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Abstract

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A STABILIZATION THEOREM FOR FELL BUNDLES OVER GROUPOIDS

MARIUS IONESCU, ALEX KUMJIAN, AIDAN SIMS, AND DANA P. WILLIAMS

ABSTRACT. We study the C^* -algebras associated to upper-semicontinuous Fell bundles over second-countable Hausdorff groupoids. Based on ideas going back to the Packer–Raeburn “Stabilization Trick,” we construct from each such bundle a groupoid dynamical system whose associated Fell bundle is equivalent to the original bundle. The upshot is that the full and reduced C^* -algebras of any saturated upper-semicontinuous Fell bundle are stably isomorphic to the full and reduced crossed products of an associated dynamical system. We apply our results to describe the lattice of ideals of the C^* -algebra of a continuous Fell bundle by applying Renault’s results about the ideals of the C^* -algebras of groupoid crossed products. In particular, we discuss simplicity of the Fell-bundle C^* -algebra of a bundle over G in terms of an action, described by the first and last named authors, of G on the primitive-ideal space of the C^* -algebra of the part of the bundle sitting over the unit space. We finish with some applications to twisted k -graph algebras, where the components of our results become more concrete.

1. INTRODUCTION

The construction of a C^* -algebra from a Fell bundle over a groupoid subsumes all the usual ways of building a C^* -algebra out of group or groupoid. Special cases include group and groupoid C^* -algebras (with or without a twisting cocycle), group and groupoid dynamical systems, C^* -algebras associated to twists over groupoids, Green’s twisted dynamical systems, and even twisted versions of groupoid dynamical systems, just to name the most common.

An important task in understanding the structure of any C^* -algebra is understanding its ideals. Quite a bit is known about the ideal structure of group and groupoid C^* -algebras, and also of crossed-products associated to various types of C^* -dynamical systems (cf., [8, 40, 41, 12, 36, 17]). So it would be very useful to be able to “lift” results about the ideal structure of groupoid crossed products to results about Fell-bundle C^* -algebras. A result to this effect for continuous Fell bundles over étale groupoids was established by the second author in [7]: for such Fell bundles, the reduced C^* -algebra is Morita equivalent

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to the reduced C^* -algebra of a groupoid dynamical system. Muhly sketched in [25] how one can extend Kumjian’s techniques to non-étale groupoids, but still only for continuous bundles. Here we elaborate on a variation of Muhly’s approach and in doing so prove a generalization of his result (Theorem 3.7): given a second-countable saturated upper-semicontinuous Fell bundle $p : \mathcal{B} \rightarrow G$ over a second-countable Hausdorff locally compact groupoid we show that there is a groupoid dynamical system (in the sense of [28]) such that the Fell bundle associated with the groupoid dynamical system is equivalent with \mathcal{B} (see [27]). Therefore the full and reduced C^* -algebras of \mathcal{B} are Morita equivalent to the full and reduced C^* -algebras, respectively, of the corresponding groupoid dynamical system. Our techniques are a generalization of Kumjian and Muhly’s arguments and special care is needed due to the upper-semicontinuity assumption. The stabilization theorem can also be derived from the results in [4] since a Fell bundle over a groupoid is an example of what the authors call a weak action.¹

Our stabilization theorem is inspired by the Packer–Raeburn “Stabilization Trick” (see [30, Theorem 3.4]) for twisted actions of groups, which in turn builds on Quigg’s version of Takai duality for such actions (see [31, Theorem 3.1]).

As an application of our results, we apply the powerful results that Renault proved in [36] about the lattice of ideals of the C^* -algebra of a groupoid dynamical system to characterize the ideal lattice and the simplicity of the C^* -algebra of a continuous Fell bundle under certain amenability assumptions (Corollaries 3.9 and 3.11). As a second application, we describe a collection of examples of Fell bundles arising from twisted k -graph C^* -algebras, and apply our main result to provide a method for calculating their primitive-ideal spaces in some special cases.

2. BACKGROUND ABOUT FELL BUNDLES

We recall next some of the definitions and facts that are needed to prove our results. In this note we assume that G is a second-countable Hausdorff locally compact groupoid endowed with a Haar system $\{\lambda^x\}_{x \in G^{(0)}}$ [34]. We write $r : G \rightarrow G^{(0)}$ for the *range* map $r(g) = gg^{-1}$ and $s : G \rightarrow G^{(0)}$ for the *source* map $s(g) = g^{-1}g$. Recall that $\text{supp } \lambda^x = G^x := r^{-1}(x)$ for all $x \in G^{(0)}$. For $x \in G^{(0)}$ we let λ_x be the image of λ^x under inversion. Therefore, the support of λ_x is $G_x := s^{-1}(x)$. For two locally compact groupoids G and H , a G – H equivalence [26, Definition 2.1] is a locally compact Hausdorff space Z which is simultaneously a free and proper left G -space and a free and proper right H -space such that the G and H actions commute and such that the moment maps $r : Z \rightarrow G^{(0)}$ and $s : Z \rightarrow H^{(0)}$ induce homeomorphisms of Z/H onto $G^{(0)}$ and $G \backslash Z$ onto $H^{(0)}$, respectively. Consequently, given $z, w \in Z$ such that $s(z) = s(w)$ there is a unique ${}_G[z, w] \in G$ such that ${}_G[z, w] \cdot w = z$. Similarly, if $r(z) = r(w)$, then there is a unique $[z, w]_H \in H$ such that $z \cdot [z, w]_H = w$. If G is a locally compact groupoid then G is trivially a G – G equivalence.

Following [27] (see also [10, 11, 20]), a Fell bundle over G is an upper-semicontinuous Banach bundle $p : \mathcal{B} \rightarrow G$ endowed with a partially defined

¹The stabilization theorem fails for Fell bundles over non-Hausdorff groupoids (see [3, §7]).

multiplication and an involution that respect p such that the fibres $A(x)$ over $x \in G^{(0)}$ are C^* -algebras and each fibre $B(g)$ is an $A(r(g))$ – $A(s(g))$ imprimitivity bimodule with respect to the inner products and actions induced by the multiplication in \mathcal{B} . Note that for $x \in G^{(0)}$ we will write both $A(x)$ and $B(x)$ for the fiber over x to distinguish its dual role as a C^* -algebra and as the trivial imprimitivity bimodule over $A(x)$. The C^* -algebra $A := \Gamma_0(G^{(0)}; \mathcal{B})$ is called the C^* -algebra over the unit space. We assume that our Fell bundles are second countable and saturated and they have enough sections (in the sense that evaluation of continuous sections at g is surjective onto $B(g)$ for each g [28, p. 16]). Since $B(g)$ is an $A(r(g))$ – $A(s(g))$ imprimitivity bimodule, the Rieffel correspondence [39, 38, 33] induces a homeomorphism $h_g : \text{Prim } A(s(g)) \rightarrow \text{Prim } A(r(g))$. By [18, Proposition 2.2] there is a continuous action of G on $\text{Prim } A$ such that

$$(2.1) \quad g \cdot (s(g), P) = (r(g), h_g(P)).$$

Suppose that G and H are locally compact Hausdorff groupoids, that $p_G : \mathcal{B} \rightarrow G$ is a Fell bundle over G and that $p_H : \mathcal{C} \rightarrow H$ is a Fell bundle over H . A \mathcal{B} – \mathcal{C} equivalence [27, Definition 6.1] consists of a G – H equivalence Z and an upper-semicontinuous bundle $q : \mathcal{E} \rightarrow Z$ such that \mathcal{B} acts on the left of \mathcal{E} , \mathcal{C} acts on the right on \mathcal{E} , the two actions commute, and there are sesquilinear maps ${}_B \langle \cdot, \cdot \rangle$ from $\mathcal{E} *_s \mathcal{E}$ to \mathcal{B} and $\langle \cdot, \cdot \rangle_C$ from $\mathcal{E} *_r \mathcal{E}$ to \mathcal{C} such that

- (a) $p_G({}_B \langle e, f \rangle) = {}_G [q(e), q(f)]$ and $p_H(\langle e, f \rangle_C) = [q(e), q(f)]_H$,
- (b) ${}_B \langle e, f \rangle^* = {}_B \langle f, e \rangle$ and $\langle e, f \rangle_C^* = \langle f, e \rangle_C$,
- (c) ${}_B \langle b \cdot e, f \rangle = b \cdot {}_B \langle e, f \rangle$ and $\langle e, f \cdot c \rangle_C = \langle e, f \rangle_C \cdot c$,
- (d) ${}_B \langle e, f \rangle \cdot g = e \cdot \langle f, g \rangle_C$,

and such that, with the \mathcal{B} and \mathcal{C} actions and with the inner products coming from ${}_B \langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_C$, each $E(z)$ is a $B(r(z))$ – $C(s(z))$ -imprimitivity bimodule. If \mathcal{B} and \mathcal{C} are equivalent Fell bundles then $C^*(G; \mathcal{B})$ and $C^*(H; \mathcal{C})$ are Morita equivalent [27, Theorem 6.4] and so are $C_{\text{red}}^*(G; \mathcal{B})$ and $C_{\text{red}}^*(H; \mathcal{C})$ [43, 20].

An important example for this note is the Fell bundle associated with a groupoid dynamical system. Let $\pi : \mathcal{A} \rightarrow G^{(0)}$ be an upper-semicontinuous C^* -bundle over $G^{(0)}$. Assume that (\mathcal{A}, G, α) is a groupoid dynamical system [35, 28]. Then one can define a Fell bundle that we denote by $\sigma : \mathcal{A} \rtimes_\alpha G \rightarrow G$ as in [27, Example 2.1]:

$$(2.2) \quad \mathcal{A} \rtimes_\alpha G := r^* \mathcal{A} = \{ (a, g) \in \mathcal{A} \times G : \pi(a) = r(g) \}$$

with multiplication $(a, g)(b, h) = (a\alpha_g(b), gh)$ whenever $s(g) = r(h)$, and the involution given by $(a, g)^* = (\alpha_g^{-1}(a^*), g^{-1})$.

If V is a right Hilbert module over a C^* -algebra A [24, 33, 39, 20], then there is a left A -module V^* with a conjugate linear isomorphism from V to V^* , written $v \mapsto v^*$ such that $av^* = (va^*)^*$ and ${}_A \langle u^*, v^* \rangle = \langle u, v \rangle_A$. Rank-one operators on V are defined via $\theta_{u,v}(w) = u \cdot \langle v, w \rangle$. The set of compact operators $\mathcal{K}(V)$ on V is the closure of the linear span of rank-one operators. Compact operators are adjointable and $\mathcal{K}(V)$ is a C^* -algebra with respect to the operator norm [24, pp. 9–10]. If V is full then V is a $\mathcal{K}(V)$ – A imprimitivity bimodule where the left $\mathcal{K}(V)$ -inner product is ${}_{\mathcal{K}(V)} \langle u, v \rangle = \theta_{u,v}$. Hence $\mathcal{K}(V)$ and A are Morita equivalent. The map $u \otimes v^* \mapsto \theta_{u,v}$ extends to an isomorphism $V \otimes_A V^* \cong \mathcal{K}(V)$ (we make this identification in the sequel without comment).

