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Goard, Joanna and Mazur, Mathew, "Stochastic volatility models and the pricing of VIX options" (2013).

Faculty of Engineering and Information Sciences - Papers: Part A. 1187.

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Keywords

pricing, vix, volatility, options, stochastic, models

Disciplines

Engineering | Science and Technology Studies

Publication Details

Goard, J. & Mazur, M. (2013). Stochastic volatility models and the pricing of VIX options. *Mathematical Finance*, 23 (3), 439-458.

Stochastic Volatility Models and the Pricing of VIX Options

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Abstract

In this paper we examine and compare the performance of a variety of continuous-time volatility models in their ability to capture the behaviour of the VIX. The ‘3/2-model’ with a diffusion structure which allows the volatility of volatility changes to be highly sensitive to the actual level of volatility is found to outperform all other popular models tested. Analytic solutions for option prices on the VIX under the 3/2- model are developed and then used to calibrate at-the-money market option prices.

Keywords: *stochastic volatility, volatility model, VIX options*

Acknowledgments: The authors are grateful for the insightful comments of the anonymous referees who have helped to improve the paper.

1 Introduction

As stock prices all over the world dramatically rise and fall, investors are continually in search for financial instruments to reduce the variability of their portfolio values. Consequently the volatility in the market receives much interest from market participants and researchers. Currently the most popular indicator of overall market volatility in the US is the CBOE volatility index, VIX, which provides a measure of the implied volatility of options with a maturity of 30 days on the S&P500 index from eight different SPX option series. It thus presents a measure of the market's expectation of volatility over the next 30 days. The index is also known as the "fear gauge" as in times of market turmoil and large price movements, the VIX index tends to rise, whereas when the market is easing upward in a long-run bull market, the VIX index remains low and steady.

The VIX was introduced by Whaley (1993) and has become of particular interest in recent years with the introduction of VIX futures contracts in 2004 and of options on the VIX in 2006. These offered investors new instruments for speculating and hedging volatility risk directly on the S&P500 index and so valuations on VIX derivatives became necessary. Whaley himself used Black's (1976) formula (see eg Hull (2002)) to price volatility options under the assumptions of a lognormal volatility process and the existence of a futures contract on volatility with futures price equal to the current index level.

Since 1993 a growing body of literature has emerged on the pricing of volatility and variance products. Based on empirical evidence that volatility, V , was mean-reverting (eg French *et al* (1987), Harvey and Whaley (1992)), Grünbichler and Longstaff (1996) valued volatility futures and options based on a mean-reverting stochastic volatility process with a square root diffusion term ie $dV = (a - bV)dt + c\sqrt{V}dZ$, $b > 0$, with the assumption that the premium for volatility risk was proportional to volatility risk. Here and in the rest of the paper dZ refers to an increment in a Wiener process Z with probability measure P .

One of the most influential papers to date on the valuation of volatility and variance swaps

is the work of Demeterfi *et al* (1999) who laid out a simple procedure to construct a portfolio which would replicate the payoff of a variance swap. This involved replicating a log contract by using a portfolio of options with weights inversely proportional to their strikes. It is the valuation procedure used in the paper by Demeterfi *et al* which has been used to construct the VIX since 2003.

Detemple and Osakwe (2000), provide valuations for European and American volatility options under the volatility models

$$(1.1) \quad dV = (a - bV)dt + cV^\gamma dZ$$

with $a = 0, \gamma = 1$ (Geometric Brownian Motion (GBMP)), $\gamma = 0$, (Mean-Reverting Gaussian (MRGP)). $\gamma = 0.5, c^2 = 4a$ (an example of a Mean-Reverting Square Root process (MRSRP)) as well as a mean-reverting log process (MRLP), $d(\ln V) = (\alpha - \lambda \ln V)dt + \sigma dZ$. Other significant papers in the area include those by Howison *et al* (2004) who present closed-form solutions for prices of volatility-average and variance swaps under a variety of diffusion models (including (1.1) with $\gamma = 1$ and 0.5 and jump-diffusion models); Carr *et al* (2005) who priced volatility options by modelling returns as pure jump processes with independent increments; Little and Pant (2001) who valued variance swaps by developing a finite difference scheme in an extended Black-Scholes framework in which the local volatility is assumed to be a known function of time and underlying asset price; Windcliff *et al* (2006) who applied a numerical approach to solve a partial integro-differential equation for pricing discretely-sampled volatility derivatives under a jump diffusion asset model; Buehler (2006) who developed a general framework for modelling a joint market of stock price and a term structure of variance swaps in an HJM type framework; Sepp (2008) who priced volatility derivatives under a square root volatility model with jumps and Albanese *et al* (2009) who applied spectral methods in a non-parametric model.

Many continuous-time stochastic volatility models have been proposed in the literature

in order to price volatility contracts. This includes models of the form (1.1) as well as the popular Heston (1993) model where $w = V^2$ follows a mean-reverting square-root process

$$dw = (a - bw)dt + c\sqrt{w}dZ$$

(see e.g. Broadie and Jain (2008) and Elliott *et al* (2007)).

One of the main aims of this paper is to empirically test many of these models, as well as the 3/2- model

$$(1.2) \quad dV = (aV + bV^2)dt + cV^{\frac{3}{2}}dZ$$

in their ability to capture the dynamics of the VIX. The novel features of model (1.2) are 1) the specification for the diffusion having a high power law of 1.5 which can reduce the heteroskedasticity of volatility and 2) a nonlinear drift so that it exhibits substantial nonlinear mean-reverting behaviour when the volatility is above its long-run mean. Hence after a large volatility spike, the volatility can potentially quickly decrease while after a low volatility period it can be slow to increase. It will also be shown that under (1.2) with $b < 0$, V will always remain positive. Other aspects of the model have been studied by Lewis (2000) and Heston (1997) who considered (1.2) as a stochastic volatility model for equity.

All of the models to be tested can be nested within the larger model

$$(1.3) \quad dV = (c_1 + \frac{c_2}{V} + c_3V \ln V + c_4V + c_5V^2)dt + kV^\gamma dZ.$$

Achieving an empirically validated model is important as the ability of the stochastic model to capture the dynamics of the VIX ultimately affects the valuations for which it is used. What we find is that the value of γ is an important feature differentiating the volatility models and its unconstrained estimate is 1.5. Model (1.2) will be shown to outperform the current popular models in capturing the behaviour of the VIX¹. Further, an analytic solution will be found for

the value of a call option on the VIX under the 3/2- volatility model.

