Amicable Hadamard matrices and amicable orthogonal designs

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Tables are given for the existence of amicable and skew-Hadamard matrices of orders $2^t q$, $t \geq 2$ an integer, q(odd)$\leq$2000. This gives further evidence to support the conjecture that "for every odd integer q there exists an integer t (dependent on q) so that there is a skew-Hadamard matrix of order $2^t q$."

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Amicable Hadamard matrices and amicable orthogonal designs

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Abstract

New constructions for amicable orthogonal designs are given. These new designs then give new amicable Hadamard matrices and new skew-Hadamard matrices. In particular we show that if p is the order of normalized amicable Hadamard matrices there are normalized amicable Hadamard matrices of order $(p-1)^u + 1, u > 0$ an odd integer.

Tables are given for the existence of amicable and skew-Hadamard matrices of orders $2^t q, t \geq 2$ an integer, q(odd)≤ 2000. This gives further evidence to support the conjecture that "for every odd integer q there exists an integer t (dependent on q) so that there is a skew-Hadamard matrix of order $2^t q$."

1 Introduction

An *orthogonal design of order n and type* $OD(n; u_1, ..., u_s)$ $(u_i > 0)$ on the commuting variables $x_1, ..., x_s$ is an $n \times n$ matrix A with entries from $\{0, x_1, ..., x_s\}$ such that

$$AA^T = \sum_{i=1}^{s} (u_i x_i^2) I_n.$$

In [2], where this was first defined and many examples and properties of such designs were investigated, it was shown that the numbers of variables, s, satisfies $s \leq \rho(n)$, where $\rho(n)$ (Radon's function) is defined as follows:

$$\text{if } n = 2^a b, \text{ where } b \text{ is odd and } a=4c+d, \text{ where } 0 \leq d \leq 4,$$
$$\text{then } \rho(n) = 8c + 2^d.$$

A powerful construction for Hadamard matrices in [17] showed that the existence of orthogonal designs in powers of two was of great import. W. Wolfe and D. Shapiro showed that the cases of the problem in powers of two are crucial to the understanding of the algebraic structure involved (see [16]).

All possible designs exist in order 2, 4, and 8. The existence problem for order 16 was solved in [7] and in [6] many designs were constructed for order 32.

M and N of order n are said to be *amicable orthogonal designs* of type $AOD(n; m_1, ..., m_p); (n_1, ..., n_q))$ if M is an $OD(n; m_1, ..., m_p)$, N is an *orthogonal design* $OD(n; n_1, ..., n_q)$ and $MN^T = NM^T$. If M comprises the variables

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$x_1, \ldots, x_p$ and N comprises the variables $y_1, \ldots, y_q$ then

$$MM^T = \sum_{i=1}^{p} m_i x_i^2 I_n, \quad NN^T = \sum_{j=1}^{q} n_j y_j^2 I_n$$

and

$$ZZ^T = (m_1 x_1^2 + \ldots + m_p x_p^2)(n_1 y_1^2 + \ldots + n_q y_q^2) I_n$$

where $Z = MN^T$. So amicability is linked with factorizing quadratic forms.

Wolfe and Shapiro (see [3]) have studied and solved the algebraic necessary conditions for amicable orthogonal designs but the sufficiency conditions are largely unresolved (see [6, 3, 11] for partial results).

An *Hadamard matrix*, H, is an orthogonal design of order n and type (n) or alternatively, a matrix with entries ± 1 satisfying $HH^T = nI_n$. H is said to be *skew-Hadamard* if $H + I$ or $H - I$ is skew-symmetric. Two Hadamard matrices $H = M + I$ and N or order n are called *amicable Hadamard matrices* if $M^T = -M$, $N^T = N$, $HN^T = NH^T$. It is shown in [3] that amicable orthogonal designs $AOD(n; (1, n-1), (n))$ give amicable Hadamard matrices (they are not the same since the orthogonal designs have no symmetry or skew symmetry conditions). *Normalized amicable Hadamard matrices* of order h can be written in the form

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ - & & & \\ \vdots & & I+S & \\ - & & & \end{bmatrix}, \qquad N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & P+R & \\ 1 & & & \end{bmatrix}$$

where

$$S^T = -S,\ P^T = P,\ R^T = R,\ PR^T + RP^T = 0,\ RR^T = I,\ SJ = PJ = 0$$

$$RJ = -J,\ SP^T = PS^T,\ SR^T = RS^T,\ SS^T = PP^T = (h-1)I - J$$

A *weighing matrix* $W(n, n-1)$ is an orthogonal design of order n and type $(n-1)$.

