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Pricing Parisian and ParAsian options analytically

Song-Ping Zhu

University of Wollongong, spz@uow.edu.au

Wen-Ting Chen

University of Wollongong, wtchen@uow.edu.au

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In this paper, two analytic solutions for the valuation of European-style Parisian and ParAsian options under the Black-Scholes framework are, respectively, presented. A key feature of our solution procedure is the reduction of a three-dimensional problem to a two-dimensional problem through a coordinate transform designed to combine the two time derivatives into one. Compared with some previous analytical solutions, which still require a numerical inversion of Laplace transform, our solutions, written in terms of double integral for the case of Parisian options but multiple integrals for the case of ParAsian options, are both of explicit form; numerical evaluation of these integrals is straightforward. Numerical examples are also provided to demonstrate the correctness of our newly derived analytical solutions from the numerical point of view, through comparing the results obtained from our solutions and those obtained from adopting other standard finite difference approaches. © 2012 Elsevier B.V.

Keywords

analytically, pricing, options, parisian, parAsian, ERA2015

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Pricing Parisian and *Parasian* options analytically

Song-Ping ZHU * Wen-Ting CHEN †

Abstract

In this paper, two exact and analytic solutions for the valuation of European-style Parisian and *Parasian* options under the Black-Scholes framework are respectively presented. To the best of our knowledge, closed-form analytic formulae have never been found for the pricing of Parisian and *Parasian* options, although quite a few approximate solutions and numerical approaches have been proposed. A key feature of our solution procedure is the reduction of a three-dimensional problem to a two-dimensional problem through a coordinate transform that has elegantly “absorbed” the directional derivative associated with the “barrier time” into the time derivative and thus resulted in two coupled, but simplified PDE (partial differential equation) systems. For Parisian options, the coupled PDE systems are then analytically solved by applying the Laplace transform technique in conjunction with the construction of “moving windows”, which are introduced to evaluate the option prices backwards, slide by slide, until the value of the option at a given time and trigger value is found for a given underlying price. On the other hand, due to the non-resetting mechanism of the *Parasian* option, the coupled PDE systems of this type of options are much more complicated than those of their Parisian counterparts, and the “moving window” technique fails in this case. Alternatively, the double Laplace transform technique is then applied to solve for the option prices in the Laplace space. However, our success of obtaining closed-form analytical solution hinges on overcoming

*School of Mathematics and Applied Statistics, University of Wollongong NSW 2522, Australia. (spz@uow.edu.au)

†School of Mathematics and Applied Statistics, University of Wollongong NSW 2522, Australia. (wc904@uow.edu.au)

the difficulty of performing Laplace inversions analytically. Finally, we have compared the results obtained from the newly-derived analytical solutions with those obtained through a numerical solution procedure. Such a comparison has not only reinforced the correctness of our newly-derived analytical solutions from numerical point of view, but also has demonstrated the efficiency of using the newly-derived analytical solutions to calculate option prices in finance practice.

AMS(MOS) subject classification.

Keywords. Parisian options, *Parasian* options, analytical solution, Laplace transform method.

1 Introduction

Parisian and *Parasian* options are simple extensions of classical barrier options, with a “trigger” device added on, mainly for the purpose of preventing option traders from deliberately manipulating the underlying asset price when it is close to the barrier, in order to gain advantage in the option position they hold. However, such a simple addition of a financial clause has caused considerable difficulties in quantitatively pricing these options, at least analytically. In this paper, two closed-form analytic formulae for the prices of Parisian and *Parasian* options are presented for the first time.

While both Parisian options and *Parasian* options share the same feature that there is a separate “clock” set up to record the total time that the underlying has passed the barrier (either above or below, depending on the type of the barrier option), the main difference between them is how the clock is reset. If one accumulates the time spent in a row and resets it to zero each time the underlying price crosses the barrier, this type is referred to as continuous Parisian options, or simply Parisian options. On the other hand, if one adds the time spent below or above the barrier without resetting the accumulated time to zero each time the underlying crosses the barrier, these options are named as cumulative Parisian

options, or simply *Parasian* options. For simplicity of reference, we may sometimes in this paper refer to these two options as “Parisian-type options” when there is no need to distinguish them.

Financially, there are at least two reasons for the introduction of Parisian-type options. Firstly, comparing with the classical barrier options, Parisian-type options are less susceptible to short-term movements of the underlying price, since a single touch of the barrier can no longer trigger the knock-in or knock-out feature of these options. Secondly, the hedging problem close to the barrier, which is usually encountered in the trading of the classical barrier options, can somehow be reduced, or at least “smoothed”, for Parisian-type contracts [2]. Of course, Parisian-type options still possess many inherent features of classical barrier options.

The valuation problem of Parisian-type options has been recognized as much more difficult than that of classical barrier options [5]. While a closed-form analytical solution for the latter has already been found [5], the former could only be solved approximately. The difficulty of pricing Parisian-type options mainly comes from the co-existence of two different barriers specified in the option contracts: a barrier of the underlying asset price and a barrier time, which is defined as the accumulated time that the underlying has spent above or below the barrier. Mathematically, the specification of the Parisian options has made a 3-D (three-dimensional) PDE (partial differential equation) system coupled with a 2-D PDE system (or two 3-D PDE systems coupled for the *Parasian* case), through the prescribed continuity of the option price and the Delta across the barrier. Such coupled systems are hard to handle either numerically or analytically. On one hand, although the method of images can be used to solve for the price of classical barrier options effectively, it is the conjunction of the two barriers that has hampered the application of this powerful method to Parisian-type options. On the other hand, traditional numerical methods, such as the Monte Carlo simulations, would be inefficient, as one needs to trace the time accumulated in the “trigger” all the time. Even within Parisian-type options themselves, the pricing of Parisian and *Parasian* options, albeit being financially similar, can be quite

different from mathematical point of view, as resetting or no-resetting of the amount of time already accumulated in the trigger has changed the solution approach required to produce analytical pricing formulae.

In the literature, several researchers have also addressed the pricing of Parisian-type options. Predominately two types of valuation techniques, the quasi-analytic approaches and the numerical methods, are well documented. Of all the quasi-analytic methods, the most influential approach was the one proposed by Chesney *et al.* [1]. They used the theory of Brownian excursions and defined the value of a Parisian option in terms of an integral expressed as an inverse Laplace transform. Afterwards, their framework has been further developed by Hartly [3], Hugonnier [4], and Schröder [8] to price and hedge the Parisian-type options. Numerical methods, as another alternative, were also intensively developed recently. A typical method in this category is the PDE approach proposed by Wilmott *et al.* [2]. In their article, two PDE systems governing the prices of Parisian and *Parasian* options are established, and then solved by using the explicit finite difference scheme. Whilst flexible and easy to implement, there are at least three major deficiencies that should be pointed out regarding their approach. Firstly, in their work [2], the singularities associated with adopting appropriate PDEs to price Parisian-type options are not explored at all; appropriately identifying these singularities and dealing with them not only make the developed PDE system (not just the PDE itself) correctly reflects what the financial clauses dictate, but also ensure any meaningful numerical scheme would lead to the correct solution. Secondly, some boundary conditions set in their framework do not seem to correctly represent the corresponding financial clauses of these options. Furthermore, we believe that at least one boundary condition connecting the pricing domains has been totally overlooked by them, and thus their pricing systems are not properly closed. Lastly, in terms of the explicit finite difference method they adopted, it is only conditionally stable, resulting in computational inefficiency and low accuracy, especially for the current 3-D problems.

In this paper, we present two closed-form analytic solutions for the valuation of Parisian

and *Parasian* options, respectively, under the Black-Scholes framework. Based on several reasonable financial arguments, two new PDE systems for the prices of Parisian and *Parasian* options have been established first, with one boundary condition added on for each system to ensure its closeness. Moreover, the singularities associated with these systems are also thoroughly discussed. The newly established PDE systems are then simplified through a coordinate transform that has elegantly “absorbed” one dimension associated with the “barrier time” into the time direction. The purely analytical procedures, adopted afterwards, differ w.r.t. (with respect to) the resetting mechanisms specified in the option contracts. For a Parisian option, the resulted simplified PDE system is solved analytically by applying the Laplace transform technique together with the construction of “moving windows” to evaluate the option prices backwards, slide by slide, until the given time has been reached, whereas for a *Parasian* option, its non-resetting mechanism has obstructed the application of the “moving window” technique, and we apply the double Laplace transform as an alternative to analytically solve for its option price. Finally, through Laplace inversions, two completely analytic closed-form solutions are obtained for the prices of Parisian and *Parasian* options, respectively. It should be pointed out that our explicit pricing formulae for pricing Parisian-type options should be valuable in both theoretic and practical senses. Theoretically, although there are several existing methods, as mentioned above, to price Parisian-type options, an explicit and closed-form exact solutions are presented for the first time¹. Practically, the final form of our solutions, written in terms of a linear combination of several integrals, can be used to price Parisian-type options accurately and efficiently. With a growing demand of trading exotic options in today’s finance industry, our solution procedures may lead to the development of pricing formulae for other exotic derivatives.

The rest of the paper is organized as follows. In Section 2, we introduce the PDE

¹A solution written in terms of the inverse Laplace transform without the inversion being carried out analytically is still of closed form. However, since numerical inversion of Laplace transform is an ill-posed problem, such kind of solutions is not truly “explicit” as far as the computation of the numerical values of an option is concerned.

systems that the prices of the Parisian and *Parasian* options must satisfy. In Section 3, we present our analytic solution procedure in detail. In Section 4, some numerical examples and discussions are presented to illustrate the performance of our analytic solutions when numerical values need to be calculated from them. Our brief concluding remarks are given in the last section.

2 PDE systems for pricing Parisian and *Parasian* options

As pointed out previously, under the Black-Scholes framework, the PDE systems for the prices of Parisian-type options have already been established in [2]. However, the complexities associated with their PDE systems have hindered the application of various analytic methods. In this section, two simplified PDE systems governing the prices of Parisian-type options are provided, which pave the way for the achievement of closed-form analytic solutions for both options. In specific, the re-establishment of the PDE system for the valuation of the Parisian options will be considered in the first subsection, while that of the *Parasian* options will be provided in the second subsection.

2.1 Parisian options

A Parisian option is a special kind of barrier options for which the knock-in or knock-out feature is only activated if the underlying remains continually in breach of the barrier \bar{S} for a pre-specified time period \bar{J} . Like classical barrier options, Parisian call options can have four different forms: down-and-out, up-and-out, down-and-in, up-and-in call (similarly four types of Parisian puts as well). Without loss of generality, we shall consider, here in this paper, the pricing of a European-style Parisian up-and-out call option as an example to demonstrate our solution approach; the extensions to other cases should be rather straightforward, based on the parity relationships established in [1].

Comparing with classical barrier options, the pricing of Parisian options requires the value of a new state variable J , the barrier time (the time recorded in the “trigger clock”, which dictates the “knock-in” or “knock-out” action once the trigger value \bar{J} is reached), which is defined as the total time the underlying has spent continually above (for up-and-out and up-and-in Parisian options) or below (for down-and-out and down-and-in Parisian options) the barrier. For the case of an up-and-out barrier, we have

$$\begin{cases} J = 0, & dJ = 0, & S \leq \bar{S}; \\ dJ = dt, & & S > \bar{S}, \end{cases}$$

where \bar{S} is a preset barrier of the underlying. The above expression states that when the underlying is beyond \bar{S} , the state variable J starts to accumulate values at the same rate as the passing time t , and when the underlying is equal to or below \bar{S} , J is reset to zero, and remains zero. If the barrier time is not reset to zero each time the underlying crosses the barrier \bar{S} , the option contract becomes *Parasian* type. It should be remarked that the resetting mechanism associated with a Parisian option contract has made a Parisian up-and-out call option always worth more than its *Parasian* counterpart, as the “out” feature of the option has been somewhat amplified by the *Parasian* specification, as a result of the risk being knocked out is now higher with the residue possibly left in the trigger clock.

When the “barrier” of a Parisian option takes some extreme values, one can easily obtain the option price, as in these cases, the Parisian option degenerates to either a classical barrier option or a vanilla option. For example, in the case of a Parisian up-and-out call option, when the trigger value \bar{J} approaches zero, the option will be immediately knocked out once the underlying touches the barrier from the below, which is the same as the specification of a classical barrier call option with up-and-out feature. On the other hand, if \bar{J} becomes infinitely large, the knock-out feature will never be activated, and thus the option remains as a European call. In terms of the barrier \bar{S} , when \bar{S} approaches zero, as long as the time to maturity exceeds the difference between the trigger value and the

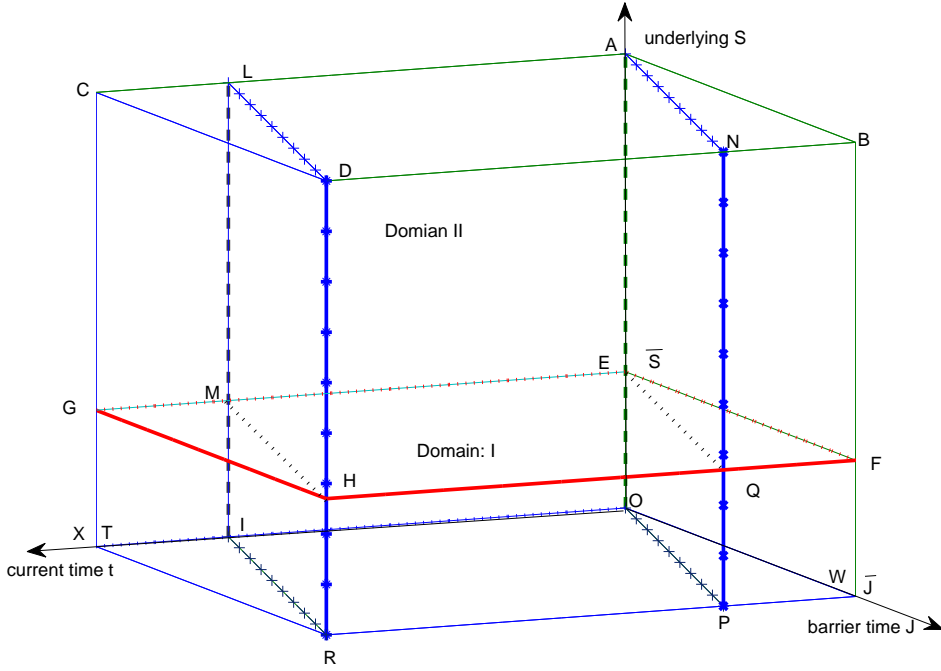


Figure 1: Pricing domain of a Parisian up-and-out option

accumulated barrier time, which means that J can reach \bar{J} for sure, the knock-out feature will certainly be activated, and thus the option price becomes zero. But, if the time to maturity is less than that difference, the barrier time J will never reach \bar{J} , and thus the option will never be knocked out. The price then is the same as that of a European call. Lastly, when \bar{S} approaches infinity, it is clear that neither the barrier nor the trigger value could be reached, and thus the option degenerates to a European call again.

For any other non-degenerate cases, the price of a Parisian option then depends on the underlying price S , the current time t and the barrier time J , in addition to other parameters such as the volatility, risk-free interest rate and the expiry time. If modeled under the Black-Scholes economy, we simply assume that the underlying asset, S , that attracts a continuous dividend payment at a rate D , follows a lognormal Brownian motion given by

$$dS = (\mu - D)Sdt + \sigma SdZ, \quad (2.1)$$

where Z is a standard Brownian motion.

Now, let $V_1(S, t)$ and $V_2(S, t, J)$ denote the option prices in the region \bar{I} and \bar{II} , respectively, with the regions \bar{I} and \bar{II} referring to the plane GEOX and the cuboid ABDCGEFH, respectively, as shown in Fig 1. Clearly, in region \bar{I} , the variable J remains unchanged, as a result of the underlying being below the barrier. By applying the Feynman-Kac theorem [9] to (2.1), $V_1(S, t)$ should satisfy the classical BS (Black-Scholes) equation

$$\frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \quad (2.2)$$

where $\mathbb{L} = \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (r - D)S \frac{\partial}{\partial S} - rI$, with I being the identity operator. The terminal condition in this region is given by the payoff function of a European call option, i.e.,

$$V_1(S, T) = \max(S - K, 0). \quad (2.3)$$

Besides the terminal condition, a set of boundary conditions along the S direction is also needed to solve for V_1 . The fact that a call option becomes worthless when the underlying price approaches zero gives

$$\lim_{S \rightarrow 0} V_1(S, t) = 0, \quad (2.4)$$

whereas the continuity of the option price across the barrier \bar{S} demands

$$\lim_{S \rightarrow \bar{S}} V_1(S, t) = \lim_{S \rightarrow \bar{S}} V_2(S, t, 0). \quad (2.5)$$

On the other hand, in region \bar{II} , the barrier time J starts to accumulate. As a result, $V_2(S, t, J)$ is governed by a modified BS equation [2]

$$\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + \mathbb{L}V_2 = 0, \quad (2.6)$$

with the operator \mathbb{L} being the same as that defined earlier. Appropriate boundary conditions are also needed to close the PDE system. From the definition of the “trigger clock”, it is clear that when the variable J reaches the trigger value \bar{J} , the option becomes worthless,

i.e.,

$$\lim_{J \rightarrow \bar{J}} V_2(S, t, J) = 0. \quad (2.7)$$

Also, due to the fact that it would take infinite amount of time for an infinitely large underlying price to fall back to the barrier \bar{S} , the option must be worth nothing when S becomes very large, i.e.,

$$\lim_{S \rightarrow \infty} V_2(S, t, J) = 0. \quad (2.8)$$

Moreover, the boundary condition at $S = \bar{S}$ is specified by the so called “reset condition”, i.e.,

$$\lim_{S \rightarrow \bar{S}} V_2(S, t, J) = \lim_{S \rightarrow \bar{S}} V_1(S, t), \quad (2.9)$$

which indicates that J is reset to zero every time the underlying S falls back to the barrier \bar{S} from above. It should be remarked that this “reset condition” is removed for *Parasian* options, as no resetting mechanism is specified in a *Parasian* option contract, a feature that distinguishes the pricing of *Parasian* options from that of their Parisian counterparts.

