

1994

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de Launey, Warwick and Seberry, Jennifer: The strong Kronecker product 1994.
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The strong Kronecker product

Abstract

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Disciplines

Physical Sciences and Mathematics

Publication Details

W. de Launey and Jennifer Seberry, The strong Kronecker product, *Journal of Combinatorial Theory, Ser. A*, 66, (1994), 192-213.

The Strong Kronecker Product

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Communicated by the Managing Editors

Received December 22, 1991; accepted in revised form September 9, 1992

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1. INTRODUCTION

We study the structure and properties of the strong Kronecker product. The results obtained are then used to obtain powerful new multiplication theorems and structure theorems for Hadamard matrices and related orthogonal matrices.

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2. COMPLEX ORTHOGONAL DESIGNS

Let i denote a square root of -1 . Let $x_1, x_2, \dots, y_1, y_2, \dots$ be commuting real indeterminates. We let \mathcal{P} and \mathcal{Q} be the sets $\{\pm x_j, \pm iy_j | j=1, 2, \dots\}$ and $\{\pm y_j, \pm iy_j | j=1, 2, \dots\}$. If $A = (a_{ij})$ is a matrix whose non-zero entries are from \mathcal{P} , we say it is a $(0, \mathcal{P})$ -matrix, and A^* denotes the matrix (a_{ji}^*) , where the aster denotes complex conjugation. If A and a $(0, \mathcal{Q})$ -matrix $B = (b_{ij})$ have the same dimensions, then $A \wedge B = (a_{ij}b_{ij})$ denotes the *Hadamard product* of A and B .

Now let $\mathbf{u} = (u_1, u_2, \dots, u_b)$ and $\mathbf{s} = (s_1, s_2, \dots, s_a)$ be integer vectors, and write

$$f = \sum_{i=1}^a s_i x_i^2 \quad \text{and} \quad g = \sum_{j=1}^b u_j y_j^2. \quad (1)$$

A *complex orthogonal design* $\text{COD}(c; \mathbf{s})$, A , is a square $(0, \mathcal{P})$ -matrix such that

$$AA^* = fI_c.$$

Such a matrix may be written uniquely as a sum

$$A = \sum_j x_j B_j + i \sum_j x_j C_j,$$

where B_j and C_j are $(0, 1, -1)$ -matrices such that $B_j B_j^T + C_j C_j^T$ is a diagonal matrix, for $j \neq i$,

$$B_j B_k^T + B_k B_j^T + C_j C_k^T + C_k C_j^T = 0,$$

and, for all j and k ,

$$B_k C_j^T + B_j C_k^T = C_k B_j^T + C_j B_k^T.$$

We consider some cases where B_j and C_j are weighing matrices. If $C_j = 0$ for all j , then the matrices B_j are weighing matrices, and A is a real matrix which is called an *orthogonal design* $\text{OD}(c; \mathbf{s})$. If $B_j(C_j)$ has elements $0, 1, -1$ then $B_j \wedge B_j(C_j \wedge C_j)$ has elements $0, 1$. Hence if the rows (columns) of $B_j(C_j)$ have constant weights, as for a weighing matrix, then $B_j \wedge B_j(C_j \wedge C_j)$ has constant row sum. Hence if B_j and C_j are weighing matrices, then the matrix $A \wedge A$ has constant row sum, and

$$D = \sum_j x_j B_j + i \sum_j y_j C_j,$$

is a COD, where each indeterminate has only real coefficients or only complex coefficients. A COD will be said to be *sharp* if every indeterminate in

the design, say z_k , has only real coefficients or only complex coefficients. If $D \wedge D$ has constant row sum $f - g$, then D is a *sharp complex orthogonal design SCOD* ($c; \mathbf{s}; \mathbf{u}$). (Note the careful use of the second semi-colon.)

Now suppose \mathcal{Q} and \mathcal{P} are disjoint and a $(0, \mathcal{Q})$ -matrix B is a $\text{COD}(d; \mathbf{u})$; then A and B are said to be *amicable* if

$$AB^* = BA^*,$$

and *anti-amicable* if

$$AB^* = -BA^*.$$

Note that A and B are amicable if and only if A and iB are anti-amicable.

A sharp COD corresponds to a set of disjoint weighing matrices which divides into two sets such that a pair of matrices chosen from the same set are antiamicable and all other pairs are amicable. More complex amicability relations could be embodied by adding more commuting square roots of -1 . A pair of (anti-)amicable SCODs corresponds to a set of weighing matrices with strong amicability and disjointness properties.

3. THE STRONG KRONECKER PRODUCT

Suppose N and M are matrices whose non-zero entries are in \mathcal{P} and \mathcal{Q} , respectively. In this paper, we let $N \times M$ denote the usual Kronecker product of N and M .

Now let M and N be respectively presented as $r \times t$ and $t \times u$ block matrices. We may write

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1t} \\ M_{21} & M_{22} & \cdots & M_{2t} \\ \vdots & \vdots & & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rt} \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1u} \\ N_{21} & N_{22} & \cdots & N_{2u} \\ \vdots & \vdots & & \vdots \\ N_{t1} & N_{t2} & \cdots & N_{tu} \end{bmatrix},$$

where each M_{ij} ($i = 1, 2, \dots, r$ and $j = 1, 2, \dots, t$) is an $m \times p$ matrix, and each N_{ij} ($i = 1, 2, \dots, t$ and $j = 1, 2, \dots, u$) is a $n \times q$ matrix. Now consider the following operation denoted by \circ ,

$$M \circ N = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1u} \\ L_{21} & L_{22} & \cdots & L_{2u} \\ \vdots & \vdots & & \vdots \\ L_{r1} & L_{r2} & \cdots & L_{ru} \end{bmatrix},$$

where each

$$L_{ij} = M_{i1} \times N_{1j} + M_{i2} \times N_{2j} + \dots + M_{it} \times N_{tj}$$

is an $mn \times pq$ matrix. We call L the $r \times t$ times $t \times u$ strong Kronecker product of M and N . It is important to note that the operation is fully determined only after the parameters r , t , and u are set. Generally, the partitioning of the matrices will be clear from the context, and then we call L the strong Kronecker product of M and N .

