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Analyticity and Mixing Properties in Classical Models Kac Type

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Abstract

In beginning of the 90's, Bernard Helffer and Jöhannes Sjostrand introduced operators serving to develop methods for the study of integrals in high dimensions of the type that appear in Statistical Mechanics and Euclidean Field Theory. In these papers, the authors studied a certain class of unbounded spin models by means of the spectrum of the Witten Laplacians. Thus, the decay of correlations, the analyticity of the free energy, the Poincaré and Log-Sobolev inequalies turned out to be the relevant tools for investigating phase transitions in certain classical continuous models. The present paper proposes a direct and more general method for investigating phase transitions in classical continuous models of Kac type. We discuss hypotheses on the source term that will result in a direct proof of the analyticity of the free energy without using the truncated correlations. We also use the Witten Laplacians to derive mixing properties and a decay of correlations that lead to the logarithmic Sobolev inequality. The novelty, as compared to previous work, is that our method is more direct and does not use the one dimensional Witten Laplacians.

1 Introduction

Phase transitions and critical points correspond to mathematical singularities in the thermodynamic potentials and other thermodynamic quantities which are related to appropriate derivatives of the free energy. For example, at the critical point of a ferromagnetic system, the spontaneous magnetization vanishes and the susceptibility diverges. It is therefore central to develop methods for calculating the thermodynamic potentials and their derivatives. There are a number of different thermodynamic potentials that can be used to describe the behavior and stability of equilibrium systems, depending on the type of constraints imposed on the system. For a system which is isolated from the world, the internal energy will be a minimum for the equilibrium state. However, if we couple the system thermally, mechanically, or chemically to the outside world, other thermodynamic potentials will be minimized at equilibrium. The energy which is stored and retrievable in the form of work is called the free energy. There are many different forms of free energy in a thermodynamic system as there are combinations of constraints. The most common ones are: internal energy, U; the enthalpy, H; the Helmholtz free energy, A; the Gibbs free energy, G; and the grand potential, Ω. These quantities play a role analogous to that of the potential energy in a spring, and for that reason they are called the thermodynamic potentials.

In the beginning of the 90's, new methods that are purely based on the Witten Laplacians have been developed by Helffer and Sjöstrand. In these papers, the authors studied a certain class of unbounded spin models by means of the spectrum of the Witten Laplacians. Thus, the decay of correlations, the analyticity of the free energy, the Poincaré and Log-Sobolev inequalies turned out to be the relevant tools for investigating the existence of phase transitions in certain classical continuous models.

In this paper, we propose a direct and more general method for investigating phase transitions in classical continuous models of Kac type. We discuss hypotheses on the source term that will result in a direct proof of the analyticity of the free energy without using the truncated correlations. We also use the Witten Laplacians to derive mixing properties and a decay of correlations that lead to the logarithmic Sobolev inequality. The novelty, as compared to previous work, is that our method is more direct and does not use the one dimensional Witten Laplacians. The paper is organized as follows:

In section 2, we give the background framework of the Witten Laplacian formalism. The readers who are interested in the connection between the operators used in this paper and the Laplacians initially introduced by Edward Witten [7] in 1882 may consult the appendix at the end of this paper.

In section 3, we discuss the models to be considered. These are generalizations of models for fluids whose Hamiltonians in the one dimensional case ($d = 1$) take the form

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2M} + \sum_{1 \leq i < j \leq N} W(x_i, x_j) + \sum_{i=1}^{N} U(x_i),$$

where $U(x_i)$ is the potential energy of the $i - th$ particle in an external field, W is a pair interaction potential.

In section 4, we discussed a new method for a direct proof of the analyticity of the free energy. The motivation for the study of the differentiability or even the analyticity of the pressure with respect to distinguished thermodynamic parameters such as temperature, chemical potential or external field comes from the fact that the analytic behavior of the pressure is the classical thermodynamic indicator for the absence or existence of phase transition. The most famous result on the analyticity of the pressure is the circle theorem of Lee and Yang. This theorem asserts the following. Consider a $\{-1, 1\}$ −valued spin system with ferromagnetic pair interaction and external field h and regard the quantity $z = e^h$ as a complex parameter, then all zeroes of all partition functions (with free boundary condition), considered as functions of z lie in the complex unit circle. This theorem readily implies that the pressure is an analytic function of h in the region $h > 0$ and $h < 0$.

There are various conventional indirect methods for proving the infinite differentiability or the analyticity of the pressure for (ferromagnetic and non ferromagnetic) systems at high temperatures, or at low temperatures, or at large external fields. Most of these take advantage of a sufficiently rapid decay of correlations and /or cluster expansion methods. The method that we develop in this section is based on a direct C^n-bound of the derivatives of the free energy and permits a study of of the decay of the truncated correlations through the free energy. A famous open problem in Coulomb gases [26] is to find a direct proof of convergence of the Mayer expansion for dipoles at low activity (which does not use a cluster expansion). The dipole-dipole interaction should be smoothed at short distance so that it is stable. Our method in this section may help solve this problem provided that one can find a regularization of the dipole potential that satisfies the assumptions of our Hamiltonian. Exact formulas of the derivative of the free energy that do not use cluster expansions of renormalization group techniques are almost inexistent in the literature. M. Kac [27] obtained a formula for the pressure in terms of irreducible distribution functions. In section 5, we propose an alternative method for obtaining the decay of correlations and consequently the Log-Sobolev inequality without reducing the dimensions of the Witten Laplacians.

2 The Witten Laplacian Formalism

In this section, we give a brief outline of the Witten-Laplacian formalism framework. These methods were introduced in 1994 by Bernard Helffer and Jöhannes Sjostrand [2]. They are generally based on the analysis of suitable differential operators

$$\mathbf{W}_{\Phi}^{(0)} = \left(-\mathbf{\Delta} + \frac{|\mathbf{\nabla}\Phi|^2}{4} - \frac{\mathbf{\Delta}\Phi}{2} \right)$$

and

$$\mathbf{W}_{\Phi}^{(1)} = -\mathbf{\Delta} + \frac{|\mathbf{\nabla}\Phi|^2}{4} - \frac{\mathbf{\Delta}\Phi}{2} + \mathbf{Hess}\Phi$$

which are in some sense, deformations of the standard Laplace Beltrami operator. These operators commonly called Witten Laplacians were first introduced by Edward Witten [7] in 1982 in the context of Morse theory for the study of some topological invariants of compact Riemannian manifolds. In 1994, Bernard Helffer and Jöhannes Sjostrand [2] introduced two elliptic differential operators

$$A_{\Phi}^{(0)} := -\mathbf{\Delta} + \mathbf{\nabla}\Phi \cdot \mathbf{\nabla}$$

and

$$A_{\Phi}^{(1)} := -\mathbf{\Delta} + \mathbf{\nabla}\Phi \cdot \mathbf{\nabla} + \mathbf{Hess}\Phi$$

sometimes called Helffer-Sjostrand operators serving to get direct methods for the study of integrals and operators in high dimensions of the type that appear in Statistical Mechanics and Euclidean Field Theory. In 1996, Jöhannes Sjostrand [4] observed that these so called Helffer-Sjostrand operators are in fact equivalent to Witten's Laplacians. Since then, there has been significant advances in the use of these Laplacians for the study of the thermodynamic behavior of quantities related to the Gibbs measure $Z^{-1}e^{-\Phi}dx$. As a simple illustration, say, one is interested in the study of the mean value $\langle g\rangle_\Lambda$where

$$\langle g\rangle_\Lambda = \int g d\mu_\Lambda.$$

and

$$d\mu_\Lambda = \frac{e^{\Phi_\Lambda}dx}{\displaystyle\int e^{\Phi_\Lambda}dx}$$

for a suitable smooth function g, one can first solve the equation

$$\boldsymbol{\nabla} g = \left(-\boldsymbol{\Delta} + \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\right)\mathbf{v} + \mathbf{Hess}\Phi\mathbf{v},$$

where the solution $\mathbf{v}$ is a suitable $C^\infty-$vector field and the operator $(-\boldsymbol{\Delta} + \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla})$ acts diagonally on each component of $\mathbf{v}$. Under certain assumptions on the Hamiltonian Φ one can see that $\mathbf{v}$ is also a solution of the system

$$g = < g >_\Lambda + \mathbf{v} \cdot \boldsymbol{\nabla}\Phi - div\mathbf{v}.$$

It turns out that if $g(0) = 0$ and 0 is a critical point of Φ then

$$< g >_\Lambda = div\mathbf{v}(0).$$

The study of the thermodynamic properties of the mean value is then reduced to estimating the derivatives of the solution $\mathbf{v}$.
Numerous techniques have been developed for the study of integrals associated to the equilibrium Gibbs state for certain unbounded spins systems [6], [10-14]. One of the most striking result is an exact formula for the covariance of two functions in terms of the Witten Laplacian on one forms leading to sophisticated methods for estimating the two-point correlation functions. This formula is in some sense a stronger and more flexible version of the Brascamp-Lieb inequality [9]. The formula may be written as:

$$\mathbf{cov}(f,g) = \int \left(A_\Phi^{(1)^{-1}} \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g\right) e^{-\Phi(x)}dx.$$