Amicable orthogonal designs, amicable Hadamard matrices, and skew-Hadamard matrices have proved difficult to find. The Kronecker produce of skew-Hadamard matrices is not a skew-Hadamard matrix. But if h_1 and h_2 are the orders of amicable Hadamard matrices then there are amicable Hadamard matrices of order $h_1 h_2$; further, if g is the order of a skew-Hadamard matrix there are skew-Hadamard matrices of orders $h_1 g$ and $h_2 g$. We list from [3] and [12] the orders for which Hadamard matrices are known.

Summary 1

AI	2^t	t a non-negative integer
AII	$p^r + 1$	p^r (prime power) $\equiv 3 \pmod 4$.
AIII	$(p-1)^u + 1$	p the order of normalized amicable Hadamard matrices, there are normalized amicable Hadamard matrices of order $(p-1)^u + 1, u > 0$ an odd integer
AIV	$2(q+1)$	$2q+1$ is a prime power, q (prime) $\equiv 1 \pmod 4$.
AV	$(\lvert t\rvert + 1)(q+1)$	q (prime power) $\equiv 5 \pmod 8$ $= s^2 + 4t^2$, $s \equiv 1 \pmod 4$, and $\lvert t\rvert + 1$ is the order of amicable orthogonal designs of type $AOD(1 + \lvert t\rvert; (1, \lvert t\rvert); (\frac{1}{2}(\lvert t\rvert + 1), \frac{1}{2}(\lvert t\rvert + 1))$.
	$2^r(q+1)$	q (prime power) $\equiv 5 \pmod 8$ $= s^2 + 4(2^r - 1)^2$, $s \equiv 1 \pmod 4$, r some integer.
AVI	S	where S is a product of the above orders.

Skew-Hadamard matrices are known for the following orders (the reader should consult [18, p451] for more details):

Summary 2

SI	$2^t \Pi k_i$	t, r_i all non-negative positive integers, $k_i - 1 \equiv 3 \pmod 4$ a prime power.
SII	$(p-1)^u + 1$	p the order of a skew-Hadamard matrix, $u > 0$ an odd integer.
SIII	$2(q+1)$	$q \equiv 5 \pmod 8$ a prime power.
SIV	$2(q+1)$	$q = p^t$ is a prime power with $p \equiv 5 \pmod 8$ and $t \equiv 2 \pmod 4$.
	$4(q+1)$	$q \equiv 9 \pmod{16}$ a prime power.
SV	$4m$	$m \in \{$ odd integers between 3 and 31, inclusive$\}$.
SVI	$mn(n-1)$	n the order of amicable orthogonal designs $AOD(n; (1, n-1); (n))$, nm the order of an orthogonal design; $OD(mn; 1, m, mn - m - 1)$; Theorem 7 of [10].
SVII	$(\lvert t\rvert + 1)(q+1)$	$q = s^2 + 4t^2 \equiv 5 \pmod 8$ a prime power and $\lvert t\rvert + 1$ the order of a skew-Hadamard matrix; from [19].

SVIII	$4(q^2+q+1)$	q a prime power and $q^2+q+1 \equiv 3,5$ or 7 (mod 8) a prime or $2(q^2+q+1)+1$ a prime power; see [13] or use Lemma 1 of [10].
SIX	$2^t q$	$q = s^2+4r^2 \equiv 5$ (mod 8) a prime power and all orthogonal designs of type $OD(2^t;1,a,b,c,c+\lvert r\rvert)$, where $1+a+b+2c+\lvert r\rvert = 2^t$, exist in 2^t; from [10].
	$2^t q$	$q \equiv 5$ (mod 8) a prime power, $\frac{1}{2}(q+1)$ prime, $t \geq [2\log_2(p-2)]$, from [10].
SX	hm	h the order of a skew-Hadamard matrix, m the the order of amicable Hadamard matrices.

2 New Results

First we give some definitions. Let Y of order n be an $OD(n; u_1, ..., u_s)$ then we say the s matrices $Y_i, i = 1, ..., s$ of order m are *suitable* ± 1 *matrices for the orthogonal design* Y or *suitable matrices* for Y if

(i) the elements of each Y_i are ± 1,

(ii) the Y_i are pairwise amicable so $Y_i Y_j^T = Y_j Y_i^T$, $i, j = 1, ..., s$,

(iii) $\sum_{i=1}^{s} Y_i Y_i^T = nmI$.