3. MAIN RESULT AND APPLICATIONS

Let G be a locally compact Hausdorff groupoid endowed with a Haar system $\{\lambda_x\}_{x \in G^{(0)}}$. Let $p : \mathcal{B} \rightarrow G$ be a second-countable saturated Fell bundle over G and let $A = \Gamma_0(G^{(0)}; \mathcal{B})$ be the C^* -algebra over $G^{(0)}$. We construct an upper-semicontinuous C^* -bundle $k : \mathcal{K}(V) \rightarrow G^{(0)}$ and an action α of G on $\mathcal{K}(V)$ such that $(\mathcal{K}(V), G, \alpha)$ is a groupoid dynamical system and such that \mathcal{B} and $\mathcal{K}(V) \rtimes_{\alpha} G$ are equivalent Fell bundles, generalizing the construction and results of [20, Section 4] and [25, Section 4]. We break our construction into a series of lemmas and propositions.

For $x \in G^{(0)}$ let $V(x)$ be the closure of $\Gamma_c(G_x; \mathcal{B})$ under the pre-inner product

$$\langle \xi, \eta \rangle_{A(x)} = \int_{G_x} \xi(\gamma)^* \eta(\gamma) d\lambda_x(\gamma).$$

Then $V(x)$ is a full right Hilbert $A(x)$ -module with the right action given by $(\xi \cdot a)(\gamma) = \xi(\gamma)a$ for $\xi \in \Gamma_c(G_x; \mathcal{B})$ and $a \in A(x)$ [33, Lemma 2.16]. Let $V := \bigsqcup_{x \in G^{(0)}} V(x)$ and let $\nu : V \rightarrow G^{(0)}$ be the projection map.

Lemma 3.1. *With notation as above, the map $x \mapsto \langle \xi, \eta \rangle_{A(x)}$ is continuous for all $\xi, \eta \in \Gamma_c(G; \mathcal{B})$. Moreover, there is a unique topology on V such that $\nu : V \rightarrow G^{(0)}$ is an upper-semicontinuous Banach bundle over $G^{(0)}$ of which $x \mapsto \hat{f}(x) = f|_{G_x}$ is a continuous section for each $f \in \Gamma_c(G; \mathcal{B})$. The space $\mathcal{V} := \Gamma_0(G^{(0)}; V)$ is then a full Hilbert A -module.*

Proof. The first assertion follows from the fact that $f^*g \in \Gamma_c(G; \mathcal{B})$ for all $f, g \in \Gamma_c(G; \mathcal{B})$ (see [27, Corollary 1.4]). Given a section $f \in \Gamma_c(G; \mathcal{B})$ define a section \hat{f} of V by $\hat{f}(x) = f|_{G_x}$. As

$$\|\hat{f}(x)\| = \|\langle \hat{f}, \hat{f} \rangle_{A(x)}\|^{\frac{1}{2}},$$

the map $x \mapsto \|\hat{f}(x)\|$ is the composition of a continuous function and an upper-semicontinuous function, and hence itself upper semicontinuous. The vector-valued Tietze Extension Theorem [27, Proposition A.5] implies that the set $\{\hat{f}(x) : f \in \Gamma_c(G; \mathcal{B})\}$ is dense in $V(x)$. The Hofmann–Fell theorem (see, for example, [10, Theorem II.13.18], [14], [15], and [18, Theorem 1.2]) implies that there is a unique topology such that V is an upper semicontinuous Banach bundle over $G^{(0)}$ and such that $\Gamma(G^{(0)}; V)$ contains $\{\hat{f} : f \in \Gamma_c(G; \mathcal{B})\}$. It is now easy to see that \mathcal{V} , equipped with the natural inner product and right A -action, is a Hilbert A -module. \square

For the next lemma let $\mathcal{K}(V) := \bigsqcup_{x \in G^{(0)}} \mathcal{K}(V(x))$ and let $k : \mathcal{K}(V) \rightarrow G^{(0)}$ be the canonical map.

Lemma 3.2. *Resume the notation of Lemma 3.1. Then $\mathcal{K}(V)$ has a unique topology such that $k : \mathcal{K}(V) \rightarrow G^{(0)}$ is an upper semicontinuous C^* -bundle admitting a $C_0(G^{(0)})$ -linear isomorphism of $\mathcal{K}(V)$ onto $\Gamma_0(G^{(0)}; \mathcal{K}(V))$, and each $\mathcal{K}(V)(x) \cong \mathcal{K}(V(x))$.*

Proof. Since $A = \Gamma_0(G^{(0)}; \mathcal{B})$ is a $C_0(G^{(0)})$ -algebra there is a continuous map $\sigma_A : \text{Prim } A \rightarrow G^{(0)}$ (see, for example, [45, Theorem C.26]) given by $\sigma(I) = x$ if and only if I contains the ideal generated by functions in $C_0(G^{(0)})$ that vanish at x . Since \mathcal{V} is an $\mathcal{K}(V)$ - A imprimitivity bimodule, the Rieffel correspondence

restricts to a homeomorphism $h : \text{Prim } \mathcal{K}(\mathcal{V}) \rightarrow \text{Prim } A$ (see, for example, [33, Proposition 3.3.3]). Therefore one obtains by composition a continuous map $h \circ \sigma_A : \text{Prim } \mathcal{K}(\mathcal{V}) \rightarrow G^{(0)}$. Theorem C.26 of [45] implies that $\mathcal{K}(\mathcal{V})$ is a $C_0(G^{(0)})$ -algebra and that $k : \mathcal{K}(\mathcal{V}) \rightarrow G^{(0)}$ is an upper-semicontinuous C^* -bundle such that there is a $C_0(G^{(0)})$ -linear isomorphism of $\mathcal{K}(\mathcal{V})$ onto $\Gamma_0(G^{(0)}, \mathcal{K}(\mathcal{V}))$. The fiber of $\mathcal{K}(\mathcal{V})$ over $x \in G^{(0)}$ is isomorphic to $\mathcal{K}(V(x))$ by construction, so $\mathcal{K}(\mathcal{V})(x)$ is isomorphic to $\mathcal{K}(V(x))$ for all $x \in G^{(0)}$. \square

The following lemma will be useful in the definition of the action of G on $\mathcal{K}(V)$ and the equivalence between Fell bundles.

Lemma 3.3. *For $g \in G$, the map $\beta_g : \Gamma_c(G_{r(g)}; \mathcal{B}) \odot B(g) \rightarrow \Gamma_c(G_{s(g)}; \mathcal{B})$ defined on elementary tensors by*

$$\beta_g(\xi \otimes b)(\gamma) = \xi(\gamma g^{-1})b$$

extends to an isometric isomorphism $\beta_g : V(r(g)) \otimes_{A(r(g))} B(g) \rightarrow V(s(g))$ of Hilbert $A(s(g))$ -modules.

Proof. Fix $g \in G$. Let $\xi, \eta \in \Gamma_c(G_{r(g)}; \mathcal{B})$ and $a, b \in B(g)$. Using left invariance at the third equality, we calculate:

$$\begin{aligned} \langle \beta_g(\xi \otimes a), \beta_g(\eta \otimes b) \rangle &= \int_{G_{s(g)}} \beta_g(\xi \otimes a)^*(\gamma) \beta_g(\eta \otimes b)(\gamma) d\lambda_{s(g)}(\gamma) \\ &= \int_{G_{s(g)}} a^* \xi^*(\gamma g^{-1}) \eta(\gamma g^{-1}) b d\lambda_{s(g)}(\gamma) \\ &= a^* \int_{G_{r(g)}} \xi^*(\gamma) \eta(\gamma) d\lambda_{r(g)}(\gamma) b \\ &= a^* \langle \xi, \eta \rangle_{A(r(g))} b \\ &= \langle \xi \otimes a, \eta \otimes b \rangle_{A(s(g))} \end{aligned}$$

Thus β_g preserves the inner-product, which implies first that it is isometric, and second—by right-linearity of the inner product—that it preserves the right $A_{s(g)}$ -action.

To check that the range of β_g is dense in $V(s(g))$ fix $\xi' \in \Gamma_c(G_{s(g)}; \mathcal{B})$. There is an approximate identity $\{c_\nu\}_\nu$ of $A(s(g))$ of the form $c_\nu = \sum_{i=1}^{n_\nu} b_{\nu,i}^* b_{\nu,i}$ where $b_{\nu,i} \in B(g)$ for all ν, i . For each ν, i , define $\xi_{\nu,i} \in \Gamma_c(G_{r(g)}; \mathcal{B})$ by $\xi_{\nu,i}(\gamma) = \xi'(\gamma g) b_{\nu,i}^*$. Then the net $\{\sum_{i=1}^{n_\nu} \beta_g(\xi_{\nu,i} \otimes b_{\nu,i})\}_\nu$ converges to ξ' . Hence β_g extends to an isometric isomorphism of Hilbert $A(s(g))$ -modules. \square

As an easy consequence of Lemma 3.3 we obtain that, for $g \in G$, the map $\beta_g^* : B(g) \odot \Gamma_c(G_{s(g)}; \mathcal{B})^* \rightarrow \Gamma_c(G_{r(g)}; \mathcal{B})^*$ defined on elementary tensors by

$$\beta_g^*(b \otimes \eta^*) = (\beta_{g^{-1}}(\eta \otimes b^*))^*$$

extends to an isometric isomorphism $B(g) \otimes V(s(g))^* \cong V(r(g))^*$ of left Hilbert $A(r(g))$ -modules. It is important in the following to keep in mind that the maps β_g and β_g^* are onto. Moreover, $\overline{\text{span}}\{\beta_g(\xi \otimes b) : \xi \in V(r(g)), b \in B(g)\} = V(s(g))$ and $\overline{\text{span}}\{\beta_g^*(b \otimes \eta^*) : b \in B(g), \eta \in V(s(g))\} = V(r(g))^*$. Therefore, for $g \in G$, the map α_g defined by

$$(3.1) \quad \alpha_g(\beta_g(\xi \otimes b) \otimes \eta^*) = \xi \otimes \beta_g^*(b \otimes \eta^*),$$

for $\xi \in V(r(g))$, $b \in B(g)$, and $\eta \in V(s(g))$, extends to an isomorphism between $\mathcal{K}(V(s(g)))$ and $\mathcal{K}(V(r(g)))$. We collect in the following lemma a few useful facts about the maps β_g and α_g .

Lemma 3.4. *Resume the notation of Lemma 3.3 and let $\{\alpha_g : g \in G\}$ be the isomorphisms defined in (3.1).*

(a) *For $g \in G$ we have*

$$(3.2) \quad \beta_{g^{-1}}(\beta_g(\xi \otimes b_1) \otimes b_2^*) = \xi \cdot_{A(r(g))} \langle b_1, b_2 \rangle = \xi \cdot (b_1 b_2^*).$$

for all $\xi \in V(r(g))$ and $b_1, b_2 \in B(g)$.

(b) *For composable $g, h \in G$, we have*

$$(3.3) \quad \beta_{gh}(\xi \otimes b_1 b_2) = \beta_h(\beta_g(\xi \otimes b_1) \otimes b_2).$$

for all $\xi \in V(r(g)) = V(r(gh))$, $b_1 \in B(g)$ and $b_2 \in B(h)$.

(c) *For $g \in G$ and $T \in \mathcal{K}(V(s(g)))$, we have*

$$(3.4) \quad T \circ \beta_g = \beta_g \circ (\alpha_g(T) \otimes \text{Id}).$$

Proof. (a) Let $g \in G$, $\xi \in \Gamma_c(G_{r(g)}; \mathcal{B})$, $b_1, b_2 \in B(g)$, and $\gamma \in G_{r(g)}$. Then

$$\beta_{g^{-1}}(\beta_g(\xi \otimes b_1) \otimes b_2^*)(\gamma) = \beta_g(\xi \otimes b_1)(\gamma g) b_2^* = \xi(\gamma) b_1 b_2^*.$$

The result follows.

