties are identified which permit the standard finite element formulation used for undamped structures to be extended to viscoelastic structures. The damping matrix in their method is explicitly calculated using a system of coordinates that has been augmented by the inclusion of an appropriate set of dissipation coordinates. However, the values of certain viscoelastic material damping parameters must be available for calculating the damping matrix.
A damped finite element analysis for viscoelastically damped structures is also proposed by Bagley and Torvik (1983). In their approach, the fractional calculus concept is used to construct a stress-strain relationship for viscoelastic structural materials. The attractive feature of their approach is that very few damping parameters are required to model the viscoelastic material. They have shown that their fractional derivative can accurately describe the mechanical properties of a variety of materials over wide ranges of frequency. However, as the expanded form of the damped equations of motion is very large in their finite element scheme, the numerical task would be prohibitive even when implemented on today’s computers.

In the literature, several other techniques have been proposed to increase the efficiency and speed of the method (Shahruz and Langari 1991, Shahruz and Packard 1991, Duncan and Taylor 1979, Clart and Venanico-Filho 1991), but these approaches are mainly focussed on creating advanced computer algorithms to save computing power and time.

2.5 DAMPED STRUCTURES

Structural damping mostly arises as a result of many energy dissipation mechanisms acting in a system. These sources of damping might be classified by considering a structure as an assemblage of elements which interact at interfaces (nodes). Material damping occurs within the elements, and can be added to joint impact and friction damping arising from the interaction of one element with another at a common interface, or from the interaction of the structure with a non-structural internal or external environment.
2.5.1 Added Structural Dampers

In recent years, both researchers and practising engineers have recognised that energy dissipation dampers can provide an efficient means for controlling the response of a structure induced by dynamic loads. Most structures can be designed to withstand severe dynamic forces by providing ductility and energy absorption capacity in the structural elements, but usually at the expense of substantial damage to the structural elements, and also to non structural elements and services.

On the other hand, by dissipating the vibratory energy in structural dampers, the risk of the structure experiencing excessive deformations or accelerations can be reduced. As a result, less ductility or inelastic energy demand is required in the structural frame. In particular, structural isolation systems can be designed essentially to limit the non-linear behaviour to the isolation devices, thereby imposing very small or no ductility demand on the structure itself.

Many different analytical methods have been proposed in the literature for dealing with passive and active structural vibration dampers (Indaudi et. al. 1993, Kasai et. al. 1993 and others). These structural dampers can not only reduce steady-state vibration levels, but they can reduce the time needed for transient vibration to settle. Passive structural dampers are easier to analyse as active structural dampers usually require a sophisticated control system. In addition, the design of stable, fast responding control systems for active structural dampers require an accurate knowledge of inherent damping behaviour of structural materials.

Recently, Makris and Constantinou (1992), have developed an analytical method for spring-viscous structural dampers. The coupled lateral-vertical-rocking dynamic response of spring-viscous structural dampers is considered in their numerical approach. The mechanical characteristics of structural viscous dampers is
described using the concept of a fractional derivative viscoelastic model, and the resulting equations of motion are derived and reduced to a form for direct solution by the discrete Fourier transform method.

Huffman (1986) also proposed a linear damping model for describing the characteristics of damping resistance of visco-dampers (i.e., GERB viscodampers). Later Schwahn et. al. (1987), presented an equivalent rheological model for visco-dampers and discussed the implication of using these models. An equivalent stiffness and damping concept was also used by Kawamata (1988) to describe the analysis of liquid mass dampers.

Among the researchers who developed analytical schemes for structural dampers are Inaudi and Kelly (1993). Their numerical scheme considers a hybrid absorption system in structures which includes rubber bearings, visco-dampers and actuators. They have developed an integrated design procedure for the passive and active components of the structural dampers. Fractional derivative Maxwell damping elements were used to model the mechanical behaviour of the visco-dampers. The active component of the isolation system was designed by considering applied forces proportional to the absolute velocity of the active structural damper. The optimal hybrid damping system was then designed by employing constraints to the deformation capacity of the structural dampers as well as constraints to the capacity of the actuators.

Tuned Mass Dampers (TMD’s) are one of the simplest and most reliable structural damper systems. The system is composed of a mass, a spring and a damper. The mechanism of suppressing structural vibrations by attaching a TMD to the structure is to transfer the vibration energy of the structure to the TMD and to dissipate the energy at the damper. Yamaguchi and Harnpornchai (1993) have proposed a multiple TMD damping system consisting of many tuned mass dampers.
with distributed natural frequencies for suppressing effectively the harmonically forced single mode response of structures.

Samali et. al. (1993a, 1993b) have also investigated the effectiveness of TMD and TLCD (tuned liquid column damper) structural dampers under non-stationary earthquake and wind-induced motions. They have shown that both TMD and TLCD structural dampers are capable of reducing the earthquake and wind-induced responses. However, their results show that the TMD structural damper seems to have a slight advantage over TLCD structural dampers in regard to its response reduction capability in earthquake-induced motions.

Viscoelastic dampers have also been incorporated in several tall buildings to reduce vibration due to wind and earthquake loading. Mahmoodi (1969) was among the first who proposed an analytical method to investigate the behaviour of these dampers. Later Aiken, Kelly and Mahmoodi (1990) conducted a series of earthquake simulation tests of a large model structure fitted with viscoelastic damping devices. For all of the earthquake tests the model response was reduced compared with the undamped structure. Furthermore, in some earthquake cases, the dampers reduced the structural accelerations and interstory drift by as much as one half. Since the installation of these dampers in 1969, many different analytical methods have been proposed to model the frequency and temperature dependent damping characteristic of dampers (Kasai et. al. 1993, Zhang and Soong 1992, Keel et. al. 1986, Aiken et. al. 1990, Blondet 1993, Crosby et. al. 1994, Oh et. al. 1992 and others).

2.5.2 Base-Isolated Structures

The concept of base-isolation is one in which flexibility and energy absorption capacity are provided by a specially designed isolation system that is
placed between the superstructure and its foundation. Many unimplemented base-isolation schemes have been proposed over the past one hundred years but the concept has become a practical one with the recent development of reliable analytical methods.

A variety of analysis and design methods are presently available for modelling the response of non-linear damping devices in the base, as well as the structure itself. A very important application of non-linear dynamics arises from base isolation analysis where the superstructure is assumed to remain elastic during seismic excitation while the isolation system undergoes non-linear motion. Excitation of structures by earthquake ground motion induces inertia forces which depend on the dynamic properties of the structural system and the characteristics of the ground excitation.

In recent years Nagarajaiah et. al. (1991) proposed a solution algorithm for the non-linear dynamic analysis of three-dimensional base-isolated structures with elastomeric and/or sliding isolation systems. The novelty of their analytical model and solution algorithm is its capability to capture the highly non-linear frictional behaviour of sliding isolation systems. Their approach is also capable of dealing with the three dimensional behaviour such as lateral-torsional response and the effect of biaxial interaction in isolation bearings on the response of base isolated structures.

The non-linear time domain method is based on assumptions that; firstly, the superstructure remains elastic at all times, secondly, there exists a rigid slab at the base level so that all isolation elements are connected, and finally, the dynamic characteristics of each floor can be modelled using three degrees of freedom in which these degrees of freedom are located at the center of mass of each floor. As the isolation system often experiences multi-directional motion under multi-
directional excitation, the method uses a biaxial hysteretic element based on the Park
differential equations (Park et. al., 1986).

However, as the non-linear time domain approach assumes three degrees of
freedom for each floor it is not capable of dealing with floor flexibility. Furthermore, the number of natural modes required for modal reduction should
always be a multiple of three, as three degrees of freedom per floor are considered.

Bilinear or trilinear models were also used to model isolation elements like
lead-rubber bearings by Miyazaki (1988), and later by Fujita et. al. (1989). Many
Japanese researchers (Yasaka et. al. 1988, Lee 1980 and others) have used the
bilinear model for modelling lead-rubber bearing and steel dampers. In particular,
Mizukoshi (1989) analysed a nuclear reactor building on a lead-rubber isolation
system, including soil-structure interaction, to study the torsional response of the
structure.

Constantinou et. al. (1990), have developed a damping model which has the
advantage of computational efficiency over hysteresis models. They have used the
differential equation model proposed by Wen et. al. (1976) to develop a
viscoelastic damping model which is capable of capturing most of the mechanical
behaviour of isolation devices. His model is capable of modelling lead-rubber
bearings, high damping elastomeric bearings, steel dampers and sliding bearings.

A general alternating frequency/time domain method (G-AFT) was
proposed by Aprile et. al. (1994) to deal with base isolated structures. Their
frequency domain non-linear dynamic analysis approach is a unifying proposal
which combines the features of time and frequency domain methods in an iterative
solver. The G-AFT method is capable of dealing with different problems such as
non-proportional damping, Coulomb damping, and particularly soil-structure and base isolation systems. The performance of the method is particularly powerful in cases when damping is not uniform within the structure or is even localised in just one component dominating the dissipation of the whole system.

### 2.6 SUMMARY AND CONCLUDING REMARKS

It is well known that the application of the finite element method has had a profound effect on the vibration analysis of structures. The idea of increasing the stability and efficiency of the conventional finite element method has been a major reason behind much research work in past decades. Conventional finite element methods treat the inertia forces induced by the mass as concentrated loads applied to the end nodes of each element. Consequently, if the structural joints are far apart, many elements should be used to model the mass distribution adequately.

As the conventional finite element method requires many elements to model the mass distribution adequately, the computer power and time needed for dynamic analysis of large complex structures is substantial. So, it seems to be advantageous to search for a procedure requiring less computer time and effort to model a structure. On the other hand, although several techniques have recently been proposed in order to find reliable approximate methods able to decouple the damped motion equations, the validity of these approximations depends on the form of the damping matrix.

As stated earlier, the solution to any dynamical problem, in general, can proceed by using two basic methodological approaches: the time domain integration method and the frequency domain integration method. The time domain integration method is usually thought to be superior, as the extension to non-linear problems can be dealt with by introducing iterative techniques (as in static analysis).
On the other hand, the use of numerical methods in the frequency domain to describe the frequency-dependent mechanical characteristic of structures is also seen to be efficient when it is noted that this non-linear behaviour can be treated linearly in the transform frequency domain instead of in the direct time domain. Hence, it seems to be advantageous to formulate a unifying proposal which combines the features of time and frequency domain methods in an efficient manner. The proposed method should satisfy the following requirements:

- The mechanical characteristics of structural materials (i.e., frequency and temperature dependency of the damping materials) should be modelled accurately without using complicated models with a large number of parameters.

- More advanced displacement functions must be formulated for individual elements so as to create an accurate model of dynamical systems where the choice of the element size is not restricted by the need to approximate the distributed mass and allows the choice to be governed by the structural connections and discontinuities.

- The method should be flexible enough so that higher order rod and beam theories (Timoshenko) can be formulated easily in the case of framed structures.

- The effect of damping should easily be adapted into the method. The method should have a complex representation so that non-proportional damping can be incorporated without any special treatment.
In essence, attention is paid herein to the alternative spectral approach which works in the frequency domain and draws its robustness from the speed and switching capabilities of the Fast Fourier Transform. As the only way to solve problems with complicated boundaries and discontinuities is to develop a matrix methodology for use on a computer, the mathematical basis of the spectral method can be used to develop a numerical method using matrix methodology. The spectrally formulated elements can be used to model different structural members.

The concept of the fractional derivative damping model, capable of representing different damping mechanisms, can be used to solve the dissipation problem in damped systems. While traditional differential operators are typically employed in a conventional formulation, the fractional operators give rise to a richer variety of functional families and hence lead to an improved integro-differential type curve fitting of constitutive representations of the damping material.

In the case of non-linear dynamical systems, an Alternative Frequency/Time domain method (AFT) can be adapted to solve highly non-linear dynamic problems. The non-linear contribution can be of the form of mass, stiffness or damping non-linearity. The AFT method is based on the concept that all non-linear contributions can be moved to the right hand side of the equation (pseudo-force term as discussed by Udwadia et. al., 1994), leaving on the left hand side the conventional dynamical system equation. The analysis can proceed by computing alternately the cited pseudo-force contribution and the system response, until the solution converges to some defined degree of accuracy.
Chapter Three

Analytical Modelling of Damping in Structures

GENERAL

To design a structure in terms of its vibration response, the desired response of the structure under dynamic loads must be clearly stated. Vibration can generally lead to a number of undesirable circumstances in which the structure can ultimately fail. Many different methods of measuring damping have been proposed so that the desired displacement, velocity or acceleration can be achieved. Passive and active damping of vibration in structures can be very important for several reasons. In terms of performance, higher damping can reduce steady-state vibration time, and also it can reduce the time needed for transient vibrations to settle. Generally, passive damping can reduce the complexity of the active control needed.

Structural damping mostly arises as the result of many energy dissipation mechanisms acting in a system. These sources of damping might be classified by considering a structure as an assemblage of elements which interact at interfaces (nodes). Material damping occurs within the elements, and can be added to joint impact and friction damping coming from the interaction of one element with another at a common interface, or from the interaction of the structure with a non-structural internal or external environment. Material damping is generally a complex
function of frequency, temperature, type of deformation, amplitude and structural geometry. Material damping is inherently a thermodynamic phenomenon which arises as the result of the mutual coupling of stress and strain with other material state variables. A new equilibrium state can be obtained in the material only through kinetic processes like diffusion or twinning as in superelastic materials.

The material scientist is mainly interested in the mechanism and accurate measurement of the intrinsic material damping. On the other hand, the structural dynamic researcher is primarily interested in predicting the damping of a structure. The main objective of this Chapter is to establish a fundamental connection between material and structural damping in a distributed-parameter system. The exact frequency equation of the system can then be obtained using the Mathematica computer package. This exact frequency equation can then be used to derive an analytical damping model in a form using complex moduli for the material. The concept of exact frequency equation and complex moduli will be used subsequently in the next Chapters to develop an exact finite element numerical procedure.

3.2 DAMPING IN ENGINEERING MATERIALS

The basic and most common equation of motion used in structural dynamics is:

\[ M\ddot{x} + C\dot{x} + Kx = f(t) \]  

(3-2-1)

which in the case of constant \( M, C \) and \( K \) is a linear differential equation and can be solved using a basic mathematical solution. In this equation \( C \) is the damping matrix with appropriate assumptions concerning the proportionality of damping with the velocity of motion. It can be shown that the damping matrix in this form has to be
positive semi-definite and symmetric. Generally, there is no known procedure for determining its elements, except in cases involving discrete viscous damping. It can also be proved that no simple, element-based damping matrix can accurately represent the behaviour of real engineering materials over a range of frequencies (Tschoegl, 1989). The most general method of damping measurement involves the use of dimensionless quantities dealing with the frictional displacement and energy loss per cycle of vibration. Some of the dimensionless measures are, $\psi$ (damping capacity), $\delta$ (log decrement), $\eta$ (loss factor), $Q$ (quality factor), $\zeta$ (damping ratio). For small damping, these measures can be converted to other units using the following scale factors (Lesieutre, 1989),

$$\zeta = \frac{\eta}{2} = \frac{\psi}{4\pi} = \frac{\delta}{2\pi} = \frac{1}{2Q} \quad (3-2-2)$$

There are many different approaches to model damping that can be found in structural dynamic text books (i.e., viscous, frequency-dependent viscous, viscoelasticity). The viscous damping model is the simplest approximate model in which the energy dissipated per cycle at a single frequency is matched with the experimental measurements. One of the most accurate damping models in structural dynamics is the viscoelastic model in which a generalised form of the constitutive stress-strain relation is written in the form of a differential equation. Although the viscoelastic damping model is capable of producing accurate results, wide application of the model has not been found due to the inherent complexity of its mathematical equations and the difficulty of incorporating it in the popular numerical solutions. In the study herein, an attempt has been made to develop an efficient numerical method to solve the dynamic problem of damped systems. The basic mathematical connection between the intrinsic material damping and structural damping in the form of a viscoelastic constitutive relation will be presented in this
Material behaviour is termed viscoelastic if the material stores part of the deformational energy elastically as potential energy, and dissipates the rest simultaneously through viscous forces. The rheological properties of a viscoelastic material are time-dependent. Although, in principle, all real materials are viscoelastic, this property becomes significant when the time required for the full development of a response is comparable with the time scale of the test performed to determine it. When a stress or a strain is impressed upon a body, rearrangement take place inside the material as it responds to the imposed excitation. In any real material these rearrangements necessarily require a finite time. As a consequence of the material rearrangement taking place on a time scale comparable to that of the test in which the response is observed, the relation between stress and strain or rate of strain cannot be expressed by material constants as in the case of purely elastic or purely viscous material.

\[
\sigma = E\varepsilon \quad (\text{elastic}) \tag{3-2-3}
\]
\[
\sigma = \eta \frac{d\varepsilon}{dt} \quad (\text{viscous}) \tag{3-2-4}
\]

where \(\eta\) is the viscosity. It has been shown (Tschoegl, 1989) that the simplest constitutive equation which adequately describes the infinitesimal deformation of a viscoelastic body is

\[
\sigma + a \frac{d\sigma}{dt} = b\varepsilon + c\frac{d\varepsilon}{dt} \tag{3-2-5}
\]

where \(a, b\) and \(c\) are constants. This linear differential equation can be expressed in another form,
where $u_n$ and $q_m$ are constant coefficients. The manipulation of this differential integral equation is facilitated by the use of integral transform methods. For the purpose herein the one-sided Laplace transformation is most suitable. Therefore, Eq. (3-2-6) can be rewritten as

$$\tilde{u}(s)\tilde{\sigma}(s) = \tilde{q}(s)\tilde{\varepsilon}(s)$$

(3-2-7)

Where $\tilde{\sigma}(s)$ and $\tilde{\varepsilon}(s)$ are the stress and strain transforms respectively, and

$$\tilde{u}(s) = \sum u_n s^n \quad \text{and} \quad \tilde{q}(s) = \sum q_m s^m$$

(3-2-8)

are polynomials in the transform variable $s$. Rearranging Eq. (3-2-7),

$$\tilde{\sigma}(s) = \frac{\tilde{q}(s)}{\tilde{u}(s)}\tilde{\varepsilon}(s)$$

(3-2-9)

$$\tilde{\varepsilon}(s) = \frac{\tilde{u}(s)}{\tilde{q}(s)}\tilde{\sigma}(s)$$

(3-2-10)

Then

$$\tilde{Q}(s) = \frac{\tilde{q}(s)}{\tilde{u}(s)} \rightarrow \tilde{\sigma}(s) = \tilde{Q}(s)\tilde{\varepsilon}(s)$$

(3-2-11)
Eqs. (3-2-11) and (3-2-12) form the basis upon which the linear theory of viscoelastic behaviour is developed. These equations also form the basis for the important correspondence or equivalence principle (Tschoegl, 1989). According to this principle, if an elastic solution to a boundary value problem (stress analysis problem) is known, substitution of the appropriate Laplace transforms for the quantities employed in the elastic analysis furnishes the viscoelastic solution in the transform plane. The time dependent viscoelastic solution is then obtained by inverting the transform. The principle can be applied if the boundaries themselves do not change with time.

As stated earlier, in the deformation of a viscoelastic body part of the total work of deformation is dissipated as heat through viscous losses but the remainder of the deformational energy is stored elastically. It is of interest to determine the amount of energy dissipated. The rate at which energy is absorbed per unit volume of a viscoelastic material during deformation is defined as being equal to the stress power (Tschoegl, 1989). The stress power at time $t$ is

$$\dot{w}(t) = \sigma(t)e'(t)$$

which is the product of the instantaneous stress and rate of strain. The energy stored $w_s(t)$, and the energy dissipated, $w_d(t)$, combine to make up the total deformational energy. Thus

$$w_t(t) = w_s(t) + w_d(t)$$
For harmonic response it is not difficult to prove (Tschoegl, 1989),

\[
\frac{\frac{W_d}{2w_s}}{\psi_w} = 2\pi \tan \theta
\]  

(3-2-15)

The ratio \(\psi_w\) is sometimes known as the specific loss, and \(\theta\) is the loss tangent, and is the angle by which the stress leads the strain in a steady-state time harmonic loading of the material. The elastic response of an isotropic material is governed by two independent elastic constants, for example Young's modulus \(E\), and the shear modulus \(G\). These constants can be replaced by complex moduli \(E_v\) and \(G_v\) in a viscoelastic material as follows,

\[E_v = E' + iE'' = |E_v|e^{i\alpha \theta_E}\]

(3-2-16)

\[G_v = G' + iG'' = |G_v|e^{i\alpha \theta_G}\]

(3-2-17)

\[\omega_v = \omega' + i\omega'' = |\omega_v|e^{i\alpha \frac{\omega'}{\omega''}}\]

(3-2-18)

\[\theta_k = \frac{k''}{k'}\]

(3-2-19)

where \(\omega', \omega''\) and \(k', k''\) are the frequency and stiffness real and imaginary parts respectively, \(\theta_E = \frac{E''}{E'}, \theta_G = \frac{G''}{G'}\) and \(\theta_k\) is a loss angle of stiffness for the structure. Recalling Eq. (3-2-15), it can be seen that (Horr and Schmidt, 1995),

\[\theta_k = \frac{k''}{k'} = 2\frac{w''}{w'} \rightarrow \theta_k = 2\theta_w\]

(3-2-20)
where \( w', w'' \) are the real and imaginary parts of the energy and \( \theta_w \) is the energy phase angle,

\[
\psi_j = 2\psi_w = 2(2\pi \tan \theta_w) = 4\pi \tan \theta_w
\]  

(3-2-21)

where \( \psi_j \) is the specific damping capacity of the structure.

In an earlier work, Kinra and Yapura (1990) obtained the viscoelastic structural damping with the use of the correspondence principle. They assumed that for most structural materials it is unrealistic to represent \( E \) with a very large phase angle \( \theta_E \). In fact, the real part of the complex modulus \( E \) is always much larger than the imaginary part for most practical cases. Fig. 3-1 shows a schematic sketch of \( E_v \) in a complex system for a realistic value of \( \theta_E \).

![Figure 3-1](image)

**Figure 3-1**

Phase Angle for Complex Modulus \( E_v \).
It can be easily assumed that

\[ |E_\nu| = E' = E \]  \hspace{1cm} (3-2-22)

\[ E_\nu = |E| e^{i \tan \theta_E} \]  \hspace{1cm} (3-2-23)

\[ \tan \theta_E = \frac{E'}{E} \quad (\theta_E < 2^\circ) \]  \hspace{1cm} (3-2-24)

Similar approximations can be made in the case of \( \omega \). The key argument is that as the imaginary parts of \( E \) and \( G \) are small, the imaginary part of \( \omega \) is necessarily small. This point allows a Taylor series expansion of the exact frequency equation about \( \omega, E \) and \( G \) to be performed.

\[ f(E_\nu,G_\nu,\omega_\nu) = f(E,G,\omega) + \Delta E_\nu \frac{\partial f}{\partial E} + \Delta G_\nu \frac{\partial f}{\partial G} + \Delta \omega_\nu \frac{\partial f}{\partial \omega} + \text{higher order terms} = 0 \]  \hspace{1cm} (3-3-25)

After neglecting the higher order terms, Eq. (3-3-25) gives

\[ \Delta \omega_\nu = \frac{G_\nu \frac{\partial f}{\partial G} - E_\nu \frac{\partial f}{\partial E}}{\frac{\partial f}{\partial \omega}} \]  \hspace{1cm} (3-2-26)

Noting that for a small damping \( \Delta E_\nu = iE^*, \Delta G_\nu = iG^* \) and \( \Delta \omega_\nu = \omega^* \), Eq.(3-2-26) can be rewritten as
\[ \omega^* = \frac{G^* \frac{\partial f}{\partial \omega} - E^* \frac{\partial f}{\partial \omega}}{\frac{\partial f}{\partial \omega}} \]  

(3-2-27)

Substituting Eq. (3-2-27) into Eq. (3-2-21) gives

\[ \psi_s = 4\pi \tan \theta_\omega = 4\pi \tan \frac{\omega^*}{\omega} = -2 \frac{\frac{\partial E}{\partial E} + G \frac{\partial G}{\partial G}}{\frac{\partial f}{\partial \omega}} \]  

(3-2-28)

\[ \psi_E = 2\pi \tan \theta_E, \quad \psi_G = 2\pi \tan \theta_G \]  

(3-2-29)

where \( \theta_E \) and \( \theta_G \) are longitudinal and shear loss factors respectively. For small damping, \( \psi_s \) (structural damping capacity) can be converted to the damping ratio, \( \zeta \), which is the most common measure of damping used by structural engineers as,

\[ \zeta = \frac{\psi_s}{4\pi} = \frac{-\frac{\partial E}{\partial E} + G \frac{\partial G}{\partial G}}{\omega \frac{\partial f}{\partial \omega} * 2\pi} \]  

(3-2-30)

One of the simplest methods for measuring damping in a material is to observe the free vibration of a clamped-free beam with a tip-mass. The vibration of the beam can be modelled using different theories of vibration (i.e. Euler-Bernoulli, Timoshenko). In the next sections the consistency of the proposed analytical damping model will be investigated using both the elementary Euler-Bernoulli and higher order Timoshenko theories. An exact solution to the Timoshenko differential equations is
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3.3 DAMPING IN VIBRATION THEORIES

As was stated earlier, one of the simplest methods for measuring damping in an engineering material is to observe the free vibration of a clamped-free beam with a tip-mass. Fig. 3-2 shows a beam carrying a tip-mass at the free end.

![Figure 3-2](image)

Clamped-Free Beam Carrying an End-Mass

To find the analytical damping capacity of the beam, it is first necessary to find the frequency equation of the beam. In what follows the exact frequency equation of the beam is derived using Euler-Bernoulli and Timoshenko beam theories. The damping capacity can then be obtained by using Eq. (3-2-28).

3.3.1 Euler-Bernoulli Beam Theory

Consider the flexural vibration of a clamped-free beam with an end-mass as shown in Fig. 3-1. The Euler-Bernoulli theory for flexural vibration of the perfectly elastic undamped beam starts with the kinetic and strain energy formulation as follows:
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\( K = \frac{1}{2} \int_{0}^{T} \rho A v^2 \, dx \) \hspace{0.5cm} \text{(kinetic energy)}; \hspace{0.5cm} V = \frac{1}{2} \int_{0}^{T} E I v'^2 \, dx \hspace{0.5cm} \text{(strain energy)} \hspace{0.5cm} (3-3-1)

Note that the effects of shear deformation and rotary inertia have been neglected in the calculation of the kinetic and strain energy. Using Hamilton's principle,

\[ EI \frac{\partial^4 v}{\partial x^4} + \rho A \frac{\partial^2 v}{\partial t^2} = 0 \hspace{0.5cm} (3-3-2) \]

Considering the beam with one end fixed, the other end free, the boundary conditions are:

\[ v(0) = 0 \hspace{0.5cm} v(\ell) = 0 ; \hspace{0.5cm} v'(0) = 0 \hspace{0.5cm} v''(\ell) = 0 \hspace{0.5cm} (3-3-3) \]

Defining the variable \( a \) as:

\[ a = \sqrt{\frac{EI}{\rho A}} \hspace{0.5cm} (3-3-4) \]

the differential Eq. (3-3-2) can be written as,

\[ a^2 \frac{\partial^4 v}{\partial x^4} = -\frac{\partial^2 v}{\partial t^2} \hspace{0.5cm} (3-3-5) \]
For this case, the frequencies of transverse vibration are solutions of the eigenvalue equation,

\[
\cos \beta \cosh \beta = -1 \tag{3-3-6}
\]

where \( \beta^4 = \omega^2 L^4 / a^2 \). Note that the frequencies are independent of the shear modulus, as expected, as shear effects have already been neglected. When the beam carries an end-mass at the free end, the eigenvalues are solutions of,

\[
\frac{M}{\rho AL} = \frac{1 + \cos \beta \cosh \beta}{\beta \left( \sin \beta \cosh \beta - \cos \beta \sinh \beta \right)} \tag{3-3-7}
\]

It is clear that when \( \frac{M}{\rho AL} \) becomes large the beam with an end-mass behaves like a single degree of freedom system with a natural frequency \( \omega = \frac{k}{\sqrt{M}} \) when \( k = \frac{3EI}{l^3} \).

For completeness, the solution of Eq.(3-3-7) for the transverse vibration of an undamped system can be in the form of,

\[
v = X( A \cos \omega t + B \sin \omega t) \tag{3-3-8}
\]

where \( X = e^x \) and coefficients \( A \) and \( B \) are arbitrary constants.
3.3.2 Timoshenko Beam Theory

The Euler-Bernoulli theory of flexural motion of elastic beams has been found to be inadequate for the prediction of higher modes of vibration, and also inadequate for those beams when the effect of cross-sectional dimensions on frequencies cannot be neglected (Clough et. al., 1975). Timoshenko beam theory takes into account the effects of rotary inertia and shear deformations in the vibration of a beam. It is not difficult to see that during flexural vibration, a typical element of a beam not only has a translatory motion but it also has rotation. With the introduction of shear deformation, the assumption of the elementary theory that plane sections remain plane is no longer valid. Consequently, the angle of rotation, which is equal to the slope of the deflection curve, is not simply obtained by differentiating the transverse displacement owing to the shear deformation. Thus, we have two independent motions, \( v(x,t) \) and \( \theta(x,t) \).

Timoshenko's corrections may be of considerable importance for studying the modes of vibration of higher frequencies when a vibrating beam is subdivided by nodal cross sections into comparatively short length portions. Timoshenko gave the equation of motion for the clamped free beam as (Timoshenko, 1921),

\[
\left( \frac{\rho A}{g} \right) \frac{\partial^2 v}{\partial t^2} - KAG \left( \frac{\partial^2 v}{\partial x^2} - \frac{\partial \theta}{\partial x} \right) = 0
\]

\[
EI \frac{\partial^2 \theta}{\partial x^2} + KAG \left( \frac{\partial v}{\partial x} - \theta \right) - \left( \frac{I_p}{g} \right) \frac{\partial^2 \theta}{\partial t^2} = 0
\]
where $K$ is a factor depending on the shape of the cross-section and $G$ is the modulus of shear rigidity. The boundary conditions of the beam (Fig. 3-2) are,

\[
\begin{align*}
    v(0) &= 0 \\
    \theta(0) &= 0
\end{align*}
\]

\[
KGA\left(\theta - \frac{\partial v(L)}{\partial x}\right) = M \frac{\partial^2 v(L)}{\partial t^2}
\]

\[
EI \frac{\partial \theta(L)}{\partial x} = -I_1 \frac{\partial^2 \theta(L)}{\partial t^2}
\]

Huang gave the solution of the differential equation as (Huang, 1961),

\[
v = v_0 e^{i\omega t}; \quad \theta = \theta_0 e^{i\omega t}
\]

Using Huang’s non-dimensional variables as,

\[
\begin{align*}
    \zeta &= \frac{x}{L} \\
    b' &= \frac{1}{EI} \frac{\rho A}{g} L^4 \omega^2 \\
    r^2 &= \frac{I}{AL^2} \\
    s^2 &= \frac{EI}{KAGL^2}
\end{align*}
\]

the boundary conditions can be rewritten as,

\[
\begin{align*}
    \zeta &= 0 \quad (x = 0) \quad v_0 = 0 \\
    \theta_0 &= 0 \\
    \zeta &= 1 \quad (x = L) \quad \frac{1}{L} \frac{d\theta_0}{d\zeta} + \frac{1}{2} b^2 \frac{MK^2}{\rho AL} \theta_0 = 0 \\
    \frac{1}{L} \frac{dv_0}{d\zeta} + \frac{M}{\rho AL} b^2 s^2 \frac{v_0}{L} = 0
\end{align*}
\]
Substituting Huang’s solutions to the governing differential equations and omitting the factor $e^{i\omega}$, Eqs. (3-3-9) and (3-3-10) can be re-written as,

\[ v_0^{(iv)} + b^2(r^2 + s^2)v_0^{'''} - b^2(1 - b^2r^2s^2)v_0 = 0 \] (3-3-15)

\[ \theta_0^{(iv)} + b^2(r^2 + s^2)\theta_0^{'''} - b^2(1 - b^2r^2s^2)\theta_0 = 0 \] (3-3-16)

The solutions of Eqs. (3-3-15) and (3-3-16) are,

\[ v_0 = c_1 \cosh b\alpha \zeta + c_2 \sinh b\alpha \zeta + c_3 \cos b\beta \zeta + c_4 \sin b\beta \zeta \] (3-3-17)

\[ \theta_0 = c_1' \sinh b\alpha \zeta + c_2' \cosh b\alpha \zeta + c_3' \sin b\beta \zeta + c_4' \cos b\beta \zeta \] (3-3-18)

where

\[ \alpha = \frac{1}{\sqrt{2}} \left\{ \sqrt{r^2 + s^2} + \left[ (r^2 - s^2) + \frac{4}{b^2} \right]^{1/2} \right\} \] (3-3-19)

and $c_1, ... , c_4, c_1', ... , c_4'$ are constants. For the clamped-free beam, there are two branches,

\[ \left[ (r^2 - s^2)^2 + \frac{4}{b^2} \right]^{1/2} > (r^2 + s^2) \] (3-3-20)
For the first branch, which considers the ratio of the shear rigidity $GAK$ to the rotary inertia $\rho I$ in the frequency equation, it can be seen that,

$$\begin{align*}
c_1 &= \frac{L}{b\alpha} \left[ 1 + b^2 s^2 (\alpha^2 + r^2) \right] c_1' \\
c_2 &= \frac{L}{b\alpha} \left[ 1 - b^2 s^2 (\alpha^2 + r^2) \right] c_2' \\
c_3 &= \frac{L}{b\beta} \left[ 1 + b^2 s^2 (\beta^2 - r^2) \right] c_3' \\
c_4 &= \frac{L}{b\beta} \left[ 1 + b^2 s^2 (\beta^2 - r^2) \right] c_4'
\end{align*} \tag{3-3-22}$$

If the second branch assumed, the following alternative formulation can be used

$$\alpha = \frac{1}{\sqrt{2}} \left\{ \left( r^2 + s^2 \right) - \left[ \left( r^2 - s^2 \right) + \frac{4}{b^2} \right] \right\}^{\frac{1}{2}} = j \alpha' \tag{3-3-23}$$

The value of $\alpha$ can then be substituted into Eqs. (3-3-17) and (3-3-18). Solutions of Eqs. (3-3-17) and (3-3-18) are the solution of the original coupled Eqs. (3-3-9) and (3-3-10). For purposes of simplicity in this work, we will follow the first branch. Substituting the solutions of Eqs. (3-3-17) and (3-3-18) into the boundary Eqs. (3-3-15) and (3-3-16) and using the variables,
\[
\begin{align*}
\kappa &= \alpha^2 + s^2 \\
m &= \frac{s^2 - \beta^2}{\alpha} \\
p^2 &= \frac{1}{2} l^4 M_t \frac{M_b}{p^2 A^2} \\
q &= \frac{M_b}{pAL}
\end{align*}
\]

\[
\begin{align*}
k_1 &= q \cosh b\alpha + \frac{b}{\alpha} \sinh b\alpha \\
k_2 &= q \sinh b\alpha + \frac{b}{\alpha} \cosh b\alpha \\
k_3 &= q \cos b\beta + \frac{b}{\beta} \sin b\beta \\
k_4 &= q \sin b\beta - \frac{b}{\beta} \cos b\beta \\
k_5 &= n(b\alpha \cosh b\alpha - p \sinh b\alpha) \\
k_6 &= n(b\alpha \sinh b\alpha - p \cosh b\alpha) \\
k_7 &= n(b\beta \cos b\beta - p \sin b\beta) \\
k_8 &= n(b\beta \sin b\beta + p \cos b\beta)
\end{align*}
\]

the following equations can be derived,

\[
\begin{align*}
c_1 + c_3 &= 0 \\
c_2 n - c_4 m &= 0 \\
k_1 c_1 + k_2 c_2 + k_3 c_3 + k_4 c_4 &= 0 \\
k_5 c_1 + k_6 c_2 + k_7 c_3 + k_8 c_4 &= 0
\end{align*}
\]

(3-3-24)

Consider the coefficients of the four equations as matrix \( C \). In order that the solutions other than zero may exist the determinant of the \( C \) matrix must be equal to zero. This leads to the frequency equation

\[
|C| = 0 \rightarrow m(k_2 k_7 - k_6 k_8 + k_1 k_6 - k_2 k_8) + n(k_5 k_8 - k_7 k_4 + k_4 k_5 - k_1 k_8) = 0
\]

(3-3-25)

or

\[
f(\omega, E, G, GEOMETRY) = 0
\]

(3-3-26)

Solving for \( \omega \), the frequency of vibration for each natural mode can be derived.
3.4 COMPARATIVE STUDY

In order to compare the performance of the Euler-Bernoulli and Timoshenko theories in the low and high frequency ranges, a numerical example is solved. In the study herein, the clamped free beam of rectangular cross section carrying a spherical shaped mass at the free end is considered. In this case, there are two elastic constants that govern the beam response: $E$ and $G$. Solving the frequency equation for the numerical data given in Table 3-1, and using the Mathematica computer package, the frequency function can be plotted against the frequency. Mathematica's log file to solve the frequency equation can be found in Appendix III.

Table 3-1: Beam Data

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$ [GPa]</td>
<td>200</td>
</tr>
<tr>
<td>$G$ [GPa]</td>
<td>87</td>
</tr>
<tr>
<td>$L$ [mm]</td>
<td>400</td>
</tr>
<tr>
<td>$A$ [mm$^2$]</td>
<td>450</td>
</tr>
<tr>
<td>$\rho$ [kg/m$^3$]</td>
<td>7800</td>
</tr>
<tr>
<td>$I$ [mm$^4$]</td>
<td>12550</td>
</tr>
</tbody>
</table>

Considering the spherical-shaped end-mass, the normalised mass, $M_n$, can be assumed as

$$M_n = \frac{4}{3} \frac{\rho_m \pi r^3}{\rho_b AL} \quad (3.4-1)$$

The normalised end-rotary inertia, $R_n$, can be assumed as the rotary inertia of the sphere about $x=L$ divided by the rotary inertia of the beam about $x=0$. 
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\[ R_n = \frac{\left( \frac{7}{5} MR^2 \right)}{\frac{1}{12} \rho_b AL(H^2 + L^2)} \]  \hspace{1cm} (3-4-2)

where \( H \) is the depth of the beam and \( R \) is the radius of gyration for the end-mass. The frequency equations for the Euler-Bernoulli and Timoshenko theories, derived in previous sections can be solved for the numerical data; \( M_n=2, \ R_n=0.25 \) and \( L/H=8 \), and the result can be plotted by using the mathematical capabilities of the *Mathematica* computer package. More details of the long and tedious calculations can be found in Horr and Schmidt (1995). Figs. 3-3 and 3-4 show the plot of the frequency equation (3-3-26) versus the frequency for both theories.

The natural frequency values show greater differences for both theories as the higher vibration modes are considered. Fig. 3-5 shows a three-dimensional plot of normalised mass, natural frequency and frequency function. By comparing Fig. 3-5(a) and Fig. 3-5(b), it can be seen that while for Timoshenko theory, with increasing normalised mass, the frequency function shows an unexpected behaviour, the Euler-Bernoulli theory shows a linear behaviour.

In fact, in the Timoshenko beam theory, with increasing normalised mass, the slope of the frequency function changes so that the third and fourth natural frequencies approach the second and third natural frequencies for the case of \( M_n=0 \) (Horr and Schmidt, 1995). In general, neglecting the shear and rotary inertia effects in Euler-Bernoulli theory causes an error in frequency results and so this theory should not be used for calculating the analytical damping capacity of materials.
Figure 3-3

Frequency Equation Versus Frequency for (a) Timoshenko and; (b) Euler-Bernoulli Theories
Figure 3-4

Frequency Equation Versus the Frequency (radian/sec) for (a) Timoshenko and; (b) Euler-Bernoulli Theories in Higher Frequency Range
Figure 3-5

Frequency Equation Versus the Frequency (radian/sec) and Normalised Mass for (a) Timoshenko; (b) Euler-Bernoulli Theories
3.5 STRUCTURAL DAMPING CAPACITY

In this section an attempt is made to calculate the damping capacity of the clamped-free beam carrying a mass at the free end (Fig. 3-2). Solving Eq. (3-2-28) for a viscoelastic Timoshenko beam (material and geometry data are given in Table 3-2), using the Mathematica computer package, damping can be plotted against frequency.