The matrices are said to be *near suitable* ± 1 *matrices for the orthogonal design* Y or *near suitable* ± 1 *matrices* for Y if (i) and (ii) hold but also

(iiia) $\sum_{i=1}^{s} Y_i Y_i^T = n(m+1)I - nJ$.

Similarly let $X = X(x_1, ..., x_t)$ and $Y = Y(y_1, ..., y_s)$ be amicable orthogonal designs $AOD(n; (u_1, ..., u_t); (v_1, ..., v_s))$ of order n. Then we say the t matrices $X_i, i = 1, ..., t$ of order m and the s matrices $Y_j, j = 1, ..., s$ of order m are *suitable* ± 1 *matrices for the amicable orthogonal designs* X and Y if

(i) the elements of each X_i and Y_j are ± 1,

(ii) the X_i and Y_j are all pairwise amicable so

$$X_i X_k^T = X_k X_i^T,\ Y_j Y_l^T = Y_l Y_j^T,\ X_i Y_j^T = Y_j X_i^T,\quad i, k = 1, ..., t,\ j, l = 1, ..., s,$$

(iii) $\sum_{i=1}^{t} X_i X_i^T = \sum_{j=1}^{s} Y_j Y_j^T = mnI$.

The matrices are said to be *near suitable* ± 1 *matrices for the amicable orthogonal designs* X and Y or *near suitable* ± 1 *matrices* for X and Y if (i) and (ii) hold but also

(iiia) $\sum_{i=1}^{t} X_i X_i^T = \sum_{j=1}^{s} Y_j Y_j^T = n(m+1)I - nJ.$

Now we have

Theorem 1 *Let* $X = X(x_1, ..., x_t)$ *and* $Y = Y(y_1, ..., y_s)$ *be amicable orthogonal designs* $AOD(n; (u_1, ..., u_t); (v_1 = 1, ..., v_s))$ *of order* n. *Suppose there are* n *near suitable* $1, -1$ *matrices of order* m, $X_1, ..., X_t$ *and* $Y_1, ..., Y_s$ *for* X *and* Y *satisfying*

(i) $eX_i^T = eY_i^T = e, \quad i = 1, ..., t,\ j = 1, ..., s,$

(ii) Y_1 *is a skew-type and* $Y_2, ..., Y_s, X_1, ..., X_t$ *are symmetric.*

Then

$$U = \begin{pmatrix} X(1,1,...,1) & X(e,e,...,e) \\ X(e^T, e^T, ..., e^T) & -X(X_1, X_2, ..., X_t) \end{pmatrix},$$

$$V = \begin{pmatrix} Y(1,1,...,1) & X(e,e,...,e) \\ -X(e^T, e^T, ..., e^T)^T & Y(Y_1, Y_2, ..., Y_t)^T \end{pmatrix},$$

are amicable Hadamard matrices of order $n(m+1)$. *If* $Y_1 = yI + Z$, *where* $eZ = 0$ *and* y *is a variable and* $v_1 = 1$,

$$U = \begin{bmatrix} X(1,1,...,1) & X(e,e,...,e) \\ X(e^T, e^T, ..., e^T) & -X(X_1, X_2, ..., X_t) \end{bmatrix},$$

$$V = \begin{bmatrix} Y(y,1,...,1) & X(e,e,...,e) \\ -X(e^T, e^T, ..., e^T)^T & Y(yI + Z, Y_2, ..., Y_t)^T \end{bmatrix},$$

are amicable orthogonal designs $AOD(nm+n; (nm+n); (1, nm+n-1))$.

Proof: By straightforward verification. □

Example:

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_3 \\ x_2 & -x_1 & x_3 & -x_3 \\ x_3 & x_3 & -x_2 & -x_1 \\ x_3 & -x_3 & -x_1 & x_2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 & y_2 & y_3 & y_3 \\ -y_2 & y_1 & y_3 & -y_3 \\ -y_3 & -y_3 & y_1 & y_2 \\ -y_3 & y_3 & -y_2 & y_1 \end{pmatrix}$$

are $AOD(4; (1,1,2); (1,1,2))$. Then

$$\begin{bmatrix} 1 & 1 & 1 & 1 & e & e & e & e \\ 1 & - & 1 & - & e & -e & e & -e \\ 1 & 1 & - & - & e & e & -e & -e \\ 1 & - & - & 1 & e & -e & -e & e \\ e^T & e^T & e^T & e^T & -X_1 & -X_2 & -X_3 & -X_3 \\ e^T & -e^T & e^T & -e^T & -X_2 & X_1 & -X_3 & X_3 \\ e^T & e^T & -e^T & -e^T & -X_3 & -X_3 & X_2 & X_1 \\ e^T & -e^T & -e^T & e^T & -X_3 & X_3 & X_1 & -X_2 \end{bmatrix}$$