We will see that this operation preserves orthogonality. We first give a result which strengthens a theorem of Seberry and Zhang [9].

LEMMA 1 (Structure Lemma). *Let $A = [A_{kj}]$ and $C = [C_{kj}]$, where A_{kj} and C_{kj} are $m \times p$ matrices, be $r \times t$ block $(0, \mathcal{P})$ -matrices, and let $B = [B_{kj}]$ and $D = [D_{kj}]$, where B_{kj} and D_{kj} are $n \times q$ matrices, be $t \times u$ block $(0, \mathcal{Q})$ -matrices. Write $(A \circ B)(C \circ D)^* = [L_{ab}]$, where $a, b = 1, \dots, r$, then*

$$L_{ab} = \sum_{j=1}^t \sum_{k=1}^t \left(A_{aj} C_{bk}^* \times \sum_{s=1}^u B_{js} D_{ks}^* \right). \tag{2}$$

In particular, if BD^ is a $t \times t$ block matrix where all off-diagonal block matrices are zero, then*

$$L_{ab} = \sum_{j=1}^t \left(A_{aj} C_{bj}^* \times \sum_{s=1}^u B_{js} D_{js}^* \right), \tag{3}$$

and, if BD^ is of the form $I_t \times E$, then*

$$L_{ab} = \left(\sum_{j=1}^t A_{aj} C_{bj}^* \right) \times E, \tag{4}$$

or, equivalently,

$$(A \circ B)(C \circ D)^* = AC^* \times E. \tag{5}$$

Moreover, if $C = A$, $D = B$, $AA^ = fI_{rm}$, and $BB^* = gI_{tn}$, where f and g are defined in Eq. (1) above, then*

$$L_{ab} = \begin{cases} 0 & \text{if } a \neq b \\ fgI_{mn} & \text{if } a = b. \end{cases} \tag{6}$$

Proof. The following calculation proves Eq. (2).

$$\begin{aligned} L_{ab} &= \sum_{s=1}^u (A_{a1} \times B_{1s} + A_{a2} \times B_{2s} + \cdots + A_{at} \times B_{ts}) \\ &\quad \times (C_{b1}^* \times D_{1s}^* + C_{b2}^* \times D_{2s}^* + \cdots + C_{bt}^* \times D_{ts}^*) \\ &= \sum_{s=1}^u \sum_{j=1}^t \sum_{k=1}^t (A_{aj} \times B_{js})(C_{bk}^* \times D_{ks}^*) \\ &= \sum_{j=1}^t \sum_{k=1}^t \left(A_{aj} C_{bk}^* \times \sum_{s=1}^u B_{js} D_{ks}^* \right). \end{aligned}$$

To prove Eq. (3), note that, if BD^* is a $t \times t$ block matrix where all off-diagonal block matrices are zero, then, for $j \neq k$,

$$\sum_{s=1}^u B_{js} D_{ks}^* = 0.$$

To prove Eqs. (4) and (5), note that

$$\sum_{s=1}^u B_{js} D_{js}^* = E,$$

and apply Eq. (3). Eq. (6) follows from Eq. (4) once it is noted that $E = gI_n$ and

$$\sum_{j=1}^t A_{aj} A_{bj}^* = \begin{cases} 0 & \text{if } a \neq b \\ fI_m & \text{if } a = b. \end{cases}$$

Note that in the Structure Lemma we find orthogonal matrices but not (complex) orthogonal designs or complex weighing matrices because we cannot guarantee that the general block matrix

$$C_{ij} = [A \circ B]_{ij} = \sum_{k=1}^t A_{ik} = B_{kj}$$

has the correct entries. In fact the greatest effort in this paper will be devoted to ensuring $A \circ B$ has appropriate entries. A little trial and error will convince the reader that only one combination always (without other special conditions) gives a useful results.

COROLLARY 1. *Suppose A and B are Hadamard matrices of order $2m$ and $2n$ respectively. Then*

$$\frac{1}{2} A \circ B$$

is a $W(2mn, mn)$.

In general though, we have:

COROLLARY 2. Suppose A and B satisfy

$$AA^* = fI_{tm} \quad \text{and} \quad BB^* = gI_{tm}.$$

Write A and B as $t \times t$ block matrices. Then $C = A \circ B$ of order tmn satisfies

$$CC^* = fgI_{tmn}.$$

4. TWO ALGEBRAIC PROPERTIES OF THE STRONG KRONECKER PRODUCT

All operations in this section are over a commutative ring.

THEOREM 1. Strong Kronecker multiplication is associative.