The structure of the paper is as follows: In Section 2, the models to be empirically tested will be described and in Section 3, the estimation technique, GMM, used to empirically test them is outlined. The data to be used in the empirical analysis is described in Section 4 and the results of the empirical analysis are given in Section 5 . Using the best model found for the VIX, the formula for the call option price on the VIX is found in Section 6 and comparisons of the call prices are made with formulae under two other VIX models. In Section 7, the model is calibrated to at-the-money market data and then used to price in-the-money and out-of-the money options. These prices are then compared to the corresponding market prices. Section 8 provides a brief conclusion.

2 The Models to be Tested

Eight volatility models were empirically tested. Each of these models were nested within (1.3), where V denotes volatility, by placing certain parameter restrictions upon the larger model (1.3) as displayed in Table 2.1.

[Table 2.1 here]

The first six models in particular have been used extensively by various authors in the literature. Specifically with variance $w = V^2$, by Itô's Lemma, Model 1 corresponds to the Mean Reverting Square Root model for w :

$$dw = (a + bw)dt + Kw^{\frac{1}{2}}dZ$$

used by authors such as Brockhaus and Long (2000) and Swishchuk (2004). With $2a \geq K^2$ it can be shown that $w > 0$. Model 1 is often referred to as the Heston model.

Model 2, which we simply call the Mean-Reverting (MR) model, was used by Howison *et al* (2004) and assumes that the variability of the proportional changes in V is constant.

Model 3 is the Mean-Reverting Square Root (MRSRP) volatility model used by Detemple and Osakwe (2000) and Grünbichler and Longstaff (1996). Model 4 corresponds to the Geometric Brownian Motion (GBMP) volatility model used by Detemple and Osakwe (2000), Hull and White (1987) and Johnson and Shanno (1987). Model 5 corresponds to the Mean-Reverting Gaussian (MRGP) volatility model used by Detemple and Osakwe (2000), Stein and Stein (1991) and Scott (1992).

By an application of Itô's Lemma it can be shown that Model 6 corresponds to the Mean-Reverting Logarithmic (MRLP) volatility model

$$d \ln V = (\alpha - \lambda \ln V)dt + \sigma dZ$$

used by Detemple and Osakwe (2000), Wiggins (1987) and Melino and Turnbull (1990). With this model, Detemple and Osakwe show that if volatility tends to infinity, the drift remains a function of V in contrast to the MRSRP and MRGP which would simply become proportional to a constant.

All of these models have a number of desirable features. With $c_4 < 0$ they are examples of mean-reverting models for either volatility or corresponding variance. This is important as from even an intuitive perspective we would expect a volatility index such as the VIX to stay within a certain range of finite values. The volatility of the market can never be expected to get really low as there is always news emerging which changes the market prices and hence sets a lower bound on the volatility. Furthermore abnormally high levels of volatility are also hard to sustain since this requires constant surprising news, which is highly unlikely to occur, and in the very unlikely event that it does occur, the public would accept this as the new norm.

Models 1-6 are also analytically tractable and easy to implement which is a major strength.

Model 7 in Table 2.1, studied by Lewis (2000) and Heston (1997), is proposed here in an effort to provide a model that may better fit the VIX data and still deliver analytic solutions to

volatility options. It has a higher power-law in the diffusion term of 1.5 which can reduce the heteroskedasticity of volatility². As well, the drift is a quadratic rather than linear function of volatility. As in this case the mean-reversion speed is a linear function of the volatility, the speed of reversion increases with volatility. This generates a balancing effect of a stronger mean reversion with higher volatility. Goard (2010) considered a time-dependent version of Model 7 to model realised variance, in order to price volatility and variance swaps.

The last model in Table 2.1, Model 8, has been included for comparisons with Model 7. It has the same diffusion term as Model 7 but has a linear drift term.

3 The Estimation Technique

We estimate the parameters in the continuous-time model (1.3) using the corresponding discrete-time econometric specification:

$$(3.1a) \quad V_{t+1} - V_t = (c_1 + \frac{c_2}{V} + c_3 V \ln(V) + c_4 V + c_5 V^2) \Delta t + \varepsilon_{t+1}$$

$$(3.1b) \quad E[\varepsilon_{t+1}] = 0$$

$$(3.1c) \quad E[\varepsilon_{t+1}^2] = k^2 V^{2\gamma} \Delta t.$$

The estimation technique used to estimate the parameters in the models and compare the models is the Generalised Method of Moments (GMM) of Hansen (1982). It will be used to test (3.1a-c) as a set of overidentifying restrictions on a set of moment equations. GMM has been used by many authors to compare continuous time models for various underlying such as interest rates (eg Chan *et al* (1992) and Raj *et al* (1997)) and temperature indices for weather derivatives (eg Hamisultane (2007)). Advantages of the method, as stated by Chan *et al* (1992) are that it makes no assumptions about the distributional nature of the changes in volatility, only that it be stationary and ergodic. As well GMM estimators and their standard errors are consistent even if the disturbances are conditionally heteroskedastic.

We let θ be the parameter vector with elements $c_1, c_2, c_3, c_4, c_5, k$ and γ and define the vector

$$f_t(\theta) = \begin{bmatrix} \varepsilon_{t+1} \otimes [1, V_t, V_t^{-1}, V_t \ln(V_t), V_t^2]^T \\ (\varepsilon_{t+1}^2 - k^2 V_t^{2\gamma} \Delta t) \otimes [1, V_t,]^T \end{bmatrix}$$

Under the null hypothesis that Equations (4a-c) are true, the orthogonality conditions, $E[f_t(\theta)] = 0$, hold. The GMM technique replaces $E[f_t(\theta)]$ with its sample counterpart $m(\theta)$ using T observations where $m(\theta) = \frac{1}{T} \sum_{t=1}^T f_t(\theta)$ and then estimates the parameters in the vector θ which minimise the quadratic form $q(\theta) = m(\theta)^T W m(\theta)$ where W is a positive definite, weighting matrix with the sample estimate adjusted for serial correlation and heteroskedasticity using the method of Newey and West (1987) with Bartlett weights. Hansen (1982) showed that setting $W = [E[f_t(\theta)f_t(\theta)^T]]^{-1}$ delivers an estimate for the vector θ with the smallest asymptotic covariance matrix for the GMM estimates of θ . For the unrestricted model, the number of unknowns is exactly equal to the number of orthogonality conditions so that the model is exactly indentified and so $q(\theta) = 0$. The parameters for the nested models (Models 1 -8) are estimated using the same weighting matrix as was found to estimate the parameters for (2.1). A hypothesis test is then used to see whether the models impose unreasonable overidentifying restrictions upon the unrestricted model, i.e. for each nested model, we conduct the hypothesis test of H_0 versus H_1 where

H_0 : The nested model does not impose overidentifying restriction and is hence not misspecified.

