

University of Wollongong

Research Online

Faculty of Engineering and Information
Sciences - Papers: Part B

Faculty of Engineering and Information
Sciences

2018

A modified Black-Scholes pricing formula for European options with bounded underlying prices

Song-Ping Zhu

University of Wollongong, spz@uow.edu.au

Xinjiang He

University of Wollongong, xinjiang@uow.edu.au

Follow this and additional works at: <https://ro.uow.edu.au/eispapers1>



Part of the [Engineering Commons](#), and the [Science and Technology Studies Commons](#)

Recommended Citation

Zhu, Song-Ping and He, Xinjiang, "A modified Black-Scholes pricing formula for European options with bounded underlying prices" (2018). *Faculty of Engineering and Information Sciences - Papers: Part B*. 1134.

<https://ro.uow.edu.au/eispapers1/1134>

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au

A modified Black-Scholes pricing formula for European options with bounded underlying prices

Abstract

In this paper, a modified Black-Scholes (B-S) model is proposed, based on a revised assumption that the range of the underlying price varies within a finite zone, rather than being allowed to vary in a semi-infinite zone as presented in the classical B-S theory. This is motivated by the fact that the underlying price of any option can never reach infinity in reality; a trader may use our new formula to adjust the option price that he/she is willing to long or short. To develop this modified option pricing formula, we assume that a trader has a view on the realistic price range of a particular asset and the log-returns follow a truncated normal distribution within this price range. After a closed-form pricing formula for European call options has been successfully derived, some numerical experiments are conducted. To further demonstrate the meaning of the proposed model, empirical studies are carried out to compare the pricing performance of our model and that of the B-S model with real market data.

Disciplines

Engineering | Science and Technology Studies

Publication Details

Zhu, S. & He, X. (2018). A modified Black-Scholes pricing formula for European options with bounded underlying prices. *Computers and Mathematics with Applications*, 75 1635-1647.

A modified Black-Scholes pricing formula for European options with bounded underlying prices

Song-Ping ZHU *

Xin-Jiang HE †

Abstract

In this paper, a modified Black-Scholes (B-S) model is proposed, based on a revised assumption that the range of the underlying asset varies within a finite zone, rather than being allowed to vary in a semi-infinite zone as presented in the classical B-S theory. This is motivated by the fact that the underlying price of any option can never reach infinity in reality; a trader may use our new formula to adjust the option price that he/she is willing to long or short. To develop this modified option pricing formula, we assume that a trader has a view on the realistic price range of a particular asset and the log-returns follow a truncated normal distribution within this price range. After a closed-form pricing formula for European call options has been successfully derived, some numerical experiments are conducted. To further demonstrate the meaning of the proposed model, empirical studies are carried out to compare the pricing performance of our model and that of the Black-Scholes model with real market data.

AMS(MOS) subject classification.

Keywords. Truncated normal distribution, Upper and lower bounds, Closed-form, Empirical studies.

*School of Mathematics and Applied Statistics, University of Wollongong NSW 2522, Australia

†Corresponding author. School of Mathematics and Applied Statistics, University of Wollongong NSW 2522, Australia.

1 Introduction

Financial derivatives have become increasingly popular among investors as well as academic researchers recently. Among these, options are one kind of the most basic and important instruments and thus option valuation receives high attention. European options, as the most fundamental ones, have received a lot of attention since it is always desirable to find an appropriate model to accurately determine their prices.

Although Black & Scholes [2] proposed the celebrated Black-Scholes (B-S) formula for pricing European options, which is still widely used in financial markets today, some fundamental assumptions made in the B-S model in order to achieve a simple and closed-form pricing formula have actually attracted critics; more and more revised B-S models and/or “modified” pricing formulae are proposed as a result. For example, the assumption of the constant volatility in the B-S model has been shown to be at odds with the so-called “volatility smile” [9] exhibited by the implied volatility of option prices. Moreover, observed returns of the underlying from financial markets are actually not normally distributed and they are usually skewed [30] and fat-tailed [32]. As a result, quite a few approaches have been proposed to modify the B-S model in order to obtain more “accurate” option prices.

In the literature, there are mainly two kinds of modifications as far as option pricing is concerned, i.e. the so-called structural models and non-structural models. To be more specific, the former provide the dynamics of the underlying price at every moment for a given period of time horizon. Apparently, the B-S model belongs to this category. There are also other models of this type. For example, stochastic volatility is adopted by Scott [35], Wiggins [39], Heston [14] and many other authors in order to alleviate the well-known “volatility smile”. Another common modification using structural models is to add components to the geometric Brownian motion or even replace the Brownian motion with other stochastic processes. For instance, jump-diffusion models [22, 27] add a jump term to the standard Brownian motion to reflect that the underlying price is discontinuous in

real markets. Moreover, the Variance-Gamma and the CGMY model were proposed by Madan [24] and Carr et al. [3] respectively to capture various characteristics shown by real market data.

On the other hand, non-structural models only specify the probability density function of the underlying at maturity conditional upon the filtration at the current time without completely describing the fine details of the stochastic process themselves at each moment. With more flexible distributions, different characteristics of the asset returns and the volatility term structure that the B-S model failed to describe properly can be captured. In particular, generalized beta distribution of the second kind was used by Bookstaber & McDonald [26] while Burr-3 distribution was adopted by Sherrick et al. [36]. Other examples include Weibull distribution used by Savickas [34], g-and-h distribution studied by Dutta & Babbel [10] and generalized gamma distribution adopted by Fabozzi [11]. In addition, a density expansion approach was firstly developed by Jarrow and Rudd [20], who proposed an integrated Edgeworth series expansion of a log-normal density in pricing theory. After that, Madan & Milne [25] gave an expansion to approximate a risk-neutral density function while Corrado & Su [7] adopted the integrated Gram-Charlier series expansion of a normal density function.

Unfortunately, all the pricing models in the existing literature, including the structural and non-structural ones, assume that the underlying price is unbounded above, i.e., the price range from zero to infinity. Although this assumption contains a clear “pitfall” as there is no way that any underlying price could reach infinity in reality, it is nevertheless a nice and elegant mathematical compromise to ensure the tractability. A “modification” to the B-S formula, which takes into account that option traders often have their own expected (finite) range of the underlying price in mind, appears to be a very reasonable and attractive idea.

In this paper, such a modification is presented, with a non-structural model being adopted under the assumption that the log-returns of the underlying asset follow a trun-

cated normal distribution during a certain period with a finite upper and lower bound.

One question that can be raised is that unlike the use of the B-S formula, in which both the writer and buyer of an option know that the underlying has been assumed to vary, albeit unreasonable, from zero to infinity, traders using our newly derived formula would not know if the opposite side of the option has taken the same view in terms of these upper and lower bounds of the underlying. But, this is not a good reason to devalue our modified formula. Even if both sides agree to adopt the original B-S formula, their views on many market factors, such as the trend of the underlying would be different anyway; otherwise there would be no “deal”. In fact, we believe that our formula could be at least used as a way to adjust the fair price of an option, after a trader adds a bit of his personal views on the range of the underlying, which he/she believes to be more reasonable to use than the $[0, \infty)$ range that was adopted in deriving the original B-S formula. For example, one could still use the original B-S formula to decide the volatility value from the historical market data. This has the effect of acknowledging that the opposite side has adopted the B-S formula. Then, he/she will stick the obtained volatility value into our new formula to obtain a “revised” option price based on his/her own view of what a reasonable price range of the underlying before the option expires should be.