Proof. Let $A = [A_{ik}]$, $B = [B_{kl}]$, and $C = [C_{lj}]$ be matrices presented as $a \times b$, $b \times c$, and $c \times d$ block matrices, respectively. The following calculation is sufficient to prove the theorem.

$$\begin{aligned} ([A_{ik}] \circ [B_{kl}]) \circ [C_{lj}] &= \left[\sum_k A_{ik} \times B_{kl} \right] \circ [C_{lj}] \\ &= \left[\sum_l \left(\sum_k A_{ik} \times B_{kl} \right) \times C_{lj} \right] \\ &= \left[\sum_{k,l} A_{ik} \times B_{kl} \times C_{lj} \right] \\ &= \left[\sum_k A_{ik} \times \left(\sum_l B_{kl} \times C_{lj} \right) \right] \\ &= [A_{ik}] \circ \left[\sum_l B_{kl} \times C_{lj} \right] \\ &= [A_{ik}] \circ ([B_{kl}] \circ [C_{lj}]). \quad \blacksquare \end{aligned}$$

COROLLARY 3. Let

$$\begin{bmatrix} H_1 + H_2 & H_1 - H_2 \\ H_3 + H_4 & H_3 - H_4 \end{bmatrix} \circ \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \circ \begin{bmatrix} K_1 + K_3 & K_2 + K_4 \\ K_1 - K_3 & K_2 - K_4 \end{bmatrix}.$$

Proof. Apply associativity to the following matrix.

$$\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} \circ \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}. \quad \blacksquare$$

THEOREM 2. Let $[H_{sr}]$, $[A_{rt}]$, $[K_{sr}]$ and $[B_{rt}]$ be presented as respectively $a \times b$, $b \times c$, $a \times b$ and $b \times c$ block matrices. Then

$$([H_{sr}] \circ [A_{rt}]) \wedge ([K_{sr}] \circ [B_{rt}]) = (H_{sr} \wedge K_{su}) \circ (A_{rt} \wedge B_{ut}).$$

Proof. Let

$$\begin{aligned} ([H_{sr}] \circ [A_{rt}]) \wedge ([K_{sr}] \circ [B_{rt}]) &= \left[\sum_r H_{sr} \times A_{rt} \right] \wedge \left[\sum_u K_{su} \times B_{ut} \right] \\ &= \left[\sum_{r,u} (H_{sr} \wedge K_{su}) \times (A_{rt} \wedge B_{ut}) \right] \\ &= [H_{sr} \wedge K_{su}] \circ [A_{rt} \wedge B_{ut}]. \quad \blacksquare \end{aligned}$$

5. DISJOINTNESS

We say a $(0, \mathcal{P})$ -matrix A and a $(0, \mathcal{Q})$ -matrix B are *disjoint* if $A \wedge B = 0$. We begin with a simple but surprising lemma.

LEMMA 2. Let A, B, C , and D be $(1, -1)$ -matrices. Then $(A \times C + B \times D)$ and $(A \times D - B \times C)$ are disjoint.

Proof. Consider the general terms

$$(a_{ij}c_{lm} + b_{ij}d_{lm}) \quad \text{and} \quad (a_{ij}d_{lm} - b_{ij}c_{lm}).$$

Observe that

$$(a_{ij}c_{lm} + b_{ij}d_{lm})(a_{ij}d_{lm} - b_{ij}c_{lm}) = c_{lm}d_{lm} - a_{ij}b_{ij} + b_{ij}a_{ij} - d_{lm}c_{lm} = 0. \quad \blacksquare$$

The next result, which follows from Theorem 2 generalises this result.

THEOREM 3. Let $[H_{sr}]$, $[A_{rt}]$, $[K_{sr}]$, and $[B_{rt}]$ be respectively $a \times b$, $b \times c$, $a \times b$, and $b \times c$ block matrices. Then $[H_{sr}] \circ [A_{rt}]$ and $[K_{sr}] \circ [B_{rt}]$ are disjoint if and only if the $a \times b^2$ times $b^2 \times c$ strong Kronecker product

$$[H_{sr} \wedge K_{su}] \circ [A_{rt} \wedge B_{ut}]$$

is zero.

EXAMPLE [Alternative Proof of Lemma 2]. Observe that

$$[(A \times C) + (B \times D)] = [A \quad B] \circ \begin{bmatrix} C \\ D \end{bmatrix},$$

and

$$[(A \times D) - (B \times C)] = [A \ B] \circ \begin{bmatrix} D \\ -C \end{bmatrix}.$$

By Theorem 3 these are disjoint if and only if

$$[A \wedge A \ A \wedge B \ B \wedge A \ B \wedge B] \circ \begin{bmatrix} C \wedge D \\ -C \wedge C \\ D \wedge D \\ -D \wedge C \end{bmatrix} = 0,$$

which is clearly true.

As a consequence of Theorem 3, we have the pretty result.

COROLLARY 4. *Let $H, A, K,$ and B be $(0, \mathcal{P})$ -matrices; then $H \circ A$ and $K \circ B$ are disjoint if and only if $K \circ A$ and $H \circ B$ are disjoint.*

COROLLARY 5. *Let $H_i, A_i, K_i,$ and B_i be $(0, \mathcal{P})$ -matrices; then*

$$\left(\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \circ \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right) \wedge \left(\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \circ \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \right) = 0$$

if and only if

$$\begin{bmatrix} H_1 \wedge K_1 & H_1 \wedge K_2 & H_2 \wedge K_1 & H_2 \wedge K_2 \\ H_3 \wedge K_3 & H_3 \wedge K_4 & H_4 \wedge K_3 & H_4 \wedge K_4 \end{bmatrix} \circ \begin{bmatrix} A_1 \wedge B_1 & A_2 \wedge B_2 \\ A_1 \wedge B_3 & A_2 \wedge B_4 \\ A_3 \wedge B_1 & A_4 \wedge B_2 \\ A_3 \wedge B_3 & A_4 \wedge B_4 \end{bmatrix} = 0.$$

In particular, suppose $H = K$ and $H_i, A_i,$ and B_i are $(1, -1)$ -matrices, then $H \circ A$ and $K \circ B$ are disjoint if $B_3 = -A_1 \wedge A_3 \wedge B_1$ and $B_4 = -A_2 \wedge A_4 \wedge B_2$. If $H_1 \neq \pm H_2$ then $B_3 = -A_1 \wedge A_3 \wedge B_1$ is necessary and similarly if $H_3 \neq \pm H_4$ then $B_4 = -A_2 \wedge A_4 \wedge B_2$ is necessary.