(b) Let $\xi \in \Gamma_c(G_{r(g)}; \mathcal{B}) = \Gamma_c(G_{r(gh)}; \mathcal{B})$ and let $\gamma \in G_{s(h)} = G_{s(gh)}$. Then

$$\beta_{gh}(\xi \otimes b_1 b_2)(\gamma) = \xi(\gamma h^{-1} g^{-1}) b_1 b_2 = \beta_g(\xi \otimes b_1)(\gamma h^{-1}) b_2 = \beta_h(\beta_g(\xi \otimes b_1) \otimes b_2)(\gamma).$$

(c) Assume that T is a rank-one operator $T = u \otimes v^*$ with $u, v \in \Gamma_c(G_{s(g)}; \mathcal{B})$ and assume that $u = \beta_g(u' \otimes b')$ for some $u' \otimes b' \in V(r(g)) \otimes B(g)$. Let $\xi \in V(r(g))$ and $b \in B(g)$. Then

$$\begin{aligned} T(\beta_g(\xi \otimes b)) &= u \cdot \langle v, \beta_g(\xi \otimes b) \rangle_{A(s(g))} \\ &= u \int_{G_{s(g)}} v^*(\gamma) \beta_g(\xi \otimes b)(\gamma) d\lambda_{s(g)}(\gamma) \\ &= u \int_{G_{s(g)}} v^*(\gamma) \xi(\gamma g^{-1}) b d\lambda_{s(g)}(\gamma) \\ &= u \int_{G_{s(g)}} v^*(\gamma) \xi(\gamma g^{-1}) d\lambda_{s(g)}(\gamma) b. \end{aligned}$$

Also,

$$\begin{aligned} \beta_g(\alpha_g(T)\xi \otimes b) &= \beta_g(\alpha_g(\beta_g(u' \otimes b') \otimes v^*)\xi \otimes b) \\ &= \beta_g((u' \otimes \beta_g^*(b' \otimes v^*)\xi) \otimes b) \\ &= \beta_g(u' \langle \beta_{g^{-1}}(v \otimes b'^*), \xi \rangle_{A(r(g))} \otimes b) \\ &= \beta_g(u' \otimes b') \int_{G_{r(g)}} v^*(\gamma g) \xi(\gamma) d\lambda_{r(g)}(\gamma) b, \end{aligned}$$

where the last equality follows from an easy computation. Using the invariance of the Haar system we deduce (3.4) for rank-one operators. The result then follows from linearity and continuity. \square

Proposition 3.5. *With $\mathcal{K}(V)$ defined in Lemma 3.2 and with α defined in (3.1), $(\mathcal{K}(V), G, \alpha)$ is a groupoid dynamical system.*

Proof. We already know that $\alpha_g : \mathcal{K}(V(s(g))) \rightarrow \mathcal{K}(V(r(g)))$ is an isomorphism. It is not obvious, however, that α_g is a *-homomorphism and we provide a proof next. Recall that using the identification $\mathcal{K}(V(s(g))) \simeq V(s(g)) \otimes_{A(s(g))} V(s(g))^*$, the product of $u_1 \otimes v_1^*$ and $u_2 \otimes v_2^* \in V(s(g)) \otimes V(s(g))^*$ is given by $u_1 \cdot \langle v_1, u_2 \rangle_{A(s(g))} \otimes v_2^*$. Let $\xi_1, \xi_2 \in \Gamma_c(G_{r(g)}; \mathcal{B})$, $b_1, b_2 \in B(g)$, and $\eta_1, \eta_2 \in \Gamma_c(G_{s(g)}; \mathcal{B})$. Then

$$\begin{aligned} & \alpha_g((\beta_g(\xi_1 \otimes b_1) \otimes \eta_1^*) \cdot (\beta_g(\xi_2 \otimes b_2) \otimes \eta_2^*)) \\ &= \alpha_g(\beta_g(\xi_1 \otimes b_1) \cdot \langle \eta_1, \beta_g(\xi_2 \otimes b_2) \rangle_{A(s(g))} \otimes \eta_2^*) \end{aligned}$$

which, using that β_g is a Hilbert module map, is

$$= \alpha_g(\beta_g(\xi_1 \otimes (b_1 \cdot \langle \eta_1, \beta_g(\xi_2 \otimes b_2) \rangle_{A(s(g))}))) \otimes \eta_2^*.$$

By the definition of α_g , this is

$$\begin{aligned} &= \xi_1 \otimes \beta_g^*(b_1 \cdot \langle \eta_1, \beta_g(\xi_2 \otimes b_2) \rangle_{A(s(g))} \otimes \eta_2^*) \\ &= \xi_1 \otimes (\beta_{g^{-1}}(\eta_2 \otimes (b_1 \cdot \langle \eta_1, \beta_g(\xi_2 \otimes b_2) \rangle_{A(s(g))})))^* \\ &= \xi_1 \otimes (\beta_{g^{-1}}(\eta_2 \otimes \langle \eta_1, \beta_g(\xi_2 \otimes b_2) \rangle_{A(s(g))}^* b_1^*))^* \\ &= \xi_1 \otimes (\beta_{g^{-1}}(\eta_2 \otimes \langle \beta_g(\xi_2 \otimes b_2), \eta_1 \rangle_{A(s(g))} b_1^*))^*. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \alpha_g(\beta_g(\xi_1 \otimes b_1) \otimes \eta_1^*) \cdot \alpha_g(\beta_g(\xi_2 \otimes b_2) \otimes \eta_2^*) \\ &= (\xi_1 \otimes \beta_g^*(b_1 \otimes \eta_1^*)) \cdot (\xi_2 \otimes \beta_g^*(b_2 \otimes \eta_2^*)) \\ &= (\xi_1 \otimes (\beta_{g^{-1}}(\eta_1 \otimes b_1^*)))^* \cdot (\xi_2 \otimes \beta_g^*(b_2 \otimes \eta_2^*)) \\ &= \xi_1 \cdot \langle \beta_{g^{-1}}(\eta_1 \otimes b_1^*), \xi_2 \rangle_{A(r(g))} \otimes (\beta_{g^{-1}}(\eta_2 \otimes b_2^*))^*. \end{aligned}$$

Since the tensor product is balanced over $A(r(g))$, this is

$$\begin{aligned} &= \xi_1 \otimes (\beta_{g^{-1}}(\eta_2 \otimes b_2^*) \cdot \langle \beta_{g^{-1}}(\eta_1 \otimes b_1^*), \xi_2 \rangle_{A(r(g))}^*) \\ &= \xi_1 \otimes (\beta_{g^{-1}}(\eta_2 \otimes (b_2^* \cdot \langle \xi_2, \beta_{g^{-1}}(\eta_1 \otimes b_1^*) \rangle_{A(r(g))})))^*. \end{aligned}$$

Using the invariance of the Haar system we have that

$$\begin{aligned} \langle \beta_g(\xi_2 \otimes b_2), \eta_1 \rangle_{A(s(g))} b_1^* &= \int_{G_{s(g)}} \beta_g(\xi_2 \otimes b_2)^*(\gamma) \eta(\gamma) d\lambda_{s(g)}(\gamma) b_1^* \\ &= b_2^* \int_{G_{s(g)}} \xi_2^*(\gamma g^{-1}) \eta(\gamma) d\lambda_{s(g)}(\gamma) b_1^* \\ &= b_2^* \int_{G_{r(g)}} \xi_2^*(\gamma) \eta(\gamma g) b_1^* d\lambda_{r(g)}(\gamma) \\ &= b_2^* \int_{G_{r(g)}} \xi_2^*(\gamma) \beta_{g^{-1}}(\eta \otimes b_1^*) d\lambda_{r(g)}(\gamma) \\ &= b_2^* \cdot \langle \xi_2, \beta_{g^{-1}}(\eta \otimes b_1^*) \rangle_{A(r(g))}. \end{aligned}$$

Therefore

$$\begin{aligned} \alpha_g((\beta_g(\xi_1 \otimes b_1) \otimes \eta_1^*) \cdot (\beta_g(\xi_2 \otimes b_2) \otimes \eta_2^*)) \\ = \alpha_g(\beta_g(\xi_1 \otimes b_1) \otimes \eta_1^*) \cdot \alpha_g(\beta_g(\xi_2 \otimes b_2) \otimes \eta_2^*). \end{aligned}$$

Take $\xi_1, \xi_2 \in V(r(g))$ and $b_1, b_2 \in B(g)$. Recall that the adjoint of $u \otimes v^* \in V(s(g)) \otimes V(s(g))^*$ equals $v \otimes u^*$. Then

$$\begin{aligned} \alpha_g(\beta_g(\xi_1 \otimes b_1) \otimes (\beta_g(\xi_2 \otimes b_2))^*)^* &= (\xi_1 \otimes \beta_g^*(b_1 \otimes (\beta_g(\xi_2 \otimes b_2))^*))^* \\ &= (\xi_1 \otimes (\beta_{g^{-1}}(\beta_g(\xi_2 \otimes b_2) \otimes b_1^*))^*)^* \\ &= \beta_{g^{-1}}(\beta_g(\xi_2 \otimes b_2) \otimes b_1^*) \otimes \xi_1^*, \end{aligned}$$

which, using (3.2), is

$$\begin{aligned} &= \xi_2 \cdot (b_2 b_1^*) \otimes \xi_1^* = \xi_2 \otimes (\xi_1 \cdot (b_1 b_2^*))^* \\ &= \xi_2 \otimes \beta_g^*(b_2 \otimes \beta_g(\xi_1 \otimes b_1)^*) \\ &= \alpha_g((\beta_g(\xi_1 \otimes b_1) \otimes (\beta_g(\xi_2 \otimes b_2))^*)^*). \end{aligned}$$

Therefore α_g is a *-isomorphism.