In fact, there are many applications of the truncated normal distribution. For example, it has been applied to the theory of queues by Pender [31], while Dey & Chakraborty [8] introduced it into the inventory model as the distribution of a fuzzy random variable. Certainly, there also exist plenty of its financial applications. Specifically, truncated normal distribution was adopted in the analysis of investments and the measurement of stock market efficiency by Norgaard & Killeen [29] and Hasan et al. [13] respectively. Recently, portfolio insurance has been another application area of the truncated normal distribution [15].

When the truncated normal distribution is chosen in option pricing, there would be a price range for the underlying. With martingale approach, a closed-form pricing formula

is derived, which does not bring any significantly extra burden compared with the B-S formula as far as the computational efficiency is concerned. Moreover, the B-S model is a special case of our model since the truncated normal distribution could degenerate to the normal distribution when the lower and upper bound approach negative and positive infinity respectively. It should also be noticed that according to the numerical results, with the two bounds varying while other parameters being the same in both models, European call option prices calculated with our formula are no greater than those obtained from B-S formula, which is consistent with our expectation since the underlying price under our setting can not go beyond a certain level, while that in the B-S model can surely take any value. This is quite useful in real markets since sellers of a call option could give up some profits by choosing a lower price with our formula if they believe that the underlying price will not exceed a certain level. To make sure that the proposed model indeed has certain advantages in finance practice, we have also conducted empirical studies, comparing the results of pricing S&P 500 Index and options with our model and the B-S model, respectively. Our results indeed show that the newly proposed model outperforms the B-S model for the tested case, implying that our model can at least act as an alternative to the B-S model in the sense that our new assumption is more realistic than its counterpart under the original B-S model.

The rest of the paper is organized as follows. In Section 2, we will firstly introduce the truncated normal distribution and then a martingale restriction will be derived for our model. After that, a closed-form pricing formula for European call options will be presented. In particular, various basic properties of the option price formula will be examined. In Section 3, numerical examples and some useful discussions will be given. In Section 4, empirical studies are carried out to compare the performance of our model and that of the Black-Scholes model, followed by some concluding remarks presented in the last section.

2 Our model

In this section, the truncated normal distribution is briefly introduced first, followed by a necessary martingale restriction that needs to be imposed in order to avoid arbitrage opportunities. Finally, we derive a closed-form pricing formula for our model and a number of basic properties of our solution are investigated.

2.1 Truncated normal distribution

If a random variable X is assumed to follow a truncated normal distribution with $X \in [a, b]$, then its probability density function can be specified as

$$f(x; \mu, \sigma, a, b) = \begin{cases} \frac{\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}, & a \leq x \leq b, \\ 0, & \textit{otherwise.} \end{cases}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ represent a standard normal density function and distribution function, respectively. Also, the expectation and variance of X can be easily calculated as

$$E(X) = \mu + \frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}\sigma, \quad (2.1)$$

$$V(X) = \sigma^2 \left[1 + \frac{\frac{a-\mu}{\sigma}\phi(\frac{a-\mu}{\sigma}) - \frac{b-\mu}{\sigma}\phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} - \left(\frac{\phi(\frac{a-\mu}{\sigma}) - \phi(\frac{b-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})} \right)^2 \right], \quad (2.2)$$

which shows that the mean and variance of the truncated normal distribution are no longer μ and σ^2 . In the following, Figure 1 exhibits the probability density function of truncated normal distribution and standard normal distribution, which further illustrates the differences between the two distributions. It can be seen clearly that with some parameters being kept the same, there would be higher probability for the truncated normal distribution in the truncated area than the standard normal distribution and the probability can become even higher if the particular area is further narrowed down.

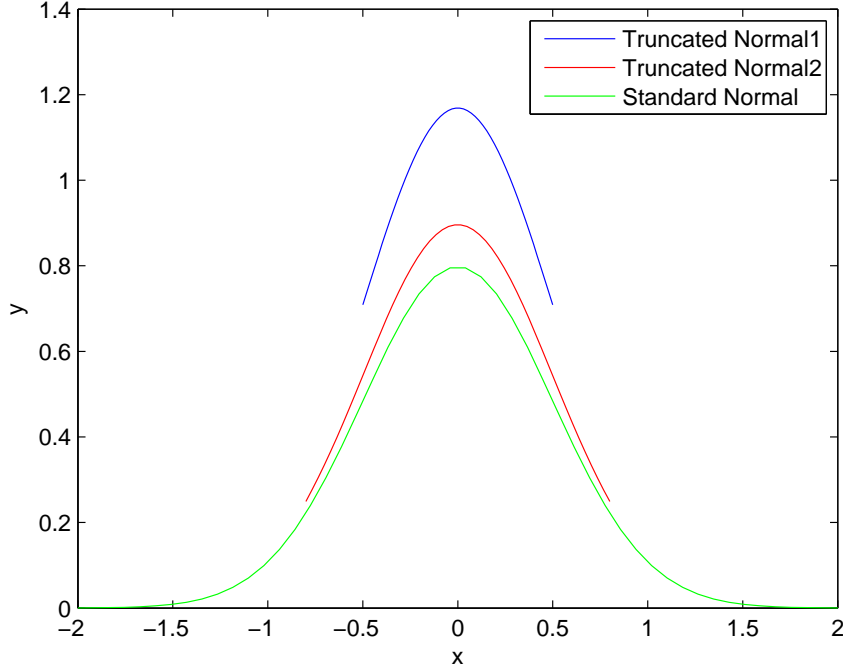


Figure 1: Differences between the two distributions. Model parameters are $\mu = 0, \sigma = 0.5, t = 1$; TN1: $a = -0.5, b = 0.5$, TN2: $a = -0.8, b = 0.8$.