To understand the idea behind the formula mentioned above, let us denote by $\langle f\rangle$ the mean value of f with respect to the measure

$$e^{-\Phi(x)}dx,$$

the covariance of two functions f and g is defined by

$$\mathbf{cov}(f,g) = \langle (f - \langle f \rangle)(g - \langle g \rangle) \rangle .$$

If one wants to have an expression of the covariance in the form

$$\mathbf{cov}(f,g) = \langle \boldsymbol{\nabla} g \cdot \mathbf{w} \rangle_{L^2(\mathbb{R}^n, \mathbb{R}^n; e^{-\Phi} dx)} ,$$

for a suitable vector field $\mathbf{w}$, we get, after observing that $\boldsymbol{\nabla} g = \boldsymbol{\nabla}(g - \langle g \rangle)$,

$$\mathbf{cov}(f,g) = \int (g - \langle g \rangle)(\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}) \cdot \mathbf{w} e^{-\Phi(x)} dx.$$

This leads to the question of solving the equation

$$f - \langle f \rangle = (\boldsymbol{\nabla}\Phi - \boldsymbol{\nabla}) \cdot \mathbf{w}.$$

Now trying to solve this above equation with $\mathbf{w} = \boldsymbol{\nabla} u$, we obtain the equation

$$\left. \begin{array}{c} f - \langle f \rangle = A_\Phi^{(0)} u \\ \langle u \rangle = 0. \end{array} \right\}$$

Assuming for now the existence of a smooth solution, we get by differentiation of this above equation:

$$\boldsymbol{\nabla} f = A_\Phi^{(1)} \boldsymbol{\nabla} u$$

and the formula is now easy to see.

3 Models To Be Considered

We shall consider systems where each component is located at a site i of a crystal lattice $\mathbb{Z}^d$; and is described by a continuous real parameter $x_i \in \mathbb{R}$. A particular configuration of the total system will be characterized by an element $X = (x_i)_{i \in \Lambda}$ of the product space $\Omega = \mathbb{R}^\Lambda$. This set is called the configuration space or phase space.

We shall denote by $\Phi = \Phi^\Lambda$ the Hamiltonian which assigns to each configuration $X \in \mathbb{R}^\Lambda$ a potential energy $\Phi(X)$: The probability measure that describes the equilibrium of the system is then given by the Gibbs measure

$$d\mu^\Lambda(X) = Z_\Lambda^{-1} e^{-\Phi(X)} dX.$$

$Z > 0$ is a normalization constant,

$$Z = Z_\Lambda = \int_{\mathbb{R}^\Lambda} e^{-\Phi(X)} dX$$

For any finite domain Λ of $\mathbb{Z}^d$; we shall consider a Hamiltonian of the phase space $\Omega = \mathbb{R}^\Lambda$; satisfying:

1. $\lim_{|X|\to\infty} |\boldsymbol{\nabla}\Phi(X)| = \infty$
2. For some M, any $\partial^\alpha\Phi$ with $|\alpha| = M$ is bounded on $\mathbb{R}^\Lambda$.
3. For $|\alpha| \geq 1$, $|\partial^\alpha\Phi(X)| \leq C_\alpha \left(1 + |\boldsymbol{\nabla}\Phi(X)|^2\right)^{1/2}$ for some $C_\alpha > 0$
4. There exist $w > 0, C > 0$ such that $X \cdot \nabla\Phi \geq C\,|X|^{1+w}$ for all $|X| \geq \frac{1}{C}$.

Here and in what follows, $\alpha = (\alpha_i)_{i=1,\ldots,m} \in \mathbb{Z}_+^{|\Lambda|}$ shall denote a multiindex. We set $|\alpha| = \sum_{i=1}^m \alpha_i$, $\alpha! = \alpha_1! \cdots \alpha_m!$. If $\beta = (\beta_i)_{i=1,\ldots,m} \in \mathbb{Z}_+^{|\Lambda|}$ and $\beta_j \leq \alpha_j$ for all $j = 1, ..., m$, then we write $\beta \leq \alpha$. For $\alpha,\ \beta \in \mathbb{Z}_+^{|\Lambda|}$ such that $\beta \leq \alpha$, we put $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$. If $\alpha = (\alpha_i)_{i=1,\ldots,m} \in \mathbb{Z}_+^{|\Lambda|}$ and $X \in \mathbb{R}^d$ we write $X^\alpha = \prod_{i=1}^m x^{\alpha_i}$, and $\partial^\alpha = \partial^{\alpha_1}/\partial x_1^{\alpha_1} ... \partial^{\alpha_m}/\partial x_m^{\alpha_m}$. The Hessian of the Hamiltonian Φ will be denoted by $\mathbf{Hess}\Phi$. Finally, if i and j are two nearest neighbor sites in $\mathbb{Z}^d$ we write $i \sim j$.

4 A Method For Proving The Analyticity Of The Free Energy

We consider the Hamiltonian given by

$$\Phi^t(x) = \Phi_\Lambda(x) - tg(x),$$

where t is a thermodynamic parameter (temperature or magnetic field), and g satisfying

$$|\partial^\alpha \boldsymbol{\nabla} g| \leq C_\alpha, \qquad \forall\alpha \in \mathbb{N}^{|\Lambda|}. \tag{1}$$

The finite volume pressure or free energy of the system is defined by

$$P_\Lambda(t) = \frac{1}{|\Lambda|} \ln\left[\int_{\mathbb{R}^\Lambda} dx e^{-\Phi^t(x)}\right].$$

We will be interested in the $k-$times differentiability of the pressure in the thermodynamic limit given by

$$P(t) = \lim_{|\Lambda|\to\infty} P_\Lambda(t).$$

We first recall some results already obtained in [5] and [16].

Theorem 1 ([5]) *If Φ_Λ satisfies assumptions 1-4 and g satisfies (1), then there exist $T^* > 0$ such that the k times derivatives of $P_\Lambda(t)$ $(k \geq 1)$ satisfies*

$$\frac{d^{(k)}P_\Lambda(t)}{dt^k} = P_\Lambda^{(k)}(t) = \frac{(k-1)!\left\langle A_g^{k-1} g\right\rangle_{t,\Lambda}}{|\Lambda|}\ ,\ \textit{for all } t \in [0, T^*) \tag{2}$$

where

$$A_g h := A_{\Phi_\Lambda^t}^{(1)^{-1}} \boldsymbol{\nabla} h \cdot \boldsymbol{\nabla} g, \tag{3}$$

$$A_g^0 g := g, \quad A_g^1 g := A_g g, \; A_g^2 g := A_g \left(A_g g\right), \quad A_g^{k-1} g := \underbrace{\left(A_g \circ A_g \circ \cdots \circ A_g\right)}_{k-1 \;\; times} g,$$

and

$$\langle \cdot \rangle_{t,\Lambda} = \frac{\int_{\mathbb{R}^\Lambda} \cdot dx e^{-\Phi^t(x)}}{\int_{\mathbb{R}^\Lambda} dx e^{-\Phi^t(x)}}.$$

The proof of this theorem can be found in [5].

Theorem 2 ([16]) *If Φ_Λ satisfies assumptions 1-4 and g satisfies (1), then*

$$\sum_{j=0}^{k} \frac{\left\langle g^j \right\rangle_{\Lambda,t} \left\langle A_g^{k-j} g \right\rangle_{\Lambda,t}}{j!} = \frac{1}{k!} \left\langle g^{k+1} \right\rangle_{\Lambda,t}, \qquad k \geq 0 \tag{4}$$

Proof. First observe that

$$\begin{aligned} & \left\langle g^p A_g h \right\rangle_{\Lambda,t} \\ = & \left\langle g^p A_{\Phi^t}^{(1)^{-1}} \boldsymbol{\nabla} h \cdot \boldsymbol{\nabla} g \right\rangle_{\Lambda,t} \\ = & \frac{1}{p+1} \left\langle A_{\Phi^t}^{(1)^{-1}} \boldsymbol{\nabla} h \cdot \boldsymbol{\nabla} g^{p+1} \right\rangle_{\Lambda,t} \\ = & \frac{1}{p+1} \mathbf{cov} \left(g^{p+1}, h\right) \\ = & \frac{1}{p+1} \left[\left\langle g^{p+1} h \right\rangle_{\Lambda,t} - \left\langle g^{p+1} \right\rangle_{\Lambda,t} \left\langle h \right\rangle_{\Lambda,t} \right], \quad p = 0, 1, ... \end{aligned}$$

Setting

$$k = p + 1 \quad \text{and} \quad h = A_g^{n-k-1} g,$$

yields

$$\left\langle g^k \right\rangle_{\Lambda,t} \left\langle A_g^{n-k-1} g \right\rangle_{\Lambda,t} = \left\langle g^k A_g^{n-k-1} g \right\rangle_{\Lambda,t} - k \left\langle g^{k-1} A_g^{n-k} g \right\rangle_{\Lambda,t}.$$

Now dividing by $k!$, summing over k and noticing that on the right hand side one obtains a telescoping sum, yields

$$\sum_{k=0}^{n-1} \frac{\left\langle g^k \right\rangle_{\Lambda,t} \left\langle A_g^{n-k-1} g \right\rangle_{\Lambda,t}}{k!} = \frac{1}{(n-1)!} \left\langle g^n \right\rangle_{\Lambda,t}.$$

■

Now observe that if

$$x_k = \frac{\left\langle g^k \right\rangle_{t,\Lambda}}{k!} \qquad \text{and} \qquad y_k = \left\langle A_g^k g \right\rangle_{t,\Lambda}, \qquad k \geq 0,$$

then (4) is equivalent to

$$\{x_k\} * \{y_k\} = (k+1)\, x_{k+1}. \tag{5}$$

where $\{x_k\} * \{y_k\}$ is the discrete circular convolution given by

$$\{x_k\} * \{y_k\} = \sum_{j=0}^{k} x_j y_{k-j}.$$

Now to find a formula for y_k, we need to find a suitable transform that will allow us to deconvolve and invert the convolution equation.