H_1 : The nested model does impose overidentifying restrictions and is hence misspecified.

The test statistic is $D = T(q(\hat{\theta}_0) - q(\hat{\theta}))$ where $q(\hat{\theta}_0)$ is the objective function for the restricted model and $q(\hat{\theta})$ is the objective function for the unrestricted model. If H_0 is true, the test statistic is asymptotically distributed χ^2 with degrees of freedom equal to the number of restrictions on the general model to obtain the nested model.

If the p -value is less than the required level of significance then it can be concluded that the

nested model is misspecified. The GMM analysis also allows for a direct comparison between different models by assessing their standard errors (also known as the root mean square error of their residuals).

4 The Data

The data used in the analysis was the VIX index values between the years of 1990 and 2009 (collected using Bloomberg). It should be noted that whereas the VIX was introduced by CBOE in 1993 to measure the market's expectation of 30-day volatility implied by at-the-money S&P 100 Index Option prices (OEX), since 2003 the CBOE has used an updated methodology to calculate the index, where it is now based on the S&P 500 (SPX) Index and estimates expected volatility by averaging weighted prices of SPX puts and calls over a wide range of strike prices (see CBOE White Paper (2009)). The original VIX under the ticker symbol VXO has data available from 1986 to the present. The “new” VIX, or the one calculated using the reproduction technique as demonstrated in Demeterfi *et al* (1999) from 1990 onwards is under the ticker symbol VIX. The data used for this paper was the “new” VIX index.

Figure 4.1 displays the historical “new” VIX values since 1990.

[Figure 4.1 here]

From Figure 4.1 it can be seen that the VIX seems to exhibit a wavelike motion around a mean of approximately 20% and has a period which lasts for many years. We also observe that the VIX value spiked up considerably in the recent past. This can be attributed to the troubles encountered during the global financial crisis and serve to justify its classification as the “fear gauge”.

The standard statistics for the VIX data are provided in Table 4.1.

[Table 4.1 here]

It can be seen from the statistics in Table 4.1 that the VIX tends to stay within a relatively

narrow range of values suggesting a mean-reverting nature.

Note that VIX values are quoted as percentages. In the following empirical analysis ³ and in the rest of the paper, we refer to the VIX as $V = VIX_{quoted}/100$.

5 Empirical Performance of Nested Models

In this section we compare the performance of the models in Table 2.1 by nesting them within the larger unrestricted model (2.1) as indicated in Section 2. The performances of the nested models are benchmarked against the larger unrestricted model.

The GMM results are presented in Table 5.1.

[Table 5.1 here]

From Table 5.1, the χ^2 values for Models 1-6 imply that they are all rejected at the 5% (and even 1%) level of significance. Hence these models are misspecified and place unreasonable restrictions on the unrestricted model.

However, Model 7 with a χ^2_4 of 5.82669 is accepted at all standard levels of significance with a p -value of 0.212. The model is thus not misspecified and the restrictions it imposes on the unrestricted model are reasonable. As well, all parameters in the model are statistically significantly different from zero.

Model 8 as well, which has the same form for the diffusion term as Model 7, cannot be rejected at the standard levels of significance. However it did not perform as well as Model 7 with a smaller p -value of 0.129.

Hence, Models 7 and 8 are the only models from the models tested that are found to be acceptable models for describing the behaviour of the VIX, and of these, Model 7 performed the best.

As a further test we compare each model's performance by their mean square error, which is a measure of the residuals squared. This then indicates how close the model is fitted to the data points. For this the best possible parameter estimates for the models (not nested) are

needed to give comparable root mean square error (RMSE) estimates. These figures can then be used to compare the models in terms of their explanatory power.

The RMSE for each model are summarised in Table 5.2 where Equation 1 and Equation 2 correspond to Equations (3.1b) and (3.1c) respectively and $E = \sqrt{RMSE_{eq1}^2 + RMSE_{eq2}^2}$.

[Table 5.2 here]

From Table 5.2, it can be seen that Model 7 has the lowest error E . This result agrees with the results found by nesting the models within a larger model and further supports Model 7 as the best of the models tested for describing the behaviour of the VIX.

We list the best parameter estimates from a GMM analysis for Models 3, 5 and 7 in Table 5.3.⁴

[Table 5.3 here]

6 Analytic Solution for Call Option Price under the 3/2-Model

In this section, an analytic solution will be found for the value of a European call option on the VIX when the VIX, V , is assumed to follow Model 7 ie

$$(6.1) \quad dV = (c_4V + c_5V^2)dt + kV^{\frac{3}{2}}dZ.$$

This is done by recognising that $w = 1/V$ follows a mean-reverting square-root process for which the probability density function is known. We note that although an explicit solution will be derived for call options, the values for put options and other volatility derivatives such as volatility swaps can be found in a similar way.

As V is not the price of a traded asset, we allow for the possibility of a non-zero market price of risk $\lambda(V, t)$ associated with the VIX. Given the real process for the VIX as (6.1), then

the risk-neutral process follows

$$(6.2) \quad dV = ((c_4V + c_5V^2) - \lambda(V, t)kV^{\frac{3}{2}})dt + kV^{\frac{3}{2}}d\tilde{Z}$$

where \tilde{Z} is a Wiener process under an equivalent risk-neutral measure Q , under which V becomes a martingale. Thus the risk-neutral probability measure Q changes the drift of the original SDE (6.1) by subtracting from it $\lambda(V, t)kV^{\frac{3}{2}}$. One way in which the market price of risk can be found is by implying it from market option prices ie choosing a λ^* that minimises the error between market and model prices. This usually involves assuming a form for the market price of risk e.g Egloff *et al* (2009) assume the Heston model for variance w and assume the market price of risk is proportional to \sqrt{w} . Similar to many authors such as Stein and Stein (1991) and Grünbichler and Longstaff (1996), we assume that the market price of risk is such that the risk-neutral process for V is of the same form as the real process (6.1). Hence we assume $\lambda(V, t) = aV^{-\frac{1}{2}} + bV^{\frac{1}{2}}$ and that the risk-neutral process ⁵ is

$$(6.3) \quad dV = (\alpha V + \beta V^2)dt + kV^{\frac{3}{2}}d\tilde{Z}$$

where $\alpha = c_4 - ak$, $\beta = c_5 - bk$ and where we assume $c_5 \leq 0$ and $\beta \leq 0$.