Fig. 3-6 shows the structural damping $\psi_s$ against the frequency $\omega$ for the clamped-free Timoshenko beam. In Fig. 3-7, the structural damping capacity is plotted against frequency and longitudinal loss factor $\theta_E$ (in Eq. 3-2-29). As can be seen from this three-dimensional plot, the damping function is very smooth and stable for $0 < \theta_E < 1.5$ in the full frequency span. In physical terms, as $\theta_E$ for most of the structural materials is far less than 1 (the real part of the complex modulus $E$ is always larger than the imaginary part), the figure shows that the method provides a stable solution for most engineering materials. Fig. 3-8 shows the structural damping capacity against frequency and shear loss factor $\theta_G$. The solution stability can also be seen in this case for $0 < \theta_G < 1$ in the full frequency span.

<table>
<thead>
<tr>
<th>Table 3-2: Viscoelastic Timoshenko Beam Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[\text{GPa}]$</td>
</tr>
<tr>
<td>$G[\text{GPa}]$</td>
</tr>
<tr>
<td>$L[\text{m}]$</td>
</tr>
<tr>
<td>$A[\text{m}^2]$</td>
</tr>
<tr>
<td>$K$</td>
</tr>
<tr>
<td>$\rho[\text{kg/m}^3]$</td>
</tr>
<tr>
<td>$\theta_E$</td>
</tr>
<tr>
<td>$\theta_G$</td>
</tr>
</tbody>
</table>
Chapter Three: Analytical Modelling of Damping in Structures

Figure 3-6

Structural Damping Capacity Versus Frequency

Figure 3-7

Structural Damping Versus Frequency and Longitudinal Loss Factor $\theta_E$
Figure 3-8

Structural Damping Versus Frequency and Shear Loss Factor $\theta_G$

Figure 3-9

Steel-rod Structural Damper with Dynamic Hysteresis Behaviour
The shear loss factor $\theta_G$ for structural materials is also far less than 1 in reality, and the method seems to be well behaved for all frequency values.

To elaborate on the efficiency of the proposed analytical damping model, the damping capacity of a steel rod damper is formulated and the result is compared with the non-linear stiffness method. Steel rod dampers are a supplementary structural damper which have been used in the isolation systems of some buildings. The damper consists of a tip mass and a steel rod which has its end restrained (Fig. 3-9). In the following sections, two different models are presented for the purpose of comparison. The first model is a non-linear stiffness model which considers the large deformation of the damper. The second model is the proposed viscoelastic non-linear damping model.

### 3.5.1 Non-Linear Stiffness Method

The usual vibration theory discusses the vibrational properties of systems with the assumption that the stiffness is proportional to their deformation, and also that the viscous damping force is proportional to velocity. Consequently, the equation of motion for such a system is a linear, second order, ordinary differential equation with constant coefficients. This equation can handle many practical problems very well, and plays a central role in linear vibration theory. But, this equation is a good model only for very small deflections, and also it has a limitation in that it can only handle the effects of viscous damping. It cannot describe larger deflections, and also cannot handle the non-linear damping characteristics of the system. The purpose of this part of the study is to determine the influence of stiffness and damping non-linearity on the vibration of the clamped free steel-rod damper carrying an end mass. Fig. 3-9 shows a typical steel rod structural damper which dissipates energy through its dynamic hysteretic behaviour.
Vibration of the end-mass causes the inertia forces shown by $F_1$ and $F_2$ in Fig. 3-9. To find the appropriate non-linear expressions for this damper, non-linear expressions need to be derived for $F_1$ and $F_2$ which satisfy the equilibrium conditions at any fixed time $t$. To solve this problem, the Euler formulation can be used. Let $s$ be the arc length of the curved beam and $k$ be the curvature. It is assumed that the length of the beam is constant and equal to $l$. The deflected curve is elastic if it minimises the potential energy as follows,

$$P = \frac{1}{2} \int_0^l EI k^2(s) ds \quad (3-5-1)$$

among all smooth curves satisfying the given constraints. Euler theory proves that this description is equivalent to the moment-curvature equation

$$M = EI k \quad \text{and} \quad k = \frac{d\theta}{ds} \quad (3-5-2)$$

for the minimiser curve. Love, (1944) has shown that after differentiation this equation can be written as,

$$\frac{d^2\theta}{ds_i^2} = F \sin\theta \quad ; \quad F = \frac{F_p l^2}{EI} \quad ; \quad s_i = \frac{s}{l} \quad (3-5-3)$$

where $F_p$ is the product of $F_1$ and $F_2$. The boundary conditions are,

$$\theta (0) = \alpha \quad (3-5-4)$$
\[
\left( \frac{d\theta}{ds_1} \right)(t) = 0 \quad (3-5-5)
\]

and the solution to this boundary value problem is presented by Wang (1969) as,

\[
\begin{align*}
\theta(s) &= \alpha + F\theta_1(s) + F^2\theta_2(s) + F^3\theta_3(s) + \ldots \\
x &= \frac{1}{3} F_1 - \frac{2}{15} F_1 F_2 - \frac{4}{105} F_1^3 + \frac{17}{315} F_1 F_2^2 + \ldots \\
y &= 1 - \frac{1}{15} F_1^2 + \frac{17}{315} F_1^2 F_2 + \ldots 
\end{align*} \quad (3-5-6)
\]

An end-mass \( m \) of a vibrating damper will affect this system with inertia forces,

\[
F_1 = -m \frac{d^2x}{dt^2} \quad F_2 = -m \frac{d^2y}{dt^2} \quad (3-5-7)
\]

If we substitute Eq. (3-5-7) into Eq. (3-5-6), and assume a non-dimensional time \( T = t \sqrt{EI / ml^3} \) for the purpose of comparison, and after some manipulation, the equation for lateral motion becomes,

\[
\ddot{x} + 3x - \frac{4}{75} \dddot{x}(\dddot{x}^2 + \dddot{x}) - \frac{4}{35} \dddot{x} = 0 \quad (3-5-8)
\]

where \( \dddot{x} \) shows the fourth order differentiation of \( x \) with respect to time. By comparing this differential equation with the linear one as,
it is seen that the non-linear model has a few more terms, and it describes \( x \) and \( y \) as polynomials of the third degree in \( F_1 \) and \( F_2 \). Solution of this fourth order differential equation can be found as,

\[
x = \varepsilon \cos(t) + \frac{81}{1400} \varepsilon^3 (\cos(t) - \cos(3t)) + \cdots
\]  

(3-5-10)

where \( \varepsilon \) is a small initial displacement and \( t \) is time. Note that this solution is not complete, and has been found using initial values for displacement and velocity only. Four initial values are needed for the fourth order differential equation, but for the case herein, which is close enough to harmonic motion, this is sufficient to satisfy the differential equation. This solution leads to the frequency equation,

\[
\omega^2 = \omega_0 + \varepsilon^2 \lambda
\]  

(3-5-11)

where \( \lambda \) is the frequency dependent response amplitude and \( \omega_0 \) is the frequency of the linearised case. It is worth noting that \( \lambda \) is not an even function of excitation frequency in the non-linear stiffness model. In fact, for the non-linear stiffness model the multi-valuedness of the frequency-response curve due to non-linearity of stiffness has a significance from a physical point of view because it leads to a jump phenomenon. In other words, the response amplitude will jump from a lower value to a higher value every time the frequency of the excitation is varied up and down through the natural frequency.
3.5.2 Viscoelastic Non-Linear Method

In this part, the frequency-dependent damping problem in the above structural damper is considered. The equations for flexural motion in the undamped model are given by,

\[
\ddot{u}(t,y) + \frac{EI}{\rho} \frac{d^{\nu} u(t,y)}{dy^{\nu}} = 0
\]

\[
\ddot{u}(t,l) + \frac{EI}{m} \frac{d^{\nu} u(t,l)}{dy^{\nu}} = 0
\]

(3-5-12)

\[u(t,0) = u'(t,0) = u''(t,l) = 0\]

where \(u(t,y)\) is the flexural displacement at any point \(0<y<l\) at time \(t>0\), and \(\rho\) is the density of the beam material. The frequency equation for this case is,

\[
f(\omega, E, G, Geo.) = \frac{1 + \cos \beta \cosh \beta}{\beta \left( \sin \beta \cosh \beta - \cos \beta \sinh \beta \right)} - \frac{m}{\rho l}
\]

(3-5-13)

where \(\beta = \sqrt{\frac{\omega^2 l^4 \rho}{EI}}\). Recalling Eq. (3-2-30), the damping ratio of the damper as a function of frequency (see Fig. 3-10) can be found using the data in Table 3-3. By using a non-linear damping model with linear stiffness the model can accurately predict the frequency-dependent damping characteristic of the system, and it does not produce any peculiar behaviour such as internal resonance, jump phenomena or any unexpected behaviour. The frequency-dependent damping behaviour of the damper can be used to find a modal damping ratio for each of the natural modes, and can be input to a finite element program for a dynamic analysis.
Table 3-3: Structural Damper Data

<table>
<thead>
<tr>
<th>$E$ [GPa]</th>
<th>200.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$ [GPa]</td>
<td>84.0</td>
</tr>
<tr>
<td>$l$ [m]</td>
<td>0.3</td>
</tr>
<tr>
<td>$\rho$ [kg/m$^3$]</td>
<td>7800</td>
</tr>
<tr>
<td>$\theta_e$</td>
<td>0.09</td>
</tr>
<tr>
<td>$\theta_o$</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Figure 3-10

Effect of the Frequency on the Damping Ratio
3.6 SUMMARY AND CONCLUDING REMARKS

A modelling method has been presented which is capable of making accurate predictions of the frequency-dependent damping characteristic of all structural members and dampers. The approach is unique, as the usual viscous treatment appears to lack a physical basis. The non-linear damping solutions were shown graphically and the stability of the method for engineering materials, has been investigated. For the sake of completeness, two different models of a steel-rod structural damper have been proposed.

The concept of using an exact frequency function and complex moduli will be used in the next Chapters to develop an efficient numerical method for damped systems in the frequency domain. However, an important point herein is that the frequency-dependence of damping in engineering materials is weaker than expected (Figs. 3-6 and 3-10). Hence it seems that instead of the first-order viscoelastic model developed in this Chapter, the fractional time derivative constitutive relation may model the damping behaviour more accurately. The relevant fractional derivative complex moduli will be developed in Chapter Five and their application to engineering materials will be investigated.
Chapter Four

Formulation of a Frequency Domain Spectral Finite Element Method

GENERAL

There has been considerable effort, to apply the method of spectral analysis to the vibration of a structure. This Chapter shows how a spectral finite element formulation (SFE) for vibration of structures can be used to obtain an accurate result with a great saving in computing power and time.

The application of the finite element (FE) method to structural dynamics is popular and also successful. However, in order to guarantee stability and accuracy of the solution, the number of elements used to model a structure may be very large indeed; more precisely, accurate results can be obtained only after a substantial computational effort. In the literature several techniques have been proposed to increase the efficiency and speed of the method, but they are mainly focussed on creating advanced computer algorithms to save computing power and time. So, it is necessary to search for a procedure requiring less computer time and effort to model a structure. In this respect, attention is paid to the spectral method which benefits from the speed and switching capabilities of the Fast Fourier Transform. In the first part of the Chapter the theoretical development of the elementary spectral finite element method is presented. In the second part, the
method is extended to the Timoshenko spectral finite element which uses the accurate Timoshenko beam theory, and finally, a comparative analysis is performed, highlighting the stability, efficiency and accuracy of the method. The spectrally formulated finite element starts at the same starting point as the conventional finite element formulation, however, it works in the frequency domain. One of the main contributions of this research work is to show the advantages of the spectral method. The length of the spectral element is not a limiting factor and it allows a huge reduction in the number of elements needed for an accurate result. The method herein, which uses an exact shape function, not only does not require a great deal of subdivision in a structure, but also, treats the mass distribution exactly, which eliminates the additional effort to model the continuous mass distribution in a structure. All the assembled equations are solved in the frequency domain, and then are transformed to the time domain using a Fast Fourier Transform.

4.2 REVIEW OF CONVENTIONAL FE METHOD

It is well known that the application of the finite element method has had a profound effect on the vibration analysis of structures. The idea of increasing the stability and efficiency of the conventional finite element method has been a major reason behind much research work in past decades. Conventional finite element methods treat the inertia forces induced by the mass as concentrated loads applied to the end nodes of elements. Consequently, even if the structural joints are far apart, many elements should be used to model the mass distribution adequately.

The conventional time domain method is based on satisfying dynamic equilibrium at selected time intervals. For the sake of clarity, consider the kinetic $[T(t)]$ and strain $[V(t)]$ energies for an axial member (Fig. 4-1) as follows,
where \( m(x) \) is the mass per unit length, and \( u(x,t) \) is the longitudinal displacement.

Using Hamilton's principle

\[
\frac{\partial}{\partial x} \left( EA(x) \frac{\partial u}{\partial x} \right) - m(x) \frac{\partial^2 u}{\partial t^2} = 0
\]

(4-2-2)

If it is assumed that the axial stiffness and density for the element are constant along the length, and all the mass is concentrated at the end nodes (lumped method), Eq. (4-2-2) can be rewritten as,

\[
\frac{\partial^2 u}{\partial x^2} = 0
\]

(4-2-3)
The simple solution for this differential equation can be found as,

\[ u(x) = a_0 + a_1 x \]  \hspace{1cm} (4-2-4)

where \( a_0 \) and \( a_1 \) are time dependent. The nodal displacements \( u_1 \) and \( u_2 \) are related to each other as,

\[ u(x) = (1 - \frac{x}{L})u_1 + \left( \frac{x}{L} \right)u_2 \]  \hspace{1cm} (4-2-5)

The relationship between the forces at the ends and the displacements can be written using the basic elasticity equation \( F = K \delta \) or \( F = EA \frac{\partial u}{\partial x} \), as,

\[ F_1 = \frac{EA}{L} u_1 - \frac{EA}{L} u_2 \quad F_2 = \frac{EA}{L} u_2 - \frac{EA}{L} u_1 \]  \hspace{1cm} (4-2-6)

or in matrix format as,

\[ [F] = [K] [U] \]  \hspace{1cm} (4-2-7)

There are two popular methods for deriving the mass matrix. The first method assumes that the inertia of the system is concentrated at each node (lumped-mass method)

\[ I_1 = -\rho AL \frac{\partial^2 u_1}{\partial t^2} \quad I_2 = -\rho AL \frac{\partial^2 u_2}{\partial t^2} \]  \hspace{1cm} (4-2-8)
or in matrix format \( [I] = [M] [\ddot{U}] \). The second method is the consistent mass method, in which all the terms in the mass matrix are obtained from the kinetic energy of the element,

\[
M_{ij} = \frac{\partial^2 T}{\partial u_i \partial u_j} \quad \text{(mass matrix)} \tag{4-2-9}
\]

but for both these methods, the number of elements used in building up the model should be sufficient to represent the distributed mass. The full system of equations is

\[
[M][\ddot{U}] + [K][U] = [F] \tag{4-2-10}
\]

which must be assembled for the whole system. A more comprehensive discussion about the axial, flexural and torsional members in the finite element method can be found in Bathe, (1983).

### 4.3 SPECTRAL SOLUTION

The spectral method of analysis concerns the synthesis of waveforms from the superposition of many frequency components. The spectral approach used in the present study is based on the exact solution of the governing differential equation for the dynamic motion of structures. However, it differs from the classical method because it uses the Fast Fourier Transform for conversion purposes. As the method uses discrete rather than continuous transforms, the frequency range is finite. Discrete points can be obtained by summing the components over frequency and
wave numbers. The goal of the present research study is the development of efficient spectral schemes for the dynamic analysis of structures.

As the conventional finite element method requires many elements to model the mass distribution adequately, the computer power and time needed for dynamic analysis of large complex structures is substantial. In essence, attention is paid to the alternative spectral approach which works in the frequency domain and draws its robustness from the speed and switching capabilities of the Fast Fourier Transform. As the only way to efficiently solve problems with complicated boundaries and discontinuities is to develop a matrix methodology for use on a computer, the mathematical basis of the spectral method can be used to develop a numerical method using matrix methodology. The spectrally formulated elements can be used to model different structural members. Moreover, an exact frequency domain approach is used to derive the element matrices. As these spectral elements treat the distributed mass exactly, only one element needs to be placed between any two joints.

The dynamic response of a structure is a general function of space and time. If the time variation of the solution is focused on at a particular point in space, then it has a spectral representation

$$u_t(x,t) = \sum_n \hat{u}_n(x,\omega_n) e^{i\omega t}$$

(4-3-1)

where $\omega$ is the angular frequency and $\hat{u}_n$ are the Fourier coefficients. At another point, it behaves as a different function $u_2$ and is represented by different Fourier coefficients. Thus, the coefficients are different at each spatial point.
The governing differential equations in structural dynamics are given in terms of both space and time derivatives. Consider the spectral representation for the time derivative as,

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \sum \hat{u}_n e^{i\omega_n t} = \sum i\omega_n \hat{u}_n e^{i\omega_n t}
\]  

(4-3-2)

The advantage of the spectral method appears from this starting point where algebraic expressions in the Fourier coefficients replace the time derivatives. However, the spectral representation of the spatial derivatives appears to exhibit no such a reduction. That is

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \sum \hat{u}_n e^{i\omega_n t} = \sum e^{i\omega_n t} \frac{\partial \hat{u}_n}{\partial x}
\]  

(4-3-3)

As the time is removed as an independent variable, the governing differential equation involves ordinary derivatives, and hence is easy to solve by integration.

The general form of the one-dimensional, homogeneous differential equation can be written as,

\[
u + c_1 \frac{\partial u}{\partial x} + c_2 \frac{\partial u}{\partial t} + c_3 \frac{\partial^2 u}{\partial x^2} + c_4 \frac{\partial^2 u}{\partial t^2} + c_5 \frac{\partial^2 u}{\partial t \partial x} + \ldots = 0
\]  

(4-3-4)

where \( c_1, c_2, \ldots \) can be functions of position but not the time. The spectral solution can be presented as,
Chapter Four: Formulation of a Frequency Domain Spectral Finite Element Method

\[ u(x,t) = \sum_n \tilde{u}_n(x, \omega_n) e^{i\omega_n t} \]  \hspace{1cm} (4-3-5)

the differential equation can be rewritten as,

\[
\left[ \sum_n \{ \tilde{u}_n + c_1 \frac{d\tilde{u}_n}{dx} + c_2 i\omega_n \tilde{u}_n + c_3 \frac{d^2\tilde{u}_n}{dx^2} + 
\right.
\left. c_4 (i\omega_n)^2 \frac{d\tilde{u}_n}{dx} + c_5 (i\omega_n) \frac{d\tilde{u}_n}{dx} + \ldots \} e^{i\omega_n t} \right] = 0 \]  \hspace{1cm} (4-3-6)

this summation must be satisfied for each value of \( n \), and hence there are \( n \) equations as,

\[
\left[ 1 + (i\omega_n) c_2 + (i\omega_n)^2 c_4 + \ldots \right] \tilde{u}_n + \left[ c_1 + (i\omega_n)c_3 + \ldots \right] \frac{d\tilde{u}_n}{dx} + \left[ c_3 + \ldots \right] \frac{d^2\tilde{u}_n}{dx^2} + \ldots = 0 \]  \hspace{1cm} (4-3-7)

which can be written as,

\[
C_1(x,\omega_n)\tilde{u}_n + C_2(x,\omega_n)\frac{d\tilde{u}_n}{dx} + C_3(x,\omega_n)\frac{d^2\tilde{u}_n}{dx^2} + \ldots = 0 \]  \hspace{1cm} (4-3-8)

where coefficients \( C_1, C_2, \ldots \) are complex values. The summation of \( n \) frequency components reconstructs the time dependency for the response. For a linear differential equation with constant coefficients, the spectral solution can have the
form $Ce^{i\kappa x}$, where $\kappa$ is called the wavenumber and it can be obtained by solving the characteristic equation,

$$C_1 + C_2 i\kappa + C_3 (i\kappa)^2 + ... = 0 \quad (4-3-9)$$

and hence, $\kappa$ has many values which satisfy this characteristic equation,

$$\kappa_m = f_m(C_1, C_2, C_3, ..., \omega_n) \quad (4-3-10)$$

where different values of $m$ correspond to the different modes. For the sake of clarity, consider the following second-order differential equation

$$q u + \frac{d^2 u}{dx^2} = 0 \quad (4-3-11)$$

the characteristic equation in this case is

$$c(q - \kappa^2) = 0 \quad (4-3-12)$$

which gives $\kappa$ and the solution as,

$$\kappa = \pm \sqrt{q} \ ; \ \tilde{u}(x) = C_1 e^{i\sqrt{q}x} + C_2 e^{-i\sqrt{q}x} \quad (4-3-13)$$
There are two solutions for this simple differential equation as there are two values for the wavenumber. The complete spectral solution of any differential equation can be obtained by the summation of the modes for each of the frequency values

$$u(x,t) = \sum_n \left( C_{1n} e^{iK_1x} + C_{2n} e^{iK_2x} + \ldots C_{mn} e^{i\kappa_{mn}x} \right) e^{i\omega_n t}$$  \hspace{1cm} (4-3-14)

Hence, the general solution for a differential equation with non-constant coefficient can be written as

$$u(x,t) = \sum_n A_n T(\kappa_{mn}x) e^{i\omega_n t}$$  \hspace{1cm} (4-3-15)

where $A_n$ is an amplitude spectrum and $T$ is the system transfer function. In what follows, the mathematical theory of the spectral approach is used to develop a spectral finite element method.

**4.4 SPECTRAL ELEMENT FOR AXIAL VIBRATION**

The idea of using the spectral solution to solve dynamic problems is general, but to make matters simple, the formulation for an axial element is presented firstly. Consider Eq. (4-2-2) in the following form,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\lambda}{E} \frac{\partial^2 u}{\partial t^2}$$  \hspace{1cm} (4-4-1)

where $\lambda = \rho/E$ is a constant coefficient. The spectral form of this differential equation can be written as
\[
\frac{d^2 \ddot{u}}{dx^2} + \omega^2 \nu \ddot{u} = 0 \quad (4-4-2)
\]
the characteristic equation in this case can be presented as
\[
C(\omega^2 \nu - \kappa^2) = 0 \quad (4-4-3)
\]
and the two roots are
\[
\kappa_1 = +\omega \sqrt{\nu}; \quad \kappa_2 = -\omega \sqrt{\nu} \quad (4-4-4)
\]
hence, the solution is
\[
u(x,t) = \sum \tilde{u}_n(x,\nu_n)e^{i\nu_t} = \ddot{u}_1(x,\nu_1)e^{i\nu_t} + \ddot{u}_2(x,\nu_2)e^{i\nu_t} + \ldots + \ddot{u}_n(x,\nu_n)e^{i\nu_t} \quad (4-4-5)
\]
where \( \ddot{u}_n \) are essentially a set of discrete Fourier coefficients. From a comparison between Eq. (4-4-5) and the time domain solution (Eq. 4-2-4), it is clear that in the spectral formulation the time variational terms are formulated in the frequency domain instead of a direct formulation in the time domain. If the spectral solution is applied to the conventional method, it gives the following relationships:
\[
[K] = [K] - \omega^2[M]; \quad [\hat{K}][U] = [F] \quad (4-4-6)
\]
where \([\hat{K}]\) is called the dynamic stiffness and is frequency dependent. In the general case, Eq. (4-2-2) can be written in spectral format as follows:
\[
\frac{d}{dx}[EA \frac{d\ddot{u}}{dx}] + \omega^2 \rho A \ddot{u} = 0 \quad (4-4-7)
\]
If it is assumed that the stiffness and density are constant, the solution of Eq. (4.4.7) is:

\[ \tilde{u}_n(x) = A_n e^{-\kappa_n x} + B_n e^{-\kappa_n x} \]  (4.4-8)

where \( \kappa_n = \omega_n \left[ \frac{p}{E} \right]^{\frac{1}{2}} \). Using Eq. (4.4-8) as a shape function, the equation of motion can be written in spectral format as:

\[ [\hat{F}_n] = \frac{EA}{L} [\hat{K}_n][\tilde{u}_n] \]  (4.4-9)

where the dynamic stiffness matrix can be derived as,

\[ [\hat{K}_n] = \begin{bmatrix} \kappa_n L & -\kappa_n L \\ \tan(\kappa_n L) & \sin(\kappa_n L) \\ -\kappa_n L & \kappa_n L \\ \sin(\kappa_n L) & \tan(\kappa_n L) \end{bmatrix} \]  (4.4-10)

which should be calculated at each frequency. The stress and strain quantities can be derived as,

\[ \sigma = \mp i \kappa E \tilde{u} \]
\[ \varepsilon = \mp i \kappa \tilde{u} \]  (4.4-11)

and the axial force is

\[ F = \mp i \kappa E A \tilde{u} \]  (4.4-12)

A very important point in this formulation is that as the shape function is an exact solution of the governing differential equation, the number of rod segments in the present formulation needs only to coincide with the number of discontinuities.
Hence, a spectral rod element can be very long. However, as the spectral rod element may be very long, the ability to calculate the response between nodes is necessary. If the exact shape function is written in terms of nodal displacements

\[
\tilde{u}(x) = \left( \frac{\sin \kappa (L - x)}{\sin \kappa L} \right) \tilde{u}_1 + \left( \frac{\sin \kappa x}{\sin \kappa L} \right) \tilde{u}_2
\]  (4-4-13)

the member load is

\[
\tilde{F}(x) = -EA\kappa \left( \frac{\cos \kappa (L - x)}{\sin \kappa L} \right) \tilde{u}_1 + EA\kappa \left( \frac{\cos \kappa x}{\sin \kappa L} \right) \tilde{u}_2
\]  (4-4-14)

As these quantities are in transform coordinates, the time history is obtained by using a FFT.

4.5 SPECTRAL ELEMENT FOR FLEXURAL VIBRATION

To apply the spectral method of analysis to solve the flexural vibration in a beam, the previous formulation can be extended to a four degree of freedom beam element (Fig. 4-2). The spectral beam element models the mass distribution exactly, and hence can be very long as in the case of the rod element.

![Figure 4-2 Typical Beam Element](image)
Based on the Euler-Bernoulli theory for flexural vibration of the perfectly elastic undamped beam element, the formulation can again be started with the kinetic and strain energy considerations,

\[ T(t) = \frac{1}{2} \int_0^l m(x) \left( \frac{\partial u(x,t)}{\partial t} \right)^2 dx \]

\[ V(t) = \frac{1}{2} \int_0^l EI(x) \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right)^2 dx \]  

where \( u(x,t) \) in this case is the lateral displacement. Note that the effects of shear deformation and rotary inertia have been neglected in the calculation of the kinetic and strain energy. Using Hamilton's principle,

\[ \frac{\partial^2}{\partial x^2} [EI(x) \frac{\partial^2 u}{\partial x^2}] + m(x) \frac{\partial^2 u}{\partial t^2} = 0 \]  

Assuming constant stiffness and density

\[ EI \frac{\partial^4 u}{\partial x^4} + \rho A \frac{\partial^2 u}{\partial t^2} = 0 \]  

The simple solution of this equation can be written as,

\[ u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \]  

where all coefficients are time dependent. The boundary conditions are,

\[ u(0) = u_1, \quad u(l) = u_2, \quad \frac{\partial u}{\partial x}(0) = \left( \frac{\partial u}{\partial x} \right)_1, \quad \frac{\partial u}{\partial x}(l) = \left( \frac{\partial u}{\partial x} \right)_2 \]  

Consider Eq. (4-5-3) in a different form as
\[
\frac{\partial^4 u}{\partial x^4} + \frac{\rho A}{EI} \frac{\partial^2 u}{\partial t^2} = 0 \rightarrow \frac{\partial^4 u}{\partial x^4} + \lambda^4 \frac{\partial^2 u}{\partial t^2} = 0 \tag{4-5-6}
\]

where again, \(\lambda = \rho A/EI\) is real and constant valued. The spectral solution of Eq. (4-5-6) is,

\[
u_n(x,t) = Ce^{-i(k_n x - \omega_n t)} \tag{4-5-7}
\]

which gives the characteristic equation as

\[
C(\kappa^4 - \omega^2 \lambda^4) = 0 \tag{4-5-8}
\]

Hence, there are four modes given by,

\[
\kappa_1 = +\lambda \sqrt{\omega} ; \quad \kappa_2 = -\lambda \sqrt{\omega} ; \quad \kappa_3 = +i\lambda \sqrt{\omega} ; \quad \kappa_4 = -i\lambda \sqrt{\omega} \tag{4-5-9}
\]

and the solution is

\[
u(x,t) = \sum C_1 e^{i(\lambda \sqrt{\omega} x + \omega t)} + \sum C_2 e^{-i(\lambda \sqrt{\omega} x - \omega t)} + \sum C_3 e^{i\omega t - \lambda \sqrt{\omega} x} + \sum C_4 e^{i\omega t + \lambda \sqrt{\omega} x} \tag{4-5-10}
\]

The displacement function can also be written as a superposition of different components as,

\[
u(x,t) = \tilde{u}_1(x,\omega_1) e^{i\omega_1 t} + \tilde{u}_2(x,\omega_2) e^{i\omega_2 t} + \ldots + \tilde{u}_n(x,\omega_n) e^{i\omega_n t} \tag{4-5-11}
\]

where \(\tilde{u}_n\) are again a set of discrete Fourier coefficients. The flexural vibration equation can be written in spectral format as follows:
Chapter Four: Formulation of a Frequency Domain Spectral Finite Element Method

One of the solutions of this differential equation is:

\[ \hat{u}_n(x) = A_n e^{-i\kappa x} + B_n e^{-i\kappa x} + C_n e^{-i\kappa_{\text{L}} x} + D_n e^{-i\kappa_{\text{L}} x} \]  

(4-5-13)

where \( \kappa = \sqrt{\omega \left[ \frac{\partial A}{\partial t} \right]_{\text{EI}}} \). Using Eq. (4-5-13) as a shape function, the governing equations of motion can be rewritten in spectral format as:

\[ \left[ \hat{\mathbf{K}} \right] = \frac{EI}{L^3} \left[ \hat{\mathbf{K}} \right] \left[ \hat{\mathbf{u}}_n \right] \]  

(4-5-14)

The \( \left[ \hat{\mathbf{K}} \right] \) matrix can be determined using the boundary conditions as

\[
\begin{bmatrix}
\hat{k}_{11} &=& \hat{k}_{33} &=& (iz_{11}z_{22} - z_{12}z_{21})\zeta^3 / \text{det} \\
\hat{k}_{12} &=& -\hat{k}_{34} &=& 0.5(1+i)(z_{11}z_{22} - z_{12}z_{21})\zeta^3 L / \text{det} \\
\hat{k}_{13} &=& \hat{k}_{31} &=& i(z_{11}z_{21} - z_{12}z_{22})\zeta^3 / \text{det} \\
\hat{k}_{14} &=& -\hat{k}_{23} &=& -(1-i)(z_{11}z_{21})\zeta^2 L / \text{det} \\
\hat{k}_{22} &=& \hat{k}_{44} &=& (z_{11}z_{22} - iz_{12}z_{21})\zeta L^2 / \text{det} \\
\hat{k}_{24} &=& -z_{11}z_{22} + iz_{12}z_{22})\zeta L^2 / \text{det}
\end{bmatrix}
\]

(4-5-15)

where

\[
\begin{align*}
\zeta &= \kappa L \\
z_{11} &= 1 - e^{-\kappa L} e^{-\xi} \\
z_{12} &= e^{-\kappa L} - e^{-\xi} \\
z_{21} &= e^{-\kappa L} + e^{-\xi} \\
z_{22} &= 1 + e^{-\kappa L} e^{-\xi} \\
\text{det} &= (z_{11}^2 + z_{12}^2) / (1 + i)
\end{align*}
\]
The stiffness matrix is symmetrical, as for the conventional finite element method, and the terms are mostly complex. The stress and strain quantities can be derived in this case as

\[
\sigma = \sum \frac{h}{2} \kappa^2 EA(e^{-i\kappa x} + ie^{-i\kappa x})e^{i\omega t} \tag{4-5-16}
\]

\[
\varepsilon = \sum \frac{h}{2} \kappa^2 A(e^{-i\kappa x} + ie^{-i\kappa x})e^{i\omega t}
\]

where \( h \) is the height of the beam element. The shear force can be evaluated as

\[
S = \sum iEI\kappa^3 A(e^{-i\kappa x} + e^{-i\kappa x})e^{i\omega t} \tag{4-5-17}
\]

The member actions and displacement at an arbitrary location along the length can be found by differentiation of the spectral shape function as

\[
\begin{Bmatrix}
\hat{u}(x) = \frac{1}{4} \left( \hat{u}_1 + \frac{i\phi_1}{\kappa} + \frac{M_1}{EI\kappa^2} - \frac{S_1}{EI\kappa^3} \right) e^{-i\kappa x} + \frac{1}{4} \left( \hat{u}_1 - i\phi_1 - \frac{M_1}{EI\kappa^2} - \frac{S_1}{EI\kappa^3} \right) e^{i\kappa x} \\
\hat{u}_2 - i\frac{\phi_2}{\kappa} - \frac{M_2}{EI\kappa^2} - \frac{S_2}{EI\kappa^3} e^{-i\kappa(L-x)} + \frac{1}{4} \left( \hat{u}_2 + i\frac{\phi_2}{\kappa} + \frac{M_2}{EI\kappa^2} - \frac{S_2}{EI\kappa^3} \right) e^{i\kappa(L-x)}
\end{Bmatrix}
\tag{4-5-18}
\]

4.6 EFFICIENCY ANALYSIS

In order to assess the computational efficiency of the spectrally formulated finite element method, parametric studies are performed and the results are compared with the conventional finite element method. One of the main advantages of the matrix methodology involved in the finite element method is that once the basic mathematical derivation of the element is established, then complicated problems can be solved simply by piecing together all these elements. The same procedure of assemblage can be used in the spectral finite element
method. However, the spectral element matrices are derived using the exact solution of the governing differential equation, and hence it should have greater efficiency than the conventional finite element method. To examine the accuracy and efficiency of the spectral finite element method, the dynamic stiffness matrix can be closely examined. When the damping is zero, $\kappa$, is real and the first term in the dynamic stiffness matrix for the spectral rod and beam elements can be written respectively as:

$$\left(\tilde{k}_{11}\right)_{\text{rod}} = \frac{(EA)(\kappa L)}{L \tan(\kappa L)}$$

(4-6-1)

$$\left(\tilde{k}_{11}\right)_{\text{beam}} = \frac{EI}{L^3} \frac{(\cos(\kappa L)\sinh(\kappa L) + \sin(\kappa L)\cosh(\kappa L))(\kappa L)^3}{1 - \cos(\kappa L)\cosh(\kappa L)}$$

(4-6-2)

The first term in the dynamic stiffness matrix for the conventional finite element formulation is

$$\left(k_{11}\right)_{\text{rod}} = k_{11} - \omega^2 m_{11} = \frac{EA}{L} - \frac{\omega^2 \rho AL}{2} = \frac{EA}{L} - \frac{(\kappa L)}{2}$$

(4-6-3)

$$\left(k_{11}\right)_{\text{beam}} = k_{11} - \omega^2 m_{11} = \frac{12EI}{L^3} - \frac{\omega^2}{35} \frac{13 \rho AL}{35} = \frac{12EI}{L^3} - \frac{13(\kappa L)^4 EI}{35L^3}$$

(4-6-4)

Figs. 4-3 and 4-4 show the first term of the dynamic stiffness matrix for the spectral rod element and the conventional finite element versus frequency, using the graphical capability of the Mathematica computer package. The Mathematica log files for evaluating these functions are presented in Appendix II. A comparison between the spectral and conventional finite element methods shows that while they both have a similar behaviour at low frequencies, the spectral dynamic stiffness intersects the zero axis a number of times in a higher range of frequencies (Fig. 4-4). However, the conventional finite element formulation shows a single intersection with the zero axis even in the higher frequency range.
Figure 4-3
(a) Spectral and; (b) FE Rod Element Dynamic Stiffness Versus Frequency (rad/sec)
Figure 4-4
(a) Spectral and; (b) FE Rod Element Dynamic Stiffness at Higher Frequency (rad/sec)
Figure 4-5

(a) Spectral and; (b) FE Rod Element Dynamic Stiffness Versus Frequency (rad/sec) and Length
Figure 4-6
(a) Spectral and; (b) FE Beam Element Dynamic Stiffness Versus Frequency (rad/sec)
Figure 4-7
(a) Spectral and; (b) FE Beam Element Dynamic Stiffness Versus Frequency (rad/sec) at Higher Frequencies
Figure 4-8

(a) Spectral and; (b) FE Beam Element Dynamic Stiffness Versus Frequency (rad/sec) and Length
Fig. 4-5(a) shows the three dimensional plot for the first term of the spectral dynamic stiffness matrix versus frequency and element length. For a constant element length in Fig. 4-5(a), as the frequency increases the number of intersections increase and it is also true for a constant frequency and various lengths. However, Fig. 4-5(b) shows a different behaviour for the finite element dynamic stiffness with a single intersection. In physical terms, it means that for the conventional finite element formulation to have the same accuracy as the spectral method, it is necessary to subdivide a member. The degree of subdivision depends on the range of required frequencies. Only with a very large number of elements would the conventional method match the spectral method in the required range of frequency. Figs. 4-6 through 4-8 show the same behaviour for the spectral and conventional dynamic stiffness for the beam element.

4.7 ELEMENT LIBRARY

It is well known that the best way to handle problems with complicated boundary and discontinuities is to use a matrix methodology on a computer. The finite element method is able to model large complex structures by assembling the properties of small elements of the structures, and the process is adaptable to matrix methods. All small elements are handled in a routine manner and then assembled into a global system of equations. As the proposed spectral approach uses the same assemblage procedure, it is necessary to develop different types of elements to be able to model different types of structures.

4.7.1 Higher Order Rod Element

The conventional formulation for the Mindlin-Hermann rod element considers only axial vibration. However, it is well known that there is a lateral vibration due to the Poisson's ratio effect. One of the best higher order theories for
the vibration of rods is the Mindlin-Herrmann theory, which gives the governing differential equations as,

\[
(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} + 2\lambda \frac{\partial v}{\partial x} - \rho \frac{\partial^2 u}{\partial t^2} = 0
\]

\[
\frac{4}{3} c_1^2 \mu r^2 \frac{\partial^2 v}{\partial x^2} - 8c_2^2 (\lambda + \mu) v - 4c_2^2 \lambda \frac{\partial u}{\partial x} - \rho r^2 \frac{\partial^2 v}{\partial t^2} = 0
\]

(4-7-1)

where \( r, \lambda, \mu \) are the rod average radius and the Lame constants as

\[
\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} ; \quad \mu = \frac{E}{2(1+\nu)}
\]

(4-7-2)

and \( \nu, \nu \) are the Poisson's ratio and the extra lateral degree of freedom respectively. The constants \( c_1 \) and \( c_2 \) are correction coefficients to account for the non-symmetry of the cross-section and the stress distribution. The spectral solution in this case can be found as

\[
u = v_0 e^{-i(\kappa x - \omega t)}
\]

(4-7-3)

which gives the characteristic equation as

\[
\kappa^4 \left[ \frac{4}{3} c_1^2 (\kappa \eta)^2 \right] + \kappa^2 \beta^2 \left[ 8c_2^2 - \frac{4}{3} c_1^2 (\kappa \eta)^2 \right] - \alpha^2 \beta^2 \left[ 8c_2^2 - (\kappa \eta)^2 \right] = 0
\]

(4-7-4)
where
\[
\alpha = \frac{\omega}{\left(\frac{E}{\rho}\right)^{\frac{1}{2}}} ; \quad \beta = \frac{\omega}{\left(\frac{2\mu + \lambda}{\rho}\right)^{\frac{1}{2}}} ; \quad \gamma = \frac{\omega}{\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}} ; \quad \eta = \frac{\omega}{\left(\frac{2\mu + \lambda}{\rho}\right)^{\frac{1}{2}}}
\]

and the shape functions are
\[
\begin{align*}
\tilde{u}_0 &= c_1e^{-i\kappa_1x} + c_2e^{-i\kappa_2x} + c_3e^{i\kappa_1x} + c_4e^{i\kappa_2x} \\
\tilde{v}_0 &= c_5e^{-i\kappa_1x} + c_6e^{-i\kappa_2x} + c_7e^{i\kappa_1x} + c_8e^{i\kappa_2x}
\end{align*}
\]

These two shape functions are related as,
\[
\tilde{v}_0 = \frac{\rho \omega^2 - (\lambda + 2\mu) \kappa^2}{i2\lambda \kappa} \tilde{u}_0
\]

and any of the two shape functions in Eq. (4-7-5) can be used.