Equations (2.2)-(2.9) constitute the differential system proposed by Wilmott *et al.* [2], the solution of which will give rise to the value of Parisian option at any underlying price S , any barrier time J , and any time t before the expiration T . Remarkably, Wilmott *et al.*'s pricing system can be viewed as substantial work in determining the price of Parisian options from the PDE point of view. However, it should also be pointed out that there are at least two fundamental flaws in their pricing system. Firstly, the boundary condition they set at $S = \infty$ is wrong for the case $\bar{J} - J > T - t$. In this case, the knock-out feature of the Parisian option will never be activated, due to the lack of enough time for J to reach the trigger value \bar{J} , and thus the price of a Parisian up-and-out call option equals that of a European call option, which will never become zero as S approaches infinity. Secondly, the closeness of Wilmott *et al.*'s PDE system is not clear. We have noticed that in their explicit finite difference approach, they must have implicitly assumed that the option price across the barrier satisfies (2.6). Without this implicit assumption, their approach could have

not produced a unique solution. Unfortunately, (2.6) is not listed explicitly as a boundary condition, which we believe to be necessary to properly close the PDE system.

Furthermore, it should also be remarked that Wilmott *et al.*'s PDE system is rather complicated, with a 2-D and a 3-D PDE system being coupled, which greatly hinders the application of various methods to solve the price of Parisian options either numerically or analytically. In fact, by realizing the following financial arguments, the original pricing domain, i.e., the regions \bar{I} and \bar{II} , can be reduced, and thus, Wilmott *et al.*'s PDE system can be further simplified. Firstly, the prism ANBFEQ (denoted by \bar{III} hereafter) should be excluded, as in this region, the elapsed time is always less than the barrier time, i.e., $0 \leq t < J$, a case that will never happen. Secondly, in the prism LCDHMG (denoted by \bar{IV} hereafter), the barrier time J has no effect on the option price, because in this domain, there is not enough time for J to reach \bar{J} , and thus the option will never be knocked out. Consequently, the option price in the prism \bar{IV} does not vary w.r.t. J , and should be the same as that of a European call option at the time to expiry $T - t$. Now, it is quite clear that the two chunks of the original pricing domain, i.e., the prisms \bar{III} and \bar{IV} , can be "cut off" when considering the valuation of Parisian options, leaving the actual pricing domain as the plane MEOI plus the domain ANDLMEQH, defined as

$$I : \{0 \leq S \leq \bar{S}, 0 \leq t \leq T - \bar{J} \mid J = 0\},$$

$$II : \{\bar{S} \leq S < \infty, J \leq t \leq J + T - \bar{J} \mid 0 \leq J \leq \bar{J}\},$$

respectively. It should be remarked that simplifying the original pricing domains has mainly resulted in two significant consequences. On one hand, by cutting the prism \bar{IV} , one no longer needs to specify the boundary condition at $S = \infty$ for $\bar{J} - J > T - t$, as this case is excluded from the new pricing domain. On the other hand, it is this simplification that has paved the way for the development of our analytical approach, as shall be discussed later.

The simplification consists of two major modifications made to Wilmott *et al.*'s pricing

system, in order to derive a properly closed PDE system on the simplified domain. Firstly, the original terminal condition in the region I , i.e., the payoff function, should be replaced by the price of a European call option at the time to expiry \bar{J} , denoted by $V_{BS}(S, \bar{J})$ as

$$V_1(S, T - \bar{J}) = V_{BS}(S, \bar{J}).$$

This is not surprising, because the value on the plane LIXC is known as a European call option price at the time to expiry $T - t$, as discussed earlier. Secondly, we should explicitly demand that the option Delta be continuous across the barrier $S = \bar{S}$, i.e.,

$$\lim_{S \rightarrow \bar{S}} \frac{\partial V_1}{\partial S}(S, t) = \lim_{S \rightarrow \bar{S}} \frac{\partial V_2}{\partial S}(S, t, 0). \quad (2.10)$$

Clearly, there are now three conditions that link the solutions in both regions effectively, with one being the reset condition (2.9), and the other two being the option price and the option Delta continuous across the barrier, i.e., (2.5) and (2.10), respectively. Hereafter, for simplicity, only (2.10) is referred to as the “connectivity condition”. It should be remarked that this additional connectivity condition is a necessity to make the pricing system properly closed. Of course, one may wonder why Wilmott *et al.* [2] could still produce a set of seemingly correct numerical results from an unclosed PDE system. This is because such a “connectivity condition” can be unintentionally satisfied when the option price has satisfied the PDE (2.6) itself across the barrier, a trick similar to the fictitious-point technique in dealing with Neumann boundary conditions [10]. However, this requires an implicit assumption that the option prices V_1 and V_2 be at least twice differentiable across the barrier, which is much stronger than the connectivity condition we have imposed. From the viewpoint that a solution produced with a strong condition taken into consideration should also satisfy a weak condition, it is therefore reasonable that the pricing system with a strong condition yields the same result as that with a weak condition, as long as the latter is well-posed.

Before finally setting up the properly closed pricing system, we should carefully examine the singularities associated with the pricing of Parisian options. In fact, exploring singularities is an indispensable step in establishing any PDE systems, since otherwise, the results produced from those systems would be incorrect at least around those singularities. In the literature, no one has discussed this issue thoroughly, and we thus believe that any previously published numerical solution approach should be revisited to make sure that these singularities have been properly taken care before the obtained results can be faithfully trusted.

The singularities of the Parisian option price mainly result from the introduction of a non-cumulative trigger for the barrier \bar{S} , i.e., the introduction of the plane NQHD (denoted by \overline{VI} hereafter) has introduced a singular line DH which is the intersection of the plane DHGC and the plane \overline{VI} . If the line DH is viewed to belong to the plane DHGC, the terminal condition should be imposed. On the other hand, If the line DH is viewed as part of the plane \overline{VI} , the knock-out condition should then be imposed. From a financially meaningful argument, we believe that the line DH should be part of the plane \overline{VI} , in order to ensure that any point on the plane LDHM (denoted by \overline{V} hereafter) may still reach \bar{J} as time further increases. Such a demand is consistent with the fact that the special point L $(\infty, T - \bar{J}, J)$ should lie on the plane \overline{V} , because it would take infinite amount of time for an infinitely large underlying price to fall back to \bar{S} , and consequently the trigger value \bar{J} would for sure be reachable when time increases by a finite amount. In fact, not only the line DH is singular, but also the entire plane \overline{V} . There should be a jump between the option price at a point very close to the plane \overline{V} , but in the prism \overline{IV} , and the option value at a point on the plane \overline{V} . Financially, one could expect that the value of the former is larger than that of the latter, because for any point on the plane \overline{V} , there is a risk that \bar{J} will be reached, whereas for any point in the prism \overline{IV} , no matter how close it is to the plane \overline{V} , it is impossible for the trigger value \bar{J} to be reached.

The existence of the plane \overline{VI} also introduces another singularity along the line QH. If we classify the line QH as part of the plane \overline{VI} , then the option price at any point $S = \bar{S}$

on the plane EQHM but not on the line QH is non-zero, because no matter how close this point is to the line QH, J will be reset to zero, when the underlying touches the barrier. However, the option price on the line QH is always zero, because on this line, the trigger value \bar{J} is reached, and the “knock-out” takes place.

Of course, in reality, which side the line DH or QH belongs to could be clearly defined in a contract. What one must realize is that any ambiguity left in a Parisian option contract on the belonging of these two boundary lines would ultimately lead to different views in terms of the value of the contract in the event that the underlying asset price and the barrier time reach these very special values.

Now summarizing what has been said, the PDE system for pricing a European-style Parisian up-and-out call option under the BS model can be written as:

$$\begin{aligned}
\mathcal{A}_1 : & \left\{ \begin{array}{l} \frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T - \bar{J}) = V_{BS}(S, \bar{J}), \\ \lim_{S \rightarrow 0} V_1(S, t) = 0, \\ \lim_{S \rightarrow \bar{S}} V_1(S, t) = \lim_{S \rightarrow \bar{S}} V_2(S, t, 0), \end{array} \right. \\
& \text{for } t \in [0, T - \bar{J}], \quad J \in [0, \bar{J}], \quad S \in [0, \bar{S}]; \\
\mathcal{A}_2 : & \left\{ \begin{array}{l} \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + \mathbb{L}V_2 = 0, \\ V_2(S, t, \bar{J}) = 0, \\ \lim_{S \rightarrow \infty} V_2(S, t, J) = 0, \\ \lim_{S \rightarrow \bar{S}} V_2(S, t, J) = \lim_{S \rightarrow \bar{S}} V_1(S, t), \end{array} \right. \\
& \text{for } t \in [J, T - \bar{J} - J], \quad J \in [0, \bar{J}], \quad S \in [\bar{S}, \infty); \\
& \text{Connectivity condition : } \lim_{S \rightarrow \bar{S}} \frac{\partial V_1}{\partial S}(S, t) = \lim_{S \rightarrow \bar{S}} \frac{\partial V_2}{\partial S}(S, t, 0), \quad \text{for } t \in [0, T - \bar{J}].
\end{aligned} \tag{2.11}$$

2.2 *Parasian* options

Contrary to the unique “resetting” feature of a Parisian option, *Parasian* options are introduced with no reset of the barrier time J ; the knock-in or knock-out is only activated if the cumulative time spent beyond or below \bar{S} exceeds some prescribed time \bar{J} . Of course, like Parisian options, *Parasian* options can also have eight different forms. For the purpose of illustration, we shall only consider the pricing of a European-style *Parasian* up-and-out call option, as it should also be straightforward to extend our work to other forms of *Parasian* options by using the parity relationships established in [1].

Apart from this main difference, a *Parasian* option behaves quite similarly to its Parisian counterpart. For extreme barrier values, it also degenerates to either a classical barrier option or a vanilla option, just the same as what the corresponding Parisian option does. For brevity, we shall not repeat the details here. For the non-degenerate cases, the pricing of *Parasian* options also requires three state variables, i.e., the current time t , the underlying S and the barrier time J . While the former two variables t and S are assumed to follow the same dynamics as those with the Parisian option, the dynamics of the latter need to be modified so that it is not reset at \bar{S} [2], i.e.,

$$dJ = \begin{cases} 0, & S \leq \bar{S}; \\ dt, & S > \bar{S}. \end{cases}$$

With these three state variables, Wilmott *et al.* [2] also established a PDE system governing

the price of the *Parasian* option as

$$\left\{ \begin{array}{l} \frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T) = \max(S - K, 0), \\ \lim_{S \rightarrow 0} V_1(S, t; J) = 0, \\ \lim_{S \rightarrow \bar{S}} V_1(S, t; J) = \lim_{S \rightarrow \bar{S}} V_2(S, t, J), \\ \text{for } S \in [0, \bar{S}]; \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + \mathbb{L}V_2 = 0, \\ V_2(S, t, \bar{J}) = 0, \\ \lim_{S \rightarrow \infty} V_2(S, t, J) = 0, \\ \text{for } S \in [\bar{S}, \infty); \end{array} \right.$$

for $t \in [0, T], J \in [0, \bar{J}]$. (2.12)

Whilst elegant and financially meaningful, it should be pointed out that the boundary condition they imposed at $S = \infty$ is at odds with the financial clause that the *Parasian* option price at the large S end would never be zero, when $T - t < \bar{J} - J$. This is because there is not enough time left for J to reach \bar{J} in this case, no matter how large the underlying is above \bar{S} , as we have already discussed for the case of Parisian options in Section 2.1. One should also notice that the PDE system (2.12) is not properly closed, just like the case of Parisian options; an additional connectivity condition in terms of Delta needs to be prescribed across the barrier.

Unfortunately, taking away the reset feature has also removed the simplicity we have for Parisian options. One should notice that if the barrier time J no longer needs to be reset, the trigger clock containing some residues will affect the option price when the underlying falls below \bar{S} . Consequently, comparing with Parisian options, the lower cuboid

EFHGXOWR (denoted by \overline{VII} hereafter), as shown in Fig 1, should also be included as part of the pricing domain for *Parasian* options, as J serves as a parameter in this particular region as well. This additional region, resulting from the non-resetting mechanism specified in a *Parasian* option contract, has no doubt added the complexity of solving for its option prices accurately and efficiently.

Despite a 3-D cuboid \overline{VII} having replaced a 2-D plane \bar{I} , the simplification of the pricing domain, which is a key step in the development of our analytic approach for finding the price of *Parasian* options, can still proceed like in the case of Parisian options. In specific, both of the prisms ANBWOP (denoted by \widetilde{III} hereafter) and LCDRIX (denoted by \widetilde{IV} hereafter) can be excluded for the same reasons stated in Section 2.1. Then, following almost the same arguments demonstrated in Section 2.1, we can re-establish a properly closed PDE system governing the price of a *Parasian* up-and-out call option as:

$$\begin{aligned}
\mathcal{A}_1 : & \left\{ \begin{array}{l} \frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T - \bar{J} + J; J) = V_{BS}(S, \bar{J} - J), \\ \lim_{S \rightarrow 0} V_1(S, t; J) = 0, \\ \lim_{S \rightarrow \bar{S}} V_1(S, t; J) = \lim_{S \rightarrow \bar{S}} V_2(S, t, J), \end{array} \right. \\
\text{for } S \in [0, \bar{S}]; & \\
\mathcal{A}_2 : & \left\{ \begin{array}{l} \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J} + \mathbb{L}V_2 = 0, \\ V_2(S, t, \bar{J}) = 0, \\ \lim_{S \rightarrow \infty} V_2(S, t, J) = 0, \\ \lim_{S \rightarrow \bar{S}} V_2(S, t, J) = \lim_{S \rightarrow \bar{S}} V_1(S, t; J), \end{array} \right. \\
\text{for } S \in [\bar{S}, \infty); & \\
\text{Connectivity condition : } & \lim_{S \rightarrow \bar{S}} \frac{\partial V_1}{\partial S}(S, t; J) = \lim_{S \rightarrow \bar{S}} \frac{\partial V_2}{\partial S}(S, t, J), \\
\text{for } t \in [J, T - \bar{J} + J], J \in [0, \bar{J}]. & \tag{2.13}
\end{aligned}$$

Clearly, there are two major differences between the newly established PDE system (2.13)

and the one proposed in [2] (i.e., the PDE system (2.12)). Firstly, the original terminal condition for V_1 in (2.12) has been replaced by a European call option price at the time to expiry $\bar{J} - J$. Secondly, to properly close the PDE system (2.12), we have further assumed that the option Delta be continuous across the barrier, i.e.,

$$\lim_{S \rightarrow \bar{S}} \frac{\partial V_1}{\partial S}(S, t; J) = \lim_{S \rightarrow \bar{S}} \frac{\partial V_2}{\partial S}(S, t, J). \quad (2.14)$$

Comparing (2.13) with (2.11), one can observe that pricing *Parasian* options is much more complicated than pricing their Parisian counterparts. The complexity mainly comes from two aspects. Firstly, when $S < \bar{S}$, the *Parasian* option price $V_1(S, t; J)$ varies w.r.t. the parameter J , whereas for a Parisian option, its price $V_1(S, t)$ is independent of J . Secondly, the connectivity condition of the *Parasian* option should be applied across the entire barrier plane $S = \bar{S}$, while for its Parisian counterpart, this condition is only specified across the line $S = \bar{S}$, $J = 0$.