$$\begin{bmatrix} y & 1 & 1 & 1 & e & e & e & e \\ - & y & 1 & - & e & -e & e & -e \\ - & - & y & 1 & e & e & -e & -e \\ - & 1 & - & y & e & -e & -e & e \\ -e^T & -e^T & -e^T & -e^T & yI+Y_1 & Y_2 & Y_3 & Y_3 \\ -e^T & e^T & -e^T & e^T & -Y_2 & yI+Y_1 & Y_3 & -Y_3 \\ -e^T & -e^T & e^T & e^T & -Y_3 & -Y_3 & yI+Y_1 & Y_2 \\ -e^T & e^T & e^T & -e^T & -Y_3 & Y_3 & -Y_2 & yI+Y_1 \end{bmatrix}$$

are $AOD(4(p+1);(4p);(1,4p-1))$ whenever near suitable X_i, $i = 1,2,3$. and Y_j, $j = 1,2,3$ of order p exist.

This is now stated as a corollary.

Corollary 2 *If there exist near suitable matrices $X_1, X_2, X_3, Y_1, Y_2, Y_3$ of order s satisfying*

(i) $X_iX_j^T = X_jX_i^T$, $\quad i \neq j,\ i,j = 1,2,3,4$

(ii) $Y_iY_j^T = Y_jY_i^T$, $\quad i \neq j,\ i,j = 1,2,3,4$

(iii) $X_1Y_1^T + X_2X_2^T + 2X_3X_3^T = Y_1Y_1^T + Y_2Y_2^T + 2Y_3Y_3^T = 4(s+1)I - 4J$
(or with y and z variables,
$Y_1Y_1^T + z^2Y_2Y_2^T + 2z^2Y_3Y_3^T = (y^2 + (4s+3)z^2)I - 4z^2J)$
(or $Y_1 = yI + zZ$, $eZ = 0$, $Z^T = -Z$),

(iv) $eX_i^T = eY_i^T = e$, $\quad i = 1,2,3,4$,

(v) $X_iY_j^T = Y_jX_i^T$, $\quad i,j = 1,2,3,4$,

(vi) Y_1 is a skew-type and other matrices are symmetric.

Then there exist amicable Hadamard matrices of order $4(s+1)$ and amicable orthogonal designs $AOD(4s+4;(4s+4);(1,4s+3))$.

Also we note

Corollary 3 *Suppose $p \equiv 5 \pmod 8 = s^2 + 36$ is a prime power then there exist $AOD(4p+4;(2p+2,2p+2),(1,4p+3))$.*

Proof: From J. Wallis [19] we have four ±1 matrices M and N where $(M-I)^T = M-I$, $N^T = N$,

$$MM^T + 3NN^T = (4p+4)I - 4J.$$

Choose $Y_1 = yI + z(M-I)$, $Y_2 = Y_3 = Y_4 = NR$ (the type two or back-circulant matrix from N). Let U and V be the ±1 incidence matrices of the quadratic residues and quadratic non-residues so

$$eU = eV = e,\ U^T = U^T,\ V^T = V,$$

$$UU^T + VV^T = 2(p+1)I - 2J.$$

Choose $X_1 = X_2 = xUR$, $X_3 = wVR$ (type two or back-circulant matrices) and the $OD(4p+4; 2p+2, 2p+2)$ as

$$\begin{bmatrix} x & x & w & w & xe & xe & we & we \\ x & -x & w & -w & xe & -xe & we & -we \\ w & w & -x & -x & we & we & -xe & -xe \\ w & -w & -x & x & we & -we & -xe & xe \\ xe^T & xe^T & we^T & we^T & -X_1 & -X_1 & -X_3 & -X_3 \\ xe^T & -xe^T & we^T & -we^T & -X_1 & X_1 & -X_3 & X_3 \\ we^T & we^T & -xe^T & -xe^T & -X_3 & -X_3 & X_1 & X_1 \\ we^T & -we^T & -xe^T & xe^T & -X_3 & X_3 & X_1 & -X_1 \end{bmatrix}$$

to obtain the result. □

Lemma 4 *If there exist suitable matrices $X_1, X_2, X_3, Y_1, Y_2, Y_3$, of order s satisfying*

$$X_1X_1^T + X_2X_2^T + 2X_3X_3^T = Y_1Y_1^T + Y_2Y_2^T + 2Y_3Y_3^T = 4sI,$$

then there are amicable Hadamard matrices of order $4s$. If $Y_1 = yI + zZ$ then there exist orthogonal designs $AOD(4s; (4s); (1, 4s-1))$.