### 4.7.3 Tapered Rod Element

Tapered members in structures can be modelled using the conventional finite element method. However, many elements have to be used to obtain accurate results. Tapered members can be treated with advantage by the spectral method as the tapered member can be considered as one element between joints. Consider the spectral form of the governing differential equation for rod members of varying stiffness or density as,
If it is assumed the area varies as,

\[ A(x) = A_0 \left( \frac{a + x}{a} \right)^\nu \]  \hspace{1cm} (4-7-8)

and substituting into Eq. (4-7-7), the governing differential equation can be written as,

\[
\frac{d^2 \ddot{u}}{dx^2} + \left( \frac{\lambda}{a + x} \right) \frac{d \ddot{u}}{dx} + \frac{\omega^2 \rho}{E} \ddot{u} = 0
\]  \hspace{1cm} (4-7-9)

This is a linear differential equation with variable coefficients

\[
\ddot{u}^* + p(x) \dot{u} + q^2 \ddot{u} = 0
\]  \hspace{1cm} (4-7-10)

where \( p(x) = \frac{\lambda}{a + x} \), \( q = \omega \sqrt{\frac{\rho}{E}} \) and it is a form of Bessel's equation. Thus the Frobenius method (Kreyszig 1993) can be used to solve the equation

\[
\ddot{u}(x) = c_1[q(a + x)]^\nu J_\nu(z) + c_2[q(a + x)]^\nu Y_\nu(z)
\]  \hspace{1cm} (4-7-11)
where \( v = \frac{1 - \lambda}{2} \), \( z = q(a + x) \). The Bessel’s functions \( J \) and \( Y \) are of the first and second kind respectively. For a linear taper where \( \lambda = 2 \) and \( v = 0.5 \), Bessel functions and the shape function can be derived,

\[
J_{0.5}(z) = \sqrt{\frac{2 \sin^2(z)}{\pi z}}
\]
\[
Y_{0.5}(z) = \sqrt{\frac{2 \cos^2(z)}{\pi z}}
\]
\[
\hat{u}(x) = \frac{1}{a + x} \left[ c_1 e^{-iq(a + x)} + c_2 e^{iq(a + x)} \right]
\]

and the shape function for a uniform rod can be obtained as a special case by letting \( \lambda = 0 \) and using the properties of the Bessel functions of half order.

4.7. Boundary Conditions for Rod Element

The boundary conditions which can be applied to the spectral model of a structure are similar to those of the conventional finite element model. However, in the dynamic analysis of very large structures, it is more efficient to partition the structure into a substructure and remote parts. The part of the structure of interest can then be analysed, which needs less computer time and power. In this case, connections between the substructure and remote parts have to be replaced with appropriate elements which conduct energy out of the local substructure. This energy removal is achieved by letting the connection members extend to infinity. The displacement function can be assumed again as
\[ \tilde{u}_n(x) = A_n e^{-\kappa_n x} + B_n e^{\kappa_n x} \]  
(4-7-14)

where \( \kappa_n = \omega_n \left[ \frac{1}{E} \right]^{1/2} \). However, as the length of the element is extended to infinity, the number of degrees of freedoms reduce to one and the governing equation of motion can be written as

\[ \vec{F}_n = \vec{K}_n \vec{u}_n \]  
(4-7-15)

where the stiffness term in this case is \( \vec{K}_n = iEA\kappa \). This element is very useful in earthquake engineering problems when the earthquake waves and energy have to be conducted out of the system (i.e., seismic analysis of piles and foundations in layered media).

### 4.7.2 Timoshenko Beam Element

To account for possibly significant shearing and rotary inertia effects, a Timoshenko beam theory is adapted into the spectral method. In the elementary theory of vibration it is assumed that the cross-sectional dimensions of the beam element are small in comparison with its length (neglecting shear and rotary inertia effects). Corrections to the theory have been given by Timoshenko (1921), for the purpose of taking into account the effects of the cross-sectional dimensions on the frequency. These corrections may be of considerable importance for studying the modes of vibration of higher frequencies (Horr and Schmidt, 1995), when a vibrating beam is subdivided into comparatively short length portions. The Timoshenko equations are,
\[
\left( \frac{\rho A}{g} \right) \frac{\partial^2 u}{\partial t^2} - KAG \left( \frac{\partial^2 u}{\partial x^2} - \frac{\partial \phi}{\partial x} \right) = 0
\]

\[
EI \frac{\partial^2 \phi}{\partial x^2} + KAG \left( \frac{\partial u}{\partial x} - \phi \right) - \left( \frac{Ip}{g} \right) \frac{\partial^2 \phi}{\partial t^2} = 0
\]

(4-7-16)

Where \( u(x,t) \), \( \phi(x,t) \) and \( K \) are the lateral displacement, an additional rotational degree of freedom (for taking into account the shear and rotary inertia effects), and a factor depending on the shape of the cross-section, respectively; \( G \) is the modulus of shear rigidity. As shown in the previous sections, the spectral solutions can be found as:

\[
u(x,t) = \tilde{u}_1(x,\omega_1)e^{-i(kx-\omega_1t)} + \tilde{u}_2(x,\omega_2)e^{-i(kx-\omega_2t)} + \ldots + \tilde{u}_n(x,\omega_n)e^{-i(kx-\omega_n t)}
\]

(4-7-17)

\[
\phi(x,t) = \tilde{\phi}_1(x,\omega_1)e^{-i(kx-\omega_1t)} + \tilde{\phi}_2(x,\omega_2)e^{-i(kx-\omega_2t)} + \ldots + \tilde{\phi}_n(x,\omega_n)e^{-i(kx-\omega_n t)}
\]

(4-7-18)

which result in the following relationships between the motions for each mode:

\[
u_0 = \frac{GAK\kappa i}{GAK\kappa^2 - \rho A\omega^2} \phi_0
\]

(4-7-19)

where

\[
\kappa = \pm \left[ \frac{1}{2} \omega^2 \left( \frac{1}{C_3} \right)^2 + \left( \frac{C_1}{C_2} \right)^2 \right] \pm \sqrt{\left( \frac{\omega}{C_2} \right)^2 + \frac{1}{4} \omega^4 \left( \frac{1}{C_3} - \left( \frac{C_1}{C_2} \right)^2 \right)^2}
\]

\[
C_1 = \sqrt{\frac{EI}{ho A}}; \quad C_2 = \sqrt{\frac{EI}{ho A}}; \quad C_3 = \sqrt{\frac{GAK}{\rho A}}
\]
As the two motions are dependent, it is possible to take one of the motions as the unknown.

\[
\hat{\phi}(x) = Ae^{-ik_1x} + Be^{-ik_2x} + Ce^{ik_1x} + De^{ik_2x}
\]

\[
\hat{u}(x) = P_1Ae^{-iK_1x} + P_2Be^{-iK_2x} - P_1Ce^{iK_1x} - P_2De^{iK_2x}
\]

where \(A, B, C, D, P_1\), and \(P_2\) are constants. Using one of the Eqs. (4-7-20) as a shape function, which are the exact solutions to the governing differential equations of motion, the unknown coefficients can be written in terms of nodal displacements. For the element of length \(L\), with no load applied between nodes, the boundary condition can be written as:

\[
u_1 = v(0); \phi_1 = \phi(0); \nu_2 = v(L); \phi_2 = \phi(L)
\]

(4-7-21)

The end shear forces and end moments can be written in terms of the displacement matrix, as in the case of the elementary spectral beam element.

\[
[\hat{F}_n] = \frac{EI}{L} [\hat{K}_n][\hat{u}_n]
\]

(4-7-22)

The \([\hat{K}]\) matrix can be determined using the boundary conditions as

\[
\begin{bmatrix}
    \hat{k}_{11} & \hat{k}_{12} & \hat{k}_{13} & \hat{k}_{14} \\
    \hat{k}_{21} & \hat{k}_{22} & \hat{k}_{23} & \hat{k}_{24} \\
    \hat{k}_{31} & \hat{k}_{32} & \hat{k}_{33} & \hat{k}_{34} \\
    \hat{k}_{41} & \hat{k}_{42} & \hat{k}_{43} & \hat{k}_{44}
\end{bmatrix}
\]

\[
\begin{align*}
\hat{k}_{11} &= \hat{k}_{33} = (\zeta_2^2 - \zeta_1^2) (c_1 z_{22} + c_2 z_{21}) L / \det \\
\hat{k}_{12} &= -\hat{k}_{34} = [-i\zeta_2 (c_1 z_{11} + c_2 z_{12}) + i\zeta_1 (c_1 z_{11} - c_2 z_{12})] L^2 / \det \\
\hat{k}_{13} &= (\zeta_1^2 - \zeta_2^2) (c_1 z_{22} + c_2 z_{21}) L / \det \\
\hat{k}_{14} &= -\hat{k}_{23} = [-i\zeta_1 (c_1 z_{11} - c_2 z_{11}) - i\zeta_2 (c_1 z_{12} + c_2 z_{12})] L^2 / \det \\
\hat{k}_{22} &= \hat{k}_{44} = (i\zeta_1 A_1 - i\zeta_2 A_2) (c_1 z_{22} - c_2 z_{21}) L^2 / \det \\
\hat{k}_{24} &= \hat{k}_{34} = (i\zeta_1 A_2 - i\zeta_2 A_1) (c_1 z_{21} - c_2 z_{22}) L^2 / \det
\end{align*}
\]

(4-7-23)

where
\[
\begin{aligned}
\zeta_1 &= \kappa_1 L ; \quad A_i = \frac{iL}{GAK} \left( \frac{EI}{L^2} \zeta_1^2 + GAK - \rho I\omega^2 \right) \\
\zeta_{11} &= 1 - e^{-i\kappa_1} e^{-i\kappa_2} ; \quad z_{12} = e^{-i\kappa_1} - e^{-i\kappa_2} ; \quad z_{21} = e^{-i\kappa_1} + e^{-i\kappa_2} \\
z_{22} &= 1 + e^{-i\kappa_1} e^{-i\kappa_2} ; \quad c_1 = (A_1 - A_2) z_{11} ; \quad c_2 = (A_1 + A_2) z_{12} \\
det &= c_1^2 - c_2^2
\end{aligned}
\]

The stiffness matrix is again symmetrical, as for the elementary spectral beam element, and the terms are mostly complex.

### 4.7.3 Tapered Beam Element

Tapered beam members in frame structures can also be modelled using the spectral finite element method. The analogy is similar to the tapered rod element and it starts with the solution of the governing differential equation for the general beam element.

Consider the spectral form of the governing differential equation for beam members of varying stiffness or density as,

\[
\frac{d^2}{dx^2} \left[ EI(x) \frac{d^2\ddot{u}}{dx^2} \right] + \omega_n^2 \rho A(x) \ddot{u} = 0 \quad (4-7-24)
\]

If it is assumed that the width is constant and only the thickness varies along the length,

\[
D = D_0 \left( \frac{a + x}{a} \right)^\lambda ; \quad A(x) = A_0 \left( \frac{a + x}{a} \right)^\lambda ; \quad I(x) = I_0 \left( \frac{a + x}{a} \right)^{3\lambda} \quad (4-7-25)
\]
Substituting into Eq. (4-7-24), the governing differential equation can be written as

\[
\alpha^2 \frac{d^4 \ddot{u}}{dx^4} + 6\lambda \alpha^{2\lambda-1} \frac{d^3 \ddot{u}}{dx^3} + \alpha^{2\lambda-2}(9\lambda^2 - 3\lambda) \frac{d^2 \ddot{u}}{dx^2} - \frac{d^2 \omega^2 \rho A_0}{EI_0} \ddot{u} = 0 \quad (4-7-26)
\]

where \(\alpha = a + x\). This is a linear differential equation with variable coefficients and it is difficult to find a generic solution. However, the solution may be found in special cases where the equation can be factorised. When \(\lambda = 1\), the equation reduces to:

\[
\left( \alpha \frac{d^2 \ddot{u}}{dx^2} + 2 \frac{d \ddot{u}}{dx} - a \omega \sqrt{\frac{\rho A_0}{EI_0}} \ddot{u} \right) \left( \alpha \frac{d^2 \ddot{u}}{dx^2} + 2 \frac{d \ddot{u}}{dx} + a \omega \sqrt{\frac{\rho A_0}{EI_0}} \ddot{u} \right) = 0 \quad (4-7-27)
\]

and the shape function which is the solution to these two differential equations can be found as (Doyle 1989)

\[
\ddot{u}(x) = \left[ c_1 J_2(z) + c_2 Y_2(z) + c_3 J_2(z) + c_4 K_2(z) \right] (z)^{-1/2} \quad (4-7-28)
\]

where \( z = 2(a^2 + ax)^{1/2} \left( \omega^2 \rho A_0 \right)^{1/4} \) and \( J, Y, I, K \) are Bessel functions of the second order. The shape function for a uniform beam can be obtained as a special case by letting \(\lambda = 0\) and using the properties of Bessel functions of second order.
4.7.4 Winkler Beam Element

One of the applications of the spectral beam element is the Winkler beam in which the beam element is supported on a continuous foundation. Railway tracks are a well known engineering application of these beam elements, where the steel rail or beam can be assumed to be continuously supported by an elastic foundation. The governing differential equation of motion in this case can be written as,

\[ EI \frac{\partial^4 u}{\partial x^4} + \rho A \frac{\partial^2 u}{\partial t^2} + Ku = P(t) \]  

(4-7-29)

where \( K \) is the elastic stiffness of the foundation. The spectral solution can be given as,

\[ \tilde{u}(x,t) = u_0 e^{-i(\kappa x - \omega t)} \]  

(4-7-30)

and the characteristic equation is

\[ EI\kappa^4 - \rho A\omega^2 + K = 0 \]  

(4-7-31)

The spectral relations are then

\[ \kappa_1 = \pm \left( \frac{\rho A}{EI} \omega^2 - \frac{K}{EI} \right)^{\frac{1}{2}} \] ; \[ \kappa_2 = \pm i \left( \frac{\rho A}{EI} \omega^2 - \frac{K}{EI} \right)^{\frac{1}{2}} \]  

(4-7-32)

and the shape function can be written as
\[\tilde{u}(x) = Ae^{-ik\xi} + Be^{-ik\xi} + Ce^{ik\xi} + De^{ik\xi}\] (4-7-33)

The problem of the stresses caused by temperature can also be considered, when a term \(T \frac{\partial^2 u}{\partial x^2}\) is added to the governing differential equation of motion (Eq. 4-7-29). However, in this case the wavenumber is

\[\kappa = \pm \left[ \frac{-T \pm \sqrt{T^2 + 4EI(\rho A \omega^2 - K)}}{2EI} \right]^\frac{1}{2} \] (4-7-34)

The elementary beam solution can be recovered by setting \(K=0\) and \(T=0\).

### 4.7.5 Boundary Beam Element

In the case of sub-structuring of a very large structure, when the system is modelled using the spectral beam element, it is necessary to have a spectral boundary beam element to conduct the vibratory energy out of the substructure to the remote parts of the overall structure. The analogy is similar to the spectral boundary rod element. However, there are two degrees of freedom in this case. Consider the shape function for a boundary beam element when the element extends to infinity, as,

\[\tilde{u}(x) = c_1e^{-ik\xi} + c_2e^{-k\xi}\] (4-7-35)
which gives the stiffness matrix for the element as,

\[
\begin{align*}
\begin{bmatrix}
\kappa^3(i-1) & i\kappa^2 \\
 i\kappa^2 & \kappa(i+1)
\end{bmatrix}
\end{align*}
\]

(4-7-36)

This matrix is always complex and can be assembled into the global stiffness matrix by using the conventional assemblage procedure.

### 4.7.6 Torsional Shaft Element

The dynamic load on a structure can also be of the form of axial twisting which causes the torsion in members. The shaft member is usually designed for this axial torsion and the behaviour is very similar to that of a rod. Consider the governing differential equation of motion for a torsional member of circular cross section as,

\[
\frac{\partial}{\partial x} \left( GJ \frac{\partial \theta}{\partial x} \right) - \rho I \frac{\partial^2 \theta}{\partial t^2} = 0
\]

(4-7-37)

where \(GJ\) is the torsional rigidity, \(\rho I\) is the rotational inertia per unit length and \(\theta(x,t)\) is the rotation about the \(x\) axis. If it is assumed that the stiffness and mass are constant along the length

\[
GJ \frac{\partial^2 \theta}{\partial x^2} - \rho I \frac{\partial^2 \theta}{\partial t^2} = 0
\]

(4-7-38)
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The spectral solution in this case can be written as

$$\hat{\Theta}(x) = c_1 e^{-i\kappa x} + c_2 e^{i\kappa x}$$

(4-7-39)

and wavenumber can be found as

$$\kappa = \omega \left( \frac{\rho I}{GJ} \right)^{1/3}$$

(4-7-40)

Using Eq. (4-7-39) as a shape function, the equation of motion can be written in spectral format as:

$$[\hat{T}_n] = \frac{GJ}{L} [\hat{K}_n] [\hat{\Theta}_n]$$

(4-7-41)

where the dynamic stiffness matrix can be derived as,

$$[\hat{K}_n] = \begin{bmatrix}
\kappa_n L & -\kappa_n L \\
\tan(\kappa_n L) & \sin(\kappa_n L) \\
-\kappa_n L & \kappa_n L \\
\sin(\kappa_n L) & \tan(\kappa_n L)
\end{bmatrix}$$

(4-7-42)

which should be calculated for each frequency. The viscosity effect can also be implemented when a term $\eta \frac{\partial \Theta}{\partial t}$ is added to the governing differential equation of motion. In this case the torsional wave number can be found as
which has a dispersive effect on the torsional vibration of a shaft.

4.7.7 Two-Dimensional Plate Elements

This section develops a two-dimensional plate element for studying dynamic behaviour of solid plate structures. The medium under consideration in this section can support multi-dimensional vibration. Consider the general governing differential equations as,

\[ \sum_j \frac{\partial}{\partial x_j} \left( 2\mu \varepsilon_{ij} + \lambda \sum_k \varepsilon_{kk} \delta_{ij} \right) + \rho \left( p_i - \frac{\partial^2 u_i}{\partial t^2} \right) = 0 \]  

(4-7-44)

where \( p_i \) is the component of the body force, \( \varepsilon_{ij}, \delta_{ij}, \mu, \) and \( \lambda \) are the strain function, Kronecker delta and Lame constants respectively, and can be calculated as

\[ 2\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \]  

(4-7-45)

\[ \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}; \quad \mu = \frac{E}{2(1+\nu)} \]  

(4-7-46)

and \( \nu \) is Poisson's ratio. The displacement formulation can be used to reduce the field equations from the tensor formulation to a system of three equations in the
three unknown displacements (Love, 1944). These equations are known as Navier's equations and can be written as

\[
(\lambda + \mu) \sum_k u_{k,kl}(x,y,z,t) + \mu \sum_k u_{k,kl}(x,y,z,t) + \rho \left( p_i - \frac{\partial^2 u_i}{\partial t^2} \right) = 0 \tag{4-7-47}
\]

in the absence of body forces, Eq. (4-7-47) can be written in another form as

\[
(\lambda + \mu) \nabla \nabla u(x,y,z,t) + \mu \nabla^2 u(x,y,z,t) - \rho \frac{\partial^2 u(x,y,z,t)}{\partial t^2} = 0 \tag{4-7-48}
\]

where \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \). A reduced set of equations can be obtained by decomposing the displacement vector into scalar and vector potentials \( \phi(x,y,z,t) \) and \( H(x,y,z,t) \). Hence, the Helmholz decomposition represents the displacements by

\[
u(x,y,z,t) = \nabla \phi(x,y,z,t) + \nabla \times H(x,y,z,t) \]

\[\nabla \cdot H(x,y,z,t) = 0 \tag{4-7-49}\]

The Navier's equations can now be converted to the following two sets of differential equations by using the Heimholtz decomposition (Love, 1944) as,

\[
(\lambda + \mu) \nabla^2 \phi(x,y,z,t) - \rho \frac{\partial^2 \phi(x,y,z,t)}{\partial t^2} = 0 \tag{4-7-50}
\]

\[
\mu \nabla^2 H(x,y,z,t) - \rho \frac{\partial^2 H(x,y,z,t)}{\partial t^2} = 0
\]
The scalar potential, $\phi$, is associated with in-plane (horizontal) vibration and the vector potentials, $H$, with distortional vibration in infinite plates. When the plate structure vibrates in the vertical plane, Eq.(4-7-50) reduces to,

$$
\nabla^2 \phi(x, y, t) - \frac{1}{\alpha^2} \frac{\partial^2 \phi(x, y, t)}{\partial t^2} = 0
$$

$$
\nabla^2 H_z(x, y, t) - \frac{1}{\beta^2} \frac{\partial^2 H_z(x, y, t)}{\partial t^2} = 0
$$

where $\alpha = \left(\frac{\lambda + 2\mu}{\rho}\right)^{\frac{1}{2}}$ and $\beta = \left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}$. The spectral solution in this case, can be found as

$$
\phi(x, y, t) = \sum_{n=1}^{N} \sum_{m=0}^{M} \phi_{nm}(x, \kappa, \omega_n)e^{i\omega_n t}
$$

$$
H_z(x, y, t) = \sum_{n=1}^{N} \sum_{m=0}^{M} H_{nm}(x, \kappa, \omega_n)e^{i\omega_n t}
$$

(4-7-52)

where $\kappa$ is the wave number and $\omega$ is the angular frequency. The associated amplitudes are

$$
\phi_{nm}(x, \kappa, \omega_n) = c_{1nm}e^{-iK_1\omega_n x}
$$

$$
H_{zm}(x, \kappa, \omega_n) = c_{2nm}e^{-iK_2\omega_n x}
$$

(4-7-53)

where
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\[ \kappa_{1nm} = \pm \left[ \left( \frac{\omega_h}{\alpha} \right)^2 - \kappa_h^2 \right]^{1/2} \; ; \; \kappa_{1nm} = \pm \left[ \left( \frac{\omega_h}{\beta} \right)^2 - \kappa_h^2 \right]^{1/2} \]  (4-7-54)

and \( \kappa_h \) is the horizontal wave number given by

\[ \kappa_h = \pm i \sqrt{\frac{\omega_h}{K}} \]  (4-7-55)

where \( h \) is the plate thickness and \( \nu \) is Poisson's ratio. The spectral representation of the displacements can be found as

\[ u(x,y,t) = \frac{\partial \phi(x,y,t)}{\partial x} + \frac{\partial H(z(x,y,t))}{\partial y} = \sum_{n=1}^{N} \sum_{m=0}^{M} \bar{u}_{nm}(x, \kappa_h, \omega_h)e^{i\omega_h t} \]  (4-7-56)

\[ v(x,y,t) = \frac{\partial \phi(x,y,t)}{\partial y} - \frac{\partial H(z(x,y,t))}{\partial x} = \sum_{n=1}^{N} \sum_{m=0}^{M} \bar{v}_{nm}(x, \kappa_h, \omega_h)e^{i\omega_h t} \]

and the shape functions are

\[ \bar{u}(x, \kappa, \omega) = -c_{1nm}(i\kappa_{1nm})e^{-i\kappa_{1nm}x} \pm c_{2nm}(\kappa_{2nm})e^{-i\kappa_{2nm}x} \]  (4-7-57)

\[ \bar{v}(x, \kappa, \omega) = \mp c_{1nm}(\kappa_{hm})e^{-i\kappa_{hm}x} \pm c_{2nm}(\kappa_{2nm})e^{-i\kappa_{2nm}x} \]

The coefficients can be related to the nodal displacement as for the case of the beam element, and the governing differential equation can be written as
\[
\begin{align*}
\mathbf{\hat{K}}_{nm} &= \mathbf{\hat{K}}_{nn} [\hat{u}_{nn}] \\
\text{and the dynamic stiffness of the element can be found as}
\end{align*}
\]

\[
\begin{align*}
\mathbf{k}_{11nm} &= \mathbf{k}_{33nm} = \frac{\mu}{\Delta_{nm}} \left(-i\kappa_{2nm} \frac{\omega_t^2}{\beta^2} \right) \kappa_{1nm} \kappa_{2nm}^2 \zeta_{12} \zeta_{21} + \kappa_{nm}^2 \zeta_{11} \\
\mathbf{k}_{12nm} &= -\mathbf{k}_{34nm} = \frac{\mu}{\Delta_{nm}} \kappa_{hnm} \kappa_{1nm} \kappa_{2nm} \left( 4 \kappa_{hm}^2 - \left( \frac{\omega_t}{\beta} \right)^2 \right) \left( 4 e^{-i \kappa_{12} L} e^{-i \kappa_{21} L} - \zeta_{12} \zeta_{21} \right) \\
\mathbf{k}_{13nm} &= \frac{\mu}{\Delta_{nm}} 2i \kappa_{2nm}^2 \left( \frac{\omega_t^2}{\beta^2} \right) \kappa_{1nm} \kappa_{2nm}^2 e^{-i \kappa_{12} L} \zeta_{21} + \kappa_{hm}^2 e^{-i \kappa_{21} L} \zeta_{11} \\
\mathbf{k}_{14nm} &= -\mathbf{k}_{35nm} = \frac{\mu}{\Delta_{nm}} 2 \kappa_{hnm} \kappa_{1nm} \kappa_{2nm} \left( \frac{\omega_t^2}{\beta^2} \right) e^{-i \kappa_{12} L} \zeta_{22} - e^{-i \kappa_{21} L} \zeta_{12} \\
\mathbf{k}_{22nm} &= \mathbf{k}_{44nm} = \frac{\mu}{\Delta_{nm}} \left(-i\kappa_{1nm} \frac{\omega_t^2}{\beta^2} \right) \kappa_{1nm} \kappa_{2nm}^2 \zeta_{12} \zeta_{22} + \kappa_{hnm}^2 \zeta_{11} \zeta_{21} \\
\mathbf{k}_{24nm} &= \frac{\mu}{\Delta_{nm}} 2i \kappa_{1nm} \left( \frac{\omega_t^2}{\beta^2} \right) \kappa_{1nm} \kappa_{2nm}^2 e^{-i \kappa_{21} L} \zeta_{11} + \kappa_{hnm}^2 e^{-i \kappa_{12} L} \zeta_{21} \\
\end{align*}
\]

(4-7-59)

where

\[
\Delta_{nm} = 2 \kappa_{hnm} \kappa_{1nm} \kappa_{2nm} \left( 4 e^{-i \kappa_{12} L} e^{-i \kappa_{21} L} - \zeta_{12} \zeta_{22} \right) - \left( \kappa_{1nm}^2 \kappa_{2nm}^2 + \kappa_{hnm}^4 \right) \zeta_{11} \zeta_{21}
\]

\[
\zeta_{11} = 1 - e^{-2i \kappa_{12} L} ; \quad \zeta_{12} = 1 + e^{-2i \kappa_{12} L} ; \quad \zeta_{21} = 1 - e^{-2i \kappa_{21} L} ; \quad \zeta_{21} = 1 + e^{-2i \kappa_{12} L}
\]

The dynamic stiffness is symmetric and it is similar to that of the spectrally formulated beam element.
4.8 FRAME STRUCTURAL ANALYSIS

The conventional finite element method treats the structure as an assemblage of smaller elements. All the local matrices are handled in a straightforward manner, and then are transformed into the global structural coordinates. Frame structures can be classified as: plane trusses, space trusses, plane frames, grillage and space frames, in which it is assumed that their members are connected either by pins, by semi-rigid or rigid joints (Fig. 4-9).

A two-dimensional member in the structural frame may be subjected to both axial and bending loads. For a small displacement, the axial and flexural displacements are uncoupled and the total stiffness matrix for the conventional plane element (Fig. 4-10) in the local coordinate system can be derived as (Bathe, 1982),

\[
[K_{ec}] = \begin{bmatrix}
12b & 6Lb & 0 & -12b & 6Lb \\
4L^2b & 0 & -6Lb & 2L^2b \\
a & 0 & 0 & 0 \\
\text{Sym.} & 12b & -6Lb & 4L^2b
\end{bmatrix}
\] (4-8-1)

where \([K_{ec}]\) is the element stiffness matrix in the local coordinates and

\[
a = \frac{EA}{L} ; \quad b = \frac{EI}{L^2}
\] (4-8-2)
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Figure 4-9
Typical Frame Structure

Figure 4-10
Typical Degrees of Freedom for a 2-D Element in (a) Global; (b) Local Coordinates System
and the governing equation of motion for the element can be written as

\[
\begin{bmatrix}
F_{1\text{Axial}} \\
F_{1\text{Lateral}} \\
M_1 \\
F_{2\text{Axial}} \\
F_{2\text{Lateral}} \\
M_2
\end{bmatrix}
= \begin{bmatrix}
\phi_1 \\
\phi_2
\end{bmatrix}
\begin{bmatrix}
K_{ec}^{11} & K_{ec}^{12} \\
K_{ec}^{21} & K_{ec}^{22}
\end{bmatrix}

(4-8-3)
\]

The same augmented procedure can be used to derive a local dynamic stiffness for a general spectral plane element as

\[
[\hat{K}_{ec}] = \begin{bmatrix}
\hat{k}_{11\text{Rod}} & 0 & 0 & \hat{k}_{12\text{Rod}} & 0 & 0 \\
\hat{k}_{11\text{Beam}} & \hat{k}_{12\text{Beam}} & 0 & \hat{k}_{13\text{Beam}} & \hat{k}_{14\text{Beam}} \\
\hat{k}_{22\text{Beam}} & 0 & \hat{k}_{23\text{Beam}} & \hat{k}_{24\text{Beam}} \\
\hat{k}_{22\text{Rod}} & 0 & 0 & \hat{k}_{33\text{Beam}} & \hat{k}_{34\text{Beam}} \\
\text{Sym.}
\end{bmatrix}
\]

(4-8-4)

where the dynamic stiffness terms can be of the elementary or higher order forms as stated earlier. It should be noted that any combination of axial, lateral and torsional actions can be augmented in the general dynamic stiffness matrix (for example, if an axial twist is acting along the length instead of the axial force, the rod terms can be replaced by torsional terms). The description of the dynamic stiffness for three-dimensional frames follows from the assumption that the response of a general member is a simple superposition of axial, lateral and twisting actions. Hence, the twelve by twelve dynamic stiffness matrix can be assembled by using the same procedure as
\[
\begin{bmatrix}
\hat{K}_{11ec} & \hat{K}_{12ec} \\
\hat{K}_{21ec} & \hat{K}_{22ec}
\end{bmatrix}
\]

where the first sub matrix of the local dynamic stiffness matrix is

\[
\begin{bmatrix}
\tilde{k}_{11,Rod} & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{k}_{11,Beam} & 0 & 0 & \tilde{k}_{12,Beam} & 0 \\
0 & \tilde{k}_{11,Shaft} & 0 & 0 & \tilde{k}_{12,Beam} & 0 \\
\tilde{k}_{21,Beam} & 0 & \tilde{k}_{12,Beam} & 0 & 0 & \tilde{k}_{22,Beam}
\end{bmatrix}
\]

The superscripts \(x, y\) and \(z\) indicate the local axis about which the moment of inertia \(I\) is calculated.

### 4.8.1 Transformation Matrix

To analyse a general frame structure, it is essential to consider the stiffness of an arbitrarily oriented member. As there may be many differently oriented members, they should have a common global coordinate system. The relation between the local and global coordinate systems is obtained by using the usual coordinate transformation for vectors. Hence, for the forces in Figs. (4-10a) and (4-10b), it can be written
The matrix form of the governing equations of motion of the element in the global coordinate system is

\[ [T][\dot{F}] = [\tilde{K}_{ec}][T][\dot{u}] \rightarrow [\dot{F}] = [T]^T[\tilde{K}_{ec}][T][\dot{u}] \]  

(4-8-8)

In the case of three dimensional frames, the general rotation matrix is more complicated than for planar frames. Fig. 4-11 shows the local and global three dimensional coordinate systems where the general rotation matrix can be constructed by using three successive rotations from the global to local coordinates. The first two rotations are the rotation about \( z \) and \( y \) axes. The final rotation consists of the rotation through the angle \( \phi \) about the \( x \) axis, causing the local \( y \) and \( z \) axes to coincide with the principal axes of the cross section. These three successive rotations give the general transformation matrix as

\[
[T_{3D}] = \begin{bmatrix}
x_i & y_i & z_i \\
-y_i \cos \phi - x_i z_i \sin \phi & x_i \cos \phi - y_i z_i \sin \phi & \lambda \sin \phi \\
\frac{y_i \sin \phi - x_i z_i \cos \phi}{\lambda} & \frac{-x_i \sin \phi - y_i z_i \cos \phi}{\lambda} & \lambda \cos \phi
\end{bmatrix}
\]  

(4-8-9)

where \( x_i, y_i, z_i \) are shown in Fig. 4-11 and \( \lambda = \sqrt{1 - z_i^2} \).
4.8.2 Boundary Conditions

To account for the prescribed forces or displacements at the nodes, the global system of equations must be rearranged and renumbered. If the displacements at some nodes are zeros (fixed boundaries) then the associated rows and columns can be eliminated and the reduced system matrices can be solved. However, when the external force is given, the corresponding displacements are unknown quantities, and the global system of equations should be solved to obtain these unknowns. Another case occurs when there are known displacements (non-zero) at some nodes in addition to external forces. The global equations for the system in this case can be partitioned according to the unknown and known displacements as,

\[
\begin{bmatrix}
\hat{\mathbf{F}}_k \\
\hat{\mathbf{F}}_u
\end{bmatrix} =
\begin{bmatrix}
\hat{\mathbf{K}}_{uu} & \hat{\mathbf{K}}_{uk} \\
\hat{\mathbf{K}}_{ku} & \hat{\mathbf{K}}_{kk}
\end{bmatrix}
\begin{bmatrix}
\hat{\mathbf{u}}_u \\
\hat{\mathbf{u}}_k
\end{bmatrix}
\] (4-8-10)
where the subscript \( k \) and \( u \) refers to known and unknown, respectively. This form of the system equations gives

\[
\begin{align*}
\{\hat{f}_k\} &= [\hat{K}_{uu}][\hat{\hat{u}}_u] + [\hat{K}_{uk}][\hat{\hat{u}}_k] \Rightarrow \{\hat{\hat{u}}_u\} = [\hat{K}_{uu}]^{-1}\{\hat{f}_k\} - [\hat{K}_{uk}][\hat{\hat{u}}_k] \\
\{\hat{f}_u\} &= [\hat{K}_{ku}][\hat{\hat{u}}_u] + [\hat{K}_{kk}][\hat{\hat{u}}_k] \Rightarrow \{\hat{\hat{f}}_u\} = [\hat{K}_{ku}][\hat{K}_{uu}]^{-1}\{\hat{f}_k\} - [\hat{K}_{uk}][\hat{\hat{u}}_k] + [\hat{K}_{kk}][\hat{\hat{u}}_k]
\end{align*}
\]

(4-8-11)

where the unknown displacements can be obtained from the first equation and the unknown external loads are obtained from the second equation.

In earthquake engineering problems where there is an imposed non-zero displacement (acceleration) at the base, a more efficient treatment is to add boundary elements. This method can be interpreted as adding an elastic spring of large stiffness at the basement nodes and specifying a load that produces the required displacements. Actually the idea of the boundary element (penalty method) adds a lot of flexibility to the matrix method. It can also be used to implement oblique supports in structures (i.e., a roller support on an inclined surface). A further discussion about the penalty method can be found in Bathe (1982).

### 4.9 COMPUTER IMPLEMENTATION

The algorithms for implementing the conventional finite element method on a computer are well known and can be found in many textbooks. They start with the derivation of local stiffness, damping and mass matrices for each element, and then they transform these matrices to global coordinates for the whole structure. The global stiffness, damping and mass matrices are assembled by associating the
appropriate nodal numbers. The zero degrees of freedoms associated with the structural supports can then be removed by taking out their associated rows and columns in the global matrices. The system of simultaneous equations are then solved by using one of the standard solvers.

There are two conventional types of dynamic analyses of structures; modal and transient analyses. The power of modal analysis is that it gives the basic dynamic characteristics of structures. It can reduce the dynamic problem to solving the partial eigenvalue problem, as it is usually only the first few modes that are of interest. The dynamic transient analysis is also of interest because many realistic loads vary with time. Hence, a direct integration scheme is normally employed in the conventional finite element to carry out a sequence of pseudo-static analyses for each time interval. The stability and accuracy of these incremental solutions depend on the numerical methods employed.

4.9.1 Computer Algorithm for the Spectral Finite Element Method

The spectral formulation, as described in earlier sections, can be implemented in a large computer program which runs under the Matlab and Mathematica computer environments. A more comprehensive discussion about the Matlab and Mathematica computer packages is presented in Appendix II. However, before the task of writing a computer program begins, the algorithm on which the program is based, should be designed. An algorithm is a sequence of steps for arriving at the required results with the use of available data. A widely used notation for describing an algorithm is the flowchart which is a graphical tool explaining the sequence, decision and loops used in the logic of the computer program.

The algorithm architecture for the spectral finite element method is almost identical to that of the conventional finite element method in terms of assemblage,
input and output, and solver. However, as the spectral finite element method is formulated in the frequency domain, all the system matrices must be assembled for each frequency step, which in programming terms means that there should be a do-loop over all of the frequency components.

The structural dynamic stiffness matrix at each frequency step is then assembled from local matrices simply by associating the appropriate nodal numbers. Those zero degree of freedoms are removed and the resulting system of equations for the frequency step are determined by using a solver. The solver is different from the conventional finite element solver in that it should handle complex algebra.

The displacement results are obtained at each frequency step, and then can be transformed to the time domain using the FFT. The response at locations that are not nodes can also be found using the shape function. Fig. 4-12 shows the top-down flowchart of the proposed method. The performance of the algorithm on a PC machine is estimated in Figs. 4-13 and 4-14 using the symbolic environment of the Mathematica computer package. In order to do that, the theoretical operation counts of both the proposed and the Newmark time domain solution are compared.