Recall that the discrete time Fourier transform of the sequence $x_k \quad k \geq 0$ is defined to be

$$X(\omega) = \sum_{k=0}^{\infty} x_k e^{-ik\omega}, \qquad \omega \in [-\pi, \pi].$$

The corresponding inverse transform is given by

$$x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{ik\omega} d\omega, \qquad k \geq 0.$$

With $x_k = \frac{\langle g^k \rangle_{t,\Lambda}}{k!}$, the input g will be selected so that the sequence $(x_k)_k$ will be of temperate growth and that $X(\omega)$ will not vanish on $[-\pi, \pi]$. As a result of this $X(\omega)$, will be a $2\pi-$periodic distribution defined as the boundary value of a holomorphic function in the open lower half plane.

If $X(\omega)$ and $Y(\omega)$ are the discrete time Fourier transform of x_k and y_k respectively, then the discrete time Fourier transform of $x_k * y_k$ is $X(\omega)Y(\omega)$. It also follows from the properties of the transform that the discrete time Fourier transform of $(k+1)\, x_{k+1}$ is $ie^{i\omega} \frac{dX(\omega)}{d\omega}$. Thus taking the transform on both sides of equation (5), we obtain

$$X(\omega)Y(\omega) = ie^{i\omega} \frac{dX(\omega)}{d\omega}.$$

Hence

$$Y(\omega) = i \frac{X'(\omega)}{X(\omega)} e^{i\omega}.$$

Now taking the inverse transform, we get

$$y_k = \frac{i}{2\pi} \int_{-\pi}^{\pi} \frac{X'(\omega)}{X(\omega)} e^{i\omega(k+1)} d\omega \tag{6}$$

and

$$|\, y_k| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{X'(\omega)}{X(\omega)} \right| d\omega.$$

Note that

$$X(\omega) := \sum_{k=0}^{\infty} \frac{\langle g^k \rangle_{t,\Lambda}}{k!} e^{-ik\omega} = \left\langle \exp\left(g e^{-i\omega}\right) \right\rangle_{t,\Lambda}, \qquad \omega \in [-\pi, \pi].$$

Thus

$$\frac{X'(\omega)}{X(\omega)} = -ie^{i\omega}\frac{\left\langle g\exp\left(ge^{-i\omega}\right)\right\rangle_{t,\Lambda}}{\left\langle \exp\left(ge^{-i\omega}\right)\right\rangle_{t,\Lambda}}.$$

Because g has bounded derivatives, one can see that

$$\int_{-\pi}^{\pi}\left|\frac{X'(\omega)}{X(\omega)}\right| d\omega = \mathcal{O}\left(|\Lambda|\right).$$

See Proposition 1.7 in [2].

We have the following result:

Proposition 3 *Suppose that the Hamiltonian Φ_Λ satisfies assumptions 1-4 and that g satisfies (1) and is g selected so that the sequence $\left(\frac{\langle g^k\rangle_{t,\Lambda}}{k!}\right)_k$ will be of temperate growth and that*

$$X(\omega) := \sum_{k=0}^{\infty}\frac{\left\langle g^k\right\rangle_{t,\Lambda}}{k!}e^{-ik\omega}, \qquad \omega\in[-\pi,\pi]$$

will not vanish on $[-\pi,\pi]$. There exist $T^*>0$ such that the infinite volume pressure $P(t)$ is $k-$times continuously differentiable on the interval $(0,T^*)$. Moreover for $k\geq 1$,

$$\left|P^{(k)}(t)\right| \leq C\,(k-1)!.$$

Proof. The result follows from Theorem 1 and equation (6). ■

Recall that a function f is real analytic on an open interval I if and only if f is real smooth and for every compact set $K\subset I$ there exists a constant C such that for every $t\in K$ and every non-negative integer k the following bound holds

$$\left|\frac{d^k f}{dt^k}(t)\right| \leq C^{k+1}k!.$$

Corollary 4 *Under the assumptions of Proposition 3, there exist $T^*>0$ such that the infinite volume pressure $P(t)$ is analytic on the interval $(0,T^*)$.*

4.1 Comparison With Previous Results

There are various ways of proving the infinite differentiability or the analyticity of the pressure for (ferromagnetic and non ferromagnetic) systems at high temperatures, or at low temperatures, or at large external fields. Most of these take advantage of a sufficiently rapid decay of correlations and /or cluster expansion methods. Here is a small sample of relevant references. Bricmont, Lebowitz and Pfister [21], Dobrushin [22], Dobrushin and Sholsman [23], [24]....

Recall that our method is suitable for investigating classical models whose Hamiltonians are of the form

$$H_{\Lambda,\gamma}\left(\sigma_\Lambda/\sigma_{\Lambda^c}\right) = -\frac{1}{2}\sum_{i,j\in\Lambda}J_\gamma(i,j)\sigma_i\sigma_j - \sum_{i\in\Lambda,j\in\Lambda^c}J_\gamma(i,j)\sigma_i\sigma_j,$$

where Λ is a finite subset of $\mathbb{Z}^2$, $\sigma_\Lambda = (\sigma_i)_{i\in\Lambda} \in \{-1,1\}^\Lambda$ with boundary condition $\sigma_{\Lambda^c} = (\sigma_i)_{i\in\Lambda^c}$.

Indeed, Marc Kac showed in [1] that when

$$J(r) = e^{-|r|},$$

this model may be studied through the transfer operator

$$K_\gamma^m = e^{-\frac{1}{2}\gamma q(x)} e^{\gamma \mathbf{\Delta}_m} e^{-\frac{1}{2}\gamma q(x)},$$

where

$$\gamma q(x) = \frac{1}{2}\tanh\left(\frac{\gamma}{2}\right)\sum_{i=1}^m x_i^2 - \sum_{i=1}^m \log\cosh\left[\sqrt{\frac{\gamma\beta}{2}}\,(x_i + x_{i+1})\right],$$

with the convention $x_{m+1} = x_1$. He proved that when γ approaches 0, the behavior of the system only depends on the Kac potential

$$q(x) = \sum_{i=1}^m \frac{x_i^2}{4} - \sum_{i=1}^m \log\cosh\left[\sqrt{\frac{\beta}{2}}\,(x_i + x_{i+1})\right].$$

Thus by reducing the two dimensional problem into a one dimensional problem, M. Kac showed that the critical temperature occurs at $\beta_c = \dfrac{1}{4}$.

The d-dimensional mean field Kac Hamiltonian

$$\Phi(x) = \frac{x^2}{2} - 2\sum_{i\sim j} \ln\cosh\left[\sqrt{\frac{\beta}{2}}\,(x_i + x_j)\right].$$

satisfies assumptions 1-4 above if $\beta < \dfrac{1}{4d}$.

Indeed, let

$$\Phi(x) = \frac{x^2}{2} + \Psi(x),$$

where

$$\Psi(x) = -2\sum_{i,j\in\Lambda, i\sim j} \ln\cosh\left[\sqrt{\frac{\beta}{2}}\,(x_i + x_j)\right].$$

We have

$$\Psi_{x_i} = -2\sum_{j:,j\sim i} \frac{\sqrt{\frac{\beta}{2}}\sinh\left[\sqrt{\frac{\beta}{2}}\,(x_i + x_j)\right]}{\cosh\left[\sqrt{\frac{\beta}{2}}\,(x_i + x_j)\right]}$$

$$\Psi_{x_i x_k} = \begin{cases} -\beta \sum_{j:,j\sim i} \dfrac{1}{\cosh^2\left[\sqrt{\frac{\beta}{2}}\,(x_i + x_j)\right]} & \text{if } k = i \\ -\dfrac{\beta}{\cosh^2\left[\sqrt{\frac{\beta}{2}}\,(x_i + x_k)\right]} & \text{if } k \sim i \\ 0 & \text{otherwise.} \end{cases}$$

It then follows that

$$|\Psi_{x_i}| \leq 4d\sqrt{\frac{\beta}{2}},$$
$$|\Psi_{x_ix_i}| \leq 2d\beta,$$

and

$$|\Psi_{x_ix_k}| \leq \beta \qquad \text{if } k \sim i.$$

Similarly, using the properties of cosh and sinh and the fact that $\sinh t \leq \cosh t$ for all t one can see that all derivatives of order greater than or equal to one are bounded.

Now we propose to find the values of β for which assumption 4 holds. i.e. there exist $w > 0, C > 0$ such that $x \cdot \nabla\Phi \geq C\,|x|^{1+w}$ for all $|x| \geq \frac{1}{C}$.