On the other hand, it should be pointed out that the two newly established PDE systems (2.11) and (2.13) are quite different from the corresponding ones used in [2]. One may wonder whether different pricing systems will yield the same option price. This issue will be further discussed in Section 4.1. It should also be remarked that although (2.11) and (2.13) are linear, they are both coupled systems, with the solution in one domain being the boundary condition for another domain. Whilst difficult, we still manage to develop two analytic approaches to solve (2.11) and (2.13), respectively. These approaches are discussed in the next section.

3 Our solution procedure

Although the newly established PDE systems are simpler, to some extent, than the original ones used in [2], they are still in 3-D, a difficulty that hampers the achievement of the analytic solutions. By taking advantage of the shape of the current pricing domain,

however, the above 3-D PDE systems (2.11) and (2.13) can be further simplified to two 2-D PDE systems, which can then be easily solved analytically by utilizing the Laplace transform technique.

In this section, we shall demonstrate our solution procedure in detail. The transformations from (2.11) and (2.13) to 2-D PDE systems will be discussed in the first subsection. Then in the second subsection, we shall explore the degenerations of the resulted 2-D PDE systems for some special cases. Finally, the details of analytically solving for the prices of the Parisian and *Parasian* options from the 2-D PDE systems will be presented in the third and fourth subsections, respectively.

3.1 “Moving windows” and the 2-D PDE systems

Clearly, to transform (2.11) to a 2-D PDE system, we only need to deal with the system governing V_2 , i.e., \mathcal{A}_2 , as the one that V_1 needs to satisfy is already in 2-D. On the other hand, to reduce dimensionality of a PDE system usually requires the application of some sorts of advanced transformation techniques, such as the Fourier transform, the Laplace transform, and so on. The main drawback of the traditional way is that either the resulted 2-D system could still be hard to solve analytically or, in the case that a solution can be found in the transformed space, the inverse of this solution could still not be found. For Parisian options, however, one additional variable in \mathcal{A}_2 can still be elegantly “absorbed” without applying any transformation methods mentioned above, which will pave the way for our final analytical solution of this problem.

One can observe that the pricing domain II is a parallelepipedon, and can be decomposed into infinite many cross-sections (which will be referred to as “slides” hereafter), all of which are of 45° angles to both of the plane $t = 0$, and $J = 0$. However, for any given state point (S, t, J) in II , there is a unique slide passing through that point. In light of this geometric characteristic of the domain II , it is clear that the option value V_2 at any given point (S, t, J) can be uniquely determined as long as enough information along the very

slide passing through that static point is known. In other words, the original 3-D problem can be decomposed into a set of 2-D problems defined on each slide, if viewed from a 45° rotated coordinate system.

Mathematically, to obtain the PDE governing V_2 in the rotated coordinate system, we can replace the sum of the two partial derivatives appearing in \mathcal{A}_2 , i.e., $\frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial J}$, by the directional derivative $\sqrt{2}\frac{\partial V_2}{\partial l}$, which represents the instantaneous rate of change of the function V_2 at the point (t, J) , in the direction of $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, i.e., a counter clockwise 45° plane viewed from the top. As a result, the governing equation in the new coordinate system can be written as

$$(\sqrt{2}\frac{\partial}{\partial l} + \mathbb{L})V_2(S, l; t) = 0. \quad (3.15)$$

Here t serves as a parameter, which identifies the slide passing through the point $(\bar{S}, t, 0)$, and thus the solution $V_2(S, l; t)$ in the new coordinate system corresponds to $V_2(S, t, J)$, with $t \in [t, \frac{l}{\sqrt{2}} + t]$ and $J = \frac{l}{\sqrt{2}}$ in the original 3-D coordinate system. In other words, the new solution $V_2(S, l; t)$ equals $V_2(S, t + \frac{l}{\sqrt{2}}, \frac{l}{\sqrt{2}})$ in the original pricing domain II .

It should be remarked that replacing the derivatives by a directional derivative along the l direction is also financially meaningful, because such a replacement corresponds to the restriction that when $S > \bar{S}$, t and J increase at the same rate as stated in the definition of the “barrier time”, and thus the movement of the underlying can be viewed as if it were along the diagonal plane between $t = 0$ and $J = 0$. It is this restriction that has reduced the 3-D problem to a 2-D problem in a rotated coordinate system. Furthermore, the constant $\sqrt{2}$ can be absorbed by rescaling l , i.e., $l' = \frac{l}{\sqrt{2}}$, and (3.15) becomes

$$(\frac{\partial}{\partial l'} + \mathbb{L})V_2(S, l'; t) = 0, \quad (3.16)$$

which is nothing but the BS equation! Such a degeneration makes perfect sense too, because for $S > \bar{S}$, the movements of the underlying are still the same as $S < \bar{S}$, except that they should be viewed from a different space, and thus the governing equation in the rotated

and rescaled coordinate system, i.e., along this 45° plane, should certainly be nothing but the BS equation.

The boundary conditions set for $V_2(S, l'; t)$ can be extracted from the corresponding boundary conditions that $V_2(S, t, J)$ needs to satisfy. The fact that the trigger value \bar{J} will definitely be crossed if the underlying price is infinitely large gives

$$\lim_{S \rightarrow \infty} V_2(S, l'; t) = 0. \quad (3.17)$$

Moreover, the knock-out condition in the rotated coordinate system now reads

$$\lim_{l' \rightarrow \bar{J}} V_2(S, l'; t) = 0, \quad (3.18)$$

which can be viewed as the “terminal condition” along the l' direction. In addition, in the new coordinate system, the reset condition and the connectivity condition become

$$V_2(S, l'; t) = V_1(S, t + l'), \quad \lim_{s \rightarrow \bar{s}} \frac{\partial V_2}{\partial S}(S, 0; t) = \lim_{s \rightarrow \bar{s}} \frac{\partial V_1}{\partial S}(S, t),$$

respectively, simply because the solution $V_2(S, l'; t)$ now corresponds to $V_2(S, t + l', l')$ in the original pricing domain II , as demonstrated earlier.

Therefore, the 2-D PDE systems that govern the price of a Parisian up-and-out call

option can be now summarized as:

$$\begin{aligned}
& \mathcal{A}_1(S \in [0, \bar{S}]) \left\{ \begin{array}{l} \frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T - \bar{J}) = V_{BS}(S, \bar{J}), \\ \lim_{S \rightarrow 0} V_1(S, t) = 0, \\ \lim_{S \rightarrow \bar{S}} V_1(S, t) = W(t), \end{array} \right. \\
& \mathcal{A}_2(S \in [\bar{S}, \infty)) \left\{ \begin{array}{l} \frac{\partial V_2}{\partial t'} + \mathbb{L}V_2 = 0, \\ V_2(S, \bar{J}; t) = 0, \\ \lim_{S \rightarrow \infty} V_2(S, l'; t) = 0, \\ \lim_{S \rightarrow \bar{S}} V_2(S, l'; t) = W(t + l'), \end{array} \right. \\
& \text{Connectivity condition : } \lim_{S \rightarrow \bar{S}} \frac{\partial V_1}{\partial S}(S, t) = \lim_{S \rightarrow \bar{S}} \frac{\partial V_2}{\partial S}(S, 0; t), \\
& \text{for } t \in [0, T - \bar{J}], l' \in [0, \bar{J}], \tag{3.19}
\end{aligned}$$

with $W(t)$ being defined as $\lim_{S \rightarrow \bar{S}} V_2(S, 0; t)$, which needs to be solved as part of the solution.

Similarly, we can also deduce the 2-D systems for the valuation of a *Parasian* up-and-out

call option as:

$$\begin{aligned}
\mathcal{A}_1(S \in [0, \bar{S}]) & \left\{ \begin{array}{l} \frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T - \bar{J} + J; J) = V_{BS}(S, \bar{J} - J), \\ \lim_{S \rightarrow 0} V_1(S, t; J) = 0, \\ \lim_{S \rightarrow \bar{S}} V_1(S, t; J) = W(t, J), \end{array} \right. \\
\mathcal{A}_2(S \in [\bar{S}, \infty)) & \left\{ \begin{array}{l} \frac{\partial V_2}{\partial t'} + \mathbb{L}V_2 = 0, \\ V_2(S, \bar{J}; t) = 0, \\ \lim_{S \rightarrow \infty} V_2(S, l'; t) = 0, \\ \lim_{S \rightarrow \bar{S}} V_2(S, l'; t) = W(t + l', l'), \end{array} \right. \\
\text{Connectivity condition : } & \lim_{S \rightarrow \bar{S}} \frac{\partial V_1}{\partial S}(S, t; J) = \lim_{S \rightarrow \bar{S}} \frac{\partial V_2}{\partial S}(S, J; t - J), \\
\text{for } t \in [0, T - \bar{J}], l' \in [0, \bar{J}], & \tag{3.20}
\end{aligned}$$

with $W(t, J)$ being defined as $\lim_{S \rightarrow \bar{S}} V_2(S, J; t - J)$.

One should notice that the coupling of V_1 and V_2 is now through a function of both t and J , i.e., $W(t, J)$, rather than a function of t only for the Parisian option. This additional variable J explicitly appearing in W is a result of the non-resetting mechanism specified in the *Parasian* option contract. It is also this additional variable that has made the solution procedure for the *Parasian* option much more complicated than and totally different from that of its Parisian counterpart.

3.2 Degeneration of the 2-D systems

An efficient way to verify the derivation of (3.19) from (2.11) and that of (3.20) from (2.13) is to investigate the degenerations of (3.19) and (3.20) with the involved parameters taking some extreme values, because whether or not the degenerate cases agree with the financial terms set for the corresponding options is a necessary condition to verify these 2-D systems.

There are four special cases that need to be examined.

The first case is associated with the zero value of the trigger time \bar{J} . One can verify that the terminal condition appearing in \mathcal{A}_1 of (3.19) becomes the payoff function of a European call option, by simply taking the limit process, i.e.,

$$V_1(S, T) = \lim_{\bar{J} \rightarrow 0} V_1(S, T - \bar{J}) = \lim_{\bar{J} \rightarrow 0} V_{BS}(S, T - \bar{J}) = \max(S - K, 0).$$

Moreover, it is also true that the variable l' approaches zero as well in the current case, because l' varies within the range $[0, \bar{J}]$. As a result, \mathcal{A}_2 in (3.19) vanishes, leaving $V_2(S, 0; t) = 0$ valid for $S \in [\bar{S}, \infty)$, $t \in [0, T]$. Combining these two points, it is clear that when $\bar{J} \rightarrow 0$, the pricing system (3.19) degenerates to

$$\left\{ \begin{array}{l} \frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T) = \max(S - K, 0), \\ \lim_{S \rightarrow 0} V_1(S, t) = 0, \\ \lim_{S \rightarrow \bar{S}} V_1(S, t) = 0, \end{array} \right. \\ \text{for } S \in [0, \bar{S}], t \in [0, T].$$

This is indeed the same as the PDE system governing the price of a barrier call option with up-and-out feature, as expected.

Secondly, when $\bar{J} \rightarrow \infty$, it is clear that for any finite option maturity value T , $T - \bar{J}$ is always less than zero, which is equivalent to saying that both \mathcal{A}_1 and \mathcal{A}_2 defined in (3.19) vanish. Geometrically, the above statement reveals that the pricing domain in the current case is in fact part of the prism \overline{TV} . Since the value in the prism \overline{TV} is defined as the price of a European call option, the degeneration as $\bar{J} \rightarrow \infty$ becomes obvious, and does indeed agree with the financial definition of a Parisian up-and-out call option contract.

The third case is when the barrier \bar{S} approaches zero. Since the option value for any $t \in (T - \bar{J}, T]$ is already defined as a European call option price, we only need to consider

the case when $t \in [0, T - \bar{J}]$. Clearly, when $\bar{S} \rightarrow 0$, \mathcal{A}_1 in (3.19) vanishes, leaving $V_1(0, t) = 0$, and thus, for any $t \in [0, T - \bar{J}]$, we obtain

$$W(t) = \lim_{S \rightarrow \bar{S}} V_1(S, t) = \lim_{S \rightarrow 0} V_1(S, t) = 0.$$

In fact, we could conclude that $W(t)$ equals zero for any $t \in [0, T]$, because in the additional region $(T - \bar{J}, T]$, $W(t)$ is defined as the value of a European call option with zero underlying price, i.e.,

$$W(t) = \lim_{S \rightarrow 0} V_{BS}(S, t) = 0.$$

Now, it is clear that $W(t + l')$ appearing in \mathcal{A}_2 equals zero, as a result of $t + l'$ varying within $[0, T]$. Consequently, \mathcal{A}_2 becomes a homogenous system defined on $t \in [0, T - \bar{J}]$, $S \in [0, \infty]$, $J \in [0, \bar{J}]$, and thus \mathcal{A}_2 only has a trivial solution. Therefore, when $\bar{S} \rightarrow 0$, the Parisian up-and-out call option price calculated from our newly defined system (3.19) is indeed equal to a European call option price if $t \in [0, T - \bar{J}]$, and zero otherwise.

Finally, when $\bar{S} \rightarrow \infty$, it is clear that \mathcal{A}_2 vanishes, and moreover, \mathcal{A}_1 degenerates to the following backward problem:

$$\begin{cases} \frac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T - \bar{J}) = V_{BS}(S, T - \bar{J}), \end{cases} \quad \text{for } S \in [0, \infty), t \in [0, T - \bar{J}], \quad (3.21)$$

the solution of which is identical to a European call option price at the time to expiry $T - t$. In other words, a European call option can be viewed as a Parisian up-and-out call option without barriers. Therefore, the result of such a degeneration mathematically indeed matches with the financial interpretation.

Although the above arguments are established for a Parisian up-and-out call option, they also hold under the framework of a corresponding *Parasian* option. Therefore, we shall not repeat the details here for the case of *Parasian* options.

From the above discussions, one can thus conclude that, with the parameters taken on some extreme values, the 2-D PDE systems (3.19) and (3.20) indeed degenerate to the ones that could be excepted from the financial point of view. This has somehow confirmed that the way of establishing (3.19) and (3.20) is valid and reasonable. Having gained confidence on the correctness of the newly established 2-D PDE systems, two purely analytic approaches will be developed in the next two subsections to solve (3.19) and (3.20), respectively.

3.3 The analytical solution for the Parisian option price

To solve the newly established pricing system (3.19) effectively, we shall first non-dimensionalize all variables by introducing dimensionless variables

$$\begin{aligned} x &= \ln \frac{S}{K}, \bar{x} = \ln \frac{\bar{S}}{K}, V_1' = \frac{V_1}{K}, V_2' = \frac{V_2}{K}, \tau = (T - \bar{J} - t) \frac{\sigma^2}{2}, W' = \frac{W}{K}, \\ \bar{J}' &= \frac{\sigma^2 \bar{J}}{2}, \tilde{l} = \frac{\sigma^2}{2}(\bar{J} - l'), \gamma = \frac{2r}{\sigma^2}, q = \frac{2D}{\sigma^2}. \end{aligned}$$

With all primes and tildes dropped from now on, the dimensionless PDE system reads:

$$\mathcal{A}_1(x \in (-\infty, \bar{x}]) \left\{ \begin{array}{l} \frac{\partial V_1}{\partial \tau} = \mathbb{L}V_1, \\ V_1(x, 0) = V_{BS}(Ke^x, \frac{2\bar{J}}{\sigma^2})/K, \\ \lim_{x \rightarrow -\infty} V_1(x, \tau) = 0, \\ \lim_{x \rightarrow \bar{x}} V_1(x, t) = W(\tau), \end{array} \right.$$

$$\begin{aligned}
& \mathcal{A}_2(x \in [\bar{x}, +\infty)) \left\{ \begin{array}{l} \frac{\partial V_2}{\partial l} = \mathbb{L}V_2, \\ V_2(x, 0; \tau) = 0, \\ \lim_{x \rightarrow \infty} V_2(x, l; \tau) = 0, \\ \lim_{x \rightarrow \bar{x}} V_2(x, l; \tau) = W(\tau - \bar{J} + l), \end{array} \right. \\
& \text{Connectivity condition : } \lim_{x \rightarrow \bar{x}} \frac{\partial V_1}{\partial x}(x, \tau) = \lim_{x \rightarrow \bar{x}} \frac{\partial V_2}{\partial x}(x, \bar{J}; \tau), \\
& \text{for } \tau \in [0, \bar{\tau}], l \in [0, \bar{J}], \tag{3.22}
\end{aligned}$$

where operator $\mathbb{L} = \frac{\partial^2}{\partial x^2} + k \frac{\partial}{\partial x} - \gamma I$, with k being equal to $\gamma - q - 1$.

From (3.22), it can be observed that once $W(\tau)$ is found, V_1 and V_2 are no longer coupled, and the corresponding solutions V_1 and V_2 can be obtained straightforwardly. While it seems quite natural to treat the determination of $W(\tau)$ as a key step of solving (3.22), it is, however, not an easy task.