Despite the solution being machine and problem dependent, the parametric study herein shows some interesting comparisons on how these two methods work. The efficiency analysis in Figs. 4-13 and 4-14 shows that although the operation count (OC) of the two solutions for small structures with a small number of degrees of freedoms is nearly the same for the two methods, for large structures the proposed spectral method outperformed the conventional finite element method (the number of the operation count for the Newmark time domain solution is 2.5 times that of the proposed method for 10000 degrees of freedoms).
Start Analysis

Input Material and Geometry Data

Fast Fourier Transform: Applied Force

Calculate the Element Dynamic Stiffness Matrix at each Frequency Step

Assemble the Global Dynamic Stiffness Matrix

Rearrangement and Renumbering Due to Boundary Conditions

Solve the Global System of Equations to Obtain Results

Check for Frequency Step

Fast Fourier Transform of Result to Obtain Time History Results

Finish analysis

Figure 4-12

Top-Down Flowchart for the Spectral Method
Figure 4-13

Operation Count (OC) for (a) Newmark and; (b) Spectral Method Versus the Number of Degrees of Freedoms (DOFs) and the Half-Band Width of Stiffness Matrix [K]
Figure 4-14

Operation Count (OC) for (a) Newmark and; (b) Spectral Method Versus the Number of Degrees of Freedoms (DOFs) and the Half-Band Width of Stiffness Matrix [K]
4.9.2 FF Transformation

The Fast Fourier Transform (FFT) is a computer algorithm for evaluating the discrete Fourier transform. The replacement of the Fourier integrations by summations in the conventional discrete Fourier approach is a further step in the numerical implementation of the continuous transform. The discrete Fourier transform has the same properties of the continuous transform. A final step in the numerical implementation of Fourier transform is the development of the computer algorithm of FFT for performing the summation of the discrete Fourier transform in an efficient manner. Consider the generic discrete Fourier forward transform given as,

\[ X_k = \frac{1}{N} \sum_{r=0}^{N-1} x_r e^{-\frac{2\pi i r k}{N}} \quad k = 0, 1, 2, \ldots, (N-1). \]  \hspace{1cm} (4-9-1)

in which the conventional direct approach would make \( N \) multiplications for each of \( N \) values of \( X_k \), and so the total work of calculating the full sequence \( X_k \) would require \( N^2 \) multiplications. The FFT reduces this work to a number of operations of the order \( N \log_2 N \) by taking advantage of the special form of the exponential terms.

In fact, it is apparent that in the expanded form of summation many of the computations used in forming one of the summations is also used in the others and it does not have to be repeated. A computer program for performing the FFT can be found in Appendix I. Fig 4-15(a) shows an arbitrary dynamic load which is plotted against time. The power spectral density, a measurement of the energy at various frequencies, for the dynamic load can be obtained using FFT. Fig 4-15(b) shows the power spectral density (which is used in the design of dynamical systems) of the time domain load where the function of spectral density is plotted versus the frequency. It can be seen that if a dynamical system is excited by this dynamical force, the maximum energy may be transferred at 120 Hz, which is an essential fact in the design of the system.
Figure 4-15
(a) Arbitrary Dynamic Load History; (b) Power Spectral Density of the Load
4.10 SUMMARY AND CONCLUDING REMARKS

The theoretical basis of the spectral finite element method of analysis has been presented. As it appears from the presented analytical methods, the main advantage of the spectral finite element method lies in its capability to model a long length of uniform section as one element.

As far as stability and efficiency are concerned, it can be assessed that even though the assemblage of global matrices has to be repeated for all frequency components in comparison with the single assemblage procedure for the conventional finite element method, the spectral finite element approach outperforms the conventional method in terms of computer time for large structures.
Chapter Five

Fractional Derivative Damping Models

5.1 GENERAL

The mathematical basis of the generalised integro-differential operators to arbitrary order can be dated back to the nineteenth century. The concept of using fractional calculus in the formulation of constitutive equations for engineering materials has been proposed during the last 50 years. In particular, a number of authors have explicitly used fractional calculus as an empirical method of describing the properties of viscoelastic materials. The traditional differential operators (Jones 1980) are typically employed in the formulations of complex viscoelastic constitutive relationships. However, the advantages of fractional operators in establishing a richer variety of functional families, and hence the possibility of improved generalised integro-differential type curve fitting of constitutive relationships, have attracted much attention in recent years (Bagley 1979; Koeller 1984).

There are several mathematical possibilities to establish a generalised integro-differential calculus of any fractional order. However, from a physical point of view, attention should be given to that method which is capable of more accurate
modelling of physical applications. Many problems in physical science and engineering, e.g. constitutive relation theory, potential theory and transport theory, can be solved by using the fractional derivative approach. Fractional-derivative stress-strain constitutive relationships for viscoelastic materials not only describe the mechanical properties of some damping materials, but lead to straightforward solutions of the finite element equations of motion for damped structures.

5.2 FRACTIONAL CALCULUS

The advantages of fractional operators, in establishing an improved generalised integro-differential type curve fitting of constitutive relationships for engineering materials, have attracted much attention in recent years (Bagley, 1983 and Koeller 1984). Previous attempts to model the mechanical properties of viscoelastic materials have not been completely successful, because the model simply has not been linked to the physical principle involved.

Before proceeding to construct fractional derivative constitutive relations for viscoelastic materials, it is appropriate to introduce the mathematical basis of these operators. To enable the development of the generalised derivative, consider a function,

\[ y = f(x) \]  \hspace{1cm} (5.2-1)

which can be expanded into a power series with positive exponents as :

\[ y = a + bx + cx^2 + \ldots \]  \hspace{1cm} (5.2-2)
If the hierarchy of formal limiting expressions of integer derivatives is recalled,

$$\frac{d^1}{dx^1}(f(x)) = \lim_{\Delta x \to 0} \left( f(x) - f(x - \Delta x) \right) (\Delta x)^{-1} \quad (5-2-3)$$

$$\frac{d^2}{dx^2}(f(x)) = \lim_{\Delta x \to 0} \left( f(x) - 2f(x - \Delta x) + f(x - 2\Delta x) \right) (\Delta x)^{-2} \quad (5-2-4)$$

Continuing the sequence of operators, using mathematical induction,

$$\frac{d^q}{dx^q}(f(x)) = \lim_{\Delta x \to 0} \left( \sum_{i=0}^{q} (-1)^i \frac{q!}{i!(q-i)!} f(x - i\Delta x) \right) (\Delta x)^{-q} \quad (5-2-5)$$

where $q$ is an integer. As,

$$(-1)^i \frac{q!}{i!(q-i)!} = \frac{(i-q-1)!}{i!(-q-1)!} = \frac{\Gamma(i-q)}{\Gamma(-q)\Gamma(i+1)} \quad (5-2-6)$$

it can be seen that the general form of Eq.(5-2-5) for any integer or irrational number $q$ can be reduced to the form (Koeller 1984)

$$\frac{d^q}{dt^q}(f(x)) = \frac{1}{\Gamma(-q)} \int_0^x (x-z)^{-q-1} f(z)dz \quad (5-2-7)$$
where $\Gamma(\cdot)$ denotes the gamma function and $z$ is a dimensionless variable. Introducing a variable $\alpha=q+1$, Eq.(5-2-7) can be rewritten in the form of a fractional derivative for $0<\alpha<1$ using the Cauchy's formula (Bagley 1979),

$$\frac{d^\alpha}{dx^\alpha} (f(x)) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int \frac{f(z)}{(x-z)^{\alpha}} dz \quad (5-2-8)$$

An interesting mathematical property of this fractional derivative operator is that the Fourier transform of the fractional derivative of the function $\sigma(t)$, under certain conditions, is equal to the transform of $\sigma(t)$ multiplied by a coefficient,

$$F\left(D^\alpha [\sigma(t)]\right) = (i\omega)^\alpha F(\sigma(t)) \quad (5-2-9)$$

where

$$F\langle x(t) \rangle = \int_{-\infty}^{\infty} x(t)e^{-i\omega t} dt \quad (5-2-10)$$

is the Fourier transform function. The first condition is that

$$\sigma(t) = 0 \text{ for } t < 0 \rightarrow F\langle x(t) \rangle = \int_{0}^{\infty} x(t)e^{-i\omega t} dt \quad (5-2-11)$$

and, the second is that the integral in Eq.(5-2-11) exists. A similar relationship exists in the Laplace transform as,
\[ L(D^a [\sigma(t)]) = s^a L(\sigma(t)) \]  
(5-2-12)

where

\[ L(x(t)) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \]  
(5-2-13)

Another important property of the fractional derivative is that it is a linear operator

\[ D^a [\sigma_1(t) + \sigma_2(t)] = D^a [\sigma_1(t)] + D^a [\sigma_2(t)] \]  
(5-2-14)

and the fractional derivative of order \( \alpha_1 \), of the fractional derivative of order \( \alpha_2 \) is the fractional derivative of order \( \alpha_1 + \alpha_2 \) of the function

\[ D^{\alpha_1} [D^{\alpha_2} (\sigma(t))] = D^{\alpha_1+\alpha_2} [\sigma(t)] \]  
(5-2-15)

In order to show the applicability of the fractional calculus, consider the following function,

\[ f(x) = Ax^n \]  
(5-2-16)

which can be expanded into a power series with positive exponents. The fractional derivative for the \( q \)th order gives,
\[
\frac{d^q}{dx^q} = A n(n-1)\ldots(n-q+1).x^{n-q} \tag{5-2-17}
\]

This operator can be rewritten as,

\[
\frac{d^q}{dx^q} = A \frac{n!}{(n-q)!} x^{n-q} \tag{5-2-18}
\]

where the value of \( q \) can be any integer or fraction number. Fig 5.1 shows a series of fractional derivatives of different orders plotted against the variable \( x \) for the function \( f(x) = x^5 \). It can be seen that the fractional functions are an interpolation between the integer values.

\[\begin{align*}
1. & \quad f(x) = x^5 \\
2. & \quad \frac{d^{0.5}}{dx^{0.5}} [f(x)] = 2.29258 x^{4.5} \\
3. & \quad \frac{d^1}{dx^1} [f(x)] = 5 x^4 \\
4. & \quad \frac{d^{1.5}}{dx^{1.5}} [f(x)] = 10.3166 x^{3.5}
\end{align*}\]

Figure 5.1

Fractional Operators of the Function \( f(x) \) Against the variable \( x \)
In the following discussion, the fraction derivative concept as stated will be used to model the damping characteristic of viscoelastic materials.

5.3 VISCOELASTIC DAMPING MODEL

Material behaviour is termed viscoelastic if the material stores part of the deformational energy elastically as potential energy, and dissipates the rest simultaneously through viscous forces. The rheological properties of a viscoelastic material are time-dependent. Although, in principle all real materials are viscoelastic, this property becomes significant when the time required for the full development of a response is comparable with the time scale of the test performed to determine it. When a stress or a strain is impressed upon a body, rearrangement take place inside the material as it responds to the imposed excitation. In any real material these rearrangements necessarily require a finite time. As a consequence of the material rearrangement taking place on a time scale comparable to that of the test in which the response is observed, the relation between stress and strain or rate of strain cannot be expressed by material constants as in the case of purely elastic or purely viscous material.

\[
\sigma = E\varepsilon \quad (elastic) \quad (5-3-1)
\]

\[
\sigma = \eta \frac{d\varepsilon}{dt} \quad (viscous) \quad (5-3-2)
\]

where \( E \) is the Young's modulus and \( \eta \) is the viscosity. It has been shown (Flugge, 1975) that the simplest constitutive equation which adequately describes the infinitesimal deformation of a viscoelastic body is,
\[
\sigma + a \frac{d\sigma}{dt} = b\varepsilon + c \frac{d\varepsilon}{dt}
\]  
(5-3-3)

where \(a\), \(b\) and \(c\) are constants. This linear differential equation can be expressed in another form,

\[
\sum_{n=0}^{\infty} u_n \frac{d^n\sigma(t)}{dt^n} = \sum_{m=0}^{\infty} q_m \frac{d^m\varepsilon(t)}{dt^m}
\]  
(5-3-4)

where \(u_n\) and \(q_m\) are constant coefficients. The manipulation of this differential-integral equation is facilitated by the use of integral transform methods (Jones 1980). For the purpose herein the one-sided Laplace transformation is most suitable. Therefore, (5-3-4) can be rewritten as,

\[
\bar{u}(s)\bar{\sigma}(s) = \bar{q}(s)\bar{\varepsilon}(s)
\]  
(5-3-5)

Where \(\bar{\sigma}(s)\) and \(\bar{\varepsilon}(s)\) are the stress and strain transforms respectively, and

\[
\bar{u}(s) = \sum_{n} u_n s^n \text{ and } \bar{q}(s) = \sum_{m} q_m s^m
\]  
(5-3-6)

are polynomials in the transform variable \(s\). Rearranging (5-3-5),

\[
\bar{\sigma}(s) = \frac{\bar{q}(s)}{\bar{u}(s)} \bar{\varepsilon}(s)
\]  
(5-3-7)
\[
\overline{\epsilon}(s) = \frac{\overline{u}(s)}{\overline{q}(s)} \overline{\sigma}(s)
\]  
(5-3-8)

Then

\[
\overline{Q}(s) = \frac{\overline{q}(s)}{\overline{u}(s)} \rightarrow \overline{\sigma}(s) = \overline{Q}(s)\overline{\epsilon}(s)
\]  
(5-3-9)

\[
\overline{U}(s) = \frac{\overline{u}(s)}{\overline{q}(s)} \rightarrow \overline{\epsilon}(s) = \overline{U}(s)\overline{\sigma}(s)
\]  
(5-3-10)

Eqs. (5-3-9) and (5-3-10) form the basis upon which the linear theory of viscoelastic behaviour is developed. These equations also form the basis for the important correspondence or equivalence principle (Flugge 1975). According to this principle, if an elastic solution to a boundary value problem (stress analysis problem) is known, substitution of the appropriate Laplace transforms for the quantities employed in the elastic analysis furnishes the viscoelastic solution in the transform plane. The time dependent viscoelastic solution is then obtained by inverting the transform. The principle can be applied if the boundaries themselves do not change with time.

As stated earlier, in the deformation of a viscoelastic body part of the total work of deformation is dissipated as heat through viscous losses, but the remainder of the deformational energy is stored elastically. It is of interest to determine the amount of energy dissipated. The rate at which energy is absorbed per unit volume of a viscoelastic material during deformation is defined as being equal to the stress power. The stress power at time \( t \) is:

\[
\dot{w}(t) = \sigma(t)\dot{\epsilon}(t)
\]  
(5-3-11)
i.e. it is the product of the instantaneous stress and rate of strain. The energy stored \( w_s(t) \), and the energy dissipated, \( w_d(t) \), combine to make up the total deformational energy. Thus

\[
w_s(t) = w_s(t) + w_d(t)
\]  
(5-3-12)

For harmonic response it is not difficult to prove (Flugge 1975):

\[
\frac{w_d}{2w_s} = \psi_w = 2\pi \tan \theta
\]  
(5-3-13)

The ratio \( \psi_w \) is sometimes known as the specific loss, \( \theta \) is the loss tangent, and is the angle by which the stress leads the strain in a steady-state time harmonic loading of the material.

### 5.4 FRACTIONAL VISCOELASTIC FORMULATION

In the classical viscoelasticity theory either differential or hereditary integral formulations can be used to represent material behaviour. These approaches (i.e., the linear viscoelastic models) can be found in textbooks on dynamics. While such models have wide use, they are restricted to the limited classical functional families; exponential, trigonometric, hypergeometric, etc. On the other hand, the use of fractional integro-differential operators extends the number of functional families to a richer variety which may be used to model the material response.
The concept of using a fractional-derivative damping model in the formulation of viscoelastic damped structures has been proposed previously (Bagley 1979), but, these attempts to solve the resulting equation of motion for the structure lead to long and tedious calculation, with long computer times and large memory requirements. One of the simplest of these models is to use a complex constant modulus in which the constitutive relationship is formulated using the complex uniaxial, biaxial or triaxial modulus.

The general form of constitutive equation (Eq. 5-3-3) can be written in fractional derivative form as,

\[ \sigma(t) + \sum_{i=1}^{M} a_i D^b [\sigma(t)] = e_0 \varepsilon (t) + \sum_{j=1}^{N} f_j D^c [\varepsilon (t)] \]  

where \( a_i, b_i, c_j, e_0 \) and \( f_j \) are model parameters. Since previous experimental tests (Bagley 1979) indicate that most viscoelastic materials can be accurately modelled using only the first fractional derivative term in each series, the result is a five parameter model expressed as,

\[ \sigma(t) + a D^b [\sigma(t)] = e_0 \varepsilon (t) + f D^c [\varepsilon (t)] \]  

(5-4-2)

Taking the Fourier transform of Eq. (5-4-2) leads to,

\[ \hat{\sigma}(i\omega) + a (i\omega)^b \hat{\sigma}(i\omega) = e_0 \hat{\varepsilon}(i\omega) + f (i\omega)^c \hat{\varepsilon}(i\omega) \]  

(5-4-3)
where the time variational terms are formulated in the frequency domain instead of a direct formulation in the time domain, and $\sigma(i\omega)$ and $\dot{\varepsilon}(i\omega)$ are the stress and strain histories in transform coordinates. The more familiar relationship between stress and strain can be obtained by rewriting Eq. (5-4-3) as,

$$\sigma(i\omega) = \frac{e + f(i\omega)\varepsilon}{1 + a(i\omega)^b} \dot{\varepsilon}(i\omega)$$

which can be written as,

$$\hat{\sigma}(i\omega) = \hat{E}(\omega)\varepsilon(i\omega), \quad \hat{E}(\omega) = \frac{e + f(i\omega)^c}{1 + a(i\omega)^b}$$

which is similar to $\sigma=\varepsilon$ for elastic materials. However, in the fractional derivative model the modulus is complex, frequency dependent, and, most importantly, it is a function of fractional powers of frequency.

5.5 MATERIALS UNDER SHEAR FORCES

The damping behaviour of viscoelastic materials subjected to shear forces can be described by a simplified linear viscoelastic law. The relationship between the shear stress $\tau$ and strain $\theta$ can be expressed as,

$$\tau(t) = \theta_0 (G'\sin\omega t + G''\cos\omega t)$$
where \( \theta_0 \) is the shear strain amplitude, \( \omega \) is the frequency, \( G' \) (the real part) is the elastic shear storage modulus, and \( G'' \) (the imaginary part) is the shear loss modulus, and these may be defined as follows:

\[
G' = \frac{\tau_0}{\theta_0} \cos \phi \quad G'' = \frac{\tau_0}{\theta_0} \sin \phi
\]  

(5-5-2)

where \( \tau_0 \) is the shear stress amplitude, and \( \phi \) is the phase angle between stress and strain under steady state conditions and is a function of excitation frequency. For the steady state response of a viscoelastic shear damper subjected to sinusoidal deformation, we have,

\[
\tau(t) = \tau_0 \sin(\omega t + \phi) \quad \theta(t) = \theta_0 \sin(\omega t)
\]  

(5-5-3)

Substituting Eq. (5-5-3) into Eq. (5-5-1), the shear stress in the viscoelastic material can be described as,

\[
\tau(t) = G'\theta(t) \pm G'' \sqrt{\theta_0^2 - \theta(t)^2}
\]  

(5-5-4)

where Eq. (5-5-4) represents a combination of a straight line representing the elastic shear component and an ellipse representing the energy dissipation component (Fig. 5.2). The equivalent viscous damping ratio, \( \xi_d \), for the viscoelastic material can be related to this hysteresis formulation as,

\[
\xi_d = \frac{E_d}{4\pi E_s} = \frac{\pi G'\theta_0^2}{4\pi \left(0.5G'\theta_0^2\right)} = \frac{G''}{2G'}
\]  

(5-5-5)
where $E_d$ is the energy dissipated during one cycle of the hysteresis loop and $E_s$ is the elastic strain energy stored in the material at peak deformation.

5.5.2 Fractional-Derivative Formulation

Consider a fractional-derivative constitutive relation for a viscoelastic material under shear stress as,

$$\tau(t) + a D^b \tau(t) = G_1 \theta(t) + G_2 D^c \theta(t)$$

(5-5-6)

Taking the Fourier transform of Eq. (5-5-6) leads to,

$$\tilde{\tau}(i\omega) + a (i\omega)^b \tilde{\tau}(i\omega) = G_1 \tilde{\theta}(i\omega) + G_2 (i\omega)^c \tilde{\theta}(i\omega)$$

(5-5-7)
where the time variational terms are formulated in the frequency domain instead of a
direct formulation in the time domain, and \( \tau(i\omega) \) and \( \dot{\theta}(i\omega) \) are the stress and strain
histories in transform coordinates. The more familiar relationship between stress and
strain can be obtained by rewriting Eq. (5-5-7) as,

\[
\tau(i\omega) = \frac{G_1 + G_2(i\omega)^c}{1 + a(i\omega)^b} \dot{\theta}(i\omega)
\]  \hspace{1cm} (5-5-8)

which can be written as,

\[
\tau(i\omega) = \hat{G}(\omega) \times \dot{\theta}(i\omega), \quad \hat{G}(\omega) = \frac{G_1 + G_2(i\omega)^c}{1 + a(i\omega)^b}
\] \hspace{1cm} (5-5-9)

which is a similar to \( \tau = G\theta \) for elastic materials. However, in the fractional
derivative model the modulus is complex, frequency dependent, and most
importantly, it is function of fractional powers of frequency.

To examine the credibility of the fractional derivative model, the hysteresis
behaviour of the model will be checked. To carry out this check, the sinusoidal
strain history can be assumed as,

\[
\theta(t) = \theta_0 \sin(\omega t)
\] \hspace{1cm} (5-5-10)

The hysteresis loop is produced if this sinusoidal strain history leads to a sinusoidal
stress history as time becomes very large. Since (Bagley, 1979),
\[
\lim_{t \to \infty} \mathcal{D}_t^{\alpha} \left( \theta_0 \sin \omega t \right) = \frac{\omega \cos \omega t}{\Gamma(1-\alpha)} \int_0^t \cos \omega t \, dt + \frac{\omega \sin \omega t}{\Gamma(1-\alpha)} \int_0^t \sin \omega t \, dt
\]  
(5-5-11)

The two integrals in Eq. (5-5-11) are the Fourier cosine and sine transformation of \( t^{-\alpha} \). Evaluating these integrals leads to,

\[
\lim_{t \to \infty} \mathcal{D}_t^{\alpha} \left( \theta_0 \sin \omega t \right) = \omega^\alpha \theta_0 \sin \left( \omega t + \frac{\alpha \pi}{2} \right)
\]  
(5-5-12)

In the limit as \( t \) becomes very large, Eq. (5-5-12) for the sinusoidal strain history can be written as,

\[
\tau(t) + a \mathcal{D}^b \left[ \tau(t) \right] = G_1 \theta_0 \sin \omega t + \int_0^t G_2 \omega^\alpha \theta_0 \sin \left( \omega t + \frac{\alpha \pi}{2} \right) dt
\]  
(5-5-13)

Since the superposition of any number of out-of-phase sine waves can be expressed as a single sine wave, the existence of a hysteresis loop is established.

### 5.6 FRACTIONAL DERIVATIVE DAMPING MODELS

The task in this section is to use the basic generalised derivative constitutive relation as the building block for constitutive relations that model the non-linear damping characteristics of engineering materials and dampers. The advantages of fractional operators in establishing an improved generalised integro-differential type curve fitting of constitutive relationships for engineering materials, have been
investigated in recent years by some researchers (Bagley, 1979, Koeller, 1984). In general, an engineering material with a reasonable damping capacity shows frequency and temperature dependencies which can be described using a complex modulus. At a low frequency, the real part of the complex modulus describing its behaviour is relatively constant, while the imaginary part of the modulus increases with frequency. At intermediate frequencies, both the real and imaginary parts of the modulus increase with increasing frequency. However, the rate of increase of the real part slowly overtakes the rate of increase of the imaginary part. At high frequencies, the real part is again constant and the imaginary part of the modulus start to decrease with increasing frequency.

The fractional derivative constitutive relations presented in previous sections can be used to formulate a non-linear damping element. The fractional derivative damping element can then be calibrated to model the usual structural dampers (i.e. viscoelastic dampers in tall structures, elastomeric dampers in base-isolated structures).

5.6.1 Material Damping

Many engineering materials exhibit damping to some degree, which proves that the strain in these materials is not a function of stress alone. In fact, material damping mechanisms by which the energy is dissipated in materials and systems are large in number and complexity. These mechanisms may depend on the type of materials, chemical composition, internal crystalline or non-crystalline structure, temperature, pre-stress, initial strain, geometry, amplitude and frequency.

Early observations of the mechanical properties of damping materials by Nutting (1921), indicated that the stress relaxation phenomenon appeared to be proportional to time raised to fractional powers. These observations have then been
confirmed by Gemant (1938), and he suggested that the frequency-dependency of the material damping be modelled using differentials of fractional order. The use of the fractional-derivative constitutive relation in modelling the response of engineering materials has an advantage over the conventional models. The major drawback of the standard viscoelastic constitutive model is that a large number of terms are required to describe a material adequately. The simplest conventional uniaxial constitutive relation for damping materials can be written in the frequency domain as,

\[ \sigma(\omega) = \dot{E}(\omega)\dot{\varepsilon}(\omega) \]  \hspace{1cm} (5-6-1)

where \( \dot{E}(\omega) \) can be measured for a set of discrete values of frequency. However, the major drawback of this method is the huge task of calculating the inverse transform for every point in time at which the value of the response is required. The general form of the conventional constitutive relation can be written as,

\[ \sigma(t) = \delta \lambda \varepsilon(t) + 2\mu \ddot{\varepsilon}(t) \]  \hspace{1cm} (5-6-2)

where \( \delta \) is the Kronecker delta, \( \lambda \) and \( \mu \) are the dilatation and shear modulus respectively, and,

\[ \varepsilon(t) = \varepsilon_{11}(t) + \varepsilon_{22}(t) + \varepsilon_{33}(t) \]  \hspace{1cm} (5-6-3)

Taking the Fourier transform of Eq. (5-6-2) gives,
\[
\hat{\sigma} (\omega) = \delta \lambda_y (\omega) \hat{\varepsilon} (\omega) + 2 \hat{\mu}_f (\omega) \hat{\varepsilon} (\omega) 
\]  
(5-6-4)

A general fractional derivative constitutive relation can be proposed as,

\[
\sigma(t) = \delta \lambda_y \varepsilon (t) + 2 \mu_f \varepsilon (t) 
\]  
(5-6-5)

where \( \lambda_y \) and \( \mu_f \) are the fractional forms of the dilatation and shear modulus, respectively. Taking the Fourier transform of Eq. (5-6-5) and using the fractional derivative mathematical properties (Eq. 5-2-9), the frequency domain model can be derived as,

\[
\hat{\sigma} (\omega) = \delta \lambda_y (\omega) \hat{\varepsilon} (\omega) + 2 \hat{\mu}_f (\omega) \hat{\varepsilon} (\omega) 
\]  
(5-6-6)

where \( \lambda_y (\omega) \) and \( \hat{\mu}_f (\omega) \) are the Fourier transforms of the fractional dilatation and shear modulus, respectively, and can be found as,

\[
\lambda_y (\omega) = \lambda_0 + \sum_{i=1}^{N} \lambda_i \omega^{\alpha_i} \cos \frac{\pi \alpha_i}{2} + i \sum_{i=1}^{N} \lambda_i \omega^{\alpha_i} \sin \frac{\pi \alpha_i}{2} 
\]  
(5-6-7)

\[
\hat{\mu}_f (\omega) = \mu_0 + \sum_{j=1}^{M} \mu_j \omega^{\beta_j} \cos \frac{\pi \beta_j}{2} + i \sum_{j=1}^{M} \mu_j \omega^{\beta_j} \sin \frac{\pi \beta_j}{2} 
\]  
(5-6-8)

where \( \lambda_0 \) and \( \mu_0 \) are positive, real parameters proportional to elastic stresses, \( \lambda_i \) and \( \mu_j \) are positive, real parameters proportional to viscoelastic stresses, and \( \alpha_i \) and \( \beta_j \) are fractional orders of the derivatives.
As stated earlier, previous experimental tests indicate that most viscoelastic materials can be accurately modelled using only the first fractional derivative term in each series. Under conditions of uniaxial stress and strain Eq. (5-6-6) can be recognised as the five parameter complex moduli formulated in the previous sections as,

\[
\dot{\sigma}(i\omega) = \dot{E}(\omega)\dot{\epsilon}(i\omega), \quad \dot{E}(\omega) = \frac{e + f(i\omega)^c}{1 + a(i\omega)^b}
\]

(5-4-5)

where the five parameter complex moduli describe the constitutive relation for a material. To investigate the non-linear characteristics of the five parameter fractional derivative damping model, the Laplace transform can be used to transform the equation as,

\[
\sigma^*(s) = \frac{e + fs^c}{1 + as^b}\dot{\epsilon}^*(s)
\]

(5-6-9)

Figs. 5-3(a) and 5-3(b) show three dimensional plots in which the sensitivity of the complex modulus for different values of the fractional orders is investigated in the case of an aluminium alloy material. The advantage of the proposed model is that the fractional derivative model and its constitutive equation has the continuity property from the ideal solid state \((E>0)\) to the ideal fluid state \((E=0)\). The material behaviour changes toward the ideal solid state as the value of fractional order \(b\) increases to one and the value of fractional order \(c\) decreases to zero. On the other hand, the fluid state of materials can be modelled by increasing the fractional order \(c\) and decreasing the fractional order \(b\). The viscoelastic state of the material is an intermediate state in which both fractional orders have values between zero and one.
Figure 5-3

Variation of the Complex Damping Model with Frequency (rad/sec) and; (a) $c$
Parameter; (b) $b$ Parameter; (c) Variation of Storage Modulus with Frequency and $e$
The five parameters are determined by a least squares fit of this model to the frequency-dependent mechanical properties of the material. For instance, the parameters of the model for an aluminium alloy are: $e = 4.15 \times 10^9 \text{ N/m}^2$, $f = 1.09 \times 10^{11} \text{ N.s/m}^2$, $a = 3.50 \text{ s}^b$, $c = 0.641$ and $b = 0.631$ (Bagley and Torvik, 1983). Fig. 5-3(c) shows the variation of the real part of the modulus (storage modulus $E_{\text{Real}}$) with the parameter $e$ (which is proportional to the elastic stresses). It can be seen that the modulus behaviour is consistent and stable for a broad range of elastic stiffness in engineering materials.

### 5.6.2 Added Structural Dampers

Typically, an elastomeric structural damper at constant, uniform temperature has complex moduli that vary with the frequency of motion. At low frequency, the damper displays rubbery behaviour in which the real part of the modulus is constant while the imaginary part increases with increasing frequency. At intermediate frequencies, both the real and imaginary parts of the modulus increase with increasing frequency, which is called the transition region. At high frequencies, the glassy behaviour dominates the response of the damper in which the real part is constant while the imaginary part decreases with increasing frequency.

The spring and dashpot model, known as the Kelvin-Voigt model, is often used as a simple approximation for the modelling of dampers in structures. However, this linear damping element is not capable of modelling the non-linear damping characteristic of damping materials. In the study herein, a non-linear complex spring and dashpot damping element is developed which is capable of forming an accurate model for structural dampers. The dashpot is replaced by a non-linear spring-pot, which is formulated using the fractional derivative calculus, and is then adapted for the spectral method using the Fast Fourier Transform. Fig.
5-4 shows the conventional and proposed models where the dashpot is replaced by a diamond-like spring-pot.

![Diagram of conventional and proposed models]

Figure 5-4
(a) Spring and Dashpot Damping Element, (b) Fractional Calculus Spring and Spring-Pot Damping Element

For the conventional parallel spring and dashpot damping element, the stress can be written as,

\[ \sigma(t) = E \epsilon(t) + \eta \frac{d \epsilon(t)}{dt} \]  \hspace{1cm} (5-6-10)

where \( \eta \) is the viscosity. This equation can be written in the frequency domain as,

\[ \hat{\sigma}(\omega) = (E + i\omega \eta) \hat{\epsilon}(\omega) \]  \hspace{1cm} (5-6-11)

The relationship between forces and displacements for the massless axial spring and dashpot element can be written using matrix format as,
\[ \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{bmatrix} = \left( \frac{EA}{L} + i\omega \frac{\eta A}{L} \right) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} \]  

The fractional derivative form of Eq. (5-6-10) can be proposed as

\[ \sigma(t) = E\dot{\varepsilon}(t) + \frac{\eta^{\alpha+1}}{E^\alpha} d^{\alpha+1} \dot{\varepsilon}(t) \]  

\[ \sigma(\omega) = E\ddot{\varepsilon}(\omega) + \frac{\eta^{\alpha+1}}{E^\alpha} (i\omega)^{\alpha+1}\ddot{\varepsilon}(\omega) = (E + E_1(i\omega)^{\alpha+1})\ddot{\varepsilon}(\omega) \]

For \( \alpha = 0 \) the fractional derivative model furnishes the same stress equation as Eq. (5-6-10). Taking the Fourier transform of Eq. (5-6-13),

The flexibility and numerical compatibility of the fractional damping element with existing dynamical methods makes it an attractive model to simulate damper behaviour in structures. For instance, the concept of the fractional derivative damping model has also been used by Makris and Constantinou (1992) to model fluid damping behaviour in structures.
5.6.3 Temperature-Frequency Equivalence Method

The damping materials used in structures are known to be frequency and temperature sensitive. The frequency dependence of the storage and dissipation moduli can change with respect to temperature in these materials. Temperature is by far the most important factor, as mechanical properties can in some cases vary by as many as five orders of magnitude over a narrow temperature range (Jones, 1980).

In general, three distinct temperature regions can be observed in viscoelastic materials, namely the glassy, the transition and the rubbery regions, as was the case in the frequency spectra. In the glassy region the real part of the complex modulus of the material is high in magnitude while the imaginary part is low. In the transition region the real part varies rapidly with temperature and the imaginary part is high, and finally, in the rubbery region, the real part varies more slowly with temperature and the imaginary part is lower than in the transition region.

As the effects of frequency and temperature on damping material behaviour are to be taken into account simultaneously herein, the reduced frequency technique known as the temperature-frequency equivalence principle can be used to model linear viscoelastic materials. If a reference temperature $t_{ref}$ is assumed at the time when the structure undergoes a vibratory motion, then the storage modulus at any other temperature can be found as,

$$E_s(\omega) \bigg|_{t_{ref}} = E_s(c\omega) \bigg|_{t_{oc}}$$  \hspace{1cm} (5-6-16)

which shows that the modulus functions are repeated in the frequency spectrum with shift of their dependence on the frequency. However, the recent experimental
test by Kasai et. al. (1993), shows that the temperature-frequency shifting factor $c$ can be approximately expressed as,

$$ c = \left( \frac{T}{T_{ref}} \right)^p $$  \hspace{1cm} (5-6-17)

where $p$ is a constant. This simplified rule is proved to be applicable at least for the temperature range of 10°C to 40°C which is of primary concern for building applications. The value of $p$ can be found by knowing the storage modulus at the reference temperature and at another defined frequency and temperature. As the effect of temperature on storage modulus is much more significant than that on the dissipation modulus (Kasai et. al., 1993), the five parameter damping model presented in previous sections can be modified as,

$$ \sigma^*(s) = \frac{e + f_i s^c}{1 + a_i s^b} \varepsilon^*(s) \hspace{1cm} (5-6-18) $$

where

$$ f_i = \frac{f_{ref}}{n^c} ; \quad a_i = \frac{a_{ref}}{n^b} \hspace{1cm} (5-6-19) $$

and $f_{ref}$ and $a_{ref}$ are the values of $f$ and $a$ at a reference temperature. Note that as the thermal conductivity of damping materials is small, the transient heat conduction transferred to another part of the damper is neglected. Another assumption is that as a damper thickness of 20 mm or larger would be typical in practice, the distribution of temperature across the damping layer thickness is neglected. Hence, the damper temperature due to temperature rise (caused by the mechanical work done by the damper) can be expressed as (Kasai et. al.,1993),
\[ T(t) = t_{\text{ref}} + \frac{1}{s \rho} \int_0^t \sigma \, d\varepsilon \quad (5-6-20) \]

where \( s \) is the specific heat and \( \rho \) is the mass per unit volume.

All these formulations are valid in the case of shear dampers when the shear stress and strain are used instead of axial values and the complex axial modulus is replaced by a shear modulus.

### 5.7 SUMMARY AND CONCLUDING REMARKS

The frequency-dependent and temperature-dependent damping characteristics of structural materials can be modelled accurately using the fractional derivative model. It is shown that the proposed method can be extended to develop a non-linear damping element which can be used to model structural dampers. The approach has an advantage over the usual viscous treatment, as it can describe the non-linear damping behaviour of materials in broad frequency and temperature ranges.

The non-linear damping solution has been shown graphically, and the consistency of the model has been investigated. Using a computer program, the proposed damping model will be implemented into the spectral finite element method developed in the previous Chapter. The new method, which combines the features of the exact spectral solution and the flexibility of the fractional damping model, will then be used to derive the dynamic response of damped structural systems.
Chapter Six

Unified Fractional-Spectral Method for Dynamic Analysis of Damped Structures

6.1 GENERAL

The use of numerical methods in the frequency domain to describe the frequency-dependent damping properties of materials is seen to be efficient when it is noted that this non-linear behaviour can be treated linearly in the transform frequency domain instead of in the direct time domain. One of the advantages of the frequency domain spectral theory, which was given earlier, is that the frequency-dependent properties can be incorporated easily. In fact, to adapt the fractional derivative damping model into the complex spectral method, the changes required are that the spectrum relations should be modified.

The frequency-dependence damping characteristic of structural materials can be modelled accurately using the fractional derivative model. It is also shown that the proposed method can be extended to develop a non-linear damping element which can be used to model structural dampers. Structural dampers have been used successfully in various types of structures for the effective reduction of dynamic responses. Structural damping capacity arises from different dissipation mechanisms, which can be classified by considering a structural system as an assemblage of elements interfacing at nodes. These sources of damping may be classified by
considering the material damping arising within the element, joint friction damping arising from the interfacing of elements, the damping arising from the interaction of the structure with the external environment, or from the damping mechanism arising from added damping devices.

The purpose of this Chapter is to present the unified fractional-spectral finite element method for dynamic analysis of three-dimensional damped structures. Based on the theory of fractional calculus and the spectral theory of vibration, a new spectrally formulated finite element method of analysis is developed which is capable of making accurate predictions of the dynamic response of structures with added dampers. As the damping treatment traces the physical implication of the intermolecular theory of materials (Zimm, 1956) in the fractional derivative damping model, the proposed method has an advantage over the usual viscous treatment.