First write

$$\begin{aligned}
\Psi_{x_i} &= \int_0^1 \frac{d}{ds}\Psi_{x_i}(sx)ds = \int_0^1 x \cdot \boldsymbol{\nabla}\Psi_{x_i}(sx)ds \\
&= \sum_{j\in\Lambda}\int_0^1 \Psi_{x_ix_j}(sx)x_jds.
\end{aligned}$$

Thus

$$x \cdot \nabla\Psi = \sum_{i,j\in\Lambda} c_{ij}x_ix_j,$$

where

$$c_{ij} = \int_0^1 \Psi_{x_ix_j}(sx)ds.$$

There is a Schur's Lemma (see[25]) that says for each rectangular array

$$(c_{ij})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$$

and each pair of sequence $(x_i)_{1\leq i\leq m}$ and $(y_j)_{1\leq j\leq n}$ we have the bound

$$\left|\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_iy_j\right| \leq \sqrt{RC}\left(\sum_{i=1}^{m}|x_i|^2\right)^{1/2}\left(\sum_{j=1}^{n}|y_j|^2\right)^{1/2}$$

where R and C are the row sum and column sum maxima defined by

$$R = \max_i \sum_{j=1}^{n}|c_{ij}| \qquad \text{and} \qquad C = \max_j \sum_{i=1}^{m}|c_{ij}|\,.$$

Now using this result , we have

$$\begin{aligned}
\sum_{j\in\Lambda}|c_{ij}| &\leq \sum_{j\in\Lambda}\int_0^1 \left|\Psi_{x_ix_j}\right|(sx)ds \\
&= \int_0^1 |\Psi_{x_ix_i}|\,(sx)ds + \sum_{j\in\Lambda, j\sim i}\int_0^1 \left|\Psi_{x_ix_j}\right|(sx)ds \\
&\leq 2d\beta + 2d\beta.
\end{aligned}$$

Thus

$$R \leq 4d\beta.$$

Similarly, we have

$$C \leq 4d\beta,$$

and

$$|x \cdot \nabla\Psi| \leq 4d\beta \, |x|^2 \, .$$

It then follows that

$$x \cdot \boldsymbol{\nabla}\Phi \geq |x|^2 \, (1 - 4d\beta) \, .$$

The assumption is satisfied if

$$\beta < \frac{1}{4d}.$$

In [17], the authors derived hypotheses on the decrease of the truncated functions to show that they do yield analyticity. They considered models whose Hamiltonians are given by

$$H_\Lambda = -\frac{1}{2} \sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - h \sum_{i\in\Lambda,} \sigma_i$$

$$Z_\Lambda = \sum_{\sigma} {}^{\Lambda} \exp\left(-\beta H_\Lambda(\sigma, h)\right)$$

$$P_\Lambda = \frac{1}{|\Lambda|} \log Z_\Lambda$$

with the thermodynamic parameter being β and $\tilde{h} = \beta h$.

After proving that

$$\frac{\partial^{q+r}}{\partial\tilde{h}^q \partial\beta^r} P_\Lambda(\beta, \tilde{h}) = \frac{1}{|\Lambda|} \sum_{\substack{i_1,\dots,i_q\in\Lambda \\ \gamma_1,\dots,\gamma_r\in\Lambda^2}} \left(\prod_{j=1}^{r} J_{\gamma_j}\right) \left\langle \sigma_{i_1,\dots,i_q,\gamma_1,\dots,\gamma_r} \right\rangle^{T_{q,r}} (\beta, \tilde{h}) \qquad \text{(See notations in [17])},$$

The authors showed that if the following bound

$$\left| \left\langle \sigma_{A_1,\dots,A_M} \right\rangle^{T(A_1,\dots,A_M)} (\beta_0, \tilde{h}_0) \right| < \prod_{i=1}^{M} \chi\left(|A_i|\right) \times \sum_{\mathcal{F}}^{(A_1,\dots,A_M)} \prod_{l=1}^{M-1} {}^{F} u\left(d\left(A_l', A_l''\right)\right)$$

where $A_1, \dots, A_M$ are finite subsets of $\mathbb{Z}^\nu$, u is integrable $\left(\sum_{i\in\mathbb{Z}^\nu} u\left(|i|\right) < \infty\right)$ (Again see notations in [17]) holds uniformly for any region Λ, then the infinite volume pressure $\lim_{|\Lambda|\to\infty} P_\Lambda$ is analytic at $(\beta_0, \tilde{h}_0)$. Analogous results in classical continuous models are also derived in [17].

In our case , If we take

$$g(x) = \sum_{i\in\Gamma} x_i \ ,$$

where $\Gamma \subsetneq \Lambda$ fixed.

We have

$$P_\Lambda(t) = \frac{1}{|\Lambda|} \log \left[\int_{\mathbb{R}^\Lambda} dx e^{-\Phi(x)+t \sum_{i\in\Gamma} x_i} \right]$$

$$P'_\Lambda(t) = \frac{1}{|\Lambda|} \sum_{i_1\in\Gamma} \langle x_{i_1} \rangle_{t,\Lambda},$$

$$\begin{aligned} P''_\Lambda(t) &= \frac{1}{|\Lambda|} \sum_{i_1\in\Gamma} \mathbf{cov}\left(x_{i_1}, \sum_{i_2\in\Gamma} x_{i_2} \right) \\ &= \frac{1}{|\Lambda|} \sum_{i_1,i_2\in\Gamma} \mathbf{cov}\left(x_{i_1}, x_{i_2} \right) \\ &= \frac{1}{|\Lambda|} \sum_{i_1,i_2\in\Gamma} \left\langle \left(x_{i_1} - \langle x_{i_1} \rangle_{t,\Lambda} \right) \left(x_{i_2} - \langle x_{i_2} \rangle_{t,\Lambda} \right) \right\rangle_{t,\Lambda} \end{aligned}$$

$$P_\Lambda^{(k)}(t) = \frac{1}{|\Lambda|} \sum_{i_1,i_2,\dots,i_k\in\Gamma} \langle x_{i_1,i_2,\dots i_k} \rangle^T_{\Lambda,t}, \quad \text{for } k \geq 2.$$

where $\langle x_{i_1,i_2,\dots i_k} \rangle^T_{\Lambda,t}$ are the usual truncated correlations (see [17]).

Rather than dealing with the truncated correlations which turn out to be heavier, we simply obtain analyticity of the pressure under the condition

$$\int_{-\pi}^{\pi} \left| \frac{X'(\omega)}{X(\omega)} \right| d\omega = \mathcal{O}(|\Lambda|)$$

where

$$X(\omega) := \sum_{k=0}^{\infty} \frac{\left\langle \left(\sum_{i\in\Gamma} x_i \right)^k \right\rangle_{t,\Lambda}}{k!} e^{-ik\omega}, \qquad \omega \in [-\pi, \pi].$$

As an important corollary to Proposition 3, we have the following cluster property

Corollary 5 *Any classical unbounded models of Kac type whose Hamiltonian satisfies assumptions 1-4 has the following precise cluster property:*

$$\frac{1}{|\Lambda|} \left| \sum_{i_1,i_2,\dots,i_n\in\Gamma} \langle x_{i_1,i_2,\dots i_k} \rangle^T_\Lambda \right| \leq C\,(k-1)!$$

for each $\Gamma \subsetneq \Lambda$.

Remark 6 *It was even mentioned in [15] that one has not been able to get precise estimates of the truncated correlations through the Helffer-Sjöstrand and Witten Laplacian technique. The result of this corollary above may be viewed as a first attempt in this direction.*

5 On The Logarithmic Sobolev Inequality.

In [13,14] the authors studied the existence of uniform logarithmic Sobolev inequalities using Zegarlinski criterion. Because of the difficulty in having a uniform lower bound for the spectrum of the Witten Laplacian, they considered models whose Hamiltonian are of the form

$$\Phi_\Lambda(X) = \sum_{j\in\Lambda} \phi(x_j) + \frac{\mathcal{J}}{2} \sum_{\substack{(\{i\}\cup\{k\})\cap\Lambda\neq\varnothing\,, \\ j\sim k}} |z_j - z_k|^2$$

under a condition of strict convexity at ∞ on ϕ. The authors first discussed uniform estimates for a family of 1-dimensional Witten Laplacians and explained how the result may be generalized to higher dimensions. In [13], Helffer and Bodineau gave a proof of the Log-Sobolev inequality for similar models but under weaker assumptions on ϕ.

In this paper, we shall consider classical continuous models whose Hamiltonians satisfy assumptions 1-4 above. This is a generalized version of the type of Hamiltonians used in [13] and [14].

We shall discuss a direct method for proving uniform decay of correlations without using the one-dimensional cases as discussed in [13] and [14]. As a consequence, we shall give a proof of the logarithmic Sobolev inequality that does not use the one dimensional Witten Laplacians.