Since the time to expiry τ is the only variable of the unknown function $W(\tau)$, one could expect $W(\tau)$ to be governed by some sort of simple equations (e.g. ODE (ordinary differential equation), integral equation, integral ODE, etc.), whose solution could be more analytically achievable, rather than to solve for $W(\tau)$ directly from the coupled system (3.22).

To find the governing equation for $W(\tau)$, we shall first find the integral representations of V_1 and V_2 , in terms of the unknown function $W(\tau)$. This can be achieved by solving \mathcal{A}_1 and \mathcal{A}_2 in (3.22) separately, as if they were not coupled. Specifically, the solution of \mathcal{A}_1 can be found by splitting the linear problem into two problems, a technique frequently used in solving linear PDEs. The solution of the first problem, which involves a homogeneous differential equation and homogeneous boundary conditions but arbitrary initial condition, can be easily worked out utilizing the Laplace transform technique as well as the Green function method for the resulted ODE, while the solution of the second problem, which involves a homogeneous differential equation and zero initial condition but non-homogeneous boundary condition at $x = \bar{x}$ can be obtained by applying the Laplace transform technique.

Without going through the lengthy derivation procedures, the integral representation of V_1 can be written as

$$V_1(x, \tau) = F(x, \tau) + \int_0^\tau W(s)g_1(x, \tau - s)ds, \quad (3.23)$$

where

$$\begin{aligned} F(x, \tau) &= \int_{-\infty}^{\bar{x}} \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{k}{2}(x-z) - (\frac{k^2}{4} + \gamma)\tau} [e^{-\frac{(x-z)^2}{4\tau}} - e^{-\frac{(x+z-2\bar{x})^2}{4\tau}}] f(z) dz, \\ g_1(x, \tau) &= -\frac{x - \bar{x}}{2\sqrt{\pi\tau}^{\frac{3}{2}}} e^{-(\frac{k^2}{4} + \gamma)\tau - \frac{(x-\bar{x})^2}{4\tau} - \frac{k}{2}(x-\bar{x})}, \\ f(z) &= V_{BS}(z, \bar{J}). \end{aligned}$$

Similarly, V_2 can be written as

$$V_2(x, l) = \int_0^l W(\tau - \bar{J} + s)g_2(x, l - s)ds, \quad (3.24)$$

where $g_2(x, l) = -g_1(x, l)$. Now, applying the connectivity condition to (3.23) and (3.24), we obtain the integral equation governing $W(\tau)$ as

$$\frac{\partial F}{\partial x}(x, \tau)|_{x=\bar{x}} + \int_0^\tau W(s) \frac{\partial g_1}{\partial x}(x, \tau - s)ds|_{x=\bar{x}} = \int_0^{\bar{J}} W(\tau - \bar{J} + s) \frac{\partial g_2}{\partial x}(x, \bar{J} - s)ds|_{x=\bar{x}}. \quad (3.25)$$

It can be observed that the LHS (left hand side) of (3.25) contains the information of $W(s)$ from the expiry ($\tau = 0$) to the current time to expiry, τ , while its RHS (right hand side) involves the value of $W(s)$, with s varying within $[\tau - \bar{J}, \tau]$, which coincides with the projection of the slide passing through $(\bar{S}, \tau, 0)$ on the plane $J = 0$. Note that for simplicity, we shall name the projection of a slide on the plane $J = 0$ as a “window” from now on. The initial window (the zeroth window) is thus the one with τ varying within $[-\bar{J}, 0]$, where $W(\tau)$ is defined as the price of a European call option at $S = \bar{S}$. Intuitively, by solving (3.25) once, it could at most be expected to have the unknown function W determined on one particular window, rather than over the entire domain $[0, \tau]$, unless $[0, \tau]$ belongs to the first window, which corresponds to the case $0 \leq \tau \leq \bar{J}$.

Consequently, for a state point (S, τ, J) , one can evaluate W forwards, window by window, until the value at the required time τ is found. Such a procedure is equivalent to treating the window as if it were moving, each time with a \bar{J} distance, from the initial window to the place containing the given τ value. Therefore, determining the corresponding W value on the window moving once from the initial window $[-\bar{J}, 0]$ becomes crucial, as the procedure can be repeated until the desired place is reached. In the following, we shall first solve for $W(\tau)$, $\tau \in [0, \bar{J}]$, which lies on the window that has moved once from the initial place. Hereafter, the window that moves n ($n \geq 1$) times from the initial window is called the n th window for simplicity.

Now defining a new variable $\xi = \tau - \bar{J} + s$, and substituting it into the RHS of (3.25), we obtain

$$\int_0^{\bar{J}} W(\tau - \bar{J} + s) \frac{\partial g_2}{\partial x}(x, \bar{J} - s) ds \Big|_{x=\bar{x}} = \int_{\tau-\bar{J}}^{\tau} W(\xi) \frac{\partial g_2}{\partial x}(x, \tau - \xi) d\xi \Big|_{x=\bar{x}}. \quad (3.26)$$

Since $W(\tau)$ is already defined for $\forall \tau \in [-\bar{J}, 0]$, it is reasonable to split (3.26) into two terms, i.e.,

$$\int_{\tau-\bar{J}}^{\tau} W(\xi) \frac{\partial g_2}{\partial x}(x, \tau - \xi) d\xi \Big|_{x=\bar{x}} = \int_{\tau-\bar{J}}^0 W(\xi) \frac{\partial g_2}{\partial x}(x, \tau - \xi) d\xi \Big|_{x=\bar{x}} + \int_0^{\tau} W(\xi) \frac{\partial g_2}{\partial x}(x, \tau - \xi) d\xi \Big|_{x=\bar{x}}, \quad (3.27)$$

with the first term already known. Substituting (3.27) into (3.25), we obtain

$$\begin{aligned} & \left[\frac{\partial F}{\partial x}(x, \tau) + \int_0^{\tau} W(s) \frac{\partial g_1}{\partial x}(x, \tau - s) ds \right] \Big|_{x=\bar{x}} \\ &= \left[\int_{\tau-\bar{J}}^0 W(\xi) \frac{\partial g_2}{\partial x}(x, \tau - \xi) d\xi + \int_0^{\tau} W(\xi) \frac{\partial g_2}{\partial x}(x, \tau - \xi) d\xi \right] \Big|_{x=\bar{x}}. \end{aligned} \quad (3.28)$$

Clearly, the unknown function $W(\xi)$ ($\xi \in [0, \tau]$) is now only involved in the two convolutions that appear on both sides of (3.28). According to the Convolution Theorem [6], the convolution integral can be further eliminated by taking the Laplace transform w.r.t. τ ,

and thus we obtain

$$\hat{H}(x, p)|_{x=\bar{x}} + \hat{W} \frac{\partial g_1}{\partial x}(x, p)|_{x=\bar{x}} = \hat{G}(x, p)|_{x=\bar{x}} + \hat{W} \frac{\partial g_2}{\partial x}(x, p)|_{x=\bar{x}}, \quad (3.29)$$

where

$$\begin{aligned} \hat{H}(x, p) &= \mathcal{L}\left[\frac{\partial F}{\partial x}(x, \tau)\right], \quad \hat{G}(x, p) = \mathcal{L}\left[\int_{\tau-\bar{J}}^0 W(\xi) \frac{\partial g_2}{\partial x}(x, \tau - \xi) d\xi\right]_{x=\bar{x}}, \quad \hat{W}(p) = \mathcal{L}[W(\tau)], \\ \hat{g}_1(x, p) &= \mathcal{L}[g_1(x, \tau)] = e^{\lambda_1(x-\bar{x})}, \quad \hat{g}_2(x, p) = \mathcal{L}[g_2(x, \tau)] = e^{\lambda_2(x-\bar{x})}, \end{aligned}$$

with

$$\lambda_1 = -\frac{k}{2} + \sqrt{\left(\frac{k^2}{4} + \gamma\right) + p}, \quad \lambda_2 = -\frac{k}{2} - \sqrt{\left(\frac{k^2}{4} + \gamma\right) + p},$$

and p being the Laplace parameter. After some simple algebraic manipulations, the unknown function W in the Laplace space can be solved as

$$\hat{W}(p) = \frac{\hat{H}(x, p)|_{x=\bar{x}}}{\lambda_2 - \lambda_1} + \frac{\hat{G}(x, p)|_{x=\bar{x}}}{\lambda_1 - \lambda_2}. \quad (3.30)$$

Although (3.30) is remarkably simple, it is unfortunately still in terms of the Laplace parameter p . In order to obtain an analytical formula for $W(\tau)$, one still needs to carry out the Laplace inversion, a formidable process that often prevents this great technique being widely used to solve PDEs. However, the significance of (3.30) should never be underestimated, even though it is in terms of the Laplace parameter p . As pointed out previously, finding the option price across the barrier, i.e., $W(\tau)$, is a key step to solve the Parisian option price. Once $W(\tau)$ is found, V_1 and V_2 are no longer coupled, and can be obtained analytically through (3.23)-(3.24).

To analytically invert (3.30) is quite difficult, but fortunately, achievable. With the derivation details left in Appendix A and Appendix B, we obtain the fully explicit analytic

expression for $W(\tau)$ as

$$\begin{aligned}
W(\tau) &= \mathcal{L}^{-1}\left[\frac{\hat{H}(x, p)}{\lambda_2 - \lambda_1}\Big|_{x=\bar{x}}\right] + \mathcal{L}^{-1}\left[\frac{\hat{G}(x, p)}{\lambda_1 - \lambda_2}\Big|_{x=\bar{x}}\right] \\
&= \int_{-\infty}^{\bar{x}} \frac{e^{-\frac{k}{2}(\bar{x}-z) - (\frac{k^2}{4} + \gamma)\tau} e^{-\frac{(\bar{x}-z)^2}{4\tau}} f(z) dz}{2\sqrt{\pi\tau}} + \frac{W(0)}{2} e^{-(\frac{k^2}{4} + \gamma)\tau} \\
&\quad - \frac{e^{-(\frac{k^2}{4} + \gamma)\bar{J}}}{2\pi\sqrt{\bar{J}}} \int_0^\tau \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau-s)}}{\sqrt{\tau-s}} W(s - \bar{J}) ds \\
&\quad - \frac{1}{\pi} \int_0^\tau \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau-s)}}{\sqrt{\tau-s}} \int_{\sqrt{s}}^{\sqrt{\bar{J}}} e^{-(\frac{k^2}{4} + \gamma)t^2} \left[\left(\frac{k^2}{4} + \gamma\right) W(s - t^2) + W'(s - t^2) \right] dt ds,
\end{aligned} \tag{3.31}$$

where the function $W(y)$, ($y \in [-\bar{J}, 0]$) involved in the RHS is already defined as the price of a European call option at the barrier \bar{S} , with the time to expiry being equal to $y + \bar{J}$. Considering the complexity of the problem, this explicit and closed-form solution for the price across the barrier is remarkably simple. Moreover, from the viewpoint that even the price of a standard barrier option with rebate involves the calculation of double integrals, it is reasonable for us to believe that (3.31) is already in its simplest form.

It should be pointed out that although (3.31) is valid only on the first window, the detailed derivation can be in fact generalized to determine W on the $(n+1)$ th window, assuming that the option price on the n th window has already been found. This can be easily achieved by realizing the following facts. Firstly, according to the semi-group property of the solution of a general heat transfer problem [7], one can solve $V_1(x, \tau)$ ($\tau \in [n\bar{J}, (n+1)\bar{J}]$) as if the diffusion started at $n\bar{J}$, with the initial condition now being equal to $f(x) = V_1(x, n\bar{J})$, and the time length being $\tau - n\bar{J}$. Secondly, solving W on the $(n+1)$ th window with the known option price on the n th window is equivalent to determining the W value on the first window defined in the new coordinate system $\tilde{\tau} = \tau - n\bar{J}$. Therefore, for $\tau \in [n\bar{J}, (n+1)\bar{J}]$, $W(\tau)$ has the same expression as (3.31), except that in this case, $f(z) = V_1(z, n\bar{J})$, and the function $W(y)$, ($y \in [(n-1)\bar{J}, n\bar{J}]$) involved in the RHS is defined as the price of a Parisian up-and-out call option at the barrier \bar{S} , with the time to

expiration being $y + \bar{J}$. More specifically, the closed-form analytic formula for the Parisian up-and-out call option at any given τ can be written as

$$W(\tau) = \begin{cases} W_0, & \tau \in [-\bar{J}, 0] \\ W_1, & \tau \in [0, \bar{J}] \\ \dots, & \vdots \\ W_n, & \tau \in [(n-1)\bar{J}, n\bar{J}] \\ W_{n+1}, & \tau \in [n\bar{J}, (n+1)\bar{J}], \end{cases}$$

where W_{n+1} can be found recursively as:

$$\begin{aligned} W_{n+1}(\tau) &= \int_{-\infty}^{\bar{x}} \frac{e^{-\frac{k}{2}(\bar{x}-z) - (\frac{k^2}{4} + \gamma)(\tau - n\bar{J})} e^{-\frac{(\bar{x}-z)^2}{4(\tau - n\bar{J})}} f_n(z) dz + \frac{W_n(n\bar{J})}{2} e^{-(\frac{k^2}{4} + \gamma)(\tau - n\bar{J})} \\ &\quad - \frac{e^{-(\frac{k^2}{4} + \gamma)\bar{J}}}{2\pi\sqrt{\bar{J}}} \int_{n\bar{J}}^{\tau} \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau-s)}}{\sqrt{\tau-s}} W_n(s - \bar{J}) ds \\ &\quad - \frac{1}{\pi} \int_{n\bar{J}}^{\tau} \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau-s)}}{\sqrt{\tau-s}} \int_{\sqrt{s-n\bar{J}}}^{\sqrt{\bar{J}}} e^{-(\frac{k^2}{4} + \gamma)t^2} \left[\left(\frac{k^2}{4} + \gamma \right) W_n(s - t^2) + W_n'(s - t^2) \right] dt ds, \end{aligned}$$

for $n = 0, 1, 2 \dots$, with

$$\begin{aligned} f_0(z) &= V_{BS}(z, \bar{J}), \\ f_n(x) &= F(x, n\bar{J}) + \sum_{i=1}^n \int_{(i-1)\bar{J}}^{i\bar{J}} W_i(s) g_1(x, n\bar{J} - s) ds, \quad n = 1, 2 \dots \\ F(x, \tau) &= \int_{-\infty}^{\bar{x}} \frac{1}{2\sqrt{\pi\tau}} e^{[-\frac{k}{2}(x-z) - (\frac{k^2}{4} + \gamma)\tau] \left[e^{-\frac{(x-z)^2}{4\tau}} - e^{-\frac{(x+z-2\bar{x})^2}{4\tau}} \right]} f_0(z) dz, \\ W_0(\tau) &= V_{BS}(x, \tau + \bar{J}), \quad \tau \in [-\bar{J}, 0]. \end{aligned}$$

The recursive nature of this analytic formula guarantees that $W(\tau)$ can be evaluated explicitly and for sure analytically, window by window, until the window containing the given τ value has been reached. Once $W(\tau)$ is solved, the price of a Parisian up-and-out call option can be calculated straightforwardly by means of (3.23) and (3.24).

3.4 The analytical solution for the *Parasian* option price

Having successfully found the analytic formula for the Parisian option price, we now move to solve for the price of its *Parasian* counterpart. Similar to the solution procedure of the Parisian option, a key step here is to determine the *Parasian* option price across the barrier, i.e., $W(\tau, J)$. Once $W(\tau, J)$ values are found, the calculation of the *Parasian* option price becomes straightforward, too. However, one should notice that $W(\tau, J)$ is now a function of both t and J , rather than a function of t only for the case of Parisian options. With an additional variable J added on, one could expect that the analytical expression of $W(\tau, J)$ would be much more complicated than and totally different from $W(\tau)$ of the Parisian option. Moreover, the involvement of the additional variable J in W has prevented the application of the exactly same solution procedure adopted in the case of pricing Parisian options for the current case; a different approach to solve for $W(\tau, J)$ needs to be explored and is the focus of this subsection.