It is necessary to realise that as V itself is not the value of a self-financing portfolio, the basic nature of our results is not affected by whether V can be expressed as a nonlinear function of other security prices. We then have the following valuation:

Theorem 6.1: The value of a call option on the VIX with strike X and expiry T , when the VIX, V , follows the risk-neutral process (6.3) with $\beta < 0$, is given by

$$(6.4) \quad C(V, t) = \frac{2\alpha e^{-r(T-t)}}{k^2 p} \exp\left[\frac{-2\alpha e^{-\alpha(T-t)}}{k^2 V p}\right] V^{-\frac{\beta}{k^2} + \frac{1}{2}} \exp\left[\alpha(T-t)\left(-\frac{\beta}{k^2} + \frac{1}{2}\right)\right] \\ \times \int_0^{\frac{1}{X}} \phi^{\frac{1}{2} - \frac{\beta}{k^2}} \left(\frac{1}{\phi} - X\right) e^{-\frac{2\alpha\phi}{k^2 p}} I_\nu\left(\frac{4\alpha\sqrt{\phi}e^{-\frac{\alpha(T-t)}{2}}}{k^2\sqrt{V}p}\right) d\phi$$

where $\nu = 1 - \frac{2\beta}{k^2}$, $p = 1 - \exp(-\alpha(T-t))$ and $I_\nu(\cdot)$ is the modified Bessel function of order ν .

Proof: Given that the VIX, V , follows the risk-neutral process (6.3), then by Itô's Lemma, $w = \frac{1}{V}$ follows the process

$$(6.65) \quad dw = ((k^2 - \beta) - \alpha w)dt - k\sqrt{w}d\tilde{Z}.$$

Hence w follows a mean-reverting square-root process such as that used by Cox *et al* (1985) to model short interest rates. Further, as explained by Feller (1951), for $k^2 \leq 2(k^2 - \beta)$, ie $\beta \leq \frac{k^2}{2}$, (and so for all negative β), w and hence V will remain positive.

With $\beta < 0$, the probability density function of w at a future time T , conditional on its value at the current time t is given by

$$(6.6) \quad f(w_T|w_t) = ce^{-u-z} \left(\frac{z}{u}\right)^{\frac{q}{2}} I_q(2\sqrt{uz})$$

where $c = \frac{2\alpha}{k^2(1-e^{-\alpha(T-t)})}$; $u = cw_t e^{-\alpha(T-t)}$; $z = cw_T$; $q = 1 - \frac{2\beta}{k^2}$ (see Cox *et al* (1985)). Using risk-neutral valuation, the value of the call option on the VIX can be found as

$$C(V, t) = e^{-r(T-t)} E^Q \left(B \max \left(\frac{1}{w_T} - X, 0 \right) \right)$$

where E^Q denotes the expectation under the risk-neutral measure Q , X is the strike price and B is a notional amount of the option measured in currency units per volatility point. For brevity we scale the option value so that the notional amount B can be taken to be one. Using (6.6) then gives

$$C(V, t) = e^{-r(T-t)} \frac{2\alpha}{k^2(1 - \exp(-\alpha(T-t)))} \exp \left[\frac{-2\alpha w_t e^{-\alpha(T-t)}}{k^2(1 - \exp(-\alpha(T-t)))} \right] \\ \times \int_0^\infty e^{\frac{-2\alpha\phi}{k^2(1 - \exp(-\alpha(T-t)))}} \left(\frac{\phi}{w_t} e^{\alpha(T-t)} \right)^{\frac{1}{2} - \frac{\beta}{k^2}} \max \left(\frac{1}{\phi} - X, 0 \right) I_\nu \left(\frac{4\alpha\sqrt{\phi}\sqrt{w_t} e^{-\frac{\alpha(T-t)}{2}}}{k^2(1 - \exp(-\alpha(T-t)))} \right) d\phi,$$

where $w_t = \frac{1}{V_t}$. This can be simplified to give the form (6.4). ##

Note 1: Other contracts on the VIX can be priced similarly e.g the price of a put option with strike price X and expiry T can be found using $P(V, t) = e^{-r(T-t)} E^Q \left(\max \left(X - \frac{1}{w_T}, 0 \right) \right)$ and the price of VIX futures with maturity T using $F(V, t) = E^Q \left(\frac{1}{w_T} \right)$.

Note 2: The value of the call option could also be found by solving the pricing partial differential equation (PDE)

$$(6.7) \quad \frac{\partial C}{\partial t} + \frac{1}{2} k^2 V^3 \frac{\partial^2 C}{\partial V^2} + (\alpha V + \beta V^2) \frac{\partial C}{\partial V} - rC = 0,$$

subject to $C(V, T) = \max(V - X, 0)$, by either letting $w = \frac{1}{V}$ and using the fundamental solution given by Feller (1951) or by reducing (6.7) to its canonical form.

Plots of the call option value as given by Equation (6.4) under the model for V given in (6.3), using parameter values as given in Table 5.3 (assuming zero market price of risk) so that $\alpha = 2.93536$, $\beta = -12.915828$, $k = 2.04727$ with $T = 1$, $r = 0.05$, $X = 0.15$ at times $t = 0, 0.5, 0.9$ are given in Figure 6.1. Similarly, using the same values for X , T and r , plots for call option values under the risk-neutral Models 3 (MRSRP) (given by Grünbichler and Longstaff (1996)) and 5 (MRGP) (given by Detemple and Osakwe (2000)) using parameter values in Table 5.3 (assuming zero market price of risk) are given in Figures 6.2a and 6.2b.

[Figures 6.1 and 6.2 here]

It can be seen from Figures 6.1 and 6.2 that all of the models have some common basic features. As expected, under each model the call option value increases with VIX, with the rate of increase increasing with time. This is to be expected as when time approaches expiry there is less time for the VIX to revert to its mean value. This also means that for larger values of the VIX, the value of the call option increases with time whereas for smaller values of the VIX the call value decreases with time, with the changeovers occurring at VIX values dependent on the exercise price and model parameters.