Proof: Use the matrices X and Y of the Example. □

If we have good matrices $X_1, X_2, X_3 = X_4$ and circulant Williamson matrices $Y_1, Y_2, Y_3 = Y_4$ then Lemma 4 will be satisfied. This is certainly true for $s = 3.5$. for example:

$s = 3$ use the circulant matrices with first rows
$X_1 = 1-1$, $X_2 = 111$, $X_3 = 1--$,
$Y_1 = 111$, $Y_2 = Y_3 = 1--$

$s = 5$ $X_1 = 11-1-$, $X_2 = 11--1$, $X_3 = 1----$,
$Y_1 = 1-11-$, $Y_2 = 11--1$, $Y_3 = 1----$.

We now give a slightly different construction which gives useful amicable orthogonal designs.

3 A powering theorem

Theorem 5 *If there exists a skew-Hadamard matrix of order $p+1$ with circulant or type one core there exist amicable Hadamard matrices of order p^u+1 and $AOD(p^u+1;(1,p^u),(1,p^u))$ for odd integers $u>0$.*

Proof: We illustrate for $u=3$. Let

$$H = \begin{bmatrix} 1 & e \\ -e^T & I+B \end{bmatrix}$$

be the skew-Hadamard matrix of order $p+1$, with B the circulant or type one core. So

$$BB^T = pI - J, \quad JB = 0, \quad B^T = B.$$

Now let $A = BR$ be the back-circulant or type two matrix obtained from H which satisfies

$$AA^T = pI - J, \; JA = 0, \; A^T = A, \; AB^T = BA^T, \quad A \pm R \text{ is } \pm 1.$$

Define

$$W_B = B \times B \times B + B \times I \times J + I \times J \times B + J \times B \times I$$

and

$$W_A = A \times A \times A + A \times R \times J + R \times J \times A + J \times A \times R$$

then

$$\begin{aligned} W_B W_B^T &= p^3 I - J,\; JW_B = 0,\; W_B^T = -W_B,\; W_B(R\times R\times R) = (R\times R\times R)W_B^T, \\ W_A W_A^T &= p^3 I - J,\; JW_A = 0,\; W_A^T = W_A, \end{aligned}$$

and

$$W_A W_B^T = AB^T \times AB^T \times AB^T + AB^T \times R \times J^2 + R \times J^2 \times AB^T + J^2 \times AB^T \times R = W_B W_A^T.$$

Writing x, y, z, w as variables

$$\begin{bmatrix} x & ey \\ -e^T y & xI + yW_B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -z & ew \\ e^T w & zR\times R\times R + wW_A \end{bmatrix}$$

are the required AOD from which the amicable Hadamard matrices are constructed by choosing $x=y=z=w=1$.

The method works for other u by choosing W_B with $B \times B \times \cdots B$ (u B's) and then replacing pairs $B\times B$ by $I \times J$, always keeping an odd number of B's. (This construction is discussed more fully in [18, pp309-312].) □

The first example of interest is 15^3+1 which does not arise in other ways.

Corollary 6 *If there exist normalized amicable Hadamard matrices of order $p+1$ there exist normalized amicable matrices of order p^u+1, u odd >0.*

4 Existence of AOD

In this section we further explore the existence of $AOD(2s+2;(1,2s+1),(s+1,s+1))$.

Theorem 7 *Let $p = 4m+3$ be a prime power and $q = 2m+1 \equiv 1 \pmod 4$ a prime. Then there exist amicable orthogonal designs $AOD(2(q+1);(1,2q+1),(q+1,q+1))$, and amicable Hadamard matrices of order $2q+2$.*

Proof: Form the $2-\{2m+1;m;m-1\}$ Szekeres difference sets X and Y of size m in the cyclic group of order $q = 2m+1$. Now X and Y have the property that $x \in X \Rightarrow -x \notin X$ and $y \in Y \Rightarrow -y \in Y$. Let M be the $(0,1,-1)$ incidence matrix of X with zero diagonal, and let N, the $(1,-1)$ incidence matrix of Y, be a diagonal of ones, both of order q. Then

$$MM^T + NN^T = (2q+1)I - 2J,\ eM = 0,\ eN = e,\ M^T = -M,\ N^T = N.$$

Let Q be the $(0,1,-1)$ incidence matrix of the quadratic residues of q. Then

$$QQ^T = qI - J,\ \ eQ = 0,\ \ Q^T = Q.$$

Now we let a, b, c and d be commuting variables and e the $1 \times q$ matrix of ones. Then, with R the back-diagonal matrix,