A consequence of this corollary is that

COROLLARY 6. *Let H_i and A_i be $(0, \mathcal{P})$ -matrices of suitable dimensions; then*

$$\left(\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \circ \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right) \wedge \left(\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \circ \begin{bmatrix} A_3 & A_4 \\ -A_1 & -A_2 \end{bmatrix} \right) = 0.$$

6. CONFERRED PROPERTIES

The concepts of amicability and anti-amicability have been of considerable importance in the study of Hadamard matrices. In this section, we show that the strong Kronecker product often preserves (anti)-amicability.

THEOREM 4 (Conferred Amicability and Anti-Amicability Theorem). *Let $A = [A_{kj}]$ and $C = [C_{kj}]$, where A_{kj} and C_{kj} are $m \times p$ matrices, be $r \times t$ block $(0, \mathcal{P})$ -matrices, and let $B = [B_{kj}]$ and $D = [D_{kj}]$, where B_{kj} and D_{kj} are $n \times q$ matrices, be $t \times u$ block $(0, \mathcal{Q})$ -matrices. Let $\alpha \in \{0, 1, -1, i, -i\}$, and set $T = A \circ D$, $U = A \circ B$, and $V = C \circ B$.*

- (1) *If $BD^* = \alpha DB^*$, then $TU^* = \alpha UT^*$.*
- (2) *If $BB^* = gI$ and $AC^* = \alpha CA^*$, then $UV^* = \alpha VU^*$.*

Proof. We prove part (2): the proof of the first part is similar and easier. Let $UV^* = (L_{ab})$ and $VU^* = (R_{ab})$, where $a, b = 1, \dots, r$. By the Structure Lemma,

$$\begin{aligned} L_{ab} &= \sum_{j=1}^t \sum_{k=1}^t \left(A_{aj} C_{bk}^* \times \sum_{s=1}^u B_{js} B_{ks}^* \right) \\ &= \sum_{j=1}^t A_{aj} C_{bj}^* \times gI. \end{aligned}$$

Similarly,

$$\begin{aligned} R_{ab} &= \sum_{j=1}^t \sum_{k=1}^t \left(C_{aj} A_{bk}^* \times \sum_{s=1}^u B_{js} B_{ks}^* \right) \\ &= \sum_{j=1}^t C_{aj} A_{bj}^* \times gI. \end{aligned}$$

Note $AC^* = \alpha CA^*$ implies that, for $a, b = 1, \dots, r$,

$$\sum_{j=1}^t A_{aj} C_{bj}^* = \alpha \sum_{j=1}^t C_{aj} A_{bj}^*.$$

So $L_{ab} = \alpha R_{ab}$ and $UV^* = \alpha VU^*$ as required. ■

7. CONSTRUCTED PROPERTIES

We now show the strong Kronecker product guarantees the existence of amicable or anti-amicable matrices.

LEMMA 3 (Constructed Amicability or Anti-Amicability Lemma). *Suppose $AA^* = fI_{tm}$ and $BB^* = gI_m$, where A and B are matrices of order tm and m , respectively, $t = 2, 4, 8$. Then there are t orthogonal matrices, C_k , $k = 1, \dots, t$ of order tm , satisfying, for $j, k = 1, \dots, t$,*

$$C_k C_k^* = fgI_{tm} \quad \text{and} \quad C_j C_k^* = C_k C_j^*.$$

Proof. For $t = 2$ the matrices are

$$\left\{ A \circ B \text{ and } A \circ \begin{bmatrix} 0 & \pm I \\ I & 0 \end{bmatrix} B \right\},$$

or

$$\left\{ A \circ B \text{ and } \begin{bmatrix} 0 & \pm I \\ I & 0 \end{bmatrix} A \circ B \right\}$$

respectively.

For $t = 4$, define

$$Q_1 = \begin{bmatrix} I & & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & I & & \\ \pm I & 0 & & \\ & & 0 & I \\ & & \pm I & 0 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & \pm I \\ \pm I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & \pm I & 0 & 0 \\ \pm I & 0 & 0 & 0 \end{bmatrix}.$$

Then the matrices are

$$\{A \circ B, A \circ Q_2 B, A \circ Q_3 B, A \circ Q_4 B\},$$

or

$$\{A \circ B, Q_2 A \circ B, Q_3 A \circ B, Q_4 A \circ B\}.$$

For $t = 8$ define Q_i analogously using an 8×8 representation of the quaternions. Then the two sets are obtained similarly.

8. AMICABLE AND ANTI-AMICABLE ORTHOGONAL DESIGNS FROM HADAMARD MATRICES

Plotkin [7] first showed that the existence of an Hadamard matrix of order $2h$ implied the existence of an $OD(2h; h, h)$. Seberry and Zhang [9]

recently obtained the same result while investigating the strong Kronecker product. We extend this result below. But first we make some remarks.

Let P be any anti-symmetric monomial matrix (i.e., $P = -P^T$ and $P \wedge P$ is a permutation matrix). Let $R = HP$, $U = \frac{1}{2}(R + H)$, and $V = \frac{1}{2}(R - H)$. Then U and V are disjoint $(0, 1, -1)$ -matrices such that $UV^T = -VU^T$ and $VV^T = UU^T = \frac{1}{4}(RR^T + HH^T) = hI_{2h}$. It follows that $Ux_1 + Vx_2$ is an $OD(2h; h, h)$.

The results of the following theorem are related to the results of Plotkin [7; 6, p. 127] that if there is an Hadamard matrix of order $2h$ then there exists an $OD(4h; h, h, h, h)$ and an $OD(8h; h, h, h, h, h, h, h, h)$.