2.2 Martingale restriction

The underlying log-price is now assumed to follow a truncated normal distribution under the martingale measure \mathbb{Q} , which can be described as

$$\ln\left(\frac{S_t}{S_0}\right) \sim f(x; \mu t, \sigma\sqrt{t}, a, b). \quad (2.3)$$

Here, S_0 represents the current underlying price, and the underlying price will always be higher than S_0e^a but lower than S_0e^b . Also, unlike the B-S model, the mean and variance of the underlying log-returns no longer take the value of μ and σ^2 directly, but take the value of (2.1) and (2.2) respectively. After the model is established, one should notice that a certain condition needs to be imposed to guarantee the non-existence of arbitrage opportunities. In fact, as we mentioned before, the so-called structural models and non-structural models are mainly two kinds of option pricing models. On one hand, when

we apply the former, we indeed specify the whole dynamics during the time period $[0, T]$ and in this case the goal of perfect hedging at any time $t \in [0, T]$ can be achieved under the martingale framework, which implies that the adopted model is arbitrage-free if we impose the condition $E^Q[e^{-rT}S_T | F_t] = e^{-rt}S_t$. However, once a process is chosen, one then has no control on the probability density function that describes the distribution of the underlying at the expiry, which may result in a mis-price of the option, the well known “volatility smile” phenomenon [33] is a typical example in this category. On the other hand, one could have a better control on the statistic properties, such as skewness and kurtosis in addition to mean and variance, of the underlying distribution at the expiry, in order to alleviate the “volatility smile”. This prompted the development of the non-structural models since it is rather difficult to find a process with the desired distribution. To be more specific, when non-structural models are adopted, we only know the information of the start date and expiry date denoted by 0 and T respectively, which stands for a two-date economy. As a result, a martingale restriction,

$$E^Q[e^{-rT}S_T | F_0] = S_0, \quad (2.4)$$

suggested by Longstaff [23], should be imposed in this kind of models. Although this is only a necessary condition, it is quite reasonable in the two-date economy since Harrison & Kreps [12] have shown that violation of the martingale restriction under the pricing measure can lead to arbitrage opportunities.

In this paper, “non-structural” approach is adopted in order to capture the property that the underlying price could not be too high or low in a certain period. This means that the martingale restriction should be imposed to avoid arbitrage opportunities. If we apply the martingale restriction in the B-S model, the drift μ could be shifted to $r - \frac{1}{2}\sigma^2$, which means a reduction in the parameter space. As a result, when we apply the condition in our model, we can also expect such a phenomenon that the expected return μ will be

represented by a function of the risk-free interest rate r and the volatility σ as well as the two bounds a and b , which will be presented in the following.

If we denote that $Y = \frac{S_t}{S_0}$, it is not difficult to find that the probability density for Y can be expressed as $\frac{1}{y}f(\ln y; \mu t, \sigma\sqrt{t}, a, b)$, leading to

$$f_Y(y) = \begin{cases} \frac{1}{y} \cdot \frac{\frac{1}{\sigma\sqrt{t}}\phi\left(\frac{\ln y - \mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)}, & e^a \leq y \leq e^b, \\ 0, & \text{otherwise.} \end{cases}$$

As a result, we can obtain

$$E[Y|F_0] = \int_{e^a}^{e^b} \frac{\frac{1}{\sigma\sqrt{t}}\phi\left(\frac{\ln y - \mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)} dy. \quad (2.5)$$

With the transform of $z = \frac{\ln y - \mu t}{\sigma\sqrt{t}}$, it can be further calculated as

$$\begin{aligned} E[Y|F_0] &= \frac{e^{(\mu + \frac{1}{2}\sigma^2)t}}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)} \int_{\frac{b - \mu t}{\sigma\sqrt{t}}}^{\frac{a - \mu t}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z - \sigma\sqrt{t})^2}{2}} dz \\ &= e^{(\mu + \frac{1}{2}\sigma^2)t} \frac{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)}. \end{aligned}$$

In order to avoid arbitrage opportunities, the martingale restriction (2.4) should be imposed, which implies that

$$E[Y|F_0] = e^{rt}. \quad (2.6)$$

Therefore, the following can be obtained

$$\frac{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)} = e^{(r - \frac{1}{2}\sigma^2 - \mu)t}, \quad (2.7)$$

which yields μ being an implicit function of given parameters and time to maturity for the target options. In other words, once a, b, σ, r, t are given, μ needs to be computed from

(2.7) as a “root finding” problem. Here, $\Phi(\cdot)$ represents the normal distribution function. It should be remarked that once an equation has been derived, the existence of the solution should be checked. In Equation (2.7), when μ approaches $+\infty$, the left hand side (LHS) and the right hand side (RHS) of the equation approaches 1 and 0 respectively, which implies that the LHS is greater than RHS. In contrast, when μ approaches $-\infty$, the LHS is still 1 while the RHS approaches $+\infty$, from which we can certainly know that the LHS is smaller than the RHS in this case. Therefore, the existence of the solution is verified.

After the martingale restriction is imposed, we are now ready to derive a closed-form pricing formula for European call options under our model with the martingale approach, which will be provided in the next subsection.

2.3 A closed-form pricing formula

In order to obtain the pricing formula for European call options, three cases with respect to the initial underlying price and the strike price should be taken into consideration. The first two cases are trivial and are illustrated in advance. When $\frac{K}{S_0} < e^a$, we should know that $S_t - K \geq 0$ always holds and thus the option price can be obtained easily as $S_0 - Ke^{-rt}$. On the other hand, when $\frac{K}{S_0} > e^b$, the underlying price will always be lower than the strike price, which tells that the option is worthless.

Now let us turn to the final case, i.e. $e^a \leq \frac{K}{S_0} \leq e^b$, the option price can be obtained according to the definition

$$\begin{aligned}
V_c &= e^{-rt} E[\max(S_t - K, 0) | F_0], \\
&= S_0 e^{-rt} \int_{-\infty}^{+\infty} \max\left(y - \frac{K}{S_0}, 0\right) f_Y(y) dy, \\
&= \frac{S_0 e^{-rt}}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} \left[\int_{\frac{K}{S_0}}^{e^b} \frac{1}{\sigma\sqrt{t}} \phi\left(\frac{\ln y - \mu t}{\sigma\sqrt{t}}\right) dy - \frac{K}{S_0 \sigma\sqrt{t} y} \phi\left(\frac{\ln y - \mu t}{\sigma\sqrt{t}}\right) dy \right], \\
&\triangleq \frac{S_0 e^{-rt}}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}}\right)} (A_1 + A_2). \tag{2.8}
\end{aligned}$$

The first integral A_1 can be easily calculated according to the derivation process of the martingale restriction in the last subsection while the second one A_2 is the integral of the probability density of the standard normal distribution after applying the transform of $x = \ln y$. As a result, the following two should hold

$$\begin{aligned} A_1 &= e^{(\mu + \frac{1}{2}\sigma^2)t} \left[\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{\ln(\frac{K}{S_0}) - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) \right], \\ A_2 &= \frac{K}{S_0} \left[\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{\ln(\frac{K}{S_0}) - \mu t}{\sigma\sqrt{t}}\right) \right]. \end{aligned}$$

Combining the martingale restriction (2.7), we can finally arrive at

$$V_c = S_0 \frac{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{\ln(\frac{K}{S_0}) - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} - K e^{-rt} \frac{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{\ln(\frac{K}{S_0}) - \mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)}. \quad (2.9)$$

With the newly derived option pricing formula, it is natural for us to consider some properties of the solution theoretically, which will be discussed in the next subsection.

2.4 Basic properties of the solution

In this subsection, various basic properties of the pricing formula would be investigated to show the rationale and validity of the solution.