$$A = \left[\begin{array}{cc|cc} a & b & be & be \\ -b & a & be & -be \\ \hline -be^T & -be^T & aI + bM & bNR \\ -be^T & be^T & -bNR & aI + bM \end{array}\right]$$

and

$$C = \left[\begin{array}{cc|cc} d & c & ce & de \\ c & -d & -de & ce \\ \hline ce^T & -de^T & (dI + cQ)R & (-cI + dQ)R \\ de^T & ce^T & (-cI + dQ)R & (-dI - cQ)R \end{array}\right]$$

are two $AOD(2q+2);(1,2q+1),(q+1,q+1))$. □

Corollary 8 *$AOD(2q+2;(1,2q+1),(q+1,q+1))$ and amicable Hadamard matrices of order $2q+2$ exist as follows:*

(i) $AOD(12;(1,11),(6,6))$,

(ii) $AOD(28;(1,27),(14,14))$,

(iii) $AOD(60;(1,59),(30,30))$,

(iv) $AOD(84;(1,83),(42,42))$,

(v) $AOD(108;(1,107),(54,54))$.

These give class AIV of the list.

Theorem 9 *Suppose amicable Hadamard matrices of order s exist. Further suppose $AOD(2p;(1,2p-1),(p,p)$ exist. Then $AOD(2ps;(1,2ps-1),(ps,ps))$ exist. In particular $AOD(2s;(1,2s-1),(s,s))$ exist.*

Proof: Let $I+S$ and P be the amicable Hadamard matrices, and $xA+yB, zC+wD$ be the $OD(1,2p-1)$ and $OD(2p;p,p)$ which are amicable x,y,z,w commuting variables. Then

$$(xI+yS)\times A+yP\times B \quad \text{and} \quad zP\times C+wP\times D$$

are the required AOD.

The second result follows by choosing $p=1$. □

In [3, Corollary 5.50, 5.52 and 5.56] it is shown that the following corollary holds.

Corollary 10 *$AOD(2^{t+1};(1,2^{t+1}-1),(2^t,2^t))$ exist for all $t\geq 0$. $AOD(p+1;(1,p),(1,p))$, $p\equiv 3 \pmod 4$ a prime power exist. $AOD(2p;(2,2p-2),(2,2p-2))$, p the order of a symmetric conference matrix exist.*

We also have

Theorem 11 *Suppose there exist $AOD(2h;(h,h),(1,2h-1))$. Let $p=s^2+4t^2\equiv 5 \pmod 8$, $2h=|t|+1$, be a prime power. Then there exist $AOD(2h(p+1);(h(p+1),h(p+1)),(1,2hp+2h-1))$.*

Proof: We use Theorem 1 with $X(2h;h,h)$ the OD on the variables x,y and $Y(2h;1,2h-1)$ the OD on the variables z and w. Let X_1,X_2,Y_1,Y_2 of Theorem 1 be $-xX_1, -yX_2$, $zI+w(Y_1-I), -zY_2$ as given in Lemma 13. □

Thus we have

Corollary 12 *There exist*

(i) $AOD(4(p+1);(2p+2,2p+2),(1,4p+3))$ *for* $p=s^2+36\equiv 5 \pmod 8$ *a prime power,*

(ii) $AOD(8(p+1);(4p+2,4p+2),(1,8p+7))$ *for* $p=s^2+188\equiv 4 \pmod 8$ *a prime power,*

(iii) $AOD(16(p+1);(8p+8,8p+8),(1,16p+15))$ *for* $p=s^2+900\equiv 5 \pmod 8$ *a prime power.*

5 The existence of sets of near suitable matrices

We explore the existence of four $(1,-1)$ matrices Y_1 (circulant or type one and skew-type), Y_2,X_1,X_2 (circulant or type one and symmetric) of order q which satisfy

$$\begin{aligned} Y_1Y_1^T+(2u+1)(Y_2R)(Y_2R)^T &= (u+1)(X_1R)(X_1R)^T+(u+1)(X_2R)(X_2R)^T \\ &= (2u+2)(q+1)I-(2u+2)J \end{aligned}$$

where R is the back-diagonal matrix.

Lemma 13 *Let* $p = 4f + 1 = s^2 + 4t^2 \equiv 5 \pmod 8$ *be a prime power. Then there are* $|t| + 1$ *near suitable matrices of order* p.