THEOREM 5. *Let $2h$ be the order of an Hadamard matrix. Then there exist four $W(2h; h)$ matrices R, S, T, U such that R, U and S, T are anti-amicable pairs, all other pairs are amicable, and R and U, R and S, S and T and T and U are disjoint pairs. In particular, there exist*

- (i) *a pair of amicable $OD(2h; h, h)$, and*
- (ii) *a pair of anti-amicable sharp $COD(2h; h; h)$.*

Proof. Let

$$W = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We note that $WW^T = XX^T = YY^T = ZZ^T = 2I_2$ and $WX^T = XW^T, WY^T = YW^T, WZ^T = -ZW^T, XY^T = -YX^T, XZ^T = ZX^T, YZ^T = ZY^T$.

Write

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix},$$

$$R = \frac{1}{2}H \circ W, \quad S = \frac{1}{2}H \circ X, \quad T = \frac{1}{2}H \circ Y,$$

$$\text{and} \quad U = \frac{1}{2}H \circ Z.$$

By the Structure Lemma $RR^T = SS^T = TT^T = UU^T = hI_{2h}$, and by the Conferred Amicability and Anti-Amicability Theorem $RS^T = SR^T, RT^T = TR^T, RU^T = -UR^T, ST^T = -TS^T, SU^T = US^T, TU^T = UT^T$. By Corollary 1, $R, S, T,$ and U are $W(2h, h)$. The disjointness properties follow directly from Corollary 6.

Now let x, y, a, b be commuting indeterminates, and put $A = xS + yT, B = aR + bU, C = xR + iyS,$ and $D = aU + ibT$. From above, A and B are $OD(2h; h, h)$ s, and C and D are sharp $COD(2h; h, h)$ s. Moreover, $AB^T = (xS + yT)(aR^T + bU^T) = BA^T,$ and $CD^* = (xR + iyS)(aU^T - ibT^T) = -DC^*$. So A and B are amicable and C and D are anti-amicable. ■

We see Plotkin's results reproved in the next corollary:

COROLLARY 7. *If there exists an Hadamard matrix of order $2h$, then there is an $OD(4h; h, h, h, h)$, disjoint amicable $AOD(4h; h, h, h, h)$, a sharp $COD(4h; h, h, h, h)$, and an $OD(8h; h, h, h, h, h, h, h, h)$.*

Proof. Let A and B be the matrices in the proof of the previous result; then

$$C = \begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

is the required $OD(4h; h, h, h, h)$. The designs

$$\begin{bmatrix} xS & yT \\ yT & xS \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} cR & dU \\ dU & cR \end{bmatrix}$$

are the disjoint amicable $AOD(4h; h, h; h, h)$ and the design

$$\begin{bmatrix} xS + icR & yT + idU \\ yT + ixS & dU + icR \end{bmatrix}$$

is the required sharp $COD(4h; h, h; h, h)$.

Now an $OD(8h; h, h, h, h, h, h, h, h)$ similar to that of Plotkin, may be written

$$W = \begin{bmatrix} xS + yT & zS + wT & -aS + bT & cR + dU \\ -zS + wT & xS - yT & -cR - dU & -aS + bT \\ aS + bT & cR + dU & xS - yT & -zS - wT \\ cR + dU & -aS - bT & -zS + wT & -xS - yT \end{bmatrix}. \quad \blacksquare$$

We note that, in the argument of the proof of Theorem 5, we may replace the Hadamard matrices, W , X , Y , and Z of order 2 with the following Hadamard matrices of order $2k$

$$W = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}, \quad X = \begin{bmatrix} C_3 & C_4 \\ C_1 & C_2 \end{bmatrix},$$

$$Y = \begin{bmatrix} C_1 & C_2 \\ -C_3 & -C_4 \end{bmatrix}, \quad Z = \begin{bmatrix} -C_3 & -C_4 \\ C_1 & C_2 \end{bmatrix}$$

to obtain designs where the parameter h is replaced by hk . Of course, the existence of designs with these parameters also follows from Theorem 5 and Agaian's result, which gives an Hadamard matrix of order $2hk$; so we will not state the apparently more general result here.

9. MULTIPLICATION THEOREMS USING 2×2 STRONG KRONECKER PRODUCT

THEOREM 6 (Multiplication Theorem). *Let $[H_1 \ H_2]$ be an Hadamard matrix of order $2a$ and let $[A_1^T \ A_2^T]$ be a $COD(2b; \mathbf{u})$ with entries from \mathcal{P} ; then the matrix*

$$D = \frac{1}{2} [H_1 \ H_2] \circ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \circ \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

is a $COD(2ab, a\mathbf{u})$.

Proof. Observe

$$D = \frac{1}{2} (H_1 + H_2) \times A_1 + \frac{1}{2} (H_1 - H_2) \times A_2.$$

Plainly, its entries are in \mathcal{P} . The Structure Theorem ensures the matrix is orthogonal, and indeed, because the entries are all in \mathcal{P} , it ensures each indeterminate appears the correct number of times in each row and column. ■

The next corollary contains some results of Agaian [1], Craigen [4], and Seberry and Zhang [9].