Figure 6.3 illustrates a comparison of call option values at $t = 0$ under Models 3, 5 and

7 for VIX values between 0.1 and 0.8 using the same parameters as above but with exercise prices a) $X = 0.1$, b) $X = 0.15$, c) $X = 0.2$ and d) $X = 0.25$.

[Figure 6.3 here]

From Figure 6.3 we see the new features of the pricing solution $C(V, t)$ of Model 7, compared to those of the simpler Models 3 and 5. For moderate values of the VIX, around double the long-term mean, at a given value of t , the solution curve of Model 7 has more upward-bulging curvature, with $-\frac{\partial^2 C}{\partial V^2}$ taking a larger value. Compared to Model 7, with its pull to the long-run mean linearly dependent on the value of the VIX, Models 3 and 5 tend to overprice at both high and low extremities of VIX values.

7 Calibration to Market Prices

For the VIX model (6.3), we let k be the given value found in Table 5.3, and fit two near at-the-money market prices using (6.4), for VIX options quoted on September 30, 2010, with a) 20 days to maturity and b) 4.5 months to maturity. The VIX value at that time was 0.237 and the interest rate 0.14%. The values thus found for α and β were for a) $\alpha = 3.169$, $\beta = -8.99$ and for b) $\alpha = 4$, $\beta = -10.3$. Thus from Section 6 and using Table 5.3, this implies a market price of risk of the form a) $\lambda(V, t) = \frac{-0.114}{\sqrt{V}} - 1.92\sqrt{V}$ and b) $\lambda(V, t) = \frac{-0.52}{\sqrt{V}} - 1.28\sqrt{V}$. [We also calculated the partial derivatives of the option price and delta with respect to the parameters fitted and found that they were $O(10^{-2})$ or less, so that small changes in these parameters would not cause large changes in the option prices or delta.] Using the calibrated α and β values, we then computed values for VIX call options using (6.4) for other strike prices and plotted these against market prices. The results are in Figure 7.1.

[Figure 7.1 here]

It can be seen from Figure 7.1a that for the very short expiry of 20 days, the market prices in-the-money and out-of-the money options slightly above those of the model prices. However for the longer expiry of 4.5 months, market and model prices are very close. To determine

whether this is usually the case and to further investigate the relationship between model and market prices, the above experiment was repeated for several quoted market call option prices from August to December, 2010, with a range of strike prices and a range of expiries from 20 to 125 days. For each option we calculated corresponding model prices, signed percentage error (ie $[\text{market price} - \text{model price}] / [\text{market price}] * 100\%$), unsigned percentage errors (ie absolute value of signed percentage errors), and moneyness M , defined by $M = \ln\left(\frac{X}{V_{\text{market}}}\right)$. Note that a negative (positive) M value denotes that the option is in (out)-of-the-money and the larger the magnitude of M , the deeper this is. Results were grouped into the range of expiries 90-125 days, 60-89 days, 40-59 days and 20-39 days and average signed and unsigned percentage errors were calculated for the various groups of moneyness. These are listed in Table 7.1 for expiries 90-125 days and 60-89 days, and Table 7.2 for expiries 40-59 days and 20-39 days. Compared to market values, model values for options within the moneyness interval (0.3,0.5) and with short expiry times of 20-39 days had large percentage errors (and small absolute errors) largely due to the small denominator in the relative error. These unstable values of relative error are not tabulated.

[Table 7.1 here]

[Table 7.2 here]

From Tables 7.1 and 7.2, it can be seen that for expiries of 40-125 days, the market consistently prices all in-the-money options slightly above those of the model prices. This was also mostly true for expiries 20-39 days, although very occasionally the market slightly underpriced near in-the-money options compared to the model. For options just out-of-the-money, the market prices are slightly lower than model prices until, for longer times to expiry of 90-125 days, the option is deep-out-the money in the range (0.3,0.6). This range of underpricing shortens as time to expiry decreases, so that for times to expiry of 40 to 59 days, it is observed in this set of data, to last to moneyness levels up to 0.2 while for times to expiry of 20-39 days, up to moneyness levels of 0.1. This can be explained by realising that what would constitute as deep-out-of-the money would start at lower levels of moneyness for shorter times to expiry.

After this period the market prices are again slightly above those of the model prices. It should be realised that while the relative percentage errors seem quite large for the (0.3,0.5) moneyness range for 40-59 day expiry times and for the (0.2,0.3) moneyness range for 20-39 day expiries, the absolute errors are in fact extremely small as the \$-value of the options are quite small.

These results mean that traders would then consider that the distribution implied by the model, understates the probability of extreme movements in the VIX. From Figure 4.1 it is obvious that the VIX can exhibit spikes making large movements in the VIX possible. This leads to higher out-of-the money call (and in-the-money put) option prices. Traders would also consider that the distribution implied by the model overstates the probability of small positive movements, implying a greater chance of positive return when it is just out-of-the-money.

Thus from this example, given that the model prices at-the-money options correctly, then the market can be seen to place heavier premiums on in- and deep-out-of the money call options, and lower premiums on near out-of-the-money call options than that implied by the model.

8 Conclusion

Since derivatives on the VIX hit the market in 2004 the number of proposals for a solution to the VIX option price has steadily grown. Using the Generalised Method of Moments (GMM) we have investigated and compared popular volatility models (Models 1-6 in Table 2.1) with less well-known models (Models 7 and 8), in terms of their ability to explain the dynamics of the VIX. The analysis revealed that Models 7 and 8 outperformed Models 1-6, with all Models 1-6 found to impose overidentifying restrictions. The key to the good fit for Models 7 and 8 is their actual diffusion term which relates to the underlying with a power of $\frac{3}{2}$. Interestingly, this was also the unconstrained estimate found by Chan *et al* (1992) in their empirical work on short interest rates.

As well, an analytic solution to call option prices under the best model (Model 7) has been provided. To the authors' knowledge, no other exactly solvable VIX option pricing model comes from using a stochastic volatility model that is statistically acceptable when compared with the data. The solution (6.4) was achieved by recognising the connection of Model 7 to the mean-reverting square root model, but could equivalently be found by reducing the pricing PDE to canonical form. The solution is easily and efficiently evaluated in terms of special functions that are routinely programmed on all mathematical software packages. The solution method could similarly be used to price put options and other derivatives such as VIX futures.