Proposition 2.1 (*Monotonicity*) *The European call option price is a monotonic increasing function of the underlying price S .*

Proof. To show the Monotonicity of the option price with respect to S , we shall just derive $\frac{\partial V_c}{\partial S}$. A direct calculation can be made as

$$\begin{aligned}\frac{\partial V_c}{\partial S} &= \frac{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{\ln\left(\frac{K}{S_0}\right) - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} + \frac{1}{\sigma\sqrt{t}} \frac{\phi\left(\frac{\ln\left(\frac{K}{S_0}\right) - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)} \\ &\quad - \frac{Ke^{-rt}}{S_0\sigma\sqrt{t}} \frac{\phi\left(\frac{\ln\left(\frac{K}{S_0}\right) - \mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)}, \\ &\triangleq M_1 + M_2 - M_3.\end{aligned}\tag{2.10}$$

Actually, M_2 can be further simplified as

$$\begin{aligned}M_2 &= \frac{1}{\sigma\sqrt{t}[\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)]} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{K}{S_0}\right) - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)^2}, \\ &= \frac{1}{\sigma\sqrt{t}[\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)]} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{K}{S_0}\right) - \mu t}{\sigma\sqrt{t}}\right)^2 + \ln\left(\frac{K}{S_0}\right) - \mu t - \frac{1}{2}\sigma^2 t}\end{aligned}\tag{2.11}$$

Substituting Equation (2.7) into (2.11) yields

$$M_2 = \frac{K}{S_0} e^{-rt} \frac{\phi\left(\frac{\ln\left(\frac{K}{S_0}\right) - \mu t}{\sigma\sqrt{t}}\right)}{\sigma\sqrt{t}[\Phi\left(\frac{b-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) - \Phi\left(\frac{a-\mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)]} = M_3.\tag{2.12}$$

On the other hand, it is well-known that the normal distribution function $\Phi(x)$ is a monotonic increasing function of x . In addition, with $e^b > \frac{K}{S_0} > e^a$, we can also obtain that

$$\frac{b - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t} \geq \frac{\ln\left(\frac{K}{S_0}\right) - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t} \geq \frac{a - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t},$$

which implies that $M_1 > 0$. Therefore, we have shown that $\frac{\partial V_c}{\partial S} > 0$. This has completed the proof.

The monotonicity of our pricing formula with respect to S is consistent with financial implications of the call options, showing the rationality of our formula. Furthermore, it

is rather important to check the asymptotic behavior of the formula to further show its validity in the following.

Proposition 2.2 (*Asymptotics*)

$$\lim_{S \rightarrow +\infty} V_c = S, \quad \lim_{S \rightarrow 0} V_c = 0. \quad (2.13)$$

Proof. The proof of this proposition is trivial. In fact, it should be noticed that when the underlying price S approaches positive infinity, S must be larger than Ke^{-a} , which means that $V = S_0 - Ke^{-rt}$. As a result, the value of Ke^{-rt} can be ignored when S is large enough and thus the first limit should hold. Similarly, when S is close to zero, the inequality, $S < Ke^{-b}$, should be satisfied, which implies that $V = 0$. This has completed the proof.

It should be noted that the asymptotic behavior of the current pricing formula is consistent with the financial settings of European call options, which verifies the correctness of our formula from one angle. On the other hand, the bounds a and b are newly introduced parameters. It should be pointed out that our price degenerates to the B-S price when the lower and upper bound approach negative and positive infinity respectively. This is clearly shown by the following proposition.

Proposition 2.3 *When a and b approach $-\infty$ and $+\infty$ respectively, our option price can degenerate to the B-S price.*

Proof. This proposition is also not difficult to verify. As we observe the newly derived

pricing formula carefully, it is not hard to find that

$$\begin{aligned}
\lim_{b \rightarrow +\infty} \Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) &= 1, \\
\lim_{a \rightarrow -\infty} \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right) &= 0, \\
\lim_{b \rightarrow +\infty} \Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) &= 1, \\
\lim_{a \rightarrow -\infty} \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right) &= 0.
\end{aligned}$$

As a result, the martingale restriction under the limitation of a and b can be simplified as

$$\mu = r - \frac{1}{2}\sigma^2, \quad (2.14)$$

which is exactly the same as the one for the B-S model. Moreover, the following can also be obtained

$$\begin{aligned}
\lim_{a \rightarrow -\infty, b \rightarrow +\infty} V_c &= S_0[1 - \Phi\left(\frac{\ln(\frac{K}{S_0}) - \mu t}{\sigma\sqrt{t}} - \sigma\sqrt{t}\right)] - Ke^{-rt}[1 - \Phi\left(\frac{\ln(\frac{K}{S_0}) - \mu t}{\sigma\sqrt{t}}\right)], \\
&= S_0\Phi\left(\frac{\ln(\frac{S_0}{K}) + \mu t}{\sigma\sqrt{t}} + \sigma\sqrt{t}\right) - Ke^{-rt}\Phi\left(\frac{\ln(\frac{S_0}{K}) + \mu t}{\sigma\sqrt{t}}\right).
\end{aligned}$$

With Equation (2.14), it is straightforward that

$$\lim_{a \rightarrow -\infty, b \rightarrow +\infty} V_c = S_0\Phi\left(\frac{\ln(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right) - Ke^{-rt}\Phi\left(\frac{\ln(\frac{S_0}{K}) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right), \quad (2.15)$$

which is exactly the B-S price. This has completed the proof.

Since the B-S formula is just a special case of our formula, it is natural for us to check whether the put-call parity under the B-S model still holds in our model. In addition, the put-call parity is a relationship between European call and put option prices and its validity is an indication that no arbitrage opportunity exists, which can reinforce that the derived martingale restriction is reasonable. Hence, its mathematical proof would be provided in

the next proposition.

Proposition 2.4 *The put-call parity holds for our model with the truncated normal distribution and its form is the same as that in B-S model, i.e.*

$$V_c - V_p = S_0 - Ke^{-rt}. \quad (2.16)$$

Proof. According to the derivation process of the call option price, we can certainly obtain

$$\begin{aligned} V_c &= S_0 e^{-rt} \int_{\frac{K}{S_0}}^{e^b} \frac{1}{y} \cdot \frac{\frac{1}{\sigma\sqrt{t}} \phi\left(\frac{\ln y - \mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)} \left(y - \frac{K}{S_0}\right) dy, \\ V_p &= S_0 e^{-rt} \int_{e^a}^{\frac{K}{S_0}} \frac{1}{y} \cdot \frac{\frac{1}{\sigma\sqrt{t}} \phi\left(\frac{\ln y - \mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)} \left(\frac{K}{S_0} - y\right) dy. \end{aligned}$$

As a result, it is straightforward that

$$V_c - V_p = S_0 e^{-rt} C_1 - K e^{-rt} C_2, \quad (2.17)$$

where

$$\begin{aligned} C_1 &= \int_{e^a}^{e^b} \frac{\frac{1}{\sigma\sqrt{t}} \phi\left(\frac{\ln y - \mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)} dy, \\ C_2 &= \int_{e^a}^{e^b} \frac{\frac{1}{\sigma\sqrt{t}y} \phi\left(\frac{\ln y - \mu t}{\sigma\sqrt{t}}\right)}{\Phi\left(\frac{b - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a - \mu t}{\sigma\sqrt{t}}\right)} dy. \end{aligned}$$

Recall the martingale restriction, it is not difficult to find that C_1 is actually equal to e^{rt} . On the other hand, if we apply the transform of $z = \frac{\ln y - \mu t}{\sigma\sqrt{t}}$, the value of C_2 can surely be worked out, which is exactly 1. Therefore, we finally arrive at the desired result

$$V_c - V_p = S_0 - Ke^{-rt}, \quad (2.18)$$

which indicates that the proof has been completed.