5.1 The Decay Of Correlations

Definition 7 *The lattice support, S_g of a function g on $\mathbb{R}^\Lambda$ is defined here to be the smallest subset Γ of $\Lambda \subset \mathbb{Z}^d$ for which g can be written as function of x_j alone with $j \in \Gamma$ For instance, if $g = x_i$; $S_g = \{i\}$.*

Recall (Theorem 1.7 -[6]) that if P is invertible and self-adjoint, the positivity of $A^{(1)}_\Phi$ is equivalent to the positivity of $PA^{(1)}_\Phi P$. If $\mathcal{P}_\Lambda$ denote by the space of all $|\Lambda| \times |\Lambda|$ diagonal matrices with positive diagonal entries.

$A^{(1)}_\Phi$ having a discrete spectrum see (Theorem 1.7 -[6]), would imply that $PA^{(1)}_\Phi P$ has a discrete spectrum for any $P \in \mathcal{P}_\Lambda$. Put

$$\rho_{A^{(1)}_\Phi} := \inf_{P\in\mathcal{P}_\Lambda} \lambda_{\min}\left[PA^{(1)}_\Phi P\right].$$

where $\lambda_{\min}$ is the lowest eigenvalue of $PA^{(1)}_\Phi P$.

Proposition 8 *Let g and h be smooth functions on $\mathbb{R}^\Gamma$,and $\mathbb{R}^{\Gamma'}$ where Γ and $\Gamma' \subsetneqq \Lambda$ with $\Gamma\cap\Gamma' = \varnothing$ denote respectively the support of g and h and assume that g and h satisfy* (1). *If the Hamiltonian Φ satisfy the assumptions 1-4 above and*

$$\rho = \rho_{A^{(1)}_\Phi} > 0$$

then

$$|\mathbf{cov}(g,h)| < Ce^{-\kappa d(S_h,S_g)}$$

where C and κ are positive constants that do not depend on Λ, but possibly dependent on the size of the supports of g and h.

Proof. It follows from Theorem 1.7 -[6] (See also [4]) that the operator

$$A_\Phi^{(1)} := -\boldsymbol{\Delta} + \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla} + \mathbf{Hess}\Phi$$

is positive on $L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx)$. i.e. there exist $c_1 > 0$ such that

$$A_\Phi^{(1)} \geq c_1.$$

Moreover, we have the formula

$$\mathbf{cov}(g,h) = Z^{-1} \int\limits_{\mathbb{R}^\Lambda} \left(A_\Phi^{(1)^{-1}} \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} h \right) e^{-\Phi(x)} dx. \tag{7}$$

where

$$Z = \int\limits_{\mathbb{R}^\Lambda} e^{-\Phi(x)} dx.$$

Let M be the diagonal matrix defined by

$$M = \left(\frac{\rho+1}{\rho} e^{\kappa d(i,S_g)} \right)_{i \in \Lambda} (\kappa > 0),$$

where $\rho = \rho_{A_\Phi^{(1)}} > 0$ and consider the equation

$$A_\Phi^{(1)} \boldsymbol{\nabla} f = \boldsymbol{\nabla} g.$$

Multiplying both sides by M^{-1}, we obtain

$$M^{-1} A_\Phi^{(1)} \boldsymbol{\nabla} f = M^{-1} \boldsymbol{\nabla} g$$

$\Longleftrightarrow$

$$M^{-1} A_\Phi^{(1)} M^{-1} M \boldsymbol{\nabla} f = M^{-1} \boldsymbol{\nabla} g.$$

Because M^{-1} is invertible and self-adjoint, the positivity of $A_\Phi^{(1)}$ is equivalent to the positivity of $M^{-1} A_\Phi^{(1)} M^{-1}$. Taking inner product with $M\boldsymbol{\nabla} f$ on both sides of this last equality and integrating with respect to $Z^{-1}e^{-\Phi}dx$, we obtain

$$\left\langle M^{-1} A_\Phi^{(1)} M^{-1} M \boldsymbol{\nabla} f, M \boldsymbol{\nabla} f \right\rangle_\Lambda = \left\langle M^{-1} \boldsymbol{\nabla} g, M \boldsymbol{\nabla} f \right\rangle_\Lambda.$$

Now using the positivity of the operator $M^{-1} A_\Phi^{(1)} M^{-1}$ on the left hand side and Cauchy Schwartz inequality on the right hand side, we get

$$\rho \left\| M \boldsymbol{\nabla} f \right\|_\Phi^2 \leq \left\| M^{-1} \boldsymbol{\nabla} g \right\|_\Phi \left\| M \boldsymbol{\nabla} f \right\|_\Phi,$$

where $\rho = \rho_{A_\Phi^{(1)}} > 0$ and $\|u\|_\Phi := \left(Z^{-1} \int\limits_{\mathbb{R}^\Lambda} |u|^2 e^{-\Phi(x)} dx \right)^{1/2}$. Due to formula (5), we have: if $\|M\nabla f\|_\Phi = 0$, then $A_\Phi^{(1)^{-1}} \nabla g = 0$ and the result follows. If $\|M\nabla f\|_\Phi \neq 0$, then we have

$$\rho \|M\nabla f\|_\Phi \leq \left\|M^{-1}\nabla g\right\|_\Phi,$$

$\Leftrightarrow$

$$\rho^2 \left(\frac{\rho+1}{\rho}\right)^2 \int\limits_{\mathbb{R}^\Lambda} \sum_{i\in\Lambda} e^{2\kappa d(i,S_g)} f_{x_i}^2 e^{-\Phi(x)} dx \leq \left(\frac{\rho}{\rho+1}\right)^2 \int\limits_{\mathbb{R}^\Lambda} \sum_{i\in\Lambda} e^{-2\kappa d(i,S_g)} g_{x_i}^2 e^{-\Phi(x)} dx.$$

$\Rightarrow$

$$\begin{aligned} \int\limits_{\mathbb{R}^\Lambda} \textstyle\sum_{i\in\Lambda} e^{2\kappa d(i,S_g)} f_{x_i}^2 e^{-\Phi(x)} dx &\leq \left(\tfrac{\rho}{\rho+1}\right)^2 \int\limits_{\mathbb{R}^\Lambda} \textstyle\sum_{i\in\Lambda} e^{-2\kappa d(i,S_g)} g_{x_i}^2 e^{-\Phi(x)} dx \\ &< \int\limits_{\mathbb{R}^\Lambda} \textstyle\sum_{i\in\Lambda} e^{-2\kappa d(i,S_g)} g_{x_i}^2 e^{-\Phi(x)} dx. \end{aligned} \tag{8}$$

Now using the fact that $g_{x_i} = 0$ if $i \notin S_g$, $d(i,S_g) = 0$ if $i \in S_g$ and (1), we obtain

$$\int\limits_{\mathbb{R}^\Lambda} \sum_{i\in\Lambda} e^{2\kappa d(i,S_g)} f_{x_i}^2 e^{-\Phi(x)} dx < ZC_g,$$

where C_g is a positive constant that only depends on the size of the support of g. Thus we finally get

$$Z^{-1} \int\limits_{\mathbb{R}^\Lambda} \sum_{i\in\Lambda} e^{2\kappa d(i,S_g)} f_{x_i}^2 e^{-\Phi(x)} dx < C_g.$$

Now we use formula (7) to get

$$
\begin{aligned}
&|\mathbf{cov}(g,h)| = Z^{-1}\left|\int\limits_{\mathbb{R}^\Lambda} \boldsymbol{\nabla} f\cdot \boldsymbol{\nabla} h e^{-\Phi(x)}dx\right| \\
\leq\ & Z^{-1}\int\limits_{\mathbb{R}^\Lambda}\sum_{i\in\Lambda}\left|f_{x_i}(x)e^{\kappa d(i,S_g)}e^{-\kappa d(i,S_g)}h_{x_i}\right|e^{-\Phi(x)}dx \\
\leq\ & Z^{-1}\int\limits_{\mathbb{R}^\Lambda}\left(\sum_{i\in\Lambda} f_{x_i}^2(x)e^{2\kappa d(i,S_g)}\right)^{1/2}\left(\sum_{i\in S_h} h_{x_i}^2(x)e^{-2\kappa d(i,S_g)}\right)^{1/2} e^{-\Phi(x)}dx \\
\leq\ & Z^{-1}\left[\int\limits_{\mathbb{R}^\Lambda}\sum_{i\in\Lambda} f_{x_i}^2(x)e^{2\kappa d(i,S_g)}d\mu^\Lambda(x)\right]^{1/2}\left[\int\limits_{\mathbb{R}^\Lambda}\sum_{i\in S_h} h_{x_i}^2(x)e^{-2\kappa d(i,S_g)}d\mu^\Lambda(x)\right]^{1/2} \\
<\ & \sqrt{C_g}\left[Z^{-1}\int\limits_{\mathbb{R}^\Lambda}\sum_{i\in S_h} h_{x_i}^2(x)d\mu^\Lambda(x)\right]^{1/2} e^{-\kappa d(S_h,S_g)} \\
<\ & \sqrt{C_g}\sqrt{C_h}e^{-\kappa d(S_h,S_g)}.
\end{aligned}
$$

Here C_g and C_h are independent of Λ. They only depend on the size of the support of g and h respectively. ■

From the of this one can easily obtain the following mixing condition:

Corollary 9 *If the Hamiltonian Φ satisfy the assumptions 1-4 above and*

$$\rho = \rho_{A_\Phi^{(1)}} > 0$$

$$|\mathbf{cov}(g,h)| \leq e^{-\kappa d(S_h,S_g)}\,\|\boldsymbol{\nabla} g\|_\Phi\,\|\boldsymbol{\nabla} h\|_\Phi \tag{9}$$

Observe that $g = h$, gives a spectral gap greater than 1.