COROLLARY 8. *If there is an Hadamard matrix of order $2a$, then*

- (i) *if there is a complex Hadamard matrix of order $2b$, there is a complex Hadamard matrix of order $2ab$;*
- (ii) *if there is an $OD(2b; u_1, u_2, \dots, u_c)$, there is an $OD(2ab; au_1, au_2, \dots, au_c)$;*
- (iii) *if there is an Hadamard matrix of order $2b$, there is an Hadamard matrix of order $2ab$;*
- (iv) *if there is a weighing matrix $W(2b, k)$ there is a $W(2ab, ak)$.*

Proof. To obtain the required designs, apply Theorem 6 with the design (A_{ij}) set equal to the initial CODs mentioned in items (i)–(iv). ■

10. A MULTIPLICATION THEOREM USING A 4×4 STRONG KRONECKER PRODUCT

The next theorem is very close to a method for “multiplying” Hadamard matrices of orders $4h$ and $4n$ together to obtain an Hadamard matrix of order $4hn$. Given Hadamard matrices of orders $4h$ and $4n$, it is possible to construct a disjoint pair of amicable $W(4hn, 2hn)$ s. If that pair had been anti-amicable, their sum would have been an Hadamard matrix of order

$4hn$. Here, we extend this result using the properties of the strong Kronecker product. Previous results give the existence of an Hadamard matrix of order $8hn$ and hence disjoint pairs of (anti-) amicable $W(8hn, 4hn)$ s. A consequence of our extended result is a set of four disjoint $W(8hn, 2hn)$ s with amicability properties. Indeed, we obtain a pair of disjoint amicable $OD(8hn; 2hn, 2hn)$ s.

THEOREM 7. *Suppose $4h$ and $4n$ are the orders of Hadamard matrices then there exist four $W(4hn, 2hn)$ s $R, S, T,$ and U such that R, S and T, U are disjoint pairs, R, U and S, T are anti-amicable pairs and all other pairs are amicable. Indeed, there exists a pair of amicable $SCOD(4hn; 2hn; 2hn)$ s, a $SCOD(8hn; 2hn, 2hn; 2hn, 2hn)$, and a pair of disjoint amicable $OD(8hn; 2hn, 2hn)$ s.*

Proof. Write the Hadamard matrices in the form

$$H = [H_1 \ H_2 \ H_3 \ H_4] \quad \text{and} \quad N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix},$$

where H_i are $4h \times h$ matrices and N_i are $n \times 4n$ matrices. Now let

$$X = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & -I_n \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 0 & 0 & I_n \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ -I_n & 0 & 0 & 0 \end{bmatrix}.$$

Set

$$R = \frac{1}{4} [H_1 \ H_2 \ H_3 \ H_4] \circ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \circ \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix},$$

$$S = \frac{1}{4} [H_1 \ H_2 \ H_3 \ H_4] \circ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \circ (NX),$$

$$T = \frac{1}{4} [H_1 \ H_2 \ H_3 \ H_4] \circ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \circ (NY),$$

and

$$U = \frac{1}{4} [H_1 \ H_2 \ H_3 \ H_4] \circ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \circ (NXY).$$

We note that

$$\begin{aligned} R &= \frac{1}{4} ((H_1 + H_2) \times N_1 + (H_1 - H_2) \times N_2 + (H_3 + H_4) \times N_3 \\ &\quad + (H_3 - H_4) \times N_4), \\ S &= \frac{1}{4} ((H_1 + H_2) \times N_1 + (H_1 - H_2) \times N_2 - ((H_3 + H_4) \times N_3 \\ &\quad + (H_3 - H_4) \times N_4)), \\ T &= \frac{1}{4} ((H_3 + H_4) \times N_2 + (H_3 - H_4) \times N_1 - ((H_1 + H_2) \times N_4 \\ &\quad + (H_1 - H_2) \times N_3)), \end{aligned}$$

and

$$\begin{aligned} U &= \frac{1}{4} ((H_3 + H_4) \times N_2 + (H_3 - H_4) \times N_1 + (H_1 + H_2) \times N_4 \\ &\quad + (H_1 - H_2) \times N_3); \end{aligned}$$

so R and S and T and U are pairs of disjoint $(0, 1, -1)$ -matrices. Note the matrices I_{4n} , X , Y and XY have amicability properties which, by the Conferred Amicability and Anti-amicability Theorem, are inherited by R , S , T , and U . Finally, by the Structure Lemma $RR^T = SS^T = TT^T = UU^T = 2hnI_{4hn}$; so R , S , T , and U are the required weighing matrices.

Now let a, b, x , and y be distinct commuting indeterminates. The designs $C = aR + ibS$ and $D = xT + iyU$ are the required amicable SCOD($4hn; 2hn; 2hn$)s. Either

$$\begin{bmatrix} iC & D \\ D & iC \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} C & D \\ D & -C \end{bmatrix}$$

is the required SCOD($8hn; 2hn, 2hn; 2hn, 2hn$), and such a design is equivalent to the required pair of disjoint amicable OD($8hn; 2hn, 2hn$)s. ■

11. CONSTRUCTED AMICABILITY

We note that Theorem 6 preserves sharpness; so it may be viewed as constructing new sets of disjoint weighing matrices with amicability properties from old. Note that just one (complex) orthogonal design is used. Our

purpose here is to show how the Conferred Amicability and Anti-amicability Theorem may be used to construct new sets of matrices with amicability properties from more than one (complex) orthogonal design. To avoid cumbersome detail, our results are stated for ODs only; similar results can be proved for (sharp) CODs.

THEOREM 8. *Suppose there is an Hadamard matrix of order $2a$ and that B and C are (anti-) amicable $OD(2b; \mathbf{u})$ and $OD(2b; \mathbf{s})$; then the designs below are (anti-) amicable $OD(2ab; \mathbf{a}\mathbf{u})$ and $OD(2ab; \mathbf{a}\mathbf{s})$.*

$$\frac{1}{2}H \circ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \circ B, \quad \frac{1}{2}H \circ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \circ C.$$

Proof. Apply the conferred Amicability and Anti-Amicability Theorem. ■

Many pairs of amicable orthogonal designs are known [10]. The simplest pair gives a simpler proof of part (1) of Theorem 5.