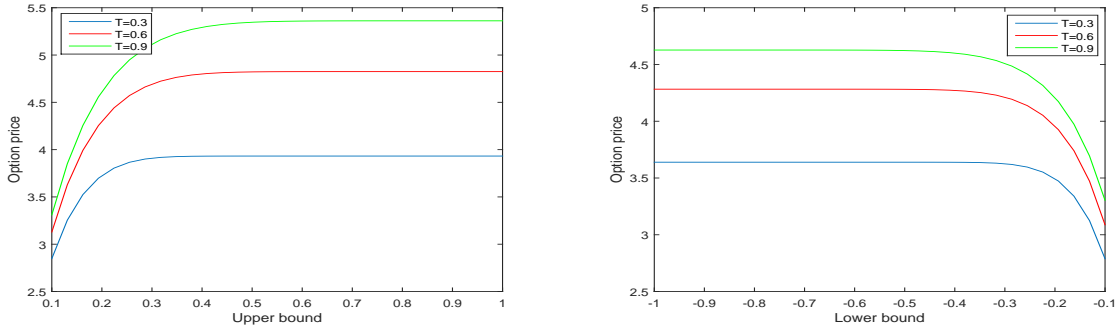
With the same put-call parity being verified, it is clear that our model can be viewed as a more general model than the B-S model. It should also be noted that the put-call parity derived here has also brought the convenience in trading practice since to obtain both of the European call and put option prices, only one price needs to be figured out; its counterpart can be easily deduced with the parity.

After these basic properties of the newly derived option pricing formula have been studied, we then focus on some numerical examples, which will be given in the next section.

3 Numerical examples and discussions

In this section, the influence of parameters a and b on European call option prices will be first studied and then a comparison of option prices obtained from the newly derived formula is made with those calculated from the B-S formula. Finally, the difference between imposing bounds on the underlying price model and the barrier option will be discussed. In terms of numerical procedure of computing option prices, a root-finding scheme needs to be adopted to find the μ value first from the martingale restriction (2.7). Then, such a value is inserted into Equation (2.9) to find the needed option value, which can be graphed into figures for the presentation purpose. It should be remarked that our model reduces back to the B-S model, with μ taking the value of $r - \frac{1}{2}\sigma^2$ and thus it is very natural to use this value as the initial guess for any root-finding procedure. Of course, the root-finding procedure would certainly consume more time than computing an option price from the B-S model. However, the actual calculation is very fast upon invoking the Matlab built-in function *fzero* with the recommended initial guess of $r - \frac{1}{2}\sigma^2$. Hereafter, the risk-free interest rate r is set to be 0.01 and the volatility σ is given the value of 0.2 for all the figures presented in the remaining part of this section.

Depicted in Figure 2 are our option prices with different bound values. To be more



(a) Option prices with variable upper bound.

(b) Option prices with variable lower bound.

Figure 2: Option pricing with changing bounds. Model parameters are $S = 100, K = 100$.

specific, it is interesting to notice that with lower bound set to be $\ln 0.85$, a higher price under our model can be expected when we increase the value of the upper bound b in Figure 2.1. It can be easily explained since the underlying price can take larger value with a higher upper bound, which can certainly give rise to a call option price. This also means that the price of a call option with a finite range of the underlying price is always lower than that obtained from the B-S formula. This does make sense financially too, because the underlying price in our model is assumed to be impossible to go beyond $S_0 e^b$, while that in the B-S model can take any value. Another important feature is that when the value of b becomes large enough, a further increase in the upper bound will make little difference to option prices, which is not difficult to understand since the right tail of the truncated normal distribution would become more similar to that of the standard normal distribution when we increase the value of b . Furthermore, higher upper bound is needed to observe this phenomenon with larger time to expiry mainly because a higher European call option price would be obtained in this case. On the other hand, Figure 2.2 exhibits that the smaller the lower bound a , the higher the call option price will be (here upper bound is $\ln 1.15$). It seemed rather confusing at first. However, we eventually understand it when we realized that the European put option price should decrease when the lower bound a takes a smaller value since the underlying price can decrease to a much lower value when a drops. As a result, with the help of the put-call parity derived in the previous section, the call

option price should also be a decreasing function of a with other parameters unchanged. In addition, a similar observation can be obtained that if the lower bound keeps decreasing, the call option price would converge to a certain price. The interpretation for it is the same as that for the case of the upper bound.

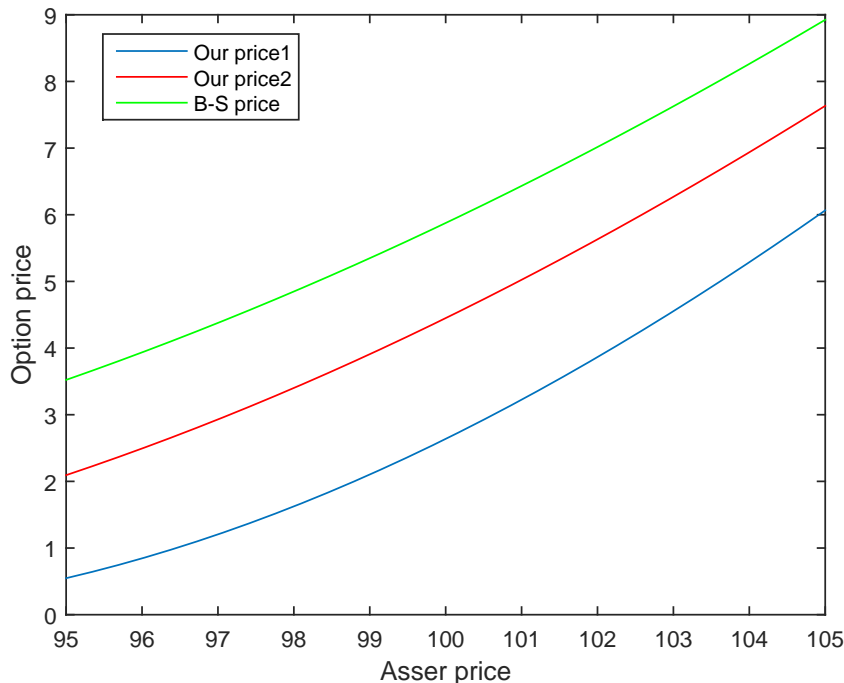
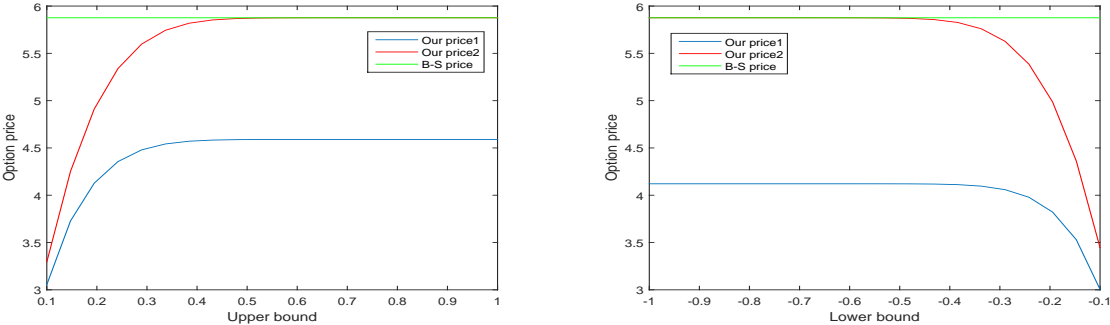