Take $g, h \in G$ such that $s(g) = r(h)$. To prove that $\alpha_{gh} = \alpha_g \circ \alpha_h$ let $\xi \in V(r(g))$, $a \in B(g)$, $b \in B(h)$, and $\eta \in V(s(h))$. Then

$$\begin{aligned} (\alpha_g \circ \alpha_h)(\beta_h(\beta_g(\xi \otimes a) \otimes b) \otimes \eta^*) &= \alpha_g(\beta_g(\xi \otimes a) \otimes \beta_h^*(b \otimes \eta^*)) \\ &= \xi \otimes \beta_g^*(a \otimes \beta_h^*(b \otimes \eta^*)) \\ &= \xi \otimes \beta_{gh}^*(ab \otimes \eta^*) \\ &= \alpha_{gh}((\beta_h(\beta_g(\xi \otimes a) \otimes b) \otimes \eta^*)). \end{aligned}$$

Finally we prove that the action of G on $\mathcal{K}(V)$ is continuous. We are going to use the criterion from [45, Proposition C.20] for the convergence of nets in an upper-semicontinuous C^* -bundle. It suffices to consider nets $\{g_i\} \subset G$ such that $g_i \rightarrow g$, $\{\xi_i\}$ indexed by a set I such that $\xi_i \in \Gamma_c(G_{r(g_i)}; \mathcal{B})$ and $\xi_i \rightarrow \xi \in \Gamma_c(G_{r(g)}; \mathcal{B})$, $\{b_i\}$ such that $b_i \in B(g_i)$ and $b_i \rightarrow b \in B(g)$, and $\{\eta_i\}$ such that $\eta_i \in \Gamma_c(G_{s(g_i)}; \mathcal{B})$ and $\eta_i \rightarrow \eta \in \Gamma_c(G_{s(g)}; \mathcal{B})$. Moreover it suffices to consider nets $\{\tilde{\xi}_i\}$, $\{\tilde{b}_i\}$, and $\{\tilde{\eta}_i\}$ in $\Gamma_c(G; \mathcal{B})$ such that $\tilde{\xi}_i \rightarrow \tilde{\xi}$, $\tilde{b}_i \rightarrow b$, $\tilde{\eta}_i \rightarrow \tilde{\eta}$, $\tilde{\xi}|_{G_{r(g_i)}} = \xi_i$, $\tilde{b}_i(g_i) = b_i$, and $\tilde{\eta}_i|_{G_{s(g_i)}} = \eta_i$. We will write $\tilde{\xi}_i(x) := \tilde{\xi}_i|_{G_x}$ and similarly for $\tilde{\eta}_i(x)$ with $x \in G^{(0)}$ in order to keep the notation simple. Fix $\varepsilon > 0$ and fix an index $j \in I$ such that $\|\tilde{\xi}_j - \tilde{\xi}\| < \varepsilon$, $\|\tilde{b}_j - b\| < \varepsilon$, and $\|\tilde{\eta}_j - \tilde{\eta}\| < \varepsilon$. Then, since $\beta_{g_i}^*$ are isometric isomorphisms,

$$\begin{aligned} \alpha_{g_i}(\beta_{g_i}(\tilde{\xi}_j(r(g_i)) \otimes \tilde{b}_j(g_i)) \otimes \tilde{\eta}_j(s(g_i))^*) &= \tilde{\xi}_j(r(g_i)) \otimes \beta_{g_i}^*(\tilde{b}_j(g_i) \otimes \tilde{\eta}_j(s(g_i))^*) \\ &\rightarrow \tilde{\xi}_j(r(g)) \otimes \beta_g^*(\tilde{b}_j(g) \otimes \tilde{\eta}_j(s(g))^*) = \alpha_g(\beta_g(\tilde{\xi}_j(r(g)) \otimes \tilde{b}_j(g)) \otimes \tilde{\eta}_j(s(g))^*). \end{aligned}$$

Therefore, if we set

$$\begin{aligned} a_i &:= \alpha_{g_i}(\beta_{g_i}(\tilde{\xi}_i(r(g_i)) \otimes \tilde{b}_i(g_i)) \otimes \tilde{\eta}_i(s(g_i))^*), \\ u_i &:= \alpha_{g_i}(\beta_{g_i}(\tilde{\xi}_j(r(g_i)) \otimes \tilde{b}_j(g_i)) \otimes \tilde{\eta}_j(s(g_i))^*), \\ a &:= \alpha_g(\beta_g(\tilde{\xi}(r(g)) \otimes \tilde{b}(g)) \otimes \tilde{\eta}(s(g))^*) \quad \text{and} \\ u &:= \alpha_g(\beta_g(\tilde{\xi}_j(r(g)) \otimes \tilde{b}_j(g)) \otimes \tilde{\eta}_j(s(g))^*), \end{aligned}$$

then the following hold: $u_i \rightarrow u$, $k(u_i) = g_i = k(a_i)$, $\|a - u\| < \varepsilon$, and $\|a_i - u_i\| < \varepsilon$ for large i . Therefore [45, Proposition C.20] implies that $a_i \rightarrow a$, that is

$$\alpha_{g_i}(\beta_{g_i}(\tilde{\xi}_i(r(g_i)) \otimes \tilde{b}_i(g_i)) \otimes \tilde{\eta}_i(s(g_i))^*) \rightarrow \alpha_g(\beta_g(\tilde{\xi}(r(g)) \otimes \tilde{b}(g)) \otimes \tilde{\eta}(s(g))^*).$$

Hence the action of G on $\mathcal{K}(V)$ is continuous and $(\mathcal{K}(V), G, \alpha)$ is a groupoid dynamical system. \square

Remark 3.6. Note that if we replace ‘‘upper semicontinuous’’ with ‘‘continuous’’ in the hypothesis of Lemma 3.1, Lemma 3.2, and Theorem 3.7, then $\nu : V \rightarrow G^{(0)}$ is a continuous Banach bundle, $k : \mathcal{K}(V) \rightarrow G^{(0)}$ is a (continuous) C^* -bundle [10, Theorem II.13.18] and, hence, $(\mathcal{K}(V), G, \alpha)$ is a continuous groupoid dynamical system (in the sense of [35, 36]).

We let $\sigma : \mathcal{K}(V) \rtimes_{\alpha} G \rightarrow G$ be the semi-direct crossed product Fell bundle. Recall from (2.2) that $\mathcal{K}(V) \rtimes_{\alpha} G = r^*\mathcal{K}(V)$ with the multiplication given by $(T, g)(S, h) = (T\alpha_g(S), gh)$ and $(T, g)^* = (\alpha_g^{-1}(T^*), g^{-1})$. Our main result shows that \mathcal{B} and $\mathcal{K}(V) \rtimes_{\alpha} G$ are equivalent Fell bundles.

Theorem 3.7. *For $g \in G$ let $E(g) = V(r(g)) \otimes_{A(r(g))} B(g)$, let $\mathcal{E} = \bigsqcup_{g \in G} E(g)$, and let $q : \mathcal{E} \rightarrow G$ be the projection map. Then $q : \mathcal{E} \rightarrow G$ is an upper-semicontinuous Banach bundle over G and a $(\mathcal{K}(V) \rtimes_{\alpha} G)$ - \mathcal{B} equivalence.*

Proof. For $\xi \in \Gamma_c(G, \mathcal{B})$ and $\eta \in \Gamma_c(G; \mathcal{B})$, define $\hat{\xi}(x) := \xi|_{G_x}$ for $x \in G^{(0)}$, and define a section $\xi \otimes \eta$ of \mathcal{E} by

$$(\xi \otimes \eta)(g) = \hat{\xi}(r(g)) \otimes \eta(g).$$

Then the set $\{\xi \otimes \eta : \xi, \eta \in \Gamma_c(G; \mathcal{B})\}$ satisfies the hypothesis of the Hofmann–Fell theorem. Hence there is a unique topology on \mathcal{E} such that $q : \mathcal{E} \rightarrow G$ is an upper-semicontinuous Banach bundle such that the above sections are continuous.

We show that \mathcal{E} is a $(\mathcal{K}(V) \rtimes_{\alpha} G)$ - \mathcal{B} equivalence. The right action of \mathcal{B} on \mathcal{E} is defined by

$$(\xi \otimes a) \cdot b = \xi \otimes (ab),$$

for $\xi \in V(r(g))$, $a \in B(g)$ and $b \in B(h)$, where $s(g) = r(h)$. It is easy to check that $q(\xi \otimes a \cdot b) = q(\xi \otimes a)p(b)$ and $(\xi \otimes a \cdot b) \cdot c = \xi \otimes a \cdot (bc)$. Moreover, it is a straightforward computation to show that $\|\xi \otimes a \cdot b\| \leq \|\xi \otimes a\| \|b\|$ using the $B(s(g))$ -inner product on $E(g)$. Therefore the analogues for right actions of axioms (a)–(c) on page 40 of [27] are satisfied by the right action of \mathcal{B} on \mathcal{E} (axiom (c) contains a typographical error, and should read $\|b \cdot e\| \leq \|b\| \cdot \|e\|$). The continuity of the action follows from a version of [45, Proposition C.20] for upper-semicontinuous Banach bundles.

To define the left action of $\mathcal{K}(V) \rtimes_{\alpha} G$ on \mathcal{E} we note that if $s(g) = r(h)$ then $V(r(h)) \otimes_{A(r(h))} B(h)$ is isomorphic to $V(r(g)) \otimes_{A(r(g))} B(gh)$. Indeed, $V(r(h)) = V(s(g))$ is isomorphic to $V(r(g)) \otimes_{A(r(g))} B(g)$ by Lemma 3.3, and multiplication induces an imprimitivity-bimodule isomorphism between $B(g) \otimes_{A(s(g))} B(h)$ and $B(gh)$ (see Lemma 1.2 of [27]). Moreover, $V(r(gh)) = V(r(g))$. Then for $(T, g) \in \mathcal{K}(V) \rtimes_{\alpha} G$, $\xi \otimes a_1 \in V(r(g)) \otimes_{A(r(g))} B(g)$, and $a_2 \in B(h)$ define

$$(T, g) \cdot \beta_g(\xi \otimes a_1) \otimes a_2 = (T\xi) \otimes (a_1 a_2) \in V(r(gh)) \otimes B(gh).$$

Then $q((T, g) \cdot \beta_g(\xi \otimes a_1) \otimes a_2) = k(T, g)q(\beta_g(\xi \otimes a_1) \otimes a_2) = gh$.

We must check that this left action of $\mathcal{K}(V)$ on \mathcal{E} is continuous and satisfies axioms (a)–(c) on page 40 of [27]. Continuity follows once again from an upper semicontinuous Banach bundle version of [45, Proposition C.20]. Axiom (a) is immediate from the definition. We use equations (3.3) and (3.4) to check axiom (b):

$$\begin{aligned} (S, t) \cdot [(T, g) \cdot \beta_{tg}(\xi \otimes b_1 b_2) \otimes b_3] &= (S, t) \cdot [(T, g) \cdot \beta_g(\beta_t(\xi \otimes b_1) \otimes b_2) \otimes b_3] \\ &= (S, t) \cdot T\beta_t(\xi \otimes b_1) \otimes b_2 b_3 \\ &= (S, t) \cdot \beta_t(\alpha_t(T)\xi \otimes b_1) \otimes b_2 b_3 \\ &= S\alpha_t(T)\xi \otimes b_1 b_2 b_3 \\ &= (S\alpha_t(T), tg) \cdot \beta_{tg}(\xi \otimes b_1 b_2) \otimes b_3. \end{aligned}$$

One can easily show that $\|(T, g) \cdot \beta_g(\xi \otimes a_1) \otimes a_2\| \leq \|T\| \|\beta_g(\xi \otimes a_1) \otimes a_2\|$ using the right $A(s(g))$ -inner product and the fact that β_g is an isometry. Therefore axiom (c) on page 40 of [27] holds for the left action of $\mathcal{K}(V)$ on \mathcal{E} .

Now we have to check that these actions of $\mathcal{K}(V)$ and \mathcal{B} on \mathcal{E} satisfy (a), (b)(i)–(b)(iv) and (c) of [27, Definition 6.1].