Also, by fitting our pricing model to at-the-money market prices and then using the computed parameters to price out- and in-the money options, it was found that compared to market prices, the market imposes slightly heavier premiums for in-the-money, deep-in-the-money and deep-out-the money options and slightly lower premiums for just out-of-the money options. Overall however, the errors between market and model prices are reasonably small, suggesting that the exact model might be a useful guide to traders.

Finally, we mention that in this paper, we value options on the VIX by directly modelling observed values of the VIX, and as the VIX itself is actually the square root of a forward-start variance swap, it would be interesting to analyse the implications on stochastic variance models implied by our model. This will be the subject of future work.

9 References

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Notes

¹However, it should be noted that a model that might not be an appropriate model for the VIX might still be a good model for realised volatility.

²See Campbell *et al* (1996) for a discussion on this for the short interest rate model of the same form.

³Note that for modelling $y = VIX$ as its quoted percentage value i.e. $y = 100 * V$ with Model 7, then by Itô's Lemma,

$$dy = (c_4y + \frac{c_5}{100}y^2)dt + \frac{k}{\sqrt{100}}y^{\frac{3}{2}}dZ.$$

⁴Note that these are not the same as those listed in Table 5.1 since the models are not being nested and so involve the optimal covariance matrices of the orthogonality conditions for the models (and not the unrestricted model).

⁵It is unclear whether with this choice of $\lambda(V, t)$, the Novikov criteria is satisfied, but as shown by Cheridito (2007), arbitrage-free models can be constructed by ensuring boundary non-attainment conditions are satisfied under both real and risk-free probability measures. As shown in the proof of Theorem 6.1, this requires $c_5 \leq \frac{k^2}{2}$ and $\beta \leq \frac{k^2}{2}$.

Model	Description	c_1	c_2	c_3	c_4	c_5	γ
1	Heston MRSRP for V^2	0		0		0	0
2	MR		0	0		0	1
3	MRSRP		0	0		0	0.5
4	GBMP	0	0	0		0	1
5	MRGP		0	0		0	0
6	MRLP	0	0			0	1
7	$\frac{3}{2}$ model with quad drift	0	0	0			1.5
8	$\frac{3}{2}$ model with linear drift		0	0		0	1.5

Table 2.1: The parameter restrictions on the unrestricted model

$$dV = (c_1 + \frac{c_2}{V} + c_3 V \ln V + c_4 V + c_5 V^2)dt + kV^\gamma dZ$$

for empirically-tested Models 1-8.

Mean	19.82789616
Standard Deviation	8.067942072
Minimum	9.31
Maximum	80.86

Table 4.1: VIX (%) Index Statistics

Model	c_1	c_2	c_3	c_4	c_5	k	γ	χ^2	DF
Unrestricted	66.0223 [.025]	-2.39462 [0.026]	269.118 [0.025]	228.220 [0.027]	328.78263 [0.036]	1.98898 <[.001]	1.47972 <[.001]	N/A	N/A
1	0	0.047608 <[.001]	0	-1.34495 0.001	0	0.180592 <[.001]	0	63.3919 <[.001]	4
2	.611602 <[.001]	0	0	-3.14312 <[.001]	0	-0.956694 <[.001]	1	24.4905 <[.001]	4
3	0.615456 <[.001]	0	0	-3.16371 <[.001]	0	0.426133 <[.001]	0.5	45.6118 <[.001]	4
4	0	0	0	0.639072 0.002	0	0.950413 <[.001]	1	57.0297 <[.001]	5
5	0.616113 <[.001]	0	0	-3.16372 <[.001]	0	0.180875 <[.001]	0	60.1120 <[.001]	4
6	0	0	-3.56258 <[.001]	-5.66506 <[.001]	0	0.958316 <[.001]	1	21.7566 <[.001]	4
7	0	0	0	3.71697 <[.001]	-16.990461 <[.001]	2.03420 <[.001]	1.5	5.92669 0.212	4
8	0.592560 <[.001]	0	0	-3.01339 <[.001]	0	2.02932 <[.001]	1.5	7.13872 0.129	4

Table 5.1: Empirical results for nesting Models 1-8 within (2.1).

Model#	RMSE for Equation 1	RMSE for Equation 2	E
1	.014635	.101885E-02	0.01467
2	.014623	0.926116E-03	0.01465
3	.014623	.981757E-03	0.01466
4	.014679	.915326E-03	0.01471
5	.014623	.101408E-02	0.01466
6	.014609	.925613E-03	0.01464
7	.014597	.884127E-03	0.01462
8	.014623	.880149E-03	0.01465

Table 5.2: The standard errors for Models 1-8.

Model	c_1	c_2	c_3	c_4	c_5	k	γ
3	0.858321 [0.015]	0	0	-4.39772 [0.023]	0	0.521295 <[.001]	0.5
5	0.858321 [.015]	0	0	-4.39772 [.023]	0	0.232128 <[.001]	0
7	0	0	0	2.93536 [.058]	-12.915828 [.067]	2.04727 <[.001]	1.5

Table 5.3: Best parameter estimates for Models 3, 5 and 7.

Moneyness M	expiry 90-125 days		expiry 60-89 days	
	av. signed %error	av. unsigned %error	av. signed %error	av. unsigned %error
(-0.9,-0.6)	3.077	3.077	3.28	3.28
(-0.6,-0.3)	4.389	4.389	3.424	3.424
(-0.3,-0.02)	2.988	2.988	2.409	2.409
(0.02,0.3)	-2.69	2.69	-1.522	2.015
(0.3,0.6)	-2.758	6.98	3.039	6.612

Table 7.1: Percentage errors in calibration of call option prices by time to expiry (90-125 days and 60-89 days) and moneyness.

Moneyness M	expiry 40-59 days		20-39 days	
	av. signed %error	av. unsigned %error	av. signed %error	av. unsigned %error
(-0.9,-0.5)	3.41	3.41	2.68	2.68
(-0.5,-0.3)	3.965	3.965	1.1	1.1
(-0.3,-0.02)	1.837	1.837	-0.74	2.11
(0.02,0.1)	-0.332	0.713	0.896	1.407
(0.1,0.2)	-0.554	0.779	8.028	8.028
(0.2,0.3)	4.73	4.73	25.148	25.148
(0.3,0.5)	11.808	11.808	-	-

Table 7.2: Percentage errors in calibration of call option prices by time to expiry (40-59 days and 20-39 days) and moneyness.