Figure 3: Our price vs B-S price with different underlying price. Our price1: $a = \ln 0.9, b = \ln 1.0$; Our price2: $a = \ln 0.8, b = \ln 1.2$.

As for Figure 3, it is clearly that our price for call options is an increasing function of the underlying price, which confirms the theoretical results obtained in the previous section. Furthermore, it is always lower than the B-S price, no matter the option is “in the money”, “at the money” or “out of money”. This is quite reasonable because the call option price would increase when we enlarge the upper bound in the sense that it would be possible for the underlying price to become larger. On the other hand, the put option price would go down if we set the lower bound to be smaller since it would be more likely for the underlying price to decrease and thus there will also be a down trend for the call option price according to the put-call parity. [However, the conclusion drawn here that our price](#)

is always lower than the B-S price is a result of assuming that the values of the volatility σ in both models are the same. It should be pointed out that this may not be true in practice since we always need to do model calibration before any mathematical model is applied in real markets and the determined values of σ in both models can be different.

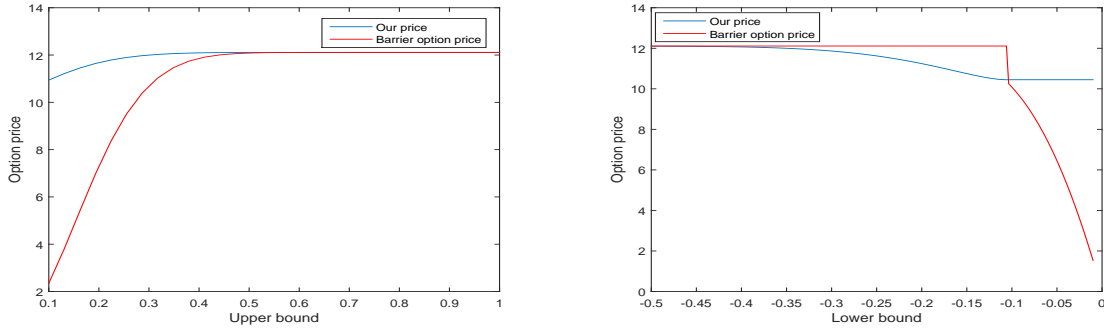


(a) Our price vs B-S price with different lower bound. Our price1: $a = \ln 0.85$; Our price2: $a = -2$.
 (b) Our price vs B-S price with with variable upper bound. Our price1: $b = \ln 1.15$; Our price2: $b = 2$.

Figure 4: Our price vs B-S price with different bounds. Model parameters are $S = K = 100$.

What can be seen in Figure 4(a) is that when the upper bound is large enough, the option price surges from approximately 4.6 to 5.8, which is rather close to the B-S price, if the lower bound a decreases from $\ln 0.85$ to -2 . This is because when the upper and lower bound are large and small enough respectively, our model will certainly become similar to the B-S model. A similar pattern appears in Figure 4(b) where our price tends to approach the B-S price when the bound range becomes larger, which is consistent with the fact that the truncated normal distribution would degenerate to the standard normal distribution if a and b are close to the negative and positive infinity respectively.

An interesting question one may raise is the difference between the option price calculated with the model presented in this paper and that from a barrier option, because both of them appear to take into consideration that the underlying of an option may only reach a certain level within a finite time horizon. Of course, the fundamental difference of the two is that the former variation of the underlying is on a finite (expected) range, while the underlying of the latter case is still allowed to vary between zero and infinity and a trader



(a) Our price vs up-and-out call option price with changing upper bound. (b) Our price vs down-and-out call option price with changing lower bound.

Figure 5: Option price vs B-S barrier option price with changing bounds. Model parameters are $S = 100, K = 90$.

would bet his expectations on the option contact being switched either on (the case of “knock-in options”) or off (the case of “knock-out options”). But, one may wish to explore the difference of option prices with these two rather different ways of acknowledging some sort of expectations from traders. In order to demonstrate this, comparisons are made so that some interesting guidelines can be provided to market traders, allowing them to have a quantitative sense how each of these can be properly used to suit their hedging and pricing purposes.

Figure 5 exhibits a comparison of our price with the barrier option price under the B-S model and it is very clear that the difference between the two kinds of option prices is distinct. As shown in Figure 5(a), with the lower bound a removed and the barrier level for the up-and-out call option being Se^b , the same as the upper bound for the underlying price, our price is always higher than the up-and-out option price, especially when the upper bound is small. This is caused by the intrinsic difference between the two prices. Although the two prices both give up some space for increase, the way to reach this goal is quite different in that it is a basic assumption in our model that the underlying price will not exceed a certain level, while the barrier level is actually introduced in the option contrast for the up-and-out options. In this case, there would be higher probability for the increase of the underlying price in our model, which leads to a higher price. In addition,

when the upper bound becomes larger, the up-and-out option price and our price become almost the same, which does make sense since the two prices eventually approach the B-S price in this case. When we turn to Figure 5(b), in which the upper bound b is removed and the barrier level is Se^a , it shows a similar pattern that our price is almost the same as the down-and-out call price and the two prices are close to the B-S price when the lower bound is small. This is reasonable since our price approaches the B-S price when the lower bound is small enough, while the barrier is meaningless when the barrier level for the down-and-out option is smaller than the strike price. However, it appears to be quite different from the first case that when the lower bound increases to some extent, there is a sudden drop in the down-and-out option price, when our price begins to become constant. This is also not difficult to understand since the barrier level for the down-and-out option starts to take effect when it is higher than the strike price, in which case the underlying price will always be higher than the strike price in our model and thus our price can be expressed as $S - Ke^{-rT}$, which is stated in Section 2 already.

4 Empirical studies

In this section, the results of a preliminary study, comparing the market performance of our model and the Black-Scholes model, are presented, in order to show whether the underlying price is better described by our model when option pricing.

4.1 Data description

This empirical study is based on the data of S&P 500 European call options from Jan 2011 to Dec 2011. As usual, the average value of bid and ask prices is regarded as the option price, and several filters need to be applied to eliminate sample noise in the estimation of parameters [1].