6.2 SPECTRUM RELATION

As it has been discussed in previous chapters, the relation between stress and strain or rate of strain in viscoelastic materials cannot be expressed by material constants as they can be in the case of purely elastic or purely viscous materials. The simplest conventional constitutive equation which adequately describes the infinitesimal deformation of a viscoelastic body is

\[ \sum_{n=0}^{\infty} u_n \frac{d^n \sigma(t)}{dt^n} = \sum_{m=0}^{\infty} q_m \frac{d^m \varepsilon(t)}{dt^m} \]  

(6-2-1)

where \( u_n \) and \( q_m \) are constant coefficients. The stress and strain in this form are related through multiple derivatives in time. One of the advantages of the spectral
method is that time-dependent effects can be incorporated in the formulation easily. The spectral form of Eq. (6-2-1) is

\[
\left[ \sum_{n=0}^{\infty} u_n(i\omega)^n \right] \sigma = \left[ \sum_{m=0}^{\infty} q_m(i\omega)^m \right] \varepsilon
\]

(6-2-2)
or, it can be written in simple form as,

\[
\sigma = \frac{\sum_{m=0}^{\infty} q_m(i\omega)^m}{\sum_{n=0}^{\infty} u_n(i\omega)^n} \varepsilon
\]

(6-2-3)

which is the viscoelastic form of the linear-elastic constitutive relation. Recalling Eq. (5-4-4) which is the fractional derivative form of the viscoelastic constitutive relation,

\[
\sigma(i\omega) = \frac{e_0 + f(i\omega)^{\alpha}}{1 + a(i\omega)^{b}} \varepsilon(i\omega)
\]

(5-4-4)

As the modulus is frequency-dependent instead of being a constant value as in the case of the linear-elastic relation, the governing differential equations of motion for any element have to be modified. Consider the governing differential equation of motion for a rod element (Eq. 4-4-2) as,

\[
\frac{d^2 \ddot{u}}{dx^2} + \omega^2 \lambda \ddot{u} = 0
\]

(4-4-2)
where \( \lambda = \rho/E \). The damped form of the equation can be written as

\[
\frac{d^2\tilde{u}}{dx^2} + \omega^2 \lambda(i\omega)\tilde{u} = 0
\]  
(6-2-4)

where in this case \( \lambda(i\omega) = \frac{\rho}{E(i\omega)} \). The only change that occurs in the spectral relation for the damped rod element is,

\[
\kappa_{ud} = \pm \omega (\lambda)^{\frac{1}{2}} \quad \text{damped} \rightarrow \kappa_d = \pm \omega [\lambda(i\omega)]^{\frac{1}{2}}
\]  
(6-2-5)

where subscripts \( ud \) and \( d \) represent the undamped and damped cases, respectively, which gives the following spectral relation when the fractional derivative constitutive relation is employed,

\[
\kappa_d = \pm \omega \left[ \frac{(1 + a(i\omega))^\mu}{e + f(i\omega)^\nu} \right]^{\frac{1}{2}}
\]  
(6-2-6)

and the damped element shape function can be written as,

\[
\tilde{u}(x) = \left( \frac{\sin \kappa_d (L - x)}{\sin \kappa_d L} \right) \tilde{t}_1 + \left( \frac{\sin \kappa_d x}{\sin \kappa_d L} \right) \tilde{t}_2
\]  
(6-2-7)

Thus the dynamic stiffness matrix which includes the damping effects is,
The effect of damping in the spectrum relation and the dynamic stiffness matrix is to decrease the amplitude of vibration in the element due to the dissipation of energy. Another important point here is that the dynamic stiffness matrix includes the damping effects, therefore there is no need to assemble an individual damping matrix for the structure.

\[ \left[ \hat{K}_d \right] = \begin{bmatrix} \frac{\kappa_d L}{\tan(\kappa_d L)} & -\frac{\kappa_d L}{\sin(\kappa_d L)} \\ -\frac{\kappa_d L}{\sin(\kappa_d L)} & \frac{\kappa_d L}{\tan(\kappa_d L)} \end{bmatrix} \quad (6-2-8) \]

6.3 FREQUENCY DOMAIN DYNAMIC ANALYSIS

The frequency domain strategy to solve dynamic problems is an alternative approach to the classical time domain method. The solution to any dynamical problem, in general, can proceed by using two basic methodological approaches: the time domain integration method and the frequency domain integration method. The time domain dynamic analysis is based on the satisfaction of static equilibrium with inertia and damping forces included only at a limited number of time intervals. The time domain integration method is usually thought to be superior, as the extension to non-linear problems can be dealt with by introducing iterative techniques (as in static analysis).

On the other hand, the dynamic analysis in the frequency domain has some advantages like spectral decomposition of the forcing function, which helps to set bounds on the dynamical problem. Also, the stability and efficiency of the Fast Fourier Transform, which allows the use of a larger time step compared with the direct integration time step is an advantage. Hence, it seems to be advantageous to find an alternative unifying approach which combines the features of time and frequency domain methods in an iterative numerical solver.
In general, it is well known that the solution of the forced vibration of a non-linear multi-degree-of-freedom system is very cumbersome when conditions which allow the equations to be uncoupled do not exist. Several numerical techniques have been proposed in recent decades to overcome the problem, but none of these methods has the generality to analyse the wide range of dynamical problems. Hence, this section is devoted to the presentation of a unifying procedure able to deal with linear and non-linear dynamical systems.

6.4 EIGENVALUE ANALYSIS

It is apparent that the dynamic analysis of large structural systems involves large systems of equations which describe the dynamical motion of different parts of structures. Many structures require numerous coordinates to describe their vibrational motion. For instance, consider a tall building with many degrees of freedom. The solution of its undamped free vibrational equations will lead to finding many resonant frequencies and the associated mode shapes, which is a huge numerical task even for today's computers. Hence it seems to be advantageous to develop a numerical method which requires lesser degrees of freedom for eigenvalue analysis.

6.4.1 Review of Conventional Eigenvalue analyses

It is seen that in the conventional finite element method based on structural material, the geometry and also the boundary conditions, a numerical procedure can be formulated which describes the dynamic properties of structures in terms of mass, damping and stiffness distribution. An important application of the finite element method is in determining natural frequencies and the associated mode shapes. The interest herein is in three-dimensional frame structures, so as a starting
point consider the governing system of equations for dynamic motion describing undamped free vibration of a three-dimensional structure as,

$$[K][u] + [M][\ddot{u}] = 0 \quad (6-4-1)$$

As the structure is in free vibration mode (harmonic motion),

$$u(t) = Ae^{i\omega t} \rightarrow \{[K] - \omega^2[M]\}[u] = 0 \quad (6-4-2)$$

which is a system of algebraic homogeneous equations where the non-trivial solutions exist when the determinant of the coefficient matrix is zero. Therefore,

$$\det\{[K] - \omega^2[M]\} = 0 \quad (6-4-3)$$

This is a standard eigenvalue problem in which the expanded form of the determinant equation can be written as

$$(\omega^2)^N + c_1(\omega^2)^{N-1} + c_2(\omega^2)^{N-2} + \ldots + c_N = 0 \quad (6-4-4)$$

where $N$ is the dimension of the system matrices and $c_1, c_2, \ldots$ are constants. The roots of this polynomial equation are the square of the natural frequencies. There are as many frequencies as the order of the system equations. The natural mode shapes (eigenvectors), which describe the deformed shape of the structure at the corresponding natural frequencies, are obtained by substituting the eigenvalues into the system of equations above. However, as the actual amplitude of the displacement depends on the magnitude of exciting force and also on the initial
conditions of the system, the natural mode shapes are represented in a normalised form. In the other words, the natural modes represent shapes of deformation rather than the absolute displacements of the structure. The process of scaling the elements of the mode shape vector is called normalisation, and the resulting scaled vectors are called orthonormal mode shape vectors.

\[ [\phi] = \text{normalised}[u] \]  

(6-4-5)

As the eigenvalues and the corresponding eigenvectors are solutions of the systems of equation of motion, each of these eigenvalues and eigenvectors satisfy the equations of motion as

\[ \sum_{i=1}^{N} \left\{ [K][\phi]_i - \omega_i^2 [M][\phi]_i \right\} = 0 \]  

(6-4-6)

which leads to the orthogonality property (Clough et. al., 1975) of the mode shapes with respect to the mass and stiffness matrices as,

\[ [\phi]_i^T [M][\phi]_j = 0 \]  

\[ [\phi]_i^T [K][\phi]_j = 0 \]  

(6-4-7)

where \( i \) and \( j \) refer to two different natural frequencies. If, the frequencies are identical, \( i=j \), then the two modes are the same and it can be written,

\[ [\phi]_i^T [M][\phi]_i = M_i \]  

\[ [\phi]_i^T [K][\phi]_i = K_i = \omega_i^2 M_i \]  

(6-4-8)
where \( M_i \) and \( K_i \) are non-zero constants, and they can be interpreted as the generalised modal mass and stiffness of the \( i \)th mode, respectively. These relations can also be expressed in matrix format by using the orthogonal properties of the mode shapes (Sehmi, 1989) as,

\[
\]

where the resultant matrices are called the generalised mass and stiffness matrices, respectively and they are both diagonal. The advantage of using these two matrices is that they can be used in a coordinate transformation of the equations of motion.

As the global stiffness and mass matrices for a system are not diagonal, the system of equations are coupled and this is called dynamic coupling. Hence, the system of equations should be uncoupled through the transformation of the displacements. If the displacements are transformed to new coordinates through the modal matrix as

\[
[u] = [\phi] [v] \quad (6-4-10)
\]

The forced equation of motion can also be solved by using modal matrices as (Clough et. al., 1975),

\[
[\bar{K}] [v] + [\bar{M}] [v] = [\phi]^T [F] \quad (6-4-11)
\]

where \([F]\) is the force vector. The system would then be uncoupled and each equation would be similar to a single degree of freedom system.
As all real structures have some type of damping (viscous, hysteretic, coulomb, ...) and the damping is usually small, then certain simplifying assumptions can be made in the conventional finite element analysis. Consider a damped vibration of a structure as

\[
[K][\ddot{u}] + [C][\dot{u}] + [M][u] = [F]
\] (6-4-12)

where \([C]\) is a damping matrix. As the introduction of damping couples the equations of motion, a simplifying assumption is usually introduced, which gives a simple proportional distribution of damping throughout the structure (Rayleigh damping model). This assumption can be written in matrix format as,

\[
[C] = \alpha [M] + \beta [K]
\] (6-4-13)

where \(\alpha\) and \(\beta\) are constants. This shows that the damping matrix can be a linear combination of the stiffness and mass matrices at each nodal point. With this simplifying assumption, the coordinate transformation using the generalised modal matrix will also diagonalise the damping matrix.

Another simplifying assumption is that the damping matrix is assumed to be proportional to the stiffness matrix only,

\[
[C] = i\lambda [K]
\] (6-4-14)

where \(\lambda\) is constant. In this case the transformation to generalised coordinates gives
There are several alternative methods available for the solution of eigenvalue problems. The efficiency of a particular method depends on the characteristic of the system matrices and also the type of solution desired. A more comprehensive discussion about eigenvalue analysis can be found in Sehmi (1989).

6.5.2 Exact Eigenvalue Analysis

The spectral approach is discussed comprehensively in Chapter Four where the basic spectral formulation and the proposed numerical finite element method have been given. The exact formulation used to formulate spectral elements can also be used to find the natural frequencies and mode shapes of structures. Consider a simple case of a beam with pin joints as shown in Fig. 6-1.

Recalling the spectral solution for the flexural vibration of a beam Eq.(4-5-13) as

\[ \tilde{u}_n(x) = A_n e^{-i\kappa_n x} + B_n e^{-\kappa_n x} + C_n e^{-i\kappa_n (L-x)} + D_n e^{-\kappa_n (L-x)} \]  

(4-5-13)
The boundary conditions at each frequency steps are

\[
x = 0, L \quad \rightarrow \quad \hat{u} = 0
\]
\[
x = 0, L \quad \rightarrow \quad EI \frac{d^2\hat{u}}{dx^2} = 0
\]

which gives the four conditions in sine and cosine format as,

\[
\begin{align*}
A + C &= 0 \\
A \cos \kappa L + B \sin \kappa L + C \cosh \kappa L + D \sinh \kappa L &= 0 \\
A - C &= 0 \\
-A \cos \kappa L - B \sin \kappa L + C \cosh \kappa L + D \sinh \kappa L &= 0
\end{align*}
\]

where \( \kappa \) is the wavenumber. It can be seen that \( A = C = 0 \) and also

\[
\begin{bmatrix}
\sin \kappa L & \sinh \kappa L \\
-\sin \kappa L & \sinh \kappa L
\end{bmatrix}
\begin{bmatrix}
B \\
D
\end{bmatrix} = 0
\]

The determinant of the matrix in this system of equations must be zero in order to have a non-trivial solution which is,

\[
\left| \begin{array}{cc}
\sin \kappa L & \sinh \kappa L \\
-\sin \kappa L & \sinh \kappa L
\end{array} \right| = 0 \rightarrow 2 \sin \kappa L \sinh \kappa L = 0
\]

As the hyperbolic term is zero only when \( \kappa L = 0 \), then
\[
\sin \kappa L = 0 \rightarrow \kappa L = n\pi \rightarrow \sqrt{\omega} \left[ \frac{\rho A}{EI} \right]^{\frac{1}{2}} L = n\pi \rightarrow \omega = \frac{n^2 \pi^2}{L^2} \left( \frac{EI}{\rho A} \right)^{\frac{1}{2}} \quad (6-4-20)
\]

which gives the exact solution for the natural frequencies of the beam. An interesting point here is that as the spectral solution uses an exact stiffness formulation there is no limit on the frequency resolution. The exact solution of the governing differential equation of motion herein gives an infinity of discrete resonant frequencies \((n=1,2,3,...)\).

To assess the efficiency of the spectral solution, consider a conventional model of a beam based on an approximate interpolation shape function. The displacement, stiffness and consistent mass matrices can be written as

\[
[u] = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad [K] = \frac{EI}{L^3} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L & 4L^2 \end{bmatrix}, \quad [M] = \frac{\rho AL}{420} \begin{bmatrix} 4L^2 & -3L^2 \\ -3L^2 & 4L^2 \end{bmatrix} \quad (6-4-21)
\]

and for free vibration, the equation of motion can be written as,

\[
\begin{bmatrix} EI & 4 \\ L^2 & 2 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} - \omega^2 \frac{\rho AL}{420} \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (6-4-22)
\]

The natural frequencies can be obtained by letting the determinant equal zero, which gives,

\[
\omega_1 = \frac{\sqrt{120}}{L^2} \sqrt{\frac{EI}{\rho A}} ; \quad \omega_2 = \frac{\sqrt{2520}}{L^2} \sqrt{\frac{EI}{\rho A}} \quad (6-4-23)
\]
The use of the interpolation shape function causes about 10% difference for the first natural mode ($\pi = 9.87$ while $120^{0.5} = 10.95$), and about 25% difference for the second natural mode ($4\pi = 39.48$ while $2520^{0.5} = 50.20$) compared with the exact solution (spectral solution) in Eq. (6-4-20). It should be noted that as there are two degrees of freedom for this model, only the first two natural frequencies can be extracted compared with the infinite numbers in the spectral method.

To expand the spectral solution into a numerical method suitable for implementation on computers, a similar procedure can be used. The natural frequencies can be determined by monitoring the behaviour of the system of equations. Consider the same beam (Fig. 6-1) when it is modelled using two spectral beam elements (Fig. 6-2).

Recalling the dynamic stiffness matrix for a spectral beam element (Eq. 4-5-15) in a sine and cosine format as

---

**Figure 6-2**

Beam modelled with Two Spectral Beam Elements
\[
\begin{bmatrix}
(S_C + CS_h)kL & SS_hk^2L^3 & -(S + S_h)k^3L^3 & (C_h - C)k^3L^3 \\
(S_C - CS_h)k^2L^3 & -(C_h - C)k^2L^3 & -(S - S_h)k^2L^3 & 0 \\
(SC_h + CS_h)kL & -SS_hk^2L^3 & -(SC_h - CS_h)k^2L^3 & 0 \\
Symm. & & & \\
\end{bmatrix}
\]

where \( S = \sin kL, C = \cos kL, S_h = \sinh kL, C_h = \cosh kL \) and \( \Delta = 2(l - CC_h) \). The reduced element dynamic stiffness for the first element is

\[
\begin{bmatrix}
(S_C - CS_h)k^2L^3 & -(C_h - C)k^2L^3 & -(S - S_h)k^2L^3 \\
(SC_h + CS_h)kL & -SS_hk^2L^3 & -(SC_h - CS_h)k^2L^3 \\
Symm. & & & \\
\end{bmatrix}
\]

and for the second element is

\[
\begin{bmatrix}
(SC_h + CS_h)k^3L^3 & SS_hk^2L^3 & (C_h - C)k^2L^3 \\
(SC_h - CS_h)k^2L^3 & -(S - S_h)k^2L^3 & 0 \\
Symm. & & & \\
\end{bmatrix}
\]

These two can be assembled into the global dynamic stiffness matrix, and the determinant of the system of equations can be solved to obtain the natural frequencies. This is shown in Fig. 6-3(a) where frequency is plotted against the response force frequency response function (which is discussed later in the Chapter). It should be noted that the number of natural frequencies possible to be computed in the spectral modal analysis is not equal to the number of degrees of freedom in the model. In the other words, as there is no upper limit on the frequency range for an exact formulation, so the number of natural frequencies extracted is infinite. However, this does not mean that the spectral model of a
dynamical system with zero degrees of freedom (i.e. fixed-fixed beam element modelled using only one spectral element) can be analysed. Fig. 6-3(b) shows the first and second natural frequencies with greater frequency resolution.

The ability to scan a narrower range of frequencies using a finer frequency increment is one of the fruitful advantages of the spectral modal analysis over the conventional finite element approach. The saving in computer time and power to carry out the frequency scan is substantial when it is noted that one of the time consuming subspace iteration or the Lanczos (Sehmi, 1989) methods are the best possible methods in the conventional partial eigenvalue problems. The mode shapes for the spectral model can easily be found by solving the system at the natural frequencies, and plotting the distribution between the nodal values.

However, as the dynamic stiffness is complex (real and imaginary), the resulting distributions are given in terms of real and imaginary values which make it easier to incorporate damping effects. This is a significant advantage over the conventional method as there is no need for a special treatment for the case of damped structures. Fig. 6-4 shows the first three natural mode shapes of the beam.

A finite element model of the beam is also created using 20 conventional beam elements. The result of a subspace iteration eigenvalue analysis is presented in Table 6-1. This analysis shows that even with 20 conventional elements, the results are not as accurate as the two element spectral model (exact solution). In fact, only with a large number of conventional elements (perhaps 100 or more), the finite element modal analysis matches the spectral modal analysis results.
Figure 6-3

Frequency Response Function Plotted Versus the Frequency for (a) First Six Natural Frequencies; (b) First Two Natural Frequencies with greater Frequency Resolution
Figure 6-4

(a) First; (b) Second; (c) Third Mode Shapes of the Beam

Table 6-1: Natural Frequencies [Hz] of the Beam

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>Spectral Solution</th>
<th>FE Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>222.14</td>
<td>222.09</td>
</tr>
<tr>
<td>2</td>
<td>888.58</td>
<td>887.71</td>
</tr>
<tr>
<td>3</td>
<td>1999.30</td>
<td>1994.92</td>
</tr>
<tr>
<td>4</td>
<td>3554.31</td>
<td>3540.70</td>
</tr>
<tr>
<td>5</td>
<td>5553.60</td>
<td>5520.86</td>
</tr>
<tr>
<td>Computer Time [s]</td>
<td>32</td>
<td>58</td>
</tr>
</tbody>
</table>
Example:

To extend the spectral modal analysis to cases of rigid-jointed frame structures, consider a simple portal frame as shown in Fig. 6-5.

![Portal Frame Structure](image)

In this case, the spectral model consists of 3 spectral frame elements which extend from one joint to another. The global dynamic stiffness matrix for the model can be assembled using the local dynamic stiffness of elements as shown in Chapter Four. All degrees of freedom at the supports are fixed, and the reduced determinant of the system of equations is solved to find the peaks. Figs. 6-6 and 6-7 show the results of analyses for a different range of frequencies.

The fourth natural mode in Fig. 6-6(b) which can hardly be observed, is shown in Fig. 6-7(a) using a finer frequency increment. Again, a finite element modal analysis was also performed using 150 conventional frame elements. A comparison between the results of spectral and conventional modal analysis is presented in Table 6-2, and the first three mode shapes are shown in Fig. 6-8.
Figure 6-6

(a) First and Second; (b) Third and Fourth Natural Frequencies (Peaks)
Figure 6-7

(a) Fourth; (b) Fifth Natural Frequencies (Peaks)
Figure 6-8

(a) First; (b) Second; (c) Third Mode Shapes of the Frame Structure
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### Table 6-2: Natural Frequencies [Hz] of the Frame

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>Spectral Solution</th>
<th>FE Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>288.82</td>
<td>288.31</td>
</tr>
<tr>
<td>2</td>
<td>1133.77</td>
<td>1132.60</td>
</tr>
<tr>
<td>3</td>
<td>1856.53</td>
<td>1853.35</td>
</tr>
<tr>
<td>4</td>
<td>1996.49</td>
<td>1994.22</td>
</tr>
<tr>
<td>5</td>
<td>3988.86</td>
<td>3973.93</td>
</tr>
<tr>
<td><strong>Computer Time [s]</strong></td>
<td><strong>58</strong></td>
<td><strong>134</strong></td>
</tr>
</tbody>
</table>

### 6.4.3 Damped Fractional-Spectral Eigenvalue Analysis

The effect of damping and viscoelasticity can easily be incorporated in the spectral method simply by substituting a complex module in the spectrum relation as discussed in previous sections. Consider the portal frame structure in the last section when the damping property of the members is modelled using a five parameter fractional damping model. Recalling the fractional derivative form of the viscoelastic constitutive relation as,

\[
\sigma(i\omega) = \frac{e_0 + f(i\omega)^c}{1 + a(i\omega)_b} \hat{\varepsilon}(i\omega) \rightarrow \hat{\sigma} = \hat{E}\hat{\varepsilon}
\]  

(5-4-4)

where the damping properties of the aluminum alloy material as given in Table 6-3, are used for calculating the complex moduli.

### Table 6-3: Damping Data for the Frame Structure

| $e_0$ [GPa] | 70.02 |
| $f$ [N.$s^{0.64}$/mm²] | 18.0 e+5 |
| $a$ [$s^{0.631}$] | 3.5 |
| $b$ | 0.631 |
| $c$ | 0.641 |
The result of a damped eigenvalue analysis for the first two natural modes is presented in Fig. 6-9 where it is compared with the undamped eigenvalue analysis. The peaks are shifted to the left, and the natural frequencies occur at lower frequencies compared with the undamped modes. It should be noted that as the spectral dynamic stiffness is complex, the resulting mode shapes are calculated in terms of real and imaginary values, and the introduction of damping will not complicate the analysis. In fact, both the real and imaginary components are treated equally in the spectral method, and this is a significant advantage over the conventional method.

Figure 6-9
First and Second Natural Frequencies for Damped and Undamped Frame Structure
6.5 SPECTRAL FORCED RESPONSE METHOD

In this section the response of a system to a variety of different dynamic forces is considered, as well as an exact matrix formulation for calculating the forced response for any applied forces, accelerations or displacements. A variety of dynamic forces are applied to systems that result in vibrational effects. For example, ocean waves provide a dynamic force to ships and offshore structures; earthquake forces can be a source for applying accelerations to structural foundations; strong winds can apply an impulsive or step loading to structures. The analysis presented in this section is useful in predicting the response of structures undergoing vibration from a variety of different loadings. Understanding the dynamic response of such structures can lead to improve design procedures and better performance.

6.5.1 Review of the Conventional Time Domain Method

A very common method of forced response analysis is the conventional time domain method. The conventional method starts with calculating the response of systems under an impulsive load and then extending it to an arbitrary general dynamic load of varying magnitude based on the concept of the impulse response. The procedure is to split the input dynamic force up into small impulses, calculate the responses to these impulses and then add these responses to calculate the total response. Consider the equation of motion for a system as

\[ ku + c\dot{u} + m\ddot{u} = F(t) \]  \hspace{1cm} (6-5-1)

The general solution of the homogenous form, \( F(t)=0 \), of this equation can be found (Clough et al., 1975) as

\[ u(t) = e^{-\alpha t}(c_1 e^{-i\omega_0 t} + c_2 e^{+i\omega_0 t}) \]  \hspace{1cm} (6-5-2)
where \( \omega_u = \sqrt{\frac{k}{m}} \) is the undamped natural frequency, \( \omega_d = \omega_u \sqrt{1 - \zeta^2} \) and \( \zeta = c/(2m\omega_0) \). The response of the system to an impulse is identical to the free response of the system subject to certain initial conditions. Fig. 6-10 shows a triangular force history where the force is plotted against time.

![Figure 6-10](image)

**Figure 6-10**

**Time History of an Arbitrary Dynamic Load**

The impulse of the force \( F(t) \) is defined by the integral

\[
F_\epsilon(\epsilon) = \int_{\tau-\epsilon}^{\tau+\epsilon} F(t) dt
\]

(6-5-3)

where \( \epsilon \) is a small fraction of time. The response to this impulse is

\[
u(t) = \frac{F_\epsilon \Delta \tau}{\omega_d} e^{-\zeta \omega_d (t-\tau)} \sin \omega_d (t - \tau)
\]

(6-5-4)
where \( F \Delta t \) is the impulse and term \((t-\tau)\) takes into account the fact that the pulse occurs at time \( \tau \) and not time zero. The total response of the system is obtained when \( \Delta t \to 0 \), and by adding the impulse responses as,

\[
\begin{align*}
   u(t) &= \frac{1}{\omega_d} e^{-\zeta \omega_d t} \left[ c_1(t) \sin \omega_d t - c_2(t) \cos \omega_d t \right] \\
   \text{where,} \\
   c_1(t) &= \zeta \omega_d u_0 + \dot{u}_0 + \frac{1}{\omega_d m} \int_0^t F(\tau) e^{\zeta \omega_d \tau} \cos \omega_d \tau \, d\tau \\
   c_2(t) &= u_0 - \frac{1}{\omega_d m} \int_0^t F(\tau) e^{\zeta \omega_d \tau} \sin \omega_d \tau \, d\tau 
\end{align*}
\]

However, as far as the numerical solution is concerned, direct integration is suitable only in the case of small damping (since in the case when a large damping capacity exists, the exponential term inside the integral becomes large and produces numerical problems). In the case of damped systems it is more convenient to use other integration schemes, such as finite differences or the Newmark method. A more comprehensive discussion about these methods can be found in Clough et. al., (1975).

### 6.5.2 Fractional-Spectral Approach

The frequency domain strategy to find the forced response of a dynamic system is an alternative approach to the conventional time domain method. As the fractional-spectral method works in the frequency domain, it seems necessary to develop an efficient and stable method capable of finding the system responses in the frequency domain. These responses can then be transformed to the time domain to
obtain the time history response. The operation count for the frequency domain analysis is similar to the classical modal superposition procedure in the time domain analysis. However, it has some advantages; firstly, spectral decomposition of the forcing function can help to set bounds on the dynamical problem. Secondly, the stability of the Fast Fourier Transform allows the use of a larger time step compared with direct integration in the time domain. Finally, the frequency-dependent properties like damping can be taken into account without the need of non-linear analysis. As a first step the Fourier series approximation of the load history must be found.

Consider a general time history dynamic load in the spectral decomposition format as,

$$ F(t) = \sum_{n=0}^{N-1} \hat{F}_n e^{i\omega_n t} $$

(6-5-7)

where \( \hat{F}_n \) is the unknown force amplitude spectrum and \( N \) is the number of frequency components. Fig. 6-11 shows the one, three and ten term Fourier series approximations of the load history (Fig. 6-10) using the Fast Fourier Transform algorithm and the mathematical capabilities of Matlab software. It can be seen that the accuracy of the frequency approximation increases as the number of terms in the series increase.

Similarly, the forced response of the system can also be decomposed as,

$$ u(t) = \sum_{n=0}^{N-1} \hat{u}_n e^{i\omega_n t} $$

(6-5-8)
Figure 6-11

Fourier Series Approximations of the Load History Using (a) One; (b) Three and (c) Ten Terms
where $\hat{u}_n$ is the unknown displacement amplitude spectrum. The governing equation of motion for a dynamic system can be written as,

$$
\sum_{n=0}^{N-1} \hat{k}_n \hat{u}_n e^{i\omega_n t} = \sum_{n=0}^{N-1} \hat{F}_n e^{i\omega_n t}
$$

(6-5-9)

where $\hat{k}$ is the dynamic stiffness term. As this equation should be true at any frequency and time, after removing the summation the equation can be written as,

$$
\hat{k}\hat{u} = \hat{F}
$$

(6-5-10)

which has a simple solution as,

$$
\hat{u} = \frac{\hat{F}}{\hat{k}} = H(\omega)\hat{F}
$$

(6-5-11)

where $H(\omega)$ is the frequency response function.

The common approach in the experimental modal analysis for measuring the frequency response function is to average several matched sets of input force time histories and output response time histories. These averages are used to produce correlation functions which are transformed to yield the corresponding power spectral densities. The magnitude of the frequency response function ($|H(\omega)|$) can then be obtained using the input force and response power spectral densities. However, because the spectral formulation is planted in the frequency domain, a number of features come naturally to it. One of the important features is the ability to find the frequency response function directly.
The time history of the displacement function for the whole system can now be written as,

\[ u(t) = \sum H(\omega)\hat{F}e^{i\omega t} = \sum \frac{\hat{F}e^{i\omega t}}{k} \]  

(6-5-12)

which shows that the frequency domain responses can be obtained (in complex form) and then transformed to the time history responses using FFT. To elaborate this, a three-dimensional five-story structure (Fig. 6-12) is analysed using the fractional-spectral method. Fig. 6-13 shows a time history of a dynamic load in which a triangular impact load of duration one second is applied to the structure.

To start the analysis, the time history force is transformed to the frequency domain using the complex FFT. A narrow frequency range of the transformed load function is shown in Fig. 6-14 where the function is presented in terms of real and imaginary values. The spectral solution can now proceed and the global dynamic stiffness matrix can be assembled using the element dynamic stiffness matrices. The frequency response function of the system can be obtained, and the system can be solved for the generalised displacement at all the nodes. The response at particular nodes or locations between the nodes is obtained by the transformation of appropriate nodal displacements to the time domain. Fig. 6-15 shows the time history response for the top of the structure.

As the fractional-spectral method is capable of finding the system transfer function directly, the eigenvalue analysis can also be performed in the forced frequency response method. Because of fewer global degrees of freedom in the spectral model compared with the conventional model, the frequency response function of the whole system can be used to find the resonant peaks. In fact, it is now meaningful to find natural frequencies from the frequency response function plot in section 6.4.2.
Chapter Six: Unified Fractional-Spectral Method for Dynamic Analysis of Damped Structures

\[ E = 2.0 \times 10^9 \, [\text{Pa}] \]
\[ \rho = 3200 \, [\text{kg/m}^3] \]
\[ A_c = 0.2 \, [\text{m}^2] \]
\[ A_b = 0.15 \, [\text{m}^2] \]
\[ (I_{zz})_c = 4.54 \times 10^{-3} \, [\text{m}^4] \]
\[ (I_{zz})_b = 1.92 \times 10^{-3} \, [\text{m}^4] \]
\[ f = 4.2 \times 10^9 \, [\text{N.(s)}^{0.64}/\text{m}^2] \]
\[ a = 3.5 \, (s)^{0.63} \]
\[ b = 0.63 \]
\[ c = 0.64 \]

Figure 6-12
Three Dimensional Five Story Structure

Figure 6-13
Time History Load Function Applied at the Top of the Structure
Figure 6-14

Transformed Load Function in Terms of Real and Imaginary Values

Figure 6-15

Response of the Top of the Structure in the Horizontal Direction
Figure 6-16
(a) Second and (b) Fourth Mode Shapes

Figure 6-17
Experimental Test Set up for the Cantilevered Beam
Fig. 6-16 shows the second and fourth natural mode shapes of the structure. It should also be noted that as the frequency response function of the dynamical system is obtained directly, only the location of the force function needs to be specified at assembly time, and the same frequency response function can be used for different force input histories.

Another advantage of the fractional-spectral method is that experimental data can be used to determine the inverse solution. In other words, the force history can be reconstructed from accelerometer and strain gage data. Fig 6-17 shows a dynamic analysis test set up for a cantilever beam. An experimental transfer function can be measured by applying a force at point 1 and measuring the response at point 1 (driving point frequency response). If the initial measurement suggests that the beam has three natural frequencies in the specified range of frequency, the system can be modelled by a three degree of freedom system. However, in order to establish enough data to determine the mode shapes from the experimental data, the beam must be measured at two other points (say points 2 and 3). Hence two additional accelerometers and a multi-channel frequency analyser can be used to obtain the required three transfer functions.

To elaborate the ability of the fractional-spectral method in determining the inverse solution, consider the equation of motion as

\[ \ddot{u} = \hat{H}\hat{F} \]  

(6-5-13)

in which the system transfer function \( \hat{H} \) is obtained analytically using the spectral solution for the cantilever beam. The response at locations 1, 2 and 3 can be found as,
\[ \hat{u}_1 = \hat{H}_1 \hat{F} \]
\[ \hat{u}_2 = \hat{H}_2 \hat{F} \]
\[ \hat{u}_3 = \hat{H}_3 \hat{F} \]

(6-5-14)

However, as the driving force is identical for the three displacement functions, the response at one location and is related to the response at another as,

\[ \hat{u}_2 = \frac{\hat{H}_2}{\hat{H}_1} \hat{u}_1 = \frac{\hat{H}_2}{\hat{H}_3} \hat{u}_3 \]

(6-5-15)

and the force history can be obtained from the response as

\[ \hat{F} = \frac{\hat{u}_1}{\hat{H}_1} = \frac{\hat{u}_2}{\hat{H}_2} = \frac{\hat{u}_3}{\hat{H}_3} \]

(6-5-16)

which allows the active use of experimental data. By having strain gage data for one location, the response at another location, the exciting force can be found using the analytical frequency response function.

### 6.6 NON-LINEAR FREQUENCY DOMAIN ANALYSIS

The evaluation of the dynamic response of a damped structural system requires the solution of a coupled system of second-order differential equations. A variety of methods are available for evaluating the response of non-linear dynamic systems. One of the most popular methods for approximating the frequency
response of non-linear systems is known as the Harmonic Balance (HB) method (Aprile, 1994). However, this method is limited to harmonically excited systems and it becomes increasingly inaccurate as higher harmonics become significant in the expansion of the non-linear force. The non-linear spectral method proposed in this section uses the Alternating Frequency/Time (AFT) method proposed by Cameron and Griffin (1989), and later extended to the G-AFT (General Alternating Frequency/Time) method proposed by Aprile, (1994).

Consider a governing equation of motion for a non-linear damped system as,

\[ m\ddot{u}(t) + c\dot{u}(t) + ku(t) + Q(t) = F(t) \]  

(6-6-1)

The non-linear contribution can be of the form of mass, stiffness or damping non-linearity. The AFT method is based on the idea that all non-linear contributions can be moved to the right hand side of the equation (pseudo-force term as discussed by Udwadia et. al., 1994), leaving on the left hand side the usual dynamical system equation. If the non-linear contributions are put to zero, the first solution can be found, and the pseudo-force may be evaluated as a function of response and its derivatives. The analysis can then proceed by recomputing alternately the cited pseudo-force contribution and the system response, until the solution converges.

A very important application of non-linear dynamics arises from base isolation analysis where the superstructure is assumed to remain elastic during seismic excitation while the isolation system undergoes a non-linear motion. In other words, base isolation is a seismic structural design strategy in which a structure is largely uncoupled from the damaging effects of a severe earthquake by a mechanism that filters the transmission of horizontal and vertical accelerations into the structure. The analysis of base-isolated structures subjected to seismic excitation is an area of great interest to structural dynamicists and engineers. A variety of
analysis and design methods are presently available for modelling the response of non-linear damping devices in the base, as well as the structure itself. To be more specific, consider the Park equation (Park et al., 1986) for a dynamic system with a biaxial non-linear hysteretic restoring force as

\[
\begin{bmatrix} m \\ c \end{bmatrix} \begin{bmatrix} \ddot{u}_x \\ \ddot{u}_y \end{bmatrix} + \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}
\] (6-6-2)

which can be used to model the damping behaviour of rubber-like materials in base-isolated structures. The non-linear restore force can be expressed as

\[
\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \alpha [k] \begin{bmatrix} u_x \\ u_y \end{bmatrix} + (1-\alpha) [k] \begin{bmatrix} Z_x \\ Z_y \end{bmatrix} = G(t) + Z(t)
\] (6-6-3)

where \( Z_x \) and \( Z_y \) satisfy two coupled Bouc-Wen differential equations (Park, 1986). If the non-linear hysteresis component, which is a function of the time history, is transferred to the right hand side of the equation of motion, Eq. (6-6-3) becomes,

\[
m\ddot{u}(t) + c\dot{u}(t) + G(t) = F(t) + Z(t)
\] (6-6-4)

which can be solved using the prescribed pseudo-force method. The pseudo-force method has been used for non-linear dynamic analysis (Stricklin, 1971, Wolf, 1988) and shows great advantages over the popular Newton-Raphson method. The method converges even in the case of severe non-linearities while the Newton-Raphson fails to converge in these cases. It also gives results of comparable accuracy with the predictor-corrector method (Gear, 1971) in a lesser number of iterations than the Newton-Raphson method.
To enable the frequency domain fractional-spectral method to solve non-linear dynamic problems, the formulations must be modified to allow the use of the AFT method. Consider the non-linear spectral form of the equation of motion as,

\[ \{\ddot{\hat{u}}_n\} + \eta(\omega, \hat{u}_n) = \{\ddot{\hat{u}}_n\} \]

(6-6-5)

where \( \eta(\omega, \hat{u}_n) \) represents the non-linear force terms. Various iterative methods are currently available for solving this non-linear algebraic equation. However, in this form, the non-linear forces depend only on \( \omega \) and displacement in transformed coordinates. One of the problems facing any method is evaluation of the non-linear term. To add these terms to \( F \) they must be known explicitly in terms of frequency, but this form is difficult to determine in the frequency domain. In cases where Coulomb friction damping for sliding isolation systems occurs, and for history-dependent hysteretic damping, the restoring force is not an explicit function of frequency. However, the inverse transform can be performed to find \( \eta(t) \) at each frequency step, and transformed back to \( \eta(\omega) \) using the FFT method.

Using the pseudo-force expedient, a non-linear iterative procedure can be performed in the alternative time/frequency domain until the solution converges. In general, the whole idea behind the proposed non-linear spectral method is that all non-linear contributions can be moved to one side of the equation (pseudo-force term) leaving the usual dynamical system equation on the other side. Then, the pseudo-force may be evaluated as a function of response and its derivatives. The computational procedure therefore requires the alternate re-computation of the cited pseudo-force contribution in the time domain and the system response in the frequency domain, until the solution converges.
To elaborate this procedure consider a two story base isolated structure where frictional sliding dampers are used to isolate the structure from the ground motion. The first natural frequency of low-to-medium rise structures often falls in the range of strong excitation frequencies induced by earthquakes (Skinner, et. al., 1993). As a result the ground vibration excites the structure in a resonant state.

A recent improved solution for design of these structures uses base isolation, where part of the vibratory energy is dissipated using external damping devices. Fig. 6-18(a) shows the base isolated structure where the friction damper at the base is modelled using an elastic-perfectly plastic mathematical model (Fig. 6-18b). An ideal model of the friction damper is shown in Fig. 6-18c, where the elastic, viscous and frictional properties of the damper are modelled using an elastic spring, a viscous dashpot and an elastic sliding element.

A spectral model of a base-isolated structure can be created using frame elements. However, to make an accurate model for the localised Coulomb or hysteretic damping devices in the basement, the non-linear equation of motion has to be solved iteratively as discussed earlier. Figs. 6-19 and 6-20 show an artificial accelerogram applied at the base of the structure, and the response of the structure, respectively.

The responses are for the Gear's Predictor-Corrector method and Alternative Frequency/Time domain methods where the displacement of the top floor in the horizontal direction is plotted against time.
Data Expressed in SI System (N,kg,m)

\[ [K] = 10^6 \times \begin{bmatrix} 5.35 & -4.5 \\ -4.5 & 4.5 \end{bmatrix} \]

\[ [M] = 10^3 \times \begin{bmatrix} 6.2 & 0 \\ 0 & 6.2 \end{bmatrix} \]

\[ [C] = 10^3 \times \begin{bmatrix} 7.4 & -4.8 \\ -4.8 & 6.7 \end{bmatrix} \]

\( K_2 \)
\( m_2 \)
\( m_1 \)
\( u(t) \)

Figure 6-18

(a) Mathematical model for a base-isolated structure; (b) Sample output for Elastic-Perfectly-Plastic Friction Damper; (c) Friction Damper System (0<e<1)
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Figure 6-19
Artificial Accelerogram

Figure 6-20
Response of the Second Floor in the Horizontal Direction
6.7 SUMMARY AND CONCLUDING REMARKS

An unifying proposal which combines the features of the frequency domain complex spectral method of analysis and non-linear properties of fractional derivatives damping models, is presented in this Chapter. The numerical aspects of the fractional-spectral finite element method of analysis has been discussed, and the applicability of the method in the case of modal and transient dynamic analyses is investigated. It can be concluded that the frequency domain equations of motion for damped three dimensional structures can be constructed and solved in a straightforward manner when the mechanical characteristics of the structural materials and dampers are portrayed using the fractional derivative model.