To solve (3.20) effectively, we shall also non-dimensionalize this system first. By using the same dimensionless variables introduced in the last subsection, and dropping all the primes and tildes afterwards, we obtain,

$$\mathcal{A}_1(x \in (-\infty, \bar{x}]) \left\{ \begin{array}{l} \frac{\partial V_1}{\partial \tau} = \mathbb{L}V_1, \\ V_1(x, -J; J) = V_{BS}(Ke^x, \frac{2(\bar{J} - J)}{\sigma^2})/K, \\ \lim_{x \rightarrow -\infty} V_1(x, \tau; J) = 0, \\ \lim_{x \rightarrow \bar{x}} V_1(x, t; J) = W(\tau, J), \end{array} \right.$$

$$\mathcal{A}_2(x \in [\bar{x}, +\infty)) \left\{ \begin{array}{l} \frac{\partial V_2}{\partial l} = \mathbb{L}V_2, \\ V_2(x, 0; \tau) = 0, \\ \lim_{x \rightarrow -\infty} V_2(x, l; \tau) = 0, \\ \lim_{x \rightarrow \bar{x}} V_2(x, l; \tau) = W(\tau - \bar{J} + l, \bar{J} - l), \end{array} \right.$$

Connectivity condition : $\lim_{x \rightarrow \bar{x}} \frac{\partial V_1}{\partial x}(x, \tau - \bar{J} + l; \bar{J} - l) = \lim_{x \rightarrow \bar{x}} \frac{\partial V_2}{\partial x}(x, l; \tau)$,

for $\tau \in [0, \bar{\tau}]$, $l \in [0, \bar{J}]$, (3.32)

where operator $\mathbb{L} = \frac{\partial^2}{\partial x^2} + k \frac{\partial}{\partial x} - \gamma I$, with k being equal to $\gamma - q - 1$.

Then, we solve for the integral representations of V_1 and V_2 , in terms of the unknown function $W(\tau, J)$ and obtain

$$V_1(x, \tau; J) = F(x, \tau; J) + \int_0^{\tau+J} W(s - J, J) g_1(x, \tau + J - s) ds, \quad (3.33)$$

$$V_2(x, l; \tau) = \int_0^l W(\tau - \bar{J} + s, \bar{J} - s) g_2(x, l - s) ds, \quad (3.34)$$

where

$$F(x, \tau; J) = \int_{-\infty}^{\bar{x}} \frac{1}{2\sqrt{\pi(\tau+J)}} e^{-\frac{k}{2}(x-z) - (\frac{k^2}{4} + \gamma)(\tau+J)} [e^{-\frac{(x-z)^2}{4(\tau+J)}} - e^{-\frac{(x+z-2\bar{x})^2}{4(\tau+J)}}] f(z) dz,$$

with the functions $g_1(x, \tau)$, $g_2(x, l)$, and $f(z)$ being the same as those defined in the last subsection. Now, applying the connectivity condition to (3.33) and (3.34), we obtain the integral equation governing $W(\tau, J)$ as

$$\begin{aligned} & \frac{\partial F}{\partial x}(x, \tau - \bar{J} + l; \bar{J} - l)|_{x=\bar{x}} + \int_0^\tau W(s + l - \bar{J}, \bar{J} - l) \frac{\partial g_1}{\partial x}(x, \tau - s) ds|_{x=\bar{x}} \\ & = \int_0^l W(\tau - \bar{J} + s, \bar{J} - s) \frac{\partial g_2}{\partial x}(x, l - s) ds|_{x=\bar{x}}. \end{aligned} \quad (3.35)$$

To facilitate the analytical procedure, we denote $\bar{W}(\tau, l) = W(\tau + l - \bar{J}, \bar{J} - l)$, and

thus rewrite (3.35) as

$$\frac{\partial F}{\partial x}(x, \tau - \bar{J} + l; \bar{J} - l)|_{x=\bar{x}} + \int_0^\tau \bar{W}(s, l) \frac{\partial g_1}{\partial x}(x, \tau - s) ds|_{x=\bar{x}} = \int_0^l \bar{W}(\tau, s) \frac{\partial g_2}{\partial x}(x, l - s) ds|_{x=\bar{x}}.$$

One can observe that this integral equation governing $\bar{W}(\tau, l)$ involves two different convolutions w.r.t. two variables, i.e., τ and l . To eliminate the two convolutions, a double Laplace transform is applied, which yields

$$\hat{H}(x, p_1, p_2)|_{x=\bar{x}} + \hat{W}(p_1, p_2)\lambda_1(p_1) = \hat{W}(p_1, p_2)\lambda_2(p_2), \quad (3.36)$$

where

$$\begin{aligned} \hat{H}(x, p_1, p_2) &= \mathcal{L}_{p_1}[\mathcal{L}_{p_2}[\frac{\partial F}{\partial x}(x, \tau - \bar{J} + l; \bar{J} - l)]], \quad \hat{W}(p_1, p_2) = \mathcal{L}_{p_1}[\mathcal{L}_{p_2}[W(\tau, l)]], \\ \lambda_1(p_1) &= -\frac{k}{2} + \sqrt{(\frac{k^2}{4} + \gamma) + p_1}, \quad \lambda_2(p_2) = -\frac{k}{2} - \sqrt{(\frac{k^2}{4} + \gamma) + p_2}, \end{aligned}$$

with p_1, p_2 being the Laplace parameters corresponding to the original variables τ and l , respectively, and the subscripts of the Laplace operator denote the corresponding Laplace transform.

From (3.36), it is clear that the unknown function \bar{W} in the Laplace space can be found as

$$\hat{W}(p_1, p_2) = \frac{\hat{H}(x, p_1, p_2)}{\lambda_2(p_2) - \lambda_1(p_1)}|_{x=\bar{x}}, \quad (3.37)$$

and thus

$$\bar{W}(\tau, l) = \mathcal{L}_{p_2}^{-1}[\mathcal{L}_{p_1}^{-1}[\frac{\hat{H}(x, p_1, p_2)}{\lambda_2(p_2) - \lambda_1(p_1)}|_{x=\bar{x}}]]. \quad (3.38)$$

Like performing Laplace inversion analytically for a single Laplace transform encountered for the case of Parisian options, analytically performing a Laplace inversion of (3.38) was initially considered as an even-more luxury and we were going to resort to a numerical inversion as did in [11]. Fortunately, we eventually managed to have carried out this

seemingly-impossible task and obtained a fully closed-form analytic expression for $\bar{W}(\tau, l)$

as

$$\begin{aligned}
\bar{W}(\tau, l) &= \frac{\sqrt{2}}{\sqrt{\pi}} e^{-(\frac{k^2}{4} + \gamma)(\tau + l)} \int_0^\infty \left\{ e^{(1 + \frac{k}{2})^2 l + \bar{x} - \frac{\eta^2}{2} - \sqrt{2\tau}(\frac{k}{2} + 1)\eta} N\left[\frac{\bar{x} - \sqrt{2\tau}\eta}{\sqrt{2l}} + \sqrt{2l}\left(\frac{k}{2} + 1\right)\right] \right. \\
&- e^{\frac{k^2}{4}l - \frac{\eta^2}{2} - \frac{\sqrt{2\tau}}{2}k\eta} N\left[\frac{\bar{x} - \sqrt{2\tau}\eta}{\sqrt{2l}} + \frac{\sqrt{2l}k}{2}\right] \left. \right\} d\eta \\
&- \frac{\sqrt{2}e^{-(\frac{k^2}{4} + \gamma)(\tau + l)}}{\pi} \int_0^{\frac{\bar{x}}{\sqrt{2(\tau + l)}}} e^{-\frac{k}{2}\sqrt{2(\tau + l)}\xi - \frac{\xi^2}{2}} (e^{\bar{x} - \sqrt{2(\tau + l)}\xi} - 1) (1 - N(\sqrt{\frac{l}{\tau}}\xi)) d\xi \\
&+ \frac{2\sqrt{2}e^{-(\frac{k^2}{4} + \gamma)(\tau + l)}}{\pi} \cdot \\
&\int_{-\infty}^{\bar{x}} \int_{\frac{\bar{x} - z}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{\eta^2}{2} - \frac{k}{2}(\bar{x} - z)} \int_{\arcsin \sqrt{\frac{2\eta^2\tau - (\bar{x} - z)^2}{2\eta^2(\tau + l) - (\bar{x} - z)^2}}}^{\frac{\pi}{2}} f_3\left(\theta, \tau - \frac{(\bar{x} - z)^2}{2\eta^2}, z\right) d\theta d\eta dz \\
&+ \frac{2\sqrt{2}e^{-(\frac{k^2}{4} + \gamma)(\tau + l)}}{\pi} \cdot \\
&\int_{-\infty}^{\bar{x}} \int_{\frac{\bar{x} - z}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{\eta^2}{2} - \frac{k}{2}(\bar{x} - z)} \int_{\sqrt{\frac{2\eta^2\tau - (\bar{x} - z)^2}{2\eta^2 l}}}^{+\infty} f_4\left(y, \tau - \frac{(\bar{x} - z)^2}{2\eta^2}, z\right) dy d\eta dz
\end{aligned} \tag{3.39}$$

where

$$\begin{aligned}
f_3(\theta, w, z) &= e^{z + (\frac{k}{2} + 1)^2(l + w) \cos^2 \theta} \left[-\left(\frac{k}{2} + 1\right)^2 N(d_1) \sqrt{l + w} \cos \theta - \left(\frac{k}{2} + 1\right) \frac{e^{-\frac{d_1^2}{2}}}{2\sqrt{\pi}} \right] \\
&+ e^{\frac{k^2}{4}(l + w) \cos^2 \theta} \left[\frac{k^2}{4} N(d_2) \sqrt{l + w} \cos \theta + k \frac{e^{-\frac{d_2^2}{2}}}{4\sqrt{\pi}} \right], \\
f_4(y, w, z) &= \frac{z}{4\sqrt{\pi}(l + w)} \left[e^{z + (\frac{k}{2} + 1)^2(l + w) \frac{1}{y^2 + 1} - \frac{d_3}{2}} - e^{\frac{k^2}{4}(l + w) \frac{1}{y^2 + 1} - \frac{d_4}{2}} \right],
\end{aligned}$$

$$\begin{aligned}
d_1 &= \frac{z}{\sqrt{2(l + w)} \cos \theta} + \left(\frac{k}{2} + 1\right) \sqrt{2(l + w)} \cos \theta, \\
d_2 &= \frac{z}{\sqrt{2(l + w)} \cos \theta} + \frac{k}{2} \sqrt{2(l + w)} \cos \theta, \\
d_3 &= \frac{z^2(1 + y^2)}{2(l + w)} + z(k + 2) + \frac{2(l + w)}{(y^2 + 1)} \left(\frac{k}{2} + 1\right)^2, \\
d_4 &= \frac{z^2(1 + y^2)}{2(l + w)} + kz + \frac{(l + w)k^2}{2(y^2 + 1)},
\end{aligned}$$

with $N(\cdot)$ being the standard cumulative distribution function. While the details of the double Laplace inversion is left in Appendix C for interested readers, it should be remarked that although the current expression (3.39) involves the evaluation of triple integrals, we believe that it is already in its simplest form, due to the complexity arising from the non-resetting mechanism associated with the *Parasian* options. Also, as a result of a simple financial switch from a resetting “trigger” to a non-resetting one, the nice recursive nature in the evaluation of (3.31) for the case of Parisian options has been destroyed; the evaluation of $W(\tau, J)$ from (3.39) requires an additional fold of integration. It is thus anticipated that the computational efficiency of adopting (3.39) to calculate numerical values of a *Parasian* option should be slightly worse off than that of adopting (3.31) to calculate numerical values of a Parisian option. Some direct comparisons of numerical efficiency are provided in the next section through some numerical examples.

4 Numerical examples and discussions

The derivations of (3.31) from (3.22) and (3.39) from (3.32) are carried out rigorously and deductively, and thus there is no need to discuss the “accuracy” of these closed-form solutions and present any calculated results. However, from the viewpoint that a comparison with previously published results (numerical solutions or solutions obtained through other approximation methods) may give readers a sense of verification of the newly found closed-form solutions, as well as help them understand the improvement in accuracy and efficiency with our exact formulae, several numerical examples are given in this section. The section is organized into two subsections, according to two important issues that should be addressed. The first subsection is to compare the results obtained from our newly proposed PDE systems (2.11) and (2.13) with the corresponding ones proposed in [2], while the second subsection is to test the numerical performance of our closed-form analytic solutions.

To help readers who may not be used to discussing financial problems with dimensionless

quantities, all results, unless otherwise stated, are now converted back to dimensional quantities in this section before they are graphed and presented.

4.1 A comparison with Wilmott *et al.*'s numerical solutions

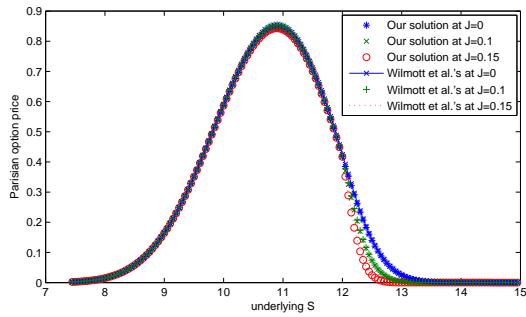
As pointed out earlier, our pricing systems (2.11) and (2.13) are primarily based on what were proposed by Wilmott *et al.* [2]. However, we found that there are some errors in [2]. It is thus interesting to have a direct comparison between the results produced with our new PDE systems and those produced with Wilmott *et al.*'s original PDE system. Of course, in addition to the financial arguments provided in Section 2, here in this subsection, we share focus on the results produced by our newly-derived analytic solution and those produced by Wilmott *et al.*'s explicit finite difference scheme.

There are two major differences between Wilmott *et al.*'s PDE system and ours presented in this paper. The first one is that our solution has been worked out from a truncated domain; apart from the results near the boundary where different boundary conditions are imposed, we expect that the results produced with our newly-derived analytic solutions should match those produced by Wilmott *et al.*'s explicit finite difference scheme. This is indeed so, as shown in Fig 2-3. In Fig 2(a-b), the Parisian and *Parasian* option prices with three different values of the time left in the trigger ($\bar{J} - J$) being less than or equal to the time to expiry, i.e., $T - t \geq \bar{J} - J$, are plotted, respectively. From these two figures, one can clearly observe that for the case $T - t \geq \bar{J} - J$, both Parisian and *Parasian* option prices calculated from our systems and those produced from Wilmott *et al.*'s systems [2] agree amazingly well with each other.

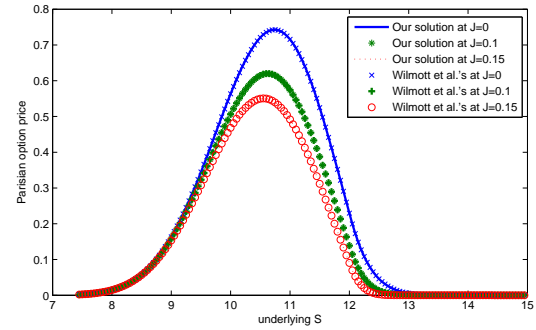
The second difference is that different boundary conditions at $S = \infty$ are proposed in the prism \overline{TV} , where J is impossible to reach \bar{J} . In this case, we should expect that the results produced by the two different PDE systems are different. A comparison of option prices for this case is shown in Figs 3(a-b) and Figs 3(c-d) for the Parisian and *Parasian* options, respectively. It can be observed that for reasonably finite values of the

underlying, our solutions seem to agree well with Wilmott *et al.*'s, whether the inequality $T - t \geq \bar{J} - J$ is held or not, as shown in Fig 3(a) and Fig 3(b). In these two figures, one may also notice that the option prices corresponding to the case $T - t \geq \bar{J} - J$ appear to have a sharp corner around \bar{S} , which seems to be contradictory to the financial clause. In fact, this is only a result of the scale with which these figures were plotted; a locally “zoom-in” plot would reveal that the “sharp” corner becomes a smooth curve as a result of imposing the continuity of the Delta between the two regions. On the other hand, since different boundary conditions at $S = \infty$ are imposed in the prism \overline{IV} , one can expect that for the case $T - t < \bar{J} - J$, the large S asymptotic behaviors of our solutions and Wilmott *et al.*'s should be different, although they agree very well with each other for reasonably finite underlying (see the option prices provided in Fig 3(a) and Fig 3(c)). To visualize the price differences at the large end of S values, we have re-plotted two sets of option price data with $J = 0$ displayed in Fig 3(a) and Fig 3(c), with a larger range ($S = 50$ instead of $S = 15$) in Fig 3(b) and Fig 3(d), respectively. Clearly, at the large S end, the option prices calculated from our systems increase to infinity at almost the same rate as the underlying, whereas Wilmott *et al.*'s option prices exhibit a dramatic decrease to zero with sufficiently large S values, as shown in Fig 3(b) and Fig 3(d) for Parisian and *Parasian* options, respectively. This is not surprising, as in the region where $T - t < \bar{J} - J$ (the prism \overline{IV}), the boundary condition at $S = \infty$ in our systems is set to S to ensure that the option price in this domain equals a European call option price, whereas in Wilmott *et al.*'s systems, it is set to zero no matter what value τ is. It should be remarked that for $T - t < \bar{J} - J$, the resulting option prices differing only for sufficiently large underlying is indeed reasonable, since the boundary condition usually only has a local effect on the final solution.