First of all, only Wednesday options data (denoted by in-sample data) is used in the

stage of parameter estimation mainly due to two reasons; the first is that Wednesday is least likely to be a holiday in a week and also less likely to be affected by day-of-the-week effect, while another is that parameter determination is time-consuming, and choosing one day a week allows us to study a relatively long period. It should be mentioned here that Thursday options data (denoted by out-of-sample data, in contrast to the concept of in-sample) will be used to assess the model performance in the prediction of option prices calculated with parameters determined by the day before. Secondly, options closed to the expiry time (less than 7 days) are discarded since these options have less time value. Also, Options with more than 120 days to expiry were also excluded because of their unpopularity caused by high trading premiums. Thirdly, if we define the moneyness as $\frac{S - K}{K}$, then options with the absolute value of moneyness over 10% are deleted since very deep in-the-money and out-of-the-money options usually have liquidity-related problems. Finally, options with prices less than \$1/8 are removed as these prices are rather volatile in real markets. It should also be pointed out that the three-month U.S. Treasury Bill Rate released daily is chosen as a proxy of risk-free interest rate [37] since the time to expiry of options used is less than 120 days.

With all the option data needed at hand, we are now ready to determine model parameters with these data, the process of which will be illustrated in the following subsection.

4.2 Parameter estimation

The first step in assessing model performance is to estimate model parameters with real market data, which is a very difficult problem and time-consuming. To find out the “optimal” parameter set that best fits the chosen market data, one common approach is minimizing the “distance” between the model and market option price, which means we need to choose an appropriate function (known as objective function) for such a distance. Following Christoffersen & Jacobs [6] and many other authors, we adopt the dollar mean-squared

errors, which is defined as

$$MSE = \frac{1}{N} \sum_{i=1}^N [C_{Market} - C_{Model}]^2, \quad (4.1)$$

with C_{Market} and C_{model} denoting the market and model price of an option respectively. N is the total number of observations used in one parameter estimation.

With the objective function chosen, we should choose a satisfactory approach to conduct parameter estimation. It should be pointed out that the objective function (4.1) is not necessarily convex, and thus local minimization algorithms will probably end up with a local minima. In this case, a global optimization is preferred, in which some stochastic factors are generally introduced in their search process so that it will not be stuck when it reaches a local minima.

The method we adopt here is the Adaptive Simulated Annealing (ASA) [19], which is one of the most popular global optimization algorithms. In fact, Simulated Annealing was firstly developed in 1983 for highly nonlinear problems [21], and it was improved in [38] with the development of the so-called Fast Simulated Annealing, which allows the global minimum can be obtained within finite time. Later on, Very Fast Simulated Reannealing [17], which is now renamed as ASA, was established to further accelerate the speed with random step selection automatically adjusted according to algorithm progress. This particular optimization method has been widely applied in many areas [4, 5], and it has already been applied in the calibration of option pricing models [18, 28].

In our study, the adopted ASA can be realized through the open-source code provided in [16], and the feedback from many users regularly assesses the source code to ensure its soundness so that it can become even more flexible and powerful. The specific procedures of our empirical study are shown in Figure 6, based on which the estimated parameters (daily averaged) are reported in the following Table 1.

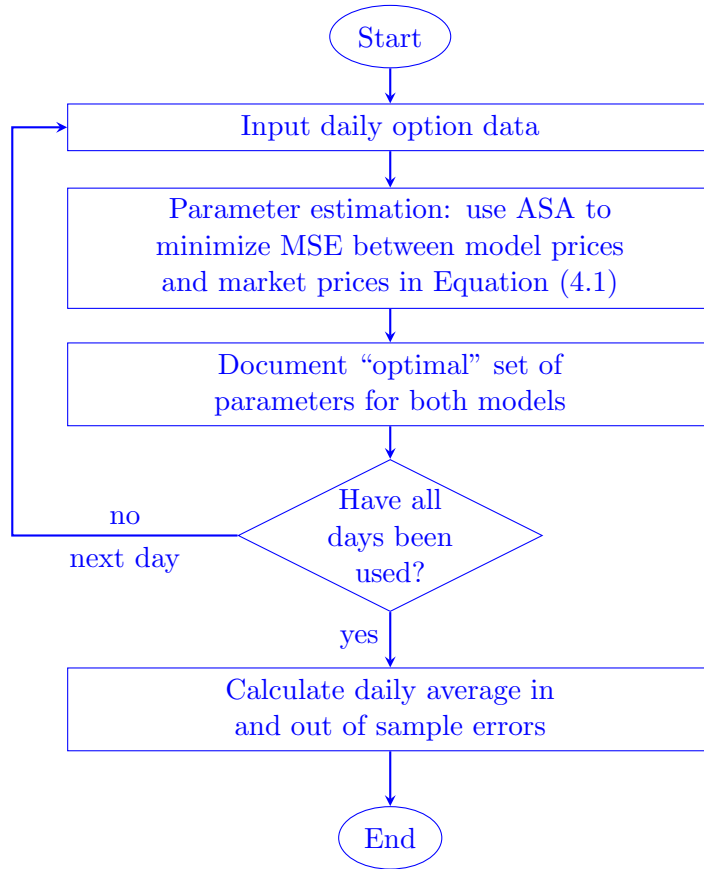


Figure 6: Flow chart on how our empirical studies are conducted.

Table 1: Estimated parameters

parameters	σ	a	b
Our model	0.2401	-1.4879	0.5932
Black-Scholes	0.2213		

4.3 Empirical comparison

Once model parameters have been obtained with real market data, it is natural for us to assess the model performance. It is widely acknowledged that the performance of a model can be regarded better if there are lower pricing differences between the calculated option prices with model and the corresponding market prices.

What are shown in Table 2 are the in-sample and out-of-sample errors under our model and the Black-Scholes model. It is not difficult to find that our model can certainly be regarded as superior over the Black-Scholes model in the tested case as our model can

Table 2: In- and out-of-sample errors for the two models

Error	In-sample	Out-of-sample
Our model	2.3905	12.7735
Black-Scholes	4.5396	14.5772
Relative difference	47.37%	12.37%

provide better fitness to both of the in-sample and out-of-sample market data. In specific, if the Black-Scholes model is replaced by our model, there is a great improvement with the relative difference being approximately 50% as far as the in-sample errors are concerned, while a 12% less errors of our model can be witnessed when out-of-sample comparison is taken into consideration.

Table 3: Out-of-sample errors for the two models with different moneyness

Error	$0.90 < S/K < 0.97(O)$	$0.97 \leq S/K \leq 1.03(A)$	$1.03 < S/K < 1.10(I)$
Our model	2.6983	16.6294	11.3016
Black-Scholes	5.0314	20.8835	12.6835
Relative difference	46.37%	20.37%	10.90%

On the other hand, options are traded with a wide range of strikes in real markets and thus it is important to check the out-of-sample performance of the two models according to different moneyness, which is presented in Table 3 with the abbreviation in the parentheses representing out of money, at the money and in the money, respectively, from the left to the right columns. It is clear that our model can provide a better data fitness, no matter options belong to which sub-category, with out-of-money options experiencing the highest relative difference of approximately 50%. The improvement in the category of at-the-money and in-the-money is relatively smaller, being around 20% and 10% respectively.