Corollary 10 *If the Hamiltonian Φ satisfy the assumptions 1-4 above and*

$$\rho = \rho_{A_\Phi^{(1)}} > 0$$

then

$$|\mathbf{cov}(x_i,x_j)| < Ce^{-\kappa d(i,j)}$$

where C and κ are positive constants that do not depend on Λ.

Proof. Take $g = x_i$ and $h = x_j$. ■

5.2 The Log-Sobolev Inequality

We shall now consider the case where the Hamiltonian is given by

$$\Phi^{\Lambda,\omega}(x) = \sum_{j\in\Lambda}\phi(x_j) + \frac{\mathcal{J}}{2}\sum_{\substack{(\{i\}\cup\{k\})\cap\Lambda\neq\varnothing,\\ j\sim k}} |z_j - z_k|^2$$

where

- $x = (x_i)_{i\in\Lambda}$, $\omega = (\omega_j)_{j\in\mathbb{Z}^d\setminus\Lambda}$, $\mathcal{J} \geq 0$,
- ϕ is a one particle phase on $\mathbb{R}$ with at least quadratic increase at ∞,
- $$\begin{array}{ll} z_j = x_j & \text{if } j\in\Lambda \\ z_j = \omega_j & \text{if } j\notin\Lambda \end{array}$$

We shall assume that ϕ is C^2 on $\mathbb{R}$ and convex at ∞. i.e. There exists $C > 0$ such that

$$\mathbf{Hess}\phi(x) \geq \frac{1}{C} \text{ for all } x\in\mathbb{R} \text{ such that } |x| \geq C.$$

It is clear that for each ω, $\Phi^{\Lambda,\omega}(x)$ satisfies the assumptions 1-4 above. For instance, when

$$\phi(x) = \frac{\lambda}{12}x^4 + \frac{\nu}{2}x^2$$

and

$$\Phi(x) = \frac{1}{h}\sum_{j=1}^{n}\left(\frac{\lambda}{12}x_j^4 + \frac{\nu}{2}x_j^2\right) + \frac{1}{h}\frac{\mathcal{J}}{2}\sum_{j=1}^{n}|x_j - x_{j+1}|^2 ,$$

where $x_{n+1} = x_1$, $h > 0$, $\mathcal{J} > 0$, and $\lambda > 0 > \nu$. These models are commonly used in Euclidean field theory. They have unbounded second derivatives and satisfy assumptions 1-4. Indeed, first observe that there exists $j \in \{1, ..., n\}$ such that $x_j \geq \frac{|x|}{\sqrt{n}}$ otherwise one would have $|x|^2 < |x|^2$. Using this, it is clear that

$$\begin{aligned} x\cdot\nabla\Phi &\geq \frac{1}{h}\sum_{j=1}^{n}\left(\frac{\lambda}{3}x_j^4 + \nu x_j^2\right) \\ &= \frac{|x|^4}{h}\left[\frac{\lambda}{3}\left(\frac{1}{\sqrt{n}}\right)^4 - |\nu|\,|x|^{-2}\right] \\ &\geq \frac{|x|^4}{C} \text{ when } |x| \geq C \text{ for some sufficiently large } C. \end{aligned}$$

Proposition 11 *Proposition 8, corollary 9 and 10 will remain valid for Hamiltonians of the form*

$$\Phi^{\Lambda,\omega}(x) = \sum_{j\in\Lambda} \phi(x_j) + \frac{\mathcal{J}}{2} \sum_{\substack{(\{i\}\cup\{k\})\cap\Lambda\neq\varnothing, \\ j\sim k}} |z_j - z_k|^2$$

where ϕ is C^2 on $\mathbb{R}$ and convex at ∞.

Proof. We only have to worry about the dependency of the ρ on ω. Unlike the proof in [13] and [14] where lots of efforts were put on the uniform estimate of the lowest eigenvalue with respect to Λ and ω, our proof only requires ρ to be strictly positive. ■

As a consequence of the decay of correlations or the mixing condition, we have the following result.

Proposition 12 *Let*

$$\Phi^{\Lambda,\omega}(x) = \sum_{j\in\Lambda} \phi(x_j) + \frac{\mathcal{J}}{2} \sum_{\substack{(\{i\}\cup\{k\})\cap\Lambda\neq\varnothing, \\ j\sim k}} |z_j - z_k|^2$$

where ϕ is C^2 on $\mathbb{R}$ and convex at ∞. If

$$\rho = \rho_{A^{(1)}_{\Phi^{\Lambda,\omega}}} > 0,$$

then there exist constant C and $\mathcal{J}_0 > 0$ such that for $\mathcal{J} \in [-\mathcal{J}_0,\ \mathcal{J}_0]$ and for any cube $\Lambda \subset \mathbb{Z}^d$, we have

$$\langle f \ln f\rangle_\Lambda - \langle f\rangle_\Lambda \ln \langle f\rangle_\Lambda \leq 2C \left\langle \left|\nabla f^{\ \frac{1}{2}}\right| \right\rangle_\Lambda$$

for all non-negative function f for which the right hand side is finite.

Remark 13 *In remark 2.5 in [14], the author mentioned the difficulty in getting a uniform estimate with respect to the boundary condition ω through the work of Bach-Jecko-Sjöstrand. Our result provides an alternative way for dealing with the dependency of spectrum of the Witten Laplacians on Λ and ω.*

5.3 Extending The Decay Result to Higher Correlations

Recall that in [7] E. Witten introduced the Witten derivative d_Φ and the Witten coderivative d^*_Φ by simply setting

$$d_\Phi = e^{-\Phi/2} d e^{\Phi/2} \qquad \text{and} \qquad d^*_\Phi = e^{\Phi/2} d^* e^{-\Phi/2}$$

where d and d^* are the exterior derivative and exterior coderivative respectively. The Witten Laplacian is then defined to be the associated second order operator

$$\begin{aligned} W_\Phi^{(k)} &= (d_\Phi + d_\Phi^*)^2 \\ &= d_\Phi d_\Phi^* + d_\Phi^* d_\Phi. \end{aligned}$$

acting on $k-$forms.

We shall now consider the operators $A_\Phi^{(k)}$ given by

$$A_\Phi^{(k)} = e^{\Phi/2} \circ W_\Phi^{(k)} \circ e^{-\Phi/2}$$

acting on the weighted spaces $L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx, \Lambda^k\mathbb{R}^\Lambda)$ (see [6] for more details), the space of $k-$smooth forms with coefficients in $L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx)$. The norm on this space is defined by

$$\begin{aligned} &\left\| \sum_{i_1<...<i_k} f_{i_1...i_k} dx^{i_1}...dx^{i_k} \right\|_\Phi \\ = &\left(Z^{-1} \int_{\mathbb{R}^\Lambda} \sum_{i_1...i_k} f_{i_1...i_k}^2(x) e^{-\Phi} dx \right)^{1/2}. \end{aligned}$$

If d_k denote the differential $k-$form operator and d_k^* its adjoint, we have

$$d_k A_\Phi^{(k)} = A_\Phi^{(k+1)} d_k. \tag{10}$$

This equality is a higher order version of

$$\nabla A_\Phi^{(0)} = A_\Phi^{(1)} \nabla$$

which is obtained when identifying $0-$forms with functions and $1-$forms with vector fields.

Now assume that the Hamiltonian $\Phi_\Lambda = \Phi$ is such that the operator $A_\Phi^{(k)}$ is positive on $L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx, \Lambda^k\mathbb{R}^\Lambda)$ $(k \geq 1)$.

Let $\mathcal{P}_\Lambda$ be the set of all multiplication operators P_ξ on $L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx, \Lambda^k\mathbb{R}^\Lambda)$ in the form

$$P_\xi \sum_{i_1<...<i_k} f_{i_1...i_k} dx^{i_1}...dx^{i_k} := \sum_{i_1<...<i_k} \xi(i_1,...,i_k) f_{i_1...i_k} dx^{i_1}...dx^{i_k}$$

where ξ is positive.

Put

$$\rho_k = \rho_{A_\Phi^{(k)}} := \inf_{P \in \mathcal{P}_\Lambda} \lambda_{\min} \left[P A_\Phi^{(k)} P \right].$$

where $\lambda_{\min}$ is the lowest eigenvalue of $P A_\Phi^{(k)} P$.

Proposition 14 *If in addition to the assumption 1-4 the Hamiltonian* Φ *is such that* $A_\Phi^{(k)}$ *is positive on* $L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx, \Lambda^k\mathbb{R}^\Lambda)$, $k \geq 1$, *then for any* g *satisfying* (1), *we have*

$$Z^{-1}\int\limits_{\mathbb{R}^\Lambda}\sum_{i_1\dots i_k} f^2_{x_{i_1}\dots x_{i_k}}(x)e^{2\kappa d(\{i_1,\dots,i_k\},S_g)}e^{-\Phi}dx \leq C_k \tag{11}$$

where C_k *is a positive constant that is independent of* Λ.