Definition 6.1(a) of [27] requires that the two actions commute, which is straightforward:

$$((T, g) \cdot \beta_g(\xi \otimes a_1) \otimes a_2) \cdot b = (T, g) \cdot (\beta_g(\xi \otimes a_1) \otimes a_2 \cdot b).$$

To check (b)(i)–(b)(iv), we must first define sesquilinear maps $\kappa_{\mathcal{K}(V) \rtimes_{\alpha} G} \langle \cdot, \cdot \rangle$ from $\mathcal{E} *_s \mathcal{E}$ to $\mathcal{K}(V) \rtimes_{\alpha} G$ and $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ from $\mathcal{E} *_r \mathcal{E}$ to \mathcal{B} . Let $g, h \in G$ such that $r(g) = r(h)$ and let $v \otimes a \in E(g)$ and $w \otimes b \in E(h)$. Since $r(g) = r(h)$, we have $v, w \in V(r(h)) = V(r(g))$. Define

$$\langle v \otimes a, w \otimes b \rangle_{\mathcal{B}} := a^* \langle v, w \rangle_{A(r(h))} b \in B(g^{-1}h).$$

Let $g, h \in G$ be such that $s(g) = s(h)$ and let $v \otimes a \in E(g)$ and $w \otimes b \in E(h)$. Notice that $w \in V(r(h)) = V(s(gh^{-1}))$ and $v \in V(r(g)) = V(r(gh^{-1}))$. Define

$$\kappa_{\mathcal{K}(V) \rtimes_{\alpha} G} \langle v \otimes a, w \otimes b \rangle := (v \otimes \beta_{gh^{-1}}^*(ab^* \otimes w^*), gh^{-1}).$$

It is a routine albeit tedious task to check that these maps satisfy Definition 6.1(b)(i)–(b)(iv) of [27]; we just prove some of them to indicate the sorts of arguments involved. For (b)(i),

$$p(\langle v \otimes a, w \otimes b \rangle_{\mathcal{B}}) = g^{-1}h$$

and

$$\sigma(\kappa_{\mathcal{K}(V) \rtimes_{\alpha} G} \langle v \otimes a, w \otimes b \rangle) = gh^{-1}.$$

The proof of (b)(ii) is relatively easy for the \mathcal{B} -valued sesquilinear form and more involved for the $\mathcal{K}(V) \rtimes_{\alpha} G$ -valued sesquilinear form; we check both. First take $g, h \in G$ such that $r(g) = r(h)$ and let $v \otimes a \in E(g)$ and $w \otimes b \in E(h)$. Then

$$\langle v \otimes a, w \otimes b \rangle_{\mathcal{B}}^* = (a^* \langle v, w \rangle_{A(r(h))} b)^* = b^* \langle w, v \rangle_{A(r(g))} a = \langle w \otimes b, v \otimes a \rangle_{\mathcal{B}}.$$

Now fix $g, h \in G$ such that $s(g) = s(h)$ and take $v \otimes a \in E(g)$ and $w \otimes b \in E(h)$. Then

$$\begin{aligned} (\kappa_{\mathcal{K}(V) \rtimes_{\alpha} G} \langle v \otimes a, w \otimes b \rangle)^* &= (v \otimes \beta_{gh^{-1}}^*(ab^* \otimes w^*), gh^{-1})^* \\ &= (\alpha_{gh^{-1}}^{-1} \left((v \otimes \beta_{gh^{-1}}^*(ab^* \otimes w^*))^* \right), hg^{-1}) \\ &= (\alpha_{gh^{-1}}^{-1} (\beta_{gh^{-1}}(w \otimes ba^*) \otimes v^*), hg^{-1}) \end{aligned}$$

which, by the definition of α , is

$$\begin{aligned} &= (\alpha_{gh^{-1}}^{-1}(\alpha_{gh^{-1}}(w \otimes \beta_{hg^{-1}}^*(ba^* \otimes v^*))), hg^{-1}) \\ &= (w \otimes \beta_{hg^{-1}}^*(ba^* \otimes v^*)), hg^{-1}) \\ &=_{\mathcal{K}(V) \rtimes_{\alpha} G} \langle w \otimes b, v \otimes a \rangle. \end{aligned}$$

The remaining axioms (b)(iii), (b)(iv) and (c) of [27, Definition 6.1] are easy to prove. Hence \mathcal{E} is a $\mathcal{K}(V) \rtimes_{\alpha} G$ - \mathcal{B} equivalence. \square

Corollary 3.8. *With the notation of Theorem 3.7, $C^*(G; \mathcal{B})$ and $C^*(G; \mathcal{K}(V) \rtimes_{\alpha} G)$ are Morita equivalent and so are $C_{\text{red}}^*(G; \mathcal{B})$ and $C_{\text{red}}^*(G; \mathcal{K}(V) \rtimes_{\alpha} G)$.*

Proof. Theorem 3.7 combined with Theorem 6.4 of [27] implies that $C^*(G; \mathcal{B})$ and $C^*(G; \mathcal{K}(V) \rtimes_{\alpha} G)$ are Morita equivalent. The second assertion follows from Theorem 3.7 and [43, Theorem 14]. \square

Our next corollary presents one of the many possible applications of Theorem 3.7 and Corollary 3.8. The extra hypothesis about continuity of the Fell bundle is needed in order to cite the results of [36] which were proved in the context of *continuous* groupoid dynamical systems. Recall that if $p : \mathcal{B} \rightarrow G$ is a continuous Fell bundle then G acts continuously on the primitive-ideal space of $A = \Gamma_0(G^{(0)}, \mathcal{B})$ with its Polish *regularized topology*² via equation (2.1) [36, Proposition 1.14]. When we say “ G acts amenably on $\text{Prim } A$ ” we require the existence of a net of functions as in [36, Remark 3.7]. For this it suffices for the Borel groupoid $\text{Prim } A \times G$ to be measurewise amenable or for G itself to be amenable. We say that the action of G on $\text{Prim } A$ is *essentially free* if the set of points with trivial isotropy is dense in every closed invariant set for the regularized topology on $\text{Prim } A$.

Corollary 3.9. *Let G be a locally compact Hausdorff groupoid and let $p : \mathcal{B} \rightarrow G$ be a continuous Fell bundle. Let A be the C^* -algebra over $G^{(0)}$. Assume that the action of G on $\text{Prim } A$ is amenable and essentially free. Then the lattice of ideals of $C^*(G; \mathcal{B})$ is isomorphic to the lattice of invariant open sets of $\text{Prim } A$.*

Proof. Since \mathcal{V} is a $\mathcal{K}(\mathcal{V})$ - A -imprimitivity bimodule, it follows that $\mathcal{K}(\mathcal{V})$ and A are Morita equivalent. Let $h : \text{Prim } A \rightarrow \text{Prim } \mathcal{K}(\mathcal{V})$ be the Rieffel correspondence (see, for example, [33, Corollary 3.3]). Then, from the definition of \mathcal{V} and $\mathcal{K}(\mathcal{V})$ and [18, Formula (1) on page 1247] it follows that if $P \in \text{Prim } \mathcal{K}(\mathcal{V})$ and $g \in G$ then $g \cdot P = h(g \cdot h^{-1}(P))$. Therefore the action of G on $\text{Prim } \mathcal{K}(\mathcal{V})$ is amenable and essentially free. The result follows from [36, Corollary 4.9], Remark 3.6 and Corollary 3.8. \square

Remark 3.10. Corollary 3.9 provides an alternative proof of the following result, which was proved under slightly stronger conditions in [22, Corollary 4.7].

Corollary 3.11. *Let G be a Hausdorff locally compact groupoid and let $p : \mathcal{B} \rightarrow G$ be a continuous Fell bundle. Assume that the action of G on $\text{Prim } A$ is amenable and essentially free. Then $C^*(G; \mathcal{B})$ is simple if and only if the action of G on $\text{Prim } A$ is minimal.*

²The regularized topology is defined in [45, Definition H.38] is Polish by [45, Theorem H.39].

Let G be an *étale* locally compact groupoid and suppose that the interior of the isotropy $\text{Iso}^\circ(G)$ is closed. Then $G/\text{Iso}^\circ(G)$ is also an *étale* locally compact groupoid [44, Proposition 2.5(d)].

Corollary 3.12. *Let G be an étale amenable locally compact groupoid, let $\sigma \in Z^2(G, \mathbb{T})$ be a continuous 2-cocycle and suppose that the interior of the isotropy $\text{Iso}^\circ(G)$ is closed. Suppose that $G/\text{Iso}^\circ(G)$ is essentially principal. Then there is a continuous Fell bundle $p : \mathcal{B} \rightarrow G/\text{Iso}^\circ(G)$ such that*

$$C^*(G/\text{Iso}^\circ(G), \mathcal{B}) \cong C^*(G, \sigma) \quad \text{and} \quad C^*(G^{(0)}, \mathcal{B}) \cong C^*(\text{Iso}^\circ(G), i^*(\sigma)).$$

The action of $G/\text{Iso}^\circ(G)$ on $\text{Prim } C^(\text{Iso}^\circ(G), i^*(\sigma))$ is essentially principal and the map which takes an ideal of $C^*(\text{Iso}^\circ(G), i^*(\sigma))$ to the ideal of $C^*(G, \sigma)$ generated by its image induces an isomorphism from the lattice of $(G/\text{Iso}^\circ(G))$ -invariant open sets of $\text{Prim } C^*(\text{Iso}^\circ(G), i^*(\sigma))$ to the lattice of ideals of $C^*(G, \sigma)$.*

The proof of Corollary 3.12 requires a lemma.

Lemma 3.13. *Let G be an étale amenable locally compact groupoid such that $G = \text{Iso}(G)$. Let \mathcal{B} be a Fell bundle over G such that $\xi \mapsto \|\xi_x\|$ is continuous for $\xi \in C^*(G^{(0)}, \mathcal{B})$. Then $C^*(G, \mathcal{B})$ is the section algebra of a continuous field of C^* -algebras over $G^{(0)}$ such that $C^*(G, \mathcal{B})_x \cong C^*(G_x, \mathcal{B})$ for $x \in G^{(0)}$.*

Proof. The central inclusion of $C_0(G^{(0)})$ in $\mathcal{M}C^*(G, \mathcal{B})$ is nondegenerate, so [45, Theorem C.27] shows that $C^*(G, \mathcal{B})$ is the section algebra of an upper-semicontinuous field with fibres $C^*(G_x, \mathcal{B})$. For lower semicontinuity, let λ be the faithful representation, induced by multiplication, of $C_r^*(G; \mathcal{B})$ on the Hilbert- $C^*(G^{(0)}, \mathcal{B})$ -module completion $L^2(\mathcal{B})$ of $\Gamma_c(G; \mathcal{B})$ for $\langle \xi, \eta \rangle := (\xi^* \eta)|_{G^{(0)}}$ (see [20, Proposition 3.2]). By [20, 3.3 and 3.4], $L^2(\mathcal{B})$ is a bundle over $G^{(0)}$ of Hilbert modules V_x , and λ determines representations $\lambda_x : C_r^*(G, \mathcal{B}) \rightarrow \mathcal{L}(V_x)$. Since G is amenable, [42, Theorem 1] gives $C^*(G, \mathcal{B}) = C_r^*(G, \mathcal{B})$, so each λ_x determines a faithful representation of $C^*(G, \mathcal{B})_x$. Fix $\xi \in C_c(G; \mathcal{B})$, $x \in G^{(0)}$, and $\varepsilon > 0$. Take $h \in L^2(\mathcal{B})$ with $\|h_x\| = \|h\| = 1$ and $\|\lambda_x(\xi)h_x\| \geq \|\xi_x\| - \varepsilon/2$. As $y \mapsto \|(\lambda(\xi)h)_y\|$ is continuous, $\|(\lambda(\xi)h)_y\| \geq \|(\lambda(\xi)h)_x\| - \varepsilon/2 \geq \|\xi_x\| - \varepsilon$ on some neighbourhood U of x . Each $\|h_y\| \leq 1$, so $\|\xi_y\| \geq \|\xi_x\| - \varepsilon$ for $y \in U$. \square

Proof of Corollary 3.12. Identify $C^*(\text{Iso}^\circ(G), i^*(\sigma))$ with the C^* -algebra of a Fell bundle \mathcal{A} over $\text{Iso}^\circ(G)$ with 1-dimensional fibres. Then $x \mapsto \|\xi_x\| = |\xi(x)|$ is continuous for $\xi \in C^*(G^{(0)}, \mathcal{A}) = C_0(G^{(0)})$. So Lemma 3.13 shows that $C^*(\text{Iso}^\circ(G), i^*(\sigma))$ is the section algebra of a continuous field of C^* -algebras over $G^{(0)}$ with fibres $C^*(\text{Iso}^\circ(G)_x, i^*(\sigma))$. Arguing as in [22, Proposition 4.2], we obtain a continuous Fell bundle $p : \mathcal{B} \rightarrow G/\text{Iso}^\circ(G)$ such that

$$C^*(G/\text{Iso}^\circ(G), \mathcal{B}) \cong C^*(G, \sigma) \quad \text{and} \quad C^*(G^{(0)}, \mathcal{B}) \cong C^*(\text{Iso}^\circ(G), i^*(\sigma)).$$

The final assertion follows from Corollary 3.9. \square

4. APPLICATIONS TO k -GRAPHS

In this section we discuss some applications of our results to computing the primitive-ideal spaces of twisted k -graph C^* -algebras.