Several remarks should be made before we discuss the accuracy and efficiency of the explicit-closed form analytic formulae, in terms of numerically evaluating the involved integrals, as is the main issue of the next subsection. Firstly, whether the prism \overline{III} (\widetilde{III}) is cut or not has no influence on the final solution for the Parisian (*Parasian*) options, due

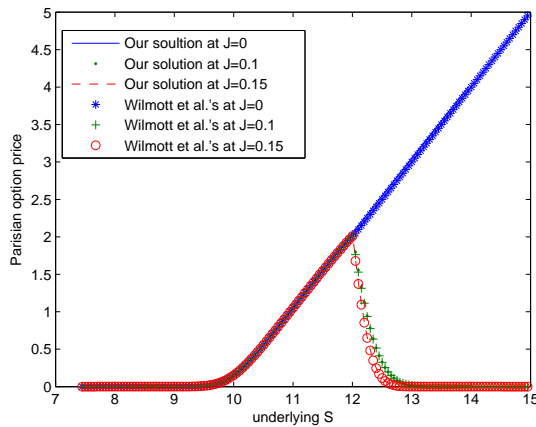


(a) Price of the Parisian up-and-out call options.

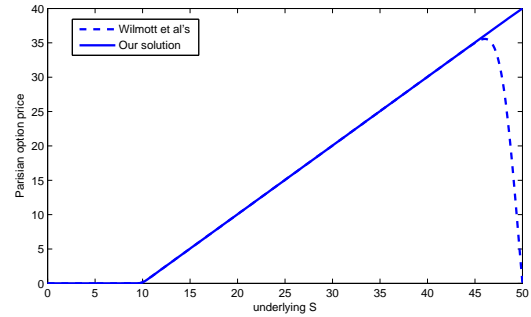


(b) Price of the Parisian up-and-out call options

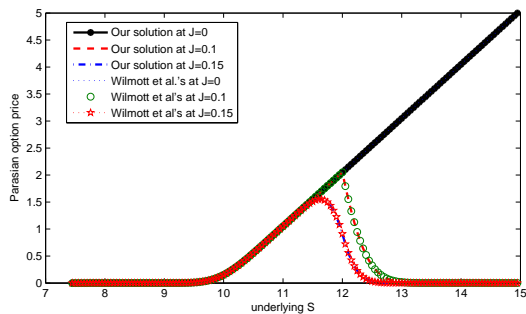
Figure 2: Wilmott *et al.*'s solutions VS our results for the case $T - t \geq \bar{J} - J$. Parameters are $\sigma = 10\%$, $r = 5\%$, $D = 0$, $K = \$10$, $\bar{S} = \$12$, $\bar{J} = 0.2(\text{year})$ and $T - t = 1(\text{year})$.



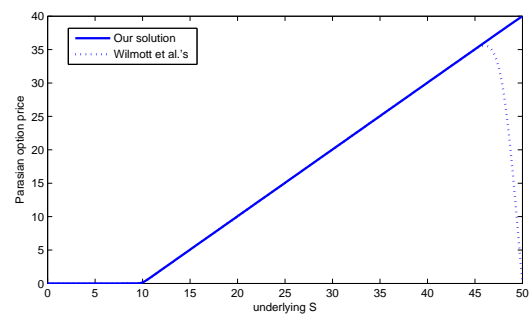
(a) Price of the Parisian up-and-out call options at finite S values.



(b) Price of the Parisian up-and-out call options at large S end.



(c) Price of the Parisian up-and-out call options at finite S values.



(d) Price of the Parisian up-and-out call options at large S end.

Figure 3: Wilmott *et al.*'s solutions VS our results for the case $T - t < \bar{J} - J$. Parameters are $\sigma = 10\%$, $r = 5\%$, $D = 0$, $K = \$10$, $\bar{S} = \$12$, $\bar{J} = 0.2(\text{year})$ and $T - t = 0.1(\text{year})$.

to the backward property of the current problem. In fact, under our pricing systems, the prism \overline{III} (\widetilde{III}) can be artificially filled in once a Parisian (*Parasian*) option with longer maturity is considered. In other words, in this region, our option prices should be identical to Wilmott *et al.*'s solutions.

Secondly, one may wonder why there is an extra condition (2.10) (or (2.14)) in our systems (2.11) (or (2.13)) whereas there is no such condition in Wilmott *et al.*'s PDE systems governing the price of a Parisian option (or *Parasian* option), yet the results from both systems agree well for reasonably small S . The reason is that when applying the explicit finite difference scheme, both of the connectivity conditions involved in the pricing systems (2.11) and (2.13) become redundant, since the option prices across the barrier are implicitly assumed to satisfy the governing equation, which is a much stronger condition than the continuity of the option Delta across the barrier. In fact, handling with the connectivity conditions in this way will not result in any difference on the final results, as clearly demonstrated in Section 2.1.

4.2 Numerical performance of our analytical solutions

To test the accuracy and efficiency of our exact solutions, in terms of numerically calculating the integrals, the best way is to compare the analytical solutions with those calculated directly from Wilmott *et al.*'s pricing systems with the utilization of their explicit finite difference method [2]. Such comparisons are shown in Table 1 and Table 2, in which the Parisian and *Parasian* up-and-out call option prices across the barrier are tabulated, respectively. Furthermore, we have considered four different times to maturity, i.e., $T - t = 0.3(\text{year})$, $T - t = 0.4(\text{year})$, $T - t = 0.5(\text{year})$, and $T - t = 1(\text{year})$. All the experiments were performed within Matlab7.8 on an Intel Pentium 4, 3GHZ machine. In these tables, columns marked with "FDM" display the results obtained from Wilmott *et al.*'s explicit finite difference scheme with extremely fine grids defined as $\Delta S = 0.05$, $\Delta \tau = \Delta J = 0.0001$, and the CPU time means the total time it takes to numerically evaluate our analytic

formulae.

From Table 1, it is clear that our analytical results for the Parisian up-and-out call options agree well with those produced by using the explicit finite difference method, with the point-wise difference between the two being in the order of $\mathcal{O}(10^{-3})$. Moreover, it only takes a few seconds to numerically carry out our exact solution, which is hundreds of folds less than what it takes to execute the code written with the explicit finite difference method for the same case, especially when options are of long maturity.

Provided in Table 2 are the corresponding results for the *Parasian* up-and-out call options. From this table, it is clear that our analytic results also agree with those calculated by using the explicit finite difference method, with the point-wise difference now being in the order of $\mathcal{O}(10^{-2})$. The loss of accuracy of the explicit finite difference method adopted to price the *Parasian* options is indeed expected, as in this case, the error along the J direction starts to accumulate when solving for both V_1 and V_2 , whereas for a corresponding Parisian option, the error along the J direction will not affect the accuracy of V_1 . In this sense, the explicit finite difference scheme used to price the *Parasian* options should be less accurate than that with the same grid size adopted to price the corresponding Parisian options. One may also notice from Table 2 that the point-wise relative errors between our exact solution and the numerical result are larger as the time to maturity becomes longer or the barrier time J is closer to the trigger value \bar{J} . This is not surprising at all, as in both cases mentioned above, the absolute value of $W(t, J)$ becomes very small, and discussing the relative error between two numbers which are quite small in absolute terms means very little. On the other hand, we should point out that although our analytic solution for $W(t, J)$ involves the evaluation of triple integrals, it is still far more efficient to implement than the explicit finite difference method, with the CPU times for the former being at least 20 folds less than those for the latter, as shown in Table 2. Therefore, from both efficiency and accuracy, points of view, our analytic formulae have clear edge over the explicit finite difference method in pricing Parisian and *Parasian* options.

As mentioned earlier, once the values of W are known, the option prices can be obtained

Table 1: The price of the Parisian up-and-out call options across the barrier, i.e., $W(t)$, with four different times to maturity. Parameters are $\sigma = 10\%$, $r = 5\%$, $D = 0$, $K = \$10$, $\bar{S} = \$12$, $\bar{J} = 0.2(\text{year})$

The time to expiry	FDM (CPU(second))	exact solution (CPU(second))
$T - t = 0.3$ (year)	1.3660 (375s)	1.3883 (1s)
$T - t = 0.4$ (year)	1.0819 (503s)	1.0934 (1s)
$T - t = 0.5$ (year)	0.8837 (631s)	0.8858 (2s)
$T - t = 1.0$ (year)	0.4246 (1269s)	0.4296 (3s)

Table 2: The price of the Parasian up-and-out call options across the barrier, i.e., $W(t, J)$, with four different times to maturity. Parameters are $\sigma = 10\%$, $r = 5\%$, $D = 0$, $K = \$10$, $\bar{S} = \$12$, $\bar{J} = 0.2(\text{year})$

At $J = 0$		
The time to expiry	FDM (CPU(second))	exact solution (CPU(second))
$T - t = 0.3$ (year)	0.9974 (439s)	0.9991 (23s)
$T - t = 0.4$ (year)	0.7055 (587s)	0.7312 (23s)
$T - t = 0.5$ (year)	0.5433 (736s)	0.5762 (23s)
$T - t = 1.0$ (year)	0.2297 (1476s)	0.2462 (23s)
At $J = 0.1$		
The time to expiry	FDM (CPU(second))	exact solution (CPU(second))
$T - t = 0.3$ (year)	0.5427 (439s)	0.5339 (23s)
$T - t = 0.4$ (year)	0.4043 (587s)	0.4050 (23s)
$T - t = 0.5$ (year)	0.3184 (763s)	0.3222 (23s)
$T - t = 1.0$ (year)	0.1388 (1476s)	0.1400 (23s)
At $J = 0.15$		
The time to expiry	FDM (CPU(second))	exact solution (CPU(second))
$T - t = 0.3$ (year)	0.3446 (439s)	0.3230 (23s)
$T - t = 0.4$ (year)	0.2599 (587s)	0.2463 (23s)
$T - t = 0.5$ (year)	0.2059 (763s)	0.1963 (23s)
$T - t = 1.0$ (year)	0.0904 (1476s)	0.0852 (23s)

straightforwardly from the integral representations (3.23)-(3.24) and (3.33)-(3.34), for the Parisian and Parasian up-and-out call options, respectively. We can thus further test the accuracy of our analytic formulae by calculating the option prices via (3.23)-(3.24) and (3.33)-(3.34), and comparing them with those produced by using Wilmott *et al.*'s method, as shown in Fig 4 and Fig 5, where the option prices are displayed as a function of the underlying S , for different J values. The excellent agreement of our analytic option prices and those calculated by the explicit finite difference scheme again provides a sense of verification of our closed-form explicit formulae.

On the other hand, Fig 4 and Fig 5 also clearly exhibit the up-and-out feature of the Parisian-type options. It can be observed that the prices of both Parisian and Parasian up-and-out call options have peaked before the barrier \bar{S} is reached. Then, as S is further

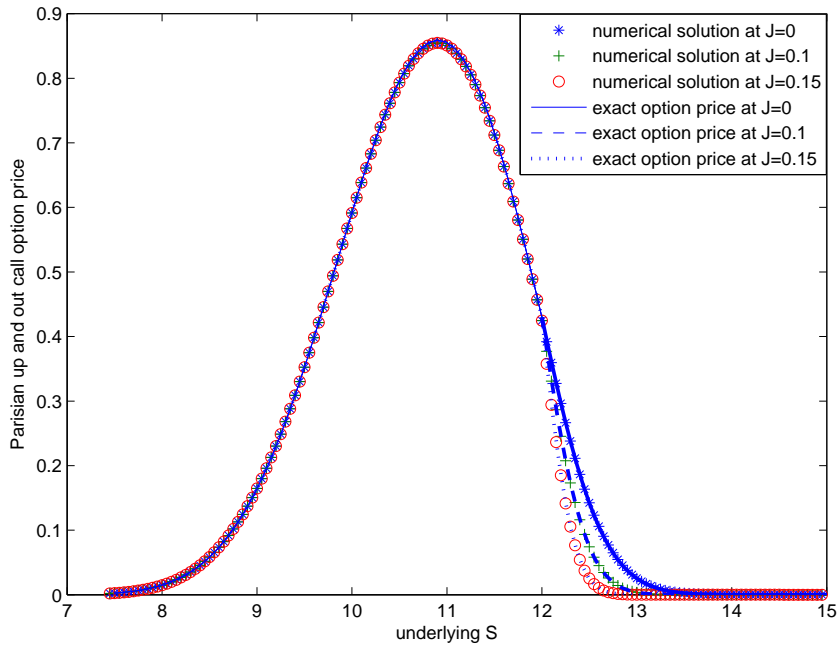


Figure 4: The price of the Parisian up-and-out call option at different J values. Parameters are $\sigma = 10\%$, $r = 5\%$, $D = 0$, $K = \$10$, $\bar{S} = \$12$, $\bar{J} = 0.2(\text{year})$ and $T - t = 1(\text{year})$

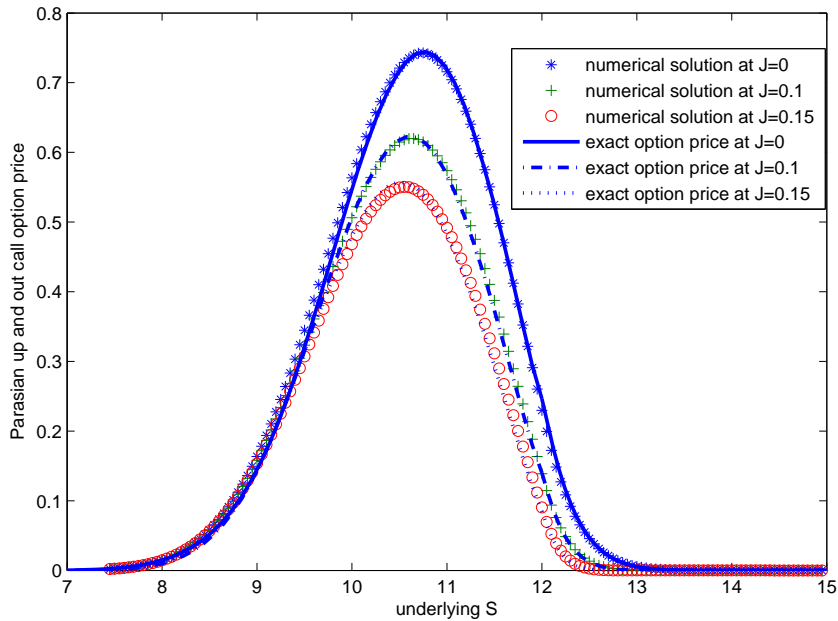


Figure 5: The price of the Parisian up-and-out call option at different J values. Parameters are $\sigma = 10\%$, $r = 5\%$, $D = 0$, $K = \$10$, $\bar{S} = \$12$, $\bar{J} = 0.2(\text{year})$ and $T - t = 1(\text{year})$

increased, the danger of being “out” has already been factorized in the price, which starts to decrease from the peak of the price and eventually tends to zero when the underlying price gets too high. These results make sense financially as once the underlying price gets closer to the barrier, market at some point will start to consider the danger of being “knocked out”. One should also notice that because of different resetting structures, the option price of a Parisian option is not affected at all by the residue left in the trigger (because the barrier time will be reset to zero once S falls back to \bar{S}), whereas the changes of the *Parasian* option price with the residue left in the trigger are “felt” in the entire domain of the underlying, as a result of the non-resetting mechanism on the trigger. It can be further observed from these two figures that when the underlying price is near \bar{S} , the Parisian specification provides a smooth transition only at $J = 0$, and moreover, its option Delta increases dramatically to infinity as the barrier time J becomes closer to the trigger value \bar{J} . This is because as long as the trigger value \bar{J} is not reached, J will be reset to zero when the barrier is touched, and furthermore, the option Delta being continuous is only specified at $J = 0$ for the Parisian option (see the connectivity condition (2.10)). In contrast, the *Parasian* specification has made the option prices smooth across the barrier \bar{S} for all non-zero J values. This is indeed a result of the option Delta of a *Parasian* option being assumed to be continuous across the whole barrier plane (see the connectivity condition (2.14)). Note that “smooth” here refers to the option Delta to be continuous.

Depicted in Fig 6 is the comparison of a Parisian up-and-out call option and the corresponding *Parasian* option. Clearly, at any given J , the Parisian option price is always higher than its *Parasian* counterpart for all underlying S . This is indeed reasonable, as the “out” feature of the *Parasian* option has been amplified by the cumulative effect on J , and thus a *Parasian* option is more vulnerable to be knocked out than its Parisian counterpart. Consequently, its value should be lower than the corresponding Parisian option price, if all the other terms are the same.