Proof. Consider the multiplication operator M_k defined on $L^2(\mathbb{R}^\Lambda, e^{-\Phi}dx, \Lambda^k\mathbb{R}^\Lambda)$ by

$$M_k\sum_{i_1<\dots<i_k} f_{i_1\dots i_k}dx^{i_1}\dots dx^{i_k} := \sum_{i_1<\dots<i_k}\frac{\rho_k+1}{\rho_k}e^{\kappa d(\{i_1,\dots,i_k\},S_g)}f_{i_1\dots i_k}dx^{i_1}\dots dx^{i_k}.$$

This operator is consequently self-adjoint and invertible. Let f be a smooth solution of

$$A_\Phi^{(0)}f = g - \langle g\rangle_{L^2(\mu^\Lambda)} \text{ in } \mathbb{R}^\Lambda.$$

Using (10), we have

$$A_\Phi^{(k)}\nabla^k f = \nabla^k g$$

where $\nabla^k u$ denote the $k-$order Hessian of u. Multiplying both sides by M_k^{-1} and taking inner product with $M_k\nabla^k f$, we obtain

$$\left\langle M_k^{-1}A_\Phi^{(k)}M_k^{-1}M_k\nabla^k f, M_k\nabla^k f\right\rangle = \left\langle M_k^{-1}\nabla^k g, M_k\nabla^k f\right\rangle.$$

It then follows from the positivity of the operator $M_k^{-1}A_\Phi^{(k)}M_k^{-1}$ and Cauchy-Schwartz inequality that

$$\rho_k\left\|M_k\nabla^k f\right\|_\Phi^2 \leq \left\|M_k^{-1}\nabla^k g\right\|_\Phi\left\|M_k\nabla^k f\right\|_\Phi. \tag{12}$$

If $\left\|M_k\nabla^k f\right\|_\Phi = 0$, then there is nothing to prove. However if $\left\|M_k\nabla^k f\right\|_\Phi \neq 0$, we have

$$\begin{aligned}&\int\limits_{\mathbb{R}^\Lambda}\sum_{i_1\dots i_k}\rho_k^2\left(\frac{\rho_k+1}{\rho_k}\right)^2 f^2_{x_{i_1}\dots x_{i_k}}(x)e^{2\kappa d(\{i_1,\dots,i_k\},S_g)}e^{-\Phi}dx \\ \leq &\int\limits_{\mathbb{R}^\Lambda}\sum_{i_1\dots i_k}\left(\frac{\rho_k}{\rho_k+1}\right)^2 g^2_{x_{i_1}\dots x_{i_k}}(x)e^{-2\kappa d(\{i_1,\dots,i_k\},S_g)}e^{-\Phi}dx.\end{aligned}$$

We have

$$\begin{aligned}\int\limits_{\mathbb{R}^\Lambda}\sum_{i_1\ldots i_k} f^2_{x_{i_1}\ldots x_{i_k}}(x)e^{2\kappa d(\{i_1,\ldots,i_k\},S_g)}e^{-\Phi}dx &\le \int\limits_{\mathbb{R}^\Lambda}\sum_{i_1\ldots i_k}\rho_k^2\left(\frac{\rho_k+1}{\rho_k}\right)^2 f^2_{x_{i_1}\ldots x_{i_k}}(x)e^{2\kappa d(\{i_1,\ldots,i_k\},S_g)}e^{-\Phi}dx \\ &\le \int\limits_{\mathbb{R}^\Lambda}\sum_{i_1\ldots i_k}\left(\frac{\rho_k}{\rho_k+1}\right)^2 g^2_{x_{i_1}\ldots x_{i_k}}(x)e^{-2\kappa d(\{i_1,\ldots,i_k\},S_g)}e^{-\Phi}dx \\ &\le \int\limits_{\mathbb{R}^\Lambda}\sum_{i_1\ldots i_k} g^2_{x_{i_1}\ldots x_{i_k}}(x)e^{-2\kappa d(\{i_1,\ldots,i_k\},S_g)}e^{-\Phi}dx \\ &\le C_{k,g}.\end{aligned}$$

where $C_{k,g}$ is a positive constant that only depends on the support of g. ■

The higher order correlation is defined as

$$\langle g_1,...,g_k\rangle := \langle (g_1-\langle g_1\rangle)\ldots(g_k-\langle g_k\rangle)\rangle\,. \tag{13}$$

Using the estimate above for the higher order Hessians of the solution f and following the same argument as in the proof of proposition 11, one can easily see that

$$|\langle g_1,g_2,g_3\rangle| \le C\left[e^{-\kappa_1 d\left(S_{g_2},S_{g_1}\right)}+e^{-\kappa_1 d\left(S_{g_3},S_{g_1}\right)}\right]$$

If $g_1=x_i$, $g_2=x_j$, and $g_3=x_k$, we obtain

$$|\langle (x_i-\langle x_i\rangle)(x_j-\langle x_j\rangle)(x_k-\langle x_k\rangle)\rangle| \le C\left[e^{-\kappa_1 d(i,j)}+e^{-\kappa_1 d(i,k)}\right].$$

Note that in the one dimensional case, we obtain a stronger exponential decay in the sense that

$$d(i,j)\to\infty \Rightarrow d(i,k)\to\infty.$$

Indeed we have

$$i\le j\le k \Longrightarrow d(i,k)=d(i,j)+d(j,k)\ge d(i,j).$$

However, this is not the case in higher dimensions. Thus if $d>1$, we obtain this weak exponential decay of the correlations in the sense that the exponential decay occurs as you simultaneously pull the spins away from a fixed one.

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Reference

1. Kac, M., Mathematical mechanism of phase transitions. Gordon & Breach, New York (1966).

2. Helffer. B and Sjöstrand. J, *On the correlation for Kac-like models in the convex case. J. of Stat. phys, 74 Nos.1/2, (1994).*

3. Lo, A., *Witten laplacian methods for the decay of correlations. Journal of Statistical Physics, Volume 132, Number 2 / July, Pages 355-396, 2008.*

4. Sjöstrand. J., *Correlation asymptotics and Witten laplacians, Algebra and Analysis 8, no. 1, 160-191, (1996).*

5. Lo, A., *On the exponential decay of the n-point correlation functions and the analyticity of the pressure, J. Math. Phys.* 48, 123506, (2007).

6. Jon, J., *On spectral properties of Witten-Laplacians, their range of projections and Brascamp-Lieb inequality* Integral Equations Operator Theory 36 (3) 288–324, (2000).

7. Witten, E., *Supersymmetry and Morse theory, J. of Diff. Geom. 17, 661-692, (1982).*

8. Lebowitz, j. L., Bounds on the correlations and analyticity properties of ferromagnetic Ising spin systems Communications in Mathematical Physics Volume 28, Number 4 , 313-321, DOI: 10.1007 (1972).

9. Brascamp, H. and Lieb, E. H., *On extensions of the Brunn-Minkowski and Prekopa-Leindler theorems including inequalities for log concave functions, and with application to the diffusion equation, J. Funct. Analysis, 22, 366-389 (1976).*

10. Bach, V., Jecko, T., Sjöstrand. J., Correlation asymptotics of classical lattice spin systems with nonconvex Hamiltonian function at low temperature. Ann. Henri Poincare 1, 59-100, (2000).

11. Bach, V., Møller, J.S., Correlation at low temperature I, exponential decay. J. Funct. Anal. 203, 93-148, (2003).

12. Bach, V., Møller, J.S., Correlation at low temperature II, asymptotics, *J. of Stat. phys, Vol 116, Nos 114, (2004).*

13. Bodineau, T., Helffer B., The Log-Sobolev Inequality for Unbounded Spin Systems. Journal of Functional Analysis 166:1, 168-178, (1999).

14. Helffer. B., Remarks on decay of correlations and Witten laplacians III. Application to logarithmic Sobolev inequalities, Ann. Inst. Henri Poincaré Vol 35, n^o 4 p. 483-508, (1999).

15. Lo, A., A short proof of the McCoy conjecture in higher-dimensional classical continuous models of Kac types Mod. Phys. Lett. B, 31, 1750111 (2017) [14 pages] .

16. Lo, A., "A direct method for the analyticity of the pressure for certain classical unbounded models," Advances in Mathematical Physics, vol. 1, no. Article 808276, Jan. 2011. 14 pages doi:10.1155/2011/808276.

17. M. Duneau, D. Iagolnitzer, and B. Souillard, “Decrease properties of truncated correlation functions and analyticity properties for classical lattices and continuous systems,” Communications inMathematical Physics, vol. 31, pp. 191–208, 1973.

18. Ruelle, D., Statistical Mechanics of a One-Dimensional Lattice Gas Commun. math. Phys. 9, 267–278 (1968) .

19. G. Gallavotti, S. Miracle-S. and Ruelle, D. Absence of phase transitions in one dimensional systems with hard-core, phys. Lett. 26A, 350 (1968).

20. G. Gallavotti, S. Miracle-S. Absence of phase transitions in hard-core one dimensional systems with long range interactions, J. Math. Phys. 11, 147, (1970).