5 Conclusion

In this paper, the underlying log-price is assumed to follow a truncated normal distribution, which is able to describe the phenomenon that there should be reasonable bounds for the underlying price in a certain period. By adopting the non-structural model, the martingale

restriction for our model is obtained and a closed-form pricing formula for European call options is derived, after which various basic properties of the newly derived formula are investigated, showing the validity of the solution. Furthermore, through numerical experiments, the influence brought by the introduction of upper and lower bounds on option prices is studied and results show that our price is an increasing function of the upper bound while it is a decreasing function with respect to the lower bound. Our price is also compared with the B-S price, and it is reasonable to find that our price will approach the B-S price if the lower bound and upper bound become small and large enough respectively since this case can be viewed as bounds removed. Finally, empirical studies show that our model greatly outperforms the Black-Scholes model in the tested data set, which can certainly lead us to the conclusion that our model can at least act as an alternative to the Black-Scholes model in real markets.

References

- [1] G. Bakshi, C. Cao, and Z. Chen. Empirical performance of alternative option pricing models. *The Journal of Finance*, 52(5):2003–2049, 1997.
- [2] F. Black and M. Scholes. The pricing of options and corporate liabilities. *The Journal of Political Economy*, pages 637–654, 1973.
- [3] P. Carr, H. Geman, D. B. Madan, and M. Yor. The fine structure of asset returns: An empirical investigation*. *The Journal of Business*, 75(2):305–333, 2002.
- [4] S. Chen, B. Luk, and Y. Liu. Application of adaptive simulated annealing to blind channel identification with hoc fitting. *Electronics Letters*, 34(3):234–235, 1998.
- [5] S. Chen and B. L. Luk. Adaptive simulated annealing for optimization in signal processing applications. *Signal Processing*, 79(1):117–128, 1999.

- [6] P. Christoffersen and K. Jacobs. The importance of the loss function in option valuation. *Journal of Financial Economics*, 72(2):291–318, 2004.
- [7] C. J. Corrado and T. Su. Skewness and kurtosis in S&P 500 index returns implied by option prices. *Journal of Financial Research*, 19(2):175–192, 1996.
- [8] O. Dey and D. Chakraborty. A single period inventory model with a truncated normally distributed fuzzy random variable demand. *International Journal of Systems Science*, 43(3):518–525, 2012.
- [9] B. Dumas, J. Fleming, and R. E. Whaley. Implied volatility functions: Empirical tests. *The Journal of Finance*, 53(6):2059–2106, 1998.
- [10] K. K. Dutta and D. F. Babbel. Extracting probabilistic information from the prices of interest rate options: Tests of distributional assumptions*. *The Journal of Business*, 78(3):841–870, 2005.
- [11] F. J. Fabozzi, R. Tunaru, and G. Albot. Estimating risk-neutral density with parametric models in interest rate markets. *Quantitative Finance*, 9(1):55–70, 2009.
- [12] J. M. Harrison and D. M. Kreps. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20(3):381–408, 1979.
- [13] M. Z. Hasan, A. A. Kamil, A. Mustafa, and M. A. Baten. Stochastic frontier model approach for measuring stock market efficiency with different distributions. *PloS one*, 7(5):e37047, 2012.
- [14] S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.
- [15] A. Hocquard, N. Papageorgiou, and B. Remillard. The payoff distribution model: an

- application to dynamic portfolio insurance. *Quantitative Finance*, (ahead-of-print):1–14, 2012.
- [16] L. Ingber. Home page of lester ingber. <https://www.ingber.com>.
- [17] L. Ingber. Very fast simulated re-annealing. *Mathematical and Computer Modelling*, 12(8):967–973, 1989.
- [18] L. Ingber. High-resolution path-integral development of financial options. *Physica A: Statistical Mechanics and its Applications*, 283(3):529–558, 2000.
- [19] L. Ingber et al. Adaptive simulated annealing (ASA): Lessons learned. *Control and Cybernetics*, 25:33–54, 1996.
- [20] R. Jarrow and A. Rudd. Approximate option valuation for arbitrary stochastic processes. *Journal of Financial Economics*, 10(3):347–369, 1982.
- [21] S. Kirkpatrick, C. D. Gelatt, M. P. Vecchi, et al. Optimization by simulated annealing. *Science*, 220(4598):671–680, 1983.
- [22] S. G. Kou. A jump-diffusion model for option pricing. *Management science*, 48(8):1086–1101, 2002.
- [23] F. A. Longstaff. Option pricing and the martingale restriction. *Review of Financial Studies*, 8(4):1091–1124, 1995.
- [24] D. B. Madan, P. P. Carr, and E. C. Chang. The variance gamma process and option pricing. *European Finance Review*, 2(1):79–105, 1998.
- [25] D. B. Madan and F. Milne. Contingent claims valued and hedged by pricing and investing in a basis. *Mathematical Finance*, 4(3):223–245, 1994.
- [26] J. B. McDonald and R. M. Bookstaber. Option pricing for generalized distributions. *Communications in Statistics-Theory and Methods*, 20(12):4053–4068, 1991.

- [27] R. C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3(1):125–144, 1976.
- [28] S. Mikhailov and U. Nögel. *Hestons stochastic volatility model: Implementation, calibration and some extensions*. John Wiley and Sons, 2004.
- [29] R. Norgaard and T. Killeen. Expected utility and the truncated normal distribution. *Management Science*, 26(9):901–909, 1980.
- [30] A. Peiro. Skewness in financial returns. *Journal of Banking & Finance*, 23(6):847–862, 1999.
- [31] J. Pender. The truncated normal distribution: Applications to queues with impatient customers. *Operations Research Letters*, 43(1):40–45, 2015.
- [32] S. T. Rachev, C. Menn, and F. J. Fabozzi. *Fat-tailed and skewed asset return distributions: implications for risk management, portfolio selection, and option pricing*, volume 139. John Wiley & Sons, 2005.
- [33] M. Rubinstein. Implied binomial trees. *The Journal of Finance*, 49(3):771–818, 1994.
- [34] R. Savickas. A simple option-pricing formula. *Financial Review*, 37(2):207–226, 2002.
- [35] L. O. Scott. Option pricing when the variance changes randomly: Theory, estimation, and an application. *Journal of Financial and Quantitative analysis*, 22(04):419–438, 1987.
- [36] B. J. Sherrick, P. Garcia, and V. Tirupattur. Recovering probabilistic information from option markets: Tests of distributional assumptions. *Journal of Futures Markets*, 16(5):545–560, 1996.
- [37] J. Shu and J. E. Zhang. Pricing S&P 500 index options under stochastic volatility with the indirect inference method. *Journal of Derivatives Accounting*, 1(2):1–16, 2004.

- [38] H. Szu and R. Hartley. Fast simulated annealing. *Physics Letters A*, 122(3-4):157–162, 1987.
- [39] J. B. Wiggins. Option values under stochastic volatility: Theory and empirical estimates. *Journal of Financial Economics*, 19(2):351–372, 1987.