COROLLARY 9. *If there is an Hadamard matrix of order $2a$, then there is a pair of amicable $OD(2a; a, a)$.*

Proof. Let $w, x, y,$ and z be distinct commuting indeterminants. The designs are

$$\frac{1}{2}H \circ \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} \circ \begin{bmatrix} x & y \\ y & -x \end{bmatrix}$$

and

$$\frac{1}{2}H \circ \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} \circ \begin{bmatrix} z & w \\ -w & z \end{bmatrix}. \quad \blacksquare$$

The designs in the next corollary can be “plugged” into orthogonal design to produce orthogonal designs on more variables.

COROLLARY 10. *Let x_1, x_2, \dots and w and y be distinct commuting indeterminates. If there is an Hadamard matrix of order $2a$, then the designs below are amicable.*

$$D_i = \frac{1}{2}H \circ \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} \circ \begin{bmatrix} x_i & x_i \\ x_i & -x_i \end{bmatrix}$$

and

$$E = \frac{1}{2}H \circ \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix} \circ \begin{bmatrix} y & w \\ -w & y \end{bmatrix}.$$

Hence, if there is an $OD(2b; u_1, u_2, \dots, u_d)$, then there is an $OD(4ab; au_1, au_1, 2au_2, \dots, 2au_d)$.

The resulting OD has one more indeterminate than the $OD(4ab; 2au_1, 2au_2, \dots, 2au_d)$ obtained by taking the usual Kronecker product with the Hadamard matrix. Moreover, we have

COROLLARY 11. *The designs D_i and E of Corollary 10 may be plugged into a pair of amicable $OD(2b; \mathbf{u})$ and $OD(2b; \mathbf{s})$ to obtain a pair of amicable $OD(4ab; 2\mathbf{as})$ and $OD(4ab; au_1, au_1, 2au_2, \dots, 2au_d)$.*

Proof. Use the design E in the $OD(2b; \mathbf{u})$ and the other designs D_i as needed. ■

12. USING DISJOINT WEIGHING MATRICES

Results in previous sections give many pairs of disjoint (anti-)amicable weighing matrices. We show how disjoint amicable weighing matrices may be used to obtain a pair of amicable orthogonal designs, and how they may be used in the strong Kronecker product with an OD to obtain another OD. Similar results hold for anti-amicable or complex weighing matrices.

THEOREM 9. *Let W_1 and W_2 be disjoint amicable $W(n, w_i)$ s. Then*

$$a \begin{bmatrix} W_1 & W_2 \\ W_2 & -W_1 \end{bmatrix} + b \begin{bmatrix} -W_2 & W_1 \\ W_1 & W_2 \end{bmatrix}$$

and

$$c \begin{bmatrix} W_1 & -W_2 \\ W_2 & W_1 \end{bmatrix} + d \begin{bmatrix} W_2 & W_1 \\ -W_1 & W_2 \end{bmatrix}$$

are amicable $OD(2n; w_1 + w_2, w_1 + w_2)$ s.

Proof. Consider the behaviour of the following matrices under multiplication.

$$M_1 = \begin{bmatrix} I_n & \\ & I_n \end{bmatrix}, \quad M_2 = \begin{bmatrix} & I_n \\ -I_n & \end{bmatrix},$$

$$M_3 = \begin{bmatrix} I_n & \\ & -I_n \end{bmatrix}, \quad M_4 = \begin{bmatrix} & I_n \\ I_n & \end{bmatrix}.$$

Note $M_i M_j^T = M_j^T M_i$ except when $(i, j) \in \{(1, 2), (3, 4)\}$, and in these two cases $M_i M_j^T = -M_j M_i^T$.

Observe that

$$X = \begin{bmatrix} W_1 & W_2 \\ W_2 & -W_1 \end{bmatrix}$$

is a $W(2n, w_1 + w_2)$ and

$$XM_1 \wedge XM_2 = XM_3 \wedge XM_4 = 0.$$

It follows that the two matrices

$$aXM_1 + bXM_2 \quad \text{and} \quad cXM_3 + dXM_4$$

are amicable $OD(2n; w_1 + w_2, w_1 + w_2)$ s. ■

THEOREM 10. *Let W_1 and W_2 be disjoint amicable $W(n, w_i)$ s, and let K be an $OD(c; \mathbf{s})$; then*

$$D = \begin{bmatrix} W_1 & W_2 \\ W_2 & -W_1 \end{bmatrix} \circ K$$

is an $OD(cn; (w_1 + w_2) \mathbf{s})$.

Proof. Write K as

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix}.$$

Then

$$D = \begin{bmatrix} W_1 \times K_1 + W_2 \times K_2 \\ W_2 \times K_1 - W_1 \times K_2 \end{bmatrix}.$$

By the Structure Lemma, D is orthogonal, and, by inspection, each row contains $\pm x_i$ precisely $(w_1 + w_2) s_i$ times. ■

A similar result holds for (sharp) CODs.

13. ORTHOGONAL PAIRS

Craigen [2] first introduced the following important idea:

DEFINITION 1. Two $(1, -1)$ -matrices C and D of order $2c$ will be called an *orthogonal pair* if

$$CC^T + DD^T = 4cI_{2c} \quad \text{and} \quad CD^T = 0.$$

If C and D have elements from $\{1, -1, i, -i\}$ they are called a *complex orthogonal pair*. If C and D are $(0, \mathcal{P})$ -matrices such that

$$CC^T + DD^T = 2fI_{2c} \quad \text{and} \quad CD^T = 0;$$

then they are called a *complex orthogonal design pair*, written $\text{CODP}(2c; \mathbf{s})$. If they are real matrices they are called *orthogonal design pairs*.

Perhaps the simplest way to construct such pairs follows.