Recall (see [5, Definition 2.1], and also [21]) that if P is a submonoid of an abelian group A with identity 0, then a P -graph is a countable small category Λ with a functor $d : \Lambda \rightarrow P$, called the *degree map*, such that whenever $d(\lambda) =$

$p+q$, there are unique $\mu \in d^{-1}(p)$ and $\nu \in d^{-1}(q)$ such that $\lambda = \mu\nu$ (this is called the factorisation property). We write $\Lambda^p = d^{-1}(p)$. The factorisation property ensures that Λ^0 is the set of identity morphisms, so we identify it with the object set, and think of the codomain and domain maps as maps $r, s : \Lambda \rightarrow \Lambda^0$. When $P = \mathbb{N}^k$, a P -graph Λ is precisely a k -graph as introduced in [21]. As a notational convention, given $v, w \in \Lambda^0$, we write $v\Lambda = \{\lambda : r(\lambda) = v\}$, $\Lambda v = \{\lambda : s(\lambda) = v\}$, $v\Lambda w = \{\lambda : r(\lambda) = v, s(\lambda) = w\}$, and so forth.

We say that Λ is *row-finite* if each $|v\Lambda^p| < \infty$ is finite, and that Λ has *no sources* if each $v\Lambda^p \neq \emptyset$. We impose both hypotheses throughout this section. We recall some facts about P -graphs and their groupoids from [5, §2] and [37, §6]. Throughout, P is a submonoid of an abelian group A as above. For more details and background, see [21, 5, 37].

Let \leq denote the partial order on P given by $p \leq q$ if and only if $q - p \in P$. As in [5, Example 2.2], there is a P -graph $\Omega = \Omega_P := \{(p, q) \in P \times P : p \leq q\}$ with degree map $d(p, q) = q - p$ and range, source and composition given by $r(p, q) = (p, p)$, $s(p, q) = (q, q)$ and $(p, q)(q, r) = (p, r)$. We have $\Omega_P^0 = \{(p, p) : p \in P\}$ and we identify it with P in the obvious way. If Λ is a P -graph, we write Λ^Ω for the collection of all functors $x : \Omega_P \rightarrow \Lambda$ that intertwine the degree maps. If $P = \mathbb{N}^k$, then Λ^Ω is precisely the infinite-path space Λ^∞ of [21, Definitions 2.1]. For $x \in \Lambda^\Omega$ we write $x(p) := x(p, p) \in \Lambda^0$ when $p \in P$ and write $r(x) := x(0)$.

Under the relative topology inherited from $\prod_{(p,q) \in \Omega} \Lambda^{q-p}$, Λ^Ω is a locally compact Hausdorff space with basic open sets $Z(\lambda) = \{x \in \Lambda^\Omega : x(0, d(\lambda)) = \lambda\}$ indexed by $\lambda \in \Lambda$ [5, page 3]. For $x \in \Lambda^\Omega$, the vertex $r(x) := x(0) \in \Lambda^0$ is the unique vertex such that $x \in Z(r(x))$. More generally, $x \in Z(x(0, p))$ for every $p \in P$. An argument like that of [21, Proposition 2.3] shows that if $\lambda \in \Lambda$ and $x \in Z(s(\lambda))$, then there is a unique element λx of Λ^Ω such that $\lambda x \in Z(\lambda)$ and $\sigma^{d(\lambda)}(\lambda x) = x$. Hence, as in [21, Remarks 2.5], there is an action of P by local homeomorphisms on Λ^Ω given by $\sigma^p(x)(q, r) = x(q+p, r+p)$. The P -graph Λ is *aperiodic* if, for each $v \in \Lambda^0$ there exists $x \in Z(v)$ such that $\sigma^p(x) \neq \sigma^q(x)$ for all distinct $p, q \in P$. It is not hard to check that Λ is aperiodic if and only if the set of x such that $\sigma^p(x) \neq \sigma^q(x)$ for all distinct $p, q \in P$ is dense in Λ^Ω .

As in [44, Lemma 3.1] (see also [6, 9, 37]), the Deaconu–Renault groupoid G_Λ associated to the action σ is the set $\{(x, p - q, y) \in \Lambda^\Omega \times A \times \Lambda^\Omega : \sigma^p(x) = \sigma^q(y)\}$, given the topology generated by the sets $Z(\mu, \nu) = \{(\mu x, d(\mu) - d(\nu), \nu x) : x \in Z(s(\mu))\}$ indexed by pairs $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$. The unit space is $\{(x, 0, x) : x \in \Lambda^\Omega\}$, which we identify with Λ^Ω (the topologies agree), and the structure maps are $r(x, g, y) = x$, $s(x, g, y) = y$, $(x, g, y)^{-1} = (y, -g, x)$ and $(x, g, y)(y, h, z) = (x, g + h, z)$. This groupoid is second countable, étale and amenable [37, Proposition 5.12 and Theorem 5.13], and the basic open sets described above are compact open bisections [5, page 4]. For more on Deaconu–Renault groupoids of the sort we study here, see [6, 9, 37].

We denote the G_Λ -orbit $\{\lambda \sigma^p(x) : p \in P, \lambda \in \Lambda x(p)\}$ of $x \in \Lambda^\omega$ by $[x]$.

We shall restrict attention to abelian monoids P which arise as the image of \mathbb{N}^k in \mathbb{Z}^k/H for some subgroup H of \mathbb{Z}^k . We begin by presenting a characterisation of the P -graphs Λ such that G_Λ is essentially principal. Before doing that, we need to introduce an order relation on Λ^Ω .

Definition 4.1. Let H be a subgroup of \mathbb{Z}^k and let P be the image of \mathbb{N}^k in \mathbb{Z}^k/H . Let Λ be a row-finite P -graph with no sources. Given $x, y \in \Lambda^\Omega$, we

write $x \preceq y$ if for every $m \in P$ there exists $n \in P$ such that $x(m)\Lambda y(n) \neq \emptyset$. We write $x \sim y$ if $x \preceq y$ and $y \preceq x$.

Lemma 4.2. *Let H be a subgroup of \mathbb{Z}^k and let P be the image of \mathbb{N}^k in \mathbb{Z}^k/H . Let Λ be a row-finite k -graph with no sources. For $x, y \in \Lambda^\Omega$, we have $x \preceq y$ if and only if $\overline{[x]} \subseteq \overline{[y]}$. In particular, $x \sim y$ if and only if $\overline{[x]} = \overline{[y]}$.*

Proof. First suppose that $x \preceq y$. We must show that $\overline{[x]} \subseteq \overline{[y]}$. Since $\overline{[y]}$ is invariant and closed, it suffices to show that $x \in \overline{[y]}$; that is, that every neighbourhood of x intersects $[y]$. Fix a basic open neighbourhood $Z(x(0, m))$ of x . Since $x \preceq y$, there exists $n \in P$ such that $x(m)\Lambda y(n) \neq \emptyset$, say $\lambda \in x(m)\Lambda y(n)$. Then $x(0, m)\lambda\sigma^n(y) \in Z(x(0, m)) \cap [y]$.

Now suppose that $\overline{[x]} \subseteq \overline{[y]}$. Then in particular $x \in \overline{[y]}$, so for fixed $m \in P$, the set $[y]$ meets the basic open neighbourhood $Z(x(0, m))$ of x , say at $z = x(0, m)z'$. By definition, we have $\sigma^p(z) = \sigma^q(y)$ for some $p, q \in P$. Choose $r \in P$ such that $r-p, r-m \in P$, and let $n = r-p+q$. Then $\sigma^n(y) = \sigma^{r-p}(\sigma^q(y)) = \sigma^{r-p}(\sigma^p(z)) = \sigma^r(z)$. So $z'(0, r-m) \in x(m)\Lambda y(n)$. Hence $x \preceq y$. \square

The next lemma characterises when a P -graph groupoid is essentially principal in terms of the order structure just discussed. Following the standard definition for k -graphs [32] (see also [37,]), we say that a subset $H \subseteq \Lambda^0$ of the vertex set of a row-finite P -graph Λ with no sources is *hereditary* if $H\Lambda \subseteq \Lambda H$ and *saturated* if whenever $v\Lambda^p \subseteq \Lambda H$, we have $v \in H$. A subset T of Λ^0 is a *maximal tail* if its complement is a saturated hereditary set and $s(v\Lambda) \cap s(w\Lambda) \neq \emptyset$ for all $v, w \in T$. We say that Λ is *strongly aperiodic* if for every saturated hereditary subset $H \subseteq \Lambda^0$, the subgraph $\Lambda \setminus \Lambda H$ is aperiodic (see [19]).

Lemma 4.3. *Let H be a subgroup of \mathbb{Z}^k and let P be the image of \mathbb{N}^k in \mathbb{Z}^k/H . Let Λ be a row-finite P -graph with no sources, and let G_Λ be the associated groupoid. Then the following are equivalent.*

- (1) *The groupoid G_Λ is essentially principal in the sense that the points with trivial isotropy are dense in every closed invariant subspace of $G_\Lambda^{(0)}$.*
- (2) *The P -graph Λ is strongly aperiodic.*
- (3) *The subgraph ΛT is aperiodic for every maximal tail T of Λ .*
- (4) *For every $y \in \Lambda^\Omega$, there is an aperiodic path $x \in \Lambda^\Omega$ such that $x \sim y$.*

Proof. (1) \implies (2) Suppose that G_Λ is essentially principal, and fix a saturated hereditary $H \subseteq \Lambda^0$. Then $(\Lambda \setminus \Lambda H)^\Omega \subseteq \Lambda^\Omega$ is a closed invariant set, and hence $G_\Lambda|_{(\Lambda \setminus \Lambda H)^\Omega}$ is topologically free. The argument of [21, Proposition 4.5] then shows that $\Lambda \setminus \Lambda H$ is aperiodic.

(2) \implies (3) If $\Lambda \setminus \Lambda H$ is aperiodic for every saturated hereditary H , then in particular, every ΛT is aperiodic because the complement of a maximal tail is saturated and hereditary.