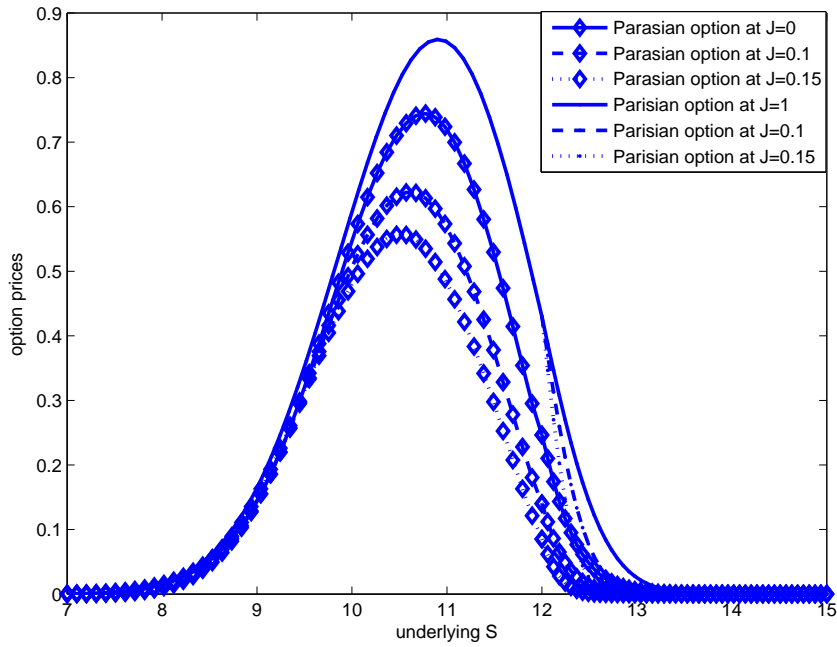


Figure 6: The Parisian up-and-out call option VS the corresponding *Parisian* option at different J values. Parameters are $\sigma = 10\%$, $r = 5\%$, $D = 0$, $K = \$10$, $\bar{S} = \$12$, $\bar{J} = 0.2(\text{year})$, $T - t = 1(\text{year})$.

5 Conclusion

In this paper, a substantial progress has been made for pricing Parisian-type options, with the successful derivation of two closed-form analytic formulae for European-style Parisian and *Parasian* options, respectively. A key step of two approaches presented here is to simplify the domain of the pricing systems, and consequently reduce a 3-D problem to two coupled 2-D systems. Through numerical examples, we have shown that our newly established pricing systems can indeed correctly replace the original 3-D ones that have been used in the literature. We have also compared the results produced from our analytical solutions with those calculated by using Wilmott *et al.*'s method, and found that our results agree with Wilmott *et al.*'s perfectly for reasonably finite values of the underlying.

The significance of our work has been illustrated from two aspects. Theoretically, although the literature provides quite a few approximate methods for pricing Parisian-type options, the closed-form solutions are presented for the first time. Practically, computational efficiency has been enormously enhanced through using the newly derived analytical formulae, which can assist market practitioners to price Parisian-type options much more quickly and accurately.

Appendix A The valuation of $\mathcal{L}^{-1}\left[\frac{\hat{H}(x,p)}{\lambda_2-\lambda_1}\Big|_{x=\bar{x}}\right]$

Since it is already known that

$$F(x, \tau) = \int_{-\infty}^{\bar{x}} \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{k}{2}(x-z) - (\frac{k^2}{4} + \gamma)\tau} [e^{-\frac{(x-z)^2}{4\tau}} - e^{-\frac{(x+z-2\bar{x})^2}{4\tau}}] f(z) dz,$$

we obtain

$$\begin{aligned} \hat{F}(x, p) &= \mathcal{L}(F(x, \tau)) \\ &= \left[\int_{-\infty}^{\bar{x}} \frac{e^{\lambda_1(x-\bar{x}) + \lambda_2(\bar{x}-z)}}{\lambda_2 - \lambda_1} + \int_{-\infty}^x \frac{e^{\lambda_2(x-z)}}{\lambda_1 - \lambda_2} + \int_x^{\bar{x}} \frac{e^{\lambda_1(x-z)}}{\lambda_1 - \lambda_2} \right] f(z) dz, \end{aligned}$$

and thus

$$\begin{aligned} \frac{\hat{H}(x, p)}{\lambda_2 - \lambda_1} \Big|_{x=\bar{x}} &= \frac{\mathcal{L}\left[\frac{\partial F}{\partial x}(x, \tau)\right] \Big|_{x=\bar{x}}}{\lambda_2 - \lambda_1} = \frac{\frac{\partial \hat{F}}{\partial x}(x, p) \Big|_{x=\bar{x}}}{\lambda_2 - \lambda_1} \\ &= \int_{-\infty}^{\bar{x}} \frac{e^{\lambda_2(\bar{x}-z)}}{\lambda_1 - \lambda_2} f(z) dz \\ &= \int_{-\infty}^{\bar{x}} \frac{e^{-\frac{k}{2}(\bar{x}-z) - \sqrt{\frac{k^2}{4} + \gamma + p}(\bar{x}-z)}}{2\sqrt{\frac{k^2}{4} + \gamma + p}} f(z) dz. \end{aligned} \tag{A.1}$$

Now, applying the Laplace inversion to (A.1), we obtain

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{\hat{H}(x, p)}{\lambda_2 - \lambda_1} \Big|_{x=\bar{x}}\right] &= \mathcal{L}^{-1}\left[\int_{-\infty}^{\bar{x}} \frac{e^{-\frac{k}{2}(\bar{x}-z) - \sqrt{\frac{k^2}{4} + \gamma + p}(\bar{x}-z)}}{2\sqrt{\frac{k^2}{4} + \gamma + p}} f(z) dz\right] \\ &= \int_{-\infty}^{\bar{x}} \frac{e^{-\frac{k}{2}(\bar{x}-z)}}{2} f(z) \mathcal{L}^{-1}\left[\frac{e^{\sqrt{\frac{k^2}{4} + \gamma + p}(\bar{x}-z)}}{2\sqrt{\frac{k^2}{4} + \gamma + p}}\right] dz \\ &= \int_{-\infty}^{\bar{x}} \frac{e^{-\frac{k}{2}(\bar{x}-z)}}{2} f(z) e^{-(\frac{k^2}{4} + \gamma)\tau} \mathcal{L}^{-1}\left[\frac{e^{-\sqrt{p}(\bar{x}-z)}}{\sqrt{p}}\right] dz \quad (\text{shift theorem}) \\ &= \int_{-\infty}^{\bar{x}} \frac{e^{-\frac{k}{2}(\bar{x}-z) - (\frac{k^2}{4} + \gamma)\tau}}{2\sqrt{\pi\tau}} e^{-\frac{(\bar{x}-z)^2}{4\tau}} f(z) dz. \end{aligned}$$

Appendix B The valuation of $\mathcal{L}^{-1}\left[\frac{\hat{G}(x,p)}{\lambda_1-\lambda_2}\Big|_{x=\bar{x}}\right]$

According to the Convolution theorem [6], we obtain

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{\hat{G}(x,p)}{\lambda_1-\lambda_2}\Big|_{x=\bar{x}}\right] &= \int_0^\tau \frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{2\sqrt{\pi(\tau-s)}} G(x,s)\Big|_{x=\bar{x}} ds \\
&= \int_0^\tau \frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{2\sqrt{\pi(\tau-s)}} \int_{s-\bar{J}}^0 W(\xi) \frac{\partial g_2}{\partial x}(x,s-\xi)\Big|_{x=\bar{x}} d\xi ds \\
&= \int_0^\tau \frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{2\sqrt{\pi(\tau-s)}} \int_0^{\bar{J}-s} W(s-\bar{J}+m) \frac{\partial g_2}{\partial x}(x,\bar{J}-m)\Big|_{x=\bar{x}} dm ds \quad (\text{B.1})
\end{aligned}$$

On the other hand, since $\frac{\partial g_2}{\partial x}(x,\tau)$ is singular at $(\bar{x},0)$, we evaluate the value $\frac{\partial g_2}{\partial x}(\bar{x},\tau)$ in the Laplace space, and then convert it back. We obtain

$$\begin{aligned}
\frac{\partial g_2}{\partial x}(\bar{x},\tau) &= \mathcal{L}^{-1}[\lambda_2] \\
&= -\frac{k}{2}\delta(\tau) - \frac{e^{-(\frac{k^2}{4}+\gamma)\tau}}{\sqrt{\pi}} \int_{0^-}^{\tau^+} \frac{\delta'(s)}{\sqrt{\tau-s}} ds. \quad (\text{B.2})
\end{aligned}$$

Here, the left limit ‘ $-$ ’ and the right limit ‘ $+$ ’ are introduced to ensure the regularity of the integration. When the kernel function is integrable, such sided limit processes are longer needed.

By substituting (B.2) into (B.1), we obtain

$$\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{\hat{G}(x,p)}{\lambda_1-\lambda_2}\Big|_{x=\bar{x}}\right] \\
&= \underbrace{-\frac{k}{4\sqrt{\pi}} \int_0^\tau \frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{\sqrt{\tau-s}} \int_0^{(\bar{J}-s)^+} W(s-\bar{J}+m) \delta(\bar{J}-m) dm ds}_I \\
&\quad \underbrace{-\frac{1}{2\pi} \int_0^\tau \frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{\sqrt{\tau-s}} \int_0^{\bar{J}-s} W(s-\bar{J}+m) e^{-(\frac{k^2}{4}+\gamma)(\bar{J}-m)} \int_{0^-}^{(\bar{J}-m)^+} \frac{\delta'(\xi)}{\sqrt{\bar{J}-m-\xi}} d\xi dm ds}_II.
\end{aligned}$$

In the following, we shall evaluate the integrals I and II separately. When evaluating I ,

we can change the order of the two integrals, and we obtain

$$I = -\frac{k}{4\sqrt{\pi}} \left[\int_0^{(J-\tau)^+} \delta(\bar{J}-m) \int_0^\tau \frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{\sqrt{\tau-s}} W(s-\bar{J}+m) ds dm \right. \\ \left. + \int_{(J-\tau)^+}^{J^+} \delta(\bar{J}-m) \int_0^{\bar{J}-m} \frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{\sqrt{\tau-s}} W(s-\bar{J}+m) ds dm \right].$$

It should be remarked that the above change of the order of the double integrals makes sense, as the inner integrated function $\frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{\sqrt{\tau-s}} W(s-\bar{J}+m)$ only has a removable singularity at $\tau = s$, as long as $W(\tau) \in \mathcal{C}^1$, which is certainly true from the financial point of view. Therefore, when $\tau > 0$,

$$I = -\frac{k}{4\sqrt{\pi}} \left[0 + \lim_{m \rightarrow \bar{J}} \int_0^{\bar{J}-m} \frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{\sqrt{\tau-s}} W(s-\bar{J}+m) ds \right] \\ = 0,$$

whereas when $\tau = 0$,

$$I = -\frac{k}{4\sqrt{\pi}} \left[\left(\lim_{\tau \rightarrow 0} \lim_{m \rightarrow \bar{J}} \int_0^\tau + \lim_{m \rightarrow \bar{J}} \int_0^{\bar{J}-m} \right) \frac{e^{-(\frac{k^2}{4}+\gamma)(\tau-s)}}{\sqrt{\tau-s}} W(s-\bar{J}+m) ds \right] \\ = 0.$$

Thus, one can conclude that the integral I equals zero.

On the other hand, to evaluate II , we also perform the change of the order of the integrals first, and we obtain

$$II = -\frac{1}{2\pi} \left[\underbrace{\int_{0^-}^{0^+} \delta'(\xi) \int_0^\tau \int_0^{\bar{J}-s} K(m, s; \xi) dm ds d\xi}_{(1)} + \underbrace{\int_{0^+}^{\tau^+} \delta'(\xi) \int_\xi^\tau \int_0^{\bar{J}-s} K(m, s; \xi) dm ds d\xi}_{(2)} \right. \\ \left. + \underbrace{\int_{0^+}^{\tau^+} \delta'(\xi) \int_0^\xi \int_0^{\bar{J}-\xi} K(m, s; \xi) dm ds d\xi}_{(3)} + \underbrace{\int_{\tau^+}^{\bar{J}^+} \delta'(\xi) \int_0^\tau \int_0^{\bar{J}-\xi} K(m, s; \xi) dm ds d\xi}_{(4)} \right],$$

where

$$K(m, s; \xi) = \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau - s)} e^{-(\frac{k^2}{4} + \gamma)(\bar{J} - m)}}{\sqrt{\tau - s} \sqrt{\bar{J} - m - \xi}} W(s - \bar{J} + m).$$

One can observe that the last three integrals, i.e., (2), (3), and (4) do not include the zero along the ξ -axis, at which the impulse function $\delta(\xi)$ has non-trivial value, and thus these three integrals all equal to zero.

Therefore,

$$\begin{aligned} II &= -\frac{1}{2\pi} \int_{0^-}^{0^+} \delta'(\xi) \int_0^\tau \int_0^{\bar{J}-s} K(m, s; \xi) dm ds d\xi \\ &= \frac{1}{2\pi} \lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} \left[\int_0^\tau \int_0^{\bar{J}-\xi} K(m, s; \xi) dm ds \right] \\ &= \frac{1}{2\pi} \int_0^\tau \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau - s)}}{\sqrt{\tau - s}} \lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} \left[\int_0^{\bar{J}-s} \frac{e^{-(\frac{k^2}{4} + \gamma)(\bar{J} - m)}}{\sqrt{\bar{J} - m - \xi}} dm \right] ds. \end{aligned} \quad (\text{B.3})$$

Now, introducing a new variable $t = \sqrt{\bar{J} - m - \xi}$, and substituting it into (B.3), we obtain

$$\begin{aligned} II &= \frac{1}{2\pi} \int_0^\tau \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau - s)}}{\sqrt{\tau - s}} \lim_{\xi \rightarrow 0} \frac{\partial}{\partial \xi} \left[\int_{\sqrt{s-\xi}}^{\sqrt{\bar{J}-\xi}} 2e^{-(\frac{k^2}{4} + \gamma)(t^2 + \xi)} W(s - \xi - t^2) dt \right] ds \\ &= \frac{1}{2\pi} \int_0^\tau \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau - s)}}{\sqrt{\tau - s}} \left\{ W(0) \frac{e^{-(\frac{k^2}{4} + \gamma)s}}{\sqrt{s}} - W(s - \bar{J}) \frac{e^{-(\frac{k^2}{4} + \gamma)\bar{J}}}{\sqrt{\bar{J}}} \right. \\ &\quad \left. - 2 \int_s^{\bar{J}} e^{-(\frac{k^2}{4} + \gamma)t^2} \left[\left(\frac{k^2}{4} + \gamma \right) W(s - t^2) + W'(s - t^2) \right] dt \right\} ds \\ &= \frac{W(0)}{2} e^{-(\frac{k^2}{4} + \gamma)\tau} - \frac{e^{-(\frac{k^2}{4} + \gamma)\bar{J}}}{2\pi\sqrt{\bar{J}}} \int_0^\tau \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau - s)}}{\sqrt{\tau - s}} W(s - \bar{J}) ds \\ &\quad - \frac{1}{\pi} \int_0^\tau \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau - s)}}{\sqrt{\tau - s}} \int_{\sqrt{s}}^{\sqrt{\bar{J}}} e^{-(\frac{k^2}{4} + \gamma)t^2} \left[\left(\frac{k^2}{4} + \gamma \right) W(s - t^2) + W'(s - t^2) \right] dt ds. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}^{-1} \left[\frac{\hat{G}(x, p)}{\lambda_1 - \lambda_2} \Big|_{x=\bar{x}} \right] &= \frac{W(0)}{2} e^{-(\frac{k^2}{4} + \gamma)\tau} - \frac{e^{-(\frac{k^2}{4} + \gamma)\bar{J}}}{2\pi\sqrt{\bar{J}}} \int_0^\tau \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau - s)}}{\sqrt{\tau - s}} W(s - \bar{J}) ds \\ &\quad - \frac{1}{\pi} \int_0^\tau \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau - s)}}{\sqrt{\tau - s}} \int_{\sqrt{s}}^{\sqrt{\bar{J}}} e^{-(\frac{k^2}{4} + \gamma)t^2} \left[\left(\frac{k^2}{4} + \gamma \right) W(s - t^2) + W'(s - t^2) \right] dt ds. \end{aligned}$$