LEMMA 4. *Suppose two $(0, \mathcal{P})$ -matrices C and D are respectively (complex) disjoint $\text{OD}(c; \mathbf{s}_1)$ and $\text{OD}(c; \mathbf{s}_2)$ such that*

$$CD^* = DC^*;$$

then $C + D$ and $C - D$ are an (complex) $\text{ODP}(c; \mathbf{s}_1 + \mathbf{s}_2)$. ■

Hence (complex) disjoint pairs of amicable or anti-amicable ODs are a fruitful source of (complex) ODPs. Another general way of constructing COSPs follows. We begin with a special case.

LEMMA 5 (Complex Orthogonal Pairs). *Suppose C and D are complex Hadamard matrices of orders $2c$ and $2d$. Then there exists a complex orthogonal pair X, Y of order $2cd$.*

Proof. Write C and D as

$$C = [C_1 \quad C_2] \quad \text{and} \quad D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

where C_i is of size $2c \times c$ and D_j is of size $d \times 2d$. Then the required pairs are $X = C_1 \times D_1$ and $Y = C_2 \times D_2$. ■

More generally, we have the following.

LEMMA 6 (Complex Orthogonal Design Orthogonal Pairs). *Suppose C is a (complex) $W(2c; d)$ and D is a (complex) $\text{OD}(2e; \mathbf{s})$. Then there exists an orthogonal design pair $\text{ODP}(2ce; d\mathbf{s})$ which is complex if either C or D is complex.*

We now indicate the power of this idea.

THEOREM 11. *Suppose there are two disjoint (complex) $W(2a, b)$ s A and B , a (complex) $W(2c; d)$ C and a (complex) $\text{OD}(2e; \mathbf{s})$ D ; then there is a (complex) $\text{OD}(4ace; bds)$ E . In particular, if A, B, C , and D are weighing matrices then so is E , and, if any of A, B, C or D is complex, then so is E .*

Proof. Apply Lemma 6 to C and D to obtain an $ODP(2ce; ds)$ X and Y . Then the required design is given by

$$A \times X + B \times Y. \quad \blacksquare$$

14. PRODUCTS OF FOUR ORTHOGONAL MATRICES

Agaian [1] has proved that, if there are Hadamard matrices of orders $4a$ and $4b$, there is an Hadamard matrix of order $8ab$. This was extended by Craigen, Seberry, and Zhang [5] who showed that, if there are Hadamard matrices of orders $4a, 4b, 4c, 4d$, there is an Hadamard matrix of order $16abcd$. Here we obtain new results in a similar vein.

THEOREM 12. *Suppose there are complex Hadamard matrices of orders $2a$ and $2b$ and Hadamard matrices of orders $4c$ and $4d$ then there is a complex Hadamard matrix of order $8abcd$.*

Proof. By Theorem 7, we have a pair of disjoint $W(4cd, 2cd)$ A and B , say. Now use Theorem 11. \blacksquare

The following corollary follows from the method of proof of the last result.

COROLLARY 12 (Product of Four Complex Hadamard Matrices). *If there exist complex Hadamard matrices of order $2a, 2b, 2c, 2d$ then there exists a complex Hadamard matrix of order $8abcd$.*

Proof. Any complex Hadamard matrix of order $2c$ gives an Hadamard matrix of order $4c$; so there exist Hadamard matrices of orders $4c$ and $4d$. Now use the argument in the proof of the previous result. \blacksquare

Similarly, a complex Hadamard matrix of order $8abcd$ can be obtained from three complex Hadamard matrices of orders $2a, 2b$, and $2c$ and one Hadamard matrix of order $4d$. We do not know whether Theorem 12 is true if the complex Hadamard matrix of order $2b$ is replaced by an Hadamard matrix of order $4b$.

Using Craigen, Seberry, and Zhang's result for 4 Hadamard matrices and Agaian's result for 2 Hadamard matrices, the best one can do is obtain a Hadamard matrix of order $64abcdef$ from Hadamard matrices of orders, $4a, 4b, 4c, 4d, 4e, 4f$. The next corollary to Theorem 12 ensures we may obtain a complex Hadamard matrix of order $32abcdef$ and hence an Hadamard matrix of the block form

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

COROLLARY 13. *Suppose there are Hadamard matrices of orders $4a$, $4b$, $4c$, $4d$, $4e$, $4f$; then there is a complex Hadamard matrix of order $32abcdef$.*

Proof. Use the first four Hadamard matrices to obtain complex Hadamard matrices of orders $4ab$ and $4cd$. Now apply Theorem 12. ■

It is worth noting that at present we do not know how to obtain a complex Hadamard matrix of order $16abcdef$ from complex Hadamard matrices of orders $2a$, $2b$, $2c$, $2d$, $2e$, $2f$. Finally, we point out the following very general method of constructing (complex) orthogonal designs.

THEOREM 13 (Two Hadamard Matrices, a (Complex) Weighing Matrix and a (Complex) Orthogonal Design). *Suppose there exist Hadamard matrices of orders $4a$ and $4b$, a (complex) $W(2c, d)$ C and a (complex) $OD(2e; \mathbf{s})$ D ; then there exists a (complex) $OD(8abce, abds)$ which is complex if either C or D is complex.*

Proof. Use the Hadamard matrices to obtain disjoint $W(4ab, 2ab)$ A and B ; then use Theorem 11.

15. CONCLUSION

We have shown the power of the strong Kronecker product and the usefulness of orthogonal pairs and disjoint weighing matrices in reducing the power of two in the multiplication of orthogonal matrices while concurrently increasing the structure.

We believe only a small advance is now needed to be able to multiply Hadamard matrices without increasing the power of two and perhaps also to obtain $OD(4h; h, h, h, h)$ from an Hadamard matrix of order $4h$.

ACKNOWLEDGMENT

We wish to thank Dr Rob Craigen for reading this paper and making many suggestions and improvements. Further work will elucidate the connections between his work and ours.

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