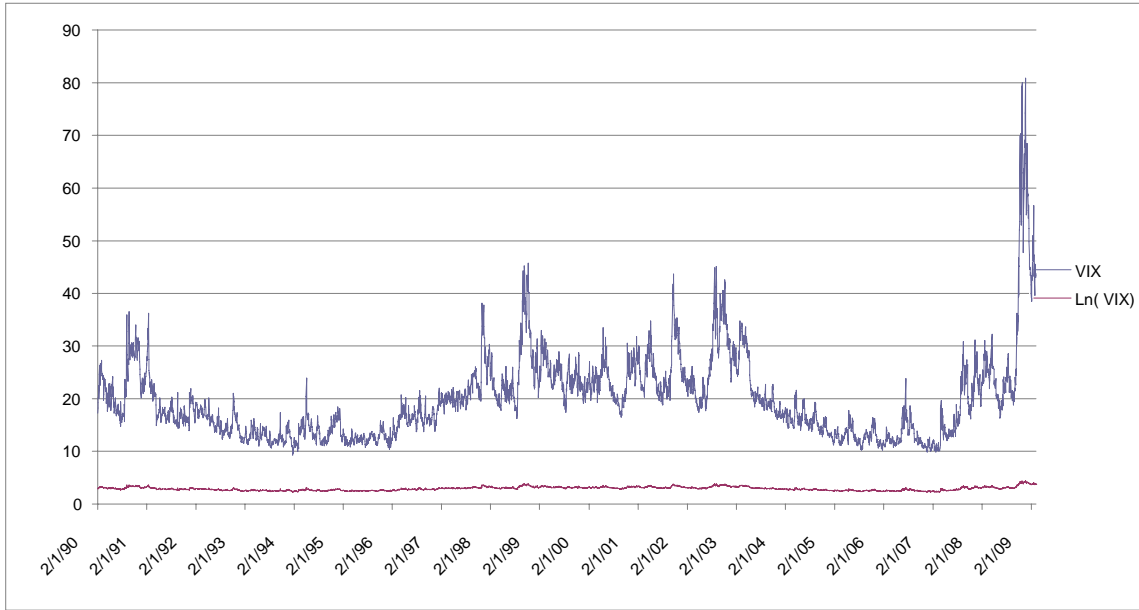


Figure 4.1: Plot of the VIX index (02/01/1990 - 02/01/2009)

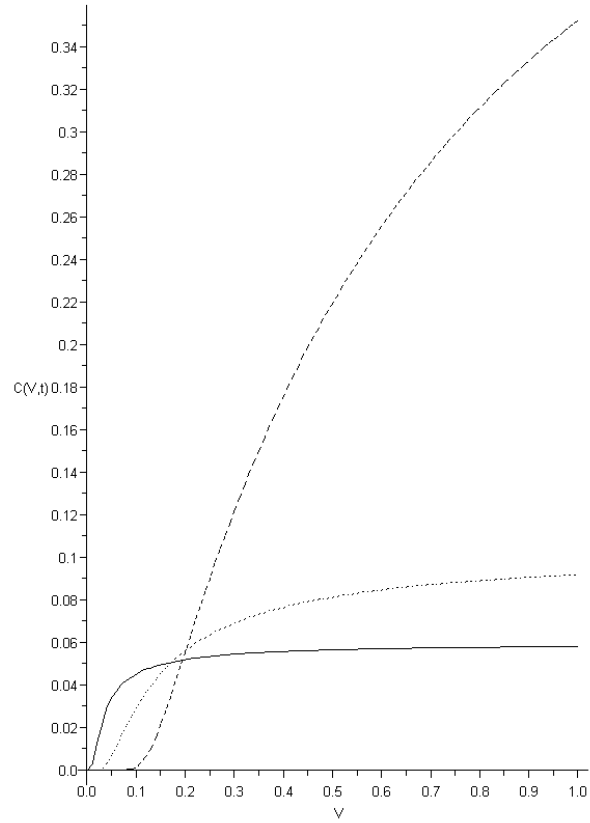
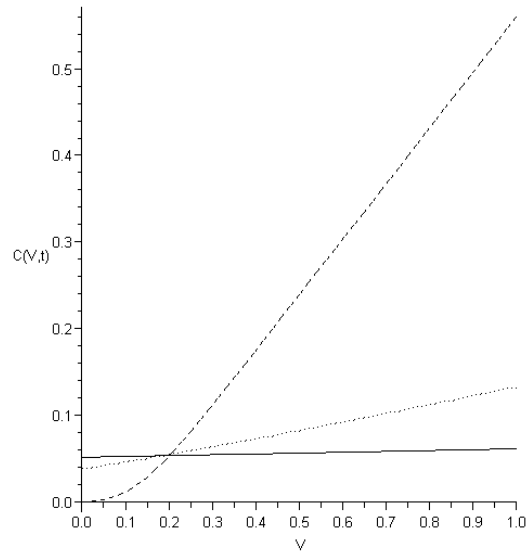


Figure 6.1: Call option values under Model 7 at times $t = 0$ (solid line), $t = 0.5$ (dotted line) and $t = 0.9$ (dashed line) with $X = 0.15$, $r = 0.05$, $T = 1$ and parameter values in Table 5.3.

a)



b)

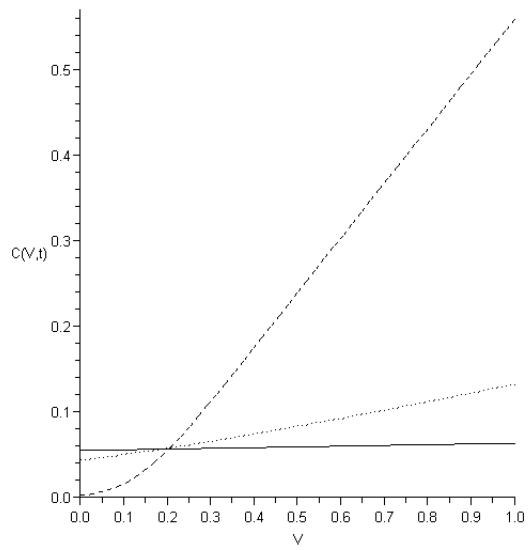
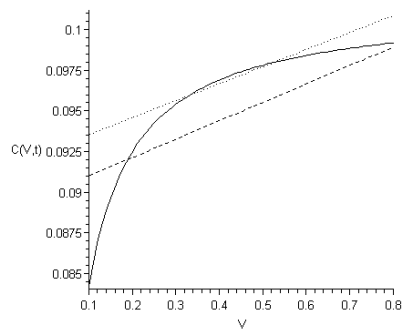
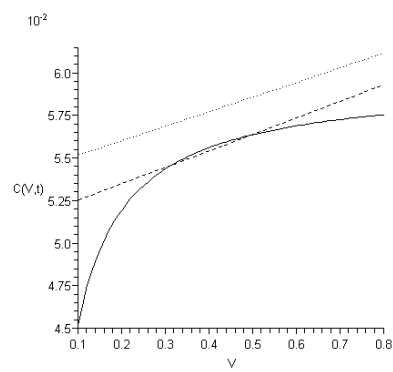