Appendix C The double Laplace inversion

In this appendix, we shall apply the double Laplace inversion to (3.37) to solve for its explicit form in the original (τ, l) space. By applying the double Laplace inversion, we obtain

$$\begin{aligned}\bar{W}(\tau, l) &= \mathcal{L}_{p_2}^{-1} \left[\mathcal{L}_{p_1}^{-1} \left[\frac{\hat{H}(x, p_1, p_2)}{\lambda_2(p_2) - \lambda_1(p_1)} \Big|_{x=\bar{x}} \right] \right] \\ &= \mathcal{L}_{p_2}^{-1} \left[\mathcal{L}_{p_1}^{-1} \left[\int_{-\infty}^{\bar{x}} \int_{-\infty}^z \frac{e^{\lambda_2(p_1)(\bar{x}-z) + \lambda_2(p_2)(z-s)}}{2\sqrt{\frac{k^2}{4} + \gamma + p_2}(\sqrt{\frac{k^2}{4} + \gamma + p_1} + \sqrt{\frac{k^2}{4} + \gamma + p_2})} \max(e^s - 1) ds dz \right. \right. \\ &\quad \left. \left. + \int_{-\infty}^{\bar{x}} \int_z^{+\infty} \frac{e^{\lambda_2(p_1)(\bar{x}-z) + \lambda_1(p_2)(z-s)}}{2\sqrt{\frac{k^2}{4} + \gamma + p_2}(\sqrt{\frac{k^2}{4} + \gamma + p_1} + \sqrt{\frac{k^2}{4} + \gamma + p_2})} \max(e^s - 1) ds dz \right] \right],\end{aligned}$$

which can be simplified as

$$\begin{aligned}\bar{W}(\tau, l) &= \int_{-\infty}^{\bar{x}} \int_{-\infty}^z \frac{1}{2} e^{-(\frac{k^2}{4} + \gamma)(\tau + l) - \frac{k}{2}(\bar{x} - s)} \mathcal{L}_{p_2}^{-1} \left[\mathcal{L}_{p_1}^{-1} \left[\frac{e^{-\sqrt{p_1}(\bar{x}-z) - \sqrt{p_2}(z-s)}}{\sqrt{p_2}(\sqrt{p_1} + \sqrt{p_2})} \right] \right] \max(e^s - 1) ds dz \\ &\quad + \int_{-\infty}^{\bar{x}} \int_z^{+\infty} \frac{1}{2} e^{-(\frac{k^2}{4} + \gamma)(\tau + l) - \frac{k}{2}(\bar{x} - s)} \mathcal{L}_{p_2}^{-1} \left[\mathcal{L}_{p_1}^{-1} \left[\frac{e^{-\sqrt{p_1}(\bar{x}-z) + \sqrt{p_2}(z-s)}}{\sqrt{p_2}(\sqrt{p_1} + \sqrt{p_2})} \right] \right] \max(e^s - 1) ds dz.\end{aligned}$$

The key step now is to evaluate $\mathcal{L}_{p_2}^{-1} \left[\mathcal{L}_{p_1}^{-1} \left[\frac{e^{-a\sqrt{p_1} - b\sqrt{p_2}}}{\sqrt{p_2}(\sqrt{p_1} + \sqrt{p_2})} \right] \right]$, where $a \geq 0$, and $b \geq 0$.

$$\begin{aligned}L_{p_2}^{-1} \left[L_{p_1}^{-1} \left[\frac{e^{-a\sqrt{p_1} - b\sqrt{p_2}}}{\sqrt{p_2}(\sqrt{p_1} + \sqrt{p_2})} \right] \right] &= \mathcal{L}_{p_2}^{-1} \left[e^{-b\sqrt{b_2}} \mathcal{L}_{p_1}^{-1} \left[e^{-a\sqrt{p_1}} \frac{1}{p_2(1 + \sqrt{p_1/p_2})} \right] \right] \\ &= \mathcal{L}_{p_2}^{-1} \left[\frac{e^{-b\sqrt{p_2}}}{\sqrt{p_2}} \int_0^\tau \frac{ae^{-\frac{a^2}{4(\tau-s)}}}{2\sqrt{\pi(\tau-s)^3} \sqrt{\pi s}} ds \right] - \mathcal{L}_{p_2}^{-1} \left[\int_0^\tau \frac{ae^{-\frac{a^2}{4(\tau-s)}}}{\pi\sqrt{(\tau-s)^3}} \int_{\sqrt{p_2 s}}^{+\infty} e^{p_2 s - b\sqrt{p_2} - w^2} dw ds \right] \\ &= \frac{1}{\pi\sqrt{\tau l}} e^{-\frac{a^2}{4\tau} - \frac{b^2}{4l}} - \frac{a}{\pi} \int_0^\tau \frac{e^{-\frac{a^2}{4(\tau-s)}}}{\sqrt{(\tau-s)^3}} \int_{\sqrt{s}}^{+\infty} \mathcal{L}_{p_2}^{-1} \left[\sqrt{p_2} e^{-b\sqrt{p_2} - p_2(\xi^2 - s)} \right] d\xi ds \\ &= \frac{1}{\pi\sqrt{\tau l}} e^{-\frac{a^2}{4\tau} - \frac{b^2}{4l}} - \frac{a}{\pi} \int_0^\tau \frac{e^{-\frac{a^2}{4(\tau-s)}}}{\sqrt{(\tau-s)^3}} \int_{\sqrt{s}}^{\sqrt{l+s}} \int_{0^-}^{(l+s-\xi^2)^+} \frac{\delta'(\eta)}{\sqrt{\pi(l+s-\xi^2-\eta)}} e^{-\frac{b^2}{4(l+s-\xi^2-\eta)}} d\eta d\xi ds.\end{aligned}$$

Therefore,

$$\begin{aligned}
\bar{W}(\tau, l) &= \frac{1}{2\pi\sqrt{\tau l}} \int_{-\infty}^{\bar{x}} \int_0^{+\infty} e^{-\frac{k}{2}(\bar{x}-s)} (e^s - 1) e^{-\frac{(\bar{x}-z)^2}{4\tau} - \frac{(z-s)^2}{4l}} ds dz \\
&- \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau+l)}}{2\pi} \int_{-\infty}^{\bar{x}} \int_0^{+\infty} e^{-\frac{k}{2}(\bar{x}-s)} (e^s - 1) (\bar{x} - z) \int_0^\tau \frac{e^{-\frac{(x-z)^2}{4(\tau-w)}}}{\sqrt{(\tau-w)^3}} \\
&\quad \int_{\sqrt{w}}^{\sqrt{l+w}} \int_{0^-}^{(l+w-\xi^2)^+} \frac{\delta'(\eta)}{\sqrt{\pi(l+w-\xi^2-\eta)}} e^{-\frac{(z-s)^2}{4(l+w-\xi^2-\eta)}} d\eta d\xi dw ds dz \\
&= \frac{1}{2\pi\sqrt{\tau l}} \int_{-\infty}^{\bar{x}} \int_0^{+\infty} e^{-\frac{k}{2}(\bar{x}-s)} (e^s - 1) e^{-\frac{(\bar{x}-z)^2}{4\tau} - \frac{(z-s)^2}{4l}} ds dz \\
&- \frac{e^{-(\frac{k^2}{4} + \gamma)(\tau+l)}}{2\pi} \int_{-\infty}^{\bar{x}} (\bar{x} - z) \int_0^\tau \frac{e^{-\frac{(x-z)^2}{4(\tau-w)}}}{\sqrt{(\tau-w)^3}} \int_{0^-}^{l^+} \delta'(\eta) \int_{\sqrt{w}}^{\sqrt{l+w-\eta}} \\
&\quad \underbrace{\int_0^{+\infty} e^{-\frac{k}{2}(\bar{x}-s)} (e^s - 1) \frac{e^{-\frac{(z-s)^2}{4(l+w-\xi^2-\eta)}}}{\sqrt{\pi(l+w-\xi^2-\eta)}} ds d\xi d\eta dw dz}_{II}.
\end{aligned}$$

By changing variables, II can be written in the form of the normal distribution function,

i.e.,

$$II = 2e^{-\frac{k}{2}(\bar{x}-z)+z+(\frac{k}{2}+1)^2(l+w-\xi^2-\eta)} N(d_1) - 2e^{-\frac{k}{2}(\bar{x}-z)+\frac{k^2}{4}(l+w-\xi^2-\eta)} N(d_2),$$

where

$$\begin{aligned}
d_1 &= \frac{z}{\sqrt{2(l+w-\xi^2-\eta)}} + \left(\frac{k}{2} + 1\right) \sqrt{2(l+w-\xi^2-\eta)}, \\
d_2 &= \frac{z}{\sqrt{2(l+w-\xi^2-\eta)}} + \frac{k}{2} \sqrt{2(l+w-\xi^2-\eta)}.
\end{aligned}$$

As a result,

$$\begin{aligned}
& \int_{0^-}^{l^+} \delta'(\eta) \int_{\sqrt{w}}^{\sqrt{l+w-\eta}} II d\xi d\eta \\
&= \frac{1}{\sqrt{l+w}} e^{-\frac{k}{2}(\bar{x}-z)} (e^z - 1) N\left(\frac{z}{0}\right) - 2e^{-\frac{k}{2}(\bar{x}-z)+z} \int_{\sqrt{w}}^{\sqrt{l+w}} e^{(\frac{k}{2}+1)^2(l+w-\xi^2)} \left\{ -\left(\frac{k}{2}+1\right)^2 N(d_1 |_{\eta=0}) \right. \\
& \quad \left. + \frac{e^{-\frac{d_1^2}{2}}|_{\eta=0}}{\sqrt{2\pi}} \left[\frac{z}{2\sqrt{2(l+w-\xi^2)^3}} - \frac{\sqrt{2}(\frac{k}{2}+1)}{2\sqrt{(l+w-\xi^2)}} \right] \right\} d\xi - 2e^{-\frac{k}{2}(\bar{x}-z)} \int_{\sqrt{w}}^{\sqrt{l+w}} e^{\frac{k^2}{4}(l+w-\xi^2)} \left\{ \right. \\
& \quad \left. \frac{k^2}{4} N(d_2 |_{\eta=0}) - \frac{e^{-\frac{d_2^2}{2}}|_{\eta=0}}{\sqrt{2\pi}} \left[\frac{z}{2\sqrt{2(l+w-\xi^2)^3}} - \frac{\sqrt{2}k}{4\sqrt{(l+w-\xi^2)}} \right] \right\} \\
&= \frac{1}{\sqrt{l+w}} e^{-\frac{k}{2}(\bar{x}-z)} (e^z - 1) N\left(\frac{z}{0}\right) - 2e^{-\frac{k}{2}(\bar{x}-z)} \left\{ \int_{\arcsin \sqrt{\frac{w}{l+w}}}^{\frac{\pi}{2}} f_3(\theta, w, z) d\theta + \int_{\sqrt{\frac{w}{l}}}^{+\infty} f_4(y, w, z) dy \right\},
\end{aligned}$$

where

$$\begin{aligned}
f_3(\theta, w, z) &= e^{z+(\frac{k}{2}+1)^2(l+w)\cos^2\theta} \left[-\left(\frac{k}{2}+1\right)^2 N(\tilde{d}_1) \sqrt{l+w} \cos\theta - \left(\frac{k}{2}+1\right) \frac{e^{-\frac{\tilde{d}_1^2}{2}}}{2\sqrt{\pi}} \right] \\
& \quad + e^{\frac{k^2}{4}(l+w)\cos^2\theta} \left[\frac{k^2}{4} N(\tilde{d}_2) \sqrt{l+w} \cos\theta + k \frac{e^{-\frac{\tilde{d}_2^2}{2}}}{4\sqrt{\pi}} \right], \\
f_4(y, w, z) &= \frac{z}{4\sqrt{\pi}(l+w)} \left[e^{z+(\frac{k}{2}+1)^2(l+w)\frac{1}{y^2+1}-\frac{d_3}{2}} - e^{\frac{k^2}{4}(l+w)\frac{1}{y^2+1}-\frac{d_4}{2}} \right],
\end{aligned}$$

$$\begin{aligned}
\tilde{d}_1 &= \frac{z}{\sqrt{2(l+w)} \cos\theta} + \left(\frac{k}{2}+1\right) \sqrt{2(l+w)} \cos\theta, \\
\tilde{d}_2 &= \frac{z}{\sqrt{2(l+w)} \cos\theta} + \frac{k}{2} \sqrt{2(l+w)} \cos\theta, \\
d_3 &= \frac{z^2(1+y^2)}{2(l+w)} + z(k+2) + \frac{2(l+w)}{(y^2+1)} \left(\frac{k}{2}+1\right)^2, \\
d_4 &= \frac{z^2(1+y^2)}{2(l+w)} + kz + \frac{(l+w)k^2}{2(y^2+1)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\bar{W}(\tau, l) &= \underbrace{\frac{1}{2\pi\sqrt{\tau l}} \int_{-\infty}^{\bar{x}} \int_0^{+\infty} e^{-\frac{k}{2}(\bar{x}-s)} (e^s - 1) e^{-\frac{(\bar{x}-z)^2}{4\tau} - \frac{(z-s)^2}{4l}} ds dz}_I \\
&\quad - \underbrace{\frac{e^{-(\frac{k^2}{4} + \gamma)(\tau+l)}}{2\pi} \int_{-\infty}^{\bar{x}} (\bar{x} - z) \int_0^{\tau} \frac{e^{-\frac{(x-z)^2}{4(\tau-w)}}}{\sqrt{(\tau-w)^3}} \frac{1}{\sqrt{l+w}} e^{-\frac{k}{2}(\bar{x}-z)} (e^z - 1) N\left(\frac{z}{0}\right) dw dz}_II \\
&\quad + \underbrace{\frac{e^{-(\frac{k^2}{4} + \gamma)(\tau+l)}}{\pi} e^{-\frac{k}{2}(\bar{x}-z)} \int_{-\infty}^{\bar{x}} (\bar{x} - z) \int_0^{\tau} \frac{e^{-\frac{(x-z)^2}{4(\tau-w)}}}{\sqrt{(\tau-w)^3}} \int_{\arcsin \sqrt{\frac{w}{l+w}}}^{\frac{\pi}{2}} f_3(\theta, w, z) d\theta dw dz}_III \\
&\quad + \underbrace{\frac{e^{-(\frac{k^2}{4} + \gamma)(\tau+l)}}{\pi} e^{-\frac{k}{2}(\bar{x}-z)} \int_{-\infty}^{\bar{x}} (\bar{x} - z) \int_0^{\tau} \frac{e^{-\frac{(x-z)^2}{4(\tau-w)}}}{\sqrt{(\tau-w)^3}} \int_{\sqrt{\frac{w}{l}}}^{+\infty} f_4(y, w, z) dy dw dz}_IV.
\end{aligned}$$

By changing variables, the first integral can be simplified as

$$\begin{aligned}
I &= \frac{\sqrt{2}}{\sqrt{\pi}} e^{-(\frac{k^2}{4} + \gamma)(\tau+l)} \int_0^{\infty} \left\{ e^{(1+\frac{k}{2})^2 l + \bar{x} - \frac{\eta^2}{2} - \sqrt{2\tau}(\frac{k}{2}+1)\eta} N\left[\frac{\bar{x} - \sqrt{2\tau}\eta}{\sqrt{2l}} + \sqrt{2l}\left(\frac{k}{2} + 1\right)\right] \right. \\
&\quad \left. - e^{\frac{k^2}{4}l - \frac{\eta^2}{2} - \frac{\sqrt{2\tau}k\eta}{2}} N\left[\frac{\bar{x} - \sqrt{2\tau}\eta}{\sqrt{2l}} + \frac{\sqrt{2l}k}{2}\right] \right\} d\eta.
\end{aligned}$$

Denoting $\eta = \sqrt{\frac{l+w}{\tau-w}}$, $\xi = \frac{\bar{x}-z}{\sqrt{2(\tau+l)}}$ and substituting them to II , we have

$$II = -\frac{\sqrt{2}e^{-(\frac{k^2}{4} + \gamma)(\tau+l)}}{\pi} \int_0^{\frac{\bar{x}}{\sqrt{2(\tau+l)}}} e^{-\frac{k}{2}\sqrt{2(\tau+l)}\xi - \frac{\xi^2}{2}} (e^{\bar{x} - \sqrt{2(\tau+l)}\xi} - 1) (1 - N(\sqrt{\frac{l}{\tau}}\xi)) d\xi.$$

Denoting $\eta = \frac{x-z}{\sqrt{2(\tau-w)}}$, and substituting it to $III - IV$, we obtain

$$\begin{aligned}
III &= \frac{2\sqrt{2}e^{-(\frac{k^2}{4} + \gamma)(\tau+l)}}{\pi} \int_{-\infty}^{\bar{x}} \int_{\frac{\bar{x}-z}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{\eta^2}{2} - \frac{k}{2}(\bar{x}-z)} \int_{\arcsin \sqrt{\frac{2\eta^2\tau - (\bar{x}-z)^2}{2\eta^2(\tau+l) - (\bar{x}-z)^2}}}^{\frac{\pi}{2}} f_3(\theta, \tau - \frac{(\bar{x}-z)^2}{2\eta^2}, z) d\theta d\eta dz, \\
IV &= \frac{2\sqrt{2}e^{-(\frac{k^2}{4} + \gamma)(\tau+l)}}{\pi} \int_{-\infty}^{\bar{x}} \int_{\frac{\bar{x}-z}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{\eta^2}{2} - \frac{k}{2}(\bar{x}-z)} \int_{\sqrt{\frac{2\eta^2\tau - (\bar{x}-z)^2}{2\eta^2 l}}}^{+\infty} f_4(y, \tau - \frac{(\bar{x}-z)^2}{2\eta^2}, z) dy d\eta dz.
\end{aligned}$$

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