Proof: Now from [19, Lemma 24] we have that if $p = 4f + 1 = s^2 + 4t^2 \equiv 5 \pmod 8$ is a prime power then the ± 1 incidence matrices of the sets

$$C_0 \cup C_1 \quad \text{and} \quad |t| \text{ copies of } C_0 \cup C_2$$

will give near suitable $(1, -1)$ matrices for Y_1 and Y_2 satisfying

$$Y_1Y_1^T + |t|Y_2Y_2^T = (|t| + 1)(p + 1)I - (|t| + 1)J.$$

Furthermore the ± 1 incidence matrices of

$$\frac{1}{2}(|t| + 1) \text{ copies of } C_0 \cup C_2 \quad \text{and} \quad \frac{1}{2}(|t| + 1) \text{ copies of } C_1 \cup C_3$$

will give near suitable $(1, -1)$ matrices X_1 and X_2 satisfying

$$\frac{1}{2}(|t| + 1)[X_1X_1^T + X_2Y_2^T] = (|t| + 1)(p + 1)I - (|t| + 1)J.$$

Note Y_2, X_1, X_2 are circulant (type one) and symmetric but we use their back-circulant (type two) form. Y_1 is skew-type circulant (type one). Hence X_1, X_2, Y_1, Y_2 are the required matrices. □

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6 Tables of Amicable Hadamard Matrices

q	t	q	t	q	t	q	t	q	t
1	2	101	.	201	3	301	5	401	.
3	2	103	3	203	2	303	3	403	6
5	2	105	2	205	4	305	5	405	2
7	2	107	.	207	2	307	.	407	2
9	3	109	9	209	4	309	4	409	3
11	2	111	2	211	4	311	.	411	4
13	3	113	8	213	4	313	3	413	4
15	2	115	2	215	2	315	4	415	3
17	2	117	2	217	5	317	6	417	2
19	3	119	4	219	7	319	3	419	4
21	2	121	3	221	2	321	2	421	7
23	4	123	2	223	3	323	2	423	4
25	3	125	2	225	4	325	5	425	2
27	2	127	.	227	2	327	2	427	4
29	4	129	3	229	3	329	6	429	4
31	3	131	2	231	3	331	3	431	2
33	2	133	3	233	4	333	3	433	3
35	2	135	4	235	3	335	7	435	4
37	.	137	2	237	2	337	.	437	2
39	3	139	4	239	4	339	3	439	3
41	2	141	2	241	.	341	5	441	3
43	3	143	2	243	2	343	6	443	6
45	2	145	5	245	4	345	4	445	3
47	4	147	2	247	6	347	.	447	2
49	4	149	4	249	4	349	3	449	.
51	4	151	5	251	6	351	4	451	3
53	2	153	3	253	6	353	4	453	2
55	3	155	2	255	2	355	4	455	5
57	2	157	5	257	4	357	2	457	.
59	.	159	4	259	5	359	4	459	3
61	3	161	2	261	3	361	3	461	.
63	2	163	3	263	2	363	2	463	7
65	4	165	2	265	4	365	2	465	3
67	5	167	4	267	4	367	2	467	2
69	4	169	5	269	8	369	4	469	7
71	2	171	2	271	7	371	2	471	3
73	7	173	2	273	2	373	7	473	5
75	3	175	3	275	5	375	2	475	4
77	2	177	.	277	5	377	7	477	2
79	3	179	8	279	4	379	.	479	.
81	3	181	3	281	2	381	2	481	3
83	2	183	4	283	.	383	2	483	2
85	4	185	2	285	4	385	3	485	4
87	2	187	4	287	4	387	5	487	5
89	4	189	3	289	3	389	.	489	3
91	3	191	.	291	2	391	5	491	.
93	3	193	3	293	2	393	2	493	3
95	2	195	3	295	5	395	2	495	2
97	9	197	2	297	2	397	5	497	2
99	4	199	3	299	4	399	3	499	3