As it appears from the presented analytical method, the principal advantage of the fractional-spectral method lies in its capability to model a long length of uniform section as one element. Furthermore, the proposed method is capable of dealing with structural dampers, with frequency-dependent damping behaviour, in a linear manner without the need of iteration. Several examples have been solved throughout the Chapter which demonstrate the capabilities of the proposed method.
Chapter Seven

Application to Damped Structural Systems

7.1 GENERAL

This Chapter presents the fractional-spectral solution to some engineering dynamical problems. Over the years, many analytical techniques have been developed for treating damped vibrating systems. While it is possible to solve large structural dynamics problems by starting with direct integration in the time domain, the task is large even for today's computers. By using the frequency domain fractional-spectral finite element method, not only can the solution procedure take advantage of many of the assembly techniques already developed for use in the time domain analysis, but the natural features of the exact formulation in the frequency domain add considerable efficiency to the method.

For linear dynamical systems the principle of superposition holds, and their solution treatments are well established. However, in the case of non-linear systems, the solution techniques are less well known, and are cumbersome to some degree. In fact, the physical explanation of the phenomena of vibration concerns the interchange between potential and kinetic energies. There are two general classes of vibration analysis; free and forced vibration. Free vibration occurs when a system is vibrated under the action of forces inherent in the system itself. In
contrast, the forced vibration occurs when the excitation of external dynamic forces vibrate the system. Vibrating systems are also subjected to damping to some degree because part of the energy is dissipated through the inherent damping mechanisms of the system or through the interface of the system with the environment. In what follows, the application of the proposed fractional-spectral method to linear and non-linear dynamical problems is investigated.

7.2 DAMPED STRUCTURES I - MATERIAL DAMPING

Material damping is generally a complex function of frequency, temperature, amplitude and structural geometry. In real engineering materials the strain is not a function of stress alone and time is required for the material to reach dynamic equilibrium in response to a change in the applied forces (relaxation). However, popular treatment of material damping in the conventional dynamic analysis methods is not physically based and is unable to model this fundamental behaviour.

In attempts to trace the physical implications of the material damping, Bagley and Torvik (1979), show that the fractional derivative of order 1/2 does arise naturally in the shear stress-strain constitutive relation of polymer solids under certain conditions. A similar related molecular theory taking into account the intermolecular hydrodynamic forces was developed by Zimm (1956), in which the order of the fractional derivative model was found as 2/3. These theoretical findings provide a link between microscopic theories of ideal engineering media and macroscopic behaviour of real materials. The fractional derivative damping model was presented in Chapter Five, and will be used to model the material damping in engineering structures.
7.2.1 Simple Damped Truss Structure

Fig 7-1 Shows a 10-bay truss structure which is a part of a satellite antenna tower. The model is built from three different types of elements: longerons, diagonals and battens. The structural members function as axial members, and all joints and other non-structural masses are neglected. The structural members are made from three different materials as shown in Table 7-1.

+----------------+-----------------+------------------+
<table>
<thead>
<tr>
<th>Battens</th>
<th>Longerons</th>
<th>Diagonals</th>
</tr>
</thead>
<tbody>
<tr>
<td>2m</td>
<td>30m</td>
<td>2m</td>
</tr>
</tbody>
</table>

Figure 7.1
10-Bay Space Truss Structure

Finite Element Model: A finite element model can be generated using the ANSYS finite element program. The model consists of 204 conventional rod elements. Two different modal analyses have been performed, undamped and damped. The eigenvalues for the undamped model have been found using the highly accurate subspace iteration technique, which internally uses the generalised Jacobi iteration algorithm. For the damped model, the complex modal analysis has been performed using the Lanczos algorithm. This complex analysis leads to the complex eigenvalues from which the imaginary part represents the damped natural frequency and, the real part is a measure of the stability of the system. A damping constant, $\beta=0.005$, is used to define Rayleigh damping, and the dominant frequency of $f=150$ Hz has been chosen to prevent overdamping (natural
frequencies higher than the dominant frequency will be damped more, and lower natural frequencies will be damped less in the Rayleigh damping approach).

ATF Method: The ATF method (Augmented Thermodynamic Fields) was presented in recent years by Lesieutre, (1989). The resulting damped ATF finite element method seems to be appropriate for modelling the behaviour of rods and structures built from rods. For the sake of completeness, a brief review of this method is presented herein.

As the ATF method is motivated by thermoelastic damping in materials and also by internal material state variables, an augmented thermodynamic field is introduced into the method to interact with the usual mechanical displacement field of continuum elasticity.

Consider a single axial element in Fig. 7-2 in which the two dependent fields, \( u \) and \( p \) are displacement and temperature fields respectively.

![Figure 7-2](image)

Typical Axial Element
The derivation of system matrices can be started by assuming the two shape functions as,

\[
\begin{align*}
    u(x) &= \left[ \left(1 - \frac{x}{L} \right) \frac{x}{L} \right] u(0) \\
    T(x) &= \left[ \left(1 - \frac{x}{L} \right) \frac{x}{L} \right] \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}
\end{align*}
\] (7-2-1)

where \( p_1 \) and \( p_2 \) are the temperatures at the nodes. This gives the coupled equations of motion as (Lesieutre, 1989)

\[
\begin{align*}
    M \ddot{\mathbf{D}} + K \mathbf{D} &= -Bp \\
    C\dot{\mathbf{p}} + H\mathbf{p} &= -G\mathbf{D}
\end{align*}
\] (7-2-2)

where \( B \) and \( G \) are the coupling submatrices, \( H \) is the thermal conductivity submatrix, \( M \) and \( K \) are the usual mass and stiffness matrices, \( C \) is the heat capacity submatrix, and \( D = \begin{bmatrix} u(0) \\ u(L) \end{bmatrix} \). The full equation of motion can now be written in matrix format as

\[
\begin{bmatrix}
    M & 0 & 0 \\
    0 & I & 0 \\
    0 & 0 & C
\end{bmatrix}
\begin{bmatrix}
    \dot{\mathbf{D}} & \dot{\mathbf{D}} & \dot{\mathbf{p}}
\end{bmatrix}
\begin{bmatrix}
    0 & K & B \\
    -I & 0 & 0 \\
    G & 0 & H
\end{bmatrix}
\begin{bmatrix}
    0 \\
    0 \\
    \mathbf{p}
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix}
\] \[\rightarrow [C_1] \dot{x} + [C_2] x = 0 \] (7-2-3)

The matrix equations of motion are formulated and can be used to carry out a modal analysis. A more comprehensive discussion about the ATF method can be found in Lesieutre (1989). The ATF damped truss model uses the same geometry and material properties as the finite element model. However, it has three different
peak damping ratios at three different frequencies (as chosen by Lesieutre, 1989) for three types of structural elements as shown in Table 7-2.

*Proposed Fractional-Spectral (FS) Model*: The fractional-spectral model consists of 41 rod elements in which only one element is needed to model each of the truss members. The modal analysis has been performed using the exact method explained in Chapter Six and using the same geometry and material data as before. However, three different sets of parameters \((a, b, c, f)\) have been used to calculate the viscoelastic modulus \(\tilde{E}(\omega)\) for the three different truss members, as shown in Table 7-3. The complex modulus used to model the damping herein can be written as,

\[
\tilde{\sigma}(i\omega) = \frac{\sigma_0 + f(i\omega)^2}{1 + a(i\omega)^b} \tilde{\epsilon}(i\omega) \rightarrow \tilde{\sigma}(i\omega) = \tilde{E}(\omega)\tilde{\epsilon}(i\omega) \tag{7-2-4}
\]

Comparison of results in Table 7-4, shows the natural frequencies for four different models: undamped and damped finite element, ATF, and the proposed fractional-spectral models. The fourth, sixth and tenth undamped natural mode shapes of the 10-Bay space truss are shown in Fig. 7-3. Fig. 7-4 shows the normalised frequency (the ratio of damped over undamped frequency) plotted against natural mode number for the FE, ATF and FS methods.

It must be noted that as the damping property is matched at single frequency in the Rayleigh damping method, some of the natural frequencies closer to that frequency (dominant frequency) are unrealistically damped more when compared with other natural frequencies (see eighth, ninth and tenth FE natural frequencies in Fig. 7-4).
Table 7-1: Truss Structure Data

<table>
<thead>
<tr>
<th></th>
<th>Longeron</th>
<th>Diagonal</th>
<th>Batten</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area, $A \text{ [m}^2 \times 10^5\text{]}$</td>
<td>31.</td>
<td>19.</td>
<td>6.3</td>
</tr>
<tr>
<td>Modulus, $E \text{ [Pa} \times 10^{10}\text{]}$</td>
<td>36.72</td>
<td>18.72</td>
<td>8.4</td>
</tr>
<tr>
<td>Density, $\rho \text{ [kg/m}^3\text{]}$</td>
<td>2200</td>
<td>1600</td>
<td>2700</td>
</tr>
</tbody>
</table>

Table 7-2: ATF Model Data

<table>
<thead>
<tr>
<th></th>
<th>Longeron</th>
<th>Diagonal</th>
<th>Batten</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peak Frequency, $\omega(\zeta)$</td>
<td>200</td>
<td>2000</td>
<td>8000</td>
</tr>
<tr>
<td>Peak damping, $\zeta$</td>
<td>0.005</td>
<td>0.01</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 7-3: Fractional-Spectral Model Data

<table>
<thead>
<tr>
<th></th>
<th>Longeron</th>
<th>Diagonal</th>
<th>Batten</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \text{ (FS Model) [s}^b\text{]}$</td>
<td>3.1</td>
<td>3.2</td>
<td>3.4</td>
</tr>
<tr>
<td>$b \text{ (FS Model)}$</td>
<td>0.52</td>
<td>0.57</td>
<td>0.6</td>
</tr>
<tr>
<td>$c \text{ (FS Model)}$</td>
<td>0.53</td>
<td>0.57</td>
<td>0.61</td>
</tr>
<tr>
<td>$f \text{ (FS Model) [N.s}^5\text{m}^2\times 10^{10}\text{]}$</td>
<td>8.5</td>
<td>8.5</td>
<td>8.5</td>
</tr>
</tbody>
</table>

Table 7-4: Natural Frequencies [Hz]

<table>
<thead>
<tr>
<th>Mode no.</th>
<th>Undamped FE</th>
<th>Damped FE</th>
<th>Damped ATF</th>
<th>Damped ATF</th>
<th>Damped FS.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>36.871</td>
<td>36.809</td>
<td>36.828</td>
<td>36.814</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>76.717</td>
<td>76.158</td>
<td>76.522</td>
<td>76.354</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>115.070</td>
<td>113.18</td>
<td>114.48</td>
<td>113.867</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>147.312</td>
<td>143.32</td>
<td>146.025</td>
<td>145.045</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>172.414</td>
<td>165.97</td>
<td>170.137</td>
<td>168.794</td>
<td>-</td>
</tr>
<tr>
<td>9</td>
<td>179.881</td>
<td>172.67</td>
<td>179.368</td>
<td>176.382</td>
<td>-</td>
</tr>
<tr>
<td>10</td>
<td>189.110</td>
<td>181.96</td>
<td>185.893</td>
<td>184.65</td>
<td>-</td>
</tr>
</tbody>
</table>

CPU Time [s] 112 486 - 287
Chapter Seven: Application to Damped Structural Systems

Figure 7-3
(a) Fourth; (b) Sixth; (c) Tenth Mode Shapes of the Truss Structure

Figure 7-4
Normalised Frequency Against Natural Mode Number
Another important point is that as the ATF method uses a single augmented thermodynamic field to describe damping properties, it underdamps some of the natural frequencies (see eighth and ninth ATF natural frequencies in Fig. 7-4). However, as the basic formulation of the proposed fractional-spectral method consistently describes two high-order curves for the frequency-dependent storage (elastic) and dissipation (viscous) modulus simultaneously, it is capable of monitoring the response with more accuracy and the unrealistic prediction of the damped natural frequencies in the FE and ATF method does not exist in the proposed model.

7.2.2 Three-Dimensional Damped Tower Structure

Fig. 7-5 shows a three-dimensional space-truss tower structure. All joint and other non-structural masses are again neglected, and the structural data is given in Table 7.5. A finite element model can again be generated using the ANSYS finite element program. Two different modal analyses have been performed, undamped and damped. The eigenvalues for the undamped model have been found using the subspace iteration technique. For the damped model, the complex modal analysis has been performed using the Lanczos algorithm. The damping constant, $\beta=0.009$, is used to define Rayleigh damping, and the dominant frequency of $f=12$ Hz has been chosen to prevent overdamping (again, natural frequencies higher than the dominant frequency will be damped more, and lower natural frequencies will be damped less in the Rayleigh damping approach).

The fractional-spectral model consists of three-dimensional spectral rod elements in which only one element is again needed to model each of the truss members. The exact modal analysis has been performed using the same geometry and material data as before.
Table 7-5: Truss Structure Data

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus, E [ Pa * 10⁹ ]</td>
<td>4.15</td>
</tr>
<tr>
<td>Density, ρ [ kg/m³ ]</td>
<td>2700</td>
</tr>
</tbody>
</table>

Table 7-6: Fractional-Spectral Model Data

<table>
<thead>
<tr>
<th>a (FS Model) [ sᵇ ]</th>
<th>3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>b (FS Model)</td>
<td>0.58</td>
</tr>
<tr>
<td>c (FS Model)</td>
<td>0.55</td>
</tr>
<tr>
<td>f (FS Model) [ (N.s/m²) * 10¹¹ ]</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Figure 7-5

Three-Dimensional Truss Structure
Figure 7-6
(a) Third; (b) Fifth Natural Modes

Figure 7-7
Normalised Frequency Against Natural Modes
For the damping parameters \((a, b, c, e, f)\) data have been given in Table 7-6. The third and fifth undamped natural mode shapes of the space truss are shown in Fig. 7-6. Fig. 7-7 shows the normalised frequency (the ratio of damped over undamped frequency) plotted against the natural mode number. It can be seen that the frequencies closer to the dominant frequency for the FE model (which uses the Rayleigh damping model) are again overdamped. However, the proposed fractional-spectral method does not suffer from this single matched-frequency damping effect.

7.3 DAMPED STRUCTURES II - ADDED DAMPING DEVICES

Structural dampers have been used successfully in various types of structures for the effective reduction of wind or earthquake induced response. Most structures can be designed to withstand dynamic forces caused by wind or severe earthquake by providing ductility and energy absorption capacity to the structural elements, but often at the expense of substantial damage in the structural elements, and also for non structural elements and services within the structures. On the other hand, by dissipating the vibratory energy in structural dampers, the risk for the structure of experiencing excessive deformations or accelerations can be reduced. As a result, less ductility or inelastic energy demand is required in the structural frame.

The purpose of this section is to investigate the efficiency of the fractional-spectral finite element method in cases of three dimensional structures with added damping devices. Using a fractional derivative model, the non-linear damping characteristics of structural dampers are modelled. Examples of a tall structure with added viscoelastic dampers and a bridge structure with added elastomeric dampers are solved and the results compared with finite element results in order to show the efficiency and stability of the proposed method.
7.3.1 Tall Structure With Added Viscoelastic Dampers

Over recent years, many experimental and analytical research studies have been carried out and it has been shown that damping devices can significantly improve the dynamic response of tall structures (Abbas and Kelly, 1993). Comprehensive parametric analyses have also shown that the displacement, velocity, acceleration, ductility and energy dissipation characteristic of viscoelastically-damped structures can be significantly influenced by the natural period of the undamped structure, ground motion excitation, brace-to-structure stiffness ratio, energy dissipation characteristic of any viscoelastic material used, and the strength ratio of the structure itself. The mechanical properties of viscoelastic materials are relatively complicated and may vary with environmental temperature and excitation frequency (frequency-dependent behaviour). One of the challenging aspects in research and implementation of viscoelastic dampers is to evaluate the damping characteristic and the effect of the dampers on structures. Viscoelastic dampers (VE) have been incorporated in several high rise buildings in the USA to reduce vibrational motion caused by earthquake excitation and wind loading. These dampers reduce amplitude and acceleration by converting part of the vibratory energy of wind and earthquake ground motion to heat.

Fig. 7-8(a) shows a three dimensional frame structure. In the study herein three different models have been considered for a frame structure: no bracing, diagonal bracing included, and viscoelastic dampers included in the diagonal bracing.

Finite Element Model: Three different finite element models of the structure (no bracing, diagonal bracing and viscoelastic bracing) can be generated using the ANSYS finite element program. The eigenvalues for the unbraced model have been
found using the subspace iteration technique, which internally uses the generalised Jacobi iteration algorithm. Figs. 7-8(b) and (c) show the second and fourth natural modes of the unbraced model. For the diagonally braced structure, the braces can be distributed uniformly through the stories (Fig. 7-8d) using conventional rod elements. However, to model the viscoelastic bracing in the third model, the damping characteristic of a viscoelastic material should be modelled.

One of the popular damping elements used in finite element analysis is the spring and dashpot parallel model, known as the Kelvin-Voigt model. It is often used as a first approximation for the study of damping characteristics of materials. The parameters of the model are usually determined by using a least-squares method. Fig. 7-9 shows a conventional spring and dashpot damping element, which can model axial or torsional damping behaviour. For the damped model herein, viscoelastic damping in braces can be modelled using the conventional spring and dashpot element with $k=4120$ kN/m and $\eta=480$ kN-s/m.

**Fractional Spectral Model:** Spectral three dimensional frame elements are used in the fractional-spectral model which benefits from the accuracy of the exact shape function. All the viscoelastic dampers can be modelled using the non-linear fractional damping element (Fig. 7-10). The matrix formulation for the three parameter damping element can be written as,

$$
\begin{bmatrix}
\hat{F}_1 \\
\hat{F}_2
\end{bmatrix} = \begin{pmatrix}
\frac{EA}{L} + (i\omega)^{\alpha+1}
\end{pmatrix}\begin{pmatrix}
\frac{E_i A}{L} & 1 & -1 \\
-1 & 1 & \hat{u}_1 \\
\end{pmatrix}\begin{pmatrix}
\hat{u}_2
\end{pmatrix}
$$

(7-3-1)

where $E_i = \frac{\eta^{\alpha+1}}{E^\alpha}$. In the study herein, a parameter $\alpha=0.57$ is adopted so as to give the best fit to available experimental data (Kasai et. al., 1993).
Table 7-7: 3-D Frame Structure Data

<table>
<thead>
<tr>
<th>E [GPa]</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>G [GPa]</td>
<td>86</td>
</tr>
<tr>
<td>v</td>
<td>0.33</td>
</tr>
<tr>
<td>A [m²]</td>
<td>0.036</td>
</tr>
<tr>
<td>$I_{yy}$ [m⁴ *10⁴]</td>
<td>91.54</td>
</tr>
<tr>
<td>$I_{zz}$ [m⁴ *10⁶]</td>
<td>91.80</td>
</tr>
<tr>
<td>$\rho$ [kg/m³]</td>
<td>7800</td>
</tr>
<tr>
<td>k [kN/m]</td>
<td>4120</td>
</tr>
<tr>
<td>$\eta$ [kN-s/m]</td>
<td>480</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.57</td>
</tr>
</tbody>
</table>

Figure 7-8

(a) 3-D Frame Structure Without Bracing; (b) Second Natural Mode; (c) Fourth Natural Mode; (d) Frame Structure With Bracing; (e) Structural Deformed Shape at $t=1.8$ (s)
Figure 7-9
Kelvin-Voigt damping element

\[ \frac{\eta^{\alpha+1}}{E^{\alpha}} \]

Figure 7-10
Non-linear damping element

\[ F(t) \text{ (kN)} \]

Fig 7-11
Impulsive load
After extracting the six first mode shapes of the structure, in the second part of the analysis, a simple impulsive load of 50 kN amplitude (Fig. 7-11) is applied at the top of the structure, and the response evaluation is carried out using the conventional time domain transient analysis (mode superposition) and the frequency domain fractional-spectral methods. It must be noted that in the mode superposition analysis \( \zeta=3\% \) damping is introduced to the first and second models (structures without bracing, and including diagonal bracing) for convergence purposes. The data for the structure is given in Table 7-7, and the maximum response in the \( x \) direction for the top of the structure is evaluated (Fig. 7-12) using the ANSYS finite element program and the proposed fractional-spectral method.

As shown in the Fig. 7-12, the maximum displacement occurs in the unbraced structure and the decay rate is very poor for both the unbraced and the braced structures, as expected. However, there is 9% difference in the responses of the linear and non-linear damping models (finite element and fractional-spectral) for the first cycle which reduces as the number of cycles increases. It seems that the linear spring and dashpot damping element overestimates the response of the structure in the first couple of cycles which may be considered in the design of structures with added viscoelastic dampers. Overall, it can be seen that there is a good agreement between the results of these two methods even with a large frequency step for the fractional-spectral models. A smaller frequency step can be used for more accurate results, as is the case for the finite element model with a smaller time step. However, considering the large size of the model, for the results shown in Fig. 7-12, the computer time consumed by the fractional-spectral model is 41% less than the time needed for the finite element model when it was run on an IBM-PC486 machine.
Figure 7-12
Response of the Structure Under the Dynamic Load

Figure 7-13
Base-isolated Bridge Deck
7.3.2 Isolated Bridge Structure

In recent years, elastomeric bearings have been used as isolation devices for the protection of bridges from seismic forces. During a large earthquake, large relative displacements and large energy absorption demands occur in elastomeric bearings. In most dynamic analyses for elastomeric bearings, linear viscous damping is assumed by researchers, and consequently it is assumed that the loss factor is linearly proportional to frequency. However, for many engineering materials the damping characteristic varies non-linearly with frequency. Hence, the linear assumption leads to unacceptable errors. Another popular model is the rate-independent damping model (hysteresis model), in which the stiffness and loss factor are assumed to have frequency-independent characteristics. However, this model is not suitable for dynamic analysis of elastomers and it cannot be used to analyse isolated structures. In what follows, the fractional derivative damping model for elastomers, developed in Chapter Five, is used to model elastomeric bearings in an isolated bridge.

Fig. 7-13 shows a cross section of a base-isolated bridge deck. A variety of designs for base-isolation systems including rubber, lead rubber, roller and frictional sliding bearings have been proposed, and some of these isolation systems have already been used in the construction of buildings and bridges in various countries around the world such as New Zealand, France and the United States (Kelly, 1986).

Fig. 7-14(a) shows a bridge structure. A finite element model (Fig. 7-14b) can be generated using the ANSYS finite element program and it consists of 238 elements (for input data see Table 7-5). The eigenvalues for the model have been found using the subspace iteration technique. Figs. 7-14(c), (d), and (e) show the first, third, and sixth natural modes for the undamped bridge.
Figure 7-14

(a) Bridge Frame; (b) FE Model; (c) First; (d) Third; (e) Sixth Natural Modes
Elastomeric dampers in the finite element model are modelled using the elasticity-based hyperelastic method which is discussed in Appendix IV. However, the non-linear damping model developed for the elastomeric materials in Chapter Five is used in the frequency domain fractional-spectral model. An artificial accelerogram, of 20 sec duration is then applied at the base of the structure (Fig. 7-15), and the response evaluation is carried out using the conventional time domain transient analysis and the frequency domain spectral method (Fig. 7-16).

Two time-domain transient analyses and one frequency-domain spectral analysis have been performed. In the first analysis, for convergence purposes a 5% structural damping ratio has been implemented in the un-isolated finite element model which consists of 148 elements. In the second analysis, rubber dampers have been placed under the deck in the finite element model. Fig. 7-17 shows the finite element results for the un-isolated and isolated models. In the third analysis (frequency domain analysis), a fractional-spectral model, consisting of 46 elements, has been generated.

Table 7-5: Bridge Structure Data

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_S$ [Gpa]</td>
<td>200</td>
</tr>
<tr>
<td>$E_C$ [Gpa]</td>
<td>22</td>
</tr>
<tr>
<td>$\nu_S$</td>
<td>0.33</td>
</tr>
<tr>
<td>$\rho_S$ [kg/m$^3$]</td>
<td>7800</td>
</tr>
<tr>
<td>$\rho_C$ [kg/m$^3$]</td>
<td>3000</td>
</tr>
<tr>
<td>$A_p$ [m]</td>
<td>0.08</td>
</tr>
<tr>
<td>$(I_{yy})_p$ [m*$10^4$]</td>
<td>180</td>
</tr>
<tr>
<td>$T_D$ [m]</td>
<td>0.22</td>
</tr>
<tr>
<td>$k$ [kN/m]</td>
<td>13420</td>
</tr>
<tr>
<td>$\eta$ [kN-s/m]</td>
<td>1960</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.64</td>
</tr>
</tbody>
</table>

S - Steel; C - Concrete; $T_D$ - Slab Thickness; $p$ - Pier
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Figure 7-15
Artificial Accelerogram

Figure 7-16
Comparison of the Fractional-Spectral and Finite Element (with Rubber Dampers) Results
Figure 7-17

FE Results for structure with (a) $\zeta=5\%$ Damping Ratio; (b) with Rubber Dampers
The frequency domain analysis has been performed using the same geometry, material and load history data as for the finite element model. The maximum responses in the $x$ direction for the top of the deck under the artificial accelerogram are evaluated for the finite element and fractional-spectral models (Fig. 7-16).

It can be seen that introducing the elastomeric dampers speeds up the decay rate of vibration. However, the rate of vibration decay for the fractional-spectral model is generally more than for the finite element model. Hence, neglecting the frequency-dependent characteristics of the elastomeric dampers would give an unrealistic response to some extent. The computer time for the fractional-spectral model in these analyses is almost 62% of the finite element solution time on an IBM486 machine.

7.4 DAMPED STRUCTURE III - BASE ISOLATION

The analysis of base-isolated structures subjected to seismic excitation is an area of great interest to structural dynamicists and engineers. A variety of analysis and design methods are presently available for modelling the response of non-linear damping devices in the base, as well as the structure itself. A very important application of non-linear dynamics arises from base isolation analysis where the superstructure is assumed to remain elastic during seismic excitation while the isolation system undergoes a non-linear motion. Excitation of structures by earthquake ground motion induces inertia forces which depend on the dynamic properties of the structural system and the characteristics of the ground excitation.

The purpose of this section is to compare the results of the non-linear spectrally formulated finite element method for a base-isolated structure with those
results derived from the conventional finite element method. Fig. 7-18 shows a five
story base-isolated structure with the added combination of rubber and frictional
sliding dampers. Three different methods have been used to analyse the structure:
the conventional finite element (FE) method, the non-linear time domain method
(NT) proposed by Nagarajaiah et al. (1991), and the proposed non-linear spectral
method (NS).

**Finite Element Model:** The FE model consists of 212 conventional frame
elements, and the recently developed plasticity-based hyperelastic element
(Appendix IV) is used to model the rubber material using the ANSYS general
purpose finite element program. However, for the non-linear frictional sliding
dampers, an approximated control element used in general finite element programs
has been used (Fig. 7-19). A more comprehensive discussion about the
conventional damping elements is presented in Appendix IV.

**Non-Linear Time Domain Model:** The non-linear time domain (NT) method of
dynamic analysis for base-isolated structures has been developed by Nagarajaiah,
(1991). It is based on assumptions that; firstly, the superstructure remains elastic at
all times, secondly, there exists a rigid slab at the base level so that all isolation
elements are connected, and finally, the dynamic characteristics of each floor can
be modelled using three degrees of freedom in which these degrees of freedom are
located at the center of mass of each floor. As the isolation system often
experiences multi-directional motion under multi-directional excitation, the NT
method uses a biaxial hysteretic element based on the Park differential equations
(Park et. al., 1985). Recall the Park equation Eq. (6-6-2) for a dynamic system
with biaxial non-linear hysteretic restoring force as,
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\[ [m] \ddot{u}_x + [c] \dot{u}_x + [q_s] = [f_x] \]  \hspace{1cm} (6-6-2)

which can be used to model the damping behaviour of rubber materials in base-isolated structures. The non-linear restoring force can be expressed as

\[ \begin{bmatrix} q_s \\ q_y \end{bmatrix} = \alpha [k] \begin{bmatrix} u_x \\ u_y \end{bmatrix} + (1 - \alpha) [k] \begin{bmatrix} Z_x \\ Z_y \end{bmatrix} = G(t) + Z(t) \]  \hspace{1cm} (6-6-3)

where \( \alpha \) is the postyielding to preyielding stiffness ratio of the damper, \( Z_x \) and \( Z_y \) satisfy two coupled Bouc-Wen differential equations (Park et al., 1986) as,

\[ \begin{align*}
\dot{Z}_x &= A \dot{u}_x - \beta |\dot{u}_x| Z_x Z_y - \gamma |\dot{u}_x| Z_x Z_y - \beta |\dot{u}_x| Z_x Z_y - \gamma |\dot{u}_x| Z_x Z_y \\
\dot{Z}_y &= A \dot{u}_y - \beta |\dot{u}_y| Z_y Z_x - \gamma |\dot{u}_y| Z_y Z_x - \beta |\dot{u}_y| Z_y Z_x - \gamma |\dot{u}_y| Z_y Z_x 
\end{align*} \]  \hspace{1cm} (7-4-1)

where \( A, \gamma \) and \( \beta \) are dimensionless quantities that control the shape of the hysteresis loop. Hence, the mobilised forces for elastomer dampers can be found as

\[ F_x = \alpha \frac{F_y}{Y} u_x + (1 - \alpha) F_y Z_x \]  \hspace{1cm} (7-4-2)

\[ F_y = \alpha \frac{F_y}{Y} u_y + (1 - \alpha) F_y Z_y \]

where \( F_y \) is the yield force. For a frictional sliding damper, the mobilised forces can be described as
\[ F_x = \mu_s W Z_x \]
\[ F_y = \mu_s W Z_y \]  
(7-4-3)

where \( \mu_s \) is the coefficient of friction, and \( W \) is the vertical load carried by the bearing. If the non-linear hysteresis component, which is a function of the time history, is transferred to the right hand side of the equation of motion,

\[ m \ddot{u}(t) + c \dot{u}(t) + G(t) = F(t) + Z(t) \]  
(6-6-4)

which can be solved using the prescribed pseudo-force method. A computer program to calculate the pseudo-force is presented in Appendix III.

**Non-Linear Spectral Method**: The non-linear spectral model consists of 40 spectral frame elements, which benefit from the accuracy of the exact shape function. Elastomeric dampers are modelled using the non-linear damping element presented in Chapter Five. However, for the frictional sliding dampers the same non-linear mobilised forces used in the NT method are used (see Chapter Six). The non-linear forces are calculated in the time domain and transferred into the frequency domain at each frequency step using the FFT.

To start the analysis, firstly, a modal analysis is performed for the undamped structure. Fig. 7-20 shows the second and fourth natural mode shapes. In the second part of analysis, an artificial accelerogram (Fig. 7-15) is applied at the base of the structure, and the response of the top floor is evaluated. In Fig. 7-21, the results of the three methods have been compared.
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$E = 2.0 \text{ [GPa]}$

$\rho = 3200 \text{ [kg/m}^3]\text{]}

$A_c = 0.2 \text{ [m}^2]\text{]}

$A_b = 0.15$

$(I_{\omega})_c = 4.54 \text{ [m}^4\times10^{-3}]$

$(I_{\omega})_b = 1.92 \text{ [m}^4\times10^{-3}]$

$\mu = 0.062$

$\eta = 480 \text{ [kN-s/m]}$

Figure 7-18

Three Dimensional Base-Isolated Structure

Figure 7-19

Conventional Control Damping Element
Figure 7-20

(a) Second; (b) Fourth Natural Mode Shapes

Figure 7-21

Response of the Top Floor
The comparative analyses show that the FE model with linear damping elements has 18% and 13% differences compared with the non-linear time domain and the non-linear spectral methods, respectively. An important point herein is that since the NT method uses the hysteretic damping model for elastomers, it cannot model the frequency-dependent characteristics of elastomeric materials. Hence, although both of the non-linear models use the same mobilised forces for the frictional sliding dampers, the results still differ as much as five percent.

7.4 SOIL-STRUCTURE INTERACTION

In the case of earthquake excitation as discussed in the previous section, it has been assumed that the earthquake motions were applied at the structural support points. Hence, all displacements at the base level were assumed to depend only on the earthquake-generation waves. However, the structural response to any earthquake excitation not only depends on the dynamic characteristics of the structure itself but it depends also on the relative mass and stiffness properties of the soil and the structure.

The analysis of soil-structure interaction problems is usually divided into two parts; firstly, obtaining the soil-surface accelerations without considering the structure (free-field motions), and secondly, obtaining the structural response to the free-field motions while considering the interaction of the structure and soil media. If the earthquake motions in the basement rock (under the soil layer) are known, the free-field motions can be determined by treating the soil layer exactly like any other structural systems with known support motions. By assuming that the earthquake input is a rigid-based translation, the response at the soil surface can be found. However, in most soil-structure interaction problems it is not reasonable to assume that the damping is proportional to velocity, and therefore it will not satisfy the modal orthogonality condition.
Damping coupling is especially significant in soil-structure interaction problems, as a very large approximate viscous damper is required to simulate the radiated-energy loss. On the other hand, as the fractional damping model formulated is based on viscoelastic theory, it is capable of producing far better results than conventional damping elements. In fact, Caputo, (1976) has shown that the fractional damping model is capable of matching the experimental results better than other known damping models. To compare the performance of the fractional damping model with that of the conventional model, a soil-structure interaction problem is solved using both the conventional and the proposed method. Fig. (7-22) shows a two storey structure built over a soft layer of soil. The same artificial accelerogram of Fig. 7-15 is applied at the bottom of the soil layer. It is assumed that the soil layer is founded on a rigid rock bed and the earthquake excitation is a rigid-based horizontal translation.

A finite element model of the damped system can be generated in which the soil media is modelled using plate elements. At the boundaries, the spring and dashpot conventional element has been used to transfer the energy of the system. A material damping option of the ANSYS program is used to introduce damping in the soil layer.

A fractional-spectral model is also generated using the spectral plate elements developed in Chapter Four. Damping is introduced into the system using a five parameter material damping model. Complex boundary elements with infinite element lengths, developed in Chapter Four, are used to transfer the energy of the system. The response of the damped system to the excitation in the $x$ direction is found using both the time domain and the frequency domain analysis methods (Fig. 7-23). As expected, the frequency-independent damping model in the finite element model overestimates the response of the system by as much as 32% compared with the FS results.
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7-32

Figure 7-22

(a) Damped Soil-Structure Interaction Media; (b) Side View

Figure 7-23

Response at the Structural Foundations
7.6 Damped Mechanical Systems

The complex stiffness concept can also be used to analyse damped mechanical systems. In fact, a common use of viscoelastic damping materials in design is as an additive damping treatment to increase the combined system's damping. Layers of viscoelastic materials are often added to mechanical systems composed of lightly damped materials which have sufficient stiffness for static loading but insufficient damping for controlling vibration.

Many mechanical systems are made of metals and alloys that have relatively little inherent damping. A viscoelastic layer is often added as a layer to the outside surface of a structure (free-layer damping treatment). In Fig. 7-24, the metal layer provides the appropriate rigidity $E_1 I_1$, while the viscoelastic layer with different modulus $E_2$ may provide enough damping for the structure. The overall rigidity of the beam can be written as,

$$EI = \alpha E_1 I_1 ; \quad \alpha = 1 + eh^3 + 3(1+h)^2 \frac{eh}{1+eh}$$

Figure 7-24
Free-Layer Damping Treatment for a Beam

where $e=E_2/E_1$ and $h=H_2/H_1$. 
7.6.1 *Aeroplane Wing*

To examine the ability of the fractional-spectral method in the case of damped mechanical systems, consider an aeroplane wing (Fig. 7-25).

![Schematic Sketch of Aeroplane Structure](image)

If a viscoelastic layer is added to increase damping in a wing, vibration of the mechanical system should be solved for the combined stiffness and damping values. Consider a spectral model of the idealised system as shown in Fig. 7-26.

![Fractional-Spectral Model of Wing](image)
An attempt is made herein to use the analytical frequency response function and demonstrate the ability of the fractional-spectral method in damped mechanical systems. The five parameter damping model developed in Chapter Five is used to model viscoelastic damping in the damping layer, and the engine masses are modelled using two concentrated masses at two lower nodes. The wing is separated from the body using the substructuring techniques discussed in previous chapters, and a boundary element with infinite length is used to transfer the energy from the system to remote parts. An impulsive flexural couple of 50000 N-m is applied at the free end of the wing (Fig. 7-27).

As the fractional-spectral method provides enough information to be able to obtain the time response of any variable at any point, the flexural velocity results of points A and B can be plotted against time (Fig. 7-28). These results show that most of the energy of the impact is absorbed through the viscoelastic layer, and the energy transferred to the main body of the aeroplane is very small compared with the energy at the impact site.

### 7.6.2 Satellite Dish

In this section the application of the fractional-spectral method to a large space structure is investigated. Fig. 7-29 shows a satellite antenna tower. Many large space structures will be required to point very accurately and manoeuvre very rapidly. Hence, a high performance control system will often be needed to satisfy such design requirements. Achievement of performance in accurate alignment and manoeuvrability is not possible unless accurate models of stiffness, mass and damping are considered in the analysis. However, because of incorrectly-modelled dynamics, the transfer function of these mechanical systems exhibits phase uncertainty with increasing frequency. Hence, a practical control system design must be based on a confidently formulated frequency-dependent model.
Figure 7-27

*Time History of Load Function Applied at Free End of Wing*

Figure 7-28

*Response at the Impact and Remote Sites*
In the structure considered herein, members function as axial members, and all joint and other non-structural masses are neglected. Two time domain transient analyses and one frequency domain spectral analysis have been performed. For the time domain analyses, a finite element model can be generated using the ANSYS finite element program, and it consists of 232 elements. The eigenvalues for the model have been found using the subspace iteration technique. Figs. 7-30 and 7-31 show the first, second, fourth and fifth natural modes for the undamped structure.

In the first time domain analysis the structure is not isolated, and only a 3% structural damping ratio has been implemented for convergence purposes. In the second analysis, four rubber dampers (see Appendix IV) have been placed in the finite element model between the tower and the dish. For the frequency domain analysis (third analysis), a fractional-spectral model consisting of 82 elements has been generated, in which the fractional derivative damping element is used to model the rubber dampers.

The artificial accelerogram used earlier (Fig. 7-15) is then applied at the base of the structure, and the response evaluation is carried out using the conventional time domain transient analysis and the frequency domain spectral method. The maximum responses in the $x$ direction for the top of the structure under the artificial accelerogram are evaluated for the finite element (with rubber dampers) and the fractional-spectral models (Fig. 7-32).

Figs. 7-33 and 7-34, show the finite element results for the structure with and without rubber dampers. It can be seen that elastomer damper again causes a substantial reduction in the transient vibration time. However, the finite element method (with frequency-independent damping elements) seems to overestimate the structural response by as much as 11% compared with the fractional-spectral results in Fig. 7-32.
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\[ E = 36.72 \text{ [ GPa] } \]
\[ v = 0.33 \]
\[ \rho = 2200 \text{ [ kg/m}^3] \]
\[ A = 31 \text{ [ m}^2 \times 10^{-5}] \]
\[ a_{(FS \text{ Model})} = 3.1 \text{ [ s}^5] \]
\[ b_{(FS \text{ Model})} = 0.52 \]
\[ c_{(FS \text{ Model})} = 0.53 \]
\[ f_{(FS \text{ Model})} = 8.5 \text{ [ N.s/m}^2 \times 10^{10}] \]
\[ \alpha = 0.64 \]

Figure 7-29

Satellite Antenna

Figure 7-30

(a) First; (b) Second Natural Modes
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Figure 7-31

(a) Fourth (b) Fifth Natural Modes

Figure 7-32

Response of the Top of the Structure for FE (with Rubber Dampers) and FS Methods
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Figure 7-33
Response of the top of the FE Model with 3% Structural Damping Ratio

Figure 7-34
Response of the Top of the FE Model with added Rubber Damper
7.7 SUMMARY AND CONCLUDING REMARKS

In the preceding sections, the application of the fractional-spectral method of analysis has been presented. This method is capable of making accurate predictions of the damping characteristics of structural members and frames. The non-linear damping model is compatible with current computational structural analysis methods and it is easy to adapt.