21. Bricmont, J, Lebowitz, J.L, and Pfister, C.E: *Low Temperature Expansion for Continuous Spin Ising Models. Commun.Math.Phys. 78, (1980), 117-135.*

22. Dobrushin, R.L: *Induction on Volume and no Cluster expansion. In: M. Mebkhout and R. Seneor (eds), VIII. Internat. Congress on Mathematical Physics, Marseille 1986, Singapore: World Scientific, pp. 73-91.*

23. Dobrushin, R.L: *Induction on Volume and no Cluster expansion. In: M. Mebkhout and R. Seneor (eds), VIII. Internat. Congress on Mathematical Physics, Marseille 1986, Singapore: World Scientific, pp. 73-91.*

24. Dobrushin, R.L and Sholsmann, S.B: *Completely analytical interactions: Constructive Description. J.Stat.Phys. 46, (1987), 983-1014.*

25. The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities (Maa Problem Books Series ISBN-10: 052154677X ISBN-13: 978-0521546775 2004.

26. Rivasseau, V., Constructive field theory and applications: Perspectives and open problems Journal of Mathematical Physics 41, 3764 (2000).

27. Kac, M., A formula for the pressure in statistical mechanics Journal of Mathematical Physics 14, 583 (1973).

6 Appendix

6.1 Witten's Laplacians

In 1982, Edward Witten published an article [7] on Supersymmetry and Morse theory; to relate some invariants of a Riemannian manifold **M** with some indices of a Morse function $\Phi \in C^\infty(\mathbf{M})$, he introduced the Witten derivative $\mathbf{d}_\Phi$ and the Witten coderivative $\mathbf{d}^*_\Phi$ by simply setting

$$\mathbf{d}_\Phi = \mathbf{e}^{-\frac{\Phi}{2}}\mathbf{d}\mathbf{e}^{\frac{\Phi}{2}} \quad \text{and} \quad \mathbf{d}^*_\Phi = \mathbf{e}^{\frac{\Phi}{2}}\mathbf{d}^*\mathbf{e}^{-\frac{\Phi}{2}}$$

where $\mathbf{d}$ and $\mathbf{d}^*$ are the exterior derivative and exterior coderivative respectively. The Witten Laplacian is then defined to be the associated second order operator

$$\begin{aligned} \mathbf{W}_\Phi &= (\mathbf{d}_\Phi + \mathbf{d}^*_\Phi)^2 \\ &= \mathbf{d}_\Phi\mathbf{d}^*_\Phi + \mathbf{d}^*_\Phi\mathbf{d}_\Phi \end{aligned}$$

acting on the exterior algebra bundle of the cotangent bundle of M as the standard Laplacian does.

Choosing a local orthonormal frame field $\mathbf{e}_1, ..., \mathbf{e}_d$ and denoting by $\mathbf{e}^1, ..., \mathbf{e}^d$ its dual coframe field, $\mathbf{d}$ and $\mathbf{d}^*$ could be easily represented in terms of the Riemannian connection $\boldsymbol{\nabla}$ as

$$\mathbf{d} = \mathbf{e}^i \wedge \boldsymbol{\nabla}_{e_i} \quad \text{and} \quad \mathbf{d}^* = -\mathbf{i}(\mathbf{e}_j)\boldsymbol{\nabla}_{ej}$$

where $\mathbf{i}$ denote the interior product. Here and in the rest of this section, we use the Einstein summation convention namely, an index occurring twice in a product is to be summed from 1 up to the space dimension. We consequently have

$$\mathbf{d}_\Phi = \mathbf{e}^i \wedge \boldsymbol{\nabla}_{e_i} + \mathbf{e}^i \frac{\Phi_{;\,i}}{2} \quad \text{and} \quad \mathbf{d}^*_\Phi = -\mathbf{i}(\mathbf{e}_j)\boldsymbol{\nabla}_{e_j} + \mathbf{i}(\mathbf{e}_j)\frac{\Phi_{;\,i}}{2}$$

where $\Phi_{;\,i_1 i_2 \ldots}$ denote the components of multiple covariant differentiation relative to the local frame field $\mathbf{e}_1, ..., \mathbf{e}_d$.

$$\Phi_{;\,ij} = \boldsymbol{\nabla}_{\mathbf{e}_j}\boldsymbol{\nabla}_{\mathbf{e_i}}\Phi - \boldsymbol{\nabla}_{\nabla_{\mathbf{e}_j}\mathbf{e}_i}\Phi.$$

Since $\mathbf{e}^i \wedge \boldsymbol{\nabla}_{e_i}$ and $\mathbf{i}(\mathbf{e}_j)\boldsymbol{\nabla}_{ej}$ do not depend on the choice of the local orthonormal frame and coframe field we may assume that $\mathbf{e}_1, ..., \mathbf{e}_d$ comes from a normal coordinate centered at an arbitrary point and consequently have

$$\boldsymbol{\nabla}_{\mathbf{e_j}}\mathbf{e}^i \wedge = \boldsymbol{\nabla}_{\mathbf{e}_i}\mathbf{i}(\mathbf{e}_j) = \mathbf{0}.$$

Now using the fact that

$$\mathbf{e}^i \wedge \mathbf{i}(\mathbf{e}_j) + \mathbf{i}(\mathbf{e}_j)\mathbf{e}^i \wedge = \delta_{ij},$$

we have

$$\mathbf{W}^{(p)}_\Phi = \boldsymbol{\Delta} - \frac{\Phi_{;\,i}\Phi_{;\,i}}{4} + \frac{\Phi_{;\,ij}}{2}(\mathbf{e}^i \wedge \mathbf{i}(\mathbf{e}_j) - \mathbf{i}(\mathbf{e}_j)\mathbf{e}^i \wedge).$$

In the case of $\mathbb{R}^n$ where covariant differentiation becomes standard differentiation, the Witten Laplacian on 0-forms acting on a smooth function f gives

$$\mathbf{W}^{(0)}_\Phi f = -\boldsymbol{\Delta} f - \frac{\Phi_{x_i}\Phi_{x_i}}{4} f - \frac{\Phi_{x_i x_i}}{2} f = \left(-\boldsymbol{\Delta} + \frac{|\boldsymbol{\nabla}\Phi|^2}{4} - \frac{\boldsymbol{\Delta}\Phi}{2} \right) f.$$

The Witten Laplacian on one-forms acting on a one form

$$u = u^k(x)dx^k$$

gives

$$\mathbf{W}_{\phi}^{(1)}u= \boldsymbol{\Delta} u- \frac{\boldsymbol{\phi}_{x_i}\boldsymbol{\phi}_{x_i}}{4}u- \frac{\boldsymbol{\phi}_{x_i x_i}}{2}u + 2\frac{\boldsymbol{\phi}_{x_k x_i}}{2}dx^i \wedge \mathbf{i}_{\frac{\partial}{\partial x_k}}u.$$

Identifying one-forms with vector fields in $\mathbb{R}^n$, we obtain

$$\mathbf{W}_{\Phi}^{(1)}\mathbf{u} = \left(-\boldsymbol{\Delta}+ \frac{|\boldsymbol{\nabla}\Phi|^2}{4} - \frac{\boldsymbol{\Delta}\Phi}{2} \right) \otimes \mathbf{u} + \mathbf{Hess}\Phi\mathbf{u}.$$

The tensor notation simply means that the operator $-\boldsymbol{\Delta}+ \frac{|\boldsymbol{\nabla}\Phi|^2}{4} - \frac{\boldsymbol{\Delta}\Phi}{2}$ acts diagonally on each component of the vector field $\mathbf{u}$. Let us also point out that the identification between forms and vectors fields is a common practice in Riemaniann geometry and is done via the metric tensor.

As first observed by Bernard Helffer and Johannes Sjötrand, these Laplacians provide new methods for solving problems coming from Statistical Mechanics. The methods are generally based on the analysis of the differential operators

$$A_{\Phi}^{(0)} := -\boldsymbol{\Delta} + \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}$$

and

$$A_{\Phi}^{(1)} := A_{\Phi}^{(0)} \otimes Id + \mathbf{Hess}\Phi.$$

These two elliptic differential operators for which a Fredholm theory can be developed are equivalent to Witten's Laplacian $W_{\Phi}^{(0)}$ and $W_{\Phi}^{(1)}$ respectively where

$$\mathbf{W}_{\Phi}^{(0)}= -\boldsymbol{\Delta}+ \frac{|\boldsymbol{\nabla}\Phi|^2}{4} - \frac{\boldsymbol{\Delta}\Phi}{2}$$

and

$$\mathbf{W}_{\Phi}^{(1)} = \left(-\boldsymbol{\Delta}+ \frac{|\boldsymbol{\nabla}\Phi|^2}{4} - \frac{\boldsymbol{\Delta}\Phi}{2} \right) \otimes \mathbf{I} + \mathbf{Hess}\Phi.$$

Indeed, it only suffices to observe that

$$W_{\Phi}^{(.)} = e^{-\Phi/2} \circ A_{\Phi}^{(.)} \circ e^{\Phi/2}$$

and the map

$$\begin{array}{rcl} U_{\Phi} & : & L^2(\mathbb{R}^{\Lambda}) \to L^2(\mathbb{R}^{\Lambda}, e^{-\Phi}dx) \\ u & \longmapsto & e^{\frac{\Phi}{2}}u. \end{array}$$