(3) \implies (4) Suppose that ΛT is aperiodic for every maximal tail T . For $y \in \Lambda^\Omega$, the set $T_y := \{z(n) : n \in P, z \in [y]\}$ is a maximal tail, and we have $\overline{[y]} = (\Lambda T_y)^\Omega$. List $P \times P \setminus \{(n, n) : n \in P\}$ as $(m_i, n_i)_{i=1}^\infty$. Let $\mathbf{1} \in P$ denote the image of $(1, \dots, 1) \in \mathbb{N}^k$ under the quotient map from \mathbb{Z}^k to \mathbb{Z}^k/H . We claim that there is a sequence $(\mu_i, q_i)_{i=0}^\infty \in r(y)\Lambda T \times P$ with the following properties:

- $d(\mu_i) \geq i \cdot \mathbf{1}$ for all $i \geq 0$;
- $\mu_{i+1} \in \mu_i\Lambda y(q_{i+1})$ for all $i \geq 0$; and

- for each $i \geq 1$ and each $1 \leq j \leq i$ there exists l such that $\mu_i(m_j, m_j + l) \neq \mu_i(n_j, n_j + l)$.

Set $\mu_0 = r(y)$ and $q_0 = 0$; this trivially has the desired properties. We construct the μ_i inductively. Given μ_i and q_i , note that $\mu_i \sigma^{q_i}(y) \in (\Lambda T_y)^\Omega$. Since ΛT_y is aperiodic there is an aperiodic infinite path x_{i+1} in $Z(\mu_i) \cap (\Lambda T)^\Omega$. Since x_{i+1} is aperiodic, we can choose $l \in P$ such that $x_{i+1}(m_{i+1}, m_{i+1} + l) \neq x_{i+1}(n_{i+1}, n_{i+1} + l)$. Choose $p \in P$ such that

$$p \geq d(\mu_i) + 1, \quad p \geq (m_{i+1} + l) \quad \text{and} \quad p \geq n_{i+1} + l.$$

Since $x_{i+1} \in (\Lambda T)^\Omega$, there exists $q_{i+1} \geq q_i + 1$ such that $x_{i+1}(p)\Lambda y(q_{i+1}) \neq \emptyset$. Now this choice of q_{i+1} and any choice of $\mu_{i+1} \in x_{i+1}(0, p)\Lambda y(q_{i+1})$ satisfies the three bullet points, completing the proof of the claim.

Let $x \in \Lambda^\Omega$ be the unique element such that $x(0, d(\mu_i)) = \mu_i$ for all i . By construction of the μ_i we have $\sigma^{m_i}(x) \neq \sigma^{n_i}(x)$ for all i , and so x is aperiodic. For each $m \in P$ we can choose i such that $d(\mu_i) \geq m$ and $q_i \geq m$. The first condition forces $x(m)\Lambda y(q_i) \neq \emptyset$, so that $x \preceq y$; and the second condition forces $y(m)\Lambda x(d(\mu_{i+1})) \neq \emptyset$, so that $y \preceq x$. Hence $y \sim x$ as required.

(4) \implies (1) Fix a closed invariant $X \subseteq G_\Lambda^{(0)}$ and $y \in X$. By (4), there is an aperiodic infinite path x such that $x \sim y$. Lemma 4.2 gives $\overline{[x]} = \overline{[y]} \subseteq X$, so there is a sequence (y_n) in $[x]$ converging to y . Each y_n is a point with trivial isotropy because x is aperiodic. So G_Λ is essentially principal. \square

For the definition of the C^* -algebras of the P -graphs considered here, see [5, Section 2]; for k -graphs, the definition appeared first in [21]. For our purposes, it suffices to recall first that $C^*(\Lambda)$ is isomorphic to $C^*(G_\Lambda)$ [5, Proposition 2.7], and second that this isomorphism intertwines the *gauge action* of $(\mathbb{Z}^k/H)^\wedge \cong H^\perp \subseteq \mathbb{T}^k$ on $C^*(\Lambda)$ with the action of $(\mathbb{Z}^k/H)^\wedge$ on $C^*(G_\Lambda)$ determined by $(\chi \cdot f)(x, g, y) = \chi(g)f(x, g, y)$ for $f \in C_c(G_\Lambda)$, $\chi \in (\mathbb{Z}^k/H)^\wedge$, and $(x, g, y) \in G_\Lambda$. An ideal of $C^*(\Lambda)$ is *gauge-invariant* if it is invariant for this gauge action.

In the situation where H is trivial in the preceding lemma so that the statement is about k -graphs, we could use this result combined with Corollary 3.9 to describe the primitive-ideal spaces of the C^* -algebras of strongly-aperiodic k -graphs. However, this result already follows from Renault's results about groupoid dynamical systems:

Corollary 4.4. *Let H be a subgroup of \mathbb{Z}^k and let P be the image of \mathbb{N}^k in \mathbb{Z}^k/H . Let Λ be a row-finite P -graph with no sources, and let G_Λ be the associated groupoid. Then the following are equivalent.*

- (1) *The P -graph Λ is strongly aperiodic.*
- (2) *The map $I \mapsto (I \cap C_0(G_\Lambda^{(0)}))^\wedge$ is a lattice isomorphism between the lattice of ideals of $C^*(G_\Lambda)$ and the lattice of open invariant subsets of $G_\Lambda^{(0)}$.*
- (3) *Every ideal of $C^*(\Lambda)$ is gauge-invariant.*

Proof. Lemma 4.3 shows that Λ is strongly aperiodic if and only if G_Λ is essentially principal. Since G_Λ is amenable [43, Lemma 3.5], (1) \implies (2) therefore follows from [36, Corollary 4.9]. Since $C_0(G_\Lambda^{(0)})$ is pointwise fixed by the gauge action on $C^*(\Lambda)$, we have (2) \implies (3). For (3) \implies (1), we argue the contrapositive. Suppose that Λ is not strongly aperiodic, and so G_Λ is not essentially principal. So there exists $x \in \Lambda^\Omega$ such that $G_\Lambda|_{\overline{[x]}}$ is not topologically principal.

By [2, Lemma 3.1] there is an open bisection U that is interior to the isotropy of $G_\Lambda|_{\overline{[x]}}$ but contains no units. By definition of the topology on G_Λ , we can assume that U is clopen and has the form $U = \{(y, p, y) : y \in K\}$ for some compact relatively open $K \subseteq \overline{[x]}$ and some $p \in \mathbb{Z}^k/H \setminus \{0+H\}$. Fix a character χ of \mathbb{Z}^k/H such that $\chi(p) \neq 1$. Choose any $a \in C_c(G_\Lambda)$ whose restriction to $G_\Lambda|_{\overline{[x]}}$ is equal to $1_U - \chi(p)1_K$. As in the proof of [5, Proposition 5.5], let π_x be the representation of $C^*(G_\Lambda)$ on $\ell^2([x])$ given by $\pi_x(f)\delta_y = \sum_{g \in (G_\Lambda)_y} f(g)\delta_{r(g)}$. Then $\pi_x(a)\delta_x = (1 - \chi(p))\delta_x \neq 0$, and $\pi_x(\chi \cdot a) = \pi_x(\chi(p)1_U - \chi(p)1_K) = \chi(p)\pi_x(1_U - 1_K) = 0$. So $\ker(\pi_x)$ is not gauge-invariant. \square

If Λ is a P -graph and $q : \mathbb{N}^k \rightarrow P$ is a homomorphism, then the pullback k -graph $q^*\Lambda$ is given by $q^*\Lambda = \{(\lambda, m) \in \Lambda \times \mathbb{N}^k : d(\lambda) = q(m)\}$ with pointwise operations and degree map $d(\lambda, m) = m$ (see [5, Definition 3.1] or [21, Definition 1.9]).

Recall from [23, Definition 3.5] that a \mathbb{T} -valued 2-cocycle on a k -graph Λ is a map $c : \{(\mu, \nu) \in \Lambda \times \Lambda : s(\mu) = r(\nu)\} \rightarrow \mathbb{T}$ such that $c(r(\lambda), \lambda) = 1 = c(\lambda, s(\lambda))$ for all λ and such that

$$c(\lambda, \mu)c(\lambda\mu, \nu) = c(\mu, \nu)c(\lambda, \mu\nu) \text{ for all composable } \lambda, \mu, \nu.$$

Again, rather than present a definition of the twisted C^* -algebra $C^*(\Lambda, c)$, we just recall from [23, Corollary 7.8] that for each 2-cocycle c on Λ there is a locally constant 2-cocycle σ on G_Λ such that $C^*(\Lambda, c) \cong C^*(G_\Lambda, \sigma)$.

Our next result, which provides a method for computing the primitive-ideal space of a twisted C^* -algebra associated to such a k -graph obtained as a pullback of a strongly aperiodic P -graph, follows easily from Corollary 3.12 and Lemma 4.6. We provide a description of the topology on $\text{Prim } C^*(\text{Iso}^\circ(G_\Lambda), \sigma)$ that will help in applying the theorem in Proposition 4.7 below.

Recall that if σ is a 2-cocycle on an abelian group H , then the symmetry group or symmetrizer subgroup of σ is

$$S_\sigma = \{t \in H : \sigma(t, s) = \sigma(s, t) \text{ for all } s \in H\}.$$

Note that S_σ is also the kernel $Z(h_\sigma)$ of the map $h_\sigma : H \rightarrow \hat{H}$ given by $h_\sigma(s)(t) = \sigma(s, t)\overline{\sigma(t, s)}$. We can view h_σ as an antisymmetric bicharacter on H . The map $\sigma \mapsto \sigma\sigma^*$, where $\sigma^*(s, t) = \overline{\sigma(t, s)}$, is an isomorphism of $H^2(H, \mathbb{T})$ with the group $X(H, \mathbb{T})$ of antisymmetric bicharacters on H (see [29, Proposition 3.2]). It is well-known (see [1] or [13, Proposition 34]) that the primitive-ideal space of the twisted group C^* -algebra $C^*(H, \sigma)$ is homeomorphic to the dual of S_σ .

Theorem 4.5. *Let H be a subgroup of \mathbb{Z}^k and let P be the image of \mathbb{N}^k in \mathbb{Z}^k/H . Let Γ be a row-finite P -graph with no sources, and let $\Lambda := q^*\Gamma$ be the pullback k -graph along the quotient map $q : \mathbb{Z}^k \rightarrow \mathbb{Z}^k/H$. Suppose that Γ is strongly aperiodic, let c be a \mathbb{T} -valued 2-cocycle on Λ and let σ be a locally constant \mathbb{T} -valued 2-cocycle on G_Λ such that $C^*(\Lambda, c) \cong C^*(G_\Lambda, \sigma)$. Then $\text{Iso}^\circ(G_\Lambda) \cong \Lambda^\infty \times H$ and is closed in G_Λ . For $x \in \Lambda^\infty$, let σ_x be the restriction of σ to $\{(x, h, x) : h \in H\} \subseteq (G_\Lambda)_x$. Then $C^*(\text{Iso}^\circ(G_\Lambda), \sigma)$ is the section algebra of a continuous field of C^* -algebras such that $C^*(\text{Iso}^\circ(G_\Lambda), \sigma)_x \cong C^*(H, \sigma_x)$ with $\text{Prim } C^*(H, \sigma_x) = \hat{S}_{\sigma_x}$. There is an action of G_Γ on $\text{Prim } C^*(\text{Iso}^\circ(G_\Lambda), \sigma)$ such that*

$$\text{Prim } C^*(\Lambda, c) \text{ is homeomorphic to } \text{Prim } C^*(\text{Iso}^\circ(G_\Lambda), \sigma)/G_\Gamma.$$

To prove the theorem, we need the following lemma about the structure of the groupoid of a pullback. Let G be an étale groupoid, let $t : A \rightarrow B$ be a homomorphism of locally compact abelian groups such that $\ker t$ is discrete, and let $c : G \rightarrow B$ be a continuous 1-cocycle. Then the pullback

$$t^*(G) = \{(g, a) \in G \times A : c(g) = t(a)\}$$

is a closed subgroupoid of $G \times A$ and is étale in the relative topology. The projection map $\pi_1 : t^*(G) \rightarrow G$ onto the first coordinate is a groupoid homomorphism. If t is surjective, then π_1 is a (surjective) local homeomorphism (see [16]).