The fractional-spectral results agree well with the best available linear and non-linear dynamic analysis methods in terms of performance. However, as the number of degrees of freedom is small compared with the conventional method, the computer time to solve any dynamics problem is less than that by the conventional method. In terms of performance, the fractional damping element seems to have noticeable modelling power, as it has an advantage over the conventional method of including non-linear damping (frequency and temperature dependent behaviour). In fact, while the fractional-spectral method enjoys some of the established conventional matrix techniques, it is additionally powerful in modelling of damping characteristics.
Chapter Eight

Summary and Conclusions

8.1 SUMMARY

The spectral finite element analysis tools developed in the previous Chapters provide a new approach for analysing damped structural systems. A more realistic damping model is also developed which is compatible with current computational structural analysis methods. Referred to as the fractional-spectral method, the capabilities of the fractional derivative damping model are combined with the efficiency of the complex spectral finite element method to create a new powerful analysis method which works in the frequency domain.

The fractional-derivative viscoelastic damping model enables a single formulation of the complex dynamic stiffness matrix of the damped system to be developed. This leads to the successful formation of the frequency domain equations of motion for a structure containing both elastic and viscoelastic components. However, although the development of the fractional-spectral method focuses primarily on damped structural frames, the results can readily be extended to damped mechanical systems. In addition, although the analysis of damped systems
are the primary object of this study, the spectral finite element method presented in Chapter Four can be used in the case of undamped dynamical systems.

In conclusion, the approach to damped dynamical systems resulting from this investigation is particularly powerful, in that not only the vibrational motions of a structure are modelled using an exact spectral solution to the governing differential equation of motion, but the general structural properties such as stiffness, mass and damping are treated in a single formulation.

8.2 MODELLING OF DAMPING

A set of the fractional-derivative damping models, capable of representing different damping mechanisms, have been derived for solving the dissipation problem in damped systems. The modelling of elastomeric and viscoelastic components in damped structures often requires complex viscoelastic representations. While traditional differential operators are typically employed in such a formulation, fractional operators give rise to a richer variety of functional families, and hence lead to an improved integro-differential type curve fitting of constitutive representations.

The versatility of the fractional-derivative damping model in the study herein is such that it can be incorporated into the frequency domain formulation of the governing equations of structural systems. There are some additional advantages for these fractional derivative models, particularly over the conventional linear damping models which are listed as follows:
Chapter Eight: Summary and Conclusions

8.3 COMPLEX-SPECTRAL FINITE-ELEMENT ANALYSIS

The main features of the complex-spectral finite-element method of analysis have been presented in preceding Chapters. This method is capable of making accurate predictions of the dynamic response of structural systems. Most structural systems can be analysed and designed by using the conventional finite element method. However, in order to guarantee stability and accuracy of the solution, the number of elements used to model the structure may be very large. Hence, it appears that, for large structures, it may be more effective to use the spectral approach presented herein.

As far as stability and efficiency are concerned, it can be assessed that even though the assemblage of global matrices has to be repeated for all frequency components in comparison with the single assemblage procedure for the conventional finite element method, the spectral finite element approach outperforms the conventional method in the case of large structures.
The inertia of the mass distribution in the conventional finite-element method is modelled using piecewise-continuous interpolation trial displacement functions which are assumed over a local region of the system being analysed. The individual element matrices for each piece (element) are computed and assembled into global system matrices. However, the number of elements required to model the inertia distribution effect adequately, is substantially large. On the other hand, the spectrally formulated elements developed in Chapter Four, treat the distributed mass exactly. Moreover, as the inertia distribution is represented in the exact spectral solution of the differential equations of motion, there is no need for assemblage of an individual mass matrix.

The alternative spectral approach which works in the frequency domain and draws its robustness from the speed and switching capabilities of the Fast Fourier Transform, has additional advantages as follows:

- The choice of the element size is not restricted by the need to approximate the distributed mass and allows the choice to be governed by the structural connections and discontinuities.

- Higher order rod and beam theories can be formulated without adding extra displacement coordinates.

- It is compatible with the experimental dynamic analysis. The inverse dynamical problems can also be analysed if the response is known at some locations.

- The effect of damping can easily be adapted into the method. The fractional derivative damping models can be incorporated simply by changing the dynamic stiffness matrix.
8.4 DAMPED STRUCTURAL RESPONSE

Based on the theory of fractional calculus and the spectral theory of vibration, a new spectrally-formulated finite-element method of analysis is developed which is capable of making accurate predictions of the dynamic response of structures with added dampers. The frequency-dependent and temperature-dependent damping characteristics of structural materials can be modelled accurately using the fractional derivative model. It is shown that the proposed method can be extended to develop a non-linear damping element which can be used to model structural dampers. The approach has an advantage over the usual viscous treatment, which appears to lack a physical basis.

As structural dampers have been used successfully in various types of structures for the effective reduction of vibratory motions, it is necessary to develop an accurate analysis tool to simulate their behaviours. The use of numerical methods in the transform domain to describe the non-linear mechanical properties of damped structural systems in this study has proved to be an efficient way of handling complex damped systems. The proposed fractional-derivative dynamic analysis has shown some additional advantages as follows:

- It is shown that the formulation of the equations of motion using derivatives of fractional order produces high order matrix equations to solve. However, the fractional derivative damping model typically requires five parameters for definition, which is a lesser number of parameters than is usually required with the corresponding standard linear viscoelastic damping model.

- The damped natural frequencies and mode shapes are obtained in a straightforward manner as the frequency response function of the system can be obtained directly.
The increase in accuracy for the fractional-spectral method is achieved by increasing the number of terms in the spectral representation, while in the conventional approach, simultaneously, an increase in the number of time steps and an increase in the system size are required to increase accuracy. The latter increase in the bandwidth of the conventional system matrices can have a crippling effect on the numerical performance.

8.5 RECOMMENDATIONS FOR FUTURE WORK

Despite the numerous advantages of the proposed fractional-spectral method, the method does still possess some limitations. The most obvious drawback is in the building of an element library. Although several different elements have been developed throughout this thesis, the task of developing elements becomes increasingly cumbersome with structures such as plates with irregular discontinuities (holes, cracks). It is recommended that these cases be treated using a hybrid approach. The speed and capabilities of the proposed fractional-spectral method can be combined with the natural features of the conventional finite element method in handling complicated geometries.

A less obvious drawback is in the development of an amplitude-dependent damping model. Some material damping mechanisms, such as dislocation motion and twining, are amplitude-dependent. These mechanisms generally involve permanent deformation of the material and should be modelled using non-linear theories. This research was limited to amplitude-independent material damping mechanisms. For the case of amplitude-dependent mechanisms, it is recommended that non-linear terms be added to the developed damping models. These non-linear terms can then be treated as pseudo-force terms which can be calculated in the transformed time domain.
One of the promising developments for speeding up numerical methods in engineering is the advent of parallel architectures. As each frequency and wave number component is solved independently of the next in the proposed fractional-spectral method, it is strongly recommended to use parallel processing machines. Moreover, as each step in the proposed method is independent of the next, the second processor immediately reduces the CPU time in half without the necessity of developing parallel algorithms. This natural parallelism in the proposed method can open a whole new window to the future of structural dynamic analysis techniques.
REFERENCES


ANSYS General Purpose Finite Element Program, Revision 5.0 (1993), Swanson Analysis Systems, Inc, Houston, PA, USA


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Mathematica Computer Package, Revision 2.0 (1992), Wolfram Research Inc., USA


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Sehmi, N. S., (1989), Large Order Structural Eigenanalysis Techniques, Ellis Horwood Limited Publisher, Chichester.

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References


Appendix I

Fast Fourier Transform

The Fast Fourier Transform (FFT) is a computer algorithm for calculating discrete Fourier transforms (DFT). The algorithm performs the summation of the discrete Fourier transform in a highly efficient manner. The FFT works by partitioning the full finite sequence into a number of shorter sequences. Instead of calculating the DFT of the original sequence, only the DFT’s of the shorter sequences are worked out. The replacement of the Fourier integrations by summations in the conventional discrete Fourier approach is a further step in the numerical implementation of the continuous transform. The discrete Fourier transform has the same properties of the continuous transform.

There are many codes available for performing the FFT (Doyle, 1986). The computer program FFT used throughout this study (Sections 4.9.2 and 6.5.2) has its source code listed herein. The program is written in standard C language on a personal computer IBM-AT compatible machine.

FFT Source Code

The FFT.c is capable of doing both the forward and inverse Fourier Transform. Its input data file can be in free format with blanks or commas used as separators. The input data file (time data for the forward transform and frequency
Appendix I: Fast Fourier Transform

data for the inverse transform) should be created using ASCII editor. Following is a complete listing of the FFT.c source code.

/* FFT.c -- (version 1.00, 1995).
Dept. of Civil Engrg., Univ. of Wollongong */
#include "f90.h"

/* Table of constant values */
static integer c__9 = 9;
static integer c__1 = 1;
static integer c__0 = 0;
static integer c__10 = 10;
static integer c__3 = 3;
static integer c__41011 = 41011;
static integer c__8200 = 8200;
static integer c__4 = 4;
static integer c__2 = 2;

/* MAIN for FFT */

/* Main program */
MAIN_()
{
  /* Format strings */
  static char fmt_4[] = "(002 @@ DATE:002,1x,i2,002-i002,i2,002-002,i1
4,6x,002TIME:002,1x,i2,002-002,i2)";

  /* System generated locals */
  olist o__1;
  alist al__1;

  /* Builtin functions */
  integer s_wsfe(), e_wsfe(), s_wsle(), do_lio(), e_wsle(), f_open(), f_rew(), do_fio();

  /* reserve master space and zero it */
  s_wsfe(&io__1);
  e_wsfe();
  s_wsle(&io__2);
  do_lio(&c__9, &c__1, "FFT Program ", 52L);
  e_wsle();
  s_wsle(&io__3);
  do_lio(&c__9, &c__1, " FFT of Complex Time Series Data ", 50L);
  e_wsle();
  s_wsle(&io__4);
  do_lio(&c__9, &c__1, " PC version , May 1995", 37L);
  e_wsle();
  s_wsle(&io__5);
do_lio(&c_9, &c_1, " ", 37L);
e_wsle();
s_wsle(&io_6);
do_lio(&c_9, &c_1, " FFT Program ", 52L);
e_wsle();
s_wsfe(&io_7);
e_wsfe();

for (n = 1; n <= 41011; ++n) {
    respec[n - 1] = (float)0;

} /* L2: */

/* open input file */
ilog = 67;
o__l.oerr = 0;
o__l.ounit = ilog;
o__l.ofmmlen = 8;
o__l.ofnm = "cfft.log";
o__l.ortal = 0;
o__l.osta = 0;
o__l.oacc = 0;
o__l.ofm = 0;
o__l.oblnk = 0;
f_open(&o__1);
al__1.aerr = 0;
al__1.aunit = ilog;
f_rew(&al__1);
io__11.ciunit = ilog;
s_wsle(&io__11);
do_lio(&c_9, &c_1, "@@ FFT PC version ", 24L);
e_wsle();

getdat(&iyr, &imon, &iday);
gettim(&ihr, &imin, &isec, &i100);
io__19.ciunit = ilog;
s_wsfe(&io__19);
do_fio(&c__1, (char *)&imon, (ftnlen)sizeof(integer));
do_fio(&c__1, (char *)&iday, (ftnlen)sizeof(integer));
do_fio(&c__1, (char *)&iyr, (ftnlen)sizeof(integer));
do_fio(&c__1, (char *)&ihr, (ftnlen)sizeof(integer));
do_fio(&c__1, (char *)&imin, (ftnlen)sizeof(integer));
e_wsfe();

L40:
s_wsle(&io__20);
do_lio(&c__9, &c__1, " ", 1L);
e_wsle();
s_wsle(&io__21);
do_lio(&c__9, &c__1, " MAIN menu: ", 12L);
e_wsle();
s_wsle(&io__22);
do_lio(&c__9, &c__1, " 0: Quit ", 19L);
e_wsle();
s_wsle(&io__23);
do_lio(&c__9, &c__1, " 1: input data ", 36L);
Appendix I: Fast Fourier Transform

2: FFT transforms

3: process transforms

...
Appendix I: Fast Fourier Transform

irl = 1;
ir2 = irl + 16400;
ir3 = ir2 + 16400;
ir4 = ir3 + 8200;
space_(&c__41011, &irl, &ir4, &ichk);
io___39.ciunit = ilog;
s_wsle(&io___39);
do_lio(&c__9, &c__1, "@@ FFT: memory used", 20L);
do_lio(&c__3, &c__1, (char *)&ir4, (ftnlen)sizeof(integer));
do_lio(&c__9, &c__1, " of available", 13L);
d_lio(&c__9, &c__1, (char *)&c__41011, (ftnlen)sizeof(integer));
e_wsle();

post_(&respce[irl - 1], &respce[ir2 - 1], &respce[ir3 - 1], &c__8200, &ilog);
}
goto L40;
} /* MAIN__ */

/* Subroutine */ int prep_(x, wk, tt, nmax, ilog)
real *x, *wk, *tt;
integer *nmax, *ilog;
{
  /* System generated locals */
  integer x_dim1, x_offset, wk_dim1, wk_offset, i__1;

  /* Built-in functions */
  integer s_wsle(), do_JioO, e_wsle(), s_wsfe(), do_fioO, e_wsfe(),
    s_rsle(), e_rsle();
  /* Subroutine */ int s_paus();

  /* Local variables */
  static integer icol;
  static real   temp, tmax;
  static integer iopt;
  static real   temp1, temp2;
  static integer i, ndata;
  extern /* Subroutine */ int store_(), input_();
  static real   df, dt;
  extern /* Subroutine */ int averag_(), scaler_(), siggen_();
  extern integer intgetj();
  extern /* Subroutine */ int interp(), viewer(), window();
  static integer npt;
  /* Parameter adjustments */
  --tt;
  wk_dim1 = *nmax;
  wk_offset = wk_dim1 + 1;
  wk = wk_offset;
  x_dim1 = *nmax;
  x_offset = x_dim1 + 1;
  x = x_offset;
Appendix I: Fast Fourier Transform

/* Function Body */

L40:

s_wsle(&io__46);
do_lio(&c_9, &c__1, " ", 1L);
e_wsle();
s_wsle(&io__47);
do_lio(&c_9, &c__1, " menu : ", 13L);
e_wsle();
s_wsle(&io__48);
do_lio(&c_9, &c__1, " 0: Return", 19L);
e_wsle();
s_wsle(&io__49);
do_lio(&c_9, &c__1, " 1: Input time domain data", 35L);
e_wsle();
s_wsle(&io__50);
do_lio(&c_9, &c__1, " 2: Interpolate", 24L);
e_wsle();
s_wsle(&io__51);
do_lio(&c_9, &c__1, " 3: Signal generator", 29L);
e_wsle();
s_wsle(&io__52);
do_lio(&c_9, &c__1, " 4: Separator (moving averager )", 41L);
e_wsle();
s_wsle(&io__53);
do_lio(&c_9, &c__1, " 5: Window data", 24L);
e_wsle();
s_wsle(&io__54);
do_lio(&c_9, &c__1, " 6: Scaler ", 34L);
e_wsle();
s_wsle(&io__55);
do_lio(&c_9, &c__1, " 7: Exchange columns", 29L);
e_wsle();
s_wsle(&io__56);
do_lio(&c_9, &c__1, " 8: Store results", 26L);
e_wsle();
s_wsle(&io__57);
do_lio(&c_9, &c__1, " 9: View results ", 26L);
e_wsle();
s_wsle(&io__58);
do_fio(&c__1," select -->", 11L);
e_wsle();
iopt = intget_(&c0, &c__10);
io__61.cunit = *ilog;
s_wsle(&io__61);
do_lio(&c__3, &c__1, (char *)&iopt, (ftnlen)sizeof(integer));
doiol(&c_9, &c__1, "::PREP", 7L);
e_wsle();
if (iopt == 0) {
    return 0;
}

if (iopt == 1) {
    input_(&wk[wk_offset], &tt[1], &ndata, &icol, ilog, nmax);
    /* transfer to 1 and 2 in case already interpolated */
i__1 = ndata;
Appendix I: Fast Fourier Transform

for (i = 1; i <= i___l; ++i) {
  x[i + x_dim1] = wk[i + wk_dim1];
  x[i + (x_dim1 <= 1)] = wk[i + (wk_dim1 <= 1)];
/* L50: */
  dt = tt[2] - tt[1];
npt = ndata;
}

if (iopt == 2) {
  s.wsle(&io___67);
  do_lio(&c___9, &c___1, INPUT: sampling dt # of lines", 33L);
  e.wsle();
  s.wsfe(&io___68);
  do_fio(&c___1, "-> ", 6L);
  e.wsfe();
  s.rsle(<&io___69);
  do_lio(&c___4, &c___1, (char *)dt, ftnlen)sizeof(real));
  do_lio(&c___3, &c___1, (char *)npt, ftnlen)sizeof(integer));
  e.rsle();
  s.wsle(&io___70);
  do_lio(&c___4, &c___1, (char *)dt, ftnlen)sizeof(real));
  do_lio(&c___3, &c___1, (char *)npt, ftnlen)sizeof(integer));
  e.wsle();
  io___71.cunit = *ilog;
  s.wsle(&io___71);
  do_lio(&c___4, &c___1, (char *)dt, ftnlen)sizeof(real));
  do_lio(&c___3, &c___1, (char *)npt, ftnlen)sizeof(integer));
  do_lio(&c___9, &c___1, ::dt, npt", 12L);
  e.wsle();
  tmax = wk[ndata + wk_dim1 * 3];
  temp = npt * dt;
  if (temp > tmax) {
    io___74.cunit = *ilog;
    s.wsle(&io___74);
    do_lio(&c___9, &c___1, "@@ zeroes added since tmax < npt*dt=", 36L);
    do_lio(&c___4, &c___1, (char *)temp, ftnlen)sizeof(real));
    e.wsle();
  } else if (temp < tmax) {
    io___75.cunit = *ilog;
    s.wsle(&io___75);
    do_lio(&c___9, &c___1, "@@ partly used since tmax > npt*dt=", 35L);
    do_lio(&c___4, &c___1, (char *)temp, ftnlen)sizeof(real));
    e.wsle();
  }
/*
  begin interpolation */
  interp_(&x[x_offset], &wk[wk_offset], &tt[1], &dt, &npt, &ndata, nmax)
  ;
}

if (iopt == 3) {
  siggen_(&x[x_offset], &npt, ilog, &dt, nmax);
}
Appendix I: Fast Fourier Transform

/* average the data */
if (iopt == 4) {
/* transfer to 1 and 2 to work array */
i_1 = npt;
for (i = 1; i <= i_1; ++i) {
    wk[i + wk_dim1] = x[i + x_dim1];
    wk[i + (wk_dim1 << 1)] = x[i + (x_dim1 << 1)];
} /* L44: */
    averag_(&wk[wk_offset], &tt[1], &npt, &icol, ilog, nmax);
}

/* window the data */
if (iopt == 5) {
/* transfer to 1 and 2 to work array */
i_1 = npt;
for (i = 1; i <= i_1; ++i) {
    wk[i + wk_dim1] = x[i + x_dim1];
    wk[i + (wk_dim1 << 1)] = x[i + (x_dim1 << 1)];
} /* L55: */
    window_(&wk[wk_offset], &npt, ilog, &dt, nmax);
}

/* scale the data */
if (iopt == 6) {
/* transfer 1 and 2 to work array */
i_1 = npt;
for (i = 1; i <= i_1; ++i) {
    wk[i + wk_dim1] = x[i + x_dim1];
    wk[i + (wk_dim1 << 1)] = x[i + (x_dim1 << 1)];
} /* L65: */
    scaler_(&wk[wk_offset], &npt, ilog, &dt, nmax);
}

if (iopt == 7) {
/* transfer from working to original */
i_1 = npt;
for (i = 1; i <= i_1; ++i) {
    temp1 = x[i + x_dim1];
    temp2 = x[i + (x_dim1 << 1)];
    x[i + x_dim1] = wk[i + wk_dim1];
    x[i + (x_dim1 << 1)] = wk[i + (wk_dim1 << 1)];
    wk[i + wk_dim1] = temp1;
    wk[i + (wk_dim1 << 1)] = temp2;
} /* L70: */
}

if (iopt == 8) {
    store_(&x[x_offset], &wk[wk_offset], &dt, &df, ilog, nmax, &c__1);
}
if (iopt == 9) {
    viewer_(&x[x_offset], &wk[wk_offset], &dt, &df, ilog, nmax, &c__1);
if (iopt == 10) {
    s_paus(" type DOS command or return", 27L);
    s_paus(" type DOS command or return", 27L);
}
    goto L40;

    return 0;
} /* prep_ */

/* Subroutine */ int cfft_(x, wk, wkl, nmax, ilog, ip)
real *x, *wk, *wkl;
integer *nmax, *ilog;
shortint *ip;
{
    /* System generated locals */
    integer x_diml, x_offset, wk_diml, wk_offset, i__l;

    /* Builtin functions */
    integer s_wsle(), do_lio(), e_wsle(), s_wsfe(), do_fio0, e_wsfe(),
            s_rsle(), e_rsle();
    /* Subroutine */ int s_paus();

    /* Local variables */
    static integer icol, nfft;
    static real temp, tmax;
    static integer iopt, nmax2, i, n;
    static real scale;
    extern /* Subroutine */ int store_(), input_();
    static real df, dt;
    static integer np;
    extern /* Subroutine */ int inreim_(), interp_(), fftprpj(), viewerj();
    static integer npt;

    /* Parameter adjustments */
    --ip;
    --wk1;
    wk_diml = *nmax;
    wk_offset = wk_diml + 1;
    wk -= wk_offset;
    x_diml = *nmax;
    x_offset = x_diml + 1;
    x -= x_offset;
    return 0;
Appendix II

Computer-Based Mathematical Programming

Through the history of computer programming, each programming language has developed a certain preferred style of good programming. While it is possible to solve the same problem in many different ways, there is usually some indication of what constitutes a good or a bad program. Mathematical programming on computers is a new way of solving complicated engineering problems.

Computer-based mathematical packages like Mathematica and Matlab are powerful analysis media that are applicable to all branches of science. This Appendix is devoted to present mathematical codes used throughout the thesis to solve complicated mathematical functions. These computer codes can only be run under the environment of the Mathematica and Matlab packages.

Timoshenko Beam Carrying an End-Mass

To solve the frequency function for the Timoshenko beam carrying an end-mass (Eq. 3-3-25),

\[ |C| = 0 \rightarrow m(k_2k_7 - k_6k_3 + k_1k_6 - k_2k_5) + n(k_3k_8 - k_4k_7 + k_4k_5 - k_4k_8) = 0 \] (3-3-25)
or

\[ f(\omega, E, G, \text{GEOMETRY}) = 0 \quad (3-3-26) \]

the following Mathematica log file can be used (The log file can only be run under the environment of the Mathematica package):

```
*************** Tim.m **********************

m1 = m/(ro*a*l)

s = Sqrt[(e*i)/((2/3)*a*g*l^2)]

b = Sqrt[(1/(e*i))*(ro*a/9.81)*(w^2)*(l^4)]

r = Sqrt[i/(a*l^2)]

alf = (l/Sqrt[2])*Sqrt[Sqrt[(r^2-s^2)^2+4/b^2]-(r^2+s^2)]

bet = (l/Sqrt[2])*Sqrt[Sqrt[(r^2-s^2)^2+4/b^2]+(r^2+s^2)]

r1 = (b/alf)*Sinh[b*alf]+m1*b^2*Cosh[b*alf]

r2 = (b/alf)*Cosh[b*alf]+m1*b^2*Sinh[b*alf]

r3 = (b/bet)*Sin[b*bet]+m1*b^2*Cos[b*bet]

r4 = -(b/bet)*Cos[b*bet]+m1*b^2*Sin[b*bet]

r5 = ((alf^2+s^2)/alf)*((b*alf+Cosh[b*alf]-1/2)*m1*(sig)^2)*b^2*Sinh[b*alf]

r6 = ((alf^2+s^2)/alf)*((b*alf+Sinh[b*alf]-1/2)*m1*(sig)^2)*b^2*Cosh[b*alf]

r7 = -(bet^2-s^2)/bet)*((b*bet+Cos[b*bet]-1/2)*m1*(sig)^2)*b^2*Sin[b*bet]

r8 = -(bet^2-s^2)/bet)*(-b*bet+Sin[b*bet]-1/2)*m1*(sig)^2)*b^2*Cos[b*bet]

f = ((alf^2+s^2)/alf)(r3*r8-r7*r4+r4*r5-r1*r8)+((bet^2-s^2)/bet)(r2*r7-r6*r3+r1*r6-r2*r5)
```
Matlab Finite Element Solver

In this section, a finite element solver code which has the capability of solving dynamic problems, is presented. The program calculates: natural frequencies of the problem in rad/s, the mode shape matrix, the diagonal matrix of natural frequencies, the reduced stiffness matrix corresponding to the unconstrained degrees of freedom, and the reduced mass matrix corresponding to unconstrained degrees of freedom only. Recalling the governing system of equations for dynamic motion describing undamped free vibration of a three-dimensional structure (Eq. 6-4-1),

\[ [K][u] + [M][\ddot{u}] = 0 \]  

which is a system of algebraic homogeneous equations where the non-trivial solutions exist when the determinant of the coefficients is zero. Therefore,

\[ \text{det} \{ [K] - \omega^2 [M] \} = 0 \]  

This is a standard eigenvalue problem which the Matlab finite element solver should be able to solve. There are as many frequencies as the order of the system equations. The natural mode shapes (eigenvectors), which describe the deformed shape of the structure at the corresponding natural frequencies, are obtained by substituting the eigenvalues into the system of equations above. Following is a complete listing of the FE.m source code.

**********FE.m**********

function [x,f]=FE(node,ncon,zero,force,conm)
clc
home
help FEHELP
disp(' ')
disp(' Hit return to continue')
pause

clc,home
if nargin==0
disp(' load datafile.')
end

end
filename=input('Enter name of file ','s');
eval(['load ','filename','.con']);
end

if nargin==1
    disp(' Loading datafile.')
    filename=node;
    eval(['load ','filename','.con']);
end
clc
home
disp(' Global Mass and Stiffness Matrices')
disp('')
snode=size(node);
k=zeros(3*snode(1));
m=zeros(3*snode(1));
sncon=size(ncon);
szero=size(zero);

% Assembly procedure
for ii=1:sncon(1)
    iis=num2str(ii);
    ke=zeros(6,6);
    me=zeros(6,6);
    n1=ncon(ii,1);
    n2=ncon(ii,2);
    x1=node(n1,1);
    y1=node(n1,2);
    x2=node(n2,1);
    y2=node(n2,2);
    theta=atan2(y2-y1,x2-x1);
    s=sin(theta);
    c=cos(theta);
    le=sqrt((y2-y1)^2+(x2-x1)^2);
    E=ncon(ii,3);
    A=ncon(ii,4);
    I=ncon(ii,5);
    G=ncon(ii,6);
    Rho=ncon(ii,7);
    if I==0
        I=1e-8*A;
    end
    R=A^2/I;
    alpha=1.5;
  theta;
  S=G*A^2/12/E/I/alpha;
  if S<1e-10
    mes1=[' E-B Beam ',iis,'.'];
    disp(mes1);
    ke(1,1)=R*c^2+12*s^2;
    ke(1,2)=c^2*(R-12);
    ke(2,2)=R*s^2+12*c^2;
    ke(1,3)=-6*I*e*s;
    ke(2,3)=6*I*e*c;
ke(3,3)=4*le^2;
ke(3,6)=2*le^2;
else
ke(1,1)=R*c^2+12/(1+1/S)*s^2;
ke(1,2)=c*s*(R-12/(1+1/S));
ke(2,2)=R*s^2+12/(1+1/S)*c^2;
ke(1,3)=-6*le*s/(1+1/S);
ke(2,3)=6*le*c/(1+1/S);
ke(3,3)=4*le^2*(1+1/S)/(1+1/S);
ke(3,6)=2*le^2*(1-1/2/S)/(1+1/S);
end
ke(1,4)=-ke(1,1);
ke(1,5)=-ke(1,2);
ke(1,6)=ke(1,3);
ke(2,4)=-ke(1,2);
ke(2,5)=ke(2,2);
ke(2,6)=ke(2,3);
ke(3,4)=-ke(1,3);
ke(3,5)=-ke(2,3);
ke(4,4:5)=ke(l,1:2);
ke(4,6)=-ke(l,3);
ke(5,5)=ke(2,2);
ke(5,6)=-ke(2,6);
ke(6,6)=ke(3,3);
ke=ke*E*I/le^3;
ke=ke+ke'-diag(diag(ke));
me(1,1)=140*c^2+156*s^2;
me(1,2)=-16*c*s;
me(1,3)=-22*le*s;
me(1,4)=70*c^2+54*s^2;
me(1,5)=16*c*s;
me(1,6)=13*le*s;
me(2,2)=140*s^2+156*c^2;
me(2,3)=22*le*c;
me(2,4)=me(1,5);
me(2,5)=70*s^2+54*c^2;
me(2,6)=-13*le*c;
me(3,3)=4*le^2;
me(3,4)=me(1,6);
me(3,5)=me(2,6);
me(3,6)=3*le^2;
me(4,4)=me(1,1);
me(4,5)=me(1,2);
me(4,6)=me(1,3);
me(5,5)=me(2,2);
me(5,6)=me(2,3);
me(6,6)=me(3,3);
me=me*Rho*le/420;
me=me+me'-diag(diag(me));
theta;
ke;
me;
\[ \begin{align*}
pl &= (n1-1)\times3+1; \\
p2 &= (n2-1)\times3+1; \\
k(p1:p1+2, p1:p1+2) &= k(p1:p1+2, p1:p1+2) + ke(1:3,1:3); \\
k(p1:p1+2, p2:p2+2) &= k(p1:p1+2, p2:p2+2) + ke(1:3,4:6); \\
k(p2:p2+2, p1:p1+2) &= k(p2:p2+2, p1:p1+2) + ke(4:6,1:3); \\
k(p2:p2+2, p2:p2+2) &= k(p2:p2+2, p2:p2+2) + ke(4:6,4:6); \\
m(p1:p1+2, p1:p1+2) &= m(p1:p1+2, p1:p1+2) + me(1:3,1:3); \\
m(p1:p1+2, p2:p2+2) &= m(p1:p1+2, p2:p2+2) + me(1:3,4:6); \\
m(p2:p2+2, p1:p1+2) &= m(p2:p2+2, p1:p1+2) + me(4:6,1:3); \\
m(p2:p2+2, p2:p2+2) &= m(p2:p2+2, p2:p2+2) + me(4:6,4:6); \\
\end{align*} \] 

end

\%
\%
concentrated masses.

sconm=size(conm);
if sconm(l)~=0
for i=1:sconm(1)
    loc1=(conm(i,1)-1)*3+1; \\
    loc2=(conm(i,1)-1)*3+2; \\
    loc3=(conm(i,1)-1)*3+3; \\
    m(loc1,loc1) = conm(i,2) + m(loc1,loc1); \\
    m(loc2,loc2) = conm(i,2) + m(loc2,loc2); \\
    m(loc3,loc3) = conm(i,3) + m(loc3,loc3);
end

\%

Zeroing stiffness matrix and mass matrix.

clc
home
disp(' Applying Boundary Conditions.')
disp('')
kl=k;
m1=m;
if length(zero)==0
    np=(zero(:,1)-D*3+zero(:,2);
    np=sort(np);
    p=1:3*snode;
    p(np)=p(np)*0;
    p=p(length(np)+1:length(p));
    k1=k(p,p);
    m1=m(p,p);
end
clc,home
selx=1:3:snode(1)*3; \\
selfy=2:3:snode(1)*3; \\
selfz=3:3:snode(1)*3; \\
answer=input(' static or dynamic analysis? (s/d) ', 's');
if answer=='s' || answer=='S'
    pf=(force(:,1)-1)*3+force(:,2);
    f=zeros(snode(1)*3,1);
    f(pf)=force(:,3);
end
fl=zeros(length(kl),1);
fl=f(p);
x1=k1v1;
x=zeros(snode(l)*3,1);
x(p)=x1;
f=k*x;
xx=[x(selx) x(sely) x(selt)];
ff=[f(selx) f(sely) f(selt)];
else
[vk,dk]=eig(kl);
flag=0;
for ij=1:length(kl)
    disp('Working')
    if dk(ij,ij)<le-14
        flag=1;
        keig=num2str(dk(ij,ij));
        dk(ij,ij)=le-14;
        disp(' Numerical roundoff error occurred')
        disp(' Eigenvalue of stiffness matrix changed from')
        disp([''' keig ' to 0'])
    end
end
if flag ==1
    disp(' Press return to continue'),pause
    disp(' Rebuilding corrected stiffness matrix'),pause(1)
    kl=vk*dk*vk';
end
[x1,wsq]=eig(kl,m1);
fl=wsq.^5;
/fs1,sfl]=sort(diag(fl));
f1=diag(fs1);
x1=x1(:,sfl);
snode;
x=zeros(snode(l)*3,snode(l)*3-szero(l));
if length(zero)--=0
    for ic=1:length(x1)
        x(p,ic)=x1(:,ic);
    end
else
    for ic=1:length(x1)
        x(:,ic)=x1(:,ic);
    end
end
ff=diag(fl);
f=ff;
xx=x;
end
clc,home

answer1=input('Save Results? (y/n) ','s');
if answer1=='y'
    if exist('filename')==0
        filename=input('Enter name of file, ','s');
    end
end
end
eval(['save ',"filename"'.out',' x',' f']);
end

answer2=input('Save Equations? (y/n) ','s');
if answer2=='y'
    if exist('filename')==0
        filename=input('Enter name of file. ','s');
    end
    p=p';
eval(['save ',"filename"'.eqn',' kl',' ml',' xl',' fl',' p']);
end

answer=input('Show results graphically? (y/n) ','s');
if answer=='y'
    FE3(node,x,zero,ncon,p,f);
end
Appendix III

Source Code Listing

This appendix contains the source code listing of FSPECN used in the non-linear complex spectral analysis. As the complete source code listing of computer programs used in the thesis (FSPEC, FSPECN, SBASIS) is lengthy and they would take many pages, only the implementation of the special topics is covered herein.