Figure 6.2: Call option values at times $t = 0$ (solid line), $t = 0.5$ (dotted line) and $t = 0.9$ (dashed line) with $X = 0.15$, $r = 0.05$, $T = 1$ and parameter values in Table 5.3 under a) Model 3 b) Model 5.

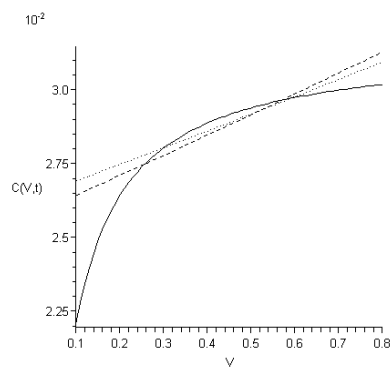
a) $X=0.1$



b) $X=0.15$



c) $X=0.2$



d) $X=0.25$

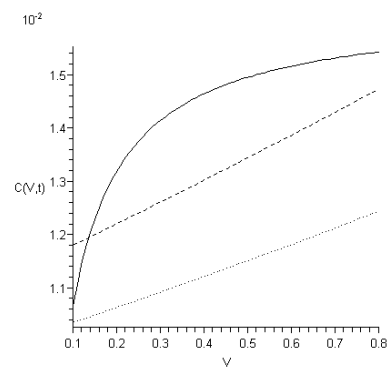
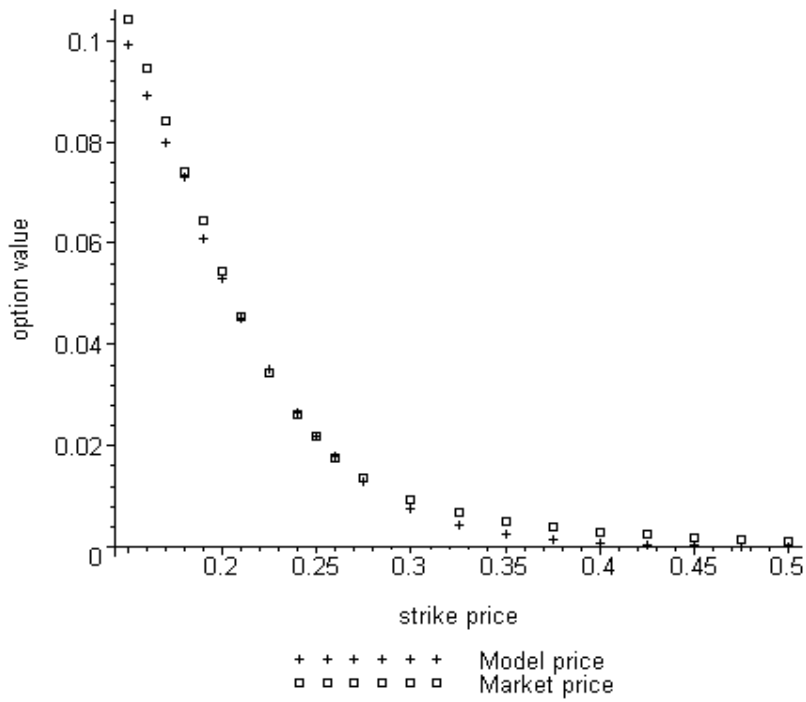


Figure 6.3: Comparison of call option values at time $t = 0$ under Model 3 (dashed line), Model 5 (dotted line) and Model 7 (solid line) for various exercise prices X and with $r = 0.05$, $T = 1$ and parameter values as in Table 5.3.

a)



b)

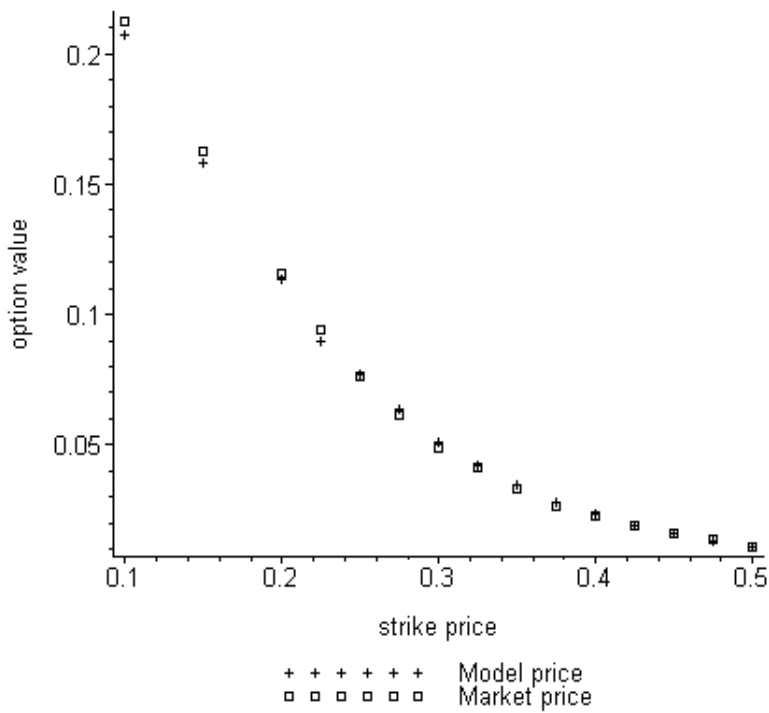


Figure 7.1: Comparison of option price model (6.4) to market data. Market and model prices are plotted using parameters values found by calibrating at-the-money market options. Time to expiry is a) 20 days b) 4.5 months.