q	t	q	t	q	t	q	t	q	t
501	2	601	5	701	2	801	2	901	3
503	2	603	2	703	3	803	9	903	4
505	.	605	4	705	2	805	4	905	4
507	2	607	5	707	6	807	4	907	5
509	.	609	3	709	.	809	.	909	4
511	5	611	6	711	2	811	5	911	2
513	4	613	3	713	2	813	2	913	4
515	5	615	2	715	4	815	2	915	2
517	6	617	2	717	4	817	6	917	4
519	4	619	3	719	4	819	3	919	3
521	2	621	3	721	5	821	6	921	6
523	7	623	4	723	3	823	.	923	2
525	2	625	3	725	6	825	2	925	4
527	4	627	4	727	.	827	2	927	4
529	3	629	.	729	4	829	.	929	.
531	7	631	.	731	5	831	2	931	5
533	2	633	2	733	.	833	2	933	.
535	2	635	2	735	2	835	3	935	2
537	5	637	5	737	7	837	2	937	5
539	4	639	4	739	.	839	.	939	4
541	3	641	6	741	2	841	8	941	.
543	5	643	.	743	2	843	2	943	6
545	2	645	2	745	6	845	7	945	2
547	2	647	.	747	4	847	4	947	6
549	3	649	7	749	.	849	3	949	3
551	2	651	5	751	3	851	.	951	2
553	3	653	2	753	2	853	3	953	.
555	4	655	4	755	2	855	4	955	3
557	.	657	5	757	5	857	4	957	5
559	5	659	.	759	4	859	3	959	4
561	2	661	.	761	.	861	4	961	3
563	2	663	3	763	11	863	4	963	.
565	3	665	2	765	3	865	4	965	4
567	2	667	8	767	2	867	2	967	.
569	4	669	3	769	3	869	4	969	4
571	3	671	2	771	5	871	3	971	6
573	3	673	7	773	.	873	2	973	6
575	4	675	2	775	3	875	2	975	5
577	.	677	2	777	4	877	.	977	2
579	5	679	3	779	5	879	4	979	5
581	4	681	4	781	3	881	6	981	2
583	3	683	2	783	3	883	.	983	2
585	2	685	3	785	7	885	2	985	4
587	2	687	2	787	5	887	2	987	2
589	6	689	5	789	3	889	5	989	4
591	4	691	3	791	2	891	3	991	3
593	2	693	4	793	3	893	2	993	4
595	3	695	4	795	3	895	3	995	4
597	4	697	4	797	2	897	5	997	.
599	.	699	3	799	6	899	7	999	5

Orders for which amicable Hadamard matrices exist.

7 Tables of orders for which skew-Hadamard matrices exist

q	t	q	t	q	t	q	t	q	t
1	2	101	10	201	3	301	3	401	.
3	2	103	3	203	2	303	3	403	5
5	2	105	2	205	3	305	4	405	2
7	2	107	.	207	2	307	2	407	2
9	2	109	9	209	4	309	3	409	3
11	2	111	2	211	2	311	.	411	2
13	2	113	8	213	4	313	2	413	4
15	2	115	2	215	2	315	2	415	2
17	2	117	2	217	4	317	6	417	2
19	2	119	4	219	4	319	3	419	4
21	2	121	3	221	2	321	2	421	2
23	2	123	2	223	3	323	2	423	4
25	2	125	2	225	4	325	5	425	2
27	2	127	7	227	2	327	2	427	2
29	2	129	3	229	3	329	6	429	3
31	2	131	2	231	2	331	3	431	2
33	2	133	3	233	4	333	2	433	3
35	2	135	2	235	3	335	7	435	4
37	3	137	2	237	2	337	18	437	2
39	3	139	2	239	4	339	2	439	2
41	2	141	2	241	.	341	4	441	3
43	3	143	2	243	2	343	6	443	6
45	2	145	5	245	4	345	4	445	3
47	4	147	2	247	6	347	.	447	2
49	4	149	4	249	4	349	3	449	.
51	2	151	5	251	6	351	2	451	3
53	2	153	3	253	4	353	4	453	2
55	2	155	2	255	2	355	2	455	4
57	2	157	3	257	4	357	2	457	.
59	.	159	2	259	5	359	4	459	3
61	2	161	2	261	3	361	3	461	17
63	2	163	3	263	2	363	2	463	7
65	4	165	2	265	4	365	2	465	3
67	5	167	4	267	4	367	2	467	2
69	3	169	5	269	8	369	4	469	3
71	2	171	2	271	2	371	2	471	2
73	2	173	2	273	2	373	7	473	5
75	2	175	2	275	4	375	2	475	4
77	2	177	.	277	5	377	6	477	2
79	2	179	8	279	2	379	2	479	.
81	3	181	3	281	2	381	2	481	3
83	2	183	2	283	.	383	2	483	2
85	2	185	2	285	3	385	3	485	4
87	2	187	2	287	4	387	2	487	5
89	4	189	2	289	3	389	15	489	3
91	2	191	.	291	2	391	4	491	.
93	3	193	3	293	2	393	2	493	3
95	2	195	2	295	5	395	2	495	2
97	9	197	2	297	2	397	5	497	2
99	2	199	2	299	4	399	2	499	2

Orders for which skew-Hadamard matrices exist.