Lemma 4.6. *Let H be a subgroup of \mathbb{Z}^k and let P be the image of \mathbb{N}^k in \mathbb{Z}^k/H . Let Γ be a row-finite P -graph with no sources, and let $\Lambda := q^*\Gamma$ be the pullback k -graph with respect to the quotient map $q : \mathbb{Z}^k \rightarrow \mathbb{Z}^k/H$. There is a homeomorphism $q^* : \Gamma^\Omega \rightarrow \Lambda^\infty$ such that $q^*(x)(m, n) = (x(q(m), q(n)), n - m)$ for all $x \in \Gamma^\Omega$ and $m \leq n \in \mathbb{N}^k$. For all $m \in \mathbb{N}^k$ and $x \in \Gamma^\Omega$ we have $\sigma^m(q^*(x)) = q^*(\sigma^{q(m)}(x))$.*

There is a groupoid isomorphism $\tilde{q} : q^(G_\Gamma) \rightarrow G_\Lambda$ such that*

$$(4.1) \quad \tilde{q}(((x, p, y), m)) = (q^*(x), m, q^*(y))$$

for all $(x, p, y) \in G_\Gamma$, $m \in \mathbb{Z}^k$ such that $q(p) = m$. The map $\pi_\infty = \pi_1 \circ \tilde{q}^{-1}$ defines a groupoid homomorphism $\pi_\infty : G_\Lambda \rightarrow G_\Gamma$ such that

$$\pi_\infty(q^*(x), m, q^*(y)) = (x, q(m), y) \quad \text{for } (q^*(x), m, q^*(y)) \in G_\Lambda,$$

and the restriction of π_∞ to $G_\Lambda^{(0)} \cong \Lambda^\infty$ is $(q^)^{-1}$.*

Proof. The map q^* is injective by construction. To see that it is surjective, fix $y \in \Lambda^\infty$. We prove that there is a well-defined map $\tilde{\pi}(y) : \Omega_P \rightarrow \Gamma$ such that $\tilde{\pi}(y)(q(m), q(n)) = \pi(y(m, n))$ for all $(m, n) \in \Omega_k$ and $q^*(\tilde{\pi}(y)) = y$. Fix $m \leq n$ and $m' \leq n'$ in \mathbb{N}^k such that $q(m) = q(m')$ and $q(n) = q(n')$. Let $\pi : \Lambda \rightarrow \Gamma$ be the quotient map $\pi(\alpha, m) = \alpha$. We have $y(m, n) = (\pi(y(m, n)), n - m)$ and $y(m', n') = (\pi(y(m', n')), n - m)$. We claim that $\pi(y(m, n)) = \pi(y(m', n'))$. To see this, let $p := n \vee n'$. Since π is a functor,

$$\pi(y(0, m))\pi(y(m, n))\pi(y(n, p)) = \pi(y(0, p)) = \pi(y(0, m'))\pi(y(m', n'))\pi(y(n', p)).$$

We have $d(\pi(y(0, m))) = q(m) = q(m') = d(\pi(y(0, m')))$, and the same calculation gives $d(\pi(y(m, n))) = d(\pi(y(m', n')))$ and $d(\pi(y(n, p))) = d(\pi(y(n', p)))$. So the factorisation property in Γ guarantees that $\pi(y(m, n)) = \pi(y(m', n'))$. It follows that there is a well-defined map $\tilde{\pi}(y) : \Omega_P \rightarrow \Gamma$ such that $\tilde{\pi}(y)(q(m), q(n)) = \pi(y(m, n))$ for all $(m, n) \in \Omega_k$. It is routine to check that $q \times q : (m, n) \mapsto (q(m), q(n))$ is a surjection from Ω_k to Ω_P , that each $\tilde{\pi}(y) \in \Gamma^\Omega$ and that each $q^*(\tilde{\pi}(y)) = y$. So q^* is surjective.

Since $(q^*)^{-1}(Z(\lambda)) = Z(\pi(\lambda))$ for $\lambda \in \Lambda$, and $q^*(Z(\gamma)) = \bigcup_{q(m)=d(\gamma)} Z((\gamma, m))$ for each $\gamma \in \Gamma$, we see that q^* is continuous and open. Moreover, for every $\gamma \in \Gamma$, $m \in \mathbb{N}^k$ and $x \in \Gamma^\Omega$ such that $d(\gamma) = q(m)$ and $s(\gamma) = r(x)$ we have $q^*(\gamma x) = (\gamma, m)q^*(x)$. Hence, for all $m \in \mathbb{N}^k$ and $x \in \Gamma^\Omega$ we have $\sigma^m(q^*(x)) = q^*(\sigma^{q(m)}(x))$.

We must next show that equation (4.1) gives a well-defined map $\tilde{q} : q^*(G_\Gamma) \rightarrow G_\Lambda$. Given $(x, p, y) \in G_\Gamma$, take $m \in \mathbb{Z}^k$ such that $q(m) = p$. We have $p = i - j$ for some $i, j \in P$ such that $\sigma^i(x) = \sigma^j(y)$. By definition of $P := q(\mathbb{N}^k)$, we can write $i = q(a_0)$ and $j = q(b_0)$ for some $a_0, b_0 \in \mathbb{N}^k$. Now $c := m - (a_0 - b_0) \in \ker(q) = H$, and so we have $c = c^+ - c^-$ for some $c^+, c^- \in \mathbb{N}^k$. Since $q(c) = 0$, we have

$q(c^+) = q(c^-)$. Now setting $a := a_0 + c^+$ and $b := b_0 + c^-$ we have $a, b \in \mathbb{N}^k$ and $m = a - b$, $q(a) = i + l$ and $q(b) = j + l$ where $l = q(c^+)$. Hence,

$$\sigma^a(q^*(x)) = q^*(\sigma^{q(a)}(x)) = q^*(\sigma^{i+l}(x)) = q^*(\sigma^{j+l}(x)) = \sigma^b(q^*(x)),$$

since $\sigma^i(x) = \sigma^j(y)$. Therefore $(q^*(x), m, q^*(y)) \in G_\Lambda$ and \tilde{q} is well defined. By construction \tilde{q} is a continuous groupoid homomorphism with inverse given by

$$G_\Lambda \ni (z, m, w) \mapsto (((q^*)^{-1}(z), q(m), (q^*)^{-1}(w)), m) \in q^*(G_\Gamma).$$

Since π_∞ is a composition of groupoid homomorphisms it is also a groupoid homomorphism. The remaining assertions are straightforward. \square

Proof of Theorem 4.5. To deduce this theorem from Corollary 3.12, we just need to establish that $\text{Iso}^\circ(G_\Lambda) \cong \Lambda^\infty \times H$ and is closed in G_Λ . Let $\pi_\infty : G_\Lambda \rightarrow G_\Gamma$ be the groupoid homomorphism of Lemma 4.6. Since G_Γ is essentially principal, we have $\text{Iso}^\circ(G_\Lambda) = \pi_\infty^{-1}(G_\Gamma^{(0)})$, which is clopen—and in particular closed—because π_∞ is continuous. The definition of π_∞ shows that $\pi_\infty^{-1}(G_\Gamma^{(0)}) = \{(x, m, x) : x \in \Lambda^\infty, m \in H\}$. By Lemma 4.6, $\pi_\infty|_{\{(x, m, x) : x \in \Lambda^\infty\}}$ is a homeomorphism onto Γ^Ω for each m , and so $\text{Iso}^\circ(G_\Lambda) \cong \Lambda^\infty \times H$ as topological spaces. \square

Proposition 4.7. *Resume the notation and hypotheses of Theorem 4.5. For each antisymmetric bicharacter ω of H , the set $C_\omega := \{x \in \Lambda^\infty : \sigma_x \sigma_x^* = \omega\}$ is a clopen invariant subset of Λ^∞ . The set Ξ of antisymmetric bicharacters such that $C_\omega \neq \emptyset$ is countable, and $\text{Prim } C^*(\text{Iso}^\circ(G_\Lambda), \sigma)$ is homeomorphic to the topological disjoint union $\bigsqcup_{\omega \in \Xi} C_\omega \times Z(\omega)^\wedge$.*

Proof. For each $x \in \Lambda^\infty$, write $\omega_x := \sigma_x \sigma_x^*$. As in the first part of the proof of [22, Lemma 3.3], the map $x \mapsto \omega_x$ is locally constant because σ is locally constant and H is finitely generated. The second part of the proof shows that the cohomology class of σ_x is constant along orbits; since $\sigma \mapsto \sigma \sigma^*$ induces an isomorphism of $H^2(H, \mathbb{T})$ with the group of antisymmetric bicharacters (see [29, Proposition 3.2]), it follows that $x \mapsto \omega_x$ is constant along orbits as well. Since every locally constant function is continuous, it follows that $x \mapsto \omega_x$ is constant on orbit closures.

Since $x \mapsto \omega_x$ is locally constant, for each bicharacter ω , the set $C_\omega = \{y \in \Lambda^\infty : \omega_y = \omega\}$ is clopen. This C_ω is also invariant because $x \mapsto \omega_x$ is constant along orbits. Choose an increasing sequence F_n of finite subsets of Λ^0 such that $\bigcup F_n = \Lambda^0$. Each $K_n := \bigcup_{v \in F_n} Z(v) \subseteq \Lambda^\infty$ is compact, so for each n the set K_n is covered by finitely many C_ω . Since the C_ω are mutually disjoint, it follows that $\Xi_n := \{\omega : C_\omega \cap K_n \neq \emptyset\}$ is finite. So $\Xi = \bigcup_n \Xi_n$ is countable.

We now have $C^*(\text{Iso}^\circ(G_\Lambda), \sigma) = \bigoplus_{\omega \in \Xi} C^*(\text{Iso}^\circ(G_\Lambda)|_{C_\omega}, \sigma)$. Fix $\omega \in \Xi$, and define $\mathcal{I}_\omega := \text{Iso}^\circ(G_\Lambda)|_{C_\omega} \cong C_\omega \times H$. The argument of the second and third paragraphs in the proof of [22, Proposition 3.1] shows that there is a 2-cocycle σ_ω of H and a 1-cochain b of \mathcal{I}_ω such that $(\delta^1 b)\sigma|_{\mathcal{I}_\omega} = 1_{C_\omega} \times \sigma_\omega$. Hence $C^*(\mathcal{I}_\omega, \sigma|_{\mathcal{I}_\omega}) \cong C_0(C_\omega) \otimes C^*(H, \sigma_\omega)$ by [34, Proposition II.1.2].

As observed earlier, $\text{Prim } C^*(H, \sigma_\omega) \cong \hat{S}_{\sigma_\omega \sigma_\omega^*} = Z(\omega)^\wedge$; the result follows. \square

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