FSPECN (Non-Linear Fractional-Spectral) Source Code

Recalling the time domain governing equation of motion for a non-linear damped system (Eq. 6-6-1),

\[ m\ddot{u}(t) + c\dot{u}(t) + ku(t) + Q(t) = F(t) \]  

(6-6-1)

where the non-linear contribution can be of the form of mass, stiffness or damping non-linearity. As discussed in Chapter Six, to enable the frequency domain fractional-spectral method to solve non-linear dynamic problems, the following non-linear equation of motion should be solved.

\[ \left[ \hat{F}_n \right] + \eta(\omega, \hat{u}_n) = \left[ \hat{K}_n \right] \left[ \hat{u}_n \right] \]  

(6-6-5)

Following is a complete listing of the FSPECN.c source code. The program is capable of doing the non-linear dynamic analysis and is written in standard C language on a personal computer IBM-AT compatible machine. The input data file can be in free format with blanks or commas used as separators.
FSPECN.c
/* FSpecn.c-- (version 1.00 1995).
   Dept. of Civil Engrg., Univ. of Wollongong */

#include "f90.h"

/* Block Declarations */

struct {
    integer kin, kout;
} forio_

#define forio_l forio_

struct {
    doublereal tsi, tsr;
} step_

#define step_l step_

struct {
    doublereal xth, time, ptsr, ulf;
    integer idat, indgacc;
} load_

#define load_l load_

struct{
    doublereal dcxth, dsxth;
} loadxt_

#define loadxt_l loadxt_

struct {
    doublereal wbet, wgam;
} hys_

#define hys_l hys_

struct{
    doublereal a1, a2, a3, a4, a5;
} conl_

#define conl_l conl_

struct {
    doublereal b1, b2, b3, b4, b5;
} con2_

#define con2_l con2_

struct para_l_ {
    doublereal c1, c2, gama, beta, y[2];
}
Appendix III: Source Code Listing

```c
#define para_l (*(struct para_l_ *) &para_)

struct {
    double real a1, a2, a3;
} conul_

#define conul_l conul_

/* Initialized data */

struct{
    doublereal e_l[2];
    doublereal fill_2[4];
} para_ = { .788675134595, -1.15470053838 };

/* Table of constant values */

static integer c__3 = 3;
static integer c__2 = 2;
static integer c__1 = 1;
static real c_b65 = (float).1;

/****************************************************BEARING*************/
/* Subroutine */ int bearingj(err, fx, fy, xp, yp, vn, vnp, an, anp, fh, it, zx, zy, fnxy, alp, yf, yd, fmax, df, pa, inelem, ne, np, delf)
integer *it;
integer *inelem, *ne, *np;
doublereal *delf;
{

/* System generated locals */
integer alp_diml, alp_offset, yf_diml, yf_offset, yd_dunl, yd_offset,
fmax_diml, fmax_offset, df_diml, df_offset, pa_diml, pa_offset,
inelem_diml, inelem_offset, _i_1;

/* Local variables */
static doublereal delf1, delf2, delf3, temp1[6] /* was [3][2] */;
temp2[3] /* was [3][1] */, temp3[3] /* was [3][1] */;
temp4[3] /* was [3][1] */, temp5[3] /* was [3][1] */;
static integer i, j;
extern /* Subroutine */ int hysub(), tmult_();
static doublereal fr[2], tp[9] /* was [3][3] */, dum1, dum2, dum3;

/****** HYSUB ****/
/* Subroutine */ int hysub_(dn, vn, vnp, fxy, i, zxx, zyy, fnxy, alp, yf, yd,
fmax, df, pa, inelem, np)
doublereal *dn, *vn, *vnp, *fxy;
integer *i;
```

integer *inelem, *np;
{
  /* System generated locals */
  integer alp_diml, alp_offset, yf_diml, yf_offset, yd_diml, yd_offset,
       fmax_diml, fmax_offset, df_diml, df_offset, pa_diml, pa_offset,
       inelem_diml, inelem_offset;
  doublereal d__1, d__2;

/* Builtin functions */
  double exp(), sqrt();

/* Local variables */
  static doublereal velc;
  extern /* Subroutine */ int uniaxial_();
  static doublereal fmewl, fmew2, v[4] /* was [2][2] */, v1, v2, yd1, yd2;
  extern /* Subroutine */ int biaxial_();
  static doublereal zxy;

/***** BEARING FORCES *************/

/* Parameter adjustments */
  dn -= 4;
  --vn;
  --vnp;
  --fxv;
  inelem_diml = *np;
  inelem_offset = inelem_diml + 1;
  inelem = inelem_offset;
  pa_diml = *np;
  pa_offset = pa_diml + 1;
  pa = pa_offset;
  df_diml = *np;
  df_offset = df_diml + 1;
  df = df_offset;
  fmax_diml = *np;
  fmax_offset = fmax_diml + 1;
  fmax = fmax_offset;
  yd_diml = *np;
  yd_offset = yd_diml + 1;
  yd = yd_offset;
  yf_diml = *np;
  yf_offset = yf_diml + 1;
  yf = yf_offset;
  alp_diml = *np;
  alp_offset = alp_diml + 1;
  alp = alp_offset;
  --fnxy;
  --zyy;
  --zxx;

/* Function Body */
  para_1.gama = (float).9;
  para_1.beta = (float).1;
para_1.y[0] = yd[*i + yd_dim1];
para_1.y[1] = yd[*i + (yd_dim1 << 1)];
v1 = (vn[1] + vn[1]) / 2;
v2 = (vn[2] + vn[2]) / 2;
v[0] = v1;
v[1] = v2;
v[2] = v1;
v[3] = v2;
if (inelem[*i + inelem_dim1] == 3) {
    biaxial_(i, v, &zxx[1], &zyy[1], np);
}
if (inelem[*i + inelem_dim1] == 1) {
    yd1 = para_1.y[0];
zxy = zxx[*i];
uniaxial_(&v1, &zxy, &yd1);
zxx[*i] = zxy;
zyy[*i] = (float)0.;
} else if (inelem[*i + inelem_dim1] == 2) {
    yd2 = para_1.y[1];
zxy = zyy[*i];
uniaxial_(&v2, &zxy, &yd2);
zyy[*i] = zxy;
zxx[*i] = (float)0.;
}
if (inelem[*i + (inelem_dim1 << 1)] == 3) {
    fxy[1] = alp[*i + alp_dim1] * yf[*i + yf_dim1] / yd[*i + yd_dim1] * 
              dn[7] + (1 - alp[*i + alp_dim1]) * yf[*i + yf_dim1] * zxx[*i];
    fxy[2] = alp[*i + (alp_dim1 << 1)] * yf[*i + (yf_dim1 << 1)] / yd[*i + 
              (yd_dim1 << 1)] * dn[8] + (1 - alp[*i + (alp_dim1 << 1)]) * 
              yf[*i + (yd_dim1 << 1)] * zyy[*i];
    if (inelem[*i + inelem_dim1] == 1) {
        fxy[2] = 0.;
    } else if (inelem[*i + inelem_dim1] == 2) {
        fxy[1] = 0.;
    }
}
if (inelem[*i + (inelem_dim1 << 1)] == 4) {
    if (inelem[*i + inelem_dim1] == 1 || inelem[*i + inelem_dim1] == 2) {
        fmew1 = fmax[*i + fmax_dim1] - df[*i + df_dim1] * exp(-pa[*i + 
              pa_dim1] * abs(vn[1]));
        fmew2 = fmax[*i + (fmax_dim1 << 1)] - df[*i + (df_dim1 << 1)] * 
              exp(-pa[*i + (pa_dim1 << 1)] * abs(vn[2]));
    } else if (inelem[*i + inelem_dim1] == 3) {
        /* Computing 2nd power */
        d__1 = vn[1];
        /* Computing 2nd power */
        d__2 = vn[2];
        velc = sqrt(d__1 * d__1 + d__2 * d__2);
        fmew1 = fmax[*i + fmax_dim1] - df[*i + df_dim1] * exp(-pa[*i + 
              pa_dim1] * abs(velc));
        fmew2 = fmax[*i + (fmax_dim1 << 1)] - df[*i + (df_dim1 << 1)] * 
              exp(-pa[*i + (pa_dim1 << 1)] * abs(velc));
    }
    fxy[1] = fmew1 * fnxy[*i] * zxx[*i];
    fxy[2] = fmew2 * fnxy[*i] * zyy[*i];
return 0;
} /* hysub_ */

/* ***************MOBILISED FORCES*******************/

/* Parameter adjustments */
v -= 3;
--zyy;
--zxx;

/* Function Body */
t = step_l.tsi;
zx[0] = zxx[*i];
zy[0] = zyy[*i];
const_(&v[3], &v[4], zx, zy);
aji[0] = para_l.c1 * t * (con2_l.b2 * 2 * zy[0] + con2_l.b3 * 2 * zy[0] +
  con2_l.b4 * zx[0] + con2_l.b5 * zx[0]) + 1;
aji[3] = para_l.c1 * t * (con1_l.a2 * 2 * zx[0] + con1_l.a3 * 2 * zx[0] +
  con1_l.a4 * zy[0] + con1_l.a5 * zy[0]) + 1;
aji[2] = -para_l.c1 * t * (con1_l.a4 * zx[0] + con1_l.a5 * zx[0]);
aji[1] = -para_l.c1 * t * (con2_l.b4 * zy[0] + con2_l.b5 * zy[0]);
for (ii = 1; ii <= 2; ++ii) {
  for (jj = 1; jj <= 2; ++jj) {
    aji[ii + (jj « 1) - 3] /= daji;
  }
}

/* Computing 2nd power */
d_1 = zx[0];
/* Computing 2nd power */
d_2 = zx[0];
zp[0] = con1_l.a1 - con1_l.a2 * (d_1 * d_1) - con1_l.a3 * (d_2 * d_2) +
  con1_l.a4 * zy[0] - con1_l.a5 * zy[0];
/* Computing 2nd power */
d_1 = zy[0];
/* Computing 2nd power */
d_2 = zy[0];
zp[1] = con2_l.b1 - con2_l.b2 * (d_1 * d_1) - con2_l.b3 * (d_2 * d_2) +
  con2_l.b4 * zy[0] - con2_l.b5 * zy[0];
for (ii = 1; ii <= 2; ++ii) {
  for (jj = 1; jj <= 2; ++jj) {
    sum = 0.;
    for (jj = 1; jj <= 2; ++jj) {
      sum += aji[ii + (jj « 1) - 3] * zp[jj - 1] * t;
    }
    rk[ii - 1] = sum;
  }
}
/* L60: */

/* Computing 2nd power */
zx[1] = zy[0] + para_l.c2 * rk[0];
zy[1] = zy[0] + para_l.c2 * rk[1];
const_(&v[5], &v[6], &zx[1], &zy[1]);
/* Computing 2nd power */
\[
d_1 = \text{zx}[1];
\]
/* Computing 2nd power */
\[
d_2 = \text{zx}[1];
\]
\[
zp[2] = \text{conl}_1.a1 - \text{conl}_1.a2 * (d_1 * d_1) - \text{conl}_1.a3 * (d_2 * d_2)
- \text{conl}_1.a4 * \text{zx}[1] * \text{zy}[1] - \text{conl}_1.a5 * \text{zx}[1] * \text{zy}[1];
\]
/* Computing 2nd power */
\[
d_1 = \text{zy}[1];
\]
/* Computing 2nd power */
\[
d_2 = \text{zy}[1];
\]
\[
zp[3] = \text{con2}_1.b1 - \text{con2}_1.b2 * (d_1 * d_1) - \text{con2}_1.b3 * (d_2 * d_2)
- \text{con2}_1.b4 * \text{zx}[1] * \text{zy}[1] - \text{con2}_1.b5 * \text{zx}[1] * \text{zy}[1];
\]
for (ii = 1; ii <= 2; ++ii) {
    sum = 0.0;
    for (jj = i; jj <= 2; ++jj) {
        sum += aji[ii + (jj « 1) - 31] * zp[jj + 1] * t;
    }
    rl[ii - 1] = sum;
} /* L100; */
/* L120: */
\[
xz[0] = \text{zx}[0] + \text{rk}[0] * (\text{float}.75 + \text{rl}[0]) * (\text{float}.25);
\]
\[
zy[0] = \text{zy}[0] + \text{rk}[1] * (\text{float}.75 + \text{rl}[1]) * (\text{float}.25);
\]
\[
\text{zxx}[*i] = \text{zx}[0];
\]
\[
\text{zyy}[*i] = \text{zy}[0];
\]
return 0;
} /* biaxial_ */

******************************************************************************
GAUSS **********
/* Subroutine */ int gaussja, b, neq, leq, 11, m)
 doublereal *a, *b;
 integer *neq, *leq, *U, *m;
 {
 /* Format strings */
 static char fmt_2000[] = "";
 /* System generated locals */
 integer a_dim1, a_offset, b_dim1, b_offset, i_1, i_2, i_3;
 /* Builtin functions */
 integer s_wsfe(), do_fio(), e_wsfe();
 /* Subroutine */ int s_stop();
 /* Local variables */
 static doublereal d;
 static integer i, j, l, n, n1;
 /* Fortran I/O blocks */
 static cilist io_64 = { 0, 6, 0, fmt_2000, 0 };

******************************************************************************
SOLVER*****************************/
/* Parameter adjustments */
  a_diml = *neq;
  a_offset = a_diml + 1;
  a -= a_offset;
  b_diml = *neq;
  b_offset = b_diml + 1;
  b -= b_offset;

/* Function Body */
if (*m == 3) {
  goto L800;
}
if (*m == 2) {
  goto L500;
}

/* --- TRIANGULARIZATION */
i__1 = *leq;
for (n = 1; n <= i__1; ++n) {
  if (n == *neq) {
    goto L400;
  }
  d = a[n + n * a_diml];
  if (d != (float)0.) {
    goto L100;
  }
  s_wsfe(&io__64);
  do_fio(&c_l, (char *)&n, (ftnlen)sizeof(integer));
  e_wsfeO;
  s_stop("", OL);
  L100:
  nl = n + 1;
  i__2 = *neq;
  for (j = nl; j <= i__2; ++j) {
    if (a[n + j * a_diml] == (float)0.) {
      goto L300;
    }
  }
  if (*neq != 1) {
    a[*neq + a_diml] = (doublereal) (*leq);
  }
  a[i + j * a_diml] = a[i + j * a_diml];
  L300:
  a[j + i * a_diml] = a[i + j * a_diml];
  L400:
  }
/* -----FORWARD REDUCTION */
L500:
if (*neq != 1) {
    *leq = (integer) a[*neq + a_dim1];
}

i_1 = *leq;
for (n = 1; n <= i_1; ++n) {
    if (n == *neq) {
        goto L650;
    }
    n1 = n + 1;
    i_2 = *11;
    for (1 = 1; 1 <= i_2; ++1) {
        i_3 = *neq;
        for (i = n1; i <= i_3; ++i) {
            /* L600: */
            b[i + 1 * b_dim1] -= a[n + i * a_dim1] * b[n + 1 * b_dim1];
        }
    }
    /* L650: */
    i_3 = *ll;
    for (1 = 1; 1 <= i_3; ++1) {
    }
    /* L675: */
    b[n + 1 * b_dim1] /= a[n + n * a_dim1];
}

/* L700: */
if (*m == 2) {
    return 0;
}

/* -----BACK-SUBSTITUTION */
L800:

n = *neq;
if (*neq != 1) {
    *leq = (integer) a[*neq + a_dim1];
}

if (*leq != *neq) {
    n = *leq + 1;
}

L810:

n1 = n;
for (n = 0) {
    /* L900: */
    b[n + 1 * b_dim1] -= a[n + j * a_dim1] * b[j + 1 * b_dim1];
    goto L810;
} /* gauss_ */
/* ****************************************** EIGENVALUE SOLVER ****************************************** */
/* -----INITIALIZATION */
/* Parameter adjustments */
-e;
  x_dim1 = n;
  x_offset = x_dim1 + 1;
  x = x_offset;
  b_dim1 = n;
  b_offset = b_dim1 + 1;
  b = b_offset;
  a_dim1 = n;
  a_offset = a_dim1 + 1;
  a = a_offset;

/* Function Body */
nt = 0;
nn = n - 1;
i_1 = nfig <= 1;
rtol = pow_r(i &c_b65, &i_1);
eps = (float)0.1;
i_1 = n;
for (i = 1; i <= i_1; ++i) {
  i_2 = n;
  for (j = 1; j <= i_2; ++j) {
    /* L20: */
    x[i + j * x_dim1] = (float)0.0;
  }
  /* L30: */
  x[i + i * x_dim1] = (float)1.0;
}
if (*n == 1) {
  goto L820;
}
/* -----Sweep off-diagonals for reduction */
i_1 = nsmax;
for (m = 1; m <= i_1; ++m) {
  ymax = (float)0.0;
i_2 = nn;
  for (j = 1; j <= i_2; ++j) {
    jji = j + 1;
i_3 = n;
    for (k = jj; k <= i_3; ++k) {
      /* ---compare with threshold value */
      if (a[k + k * a_dim1] <= (float)0.) {
        goto L1000;
      }
      if (b[k + k * b_dim1] <= (float)0.) {
        goto L1000;
      }
ea = (d_1 = a[j + k * a_dim1] / a[j + j * a_dim1] * (a[j + k * a_dim1] / a[k + k * a_dim1]), abs(d_1));
  eb = (d_1 = b[j + k * b_dim1] / b[j + j * b_dim1] * (b[j + k * b_dim1] / b[k + k * b_dim1]), abs(d_1));
  y = ea + eb;
if (y > ymax) {
    ymax = y;
}
if (y < eps) {
    goto L700;
}

/************************ DISPLACEMENTS ********/
/* Subroutine */ int driftjtime, xdl, ydl, rdl, xd2, yd2, rd2, pxdl, pydl,
    * pydl, * cx1, * cy1;
{
    /* System generated locals */
    double d_l;
    /* Local variables */
    static double axdl, aydl;
    /* ****************************
    axdl = (d_l = * xdl + * rdl * * cx1 - * xd2 - * rd2 * * cx1, abs(d_l));
    aydl = (d_l = * ydl + * rdl * * cx1 - * yd2 - * rd2 * * cx1, abs(d_l));
    if (axdl > * pxdl) {
        * pxdl = axdl;
        * pxdtl = * time;
    }
    if (aydl > * pydl) {
        * pydl = aydl;
        * pydtl = * time;
    }
    return 0;
} /* drift */

/************************ ACCL *********/
/* Subroutine */ int accljtime, f, nf, knf, sfxl, sfyl, sfxtl, sfytl)
    * time, * f;
    integer * nf, * knf;
    double * sfxl, * sfyl, * sfxtl, * sfytl;
{
    /* System generated locals */
    integer i_1;
    /* Local variables */
    static integer i, j;
    static double sum, fxdl, fydl;
    /* Parameter adjustments */
    --f;
    /* Function Body */
    sum = (float)0.;
    i_1 = * knf;
for (i = 1; i <= i_l; ++i) {
    j = (i - 1) * 3 + 1;
    sum += f[j];
} /* L1: */

fxdl = sum;
sum = (float)0.;
i__l = *knf;
for (i = 1; i <= i__l; ++i) {
    j = (i - 1) * 3 + 2;
    sum += f[j];
} /* L2: */

fydl = sum;
if (abs(fxd1) > abs(*sfx1)) {
    *sfx1 = fxd1;
    *sfxtl = *time;
}
if (abs(fyd1) > abs(*sfy1)) {
    *sfy1 = fyd1;
    *sfytl = *time;
}
return 0;
} /* accl_ */
Appendix IV

Hyperelastic Modelling of Rubber Like Materials

The mathematical modelling of the large elastic strain of a rubber-like material has been considered by many authors (Hill et. al. 1989, Rivlin 1984 and others). The load-deflection relation is derived assuming an isotropic incompressible Mooney material which undergoes a plane strain deformation. In engineering, finite rectangular or circular rubber blocks are used as damping devices in structures. Typically, such materials have bonded metal plates on their upper and lower surfaces, and the three principal deformation modes of concern are compression, shearing and rotation, which are effected by fixing the lower metal plate and applying forces or moments to the upper plate or vice versa.

One of the mathematical models of rubber like materials which can be adapted into the finite element method is a hyperelastic model. Hyperelasticity refers to materials whose stresses are derived from their total strains using a strain energy density function. These materials are assumed to be isotropic and elastic, so as no permanent deformation occurs.

Consider the stress component of the second Piola-Kirchhoff stress tensor for a rubber material as,
Appendix IV: Hyperelastic Modelling of Rubber Like Materials

\[ S_{ij} = \frac{\partial P}{\partial E_{ij}} \]  (IV-1)

where \( P \) is the strain energy function per unit of undeformed volume, and \( E_{ij} \) are the components of the Lagrangian strain tensor (Rivilin, 1984). The Lagrangian strain may be expressed as follows,

\[ E_{ij} = \frac{1}{2} (C_{ij} - \delta_{ij}) \]  (IV-2)

\[ C_{ij} = f_{ki} f_{kj} \]

where \( f_{ki} \) and \( f_{kj} \) are components of the right Cauchy-Green deformation tensor and \( \delta_{ij} \) is the Kroneker Delta. The strain energy density function can now be written as (Rivilin 1984, Mooney 1940),

\[ P = A(J_1 - 3) + B(J_2 - 3) + C(J_1 - 3)^2 + D(J_1 - 3)(J_2 - 3) + E(J_2 - 3)^2 + F(J_1 - 3)^3 + G(J_1 - 3)^2(J_2 - 3) + H(J_1 - 3)(J_2 - 3)^2 + I(J_2 - 3)^3 \]

\[ + \frac{A + B}{1 - 2\nu} (J_3 - 1)^2 \]  (IV-3)

where \( J_1, J_2 \) and \( J_3 \) are,

\[ J_1 = C_{ij} [\det C_{ij}]^{-\frac{1}{3}} \quad J_2 = \left[ 0.5(C_{ij}^2 - C_{ij}^2) [\det C_{ij}]^{-\frac{2}{3}} \quad J_3 = [\det C_{ij}] \right] \]  (IV-4)

and \( A \) through \( I \) are constants of the nine parameter cubic Mooney-Rivlin relationship and \( \nu \) is the Poisson’s ratio. Stresses can now be computed from the second Piola-Kirchoff stresses as,

\[ \sigma_{ij} = \frac{f_{ik} S_{kj} f_{jl}}{\sqrt{\det C_{ij}}} \]  (IV-5)

and strains can be computed using Eq. (IV-2). It should be noted that the Mooney-Rivlin constants can be determined by writing a computer program (macro) in
commercial finite element packages (ANSYS for the study herein). An ANSYS macro for calculating Mooney-Rivlin constants will be presented next.

**Hyper.mac**

```
! READ EXPERIMENTAL DATA
!
mooney=ARG3
*IF.mooney.ge.n,THEN
! ***************
*MSG,ERROR,n,mooney
The Number of Data Points ( %I ) Is Not Greater Than the Requested&%/Number of Mooney Rivlin Constants ( %I ).
*ENDIF
*IF,n.lt,(2*mooney),THEN
!***************
*MSG,WARN,n,mooney
The Number of Data Points ( %I ) Is Not At Least Twice the Requested&%/Number of Mooney Rivlin Constants ( %I ). %/Check Results Carefully.
*ENDIF
data=ARGl
*IF.data.gt,2,THEN
!***************
*MSG,ERROR,data
%I is not a valid data type number
*ENDIF
material=ARG4
*VREAD,z2(1),HYPER,DAT
(*f12.3)
*MFUN,x2(1),COPY,z2(1)
*MFUN,y2(1),COPY,z2(n+1)
*IF.data,EQ,2.,THEN
*VOPER,x(l),x2(l),ADD,l
*ELSE
*MFUN,x(1),COPY,x2(1)
*MFUN,y(1),COPY,y2(1)
*ENDIF
```

The following are the lambda input values

```
! ***************
*STATUS,x(1)
*MSG,INFO
*MSG,INFO
```

The following are the true strain input values

```
! ***************
*STATUS,y(1)
```
**Appendix IV: Hyperelastic Modelling of Rubber Like Materials**

**DETERMINE MATRIX VALUES**

*DO,i,l,n
count(i)=i

\[ z=x(i) \]
\[ f(i)=2*z**2-2/z \]
\[ g(i)=2*z**2 \]
\[ h(i)=f(i)*(2*z**2-6+4/z) \]

\[ px(i)=f(i)*(3*z**3-3+z**2+3/z**2) \]
\[ q(i)=f(i)*(4-6/z+2/z**3) \]
\[ r1=3*z**4-18*z**2+12*z \]
\[ r(i)=f(i)*(r1+27-36/z+12/z**2) \]
\[ s1=5*z**3-6*z**2+2-18*z+32 \]
\[ s(i)=f(i)*(s1-3/z+2+4/z**3) \]
\[ t1=8*z**2+3+2-18/z**3 \]
\[ t(i)=f(i)*(t1-18/z**2-6/z**3+5/z**4) \]
\[ u1=12+36/z**3+12/z**2 \]
\[ u(i)=f(i)*(u1-18/z**2+3/z**3) \]

\[ sumf=f(i)+sumf \]
\[ sumg=g(i)+sumg \]
\[ sumh=h(i)+sumh \]
\[ sump=px(i)+sump \]
\[ sumq=q(i)+sumq \]
\[ sumr=r(i)+sumr \]
\[ sums=s(i)+sums \]
\[ sumt=t(i)+sumt \]
\[ sumu=u(i)+sumu \]
\[ sumfsq=f(i)**2+sumfsq \]
\[ sumgsq=g(i)**2+sumgsq \]
\[ sumhsq=h(i)**2+sumhsq \]

**ENDDO**

**SOLVE NINE PARAMETER MOONEY-RIVLIN MODEL**

*ELSE*

\[ A3(l,l)=sumfsq \]
\[ A3(2,2)=sumgsq \]
\[ A3(3,3)=sumhsq \]
\[ A3(4,4)=sumpsq \]
\[ A3(5,5)=sumqsq \]
\[ A3(6,6)=sumrsq \]
\[ A3(7,7)=sumssq \]
\[ A3(8,8)=sumtsq \]
\[ A3(9,9)=sumusq \]
\[ A3(1,2)=sumtq \]
\[ A3(1,3)=sumur \]
\[ A3(1,4)=sumur \]
\[ A3(1,5)=sumur \]
\[ A3(1,6)=sumur \]
\[ A3(1,7)=sumur \]
\[ A3(1,8)=sumur \]
A3(1,9)=sumfu
A3(2,6)=sumgr
A3(2,7)=sumgs
A3(2,8)=sumgt
A3(2,9)=sumgu
A3(3,6)=sumhr
A3(3,7)=sumhs
A3(3,8)=sumht
A3(3,9)=sumhu
A3(4,6)=sumpr
A3(4,7)=sumps
A3(4,8)=sumpt
A3(4,9)=sumpu
A3(5,6)=sumqr
A3(5,7)=sumqs
A3(5,8)=sumqt
A3(5,9)=sumqu
A3(6,6)=sumrsq
A3(6,7)=sumrs
A3(6,8)=sumrt
A3(6,9)=sumru
A3(7,7)=sumssq
A3(7,8)=sumst
A3(7,9)=sumsu
A3(8,8)=sumtsq
A3(8,9)=sumtu
A3(9,9)=sumusq
*DO,i,l,9
  A30,i)=A3(i,j)
  *ENDDO
*ENDDO
B3(1)=sumyf
B3(2)=sumyg
B3(3)=sumyh
B3(4)=sumyp
B3(5)=sumyq
B3(6)=sumyr
B3(7)=sumys
B3(8)=sumyt
B3(9)=sumyu
*MOPER,C3(1),A3(1,1),SOLVE,B3(1)
/out
*MSG,INFO
  *MSG,INFO
  ***** THE DETERMINED MOONEY
RIVLIN CONSTANTS ARE: *****
*MSG,INFO,C3(1)
  C11 (D) = %G
  *MSG,INFO,C3(2)
  C02 (E) = %G
  *MSG,INFO,C3(3)
  C30 (F) = %G
  *MSG,INFO,C3(4)
  C20 (C) = %G
  *MSG,INFO,C3(5)
  C03 (I) = %G
/out,SCRATCH
test1=C3(1)+C3(2)
test2=C3(4)+C3(3)
test3=C3(4)+4*C3(6)
test4=C3(5)-9*C3(6)
*IF,test1,lt,0,then
  /out
    *MSG,WARN,test1
    For the Mooney-Rivlin Constants: C10 +
    C01 (A + B) is negative
    /out,SCRATCH
  *ENDIF
  *IF,test2,gt,0,then
    /out
      *MSG,WARN,test2
      For the Mooney-Rivlin Constants: C20 +
      C11 (C + D) is positive
      /out,SCRATCH
  *ENDIF
  *IF,C3(6),lt,0,then
    /out
      *MSG,WARN,test2
      For the Mooney-Rivlin Constants: C30
      (D) is negative
      /out,SCRATCH
  *ENDIF
  *IF,test4,gt,0,then
    /out
      *MSG,WARN,test2
      For the Mooney-Rivlin Constants: C20 -
      9*C30 (E - 9*F) is positive
      /out,SCRATCH
  *ENDIF
  tb,hyper,material
  tbdat,1,C3(1)
tbdat,2,C3(2)
tbdat,3,C3(3)
Appendix IV: Hyperelastic Modelling of Rubber Like Materials

tbdat,4,C3(4)
tbdat,5,C3(5)
tbdat,6,C3(6)
tbdat,7,C3(7)
tbdat,8,C3(8)
tbdat,9,C3(9)
*DO,i,l,
ycl=C3(1)*f(i)+C3(2)*g(i)+C3(3)*h(i)
cy2=C3(4)*px(i)+C3(5)*q(i)+C3(6)*r(i)
cy3=C3(7)*s(i)+C3(8)*t(i)+C3(9)*u(i)
yc(i)=ycl+yc2+yc3
*IF,y(i),ne,0,then
  error(i)=100*(y(i)-yc(i))/y(i)
*ELSE
  error(i)=0
*ENDIF
ycsq(i)=(y(i)-yc(i))**2
ycsum=ycsq(i)+ycsum
ymean=sumy/n
ymsq(i)=(y(i)-ymean)**2
ymsum=ymsq(i)+ymsum
*ENDDO

Following is a complete listing of the input data file for the Hyper.mac program.

*****Input Data File***********

<table>
<thead>
<tr>
<th>Data Point</th>
<th>Stretch Ratio (Lambda)</th>
<th>True Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>1.3</td>
<td>7.6</td>
</tr>
<tr>
<td>3</td>
<td>2.1</td>
<td>15.3</td>
</tr>
<tr>
<td>4</td>
<td>2.7</td>
<td>21.0</td>
</tr>
<tr>
<td>5</td>
<td>3.4</td>
<td>27.3</td>
</tr>
<tr>
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The Coefficient of Determination= %G
The Following Hyperelastic Data Table Was Generated

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Appendix V

Further Review of Past Work

V.1 ADDED VISCOELASTIC DAMPERS

Viscoelastic dampers, made from acrylic copolymers materials, have been extensively used in wind vibration control, whereas application in earthquake-resistant design are mostly at experimental and analytical stages. Bergman and Hanson (1986), demonstrated with an extensive experimental study that the damping and stiffness of viscoelastic dampers are strongly dependent on excitation frequency, shear strain amplitude and cumulative energy dissipated. Viscoelastic dampers are required to undergo higher strains in seismic application and must provide high damping as compared with 2% to 4% of critical damping required for wind vibration control. However, they concluded that viscoelastic dampers can be used to provide significant damping to the structure to improve its response in an earthquake.

Mahmoodi and Keel (1986), conducted an extensive study aimed at investigating the feasibility of viscoelastic dampers for enhancing earthquake resistance of structures in a large-scale experimental and analytical program. The dampers proved to be effective in controlling earthquake induced vibration by providing increased damping to the structure, and exhibited good aging and environmental properties. These viscoelastic dampers (installed in the bracing) have been successfully used for the control of vibration motions in the World Trade Center building in New York City, the Columbia SeaFirst building in Seattle and the Number Two Union Square building in Seattle.
dynamic characteristics and the seismic response of viscoelastically-damped structures can be simply and accurately predicted by using the modal strain energy method, and that conventional linear dynamic analysis can be used at all levels of ambient temperature and earthquake ground motions.

Zhang and Soong (1992), studied the design process for viscoelastic dampers with emphasis on the optimal placement of dampers in the structure. Viscoelastic dampers were designed assuming predominate first mode response of the structure system, with the dampers distributed over the stories using an assumed linear or actual mode shape. The methodology for optimal placement of dampers in the structure uses a controllability index, based on maximum relative displacement response of the structure, leading to a sequential procedure for optimal placement of dampers. Based on experimental data, linear material properties were assumed for the shear storage and shear-loss stiffness of the viscoelastic material. They found that optimally located dampers can produce a response reduction which is equal to or less than that for the configuration with dampers provided uniformly over the height of the structure.

The problem of optimal design of viscoelastic dampers has also been addressed by Kelly (1991) and Aiken et al. (1991), who obtained a closed-form solution for minimum base shear response. A simple design approach assuming a dominant first mode response of the structure was used, employing constant velocity spectra and empirical damping modification relationships. The design methodology considers the dependency of mechanical properties of the viscoelastic material on temperature, frequency and strain amplitude.

A detailed design procedure for viscoelastically-damped structures using a modal strain energy method have also been proposed by Chang et al. (1993). The
The feasibility of using viscoelastic dampers to reduce earthquake induced structural response, including the effects of temperature on the material characteristics, have also been studied by Soong et al. (1990). A three story steel frame was subjected to earthquake motions on a shaking table, with the top two stories rigidly braced to simulate single degree-of-freedom behaviour. Compared with the undamped structure case, reductions in the relative displacements ranged from 75% to 80% with a corresponding decrease in the absolute acceleration of approximately 60%. Special attention was paid to the temperature of the environment and damper positioning. Significant temperature dependency of the viscoelastic material was reported. As the temperature rises, the damping factor increases and the natural frequency of the system decreases. The response reductions produced are a combined consequence of the variations of the damping and natural frequency. Results showed that ambient temperature must be taken into account when designing viscoelastic dampers for earthquake-resistant design. The viscoelastic dampers proved to be very effective in reducing the seismic response.

A more extensive analytical and experimental study, with particular attention to the proper design of viscoelastic dampers considering such important factors as excitation frequencies and the ambient temperature, have been conducted by Chang et al. (1991). The experimental program was conducted on a 2/5 scale five story steel frame and a full-size prototype structure, under a variety of precisely controlled ambient temperature and recorded ground motions. Results from the study showed that the viscoelastic dampers were effective in reducing the seismic structural response at all levels of earthquake ground motion, and that their energy dissipation capacity decreased as the temperature increased, with the loss factor relatively insensitive to moderate changes in frequencies and ambient temperature.

However, the dampers were effective in reducing excessive vibrations of the test structure at all temperature tested in the research program. It was concluded that the
dynamic characteristics and the seismic response of viscoelastically-damped structures can be simply and accurately predicted by using the modal strain energy method, and that conventional linear dynamic analysis can be used at all levels of ambient temperature and earthquake ground motions.

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A detailed design procedure for viscoelastically-damped structures using a modal strain energy method have also been proposed by Chang et al. (1993). The
design procedure is iterative and requires an analysis of the primary structure to establish the dominant frequency and the associated mode shape. The required damping ratio is used as the primary parameter for adding viscoelastic dampers to the structure. The addition of viscoelastic dampers adds both stiffness and damping to the structure. The addition of stiffness reduces the structural period thus attracting higher inertial forces, while the addition of damping reduces the dynamic response of the structure.

The research study showed that the addition of viscoelastic material generally results in reducing the forces on the structure, however, beyond a certain limit, increase in viscoelastic material may result in attracting higher forces. The design issue is to establish a combination of stiffness and damping which produces optimal structural response. The optimal design may be associated with minimum base shear, minimum displacement, or maximum viscoelastically-dissipated energy.

The performance of viscoelastic dampers in the World Trade Center building has been studied by Mahmoodi et al. (1989). The research study presents information from the dynamic monitoring program, reports on the methods used to evaluate the effectiveness of the dampers, and compares overall structure damping with that calculated during structural design. In order to evaluate the effectiveness of the damping system, and in conjunction with an overall structural integrity program, four dampers on the 105th floor were instrumented with linear variable differential transformers to measure the shear deformation in the viscoelastic element of the dampers. During wind-induced building oscillation, the voltage output of these measuring devices were recorded by a 16 track analog tape recorder.

Once the overall system was in place and tested, the building experienced oscillations above the activation acceleration twice due to local storms. The
acceleration of the building and the displacement of the dampers were measured simultaneously, and the energy dissipated contributed by the dampers was found to agree well with theoretical values calculated during the course of the design. The total damping of the building was determined from the concepts and the experimental work, and found to be in the range of 2.5 to 3.0 percent of critical damping.

Among other recent researchers, working on the behaviour of viscoelastic dampers were Nielsen et al. (1994). Their study has demonstrated the benefits of viscoelastic dampers for both steel and concrete frame structures. Viscoelastic dampers, installed in a 13 story San Jose building, were studied, and the dynamic testing conditions corresponding to the different levels of seismic demand were evaluated. Failure occurred at very large strain levels (larger than 300%), by tearing of the viscoelastic material along planes from the top of the side plates to the bottom central plate.

Oh et al. (1992), have also summarised an experimental and analytical study on the application of viscoelastic dampers as energy dissipation devices in structural applications. They concluded that viscoelastic dampers are effective in reducing excessive vibrations of structures under strong earthquake ground motions. It was also found that the modal strain energy method can be used to predict reliably the equivalent structural damping, and the seismic response of a viscoelastically damped structure can be accurately estimated by conventional modal analysis techniques. Based on their study, a design procedure for viscoelastically damped structures was presented. This design procedure fits naturally into the conventional structural design flow chart by including damping ratio as an additional design parameter.

A recent study on utilising viscoelastic dampers in the seismic retrofit of a thirteen story steel framed building has been conducted by Crosby et al. (1993). The
Further Review of Past Work

A study was aimed to determine the reason why this structure performed differently from similar structures in the adjacent area. A lack of inherent damping was found to be the primary cause for poor dynamic behaviour. Three seismic dampers were studied for use in the structure to reduce the building response. The design approach and parameters are presented in their paper, and the effectiveness of the chosen damper to various dynamic excitations is compared with the approximately 1% damped existing structure. Their results show that the building response can be significantly reduced by the use of viscoelastic dampers.

A new design procedure for structures with added viscoelastic dampers have also been proposed by Chang et al. (1992a & 1992b). Some practical issues concerning the application of viscoelastic dampers to building structures for seismic performance enhancement are discussed. Results of their research study showed that seismic response of a structure can be significantly improved with added dampers at all levels of earthquake excitations and ambient temperature. A design procedure for viscoelastically damped structures by taking into account the effect of ambient temperature is proposed following a modal strain energy approach.

Numerical simulations on equivalent structural damping and structural response confirm that the dynamic behaviour of structures with added viscoelastic dampers can be satisfactorily predicted by conventional linear analytical tools. In addition, the design of structures with added viscoelastic dampers can be easily incorporated into conventional design processes.

A new viscoelastic-type connection isolator, made from elastomeric material, was proposed by Sheng-Yung and Fafitis (1992). Based on experimental results, a Kelvin-Voigt type model was developed to simulate the behaviour of the connection. A comparison analysis of the stress-strain curves of the model with those obtained
from the tests was carried out. It was found that the durability of the elastomer is the predominant material parameter. An analytical method for the analysis of unbraced and braced frame structures equipped with those viscoelastic-type connections is presented. The effectiveness of these damper connections is studied numerically by comparing the responses to various dynamic excitations of structures equipped with these dampers with the responses of the structures without dampers. They concluded that the damper connections provide significant improvement by reducing the lateral dynamic displacement of the structures.

An overview of the existing design and analysis of viscoelastic vibration dampers for structures has been presented by Mahmoodi (1988). The data presented in his paper, which are based on experiments, show that most of the energy is dissipated to surrounding media through convention, conduction and radiation. It has been shown that the decrease in the total energy loss after a number of cycles, will continue until the temperature reaches a maximum. At this temperature all the damped energy will be dissipated to the outside environment.

Mahmoodi concluded that a very important factor for design consideration is the functional dependence of specific or total energy loss on temperature, or the related number of continuous cycles of oscillation that the damper will experience. Dampers based on their frequency and temperature functions, exhibit different behaviour, therefore it is essential to establish experimentally the damping capacity and its minimum value for each as a function of temperature and the number of cycles.

Mahmoodi and Keel (1990), have also proposed an analytical method for analysis of multilayer viscoelastic dampers. The design of multilayer dampers is essential where a large static deformation in the damper is anticipated. In mulilayer
dampers the viscoelastic layers are rigidly bonded (epoxy bond) to center plates. The model in their research work was mounted on a strain gauge balance and tested in appropriate wind flow conditions. The wind tunnel generated graphs which presented peak acceleration verses the level of the wind storm for various levels of damping. Two major conclusions have been drawn from the research study, firstly, the total energy loss per cycle calculated by both of their two proposed methods is quite close, and secondly, the same is true for any strain or temperature.

At higher temperature, it can be concluded from their research work that the layers experience uniform temperature and strain. Based on these observations and the fact that strain measurements of each layer is quite cumbersome, it is better to calculate the total energy loss by the measurement of the overall strain of all the layers. In the case of multilayer dampers the assessment of the functional dependency or total energy loss due to temperature change is not simple, however, measurement of the temperature of the center layer, as an indicator of the overall damper temperature, leads to reasonable results.

Seismic response of steel frame structures with added viscoelastic dampers have also been studied by Zhang et al. (1989). The feasibility of using viscoelastic dampers to mitigate earthquake-induced structural response was studied in their paper. The properties of viscoelastic dampers are briefly described and a procedure for evaluating the viscoelastic damping effect when added to a structure is proposed. The damping effects of viscoelastic dampers is incorporated into modal ratios through an energy approach. Computer simulation of the damped response of a multi-storey steel frame structure shows significant reduction in floor displacement levels.

A correlation of experimental results with predictions of viscoelastic damping for a model structure has been presented by Soong and Lai (1991). The strain energy
method in conjunction with the finite element method were used to evaluate the
damping ratio of a 1/2.5 scale steel frame building with viscoelastic dampers installed
in the bracing. Computed damping ratios are found to be in very good agreement
with the experimental results. Six cases were used to verify the modal strain energy
method for computing the modal damping ratios of the first mode. Experimental
verification was conducted on the shaking table at the State University of New York
at Buffalo.

In their work, finite element methods were used to abstract the system mass and
stiffness matrices. A spring element was used to represent the damper stiffness. From
the theoretical and experimental studies it was concluded that the estimated damping
ratios of a 1/2.5 scaled model were verified by the tests. In addition, the viscoelastic
dampers provided a very efficient way to increase the damping and survivability for
building under earthquake excitation.

V.2 COMPLEX SPECTRAL METHOD

The dynamic analysis in the frequency domain has some advantages such as
spectral decomposition of the forcing function, which helps to set bounds on the
dynamical problem. Also, the stability and efficiency of the Fast Fourier Transform
allows the use of a larger time step compared with the direct integration time step,
which is an advantage. Moreover, for systems with frequency dependent parameters,
and also in those cases with long time duration or high stiffness coupling, frequency
domain analysis may be the only effective means of determining the dynamic
response. Gopalakrishnan et al. (1992), Farris et al. (1989) and Doyle et al. (1986,
1988, 1989, 1990a and 1990b), in particular have proposed the frequency domain
spectral matrix methodology for dynamic analysis which concerns the synthesis of
waveforms from the superposition of many frequency components.
The complex spectral approach is based on the spectral solution of the governing differential equations for the dynamic motion of structures. However, the approach differs from the classical method because it uses the Fast Fourier Transform for conversion purposes. As the method uses discrete rather than continuous transforms, the frequency range is finite. Discrete points can be obtained by summing the components over frequency and wave numbers.

V.3 SUMMARY AND CONCLUDING REMARKS

The dynamical methods for solving structures with added viscoelastic dampers, described in the previous sections, have their own advantages and disadvantages. Although the superiority of one dynamical method compared with others is strongly problem-dependent, the following general shortcomings can be found by carefully reviewing past work:

- The mechanical characteristics of viscoelastic dampers (i.e., frequency and temperature dependency of the damping materials) can only be modeled accurately in these conventional methods by using complicated models with a large number of parameters.

- These methods are not so flexible that the mechanical properties of viscoelastic dampers can be incorporated in a straight forward manner (i.e., A non-linear solution should be developed for the frequency dependency of dampers, while it can be dealt with linearly in the frequency domain).

Therefore, it seems to be advantageous to search for a damping modeling requiring a fewer number of parameters. In essence, attention is paid herein to the alternative frequency domain dynamical modeling which draws its robustness from the speed and switching capabilities of the Fast Fourier Transform.
V.4 REFERENCES


